DELT UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF AEROSPACE ENGINEERING

Memorandum M-403

BUCKLING OF MULTI-BAY PANELS
WITH RIVETED Z-STRINGERS

by

A. van der Neut

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1. The special problem with riveted stringers

The overall buckling mode of Z-stiffened panels in compression has, together with the usual deflexion normal to the plate, sideways deflexion of the free stringer flange. The restraint against this latter component of the mode is being supplied by the stringer web, which at its base is restrained against rotation by the plate. This buckling problem has been analyzed in [1]. Numerical application to a structure representative of aircraft wing panels showed that the buckling stress is affected very much by the stiffness of restraint at the web base. With stringers bonded to the plate this stiffness is great. With riveted stringers the stiffness is much smaller and so is the buckling stress. Moreover in this range of stiffnesses the buckling stress is very sensitive to variations in stiffness of restraint. Therefore accurate knowledge of the stiffness of restraint is needed. Unfortunately the riveted joint is not well suited for accurate analysis; more or less doubtful assumptions have to be made.

The analysis of [1] assumes that the local joints around rivets may be replaced by a continuous joint along the rivet line, where the deflexion and rotation of plate and stringer flange are equal. No other contact between plate and stringer occurs. With the numerical example of [1] the bonded joint has the restraint coefficient $\alpha = 0.30$ and yields the buckling coefficient $k = 0.75$. The riveted joint, behaving as assumed, has $\alpha = 2.3$ and yields the much smaller coefficient $k = 0.59$. This small value of $\alpha$ is mainly due to the deformation of the stringer flange between rivet line and web root. However, when the direction of rotation of the web root is such that the flange would penetrate into the plate, the assumption on the kind of contact between flange and plate must be supplemented with the assumption of equal deflexion of flange and plate at the web root. Then, as established by [1], with the numerical example $\alpha = 6.4$, which yields $k = 0.69$.

With multi-bay panels the deflexions of successive bays are alternating; a bay with plate-flange contact only at the rivet line has as its neighbours bays with in addition contact at the web root. The restraint coefficients with the chosen structure are alternatingly 2.3 and 6.4. In spite of this inequality of restraint the several bays buckle simultaneously, which is being accomplished by unequal half-wave
lengths. Bays, indicated with \( A \), where \( \alpha_A = 2.3 \) have half-wave length \( \lambda_A < L \) (frame spacing) and bays \( B \), where \( \alpha_B = 6.4 \), have \( \lambda_B > L \). It means that the points of inflexion move from the plane of the frames into the bays with smallest restraint. Of course \( \frac{\lambda_A}{\lambda_B} = 2 \), hence with \( n = L/\lambda \) \( n_A \) and \( n_B \) are related by

\[
\frac{1}{n_A} + \frac{1}{n_B} = 2.
\]  

(1)

The value of \( n_A \) at which \( k_A = k_B \) has to be established.

The implicit assumption that at the transition of the unequal half waves compatibility and equilibrium conditions are satisfied will be discussed in section 3.

The numerical procedure is as follows.

1. Assume a value of \( n_A > 1 \), together with some values of \( k \) in the range \( (k_A)_{n=1} < k < (k_B)_{n=1} \) and solve for \( \alpha \) pertaining to \( n_A \) and \( k \) as described in [1] section 4.

2. Establish by interpolation \( k \) at \( \alpha_A = 2.3 \).

3. Solve \( \alpha_B \) from \( n_B = (2 - \frac{1}{n_A})^{-1} \), and \( k \) just established.

4. Repeat the former operations for other values of \( n_A \), yielding \( k \) as function of \( \alpha_B \) and interpolate \( k \) for \( \alpha_B = 6.4 \).

Results are given in the tables, presented in the sequence used in the actual calculations.

<table>
<thead>
<tr>
<th>( n_A ) = 1.1</th>
<th>( n_A ) = 1.2</th>
<th>( n_A ) = 1.174</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>( \alpha_A )</td>
<td>( k )</td>
</tr>
<tr>
<td>0.61</td>
<td>2.45</td>
<td>0.61</td>
</tr>
<tr>
<td>0.63</td>
<td>2.78</td>
<td>0.63</td>
</tr>
<tr>
<td>0.65</td>
<td>3.25</td>
<td>0.65</td>
</tr>
<tr>
<td>0.60</td>
<td>2.29</td>
<td>0.603</td>
</tr>
</tbody>
</table>

Interpolation to \( \alpha_A = 2.3 \) yields

<table>
<thead>
<tr>
<th>( n_A )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>0.6012</td>
</tr>
<tr>
<td>1.2</td>
<td>0.6027</td>
</tr>
<tr>
<td>1.174</td>
<td>0.6023</td>
</tr>
</tbody>
</table>

In view of the apparent insensitivity of \( k \) for variation of \( n_A \) the calculation for \( n_A = 1.16 \) has been skipped. In order to use nevertheless \( n_A = 1.16 \) for establishing \( \alpha_B \) the value of \( k \) for \( n_A = 1.16 \) has been derived by interpolation, \( k = 0.6020 \). These four values of \( n_A \) yield \( \alpha_B \) given in the following table.
2. **Buckling coefficient k as function of wave length**

One would have expected that k would be somewhere halfway between 0.59 and 0.69, the values of k for \( n_A = n_B = 1 \). However, k appears to be very close to the lower bound of this range. The reason is that \( k_A \) is almost constant over a range of \( n_A \) around \( n_A = 1.17 \). This peculiar behaviour of \( k(n) \) asks for explanation. To this end \( k(n) \) has been established over the wider range \( 0.7 < n < 2 \), both for \( \alpha = 2.3 \) and \( \alpha = 6.4 \) (Figure 1).

It appears that at the lower end \( n = 0.7 \) (large wave length) k is almost equal for the two values of \( \alpha \) and varies very much with n. The explanation of these facts appears from the ratio \(-V/W\) given in Figure 2 (V is the sideways deflexion of the stringer flange, W is the deflexion normal to the plate). With \( n \leq 0.7 \) \(-V/W < 0.3\) the sideways deflexion is almost negligible at \( n = 0.7 \) and vanishes with decreasing n. It means that the mode deviates negligibly little from the Euler mode where \( k = n^{-2} \). Allowing for shear deflexion

\[
k = \left[ \frac{1}{n^2} + \frac{1}{(k_S)_{v=0}} \right]^{-1}
\]

With \((k_S)_{v=0} = 7.32\) the dotted curve of Figure 1 gives this value of k. Indeed \( k(n) \) approaches this curve very closely for \( n < 0.7 \). Of course, when \( V \) is negligible, the amount of restraint \( \alpha \) affects k negligibly.

With \( n > 1.5 \) (small wave length) n affects k clearly. As appears from Figure 2 \(-V/W\) is about 3, which means that the mode component flange
buckling dominates over the normal deflexion \( W \). So the mode is characterized as buckling of a bar on continuous elastic foundation. This explains the increase of \( k \) with decreasing wave length; also that the effect of \( n \) on \( k \) is great.

These two different mode types for \( n < 0.7 \) and \( n > 1.5 \) would yield equal \( k \) at two values \( n_1 \) and \( n_2 \), wide apart. Then \( k \) had to be constant between \( n_1 \) and \( n_2 \). However, physically it should be expected that the transition from the Euler case to the case of the bar on elastic foundation is a continuous one, as appears from Figure 2, where \(-V/W\) changes from 0.3 to 3; also the discontinuity of \( dk/dn \) at the ends of the transition range is replaced by a continuous curve \( k(n) \) which however is almost constant.

Figure 3 is an enlarged portion of Figure 1. It depicts the equality \( k_A = k_B = 0.6023 \) at \( n_A = 1.17 \), \( n_B = 0.873 \). For \( \alpha = 6.4 \) the slope of \( k(n) \) at \( n_B \) is very steep since the mode is close to the Euler mode. Consequently a small variation of \( n_B \) (or \( n_A \)) yields a large variation of \( k \). On the other hand with \( \alpha = 2.3 \) a variation of \( n_A \) has little effect on \( k \). Then \( k_B \) will adjust to the almost constant value of \( k_A \) by selecting the appropriate wave length \( (n_B) \).

3. Discussion of the assumptions

The modes in the domains \( A \) and \( B \) have implicitly been supposed to be sinusoidal half waves. Therefore at the transition of \( A \) and \( B \) the displacements \( V \) and \( W \) vanish. Compatibility requires that the cross section in \( x_A = 0 \) and \( x_B = 0 \) have equal rotation \( W'_A \) and \( V' \). The first condition can be satisfied by choosing

\[
-(W_{DA}) = (W_{DB})_B \tag{3}
\]

yielding \( W_{DA} = -0.746 W_{BB} \).
According to [1] (3.13)

\[ \frac{V}{W_b} = - \frac{I}{\bar{I}_s} \left( 1 - \frac{k}{k_s} \frac{n^2}{1 - k/k_s} \right). \]

Hence

\[ \frac{V_A'}{V_A} - \frac{V_B'}{V_A} = 1 - \left( \frac{V_b'}{W_b} \right)_B / \left( \frac{V_b'}{W_b} \right)_A. \] (4)

\(-V/W_b\) can be read from Figure 2: \((-V/W_b)_A = 2.425; (-V/W_b)_B = 0.675,
yielding \((V_A' - V_B')/V_A' = 0.722.\)
The rotations \(V'\) appear to be far from compatible due to the inequality
of \(n_A\) and \(n_B\). It means that a restraint between the two parts of the
flange has to be superimposed upon the assumed sinusoidal flange def-
flexion. Restraints increase the buckling load.

Another restraint stems from the fact that the flange deflection
in \(x_b = \frac{L}{k_b - L}\) must vanish when as usual the flange is fastened
to the frame. The deflexion with the sinusoidal mode is 0.198 \(v_B\). It
is small but its suppression yields some increase of the buckling load.

Due to the sinusoidal deflexions the cross sections in \(x_A = L_A,'\)
\(x_B = 0\) are without normal stresses and carry only shear stresses.
Then equilibrium requires that the shear forces at the two sides of
the transition are equal: the force \(D\) in the total cross section normal
to the plate and the force \(D_f\) in the stringer flange.

From [1] (3.6) \(D = PW' = PW_b' \frac{1}{1 - k/k_s}.\)
The difference \(D_A - D_B\) does not vanish because \(k_{SA} = 6.18\) and \(k_{SB} = 7.12.\)
Then \((D_A - D_B)/D_A = -0.0145,\) which means that the error is negligible.

\(D_f\) is given in [1] section 3.2 as

\[ D_f = -M_f' - \frac{1}{A_f} \cdot V'. \]
Eliminating $V$ by using [1] (3.13)

$$D_f = W_b \frac{n}{L} P_E \left[ \frac{n^2 I_S}{I} + \left( \frac{S^2}{A I_S} - \frac{I_f}{I} \right) \frac{k}{I} \right] \times \left( 1 - \frac{k/n}{1 - k/k_s} \right) \cos \frac{\pi x}{L}$$

Using (2)

$$D_{FA} = (W_b n) \frac{\pi}{L} P_E 0.1764$$

$$D_{FB} = (W_b n) \frac{\pi}{L} P_E 0.1696$$

The error is $(D_{FA} - D_{FB})/D_{FA} = 0.0384$, again a small one.

In order to restore equilibrium the forces $D_A - D_B$ and $D_{FA} - D_{FB}$ have to be applied to the transition points and at everyone in the same direction. The supporting frames take the reaction. Since the distance between the transition point and the frame is only $\frac{1}{4} L (1 - 1/n_A) = 0.073 L$ the effect of the error of the shear forces will be negligibly small.

The conclusion is that the principal error stems from the discontinuity of the slope of the flange at the transition and that this error causes some underestimation of the buckling load.

Reference

Fig. 1: Buckling coefficient $k$ as function of half wave length $\lambda = L/n$ for two coefficients of restraint $\alpha$ at the web base.

Fig. 2: The ratio of sideways flange deflexion $V$ to the normal deflexion $W$ as function of half wave length $\lambda = L/n$. 
Fig. 3: Enlarged portion of Fig. 1, establishing the wave length at which the bays with unequal restraint have equal buckling coefficient k.