APPLICATION OF LEAST-SQUARES SPECTRAL ELEMENT METHODS TO POLYNOMIAL CHAOS

P.E.J. Vos*, M.I. Gerritsma†

*Delft University of Technology, Faculty of Aerospace Engineering
Kluyverweg 1, 2629 HS Delft, The Netherlands
e-mail: P.E.J.Vos@student.tudelft.nl
†Delft University of Technology, Faculty of Aerospace Engineering
Kluyverweg 1, 2629 HS Delft, The Netherlands
e-mail: M.I.Gerritsma@tudelft.nl

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Abstract. This paper describes the use of the Least-Squares Spectral Element Method to polynomial Chaos to solve stochastic partial differential equations. The method will be described in detail and a comparison will be presented between the least-squares projection and the conventional Galerkin projection.

1 INTRODUCTION

Recently, new algorithms have been developed to model the input uncertainty and its propagation in flow simulations. To solve the corresponding stochastic differential equations, the stochastic input is represented by a generalized polynomial chaos expansion. This spectral representation employs the orthogonal polynomial functions of the Askey family as the trial basis to expand the stochastic input. A standard Galerkin projection is applied in the random space to obtain the equations in the weak form. The resulting system of deterministic equations is then solved with standard methods to obtain the solution for each random mode.

In most engineering applications, one aims to solve physical problems by converting it into a deterministic mathematical model. This is a rough approximation of reality, as many physical input parameters describing the problem are fixed through this conversion. In reality however, these parameters, like material properties or boundary conditions for example, show some randomness which definitely influence the behaviour of the solution. This randomness is not incorporated in the deterministic model.

In order to include this uncertainty in the mathematical model, probabilistic methods have been developed. Next to statistical approaches, which use a large sample of (pseudo-) random numbers and therefore can turn out to be very costly (eg Monte Carlo simulation),
extensive research has been done on nonstatistical (deterministic) approaches. Recently, a new nonstatistical approach, called polynomial chaos, has been developed\(^5\). This approach, based on Wiener’s concept of homogeneous chaos\(^6\), has proved to be efficient in engineering applications\(^5\). The original form of polynomial chaos is a spectral expansion based on the Hermite orthogonal polynomials in terms of Gaussian random variables. Karniadakis and co-workers extended this approach into a broader framework called ”generalized Polynomial Chaos (gPC)”\(^2\). Within this framework, the close connection between the probability functions of certain random variables and the weighting function in the orthogonality relationship of certain orthogonal polynomials is used to represent non-Gaussian processes. More precisely, it has been realized that spectral expansions based on the orthogonal polynomials of the Askey-scheme\(^7\) can be efficiently employed to represent a broad range of ”standard” distributions such as the uniform distribution, the Gamma and Beta distribution, see for instance\(^10\).

Although gPC has been successfully applied in different cases showing an exponential convergence in approximating the solution\(^1, 2\), it can be inefficient or convergence may even fail for problems requiring long-term integration or problems involving discontinuities induced by random inputs\(^8, 9\). It is believed that employing a least-squares formulation of the problem instead of the Galerkin projection used in gPC, is a way to overcome these problems, since a least-squares formulation always weights the higher order modes in the differential equation as will be shown in this paper.

To this end, in this paper, we apply the procedure of gPC to the solution of a stochastic differential equation, but instead of deriving the equation in weak form by a Galerkin projection in random space, a least-squares formulation is employed to accomplish this. We will describe the general outline of this combination of polynomial chaos and a least-squares formulation, and we will also show that this method shows the expected exponential convergence for a stochastic ordinary differential equation (ODE).

The combination of a least-squares formulation combined with a spectral expansion base, referred to as a least-squares spectral element method (LS-SEM), has been successfully applied recently. The LS-SEM was first presented by Proot and Gerritsma\(^4, 11, 12\). Independently, the method was developed and investigated by Pontaza and Reddy\(^13, 14\). The LSQSEM is beneficial when solving linear partial differential equations in the sense that the least-squares formulation always leads to symmetric positive-definite matrices. These matrices can be efficiently solved by matrix free iterative methods, such as the preconditioned conjugate gradient method\(^3\).

This paper is organized as follows: In Section 2 a brief introduction into polynomial chaos is given. In Section 3 the least-squares spectral element method is described. In Section 4 the combination of polynomial chaos and LSQSEM is discussed and applied to a stochastic growth model. Finally, conclusions are drawn in Section 5.
2 GENERALIZED POLYNOMIAL CHAOS

Generalized polynomial chaos (gPC) is employed to represent stochastic processes. Stochastic processes can be seen as the solution of a stochastic mathematical model. This mathematical model is often expressed as a stochastic differential equation. Stochastic mathematical models are based on a probability space \((\Omega, \mathcal{F}, \mathcal{P})\) where \(\Omega\) is the sample space, \(\mathcal{F} \subset 2^\Omega\) its \(\sigma\)-algebra of events, and \(\mathcal{P}\) its probability measure.

In general, stochastic processes can be seen as random fields \(X(\omega)\), i.e. mappings \(X : \Omega \rightarrow V\). If the elements \(\omega\) of \(\Omega\) are mapped on the real line \(V = \mathbb{R}\) through \(X\), one speaks of a random variable \(X(\omega)\). If the elements are mapped onto a function space \(V\), \(X(\omega)\) is considered as a stochastic process. The function space \(V\) then consists of functions \(V : D \rightarrow I\) in which \(D \subset \mathbb{R}^d \times T\) \((d = 1, 2, 3)\) is a physical domain which can be a combination of spatial and temporal dimensions, and in many cases \(I = \mathbb{R}\).

Wiener was the first to represent stochastic processes by orthogonal polynomial expansions\(^6\). To accomplish this, he used Hermite polynomials in terms of Gaussian random variables to represent Gaussian processes, which is referred to as homogeneous chaos. According to the Cameron-Martin theorem\(^{15}\), this expansion converges to any \(L^2(\Omega)\) functional in the \(L^2(\Omega)\) sense. This implies that the application of homogeneous chaos is restricted to those stochastic processes yielding

\[
\int_{\omega \in \Omega} \|X\|^2 d\mathcal{P}(\omega) < \infty
\]

where \(\|\circ\|\) is the norm, corresponding to the inner product on the Hilbert space \(V\). As a result, homogeneous chaos and gPC are restricted to second order stochastic processes, i.e. processes with finite second-order moments. These are processes with finite variance, and this applies to most physical processes.

In order to deal with a broader range of stochastic processes, the Wiener homogeneous chaos (optimal for Gaussian random processes) is generalized to the gPC\(^{16}\), also referred to as Wiener-Askey polynomial chaos\(^2\).

Generalized polynomial chaos is a means of representing second-order stochastic processes \(X(\omega)\) parametrically through a set of random variables \(\{\xi_j(\omega)\}_{j=1}^N, N \in \mathbb{N}\), through the events \(\omega \in \Omega\):

\[
X(\omega) = \sum_{k=0}^{\infty} a_k \Phi_k(\xi(\omega))
\]

Here \(\{\Phi_j(\xi(\omega))\}\) are orthogonal polynomials from the Askey-scheme in terms of a zero-mean random vector \(\xi = \{\xi_j(\omega)\}_{j=1}^N\). Since each of the polynomials of the Askey-scheme forms a complete basis in the Hilbert space determined by their corresponding support, it is expected, according to Xiu et al.\(^2\), that each type of Wiener-Askey expansion converges to any \(L^2(\Omega)\) functional in the \(L^2(\Omega)\) sense in the corresponding Hilbert functional space.
as a generalized result of Cameron-Martin theorem. The polynomials of the Askey-scheme satisfy following orthogonality relation

$$\langle \Phi_i \Phi_j \rangle = \langle \Phi_i^2 \rangle \delta_{ij}$$  \hfill (3)

where $\delta_{ij}$ is the Kronecker delta and $\langle \cdot , \cdot \rangle$ denotes the ensemble average. The inner product in (3) is in the Hilbert space determined by the measure of the random variables

$$\langle f(\xi)g(\xi) \rangle = \int_{\omega \in \Omega} f(\xi)g(\xi) d\mathcal{P}(\omega) = \int f(\xi)g(\xi)w(\xi)d\xi$$  \hfill (4)

with $w(\xi)$ the corresponding weighting function and with integration in the last integral taken over the support of $\xi$.

In order to achieve optimal performance of the gPC chaos expansion, it is important to choose the appropriate type of orthogonal polynomials from the Askey-scheme dependent of the distribution of the random variables $\xi$. It seems that for certain random variables, their probability distribution function uniquely corresponds to one of the weighting functions $w(\xi)$ of the different orthogonal polynomials of the Askey-scheme. Choosing the corresponding Wiener-Askey polynomial chaos leads to an optimal solution. An overview of this correspondence can be found in\textsuperscript{2}, where it is also computationally shown that exponential convergence is achieved when applying the optimal gPC through a Galerkin projection in random space on an stochastic ordinary differential equation.

3 THE LEAST-SQUARES SPECTRAL ELEMENT METHOD

3.1 The least-squares formulation

Consider a global formulation of an arbitrary set of partial differential equations and boundary conditions:

$$\mathcal{L}u = f \quad \text{in} \quad \Gamma$$  \hfill (5)

with $\mathcal{L}$ a linear partial differential equation. If it is assumed that the system is well-posed and that the operator $\mathcal{L}$ is a continuous mapping from the underlying function space $X(\Gamma)$ onto the space $Y(\Gamma)$, it can be shown that the following relation is satisfied:

$$C_2 \| u \|_X \leq \| \mathcal{L}u \|_Y \leq C_1 \| u \|_X \quad \forall u \in X$$  \hfill (6)

Due to the coercivity in this relation, minimizing the norm of the residual is equivalent to minimizing the error, i.e.

$$\| \mathcal{L}u - f \|_Y \rightarrow 0 \quad \Rightarrow \quad \| u - u_{ex} \|_X \rightarrow 0$$  \hfill (7)

The least squares formulation is based on this observation and seeks to minimize the residual of (5) in the $Y$-norm. The norm-equivalent functional to be minimized then becomes
\[ J(u) = \frac{1}{2} \| Lu - f \|_Y^2 \]  

(8)

Notice that minimizing \( J(u) \) and solving (5) is equivalent, and hence, leads to the same result.

Minimizing the functional \( J \) for \( u \) means: find \( u \in X \) such that

\[ \lim_{\epsilon \to 0} J(u + \epsilon v) = 0 \quad \forall v \in X \]  

(9)

Applying variational analysis leads to the following expression: find \( u \in X \) such that

\[ B(u, v) = F(v) \quad \forall v \in X \]  

(10)

where \( B(u, v) = (Lu, Lv)_Y \) is a symmetric, continuous bilinear form and \( F(v) = (f, Lv)_Y \) is a continuous linear functional.

The boundary conditions imposed to the solution \( u \) can be enforced strongly by constraining the minimization spaces \( X \) to those functions who already satisfy the boundary conditions.

Furthermore, from a computational and implementational point of view, the \( H^1 \)- and \( L^2 \)-spaces are particularly suitable as function spaces \( X \) and \( Y \) respectively.

In the LS-SEM, one tries to seek for minimum of the norm equivalent functional \( J(u) \) in a finite dimensional subspace \( X^h \subset X \). This finite dimensional subspace \( X^h \) is parameterized by \( h \), which may refer to a combination of a characteristic mesh width and polynomial degree. The discrete variational problem is then formulated as follows: find \( u^h \in X^h \) such that

\[ B(u^h, v^h) = F(v^h) \quad \forall v^h \in X^h \]  

(11)

This finite dimensional formulation leads to a finite system of symmetric, positive definite algebraic equations, which can be solved by well-established linear algebra techniques.

### 3.2 Spectral elements

In order to obtain a discrete variational formulation, the domain \( \Gamma \) is decomposed into a finite set of non-overlapping elements \( \Gamma_e \) with \( \Gamma = \bigcup_{e=1}^{N_{el}} \Gamma_e \) and the function space \( X^h \) is chosen such that the approximate solution can be expanded spectrally in a polynomial expansion as follows:

\[ u^h(x) = \sum_{e=1}^{N_{el}} \sum_{p=0}^{P} u^e_p \Theta^e_p(x) \]  

(12)

where the basisfunctions \( \Theta^e_p(x) \) are a set of orthogonal polynomials that spans the function space \( X^h \).

An extended description of spectral elements can be found in\(^{17}\).
4 APPLICATION OF THE LEAST-SQUARES SPECTRAL ELEMENT METHOD TO POLYNOMIAL CHAOS

In this section, the least squares formulation combined with the gPC is applied to stochastic differential equations. First, this technique is applied on a global formulation of a stochastic differential equation. We then solve a specific stochastic ordinary differential equation with different type of random inputs and we will analyze the results.

4.1 General Procedure

Stochastic differential equations represent a stochastic mathematical model. This model is based on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). As domain, consider the \((d + 1)\)-dimensional bounded domain \(D \subset \mathbb{R}^{d+1}\) \((d = 1, 2, 3)\) with coordinates \(x = (x_1, ..., x_d, t)\).

We consider the following problem: find the stochastic function \(u(x; \omega) : \Gamma = D \times \Omega \rightarrow \mathbb{R}\), which solves the stochastic differential equation:

\[
\mathcal{L}(x, \omega; u) = f(x; \omega) \quad \text{in} \quad \Gamma
\]  

where \(\mathcal{L}\) is a linear differential operator and where this operator, as well as the source term \(f\), can both have random components. The solution, represented by the stochastic function \(u(x; \omega)\), can also be seen as a stochastic process \(u(\omega)\), mapping the elements \(\omega \in \Omega\) onto the function space \(V(D)\), i.e. \(u : \Omega \rightarrow V(D)\).

Assuming that the problem is well-posed and that \(\mathcal{L}\) is a continuous mapping from the underlying function space \(X(\Gamma)\) onto the space \(Y(\Gamma)\), the solution of problem (13) can be found by minimizing the norm of the residual. This least-squares approach leads to the following weak form of equation (13):

Find \(u(x, \omega) \in X\) such that

\[
\left( \mathcal{L}u(x, \omega), \mathcal{L}v(x, \omega) \right)_Y = \left( f, \mathcal{L}v(x, \omega) \right)_Y \quad \forall v \in X
\]  

Notice that the least-squares formulation not only projects the solution onto the random space, but it approaches the different dimensions in one similar way.

A finite dimensional solution \(u^h\) can be found by applying the procedure of gPC. First of all, the infinite-dimensional probability space is discretized by characterizing it by a finite number of random variables \(\{\xi_j(\omega)\}_{j=1}^N, N \in \mathbb{N}\). This can be seen as assigning a finite number of coordinates \(\{\xi_j\}_{j=1}^N\) to the probability space reducing it to a finite dimensional space \(\Lambda \subset \mathbb{R}^N\). In this paper, we assume that the stochastic processes are already characterized by a known set of random variables. This translates the problem into following formulation: Find \(u(x, \xi) \in X\) such that

\[
\left( \mathcal{L}u(x, \xi), \mathcal{L}v(x, \xi) \right)_{Y(\Gamma^*)} = \left( f, \mathcal{L}v(x, \xi) \right)_{Y(\Gamma^*)} \quad \forall v \in X(\Gamma^*)
\]  

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where $\Gamma^* = (\Lambda \times D) \subset \mathbb{R}^{N+(d+1)}$.

Next, the procedure of gPC is further applied together with the spectral element method in order to find the least-squares formulation (15) in a finite dimensional subspace $X^h(\Gamma^*) \subset X(\Gamma^*)$. This can be accomplished by first expanding the solution $u(\mathbf{x}, \xi)$ by the Wiener-Askey polynomial chaos as

$$u(\mathbf{x}, \xi) = \sum_{i=0}^{Q} u_i(\mathbf{x}) \Phi_i(\xi) \quad (16)$$

The total number of expansion terms $(Q+1)$ is determined by the number $N$ of random variables $\xi$ and the highest order $p$ of the polynomials $\Phi_i$

$$(Q + 1) = \frac{(N+1)!}{N!p!} \quad (17)$$

Next, the coefficients $u_i(\mathbf{x})$ can be expanded by the spectral element method such that

$$u(\mathbf{x}, \xi) = \sum_{i=0}^{Q} \sum_{\mathbf{x}=0}^{N_{\mathbf{x}}} \sum_{p=0}^{P} u_{i,p}^\mathbf{x} \Theta(\mathbf{x}) \Phi_i(\xi) \quad (18)$$

Upon substituting this expression in the governing equation (15) and solving the resulting system, the approximate solution $u(\mathbf{x}, \xi)$ can be found.

### 4.2 Stochastic Ordinary Differential Equation

Consider the following stochastic ordinary differential equation, which can be seen as a simple model representing exponential population growth

$$\frac{du(t)}{dt} + ku(t) = 0, \quad u(0) = 1 \quad (19)$$

The reproduction rate $k$ is considered to be a random variable $k = k(\omega)$. Therefore, the solution $u(t)$ of the above equation will be a stochastic process $u(t, \omega)$. It is assumed that the stochastic processes and random variables appearing in this problem can be parameterized by a single zero-mean random variable $\xi$. This implies that equation (19) can be written as, find $u(t, \xi)$ such that it satisfies

$$\frac{du(t, \xi)}{dt} + k(\xi)u(t, \xi) = 0 \quad \text{in} \quad \Gamma = T \times S \quad (20)$$

and the initial condition $u(t = 0) = 1$. The domain $\Gamma$ consists of the product of the temporal domain $T = [0, t_{end}]$ and the domain $S$, being the support of the random variable $\xi$.

In\textsuperscript{19}, it has been shown that for this problem, two constants $C_1$ and $C_2$ can be found such that:
\[ C_2 \| u \|_{H^1} \leq \| \frac{du}{dt} + ku \|_{L^2} \leq C_1 \| u \|_{H^1} \quad \forall u \in H^1(\Gamma) \quad (21) \]

such that minimizing the residual leads to a minimization of the error and the least squares formulation can be applied. In least-squares terms, the problem is then formulated as:

\[
\left( \frac{du(t, \xi)}{dt} + k(\xi)u(t, \xi), \frac{dv(t, \xi)}{dt} + k(\xi)v(t, \xi) \right)_{L^2(\Gamma)} = 0 \quad \forall v \in H^1(\Gamma) \quad (22)
\]

Applying the Wiener-Askey polynomial chaos by expanding the solution \( u(t, \xi) \) and random input \( k(\xi) \) in terms of orthogonal polynomials of the Askey-scheme gives:

\[
u(t, \xi) = \sum_{i=0}^{Q} u_i(t) \Phi_i(\xi), \quad k(\xi) = \sum_{i=0}^{Q} k_i \Phi_i(\xi) \quad (23)\]

where \( Q \) corresponds to the highest order of polynomials \( \Phi_i \).

Next, the time dependent coefficients \( u_i(t) \) are expanded in time using linear trial functions, a first order polynomial expansion.

\[ u_i(t) = \sum_{j=0}^{1} w^j_i \Theta_j(t) \quad (24) \]

This low order expansion in time can be justified knowing that a semi-implicit approach in time is employed. In order to avoid a fully implicit method which solves the solution in the whole domain \( \Gamma \) at once, the temporal domain is divided in small time intervals. In each time interval \( [t, t+\Delta t] \), we solve the system for the unknown \( (Q+1) \) coefficients \( u_i(t+\Delta t) \), based on the value of these coefficient \( u_i(t) \) at the initial time. In this way, we are able to march through the temporal domain \( T \) starting from the given initial condition at \( t = 0 \). In fact, we solve the problem consecutively in different sub-domains \( \Gamma_n = T_n \times S \). See for further details on the space-time least-squares formulation, for instance, De Maerschalck et al.\(^{18}\).

In each subdomain \( \Gamma_n \), the problem can be solved by substituting the expansions (23) and (24) into equation (22). This results in the following system of equations:
\[
\sum_{i=0}^{Q} \left( \frac{d \Theta_1}{dt} \frac{d \Theta_1}{dt} \right) e_{ij} + \left( \frac{d \Theta_1}{dt}, \Theta_1 \right) \sum_{j=0}^{Q} k_j e_{ijm} + \\
\left( \Theta_1, \frac{d \Theta_1}{dt} \right) \sum_{j=0}^{Q} k_j e_{ijm} + \left( \Theta_1, \Theta_1 \right) \sum_{j=0}^{Q} \sum_{i=0}^{Q} k_j k_l e_{ijlm} \right) u_i^{n+1} =
\]
\[
\sum_{i=0}^{Q} \left( \frac{d \Theta_0}{dt} \frac{d \Theta_0}{dt} \right) e_{ij} + \left( \frac{d \Theta_0}{dt}, \Theta_0 \right) \sum_{j=0}^{Q} k_j e_{ijm} + \\
\left( \Theta_1, \frac{d \Theta_0}{dt} \right) \sum_{j=0}^{Q} k_j e_{ijm} + \left( \Theta_1, \Theta_0 \right) \sum_{j=0}^{Q} \sum_{i=0}^{Q} k_j k_l e_{ijlm} \right) u_i^n \quad \text{for} \quad m = 0, ..., Q
\]

where \( e_{ijlm} = \langle \Phi_i, \Phi_j, \Phi_l, \Phi_m \rangle \) denotes the ensemble average, defined as (4) and \( \langle \Theta_i, \Theta_j \rangle \) is the inner product defined as

\[
\langle \Theta_i, \Theta_j \rangle = \int_{T_n} \Theta_i(t)\Theta_j(t)dt
\]  

Solving this system for the unknown coefficients \( u_i^{n+1} \) gives the desired result.

Note that if we compare the least-squares formulation with the Galerkin projection as described in Xiu et al.\(^1\), we see the appearance of the higher order term \( e_{ijlm} \). The population growth model is non-linear in its stochastic part, due to the product of the two stochastic variables \( k \) and the solution \( u_i \). The Galerkin projection only projects onto the first \( Q \) basis function and ignores all higher modes. This leads to biasing where higher order modes are represented as lower order modes, thus deteriorating the lower order statistics. The least-squares formulation weights these higher order modes with the same higher order modes, which then leads to the inclusion of the term \( e_{ijlm} \). It is expected that a proper weighting of these higher order modes resolves the problems encountered for non-smooth problems and long-time integration.

### 4.3 Numerical Results

In this section, we present the numerical results of the combined application of the LS-SEM and the gPC applied to the stochastic ordinary differential equation introduced in the previous section. The computations are done for different types of input parameter \( k \). We will distinguish the following cases for the distribution of the reproduction rate \( k \): a Gaussian distribution, a Gamma distribution and a Beta distribution. This standard type of distributions allow for a first order gPC expansion of the parameter \( k \). Dependent on this type of distribution, the corresponding optimal Wiener-Askey polynomial chaos is employed. This means:

- Hermite-Chaos for the Gaussian distribution
• Laguerre-Chaos for the Gamma distribution
• Jacobi-Chaos for the Beta distribution

For every case, we will show the results of each random mode $u_i$ with respect to time, and we will also show the convergence behaviour of the error in function of the gPC expansion order. To this end, we define two error measures: one for the mean and one for the variance of the solution

$$
\varepsilon_{\text{mean}}(t) = \left| \frac{\bar{u}(t) - \bar{u}_{\text{exact}}(t)}{\bar{u}_{\text{exact}}(t)} \right|, \quad \varepsilon_{\text{var}}(t) = \left| \frac{\sigma(t) - \sigma_{\text{exact}}(t)}{\sigma_{\text{exact}}(t)} \right|
$$

(27)

where $\bar{u}(t)$ is the mean and $\sigma(t)$ is the variance of the solution $u$. In the presented results, the values of the error measures are calculated at the endtime, being $t = 1$.

**Case 1: Hermite-Chaos for a Gaussian distribution**  The random input parameter $k$ is assumed to be a Gaussian random variable with zero mean and unit variance, with probability density function given as

$$
f(k) = \frac{1}{\sqrt{2\pi}} e^{-k^2/2}
$$

(28)

The solution, obtained by the presented method, of the stochastic ordinary differential equation is shown in Figure 1. The different random modes $u_i$ of the solution are shown

![Figure 1](image)

(a) Solution of each random mode  (b) Error convergence of the mean and variance

Figure 1: Solution with Gaussian random input by 4th order Hermite-Chaos

for a 4th order Hermite-Chaos expansion. Compared to the deterministic solution which
is stationary, it can be seen that the randomness of the input parameter $k$ causes the mean $\bar{u} = u_0$ of the stochastic solution to be time-dependent. It can also be seen that both the error measures converge exponentially with respect to the gPC expansion order. Compared to the results of the Galerkin gPC as presented in\textsuperscript{2}, we see no difference with the LS-SEM approach for this stochastic ODE. Consequently, the LS-SEM approach can be considered as a worthy alternative of the gPC based on a Galerkin projection.

**Case 2: Laguerre-Chaos for the Gamma distribution** In this case, the random input parameter $k$ is assumed to be a random variable with Gamma distribution, characterized by the probability density function given as

$$f(k) = \frac{e^{-k}k^\alpha}{\Gamma(\alpha + 1)} \quad 0 \leq k < \infty, \quad \alpha > -1$$

(29)

The specific case $\alpha = 0$ corresponds to the exponential distribution. For this gamma distributed input parameter $k$, the numerical solution is shown in Figure 2. A similar behaviour is established, and also for this case, an exponential convergence rate of the error is achieved. Again, the results are similar to those obtained by Xiu et al.\textsuperscript{2} employing a Galerkin projection.

**Case 3: Jacobi-Chaos for the Beta distribution** For this last case, we assume the distribution of the random input parameter $k$ to be a Beta distribution with probability density function.
\[ f(k) = \frac{(1-k)^\alpha(1+k)^\beta}{2^{\alpha+\beta+1}B(\alpha+1, \beta+1)} \quad -1 < k < 1, \quad \alpha, \beta > -1 \]  

where \( B(\alpha + 1, \beta + 1) \) is the Beta function defined as \( B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \). In the specific case \( \alpha = \beta = 0 \), the distribution becomes the uniform distribution and de Jacobi-Chaos becomes the Legendre-Chaos. Figure 3 shows the solution by the LS-SEM Jacobi-Chaos solution. Also for this case, the exponential convergence rate of the error is achieved, and this for different set of values of \( \alpha \) and \( \beta \). And as expected, also in this case, the LS-SEM approach leads to similar results as presented in the literature.

5 CONCLUSIONS

We presented the approach of solving a stochastic differential equation using a combination of the least-squares spectral element method (LS-SEM) and the general polynomial chaos (gPC). It has been shown, that for a stochastic ordinary differential equation, using this approach leads to similar results, meaning exponential error convergence, as presented in the literature, where a Galerkin formulation has been employed. This strengthens the confidence that the LS-SEM applied to polynomial chaos can be used next to the original Galerkin polynomial chaos as presented in the literature, and probably can be applied in cases where this gPC fails to generate satisfying results, such as for example, long-term integration. Particularly, it is expected that the appearance of higher order terms in the LS-SEM approach can possibly resolve these problems.
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