Hankel-norm approximation and model reduction of time-varying systems

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We consider the Hankel-norm approximation problem for (bounded) upper triangular operators: generalized $\ell_2$-operators $T$ with matrix representations $[T_{ij}]_{-\infty}^\infty$ such that $T_{ij} = 0 (i > j)$. Here, each $T_{ij}$ is a matrix with dimensions $M_i \times N_j$, where $M_i, N_j$ are finite integers (possibly zero). An upper operator can be viewed as the transfer operator of a causal linear time-varying system.

Associated to $T$ is a sequence of ‘Hankel’-operators $H_k = [T_{k+i+k+j}]_{10}^\infty (k = -\infty \cdots \infty)$, which are submatrices of $T$ corresponding to its top-right parts. The rank of these operators plays an important role in realization theory: if $d_k = \text{rank} H_k < \infty$, then there exist minimal time-varying realizations with system order $d_k$ at point $k$:

$$
\begin{align*}
  x_{k+1} &= x_kA_k + u_kB_k & A_k : d_k \times d_{k+1}, & B_k : M_k \times d_{k+1} \\
  y_k &= x_kC_k + u_kD_k & C_k : d_k \times N_k, & D_k : M_k \times N_k
\end{align*}
$$

(1)

such that $[\cdots y_0 \ y_1 \ \cdots] = [\cdots u_0 \ u_1 \ \cdots]T$.

The Hankel norm of $T$ is defined to be $\|T\|_H = \sup \|H_k\|$. This definition is a generalization of the time-invariant Hankel norm and reduces to it if all $H_k$ are the same. Let $\Gamma = \text{diag} (\gamma_i)$ be an acceptable approximation tolerance, with $\gamma_i > 0$. If an operator $T_{\Delta}$ is such that

$$
\|\Gamma^{-1}(T - T_{\Delta})\|_H \leq 1,
$$

(2)

then $T_{\Delta}$ is called a Hankel norm approximant of $T$, parameterized by $\Gamma$. We are interested in Hankel norm approximants of minimal system order. In [1], we proved that if the number of Hankel singular values of $\Gamma^{-1}T$ that are larger than 1 is equal to $N_k$ at point $k$, then there exists a Hankel norm approximant $T_{\Delta}$ whose system order is equal to $N_k$ at point $k$, assuming none of the singular values are equal to 1.

In the construction of such Hankel-norm approximants $T_{\Delta}$, two additional operators play a role. The first is $U$: the inner (i.e. upper and unitary) factor of $T$ in a coprime factorization $T = \Delta^* U$ (where $\Delta$ is upper). The second is $\Theta$: a $J$-unitary operator $(\Theta^* J_1 \Theta = J_2, \Theta J_2 \Theta^* = J_1$, where $J_1, J_2$ are signature matrices) such that

$$
[U^* - \Gamma^{-1}T^*] \Theta = [A' - B']
$$

(3)

consists of two upper operators $A', B'$. This equation describes an interpolation problem. $\Theta$ exists under certain conditions and can be constructed explicitly, and the resulting signature matrices are determined by the singular values of the $H_k$. In fact, the system order of the strictly upper part of $\Theta^{-1}_{22}$ is at each point $k$ equal to $N_k$.

Let $S_L$ be an upper contractive operator. Then

$$
S = (\Theta_{11} S_L - \Theta_{12}) (\Theta_{22} - \Theta_{21} S_L)^{-1}
$$

(4)

is contractive, and the strictly upper part of $T' = T + \Gamma S^* U$ is a Hankel norm approximant. Conversely, for each $T'$ of which the strictly upper part $T_{\Delta}$ is a Hankel norm approximant, there is an upper contractive operator $S_L$ such that $\Gamma^{-1}(T' - T) = S^* U$, where $S$ is given by the above expression.