STELLINGEN
Behorende bij het proefschrift
A Mechanics based Computational Platform for Pavement Engineering
van
A. Scarpas

1. Vakken behorende tot het ‘Master of Science’ diploma zouden onder de bevoegdheid van de Nederlandse onderzoekscholen moeten vallen, gezien hun doctorale karakter. Dit zou een doorgaande bron van financiering voor de Scholen verzekeren en tevens een betere coördinatie van de onderzoekstructuur bewerkstelligen.

2. Hogere vakken in de studie Civiele Techniek vereisen wiskundige kwaliteiten op het niveau van gevorderden. Het is van nationaal belang dat de huidige trend van minimalisatie van het aantal gevorderde wiskundevakken wordt teruggedraaid. De onderzoekscholen moeten hierin een belangrijke rol gaan spelen door het beschikbaar stellen van de benodigdheden en de interuniversitaire expertise.

3. Het positieve van het ‘Poldermodel’ is, dat het discussie stimuleert in apriori een beslissing wordt genomen, en niet posteriore ten behoeve van haar minst pijnlijke implementatie.

4. Boekhoudkwaliteiten zijn nodig maar kunnen niet de enige kwalificatie van een leider met visie zijn.

5. Samenwerking moet worden uitgezet, niet afgedwongen.

6. Als meer Europese universiteiten de Bologna Conventie implementeren, zullen de voordelen zwaarder gaan wegen dan de implementatiekosten.

7. Het individuele carrière-ontwikkelingsplan, toegepast door de Amerikaanse en de meeste Europese universiteiten buiten Nederland, is bedoeld om personeelsmotivatie te bevorderen en stagnatie van carrière-ontwikkeling te voorkomen.


\[ \bar{b}_t = \exp \left[ -2 \Delta \gamma \partial \phi \left( \sigma^\tau \right) \right] b^\tau \]

In hetzelfde artikel, de correcte formulering van formule (3.16c) is tevens:

\[ b^\tau = \exp \left[ -2 \Delta \gamma \partial \phi \left( C^{-1} N \right) P^{-1} b^\tau \right] \]


\[ \sigma_{n+1} = R_{n+1} \left( T_n + \int_{t_n}^{t_{n+1}} T \, dt \right) R^T_{n+1} = R_{n+1} \left( T_n + \int_{t_n}^{t_{n+1}} C^{\text{ep}} : D \, dt \right) R^T_{n+1} \]

10. In Paragraaf 2.6 van J.C. Simo & T.J.R. Hughes, Computational Inelasticity, Springer, 1998, formule (2.6.4) is, om consistent te zijn met de rest van de formulering in dat hoofdstuk:

\[ \mathbb{K}^p := \{ \gamma \in L^2(B) | \gamma \geq 0 \} \]

Tevens, formule (2.6.5) moet zijn:

\[ \mathcal{L}^p (\tau, \gamma; e^p) := -\tau : e^p + \gamma f(\tau) \]

11. In dezelfde referentie als bovenstaand, de correcte formulering van formule (8.3.6a) is:

\[ \bar{e}_{n+\alpha} = f_{n+\alpha} e_{n+\alpha} \]

Deze stellingen worden verdedigbaar geacht en zijn als zodanig goedgekeurd door de promotoren Prof. dr. ir. J. Blauwendraad and Prof. N. Aravas, BSc., MSc., PhD.
PROPOSITIONS

Associated with the thesis
A Mechanics based Computational Platform for Pavement Engineering
of
A. Scarpas

1. Considering their post-graduate nature, studies for the Master degree should be under the jurisdiction of the Dutch schools of Graduate Studies, now commonly known as Research Schools. This would ensure a continuous source of financing for the Schools and a more efficient coordination of the research infrastructure.

2. Advanced studies in civil engineering require advanced mathematical skills. It is of national importance that the current shrinkage of advanced mathematical courses in engineering curricula is reversed. The Research Schools must play an important role in this aspect by providing the means and the necessary inter-university expertise.

3. The positive aspect of the 'Polder model' is that it encourages discussion a priori to a decision being made and not postoriori for its least painful implementation.

4. Accounting skills are necessary but cannot be the only qualification of a visionary leader.

5. Cooperation should be encouraged not dictated.

6. As more European Universities implement the Bologna convention, its benefits will outweigh the implementation costs.

7. The personal career development plan adopted by American and most European Universities outside The Netherlands is meant to encourage staff motivation and prevent career development stagnation.


\[ \delta_t^e = \exp \left[ -2 \Delta \gamma_t \partial_t \phi \left( \dot{\tau}_t, \dot{\theta}_t \right) \right] b_t^{e \text{tr}} \]

Also, in the same article, the correct form of Equation (3.16) is:

\[ \delta_t^e = \exp \left[ -2 \Delta \gamma_t \partial_t \psi \left( \dot{\tau}_t \right) \right] b_t^{e \text{tr}} \]


\[ \sigma_{n+1} = R_{n+1} \left( T_n + \int_{t_n}^{t_{n+1}} T \, dt \right) R_{n+1}^T = R_{n+1} \left( T_n + \int_{t_n}^{t_{n+1}} C^{ep} : D \, dt \right) R_{n+1}^T \]

10. In Section 2.6 of J.C. Simo & T.J.R. Hughes, Computational Inelasticity, Springer, 1998, for consistency with the rest of the developments in that Chapter, Equation (2.6.4) should read:

\[ \mathbb{K}^p := \left\{ \gamma \in L^2(B) \mid \gamma \geq 0 \right\} \]

Also, Equation (2.6.5) should read:

\[ \mathcal{L}^p \left( \tau, \gamma; e^p \right) := -\tau : e^p + \gamma f(\tau) \]

11. In the same reference as above, the correct form of Equation (8.3.6a) is:

\[ \dot{e}_{n+\alpha} := f_{n+\alpha} e_{n+1} + f_{n+\alpha} \]

These propositions are considered defendable and as such have been approved by the supervisors Prof. dr. ir. J. Blauwendraad and Prof. N. Aravas, BSc., MSc., PhD.
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To Johan Blauwendraad
my teacher, my mentor, my friend
Acknowledgements

Prof. Dr. ir. André Molenaar and Prof. dr. ir. Johan Blaauwendaal were the first who identified the need for the development of a computational platform focused on pavement engineering and who offered me the opportunity to develop it in what now constitutes the CAPA-3D finite element system. They established a spirit of close cooperation between the Road & Railroad Laboratory and the Section of Structural Mechanics of Delft University of Technology and entrusted me to create the conditions and the environment for successful joint research in the field of mechanics of pavements.

In the course of the work I consider myself privileged for having had the opportunity to work closely for many years with dr. ir. Arian de Bondt and dr. ir. Sandra Erkens at the time when they were both working towards the completion of their own PhD research work at the Road & Railroad Laboratory. Many of the current features of CAPA-3D were inspired and developed in response to their research needs. In the process, they also became coffee addicts.

Since the mid-nineties, ir. C. Kasbergen joined what has become known as the CAPA-3D development team, bringing with him expertise in computer programming and mathematics. He soon took over control of the design/maintenance of the user interface and the coordination of the programming efforts. Cor has contributed the most in designing a finite element system which enables many individuals to work concurrently in their own domain without interfering with each other's development work. He has also been our unequivocal ambassador with the several guests we host every year. The fact that many of them come back to visit him is a testimony to his efforts for which I am grateful.

Dr. Xueyan Liu also joined the development team since the beginning of his PhD thesis at the Section of Structural Mechanics. Over the years he concentrated on constitutive model development and implementation. This brought him into contact with many of our other PhD researchers and research guests and encouraged him to work with a great variety of engineering materials, constitutive models and field cases. By now, his name appears in the acknowledgements of at least as many PhD theses as mine... I am thankful for his support in outreaching to so many different disciplines and individuals.

A special thank you note is due to the efforts and contribution of ing. Frank Custers. Despite the continuously increasing number of hardware and internal/external users, Frank always managed the seamless integration and operation of all resources.

Several other of my students and colleagues have contributed to the verification and the improvement of the system and can be traced in the corresponding references of our joint publications. Among many, the contributions of Prof. Andrew Collop, dr. ir. Rien Huurman and of Prof. Jacob Uzan deserve special mention.

Over the years, I have learned and benefited a lot from the writings of my co-promotor, Prof. Nikos Aravas. The plasticity reduction algorithms in CAPA-3D are clearly based on concepts that he initiated. For the time and effort he spent in meticulously reviewing and correcting early versions of this document, I shall always be grateful.

Maureen, my wife, has always been a member of the CAPA-3D development team and is familiar with and to so many individuals within the pavement mechanics engineering community. This is the best recognition and reward for her contribution.

Tom Scarpa
Delft, November 2004
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Introduction

Most current design methodologies against fatigue and permanent deformation of road pavements rely on experience and are based mainly on information obtained from simple, mostly uniaxial, laboratory tests. Nevertheless, it is common knowledge that when a traffic load is imposed on a pavement, a non-uniform displacement field develops giving rise to a complicated state of triaxial stresses, Fig. 1.1 (exaggerated).

![Fig. 1.1 Stress states in a pavement due to traffic load](image)

Triaxiality has been known to significantly influence the response of pavement materials. For example, as shown schematically in Fig. 1.2 for the case of a cohesive material, in the presence of increasing amounts of lateral tension both, the stiffness and the compressive strength of the material in the perpendicular direction decrease.

![Fig. 1.2 Influence of lateral tension on compressive strength and stiffness](image)
Introduction

Fig. 1.3 Influence of temperature and strain rate on material response

Also, for a wide range of pavement engineering materials temperature and strain rate have influences on both the strength and the stiffness characteristics, Fig. 1.3.

Most of the current pavement design methodologies ignore the above fundamental features of material response and are based on simplified elastic theories. In spite of this inconsistency, fairly good pavements could be constructed until recently. This was mainly because the lack of appropriate material models could be backed up by practical experience.

Relying on experience is however not acceptable anymore because of the rapidly changing conditions of the international road network. First of all, the number of trucks has increased beyond expectation. Next to that there is a strong increase in wheel loads as well as in tyre pressures as a consequence of the need to increase the road transport efficiency for environmental reasons. Furthermore, the expected introduction of aramide reinforced truck tyres in combination with foreseeable changes in wheel configurations will result in extremely high wheel pressures.

All these, together with the need to use secondary recycled building materials, the need for "maintenance free" roads and the fact that pavements must increasingly fulfil non-structural demands (eg. noise levels, water permeability) result into the question how to design our future roads and asphalt mixes in order to make them resistant to the very heavy loads that will occur.

Because these developments are far beyond our experience, one needs to address these questions by applying proper design and evaluation models, which are based on a sound engineering approach towards material behaviour. For this reason, interest in mechanics based approaches for road engineering design has recently grown considerably, both nationally and internationally. This is partially due to the increased availability of computing power, which has led to the increased popularity of versatile computational tools such as the finite element method.

By accounting for the above mentioned idiosyncrasies of material response and by enabling the visualisation of the internal distributions of stresses and strains in the body of a pavement, the finite element method constitutes a valuable tool in understanding the mechanisms and the processes leading to pavement deterioration. In addition, the method enables the quantification of the interaction between the material and the geometric characteristics of a pavement.

Unfortunately, because of the dependence of pavement engineering materials on the state of stress, on the rate of loading and on the temperature, they constitute some of the most difficult materials for finite element simulation. Nevertheless, when such models are available, utilisation of the method can result to significant time and financial savings in laboratory and field-testing.
Since the early 90s, the Section of Structural Mechanics and the Laboratory of Road & Railroad Engineering of the Faculty of Civil Engineering and Geosciences of TU Delft have been closely cooperating towards the development of tools and procedures capable of addressing realistically the response characteristics of a wide range of pavement engineering materials.

In the framework of this cooperation, CAPA-3D has been developed as a finite element based platform to serve the computational needs of the joint research team at TU Delft, and of some international teams which cooperate with Delft.

Over the years, CAPA-3D has evolved into a fully fledged finite element system for static or dynamic analysis of very large scale three dimensional pavement and soil engineering models. It consists of a sophisticated user interface, a powerful band-optimising mesh generator, high quality user controlled graphical output, several material and element types, and a variety of specialised algorithms for the more efficient analysis of pavement constructions. Among others, these include a moving load simulation algorithm and a contact algorithm.

Ever since its inception in the early 90s, the system has been under continuous update and development. Invariably developments were dictated by the research needs of the progressively growing groups of researchers/users. In order to minimise interference between independently working groups, a hierarchical structure has been chosen for the system layout. This has enabled the utilisation of the system as the computational platform of several concurrent PhD theses within TU Delft and internationally.

In pavement engineering very few, if any, realistic situations can be encountered which are truly two-dimensional. Nevertheless and despite the recent developments in computer software and hardware, the use of three-dimensional Finite Element models is both, time and resource consuming. Especially so if the non-linear nature of the materials and the processes concerned is considered.

In the development of the CAPA-3D system the need for powerful, albeit expensive, computational facilities has been addressed by segmenting the system into three subsystems:

1. the ‘input subsystem’, consisting of a user interface for structured input of the geometric and the material data, a band minimising mesh generator and a graphical facility for mesh visualisation.

2. the ‘computational engine’, in which the main finite element operations are performed (e.g. matrix handling operations, state of stress determination). The system can perform static or dynamic analysis of very large scale three dimensional models as those typically encountered in pavement and soil engineering.

3. the ‘output subsystem’, consisting of a user interface for the exploitation/visualisation of the results of the finite element analyses.

All three subsystems can run in an Intel based personal computer. However, if optimum performance is to be obtained more powerful computers can be utilised to run the computational engine subsystem. Access to these computers can be attained either directly or via the Internet. As such, research teams who do not have their own powerful computing facilities can still access the system.

All subsystems provide their own integrated Windows based user guides. This monograph focuses exclusively on those aspects of continuum mechanics that were necessary for the development of a range of generic constitutive models and finite element types. These can be utilised by the user for the development of additional materials and element types.
Introduction

Over the years, many of my students and colleagues contributed to the verification and the improvement of the system. Among many, the contributions of A. de Bondt, C. Kasbergen, X. Liu, A. Collop, M. Huurman and J. Uzan deserve special mention and can be traced in the corresponding references of our joint publications. In this context, several of the significant characteristics of the system, like solution algorithms, the simulation of contact and of multiphase media will not be addressed. For these, the reader is referred to the various PhD theses of my past and current students.

This document is structured as follows. In Chapter 2, the necessary fundamental non-linear continuum mechanics theories and their implementation in the context of the Finite Element Method are presented. This Chapter is of paramount importance for understanding the material included in the following Chapters. Most of the terminology is also established in this Chapter.

The rest of the document is divided in two sections. In Section I, several generic theories of material constitutive response are presented in progressively increasing complexity. The last Chapter of this section, Chapter 6 provides a methodology which enables the combination of the earlier presented generic theories for the development of more complex constitutive models.

In Section II of the document, the characteristics of a variety of finite element types, necessary for efficient analyses of pavement and soil engineering problems, are discussed. The domain of element application and its proper utilisation are addressed.

As mentioned earlier, current pavement design is based primarily on empirical rules. However, the advent of powerful computational hardware systems, will make tools like CAPA-3D accessible to a wider research and engineering audience. This will open the way for a new generation of pavement design techniques based on rational mechanics principles and known as ‘mechanistic’ based techniques, Yoder and Witzczak [1975].

A typical example of this change in design philosophy is the recent NCHRP 1-37A [2004] report on future pavement design, in which the finite element method in combination with advanced material constitutive models and characterisation techniques constitute the backbone of the whole design process.
2.1 Introduction

In classical continuum mechanics several physical quantities are defined on the basis of the "representative volume" element (RVE). Typical examples are the density, the porosity etc. Furthermore, physical measures of deformation or force are defined by considering an averaging procedure over such a "representative volume".

![Representative volume element](image)

Fig. 2.1 Representative volume element

A deformable body having volume \( V \) and surface area \( A \) can be assumed to be consisted of an assemblage of "representative volume" elements, Fig. 2.1. The classical definition of a continuum is obtained by allowing \( (V, A) \rightarrow 0 \).

2.2 Kinematics

In a classical continuum, the location of a point before deformation is determined by the position vector \( \mathbf{X} \) defined in a Cartesian basis system \( \{ \mathbf{E}_i ; i = 1, 2, 3 \} \), Fig. 2.2. After deformation, the coordinates of point \( P \) are located by position vector \( \mathbf{x} \) defined w.r.t. an alternative Cartesian basis system \( \{ \mathbf{e}_i ; i = 1, 2, 3 \} \). The motion of a particle from the reference to the current position can be viewed as a vector mapping

\[
\mathbf{x} = \phi(\mathbf{X})
\]
Fig. 2.2  Reference and current configurations of a deformable body

Without loss of generality, in the remainder of this document, the two base systems will be assumed to coincide, however, notational differences will be maintained. Capital letters will be utilised to indicate quantities in the undeformed, or as it is commonly termed, the “reference” configuration at time \( t = 0 \) and, lower case letters to indicate the deformed, or as it is commonly termed, the “current” configuration at time \( t \).

Eq. 2.1 can be generalized to read

\[
x = \phi(X,t)
\]

2.2

The relative position vector \( dX \) of two material points \( P \) and \( Q \) in the reference configuration, Fig. 2.3, can be related to the relative position vector \( dx \) in the current configuration as

\[
dx = F \, dX
\]

2.3

in which \( F \) is known as the deformation gradient tensor defined by

\[
F = \frac{\partial x}{\partial X} = \begin{bmatrix}
\frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\
\frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\
\frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3}
\end{bmatrix}
\]

2.4
Fig. 2.3 Mapping of relative position vector from the reference to the current configuration

In the mechanics literature the operation indicated by Eq. 2.3 is termed as a *push forward* operation. The formalism

\[
dx = \phi_* \left[ dX \right]
\]

is frequently encountered. The condition \( \det F > 0 \) and the existence of a unique inverse of \( F \) express the physical requirements of *continuity, indestructibility* and *impenetrability* of matter, Malvern [1969]. Then it is also valid

\[
dX = F^{-1} dx
\]

The operation defined by Eq. 2.6 is termed as *pull back*. Formally it is indicated as

\[
dX = \phi_*^{-1} [dx]
\]

### 2.2.1 Volume and Area Transformations

The individual vector components of the relative position vector \( dx \) can be expressed by means of the current configuration basis vectors \( e_i \) and the components of the relative position vector \( dX \) as

\[
dx_i = \sum_{j=1,3} \left( \frac{\partial x_i}{\partial X_j} dX_j \right) e_j
\]

The differential volume \( dv \) in the current configuration is

\[
dv = dx_1 \cdot (dx_2 \times dx_3)
\]
Fundamentals

Substituting for the individual vector components on the basis of Eq. 2.8 it results

\[ dv = \det F \left( dX_1 dX_2 dX_3 \right) = J \, dV \]  

By means of the postulate of mass conservation, the material density \( \rho_0 \) in the reference configuration can be related to the density \( \rho \) in the current configuration via

\[ dm = \rho_0 \, dV = \rho \, dv = J \rho \, dV \]  

in which \( dm \) is an element of mass.

On the basis of Eq. 2.10 an interrelationship can be also obtained between an infinitesimal element of area \( dA \) in the reference configuration and \( da \) the corresponding infinitesimal element of area in the current configuration

\[ da = J F^{-T} dA \]  

2.2.2 Strain

Let \( dS \) denote the magnitude of the relative position vector \( dX \) at the reference configuration and \( ds \) the magnitude of the same in the current configuration. Then a measure of the change of length between the two configurations can be expressed as

\[ (ds)^2 - (dS)^2 = dx \cdot dx - dX \cdot dX \]
\[ = dX \cdot F^T F dX - dX \cdot dX \]
\[ = dX \cdot (F^T F - I) dX \]
\[ = dX \cdot (C - I) dX \]  

The tensor \( C = F^T F \) is known as the right Cauchy-Green deformation tensor Malvern [1969], Bonet & Wood [1999], Holzapfel [2000]. On the basis of \( C \) the Lagrangian-Green strain tensor is defined as

\[ E = \frac{1}{2} (C - I) \]  

Defining \( x = X + u \) in which \( u \) represents the displacement vector from \( X \) to \( x \), on the basis of Eq. 2.4

\[ F = \left[ \frac{\partial x}{\partial X} \right] = \left[ \frac{\partial u}{\partial X} \right] + I \]  

---

* The notation \( F^{-T} \) is utilised
while \( C \) can be expressed as

\[
C = F^T F = \begin{bmatrix} \frac{\partial u}{\partial X}^T \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial X} + I \end{bmatrix}
\]

\[
= \frac{\partial u}{\partial X}^T \frac{\partial u}{\partial X} + \frac{\partial u}{\partial X}^T + \frac{\partial u}{\partial X} + I
\]

In case of small displacements, the second order terms in Eq. 2.18 can be dropped and the familiar, small strain tensor \( \varepsilon_0 \) is obtained

\[
\varepsilon_0 = \frac{1}{2} \left( \frac{\partial u}{\partial X}^T + \frac{\partial u}{\partial X} \right)
\]

Alternatively, a measure of strain can be obtained on the basis of the current configuration by expressing the product \( dX \cdot dX \) as

\[
dX \cdot dX = (F^{-1} dx)^T (F^{-1} dx)
\]

\[
= dx^T (F^{-T} F^{-1}) dx
\]

\[
= dx \cdot b^{-1} dx
\]

Tensor \( b \) is commonly known as the \textit{left Cauchy-Green} or \textit{Finger} tensor, Malvern [1969], Bonet & Wood [1999], Holzapfel [2000]. Eq. 2.13 can now be expressed in terms of \( dx \)

\[
(ds)^2 - (dS)^2 = dx \cdot dx - dX \cdot dX
\]

\[
= dx \cdot dx - dx \cdot b^{-1} dx
\]

\[
= dx \cdot (I - b^{-1}) dx
\]

The \textit{Eulerian-Almansi strain tensor} is then defined as

\[
e = \frac{1}{2} (I - b^{-1})
\]

The push forward \( \phi_\ast \) and pull back \( \phi^{-1}_\ast \) operations that were defined in the above can also be extended to the tensorial measures of strain. From Eq. 2.13 and Eq. 2.21 it results

\[
dX_1 \cdot E dX_2 = dx_1 \cdot e dx_2
\]

hence

\[
E = F^T e F \quad ; \quad e = F^{-T} E F^{-1}
\]

The first of Eq. 2.24 can be viewed as the mapping of the strain tensor from the current to the reference configuration and, the second of Eq. 2.24 as the reverse procedure. In terms of the operator \( \phi_\ast \):

\[
E = \phi^{-1}_\ast [e] \quad ; \quad e = \phi_\ast [E]
\]
Fundamentals

2.2.3 Polar Decomposition

The deformation gradient \( F \) is a second order tensor. As such it can be decomposed as the product of an orthogonal tensor \( R \) times a symmetric tensor \( U \)

\[
F = RU
\]

In literature \( U \) is known as the \textit{material stretch tensor} and \( R \) as the \textit{rotation tensor}. On the basis of Eq. 2.13

\[
C = F^T F = U^T R^T RU
\]

\[
= U^2
\]

since \( R^T R = I \) and \( U^T = U \).

Because \( C \) is symmetric and positive definite, by means of the spectral decomposition theorem

\[
C = \sum_{i=1}^{3} \Lambda_i^2 L_i \otimes L_i
\]

in which \( \Lambda_i^2 > 0 \) are the eigenvalues and \( L_i \) are the corresponding orthonormal eigenvectors of \( C \). Also it can be shown that

\[
C^{-1} = \sum_{i=1}^{3} \Lambda_i^{-2} L_i \otimes L_i
\]

The eigenvalues \( \Lambda_i \) are also known as the \textit{material stretches} along the \textit{principal directions} \( L_i \). On the basis of Eq. 2.27 it is also valid

\[
U = \sum_{i=1}^{3} \Lambda_i L_i \otimes L_i
\]

Hoger & Carlson [1984] have shown that once the eigenvalues \( \Lambda_i \) are known, \( U \) can be computed directly without need for computation of the principal directions \( L_i \). Details are provided in Appendix 2.1.

Then

\[
R = FU^{-1}
\]

A physical interpretation of the role of tensors \( U \) and \( R \) in transforming the vector \( dX \) from the reference configuration to \( dx \) in the current can be obtained by combining Eq. 2.3 and Eq. 2.26

\[
dx = R \left( UdX \right)
\]

In this, vector \( dX \) is first stretched to \( UdX \) in the reference configuration and, subsequently rotated to the current configuration by \( R \), Fig. 2.4.
Fig. 2.4 Polar decomposition

\[ \mathbf{F} \text{ can be also decomposed as the product of a symmetric tensor } \mathbf{V} \text{ times the same as above tensor } \mathbf{R} \text{ so that} \]

\[ \mathrm{d}x = \mathbf{F}\mathrm{d}X = \mathbf{VR}\mathrm{d}X \quad 2.33 \]

In this, the material vector \( \mathrm{d}X \) is first rotated via \( \mathbf{R} \) and then stretched to \( \mathrm{d}x \). The symmetric Finger tensor \( \mathbf{b} \) can now be expressed as

\[ \mathbf{b} = (\mathbf{VR})(\mathbf{VR})^T = \mathbf{V}^2 \quad 2.34 \]

Also on the basis of Eq. 2.32 and Eq. 2.33 it results

\[ \mathbf{V} = \mathbf{RUR}^T \quad 2.35 \]

By means of the spectral theorem

\[ \mathbf{b} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{l}_i \otimes \mathbf{l}_i \quad 2.36 \]

in which \( \mathbf{l}_i \) are the orthogonal eigenvectors of \( \mathbf{b} \) and \( \lambda_i \) the spatial stretches.
Hence
\[ V = \sum_{i=1}^{3} \lambda_i l_i \otimes l_i \]  
2.37

Combining Eq. 2.35 and Eq. 2.30 the relationship between the material and spatial stretches and the corresponding principal directions is obtained as
\[ \Lambda_i = \lambda_i \quad ; \quad l_i = RL_i \]  
2.38

In anticipation of developments in later Chapters, an expression can be obtained for the Lagrangian-Green strain tensor in terms of the principal material stretches by means of Eq. 2.15 and Eq. 2.28 as
\[ E = \sum_{i=1}^{3} \frac{1}{2}(\Lambda_i^2 - 1) l_i \otimes l_i \]  
2.39

2.2.4 Volumetric/Deviatoric Multiplicative Split

In Eq. 2.26 the deformation gradient was decomposed as the product of an orthogonal \( R \) and a symmetric tensor \( U \). Another possible decomposition can be constructed if it is assumed that the overall deformation of a solid can be decomposed into volumetric and deviatoric (hence volume preserving) components.

Starting from the reference configuration, let
\[ dx_{iso} = F_{iso} dX \]  
2.40
denote the volume preserving (commonly termed as \textit{isochoric}) part of the total deformation.

Also, let
\[ dx = F_{vol} dx_{iso} \]  
2.41
denote the deformation necessary to recover the total deformation \( dx = F dX \).

From the above it results
\[ F = F_{vol} F_{iso} \]  
2.42

Eq. 2.42 is typical of a whole range of \( F \) decompositions typically known as \textit{multiplicative splits}, or \textit{multiplicative decompositions}, Simo & Hughes [1998], Belytschko et al. [2000]. They shall constitute in later Chapters the foundation for the development of some of the inelastic material formulations of CAPA-3D.

From Eq. 2.10, for a volume preserving deformation \( \det F_{iso} = J = 1 \). For this condition to be satisfied
\[ F_{iso} = J^{-\frac{1}{3}} F \]  
2.43

which provides a convenient expression for the computation of the deviatoric components of deformation. Then from Eq. 2.42 it becomes evident that
\[ F_{vol} = J^{\frac{1}{3}} F \]  
2.44
Once $\mathbf{F}_{iso}$ and $\mathbf{F}_{vol}$ are known, associated deviatoric and volumetric measures of deformation can be defined e.g.

$$
\mathbf{C}_{iso} = \mathbf{F}_{iso}^T \mathbf{F}_{iso} = J^{\frac{3}{2}} \mathbf{C} \quad ; \quad \mathbf{E}_{iso} = \frac{1}{2} \left( \mathbf{C}_{iso} - \mathbf{I} \right)
$$

$$
\mathbf{C}_{vol} = \mathbf{F}_{vol}^T \mathbf{F}_{vol} = J^{\frac{3}{2}} \mathbf{C} \quad ; \quad \mathbf{E}_{vol} = \frac{1}{2} \left( \mathbf{C}_{vol} - \mathbf{I} \right)
$$

### 2.2.5 Space and Time Derivatives

A linear approximation to the increment of a nonlinear scalar valued function $\Psi(X)$ in the direction of vector $\mathbf{u}$ can be obtained by means of a truncated Taylor series as

$$
\Psi(X + \varepsilon \mathbf{u}) \approx \Psi(X) + \frac{d}{d\varepsilon} \Psi(X + \varepsilon \mathbf{u}) \bigg|_{\varepsilon=0} = \Psi(X) + D_u \Psi(X)
$$

The differential term on the right-hand side of Eq. 2.46 is known from calculus as the directional derivative of $\Psi(X)$ in the direction of $\mathbf{u}$. A pictorial representation in a 2 dimensional space $\{X_1, X_2\}$ is shown in Fig. 2.5.

![Fig. 2.5  Linearisation of a function](image)

The material time derivative of an operator $\Psi$ in the reference configuration is defined as

$$
\frac{D\Psi(X,t)}{Dt} = \dot{\Psi}(X,t) = \frac{\partial \Psi(X,t)}{\partial t} \bigg|_X
$$

in which the subscript $X$ indicates differentiation exclusively with respect to time.
If the operator is defined in the current configuration then, by means of the chain rule of differentiation

\[ \dot{\psi}(x, t) = \frac{D\psi(x, t)}{Dt} \]

\[ = \frac{\partial \psi(x, t)}{\partial t} \bigg|_x + \frac{\partial \psi(x, t)}{\partial x} \bigg|_t \frac{\partial \phi_*[X, t]}{\partial t}_{|X = \phi_*^{-1}[x, t]} \]

\[ = \frac{\partial \psi(x, t)}{\partial t} \bigg|_x + \nabla \psi \bigg|_t \bigg|_{X = \phi_*^{-1}[x, t]} \]

In the above \( \frac{\partial \psi(x, t)}{\partial t} \) is commonly termed the *spatial time derivative* of \( \psi \) and \( \mathbf{v} \) is termed the *velocity vector*.

### 2.2.6 Rate of Deformation Tensors

The *spatial velocity gradient tensor* \( \mathbf{I} \) is defined by considering the relative velocity \( \mathbf{d} \mathbf{v} \) between two particles in the current configuration

\[ \mathbf{d} \mathbf{v} = \frac{\partial \mathbf{v}(x, t)}{\partial x} \mathbf{d}x = \nabla \mathbf{v} \mathbf{d}x = \mathbf{I} \mathbf{d}x \]

\( \mathbf{I} \) can also be defined on the basis of the deformation gradient \( \mathbf{F} \). Considering the material time derivative of \( \mathbf{F} \)

\[ \frac{D\mathbf{F}(X, t)}{Dt} = \ddot{\mathbf{F}}(X, t) = \frac{\partial}{\partial t} \left( \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right) = \frac{\partial}{\partial \mathbf{X}} \left( \frac{\partial \mathbf{x}}{\partial t} \right) = \frac{\partial \mathbf{v}}{\partial \mathbf{X}} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = I \mathbf{F} \]

so that

\[ I = \ddot{\mathbf{F}} \mathbf{F}^{-1} \]

The term *material velocity gradient* is commonly used for \( \ddot{\mathbf{F}} \), Holzapfel [2000].

A useful relationship can be obtained by associating the material time derivative of \( \mathbf{F} \) with the concept of the directional derivative. Applying the concept of directional derivative to the deformation gradient along the direction of the velocity vector \( \mathbf{v} \)

\[ D_{\mathbf{v}} \mathbf{F}(X) = \frac{d}{d\varepsilon} \left[ \frac{\partial (X + \varepsilon \mathbf{v})}{\partial \mathbf{X}} \right]_{\varepsilon \rightarrow 0} = \frac{d}{d\varepsilon} \left[ \frac{\partial \mathbf{x}}{\partial \mathbf{X}} + \varepsilon \frac{\partial \mathbf{v}}{\partial \mathbf{X}} \right]_{\varepsilon \rightarrow 0} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}} \]

Comparison of the above with Eq. 2.50 it results

\[ \ddot{\mathbf{F}}(X) = D_{\mathbf{v}} \mathbf{F}(X) \]
In Section 2.2.2 strain was defined by the change of length of a vector between the reference and the current configurations. Similarly, the strain rate can be defined as the time change of the magnitude of vector $dx$ in the current configuration

$$\frac{D}{Dt}[(ds)^2] = \frac{D}{Dt}[dx \cdot dx]$$

Expressing $dx$ on the basis of Eq. 2.3 (and noting that $dX$ is time independent)

$$\frac{D}{Dt}[(ds)^2] = dX \cdot (\dot{F}^T F + \dot{F}^T \dot{F}) dX$$

$$= dX \cdot \dot{C} \ dX$$

$$= 2dX \cdot \dot{\mathbf{E}} dX$$

in which the tensor

$$\dot{\mathbf{E}} = \frac{1}{2} \dot{\mathbf{C}} = \frac{1}{2} \left( \dot{F}^T F + \dot{F}^T \dot{F} \right)$$

is commonly known as the Lagrangian material strain rate tensor. It expresses the rate of change of $dx$ on the basis of its definition in the reference configuration.

Alternatively, $\dot{\mathbf{E}}$ can be expressed in terms of the principal material stretches $\Lambda_i, i=1,2,3$ by considering the time derivative of Eq. 2.39

$$\dot{\mathbf{E}} = \sum_{i=1}^{3} \Lambda_i \dot{\Lambda}_i \otimes L_i + \sum_{i=1}^{3} \frac{1}{3} \Lambda_i^2 \left( \dot{L}_i \otimes L_i + L_i \otimes \dot{L}_i \right)$$

The rate of change of the magnitude of $dx$ in the current configuration can be obtained by substituting in the last of Eq. 2.55 $dX = F^{-1} dx$

$$\frac{D}{Dt}[(ds)^2] = 2dx \cdot [F^{-T} \dot{F} F^{-1}] dx$$

$$= dx \cdot [F^{-T} \dot{F}^T + \dot{F} F^{-1}] dx$$

$$= 2dx \cdot d(dx)$$

The tensor

$$d = F^{-T} \dot{F} F^{-1}$$

is commonly known as the rate of deformation tensor. For a rigid body motion $d = 0$.

Substituting $\dot{F}$ from Eq. 2.50 into Eq. 2.56 and utilizing Eq. 2.59 it results that the symmetric component of the velocity gradient tensor constitutes the rate of deformation tensor

$$d = \frac{1}{2} (I + I^T) \quad ; \quad d^T = d$$

while the antisymmetric component constitutes a tensor commonly known as the spin tensor

$$w = \frac{1}{2} (I - I^T) \quad ; \quad w^T = -w$$
Its components are frequently defined on the basis of the associated angular velocity vector
\[ \omega = (\omega_1, \omega_2, \omega_3)^T \]
as
\[ w = \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix} \]
(2.62)

For use in later Chapters on material response simulation, an explicit relation between the spin
tensor and the rotation tensor can be obtained by substituting Eq. 2.51 in Eq. 2.61
\[ w = \dot{R}R^T + \frac{1}{2} R \left( \dot{U}U^{-1} - U^{-1}\dot{U} \right) R^T \]
(2.63)
which, for rigid body motions reduces to
\[ w = \dot{R}R^T \]
(2.64)

2.2.7 Lie Time Derivative
The Lie time derivative \( \mathcal{L} \) is a powerful conceptual tool, which will be frequently utilized in
the following Sections in operations involving the transformation of various mechanical
quantities between the current and the reference configurations.

If \( g = g(x, t) \) denotes a function in the current configuration the Lie time derivative is
obtained by means of the following operations:

i. perform a pull back operation on \( g \) to obtain \( G(X, t) = \phi_*^{-1}[g(x, t)] \)
ii. compute the material derivative \( \dot{G}(X, t) \)
iii. perform a push forward operation on \( \dot{G} \) to obtain \( \mathcal{L}(g) = \phi_* \left[ \dot{G}(X, t) \right] \)

Since, in the reference configuration, the material derivative is equivalent to the directional
derivative along the velocity vector \( v \) (i.e. Eq. 2.53) it holds
\[ \dot{G}(X, t) = D_v G(X, t) \]
(2.65)
so that \( \mathcal{L}(g) \) can also be expressed as
\[ \mathcal{L}(g) = \phi_* \left[ \dot{G}(X, t) \right] = \phi_* \left[ D_v G(X, t) \right] \]
(2.66)

2.2.8 Objectivity
Objectivity constitutes one of the most fundamental principles of mechanics, Malvern [1969],
Simo & Hughes [1998]. Quantities which are used to describe material behavior must be
objective. The notion of objectivity can be physically illustrated by imposing an orthogonal
transformation \( R \) on a vector \( dx = FdX \). The resulting vector can be expressed as
\[ d\dot{x} = Rdx \]
(2.67)
By means of Eq. 2.13 it can be shown that the magnitudes of $\text{d}x$ and $\text{d}\bar{x}$ are equal. In the mechanics literature, vectors which under rigid body rotations transform according to Eq. 2.67 are termed *objective*.

In contrast to many second order tensors whose behavior under orthogonal transformations will be examined in the following, it is worth noticing that the deformation gradient tensor $\mathbf{F}$ under orthogonal transformations transforms according to Eq. 2.67 since

$$\bar{\mathbf{F}} = \frac{\partial \bar{\mathbf{x}}}{\partial \mathbf{x}} = \frac{\partial (\mathbf{R} \mathbf{x})}{\partial \mathbf{x}} = \mathbf{R} \left( \frac{\partial \mathbf{x}}{\partial \mathbf{x}} \right) = \mathbf{R} \mathbf{F}$$  \hspace{1cm} 2.68

The notion of objectivity can be extended to tensors of any order. In particular, if an orthogonal transformation $\mathbf{R}$ is imposed to a second order tensor $\mathbf{S} = \mathbf{d}x_1 \otimes \mathbf{d}x_2$, the rotated tensor $\bar{\mathbf{S}} = \mathbf{d}\bar{x}_1 \otimes \mathbf{d}\bar{x}_2$ can be expressed by means of Eq. 2.67 as

$$\bar{\mathbf{S}} = \mathbf{d}\bar{x}_1 \otimes \mathbf{d}\bar{x}_2$$

$$= (\mathbf{R} \mathbf{d}x_1) \otimes (\mathbf{R} \mathbf{d}x_2)$$

$$= \mathbf{R} (\mathbf{d}x_1 \otimes \mathbf{d}x_2) \mathbf{R}^T$$

$$= \mathbf{R} \mathbf{S} \mathbf{R}^T$$  \hspace{1cm} 2.69

In the mechanics literature second order tensors which transform according to Eq. 2.69 are called *objective*, Simo & Hughes [1998], Holzapfel [2000]. Several of the deformation measures introduced in the above like $\mathbf{C}$, $\mathbf{E}$, $\mathbf{d}$ and $\mathbf{e}$ are objective. In Appendix 2.2 a proof for the objectivity of $\mathbf{e}$ is included. Similar steps can be followed for other tensors.

Examples of non-objective quantities are the velocity vector $\mathbf{v}$ and the velocity gradient tensor $\mathbf{I}$. More examples will be encountered in subsequent sections.

### 2.2.9 Volume Changes

The rate of volume change can be computed by differentiating with respect to time Eq. 2.10

$$\frac{d}{dt} \dot{\mathbf{v}} = \frac{d}{dt} (\mathbf{J} \mathbf{d}V) = \dot{\mathbf{J}} \mathbf{d}V + \mathbf{J} \frac{d}{dt} \mathbf{d}V = \dot{\mathbf{J}} \mathbf{d}V = \frac{\dot{\mathbf{J}}}{\mathbf{J}} \mathbf{d}V$$  \hspace{1cm} 2.70

The relationship between time and directional derivative indicated by Eq. 2.53 can be now utilised to compute the material derivative of $\mathbf{J}$ by computing the directional derivative along the velocity vector $\mathbf{v}$

$$\dot{\mathbf{J}} = \mathbf{D}_\mathbf{v} \mathbf{J} = \mathbf{J} \text{ tr}(\mathbf{I}) = \mathbf{J} \text{ tr}(\mathbf{d} + \mathbf{w})$$

$$= \mathbf{J} \text{ tr}(\mathbf{d})$$  \hspace{1cm} 2.71

### 2.3 Stress Tensor

When an infinitesimal force $\text{d}q$ is acting on an infinitesimal element of surface area $\text{d}a$ perpendicular to vector $\mathbf{n}$, the traction vector is defined as
\[ t = \lim_{da \to 0} \left( \frac{dq}{da} \right) \tag{2.72} \]

In the current configuration, the Cartesian components of the traction vector \( t \), Fig. 2.6(a), are related to the Cartesian components of the true or Cauchy stress tensor \( \sigma \), Fig. 2.6(b), via

\[ t = \sigma n \tag{2.73} \]

with

\[ t = \sum_{i=1}^{3} t_i e_i ; \quad \sigma = \sum_{i,j=1}^{3} \sigma_{ij} e_i \otimes e_j ; \quad n = \sum_{i=1}^{3} n_i e_i \tag{2.74} \]

Fig. 2.6 Traction and stresses acting on an infinitesimal element

Objectivity of the Cauchy stress tensor can be demonstrated by considering the effect of imposing an orthogonal rotation \( R \) to the traction vector \( t \)

\[ \tilde{t} = R t \tag{2.75} \]

By means of Eq. 2.73, Eq. 2.75 and the relation \( \tilde{n} = R n \) it can be easily shown that the rotated Cauchy stress \( \tilde{\sigma} \) conforms with the definition of objectivity expressed by Eq. 2.69

\[ \tilde{\sigma} = R \sigma R^T \tag{2.76} \]

It is also possible to define a traction vector \( T \) with respect to the reference configuration. If \( N \) is the vector normal to the infinitesimal area \( dA \) then

\[ T = \lim_{dA \to 0} \left( \frac{dq}{dA} \right) \tag{2.77} \]
The corresponding stress tensor is known as the first Piola-Kirchhoff stress tensor \( \mathbf{P} \)

\[
\mathbf{T} = \mathbf{P} \mathbf{N}
\]

2.78

The interrelationship between \( \mathbf{\sigma} \) and \( \mathbf{P} \) can be determined by exploiting the observation

\[
d\mathbf{q} = \mathbf{T} \, dA = t \, da
\]

2.79

Introducing Eq. 2.73 and Eq. 2.78 in the above and utilizing Eq. 2.12 it results

\[
d\mathbf{q} = J \mathbf{\sigma} \mathbf{F}^{-T} \, dA
\]
\[
= \mathbf{P} \, dA
\]

2.80

from which it can be deduced that \( \mathbf{P} \) is a non symmetric stress tensor.

An artificial infinitesimal load vector \( d\mathbf{Q} \) can be defined by the pull-back operation

\[
d\mathbf{Q} = \mathbf{F}^{-1} \, d\mathbf{q}
\]

2.81

and on the basis of Eq. 2.80

\[
d\mathbf{Q} = \mathbf{F}^{-1} \, J \mathbf{\sigma} \mathbf{F}^{-T} \, dA
\]
\[
= \mathbf{S} \, dA
\]

2.82

The symmetric tensor \( \mathbf{S} \) defined in the reference configuration is commonly known as the second Piola-Kirchhoff stress tensor. It has the advantage of being objective with respect to superimposed rotations.

In terms of the push-forward and pull-back terminology, \( \mathbf{S} \) can be considered as the pull-back equivalent of \( \mathbf{\sigma} \) i.e.

\[
\mathbf{S} = J \phi_0^{-1} [\mathbf{\sigma}] = J \mathbf{F}^{-1} \mathbf{\sigma} \mathbf{F}^{-T} = \mathbf{F}^{-1} \mathbf{P}
\]

2.83

Equivalently

\[
\mathbf{\sigma} = J^{-1} \phi_* [\mathbf{S}] = J^{-1} \mathbf{F} \mathbf{S} \mathbf{F}^T
\]

2.84

Pull-back and/or push-forward operations combined with volume scaling are termed Piola transformations in literature, Marsden & Hughes [1994].

2.3.1 Rate of Stress Tensors

In similarity to the rates of deformation tensors developed in Section 2.2.6, various rate of stress tensors can also be developed. By its nature, the rate of the second Piola-Kirchhoff stress tensor \( \dot{\mathbf{S}} \) is trivially objective. On the other hand, the material derivative of the Cauchy stress tensor \( \dot{\mathbf{\sigma}} \) is not objective because it can be expanded as

\[
\dot{\mathbf{\sigma}} = \mathbf{R} \mathbf{\sigma} \mathbf{R}^T + \mathbf{R} \dot{\mathbf{\sigma}} \mathbf{R}^T + \mathbf{R} \mathbf{\sigma} \mathbf{R}^T
\]

2.85

In the current configuration, a stress rate tensor can be defined by means of the Piola transformation as the push-forward operation on \( \dot{\mathbf{S}} \)
\[
\dot{\sigma}_T = J^{-1} \phi_* [\dot{\mathbf{S}}]
\]

On the basis of Eq. 2.83

\[
\dot{\mathbf{S}} = \frac{d\mathbf{S}}{dt} = \frac{d}{dt} \left[ J \mathbf{F}^{-1} \sigma \mathbf{F}^{-T} \right]
\]

The push-forward operation on \( \dot{\mathbf{S}} \) is

\[
\phi_* [\dot{\mathbf{S}}] = \mathbf{F} \left( \frac{d}{dt} \left[ J \mathbf{F}^{-1} \sigma \mathbf{F}^{-T} \right] \right) \mathbf{F}^T
\]

By means of Eq. 2.51

\[
\dot{\mathbf{F}}^{-1} = \frac{d}{dt} (\mathbf{F}^{-1}) = -\mathbf{F}^{-1} \mathbf{l}
\]

and similarly

\[
\dot{\mathbf{F}}^{-T} = \frac{d}{dt} (\mathbf{F}^{-T}) = -\mathbf{l}^T \mathbf{F}^{-T}
\]

Also from Eq. 2.71 \( \dot{\mathbf{J}} = J \mathbf{tr}(\mathbf{l}) \). Substituting successively in Eq. 2.88 and Eq. 2.86, after rearrangement, the \textit{Truesdell rate of Cauchy stress tensor} is obtained as

\[
\dot{\sigma}_T = \dot{\sigma} - \dot{\mathbf{l}} \sigma - \sigma \dot{\mathbf{l}}^T + \mathbf{tr}(\mathbf{l}) \sigma
\]

The operations defined by Eq. 2.87 and Eq. 2.88 constitute the basis of the Lie derivative as presented in Section 2.2.7. As such

\[
\dot{\sigma}_T = J^{-1} \mathcal{L}(\sigma) = J^{-1} \phi_* \left[ \frac{d}{dt} \left( \phi_*^{-1}(\sigma) \right) \right]
\]

Marsden & Hughes [1994] demonstrate that several objective rates of stress tensors can be constructed by its utilization.

### 2.4 Mechanical Balance Laws

The balance laws constitute fundamental tools in addressing the translational and the rotational equilibrium of a body subjected to a set of actions. In addition to enabling the establishment of the equilibrium equation for the body, they provide insight into the characteristics of the stress tensor of the constituent material.

Since attention is focussed on isothermal processes, only mechanical balance laws are considered.

#### 2.4.1 Balance of Momentum

In the current configuration the momentum of a deformable body in the current configuration is defined as

\[
m = \int_{V} \rho \mathbf{v} \, dv
\]
in which $\rho$ is the density of the material and $\mathbf{v}$ is the velocity vector defined in Section 2.2.4. The principle of balance of momentum states that

$$\frac{Dm}{Dt} = \mathbf{F}$$  \hspace{1cm} 2.94

in which $\mathbf{F}$ is the total force acting on the body.

![Diagram of body actions]

Fig. 2.7 Body actions

For the deformable body shown in Fig. 2.7 it is expressed as

$$\frac{D}{Dt} \int_{\mathbf{v}} \rho \mathbf{v} d\mathbf{v} = \int_{\mathbf{a}} \mathbf{t} d\mathbf{a} + \int_{\mathbf{v}} \mathbf{f} d\mathbf{v}$$  \hspace{1cm} 2.95

in which $\mathbf{f}$ represents the body forces per unit volume and $\mathbf{t}$ the tractions acting on the boundary per unit surface area with normal $\mathbf{n}$. Defining $\mathbf{dv}/dt = \mathbf{a}$ and taking into account Eq. 2.73

$$\int_{\mathbf{v}} \rho \mathbf{a} d\mathbf{v} = \int_{\mathbf{a}} \mathbf{\sigma} n d\mathbf{a} + \int_{\mathbf{v}} \mathbf{f} d\mathbf{v}$$  \hspace{1cm} 2.96

which by means of the Gauss theorem is transformed to

$$\int_{\mathbf{v}} (\text{div} \mathbf{\sigma} + f - \rho \mathbf{a}) d\mathbf{v} = 0$$  \hspace{1cm} 2.97

For Eq. 2.97 to be valid for any arbitrary volume $\mathbf{v}$ of the body, the necessary and sufficient condition is the vanishing of the integrand at any point of the body, hence,

$$\text{div} \mathbf{\sigma} + f - \rho \mathbf{a} = 0$$  \hspace{1cm} 2.98

which is the classical local spatial equilibrium equation for a deformable body.
Within the context of nonlinear analysis, convergence to the desired solution is equivalent to local fulfillment of Eq. 2.98. During the iterative process, discrepancies between the internal and the external forces per unit volume give rise to residual or unbalanced forces

\[ \mathbf{r} = \text{div} \sigma + \mathbf{f} - \rho \mathbf{a} \neq 0 \]  

which must be redistributed via additional iterations.

### 2.4.2 Balance of Moment of Momentum

The total moment of momentum of a deformable body in the current configuration is

\[ \mathbf{M} = \int_v \rho \mathbf{x} \times \mathbf{v} \, dv \]  

The principle of balance of moment of momentum for a deformable body can be expressed as

\[ \frac{D\mathbf{M}}{Dt} = \mathbf{M} \]  

For the deformable body shown in Fig. 2.7 it is expressed as

\[ \int_v \rho \mathbf{x} \times \mathbf{a} \, dv = \int_v \mathbf{x} \times \text{div} \sigma \, dv + \int_v \mathbf{x} \times \mathbf{f} \, dv \]  

By means of the Gauss theorem Eq. 2.102 can be transformed, Bonnet & Wood [1997], to

\[ \int_v \rho \mathbf{x} \times \mathbf{a} \, dv = \int_v \mathbf{x} \times \text{div} \sigma \, dv + \int_v \mathbf{E} : \sigma^T \, dv + \int_v \mathbf{x} \times \mathbf{f} \, dv \]  

in which \( \mathbf{E} \) is the third order permutation tensor whose terms are defined as

\[ E_{ijk} = e_i \cdot (e_j \times e_k) \]  

Rearranging and taking into account Eq. 2.97

\[ \int_v \mathbf{E} : \sigma^T \, dv = \int_v \mathbf{x} \times \{\text{div} \sigma + \mathbf{f} - \rho \mathbf{a}\} \, dv = 0 \]  

For Eq. 2.105 to be valid for any arbitrary volume \( v \) of the body, the necessary and sufficient condition is the vanishing of the integrand, from which it results

\[ \mathbf{E} : \sigma^T = 0 \]  

In Appendix 2.3 it is shown that Eq. 2.106 can be elaborated to

\[ \mathbf{E} : \sigma^T = \begin{bmatrix} \sigma_{32} - \sigma_{23} \\ \sigma_{13} - \sigma_{31} \\ \sigma_{21} - \sigma_{12} \end{bmatrix} = 0 \]  

which establishes the symmetric format of the Cauchy stress tensor.
2.5 Variational Concepts

Variational concepts provide a valuable tool for discretization of the balance laws and the formulation of finite element techniques.

2.5.1 Virtual Displacements and their Variations

Given a deformable body \( V \) at position \( x \), a new, completely independent of \( u \) displacement field \( z \) can be defined, Fig. 2.8, which results to a translation of the body to

\[
\bar{u} = u + \varepsilon z
\]

Fig. 2.8 Virtual configuration in the vicinity of \( x \)

Postulating \( \varepsilon \to 0 \), the displacement field defined by

\[
\delta u = \bar{u} - u = \varepsilon z
\]

is termed in the mechanics literature as a *virtual displacement field*. In contrast to actual infinitesimal displacement changes \( du \) which are time dependent, \( \delta u \) is postulated to occur at a fixed instant of time.

On the basis of Eq. 2.109 the following useful relation can be proven

\[
\delta \nabla u = \nabla \delta u
\]

Let \( G(U,t) \) be a vector function in the reference configuration. The *first variation* of \( G(U,t) \) is defined as the directional derivative of the function evaluated at a fixed \( U \) and along the direction \( \delta U \)
Fundamentals

\[ \delta G(U, \delta U) = D_{\delta U} G(U) = \left. \frac{d}{d\varepsilon} G(U + \varepsilon \delta U) \right|_{\varepsilon=0} \]

The **first variation in the current configuration** of a vector function \( g(u) \) at a fixed \( u \) in the direction \( \delta u \) can be elegantly obtained by means of an operation similar to the Lie time derivative in which the time derivative has been replaced by the directional derivative \( D_{\delta u} \).

\[
\delta g(u, \delta u) = \phi_* \left[ D_{\delta U} \phi_*^{-1} g(u, \delta u) \right] \\
= \phi_* \left[ D_{\delta U} G(U) \right]
\]

Then, because of Eq. 2.111

\[ \delta g(u, \delta u) = \phi_* \left[ \delta G(U, \delta U) \right] \]

2.5.2 Virtual Work

In Section 2.4.1 it was indicated that within the context of nonlinear analysis, the vector sum \( r \) indicated by Eq. 2.99 represents a residual force vector per unit volume. The virtual work done by this force during a virtual motion \( \delta u \) can be expressed by the dot product \( r \cdot \delta u \).

In order to preempt discretization in later sections in both the space and the time domains, the virtual work per unit volume and unit time can be expressed as

\[ \delta W = r \cdot \delta v \]

Integrating over the volume of the body, from Eq. 2.98, at equilibrium,

\[ \delta W = \int \left( \text{div} \sigma + f - \rho a \right) \cdot \delta v \, dv = 0 \]

In Appendix 2.4 it is shown that the above equation is equivalent to

\[ \int \sigma : \delta \mathbf{d} \, dv = \int \mathbf{f} \cdot \delta \mathbf{v} \, dv + \int \mathbf{t} \cdot \delta \mathbf{v} \, da - \int \rho \mathbf{a} \cdot \delta \mathbf{v} \, dv \]

This equation expresses the equilibrium of the system in the current configuration under the applied set of actions. It is independent of material type and as such it is valid for both elastic and inelastic materials.

Using Eq. 2.116 as a basis, equilibrium of the structural system in the reference configuration can also be established. From Eq. 2.82, the relation between the Cauchy stress tensor \( \sigma \) defined in the current configuration and the second Piola-Kirchhoff stress tensor \( S \) defined in the reference configuration is

\[ \sigma = J^{-1} F S F^T \]
The concept of the Lie time derivative can be utilized for the elegant transformation of the virtual rate of deformation tensor \( \delta \mathbf{d} \) to the reference configuration. According to the steps delineated in Section 2.2.7:

\[
\delta \mathbf{d} = \phi_* \left[ D_{\delta u} \phi_*^{-1} \mathbf{d} \right] = \phi_* \left[ D_{\delta u} \left( \mathbf{F}^T \mathbf{dF} \right) \right] \\
= \phi_* \left[ D_{\delta u} \dot{\mathbf{E}} \right] = \phi_* \left[ \delta \dot{\mathbf{E}} \right] \\
= \mathbf{F}^{-T} \delta \dot{\mathbf{E}} \mathbf{F}^{-1}
\]

The first term of Eq. 2.116 then can be rewritten as

\[
\int_V \sigma : \delta \mathbf{d} \, dV = \int_V J \sigma : \left( \mathbf{F}^{-T} \delta \dot{\mathbf{E}} \mathbf{F}^{-1} \right) \, dV = \int_V \text{tr} \left[ J \sigma \mathbf{F}^{-T} \delta \dot{\mathbf{E}} \mathbf{F}^{-1} \right] \, dV \\
= \int_V \text{tr} \left[ J \mathbf{F}^{-1} \sigma \mathbf{F}^{-T} \delta \dot{\mathbf{E}} \right] \, dV = \int_V \left( J \mathbf{F}^{-1} \sigma \mathbf{F}^{-T} \right) : \delta \dot{\mathbf{E}} \, dV \\
= \int_V \mathbf{S} : \delta \dot{\mathbf{E}} \, dV
\]

Similarly, utilizing \( f_0 = J \mathbf{f} \) and considering the fact that \( dA \) is related to \( da \) via Eq. 2.12, Eq. 2.116 can be finally recast in the reference system as

\[
\int_V \mathbf{S} : \delta \dot{\mathbf{E}} \, dV = \int_V f_0 : \delta \mathbf{v} \, dV + \int_A t_0 : \delta \mathbf{v} \, dA - \int_V \rho_0 \mathbf{a} : \delta \mathbf{v} \, dV
\]

Symbolically

\[
\delta W = \delta W_{\text{int}} - \delta W_{\text{ext}} = 0
\]

with

\[
\delta W_{\text{int}} = \int_V \mathbf{S} : \delta \dot{\mathbf{E}} \, dV
\]

and

\[
\delta W_{\text{ext}} = \int_V f_0 : \delta \mathbf{v} \, dV + \int_A t_0 : \delta \mathbf{v} \, dA - \int_V \rho_0 \mathbf{a} : \delta \mathbf{v} \, dV
\]

### 2.5.3 Clausius-Planck Inequality

Pairs such as \( \sigma \) and \( \mathbf{d} \) in Eq. 2.116, \( S \) and \( \dot{\mathbf{E}} \) in Eq. 2.120, \( \mathbf{P} \) and \( \dot{\mathbf{F}} \) etc. are termed work conjugate in the sense that their double contraction results to work per unit volume. On the basis of the 2\(^{\text{nd}}\) law of thermodynamics, for the case of a purely mechanical theory\(^1\), the Clausius-Planck inequality states that, at any point in a body, at all times, the internal dissipation \( D \) should be non-negative

\[
\mathcal{D} = \mathbf{P} : \dot{\mathbf{F}} - \dot{\Psi} \geq 0
\]

\( \dot{\Psi} \) is known as the Helmholz free-energy. In the special case where \( \dot{\Psi} \) is solely a function of \( \mathbf{F} \) it is also known as the strain energy or the stored energy.

---

\(^1\) i.e. if thermal effects are ignored
Fundamentals

A mechanical process is termed reversible if \( D = 0 \). This condition is true for elastic materials, that is, materials for which the stress at any time is a function only of the state of deformation \( F(X) \) (and the temperature).

If \( D > 0 \), the process is termed irreversible and the corresponding material dissipative. Plastic materials are typical examples.

In contrast to the non-dissipative case, \( \Psi \) is not, anymore, exclusively a function of \( F \) but it depends also on a number of additional variables associated with the memory properties of the material. They represent the irreversible physical processes within the microstructure of the material, which occur throughout the deformation history and which are the causes of nonlinearity. Then \( \Psi \) can be typically expressed as

\[
\Psi = \Psi \left( F(X), \xi_1, \xi_2, \ldots, \xi_n \right)
\]

The term internal variables is commonly utilised for the \( \xi \)'s. Depending on the particular characteristics of a material, their number \( n \) and their individual nature can vary.\(^2\)

Substituting in Eq. 2.124 \( \dot{\Psi} \) with

\[
\dot{\Psi} = \dot{\Psi} \left( F(X), \xi_1, \xi_2, \ldots, \xi_n \right) = \frac{\partial \Psi}{\partial F} : \dot{F} + \sum_{i=1}^{n} \frac{\partial \Psi}{\partial \xi_i} : \dot{\xi}_i
\]

the dissipation inequality for a dissipative material is obtained as

\[
D = P : \dot{F} - \dot{\Psi} = \left[ P - \frac{\partial \Psi}{\partial F} \right] : \dot{F} - \sum_{i=1}^{n} \frac{\partial \Psi}{\partial \xi_i} : \dot{\xi}_i \geq 0
\]

For \( D \) to be non-negative for every choice of \( \dot{F} \) the necessary conditions, Coleman & Noll [1963], Coleman & Gurtin [1967], are

\[
P = \frac{\partial \Psi}{\partial F} ; \quad - \sum_{i=1}^{n} \frac{\partial \Psi}{\partial \xi_i} : \dot{\xi}_i \geq 0
\]

In order to describe the development of the irreversible processes in the material, evolution laws must be specified for each of the internal variables

\[
\dot{\xi}_i = \Phi \left( F(X), \xi_1, \xi_2, \ldots, \xi_n \right)
\]

In general they form a system of ordinary first order differential equations. The dissipation inequality will be utilised in later Chapters for the derivation of some fundamental characteristics of elastoplastic and viscoelastic materials.

\(^2\) Without a loss of generality and for reasons of notational convenience only, the internal variables in this Section have been assumed as second order tensors.
2.6 Linearized Equilibrium Equation

The equilibrium expression in the form of Eq. 2.121 is, in general, a nonlinear equation. Nonlinearities may arise from the material response and/or the geometry. Within the context of the nonlinear analysis, the equilibrium configuration \( \phi \) is sought by means of an iterative type technique. Over the years several variants of Newton’s technique have been developed. A common characteristic of all these techniques is the replacement of the nonlinear equation with a linearized one in the vicinity of the expected solution.

According to Section 2.2.5, in the vicinity of a trial solution \( \mathbf{x} = \mathbf{x}_0 \) of \( \delta W(\mathbf{x}) \), a linear approximation to \( \delta W \) in the direction of an increment \( \Delta \mathbf{u} \) is

\[
\delta W(\mathbf{x}_0) + \mathbf{D}_{\Delta \mathbf{u}} \delta W(\mathbf{x}_0) = 0
\]

2.130

From Eq. 2.121

\[
\mathbf{D}_{\Delta \mathbf{u}} \delta W = \mathbf{D}_{\Delta \mathbf{u}} \delta W_{\text{int}} - \mathbf{D}_{\Delta \mathbf{u}} \delta W_{\text{ext}}
\]

2.131

in which \( \delta W_{\text{int}} \) is defined in Eq. 2.122 and \( \delta W_{\text{ext}} \) in Eq. 2.123.

In the following, the steps necessary for computation of the terms of Eq. 2.131 will be presented.

2.6.1 Linearized Internal Virtual Work

On the basis of Eq. 2.122 the linearization of \( \delta W_{\text{int}} \) along \( \Delta \mathbf{u} \) leads to

\[
\mathbf{D}_{\Delta \mathbf{u}} \delta W_{\text{int}} = \int_V \mathbf{D}_{\Delta \mathbf{u}} \left( \mathbf{S} : \delta \dot{\mathbf{E}} \right) dV
\]

\[
= \int_V \mathbf{D}_{\Delta \mathbf{u}} \mathbf{S} : \delta \dot{\mathbf{E}} dV + \int_V \mathbf{S} : \mathbf{D}_{\Delta \mathbf{u}} \delta \dot{\mathbf{E}} dV
\]

2.132

Considering the linearization of \( \mathbf{S} \) along \( \Delta \mathbf{u} \)

\[
\mathbf{D}_{\Delta \mathbf{u}} \mathbf{S} = \frac{d}{d\varepsilon} \left[ \mathbf{S} \left( \mathbf{E} \left( \phi + \varepsilon \Delta \mathbf{u} \right) \right) \right] = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} : \left. \frac{\partial \mathbf{E}}{\partial \varepsilon} \right|_{\varepsilon \rightarrow 0} = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} : \mathbf{D}_{\Delta \mathbf{u}} \mathbf{E}
\]

2.133

The term \( \left( \frac{\partial \mathbf{S}}{\partial \mathbf{E}} \right) \) represents the well known 4th order material or Lagrangian elasticity tensor \( \mathbf{C} \)

\[
\mathbf{C} = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} = \frac{\partial \mathbf{S}}{\partial \mathbf{C}} \frac{\partial \mathbf{C}}{\partial \mathbf{E}} = 2 \frac{\partial \mathbf{S}}{\partial \mathbf{C}}
\]

2.134

with individual terms

\[
\mathbf{C}_{ijkl} = \frac{\partial S_{ij}}{\partial E_{kl}} = 2 \frac{\partial S_{ij}}{\partial C_{kl}}
\]

2.135

then

\[
\mathbf{D}_{\Delta \mathbf{u}} \mathbf{S} = \mathbf{C} : \mathbf{D}_{\Delta \mathbf{u}} \mathbf{E}
\]

2.136
Fundamentals

From Eq. 2.56
\[ \delta \dot{E} = \frac{1}{2} \delta \left( \dot{F}^T F + F^T \dot{F} \right) \]

Since the current configuration is kept constant during the velocity field variation
\[ \delta \mathbf{F} = D_{\delta v} \left( \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \right) = 0 \]
so that
\[ \delta \dot{E} = \frac{1}{2} \left( \delta \dot{F}^T F + F^T \delta \dot{F} \right) \]

Additionally
\[ \delta \dot{F} = D_{\delta v} \left( \frac{\partial \mathbf{v}}{\partial \mathbf{X}} \right) = \frac{d}{dx} \left[ \frac{\partial \mathbf{v} + \varepsilon \delta \mathbf{v}}{\partial \mathbf{X}} \right] \bigg|_{x=0} = \frac{\partial \delta \mathbf{v}}{\partial \mathbf{X}} = \nabla \delta \mathbf{v} \]

Hence for the term \( D_{\Delta u} \delta \dot{E} \) of Eq. 2.132 it holds
\[ D_{\Delta u} \delta \dot{E} = \frac{1}{2} D_{\Delta u} \left( \delta \dot{F}^T F + F^T \delta \dot{F} \right) \]
\[ = \frac{1}{2} \left[ D_{\Delta u} \delta \dot{F}^T F + \delta \dot{F}^T D_{\Delta u} F + D_{\Delta u} F^T \delta \dot{F} + F^T D_{\Delta u} \delta \dot{F} \right] \]

From Eq. 2.140
\[ D_{\Delta u} \delta \dot{F} = D_{\Delta u} \left( \frac{\partial \delta \mathbf{v}}{\partial \mathbf{X}} \right) = 0 \]

because the virtual velocities field is independent of the displacement field. On the basis of the above and Eq. 2.140, Eq. 2.141 is simplified to
\[ D_{\Delta u} \delta \dot{E} = \frac{1}{2} \left[ \delta \dot{F}^T D_{\Delta u} F + D_{\Delta u} F^T \delta \dot{F} \right] \]
\[ = \frac{1}{2} \left[ (\nabla \delta \mathbf{v})^T \nabla \Delta \mathbf{u} + (\nabla \Delta \mathbf{u})^T \nabla \delta \mathbf{v} \right] \]
\[ = \text{sym} \left[ (\nabla \Delta \mathbf{u})^T \nabla \delta \mathbf{v} \right] \]

Substituting Eq. 2.136 and Eq. 2.143 into Eq. 2.132 and utilizing the properties of the trace \( A : B = \text{tr}(A^T B) = \text{tr}(AB^T) \) and the fact that \( \mathbf{S} \) is symmetric
\[ D_{\Delta u} \delta W_{\text{int}} = \int_V \delta \dot{E} : \mathbf{C} : D_{\Delta u} \mathbf{E} dV + \int_V \mathbf{S} : \text{sym} \left[ (\nabla \Delta \mathbf{u})^T \nabla \delta \mathbf{v} \right] dV \]

The 1st term of Eq. 2.144 can be elaborated further. On the basis of Eq. 2.140
\[ \delta \dot{F} = \nabla \delta \mathbf{v} = D_{\delta v} \mathbf{F} \]

hence, from Eq. 2.139, \( \delta \dot{E} \) can be also expressed as
\[
\delta \dot{\mathbf{E}} = \frac{1}{2} [D_{\delta v} \mathbf{F}^T \mathbf{F} + \mathbf{F}^T D_{\delta v} \mathbf{F}] = D_{\delta v} \mathbf{E}
\]

so that Eq. 2.144 can be written as

\[
D_{\Delta u} \delta W_{\text{int}} = \int_V D_{\delta v} \mathbf{E} : \mathbf{C} : D_{\Delta u} \mathbf{E} dV + \int_V \mathbf{S} : \text{sym} \left[ (\nabla_0 \Delta u)^T \nabla_0 \delta v \right] dV
\]

with

\[
D_{\Delta u} \mathbf{E} = \frac{1}{2} \left[ D_{\Delta u} \mathbf{F}^T \mathbf{F} + \mathbf{F}^T D_{\Delta u} \mathbf{F} \right] = \frac{1}{2} \left[ (\nabla_0 \Delta u)^T \mathbf{F} + \mathbf{F}^T \nabla_0 \Delta u \right] = \text{sym} \left[ \mathbf{F}^T \nabla_0 \Delta u \right]
\]

Linearization in the current configuration can be elegantly performed by simple push forward and pull back operations of the various terms of Eq. 2.147, Bonet & Wood [1999], Holzapfel [2000].

Applying a push forward operation on \( D_{\Delta u} \mathbf{E} \) and utilizing the relation \( \nabla_0 \Delta u = \nabla \Delta u \mathbf{F} \)

\[
\phi_* \left[ D_{\Delta u} \mathbf{E} \right] = \mathbf{F}^{-T} D_{\Delta u} \mathbf{E} \mathbf{F}^{-1}
\]

\[
= \frac{1}{2} \left[ \mathbf{F}^{-T} (\nabla_0 \Delta u)^T + \nabla_0 \Delta u \mathbf{F}^{-1} \right]
\]

\[
= \frac{1}{2} \left[ (\nabla_0 \Delta u)^T + \nabla_0 \Delta u \right] \Rightarrow D_{\Delta u} \mathbf{E} = \mathbf{F}^T \varepsilon \mathbf{F}
\]

Similarly, the push forward operation of \( D_{\delta v} \mathbf{E} \) is

\[
\phi_* \left[ D_{\delta v} \mathbf{E} \right] = \phi_* [\delta \dot{\mathbf{E}}] = \mathbf{F}^{-T} \delta \dot{\mathbf{E}} \mathbf{F}^{-1}
\]

\[
= \frac{1}{2} \left[ \mathbf{F}^{-T} \delta \ddot{\mathbf{F}}^T + \delta \dot{\mathbf{F}} \mathbf{F}^{-1} \right] = \frac{1}{2} \left[ \mathbf{F}^{-T} (\nabla_0 \delta \mathbf{v})^T + (\nabla_0 \delta \mathbf{v}) \mathbf{F}^{-1} \right]
\]

However, from Eq. 2.60

\[
\delta \mathbf{d} = \frac{1}{2} \left[ (\nabla \delta \mathbf{v})^T + \nabla \delta \mathbf{v} \right]
\]

Hence

\[
\phi_* \left[ D_{\delta v} \mathbf{E} \right] = \mathbf{F}^{-T} D_{\delta v} \mathbf{E} \mathbf{F}^{-1}
\]

\[
= \mathbf{F}^{-T} \delta \dot{\mathbf{E}} \mathbf{F}^{-1} = \delta \mathbf{d} \Rightarrow D_{\delta v} \mathbf{E} = \mathbf{F}^T \delta \mathbf{d} \mathbf{F}
\]
In Appendix 2.5 it is shown that on the basis of Eq. 2.149 and Eq. 2.152, the term in the first integral of the linearized virtual work expression in the reference configuration can be expressed in the current configuration as
\[
D_{\delta v} E : C : D_{\Delta u} E \, dV = \delta d : c : \varepsilon \, dv
\]
2.153
in which \( c \) is the spatial or Eulerian elasticity tensor.

Regarding the second integral of Eq. 2.147, the reference configuration gradient terms can be replaced by equivalent terms in the current configuration
\[
\nabla_0 \Delta u = \nabla \Delta u \mathbf{F} \quad ; \quad \nabla_0 \delta \mathbf{v} = \nabla \delta \mathbf{v} \mathbf{F}
\]
2.154
while \( S \) can be substituted from Eq. 2.83 and \( dV \) from Eq. 2.10 resulting to
\[
S : \left[ \left( \nabla_0 \Delta u \right)^T \left( \nabla_0 \delta \mathbf{v} \right) \right] \, dV = \sigma : \left[ \left( \nabla \Delta u \right)^T \left( \nabla \delta \mathbf{v} \right) \right] \, dv
\]
2.155
By means of Eq. 2.153 and Eq. 2.155 the linearized internal virtual work expression can be expressed in the current configuration as
\[
D_{\Delta u} \delta W_{\text{int}} = \int_V \delta d : c : \varepsilon \, dv + \int_V \sigma : \left[ \left( \nabla \Delta u \right)^T \left( \nabla \delta \mathbf{v} \right) \right] \, dv
\]
2.156

### 2.6.2 Linearized External Virtual Work

On the basis of Eq. 2.123 the linearization of \( \delta W_{\text{int}} \) along \( \Delta u \) leads to
\[
D_{\Delta u} \delta W_{\text{ext}} = D_{\Delta u} \left[ \int_V f \cdot \delta \mathbf{v} \, dv + \int_a t \cdot \delta \mathbf{v} \, da - \int_V \rho \mathbf{a} \cdot \delta \mathbf{v} \, dv \right]
\]
2.157
\[
= D_{\Delta u} \delta W_{\text{ext,f}} + D_{\Delta u} \delta W_{\text{ext,t}} - D_{\Delta u} \delta W_{\text{ext,a}}
\]
Utilizing the relation \( \rho_0 = J \rho \) the body forces component can be expressed as
\[
\delta W_{\text{ext,f}} = \int_V f \cdot \delta \mathbf{v} \, dv = \int_V \rho \mathbf{g} \cdot \delta \mathbf{v} \, dv = \int_V \rho_0 \mathbf{g} \cdot \delta \mathbf{v} \, dV
\]
2.158
Then
\[
D_{\Delta u} \delta W_{\text{ext,f}} = \rho_0 \int_V D_{\Delta u} \left[ \mathbf{g} \cdot \delta \mathbf{v} \right] \, dV = 0
\]
2.159
because both \( \mathbf{g} \) and \( \delta \mathbf{v} \) are independent of the displacement field \( \Delta u \).

For the external forces component \( \delta W_{\text{ext,t}} \) it holds
\[
\delta W_{\text{ext,t}} = \int_a t \cdot \delta \mathbf{v} \, da = \int_a p \mathbf{n} \cdot \delta \mathbf{v} \, da
\]
2.160
in which \( p \) is the applied pressure over an infinitesimal area \( da \) with normal \( \mathbf{n} \). A follower type external force is assumed.
Simo et al. [1991] have proposed an elegant methodology for evaluation of Eq. 2.160. It consists of defining a local set of coordinate axes \((\xi, \eta)\) on the basis of the boundaries of the surface area \(a\) and in a manner analogous to that typically utilized in isoparametric elements. Then it holds

\[
\mathbf{n} = \frac{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \end{vmatrix}}{\left| \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \end{vmatrix} \right|} ; \quad da = \left| \frac{\partial x}{\partial \xi} \times \frac{\partial x}{\partial \eta} \right| d\xi d\eta \tag{2.161}
\]

and hence

\[
\delta W_{\text{ext}, t} = \int_{a_{\xi}} p \delta v \left( \frac{\partial x}{\partial \xi} \times \frac{\partial x}{\partial \eta} \right) d\xi d\eta \tag{2.162}
\]

The directional derivative is

\[
D_{\Delta u} \delta W_{\text{ext}, t} = \int_{a_{\xi}} p \left( \left( D_{\Delta u} \delta v \right) \left( \frac{\partial x}{\partial \xi} \times \frac{\partial x}{\partial \eta} \right) + \delta v \cdot D_{\Delta u} \left( \frac{\partial x}{\partial \xi} \times \frac{\partial x}{\partial \eta} \right) \right) d\xi d\eta \tag{2.163}
\]

Utilizing the aforementioned condition \(D_{\Delta u} \delta v = 0\) and since

\[
D_{\Delta u} \left( \frac{\partial x}{\partial \xi} \right) = \frac{d}{d\varepsilon} \left( \left. \frac{\partial (x + \varepsilon \Delta u)}{\partial \xi} \right|_{\varepsilon=0} \right) = \frac{\partial \Delta u}{\partial \xi} \tag{2.164}
\]

and similarly

\[
D_{\Delta u} \left( \frac{\partial x}{\partial \eta} \right) = \frac{\partial \Delta u}{\partial \eta} \tag{2.165}
\]

Eq. 2.163 can then be written as

\[
D_{\Delta u} \delta W_{\text{ext}, t} = \int_{a_{\xi}} p \delta v \left( \frac{\partial \Delta u}{\partial \xi} \times \frac{\partial x}{\partial \eta} + \frac{\partial x}{\partial \xi} \times \frac{\partial \Delta u}{\partial \eta} \right) d\xi d\eta \tag{2.166}
\]

\[= \int_{a_{\xi}} p \left[ \frac{\partial x}{\partial \xi} \left( \frac{\partial \Delta u}{\partial \eta} \times \delta v \right) - \frac{\partial x}{\partial \eta} \left( \frac{\partial \Delta u}{\partial \xi} \times \delta v \right) \right] d\xi d\eta \]

Eq. 2.166 is unsymmetric hence discretization will result to an unsymmetric tangent matrix, Simo et al. [1991], Holzapfel [2000]. By utilizing the relations between surface and line integrals Bonet & Wood [1999] propose an alternative but symmetric expression for \(D_{\Delta u} \delta W_{\text{ext}, t}\) as
\[ D_{\Delta u} \delta W_{\text{ext},t} = \frac{1}{2} \int_{\alpha_\xi} \rho \left( \frac{\partial x}{\partial \xi} \left[ \left( \frac{\partial \Delta u}{\partial \eta} \times \delta v \right) + \left( \frac{\partial \delta v}{\partial \eta} \times \Delta u \right) \right] \right) \] 

\[ - \frac{\partial x}{\partial \eta} \left[ \left( \frac{\partial \Delta u}{\partial \xi} \times \delta v \right) + \left( \frac{\partial \delta v}{\partial \xi} \times \Delta u \right) \right] \right) d\xi d\eta \]  

It is this expression which will be utilized in the following for derivation of the contribution of surface forces to the stiffness matrix.

2.7 Rate Constitutive Equations

The relation between the time rates of the second Piola-Kirchhoff stress tensor \( \dot{\mathbf{S}} \) and the Lagrangian strain tensor \( \dot{\mathbf{E}} \) can be evaluated by means of linearization of \( \mathbf{S} \) along \( \Delta \mathbf{v} \) because then

\[ D_{\Delta v} \mathbf{S} = \frac{d}{d\varepsilon} \left[ \mathbf{S} \left( \mathbf{E} + \varepsilon \Delta \mathbf{v} \right) \right] = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} : \frac{\partial \mathbf{E}}{\partial \varepsilon} = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} : D_{\Delta v} \mathbf{E} \]  

2.168

but, on the basis of Section 2.2.5, the directional derivatives of \( \mathbf{S} \) and \( \mathbf{E} \) along \( \Delta \mathbf{v} \) are equal to their time rates

\[ D_{\Delta v} \mathbf{S} = \dot{\mathbf{S}} ; \quad D_{\Delta v} \mathbf{E} = \dot{\mathbf{E}} \]  

2.169

and, on the basis of Section 2.6, the 4th order tensor \( \partial \mathbf{S}/\partial \mathbf{E} \) corresponds to the elasticity tensor \( \mathbf{C} \), hence

\[ \dot{\mathbf{S}} = \mathbf{C} : \dot{\mathbf{E}} \]  

2.170

or in indicial form

\[ \dot{S}_{ij} = \sum_{K,L} C_{IKKL} \dot{E}_{KL} \]  

2.171

In the current configuration an equivalent to Eq. 2.170 relation can also be obtained. From Section 2.3.1, the relation between the Truesdell rate of Cauchy stress tensor \( \dot{\sigma}_{\text{T}} \) and \( \dot{\mathbf{S}} \) is

\[ \dot{\sigma}_{Tij} = J^{-1} \sum_{ij} F_{il} \dot{S}_{lj} F_{jl} \]  

2.172

Also from Section 2.2.6

\[ \dot{E}_{KL} = \sum_{k,l} F_{kk} \dot{d}_{kl} F_{IL} \]  

2.173

Substituting Eq. 2.171 and Eq. 2.173 into Eq. 2.172
\[
\dot{\sigma}_{Tij} = J^{-1} \sum_{I,J} F_{il} \left( \sum_{K,L} C_{ijkl} \dot{E}_{KL}^* \right) F_{jJ}
\]

\[
= J^{-1} \sum_{I,J} F_{il} \left( \sum_{K,L} C_{ijkl} \left( \sum_{k,l} F_{kK}^* d_{kl} F_{lL}^* \right) \right) F_{jJ}
\]

\[
= J^{-1} \sum_{I,J,K,L} F_{il} F_{jJ} C_{ijkl} F_{kK}^* F_{lL} \left( \sum_{k,l} d_{kl} \right)
\]

Eq. A.2.6.4 indicates that

\[
c_{ijkl} = J^{-1} \sum_{I,J,K,L=1}^3 F_{il} F_{jJ} C_{ijkl} F_{kK}^* F_{lL}^* \tag{2.175}
\]

i.e. the spatial elasticity tensor.

Hence

\[
\dot{\sigma}_{Tij} = \sum_{k,l} c_{ijkl} d_{kl} \tag{2.176}
\]

or

\[
\dot{\sigma}_T = c : d \tag{2.177}
\]

Pinsky et al. [1983] comment that the Truesdell rate of Cauchy stress is the only stress rate having a linear dependence on the rate of deformation tensor through the spatial elastic modulus tensor which is a function only of the deformation.

### 2.8 Discretization

In the following some aspects of the finite element discretization of the linearised equilibrium equation will be presented.

#### 2.8.1 Element Geometry Interpolation

Following a typical isoparametric formulation, the matrix \( N \) of interpolation functions for an \( n \)-noded element can be defined as:

\[
N = \begin{bmatrix} N_1 & N_2 & \ldots & N_n \end{bmatrix}
\]

in which \( N_k = N_k(\xi) \) and \( \xi = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix}^T \).

If the vector of nodal coordinates \( X_k \) is defined as:

\[
X_k = \begin{bmatrix} X_{k1} & X_{k2} & X_{k3} \end{bmatrix}^T
\]

then the initial configuration of any point within the element can be interpolated in terms of the corresponding nodal coordinates as:
Fundamentals

\[ \mathbf{X} = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}^T = \sum_{i=1}^{3} \left( \sum_{k=1}^{n} N_k X_{ki} \right) \mathbf{e}_i \]

\[ = \sum_{k=1}^{n} N_k \left( \sum_{i=1}^{3} X_{ki} \mathbf{e}_i \right) = \sum_{k=1}^{n} N_k \mathbf{x}_k \]

Anticipating an isoparametric formulation, at any subsequent time, the current configuration \( \mathbf{x} \) is interpolated in terms of the corresponding current nodal quantities \( \mathbf{x}_k \) as:

\[ \mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T = \sum_{i=1}^{3} \left( \sum_{k=1}^{n} N_k x_{ki} \mathbf{e}_i \right) \]

\[ = \sum_{k=1}^{n} N_k \mathbf{x}_k \]

### 2.8.2 Field Variables Interpolation

Similarly, the displacements can be expressed in terms of the current nodal displacements \( \mathbf{u}_k \):

\[ \mathbf{u} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}^T = \sum_{i=1}^{3} \left( \sum_{k=1}^{n} N_k u_{ki} \right) \mathbf{e}_i \]

\[ = \sum_{k=1}^{n} N_k \mathbf{u}_k \]

The deformation gradient tensor \( \mathbf{F} \) is interpolated over an element by differentiating Eq. 2.181 with respect to the reference system coordinates. Utilizing indicial notation

\[ \mathbf{F} = \sum_{i,j=1}^{3} \frac{\partial x_i}{\partial X_j} \mathbf{e}_i \otimes \mathbf{e}_j \]

\[ = \sum_{i,j=1}^{3} \left[ \frac{\partial}{\partial X_j} \left( \sum_{k=1}^{n} N_k x_{ki} \right) \right] \mathbf{e}_i \otimes \mathbf{e}_j \]

\[ = \sum_{i,j=1}^{3} \left[ \sum_{k=1}^{n} \frac{\partial N_k}{\partial X_j} x_{ki} + \sum_{k=1}^{n} \frac{\partial x_{ki}}{\partial X_j} N_k \right] \mathbf{e}_i \otimes \mathbf{e}_j \]

\[ = \sum_{i,j=1}^{3} \sum_{k=1}^{n} \frac{\partial N_k}{\partial X_j} x_{ki} \mathbf{e}_i \otimes \mathbf{e}_j \]

\[ = \sum_{k=1}^{n} \sum_{i,j=1}^{3} \frac{\partial N_k}{\partial X_j} x_{ki} \mathbf{e}_i \otimes \mathbf{e}_j \]

\[ = \sum_{k=1}^{n} \sum_{i,j=1}^{3} \frac{\partial N_k}{\partial X_j} x_{ki} \mathbf{e}_i \otimes \mathbf{e}_j \]

\[ = \sum_{k=1}^{n} x_k \otimes \frac{\partial N_k}{\partial \mathbf{x}} = \sum_{k=1}^{n} x_k \otimes \nabla_0 N_k \]

In order to evaluate the individual Cartesian derivatives \( \nabla_0 N_k \) in Eq. 2.183, the following transformation can be utilized
\[
\left( \frac{\partial N_k}{\partial \xi} \right) = J^T \nabla_0 N_k
\]  
\[\text{2.184}\]

in which \( J \) is known as the coordinate Jacobian matrix:

\[
J = \sum_{i,j=1}^{3} \frac{\partial X_i}{\partial \xi_j} E_i \otimes E_j = \begin{bmatrix}
\frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_1}{\partial \xi_3} \\
\frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_3} \\
\frac{\partial X_3}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_3}
\end{bmatrix}
\]  
\[\text{2.185}\]

On the basis of Eq. 2.180 the individual terms of \( J \) can be computed as:

\[
\left( \frac{\partial X_i}{\partial \xi_j} \right) = \sum_{k=1}^{a} \frac{\partial N_k}{\partial \xi_j} X_{ki}
\]  
\[\text{2.186}\]

In CAPA-3D \( J \) and \( J^{-T} \) are evaluated at every integration point of the element. Then, by means of Eq. 2.184

\[
\left( \frac{\partial N_k}{\partial X} \right) = J^{-T} \left( \frac{\partial N_k}{\partial \xi} \right)
\]  
\[\text{2.187}\]

and hence the individual terms of the deformation gradient tensor \( F \) can be computed.

In similarity to Eq. 2.185 the Jacobian on the basis of the current configuration is defined as

\[
j = \sum_{i,j=1}^{3} \frac{\partial X_i}{\partial \xi_j} E_i \otimes E_j
\]  
\[\text{2.188}\]

Once \( F \) is known several other deformation measures can be computed.

### 2.8.3 Discretized Spatial Equilibrium Equation

The expression for the spatial virtual work is given by Eq. 2.116 as

\[
\delta W = \int_{V} \sigma : \delta \mathbf{d} \, dv - \int_{V} \mathbf{f} \cdot \delta \mathbf{v} \, dv - \int_{\Omega} \mathbf{t} \cdot \delta \mathbf{v} \, da + \int_{\Gamma} \rho \mathbf{a} \cdot \delta \mathbf{v} \, dv
\]  
\[\text{2.189}\]

In this Section the above will be utilized to express the virtual work contributed by a finite element whose nodes are subjected to the virtual velocity field \( \delta \mathbf{v} \). In doing so, the various tensors will be substituted with their discretised equivalents. By summing up the contribution of all elements, a statement will be obtained for equilibrium of the continuum in discretised form.

The rate of deformation tensor \( \mathbf{d} \) has been defined earlier in Eq. 2.60 as

\[
\mathbf{d} = \frac{1}{2} \left[ (\nabla \mathbf{v})^T + \nabla \mathbf{v} \right]
\]  
\[\text{2.190}\]
Fundamentals

In terms of the element nodal values and the element shape functions

\[
\nabla \mathbf{v} = \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j \\
= \sum_{i,j=1}^{3} \left( \frac{\partial}{\partial x_j} \left( \sum_{k=1}^{n} N_k v_{ki} \right) \right) \mathbf{e}_i \otimes \mathbf{e}_j \\
= \sum_{k=1}^{n} \sum_{i,j=1}^{2} \frac{\partial N_k}{\partial x_j} v_{ki} \mathbf{e}_i \otimes \mathbf{e}_j \\
= \sum_{k=1}^{n} \mathbf{v}_k \otimes \frac{\partial N_k}{\partial x} = \sum_{k=1}^{n} \mathbf{v}_k \otimes \nabla N_k
\]

hence

\[
d = \frac{1}{2} \left[ \left( \sum_{k=1}^{n} \mathbf{v}_k \otimes \nabla N_k \right)^T + \sum_{k=1}^{n} \mathbf{v}_k \otimes \nabla N_k \right]
\]

\[
= \frac{1}{2} \sum_{k=1}^{n} \left[ \nabla N_k \otimes \mathbf{v}_k + \mathbf{v}_k \otimes \nabla N_k \right]
\]

from which

\[
\delta d = \frac{1}{2} \sum_{k=1}^{n} \left[ \nabla N_k \otimes \delta \mathbf{v}_k + \delta \mathbf{v}_k \otimes \nabla N_k \right]
\]

Also

\[
\delta \mathbf{v} = \sum_{k=1}^{n} N_k \delta \mathbf{v}_k
\]

Substituting into Eq. 2.189

\[
\delta W_{\text{ielem}} = \int_{V} \sigma : \frac{1}{2} \sum_{k=1}^{n} \left[ \nabla N_k \otimes \delta \mathbf{v}_k + \delta \mathbf{v}_k \otimes \nabla N_k \right] dV
\]

\[
- \int_{V} f \cdot \sum_{k=1}^{n} N_k \delta \mathbf{v}_k dV - \int_{a} t \cdot \sum_{k=1}^{n} N_k \delta \mathbf{v}_k da + \int_{V} \rho a \cdot \sum_{k=1}^{n} N_k \delta \mathbf{v}_k dV
\]

By exploiting the symmetry of the Cauchy stress tensor it holds

\[
\sigma : \delta \mathbf{d} = \sigma : \frac{1}{2} \sum_{k=1}^{n} \left[ \nabla N_k \otimes \delta \mathbf{v}_k + \delta \mathbf{v}_k \otimes \nabla N_k \right]
\]

\[
= \sigma : \sum_{k=1}^{n} \left[ \delta \mathbf{v}_k \otimes \nabla N_k \right] = \sum_{k=1}^{n} \left[ \delta \mathbf{v}_k : \sigma \nabla N_k \right]
\]

Substituting in Eq. 2.195, a typical term representing the contribution of element node \( k = a \) to the total virtual work of the element due to the application of a single virtual velocity \( \delta \mathbf{v}_a \) at node \( a \) is
\[
\delta W_{k=a}^{\text{elem}} = \delta \mathbf{v}_a \cdot \left( \int_v \sigma \nabla N_a \, dv - \int_v N_a \mathbf{f} \, dv - \int_a N_a \mathbf{t} \, da + \int_v \rho N_a \mathbf{a} \, dv \right) \\
= \delta \mathbf{v}_a \cdot \left( \mathbf{E}^{\text{elem}}_a - \mathbf{F}^{\text{elem}}_a \right)
\]

in which
\[
\mathbf{E}^{\text{elem}}_a = \int_{v} \sigma \nabla N_a \, dv \delta \mathbf{v}_a \\
\mathbf{F}^{\text{elem}}_a = \int_{v} N_a \mathbf{f} \, dv + \int_{a} N_a \mathbf{t} \, da - \int_{v} \rho N_a \mathbf{a} \, dv
\]

The virtual work contributed by all elements connected at node \( a \) is
\[
\delta W_{k=a} = \sum_{\text{elem}=1}^{\text{kelem}} \delta W_{k=a}^{\text{elem}} = \delta \mathbf{v}_a \cdot \left( \sum_{\text{elem}=1}^{\text{kelem}} \left( \mathbf{E}^{\text{elem}}_a - \mathbf{F}^{\text{elem}}_a \right) \right)
\]

in which \( \text{kelem} \) is the number of elements connected at node \( a \), \( \mathbf{E}_a \) is the total nodal force at node \( a \) due to internal element stresses and \( \mathbf{F}_a \) is the total external force applied at node \( a \).

It then follows that the virtual work contributed by all nodes is
\[
\delta W = \sum_{k=1}^{\text{nodes}} \delta W_{k=a} = \sum_{k=1}^{\text{nodes}} \left( \sum_{\text{elem}=1}^{\text{kelem}} \delta W_{k=a}^{\text{elem}} \right) = \sum_{k=1}^{\text{nodes}} \delta \mathbf{v}_k \cdot \left( \mathbf{E}_k - \mathbf{F}_k \right)
\]

in which \( \text{nodes} \) is the total number of nodes.

Eq. 2.200 constitutes the discretized equivalent of Eq. 2.114 in Section 2.5.2. At equilibrium
\[
\delta W = \sum_{k=1}^{\text{nodes}} \delta \mathbf{v}_k \cdot \left( \mathbf{E}_k - \mathbf{F}_k \right) = 0
\]

and since Eq. 2.201 must be satisfied for any arbitrary set \( \delta \mathbf{v}_k \), it can be concluded that
\[
\mathbf{E}_k - \mathbf{F}_k = 0 \quad \forall \, k = 1, \ldots, \text{nodes}
\]

In nonlinear analysis, determination of the nodal displacements, which correspond to the equilibrium solution, is commonly achieved by means of a Newton type iterative technique. As mentioned in Section 2.6, a common characteristic of these techniques is the replacement of the non-linear equation, in the vicinity of a trial solution \( \mathbf{x} = \mathbf{x}_0 \), by a linear approximation in the direction of a displacement increment \( \Delta \mathbf{u} \)

\[
\delta W(\mathbf{x}_0) + D_{\Delta \mathbf{u}} \delta W(\mathbf{x}_0) = 0
\]

with
\[
D_{\Delta \mathbf{u}} \delta W = D_{\Delta \mathbf{u}} \delta W_{\text{int}} - D_{\Delta \mathbf{u}} \delta W_{\text{ext}}
\]
and $\delta W_{\text{int}}$ defined by Eq. 2.122 and $\delta W_{\text{ext}}$ by Eq. 2.123.

It is the directional derivative term in Eq. 2.203 which provides the commonly known tangent stiffness operator, a key ingredient in Newton-Raphson type techniques. This can be demonstrated by presuming that a single displacement increment $\Delta u_b$ is applied at node $b$. Then, the directional derivative of the work contributed by node $a$ is, according to Eq. 2.199

$$D_{\Delta u_b} \delta W \bigg|_{k=a} = D_{\Delta u_b} \left[ \delta v_a \cdot (E_a - F_a) \right] = \delta v_a \cdot D_{\Delta u_b} (E_a - F_a) \tag{2.205}$$

Let $R_a = E_a - F_a$. Then

$$D_{\Delta u_b} (R_a) = \frac{d}{d\varepsilon} \left[ R_a \left( x, \delta_{i_{ab}} \varepsilon \Delta u_b \right) \right]_{\varepsilon=0} = \sum_{i=1}^{\text{n nodes}} \left[ \frac{\partial R_a}{\partial x_i} \right]_{x=x_{0,i}} \left[ \frac{d \left( x, \delta_{i_{ab}} \varepsilon \Delta u_b \right)}{d\varepsilon} \right]_{\varepsilon=0} \tag{2.206}$$

The differential term in Eq. 2.206 expresses the change in forces at node $a$ due to changes in the current position of node $b$ which is precisely the definition of a component $k_{ab}$ of the stiffness matrix in finite element analyses

$$D_{\Delta u_b} \delta W \bigg|_{k=a} = \delta v_a \cdot k_{ab} \Delta u_b \tag{2.207}$$

It can thus be concluded that the tangent stiffness operator necessary for Newton-Raphson type iterative techniques results from the linearization of the virtual work expression. In the following Section, the linearised virtual work expressions determined in Section 2.6 will be discretised so that they can be utilized for computation of the components of the stiffness matrix of the discretised continuum.

### 2.8.4 Discretization of the Linearized Equilibrium Equation

The linearised internal virtual work expression in the current configuration has been expressed in Section 2.6.1 as

$$D_{\Delta u} \delta W_{\text{int}} = \int_V \delta d : c : \varepsilon \, dv + \int_V \sigma : \left[ (\nabla \Delta u)^T (\nabla \delta v) \right] \, dv \tag{2.208}$$

On the basis of the isoparametric formulation presented in Section 2.8.2 $\nabla u$ and $\nabla \delta v$ can be interpolated as

$$\nabla \Delta u = \sum_{k=1}^{n} \Delta u_k \otimes \nabla N_k \tag{2.209}$$

and

$$\nabla \delta v = \sum_{k=1}^{n} \delta v_k \otimes \nabla N_k \tag{2.210}$$

38
Also

\[
\varepsilon = \frac{1}{2} \sum_{k=1}^{n} \left( \nabla N_k \otimes \Delta u_k + \Delta u_k \otimes \nabla N_k \right)
\]

By means of these discretised expressions the individual integral expression of the virtual work equation Eq. 2.208 can be written as

\[
\int_{\Omega} \delta \mathbf{d} : \varepsilon \, d\Omega = \int_{\Omega} \frac{1}{2} \sum_{k=1}^{n} \left[ \nabla N_k \otimes \delta \mathbf{v}_k + \delta \mathbf{v}_k \otimes \nabla N_k \right] : \mathbf{c} \cdot \left[ \nabla N_b \otimes \Delta \mathbf{u}_b + \Delta \mathbf{u}_b \otimes \nabla N_b \right] \, d\Omega
\]

A typical term of the product under the integral of Eq. 2.212 is

\[
Q_{ab} = \frac{1}{4} \left[ \nabla N_a \otimes \delta \mathbf{v}_a + \delta \mathbf{v}_a \otimes \nabla N_a \right] : \mathbf{c} \cdot \left[ \nabla N_b \otimes \Delta \mathbf{u}_b + \Delta \mathbf{u}_b \otimes \nabla N_b \right]
\]

Expressing \( \delta \mathbf{v}_a \otimes \nabla N_a \) and \( \nabla N_b \otimes \Delta \mathbf{u}_b \) in index notation

\[
\delta \mathbf{v}_a \otimes \nabla N_a = \sum_{m,n=1}^{3} \delta v_{am} \nabla N_{an} e_m \otimes e_n
\]

\[
\nabla N_b \otimes \Delta \mathbf{u}_b = \sum_{m,n=1}^{3} \nabla N_{bm} \Delta u_{bn} e_m \otimes e_n
\]

the double contraction of the 4th order tensor \( \mathbf{c} \) with \( \nabla N_b \otimes \Delta \mathbf{u}_b \) produces a second order tensor which can be written as

\[
\mathbf{c} \cdot \left[ \nabla N_b \otimes \Delta \mathbf{u}_b \right] = \sum_{i,j,k,m,n=1}^{3} c_{ijkl} \nabla N_{bm} \Delta u_{bn} \left( e_i \otimes e_j \otimes e_k \otimes e_l \right) : \left( e_m \otimes e_n \right)
\]

Performing also the double contraction of \( \delta \mathbf{v}_a \otimes \nabla N_a \) with the above tensor it results to

\[
\left[ \delta \mathbf{v}_a \otimes \nabla N_a \right] : \mathbf{c} \cdot \left[ \nabla N_b \otimes \Delta \mathbf{u}_b \right] = \sum_{i,j,k,m,n=1}^{3} \delta v_{am} \nabla N_{an} c_{ijkl} \nabla N_{bk} \Delta u_{bl} \left( e_m \otimes e_n \right) \left( e_i \otimes e_j \right)
\]

On the basis of Eq. 2.217 \( Q_{ab} \) in Eq. 2.213 can be written as

\[
Q_{ab} = \frac{1}{4} \sum_{i,j,k,l=1}^{3} \delta v_{ai} \nabla N_{aj} c_{ijkl} \nabla N_{bk} \Delta u_{bl}
\]

which can be recast as
Fundamentals

\[ Q_{ab} = \delta v_a \cdot K_{ab}^c u_b \]  \hspace{1cm} 2.219

with

\[ [K_{il}]_{ab}^{c} = \int_{\Omega} \sum_{i=1}^{3} \nabla N_{aj} \delta c_{ijkl} \nabla N_{bk} \, \, dv \quad ; \quad i, l \in [1, 2, 3] \]  \hspace{1cm} 2.220

The term \( K_{ab}^c \) has been termed the constitutive component matrix of the element tangent stiffness matrix Holzapfel [2000], Belytschko [2000].

Discretization of the second integral term of Eq. 2.208 leads to

\[ \int_{\Omega} \sigma: \left[ (\nabla \Delta u)^T (\nabla \delta v) \right] \, dv = \int_{\Omega} \sigma: \left[ \left( \sum_{k=1}^{n} \Delta u_k \otimes \nabla N_k \right)^T \left( \sum_{k=1}^{n} \delta v_k \otimes \nabla N_k \right) \right] \, dv \]  \hspace{1cm} 2.221

A typical term of the product within the brackets of Eq. 2.221 is

\[ (\Delta u_b \otimes \nabla N_b)^T (\delta v_a \otimes \nabla N_a) = (\delta v_a \cdot \Delta u_b) \nabla N_b \otimes \nabla N_a \]  \hspace{1cm} 2.222

hence

\[ \int_{\Omega} \sigma: \left[ (\delta v_a \cdot \Delta u_b) \nabla N_b \otimes \nabla N_a \right] \, dv = (\delta v_a \cdot \Delta u_b) \int_{\Omega} \sigma: (\nabla N_b \otimes \nabla N_a) \, dv \]  \hspace{1cm} 2.223

Noticing that the integral term in the above equation is a scalar, the dot product can be further rearranged to read

\[ (\delta v_a \cdot \Delta u_b) \int_{\Omega} \sigma: (\nabla N_b \otimes \nabla N_a) \, dv = \delta v_a \cdot \left[ \int_{\Omega} \sigma: (\nabla N_b \otimes \nabla N_a) \, dv \right] \Delta u_b \]  \hspace{1cm} 2.224

The term

\[ K_{ab}^{\sigma} = \left[ \int_{\Omega} \sigma: (\nabla N_b \otimes \nabla N_a) \, dv \right] \]  \hspace{1cm} 2.225

associating nodes \( a \) and \( b \) of the element/structure has been termed the geometrical stress component matrix of the element tangent stiffness matrix Holzapfel [2000], Belytschko [2000].

The linearized external virtual work expression in the current configuration has been expressed in Section 2.6.2 as

\[ D_{\Delta u} \delta W_{ext,t} = \frac{1}{2} \int_{\Omega} p \left[ \partial x \cdot \left( \frac{\partial \Delta u}{\partial \xi} \times \delta v \right) + \left( \frac{\partial \delta v}{\partial \eta} \times \Delta u \right) \right] \]  

\[ - \partial x \cdot \left[ \left( \frac{\partial \Delta u}{\partial \xi} \times \delta v \right) + \left( \frac{\partial \delta v}{\partial \xi} \times \Delta u \right) \right] \, d\xi \, d\eta \]  \hspace{1cm} 2.226

It holds

\[ \delta v = \sum_{i=1}^{n} N_i \delta v_i \quad ; \quad \Delta u = \sum_{k=1}^{n} N_k \Delta u_k \]  \hspace{1cm} 2.227
\[
\frac{\partial \delta \mathbf{v}}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} \delta \mathbf{v}_i; \quad \frac{\partial \delta \mathbf{v}}{\partial \eta} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} \delta \mathbf{v}_i
\]

and
\[
\frac{\partial \Delta \mathbf{u}}{\partial \xi} = \sum_{k=1}^{n} \frac{\partial N_k}{\partial \xi} \Delta \mathbf{u}_k; \quad \frac{\partial \Delta \mathbf{u}}{\partial \eta} = \sum_{k=1}^{n} \frac{\partial N_k}{\partial \eta} \Delta \mathbf{u}_k
\]

so that
\[
\frac{\partial \Delta \mathbf{u}}{\partial \eta} \times \delta \mathbf{v} = \sum_{k=1}^{n} \frac{\partial N_k}{\partial \eta} \Delta \mathbf{u}_k \times \sum_{i=1}^{n} N_i \delta \mathbf{v}_i
\]
\[
\frac{\partial \Delta \mathbf{u}}{\partial \xi} \times \delta \mathbf{v} = \sum_{k=1}^{n} \frac{\partial N_k}{\partial \xi} \Delta \mathbf{u}_k \times \sum_{i=1}^{n} N_i \delta \mathbf{v}_i
\]
\[
\frac{\partial \delta \mathbf{v}}{\partial \xi} \times \Delta \mathbf{u} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} \delta \mathbf{v}_i \times \sum_{k=1}^{n} N_k \Delta \mathbf{u}_k
\]
\[
\frac{\partial \delta \mathbf{v}}{\partial \eta} \times \Delta \mathbf{u} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} \delta \mathbf{v}_i \times \sum_{k=1}^{n} N_k \Delta \mathbf{u}_k
\]

A typical term of the product under the integral of Eq. 2.226 is
\[
Q_{ab} = \frac{1}{2} (\delta \mathbf{v}_a \times \Delta \mathbf{u}_b) : \int_{a}^{\partial} \mathbf{p} \left[ \frac{\partial \mathbf{x}}{\partial \xi} \left( \frac{\partial N_a}{\partial \eta} N_b - \frac{\partial N_b}{\partial \eta} N_a \right) \right] d\xi d\eta
\]

The term
\[
K^t_{ab} = \frac{1}{2} \int_{a}^{\partial} \mathbf{p} \left[ \frac{\partial \mathbf{x}}{\partial \eta} \left( \frac{\partial N_a}{\partial \xi} N_b - \frac{\partial N_b}{\partial \xi} N_a \right) \right] d\xi d\eta
\]

associating nodes a and b of the element/structure has been termed the \textit{external force component matrix} of the element tangent stiffness matrix Bonet & Wood [1999], Belytschko et al. [2000].

The element tangent stiffness matrix can now be assembled by the contributions of the constitutive, stress and force component matrices as
\[
K_{ab} = K^c_{ab} + K^s_{ab} + K^t_{ab}
\]

For computing purposes matrix notation (formally known also as Voigt notation) is frequently utilized in finite element codes. The transformation rules from tensor to matrix notation are presented in Appendix 2.6.
Appendix 2.1

Polar Decomposition

Hoger & Carlson [1984] have shown that $U^{-1}$ can be computed on the basis of the following methodology:

1. Compute (a) the right Cauchy-Green deformation tensor $C = F^T F$ and,
   (b) $C^2 = C^T C$

2. Compute the eigenvalues $\lambda_1^2; \lambda_2^2; \lambda_3^2$ of $C^2$.

3. Compute the invariants of the right stretch $U$. On the basis of Eq. 2.27 these are
   \[
   I_1 = \lambda_1 + \lambda_2 + \lambda_3 \\
   I_2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \\
   I_3 = \lambda_1 \lambda_2 \lambda_3
   \]

1. Compute $U^{-1}$ by means of the equation
   \[
   U^{-1} = c_1 \left[ c_2 I + c_3 C + c_4 C^2 \right]
   \]
   in which
   \[
   c_1 = \frac{1}{I_3 (I_1 I_2 - I_3)} \\
   c_2 = I_1 I_2^2 - I_3 (I_1^2 + I_2) \\
   c_3 = -I_3 - I_1 (I_1^2 - 2I_2) \\
   c_4 = I_1
   \]
Appendix 2.2

Objectivity of $e$

A typical example of a tensor which transforms according to Eq. 2.69 is the aforementioned Eulerian/Almansi strain tensor $e$. Reformulating Eq. 2.21 in terms of the rotated vector $\tilde{dx}$

$$\left(\frac{d\tilde{S}}{dS}\right)^2 = \tilde{dx} \cdot \tilde{dx} - d\tilde{x} \cdot d\tilde{x} = \tilde{dx} \cdot \tilde{dx} - \tilde{dx}^T \left(\tilde{F}^{-T} \tilde{F}^{-1}\right) \tilde{dx}$$  \hspace{1cm} (A.2.2.1)

Comparison of the above with Eq. 2.20 it reveals that $\tilde{F}^{-T} \tilde{F}^{-1} = \tilde{b}^{-1}$. Utilizing Eq. 2.68

$$\tilde{b}^{-1} = \tilde{F}^{-T} \tilde{F}^{-1} = \tilde{R} \tilde{F}^{-T} \tilde{F}^{-1} \tilde{R}^T$$  \hspace{1cm} (A.2.2.2)

$$= \tilde{R} \tilde{b}^{-1} \tilde{R}^T$$

which can be recast in the form required by Eq. 2.69

$$\tilde{b} = \tilde{R} \tilde{b} \tilde{R}^T$$  \hspace{1cm} (A.2.2.3)

The magnitude of the rotated vector $\tilde{dx}$ implied by Eq. A.2.2.1 can be further elaborated as

$$\left(\frac{d\tilde{S}}{dS}\right)^2 = \tilde{dx} \cdot \tilde{dx} - \tilde{dx}^T \left(\tilde{F}^{-T} \tilde{F}^{-1}\right) \tilde{dx}$$  \hspace{1cm} (A.2.2.4)

which is identical to Eq. 2.21. Therefore it can be concluded that utilization of the tensor $b$ as a deformation measure produces results that are independent of orthogonal transformations and hence objective.
Appendix 2.3

Cauchy Stress Symmetry

Symmetry of the Cauchy stress tensor can be established by expanding Eq. 2.107

$$\mathbf{E} : \sigma^T = \sum_{i,j,k=1}^{3} \mathbf{E}_{ijk} \sigma_{jk}^T \mathbf{e}_i$$

$$= \left( \mathbf{E}_{123} \sigma_{23}^T + \mathbf{E}_{132} \sigma_{32}^T \right) \mathbf{e}_1 +$$

$$\left( \mathbf{E}_{231} \sigma_{31}^T + \mathbf{E}_{213} \sigma_{13}^T \right) \mathbf{e}_2 +$$

$$\left( \mathbf{E}_{312} \sigma_{12}^T + \mathbf{E}_{321} \sigma_{21}^T \right) \mathbf{e}_3$$

$$= (\sigma_{32} - \sigma_{23}) \mathbf{e}_1 + (\sigma_{13} - \sigma_{31}) \mathbf{e}_2 + (\sigma_{21} - \sigma_{12}) \mathbf{e}_3$$

$$= \begin{bmatrix} \sigma_{32} - \sigma_{23} \\ \sigma_{13} - \sigma_{31} \\ \sigma_{21} - \sigma_{12} \end{bmatrix}$$

A.2.3.1

Imposing the condition $\mathbf{E} : \sigma^T = 0$ establishes the symmetric nature of the Cauchy stress tensor.
Appendix 2.4

Virtual Work

Before recasting Eq. 2.115, the virtual work expression, to a form suitable for finite element implementation some mathematical preliminaries will be reviewed. If $\mathbf{\sigma}$ represent the second order stress tensor and $\mathbf{\delta v}$ the virtual velocity vector, the divergence of $(\mathbf{\sigma} \mathbf{\delta v})$ is defined as

$$ \text{div}(\mathbf{\sigma} \mathbf{\delta v}) = \nabla (\mathbf{\sigma} \mathbf{\delta v}) : \mathbf{I} \quad \text{A.2.4.1} $$

It holds

$$ \nabla (\mathbf{\sigma} \mathbf{\delta v}) = \sum_{i,j=1}^{3} \frac{\partial (\mathbf{\sigma} \mathbf{\delta v})}{\partial x_j} e_i \otimes e_j \quad \text{A.2.4.2} $$

$$ \mathbf{I} = \sum_{l,m=1}^{3} \delta_{lm} \mathbf{e}_l \otimes \mathbf{e}_m \quad \text{A.2.4.3} $$

hence

$$ \text{div}(\mathbf{\sigma} \mathbf{\delta v}) = \sum_{i,j=1}^{3} \frac{\partial (\mathbf{\sigma} \mathbf{\delta v})}{\partial x_j} e_j \otimes e_j : \sum_{l,m=1}^{3} \delta_{lm} \mathbf{e}_l \otimes \mathbf{e}_m $$

$$ = \sum_{i,j,l,m=1}^{3} \frac{\partial (\mathbf{\sigma} \mathbf{\delta v})}{\partial x_j} \delta_{lm} \left( e_i \otimes e_j \right) : \left( e_l \otimes e_m \right) $$

$$ = \sum_{i,j,l,m=1}^{3} \frac{\partial \left( \sum_{k=1}^{3} \sigma_{ik} \mathbf{\delta v}_k \right)}{\partial x_j} \delta_{lm} \left( e_i \cdot e_j \right) \left( e_l \cdot e_m \right) $$

$$ = \sum_{i,j,l,m=1}^{3} \frac{\partial \left( \sum_{k=1}^{3} \sigma_{ik} \mathbf{\delta v}_k \right)}{\partial x_j} \delta_{lm} \delta_{il} \delta_{jm} $$

$$ = \sum_{l,m=1}^{3} \left( \sum_{k=1}^{3} \frac{\partial \sigma_{lk}}{\partial x_k} \delta v_k \right) \delta_{lm} + \sum_{k=1}^{3} \sigma_{lk} \frac{\partial \delta v_k}{\partial x_m} \delta_{lm} $$
Utilizing Eq. A.2.4.4, the term \( \text{div}\sigma \cdot \delta v \) in Eq. 2.115 can be replaced by
\[
\text{div}\sigma \cdot \delta v = \text{div}\left(\sigma \delta v\right) - \sigma : \nabla \delta v
\]
A.2.4.5

to yield
\[
\delta W = \int v \text{div}(\sigma \delta v)\delta a - \int v \sigma : \nabla \delta v \delta v + \int v f \cdot \delta v \delta v - \int v \rho a \cdot \delta v \delta v = 0
\]
A.2.4.6

By means of Gauss theorem
\[
\int v \text{div}(\sigma \delta v)\delta v = \int a n \cdot \sigma \delta v \delta a
\]
A.2.4.7

so that
\[
\delta W = \int a n \cdot \sigma \delta v \delta a - \int v \sigma : \nabla \delta v \delta v + \int v f \cdot \delta v \delta v - \int v \rho a \cdot \delta v \delta v = 0
\]
A.2.4.8

On the basis of Eq. 2.110 the gradient of the virtual velocity can be rewritten as
\[
\nabla \delta v = \delta l
\]
A.2.4.9

Also, on the basis of Eq. 2.73 and by taking into account the symmetry of Cauchy stress
\[
t^T = n^T \sigma^T = n^T \sigma
\]
A.2.4.10

so that the term \( n \cdot \sigma \delta v \) can be expressed as
\[
n \cdot \sigma \delta v = n^T \sigma \delta v = t^T \delta v = t \cdot \delta v
\]
A.2.4.11

Substituting these into Eq. A.2.4.8
\[
\delta W = \int a t \cdot \delta v \delta a - \int v \sigma : \delta l \delta v + \int v f \cdot \delta v \delta v - \int v \rho a \cdot \delta v \delta v = 0
\]
A.2.4.12

From Eq. 2.60 and Eq. 2.61
\[ \sigma : \delta l = \sigma : (\delta d + \delta w) \] 

but
\[ \sigma : \delta w = \text{tr} \left( \sigma^T \delta w \right) = \sum_{i,j=1}^{3} \sigma_{ij} \delta w_{ij} = 0 \]

because \( \sigma \) is symmetric and \( w \) antisymmetric.

Hence Eq. A.2.4.12 can be expressed as
\[ \int \sigma : \delta d \delta v = \int f \cdot \delta v \delta v + \int t \cdot \delta v \delta a - \int \rho \cdot \delta v \delta v \]
Expressing the first term of Eq. 2.153 in indicial notation

\[ D_{\varepsilon v} \mathbf{E} : \mathbf{C} : D_{\Delta u} \mathbf{E} \, dV = \sum_{I,J,K,L} D_{\varepsilon v} E_{I J} C_{IJKL} D_{\Delta u} E_{KL} \, dV \]  

A.2.5.1

From Eq. 2.149 and Eq. 2.152

\[ D_{\varepsilon v} E_{I J} = \sum_{i,j} F_{i I} \delta d_{ij} F_{j J} ; \quad D_{\Delta u} E_{KL} = \sum_{k,l} F_{k K} \varepsilon_{kl} F_{l L} \]  

A.2.5.2

Substituting into Eq. A.2.5.1

\[ D_{\varepsilon v} \mathbf{E} : \mathbf{C} : D_{\Delta u} \mathbf{E} \, dV = \sum_{I,J,K,L} \left( \sum_{i,j} F_{i I} \delta d_{ij} F_{j J} \right) C_{IJKL} \left( \sum_{k,l} F_{k K} \varepsilon_{kl} F_{l L} \right) J^{-1} \, dV \]  

A.2.5.3

\[ = \sum_{i,j,k,l} \delta d_{ij} \left( J^{-1} \sum_{I,J,K,L} F_{i I} F_{j J} F_{k K} F_{l L} C_{IJKL} \varepsilon_{kl} \right) \, dV \]

Setting

\[ c_{ijkl} = J^{-1} \sum_{I,J,K,L=1}^{3} F_{i I} F_{j J} F_{k K} F_{l L} C_{IJKL} \]  

A.2.5.4

it results to

\[ D_{\varepsilon v} \mathbf{E} : \mathbf{C} : D_{\Delta u} \mathbf{E} \, dV = \sum_{i,j,k,l} \delta d_{ij} c_{ijkl} \varepsilon_{kl} \, dV = \delta \mathbf{d} : \mathbf{c} : \mathbf{e} \, dV \]  

A.2.5.5
Appendix 2.6

Matrix Notation

The transformation from tensor to matrix notation is accomplished by the index transformations shown in Table A.2-1.

<table>
<thead>
<tr>
<th>Tensor Index</th>
<th>Matrix Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>22</td>
<td>3</td>
</tr>
<tr>
<td>33</td>
<td>4</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
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<tr>
<td>23</td>
<td>6</td>
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<tr>
<td>21</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td></td>
</tr>
</tbody>
</table>

Table A.2-1  Transformation of indices from tensor to matrix notation

Accordingly $\delta d$ can be expressed as

$$\delta d = \begin{pmatrix} \delta d_{11} & \delta d_{22} & \delta d_{33} & 2\delta d_{12} & 2\delta d_{23} & 2\delta d_{13} \end{pmatrix}^T$$  \hspace{1cm} \text{A.2.6.1}

In Eq. A.2.6.1 a factor of 2 has been included in the off-diagonal terms to ensure equivalence between double contraction tensor operations and the corresponding vector/matrix multiplication operations.

On the basis of matrix notation $\delta d$ can be computed from

$$\delta d = \sum_{k=1}^{n} B_k \delta v_k$$  \hspace{1cm} \text{A.2.6.2}

$$= B \delta v$$

with $B_k$ the familiar in finite element literature matrix:
The small strain tensor $\varepsilon_0$ (Eq. 2.211) can be expressed as or by means of matrix notation in vector format as

$$
\varepsilon_0 = \begin{bmatrix}
\varepsilon_{0_{11}} & \varepsilon_{0_{22}} & \varepsilon_{0_{33}} & 2\varepsilon_{0_{12}} & 2\varepsilon_{0_{13}} & 2\varepsilon_{0_{23}}
\end{bmatrix}^T
$$

Applying the transformation of indices from tensor to matrix notation indicated in Table A.2-1 to the elements of the constitutive tensor $C$ of Eq. 2.153 it results to the more familiar format

$$
D = \frac{1}{2}
\begin{bmatrix}
2C_{1111} & 2C_{1122} & 2C_{1133} & C_{1112} + C_{1211} & C_{1123} + C_{1311} & C_{1132} + C_{1231} \\
2C_{2222} & 2C_{2233} & 2C_{2323} & C_{2212} + C_{2122} & C_{2223} + C_{2322} & C_{2231} + C_{2132} \\
2C_{3333} & 2C_{3323} & 2C_{3233} & C_{3312} + C_{3232} & C_{3322} + C_{3232} & C_{3313} + C_{3231} \\
C_{1212} & C_{1222} & C_{1233} & C_{1213} + C_{1312} & C_{1223} + C_{1323} & C_{1231} + C_{1321} \\
C_{2332} & C_{2323} & C_{2313} & C_{2312} + C_{1323} & C_{2322} + C_{1322} & C_{2331} + C_{1331} \\
C_{1313} & C_{1323} & C_{1331} \\
\end{bmatrix}
$$

In terms of the vector/matrix expressions for the tensors the first term of Eq. 2.208 can be written as

$$
\int \dd v : \varepsilon_0 \dd v = \int (\dd \dd)^T D \varepsilon_0 \dd v = (\dd v)^T \left[ \int B^T D \dd B \dd v \right] u
$$
Section I

Constitutive Theories
Chapter 3

Hyperelasticity

3.1 Introduction

In Chapter 2 it was stated that a material is termed \textit{elastic} if the stress at any time is a function only of the state of deformation (and the temperature). It was also stated that elastic materials have zero internal dissipation for every admissible process.

Introducing the additional requirement that the \textit{Helmholtz free energy} $\Psi$ is solely a function of the deformation gradient $F$ so that

$$\dot{\Psi} = \frac{\partial \Psi(F)}{\partial F} : \dot{F} \quad 3.1$$

then, from the Clausius-Planck relation for a non-dissipative material

$$\mathcal{D} = P : \dot{F} - \dot{\Psi} = \left( P - \frac{\partial \Psi(F)}{\partial F} \right) : \dot{F} = 0 \quad 3.2$$

Since $F$ is arbitrary it results

$$P = \frac{\partial \Psi(F)}{\partial F} \quad 3.3$$

Elastic materials for which the stress tensor can be defined on the basis of Eq. 3.3 constitute a subclass of elastic materials termed \textit{hyperelastic}.

Considering that the double contraction $P : \dot{F}$ expresses work per unit reference volume, it can be concluded that, for a hyperelastic material, the strain energy function corresponds to the work done by the stresses from the initial to the final configuration

$$\Psi = \int_{t_0}^{t} P : \dot{F} \, dt \quad 3.4$$

The strain energy $\Psi$ resulting from the motion $x = \phi(X)$ must be objective. According to Section 2.2.8 this implies that for every orthogonal rotation tensor $Q$

$$\Psi(F) = \Psi(QF) = \Psi(F) \quad 3.5$$

Since $Q$ is arbitrary, it can be replaced by the transpose of the rotation tensor $R$ resulting from the polar decomposition of $F$

$$\Psi(F) = \Psi(R^T F) = \Psi(R^T RU) = \Psi(U) \quad 3.6$$

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Hyperelasticity

This relation indicates the necessary and sufficient condition for the strain energy to be objective, Holzapfel [2000]. Then, on the basis of Eq. 2.26 and Eq. 2.14 it is also valid

$$\Psi(F) = \Psi(C) = \Psi(E)$$

3.7

Considering the time derivative of Eq. 3.7

$$\dot{\Psi} = \frac{\partial \Psi}{\partial F} \cdot \frac{\partial F}{\partial t} = \frac{\partial \Psi}{\partial F} : \dot{\mathbf{F}}$$

$$= \frac{\partial \Psi}{\partial C} \cdot \frac{\partial C}{\partial t} = 2 \frac{\partial \Psi}{\partial C} : \mathbf{F}^T \dot{\mathbf{F}} = 2 \mathbf{F} \frac{\partial \Psi}{\partial C} : \dot{\mathbf{F}}$$

3.8

from which

$$\frac{\partial \Psi}{\partial F} = 2 \mathbf{F} \frac{\partial \Psi}{\partial C}$$

3.9

and hence

$$\mathbf{P} = 2 \mathbf{F} \frac{\partial \Psi}{\partial C}$$

3.10

Then on the basis of Eq. 2.80 and Eq. 2.81

$$\mathbf{S} = 2 \frac{\partial \Psi}{\partial C} \quad ; \quad \sigma = 2 J^{-1} \mathbf{F} \frac{\partial \Psi}{\partial C} \mathbf{F}^T$$

3.11

3.2 Isotropic Hyperelastic Material Response

The requirement of objectivity of the strain energy function as expressed by Eq. 3.6 is a general requirement necessary for the response of any hyperelastic material. The notion of isotropy expresses the observation that, many engineering materials exhibit during testing the same physical response irrespective of the direction of testing.

Isotropy imposes limitations on the nature of the strain energy function. Let \( \mathbf{x} = \phi(\mathbf{X}) \) denote the motion of a point of a hyperelastic continuum. Another motion is now considered consisting first of a rigid body rotation of the continuum \( \mathbf{X} = \mathbf{QX} \) and then, of a motion which moves the continuum to the same location as \( \phi(\mathbf{X}) \) i.e. such that

$$\mathbf{x} = \phi(\mathbf{X}) = \tilde{\phi}(\tilde{\mathbf{X}})$$

3.12

It holds

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \tilde{\mathbf{X}}} \mathbf{Q} = \tilde{\mathbf{F}} \mathbf{Q}$$

3.13

A hyperelastic material is termed \textit{isotropic with respect to the reference configuration} if the following relation holds

$$\Psi(F) = \Psi(\tilde{\mathbf{F}}) = \Psi(F \mathbf{Q}^T)$$

3.14

In similarity to Eq. 3.7, it can also be written

$$\tilde{\mathbf{X}}$$ can be viewed as a new reference system with deformation gradient \( \tilde{\mathbf{F}} = \partial \mathbf{x} / \partial \tilde{\mathbf{X}} \).
Hyperelasticity

\[ \Psi(\hat{\mathbf{F}}) = \Psi(\hat{\mathbf{C}}) = \Psi(\mathbf{F}^T \hat{\mathbf{F}}) = \Psi(Q \mathbf{F}^TQ^T) = \Psi(Q \mathbf{C}Q^T) \]  

so that from Eq. 3.14

\[ \Psi(\mathbf{C}) = \Psi(\hat{\mathbf{C}}) = \Psi(Q \mathbf{C}Q^T) \]  

By replacing \( Q \) with the orthogonal rotation tensor \( R \), the strain energy function may also be expressed in terms of the left Cauchy-Green tensor

\[ \Psi(\mathbf{C}) = \Psi(R \mathbf{C} R^T) = \Psi(\mathbf{b}) \]

Functions fulfilling Eq. 3.16 for any symmetric tensor \( \mathbf{C} \) and orthogonal tensor \( Q \) are termed \textit{scalar valued isotropic tensor functions}, Rivlin and Ericksen [1955]. They possess the important characteristic that they can be expressed equivalently via the invariants of their arguments i.e.

\[ \Psi(\mathbf{C}) = \Psi(I_1, I_2, I_3) = \Psi(I_1(b), I_2(b), I_3(b)) \]

with

\[
I_1(\mathbf{C}) = \mathbf{C} \cdot \mathbf{C} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \mathbf{b} : \mathbf{I} = I_1(b) \\
I_2(\mathbf{C}) = \frac{1}{2} |\mathbf{I}^2 - \mathbf{C} : \mathbf{C}| = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_1^2 \lambda_3^2 = \frac{1}{2} \left[ I_1^2 - \mathbf{b} : \mathbf{b} \right] = I_2(b) \\
I_3(\mathbf{C}) = \det \mathbf{C} = J^2 = \lambda_1^2 \lambda_2^2 \lambda_3^2 = \det \mathbf{b} = I_3(b)
\]

in which \( \lambda_i \ i = 1,2,3 \) are the principal stretches.

Several forms of strain energy functions can be encountered in literature. The \textit{Blatz-Ko} function originally proposed for a compressible material Blatz & Ko [1962] was based on theoretical arguments and was substantiated by experimental data. It has the form

\[ \Psi(I_1, I_2, I_3) = \frac{c \mu}{2} \left[ \left( I_1 - 3 \right) + \frac{1}{\alpha} \left( I_3^{-\alpha} - 1 \right) \right] + (1-c) \frac{\mu}{2} \left[ \frac{I_2}{I_3 - 3} \right] + \frac{1}{\alpha} \left( I_3^{-\alpha} - 1 \right) \]

\[ \alpha = \frac{\lambda}{2\mu} = \frac{\nu}{1-2\nu} \quad ; \quad c \in [0,1] \]

in which \( \lambda \) and \( \mu \) are the \textit{Lamé} material constants and \( \nu \) is the Poisson ratio.

By setting in the above \( c = 1 \) the strain energy function describing what has become known as the compressible \textit{neo-Hookean} model is obtained Lurie [1990], Lubarda [2002]

\[ \Psi(I_1, I_2, I_3) = \frac{\mu}{2} \left[ \left( I_1 - 3 \right) + \frac{1}{\alpha} \left( I_3^{-\alpha} - 1 \right) \right] \quad ; \quad \alpha = \frac{\lambda}{2\mu} = \frac{\nu}{1-2\nu} \]

\[ ^* \text{The definition } I_2(\mathbf{C}) = \mathbf{C} : \mathbf{C} = \lambda_1^4 + \lambda_2^4 + \lambda_3^4 = \mathbf{b} : \mathbf{b} = I_2(b) \text{ is also encountered in literature} \]
In the previous Section, the various stress tensors were expressed in terms of the tensor $\partial \Psi/\partial C$. On the basis of Eq. 3.18

$$\frac{\partial \Psi(C)}{\partial C} = \sum_{i=1}^{3} \frac{\partial \Psi(I_1, I_2, I_3)}{\partial I_i} \frac{\partial I_i}{\partial C} \sum_{i=1}^{3} \partial I_i \Psi \partial C I_i$$

The first two derivatives $\partial C I_1$ can be easily computed as

$$\partial C I_1 = \frac{\partial I_1}{\partial C} = \frac{\partial (I: C)}{\partial C} = I$$

$$\partial C I_2 = \frac{\partial I_2}{\partial C} = \frac{1}{2} \left[ 2(I: C) I - \frac{\partial (I: C)}{\partial C} \right] = I I - C$$

For computation of the third derivative, $\partial C (\det C)$, the concept of the directional derivative can be utilized, Bonet and Wood [1999], Holzapfel [2000]. Accordingly it results

$$\partial C (\det C) = J^2 C^{-1}$$

Introducing the above into Eq. 3.111 enables the stress tensor to be evaluated as

$$S = 2 \left[ \left( \partial I_1 \Psi + I_1 \partial I_2 \Psi \right) I - \partial I_2 \Psi C + I_3 \partial I_3 \Psi C^{-1} \right]$$

$$= s_1 I + s_2 C + s_3 C^{-1}$$

in which the coefficients $s_1, s_2, s_3$ for the Blatz-Ko and the neo-Hookean compressible isotropic hyperelastic models are shown in Table 3-1.

<table>
<thead>
<tr>
<th></th>
<th>Blatz-Ko</th>
<th>Neo-Hookean</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$\mu \left[ c + (1-c) \left( \frac{I_1}{I_3} \right) \right]$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$-\frac{\mu(1-c)}{I_3}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>$-\mu \left[ c I_3^{-\alpha} + (1-c) \left( \frac{L_2}{I_3} - I_3^{\alpha} \right) \right]$</td>
<td>$-\mu I_3^{\alpha}$</td>
</tr>
</tbody>
</table>

Table 3-1 Material model coefficients for Eq. 3.25

An alternative expression for the Cauchy stress tensor can be obtained by means of Eq. 3.112 as

$$\sigma = 2J^{-1} \left[ I_2 \partial I_2 \Psi + I_3 \partial I_3 \Psi \right] I + \partial I_1 \Psi b - I_3 \partial I_2 \Psi b^{-1}$$

$$3.26$$
where the relation $FF^TFF^T = bb = b^2$ has been utilized.

### 3.2.1 Elasticity Tensor

In Chapter 2 the 4th order elasticity tensor $C$ was defined as

$$C = \frac{\partial S}{\partial E} = 2 \frac{\partial S}{\partial C}$$  \hspace{1cm} 3.27

or equivalently by means of Eq. 3.111 and Eq. 3.22

$$C = 2 \frac{\partial S}{\partial C} = 4 \frac{\partial^2 \Psi (I_{11}, I_{12}, I_{13})}{\partial C \partial C} \hspace{1cm} ; \hspace{1cm} C_{ijkl} = \frac{\partial^2 \Psi (I_{11}, I_{12}, I_{13})}{\partial C_{ij} \partial C_{kl}}$$  \hspace{1cm} 3.28

Expanding Eq. 3.28 (Appendix 3.1), an expression convenient for computation and numerical implementation is obtained for the elasticity tensor in terms of the invariant set $(I_{11}, I_{12}, I_{13})$:

$$C = 4 \left[ c_1 I \otimes I + c_2 (I \otimes C + C \otimes I) + c_3 (I \otimes C^{-1} + C^{-1} \otimes I) + c_4 C \otimes C + c_5 (C \otimes C^{-1} + C^{-1} \otimes C) + c_6 C^{-1} \otimes C^{-1} + c_7 C^{-1} \otimes C^{-1} + c_8 \mathbb{I} \right]$$  \hspace{1cm} 3.29

with

$$c_1 = \left( \partial^2_{11} \Psi + 2I_{11} \partial^2_{11} \Psi + I_{11} \partial^2_{12} \Psi + I^2_{11} \partial^2_{12} \Psi \right)$$

$$c_2 = -\left( \partial^2_{11} \Psi + I_{11} \partial^2_{12} \Psi \right)$$

$$c_3 = \left( I_{11} \partial^2_{13} \Psi + I_{11} \partial^2_{13} \Psi \right)$$

$$c_4 = \partial^2_{12} \Psi$$

$$c_5 = -I_{12} \partial^2_{12} \Psi$$

$$c_6 = \left( I_{13} \partial^2_{13} \Psi + I^2_{13} \partial^2_{13} \Psi \right)$$

$$c_7 = -I_{13} \partial^2_{13} \Psi$$

$$c_8 = -\partial_{12} \Psi$$

$I$ is the 4th order identity tensor defined as $I_{ijkl} = \delta_{ik} \delta_{jl}$. The 4th order tensor $C^{-1} \otimes C^{-1}$ is defined, Holzapfel [2000], Lubarda [2002], as

$$C^{-1} \otimes C^{-1} = -\frac{\partial C^{-1}}{\partial C} \hspace{1cm} ; \hspace{1cm} (C^{-1} \otimes C^{-1})_{ijkl} = \frac{1}{2} \left( C^{-1}_{ik} C^{-1}_{jl} + C^{-1}_{il} C^{-1}_{jk} \right)$$  \hspace{1cm} 3.31
Hyperelasticity

Coefficients for the Blatz-Ko and the neo-Hookean compressible isotropic hyperelastic models can be found in Table 3-2.

<table>
<thead>
<tr>
<th></th>
<th>Blatz-Ko</th>
<th>Neo-Hookean</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$\frac{\mu(1-c)}{2I_3}$</td>
<td>0</td>
</tr>
<tr>
<td>$c_2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$c_3$</td>
<td>$-\frac{1}{2} \mu (1-c) \left( \frac{I_1}{I_3} \right)$</td>
<td>0</td>
</tr>
<tr>
<td>$c_4$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$c_5$</td>
<td>$\frac{\mu(1-c)}{2I_3}$</td>
<td>0</td>
</tr>
<tr>
<td>$c_6$</td>
<td>$\frac{1}{2} \mu \left[ \frac{c\alpha}{I_3^\alpha} + (1-c) \left( \frac{I_2}{I_3^\alpha} + \alpha I_3^\alpha \right) \right]$</td>
<td>$\frac{\mu\alpha}{2I_3}$</td>
</tr>
<tr>
<td>$c_7$</td>
<td>$\frac{1}{2} \mu \left[ \frac{c}{I_3^\alpha} + (1-c) \left( \frac{I_2}{I_3^\alpha} - I_3^\alpha \right) \right]$</td>
<td>$\frac{\mu}{2I_3^\alpha}$</td>
</tr>
<tr>
<td>$c_8$</td>
<td>$-\frac{\mu(1-c)}{2I_3}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3-2  Material model coefficients for Eq. 3.29

It is worth noticing that in the case of small strains when $I_3 \rightarrow 1$, the coefficients $c_6$ and $c_7$ of the neo-Hookean material are associated with the Lamé constants of small strain elasticity.

Once the material elasticity tensor is available, the spatial elasticity tensor can be computed as shown in Chapter 2.
3.3 Isotropic Elasticity in Principal Directions

From Eq. 3.19 it becomes apparent that scalar valued isotropic tensor functions like the strain energy can also be expressed in terms of the principal values of their arguments i.e.

$$\Psi(C) = \Psi(\lambda_1, \lambda_2, \lambda_3)$$  \hspace{1cm} 3.32

This enables re-expressing the stresses in terms of the principal stretches $\lambda_i$, $i = 1, 2, 3$. Substituting into Eq. 3.25 the identity

$$I = \sum_{i=1}^{3} L_i \otimes L_i$$  \hspace{1cm} 3.33

in which $L_i$ represent the orthonormal eigenvectors of $C$ and Eq. 2.28 and Eq. 2.29 it results

$$S = \sum_{i=1}^{3} \left[ 2 \partial_1 \Psi \left( \lambda_i - \lambda_i^2 \right) + 2 J^2 \partial_2 \Psi \lambda_i^{-2} \right] L_i \otimes L_i$$  \hspace{1cm} 3.34

which shows that for isotropic materials, the principal directions of the 2nd Piola-Kirchhoff stress tensor and the Lagrangian strain tensor coincide. Furthermore, substituting into Eq. 3.34 the following relations (which are obtained on the basis of Eq. 3.19)

$$1 = \frac{\partial I_1}{\partial \lambda_1^2} ; \quad I_1 - \lambda_1^2 = \frac{\partial I_2}{\partial \lambda_1^2} ; \quad \frac{J^2}{\lambda_i^2} = \frac{\partial I_3}{\partial \lambda_i^2}$$  \hspace{1cm} 3.35

the stress tensor can be expressed in terms of its principal components as

$$S = \sum_{i=1}^{3} \left( 2 \partial_1 \Psi \frac{\partial \lambda_1^2}{\partial \lambda_i} I_1 + 2 \partial_2 \Psi \frac{\partial \lambda_2^2}{\partial \lambda_i} I_2 + 2 \partial_3 \Psi \frac{\partial \lambda_3^2}{\partial \lambda_i} I_3 \right) L_i \otimes L_i$$

$$= \sum_{i=1}^{3} \left( 2 \partial_2 \Psi \right) L_i \otimes L_i$$  \hspace{1cm} 3.36

$$= \sum_{i=1}^{3} S_i L_i \otimes L_i$$

Alternatively from Eq. 3.111

$$S = 2 \frac{\partial \Psi}{\partial C} = 2 \sum_{i=1}^{3} \frac{\partial \Psi}{\partial \lambda_i^2} \frac{\partial \lambda_i^2}{\partial C} = 2 \sum_{i=1}^{3} \frac{\partial \Psi}{\partial \lambda_i} L_i \otimes L_i$$  \hspace{1cm} 3.37

From a comparison of Eq. 3.36 with Eq. 3.37 it can be concluded that the magnitude of the $i$-th principal stress component is

$$S_i = 2 \frac{\partial \Psi}{\partial \lambda_i^2} = \frac{\partial \Psi}{\partial \lambda_i} \frac{1}{\lambda_i}$$  \hspace{1cm} 3.38
Hyperelasticity

Also, a comparison with Eq. 2.28 and/or Eq. 2.39 reveals that for isotropic materials, $S$ has the same principal directions as $C$ and $E$.

For the Cauchy stress it holds

$$\sigma = J^{-1} F S F^T$$

$$= J^{-1} \sum_{i=1}^{3} \frac{\partial \Psi}{\partial \lambda_i} \frac{1}{\lambda_i} \left( F L_i \right) \otimes \left( F L_i \right)$$

$$= J^{-1} \sum_{i=1}^{3} \frac{\partial \Psi}{\partial \lambda_i} \frac{1}{\lambda_i} \left( \lambda_i L_i \right) \otimes \left( \lambda_i L_i \right) \quad \#$$

$$= \sum_{i=1}^{3} \lambda_i \frac{\partial \Psi}{\partial \lambda_i} L_i \otimes L_i$$

$$= \sum_{i=1}^{3} \sigma_i L_i \otimes L_i$$

which shows that for isotropic materials, the principal directions of the Cauchy stress tensor and the Eulerian strain tensor coincide.

3.3.1 Elasticity Tensor

In Chapter 2, $\dot{E}$ was expressed as

$$\dot{E} = \sum_{i=1}^{3} \dot{\Lambda}_i \dot{L}_i \otimes L_i + \sum_{i=1}^{3} \frac{\Lambda_i^2}{2} \left( \dot{L}_i \otimes L_i + L_i \otimes \dot{L}_i \right)$$

According to Eq. 2.38, each of the orthonormal eigenvectors can be expressed on the basis of the orthonormal Cartesian basis vectors $E_i$ as

$$L_i = R E_i$$

in which $R$ is an orthogonal tensor. Introducing the identity $R^T R = I$ and considering the fact that the basis vectors $E_i$ are fixed in space

$$\dot{L}_i = R \dot{E}_i = (R R^T) R E_i = \omega L_i$$

from which

$$\omega = \sum_{i=1}^{3} \dot{L}_i \otimes L_i$$

---

* Note: $FL_i = RUL_i = R \left( \sum_{j=1}^{3} \Lambda_j L_j \otimes L_j \right) L_i = R \left( \sum_{i=1}^{3} \Lambda_i L_i L_i^T \right) L_i = \Lambda_i RL_i$, because of the orthogonality of $L_i$-s. Utilizing Eq. 2.38 it results $FL_i = \Lambda_i L_i$.  

60
Substituting Eq. 3.42 into Eq. 3.42 and considering that \(\omega\) is a skew tensor

\[
\dot{\mathbf{E}} = \sum_{i=1}^{3} \Lambda_i \dot{\Lambda}_i \mathbf{L}_i \otimes \mathbf{L}_i + \frac{1}{2} \sum_{i=1}^{3} \Lambda_i^2 \left( \omega \mathbf{L}_i \otimes \mathbf{L}_i + \mathbf{L}_i \otimes \omega \mathbf{L}_i \right) \\
= \sum_{i=1}^{3} \Lambda_i \dot{\Lambda}_i \mathbf{L}_i \otimes \mathbf{L}_i + \frac{1}{2} \sum_{i=1}^{3} \Lambda_i^2 \left( -\omega^T \mathbf{L}_i \otimes \mathbf{L}_i - \mathbf{L}_i \otimes \mathbf{L}_i \omega \right) \\
= \sum_{i=1}^{3} \Lambda_i \dot{\Lambda}_i \mathbf{L}_i \otimes \mathbf{L}_i - \frac{1}{2} \left( \omega^T \mathbf{C} + \mathbf{C} \omega \right)
\]

As shown in Appendix 3.2, the last term of Eq. 3.44 can be further simplified to

\[
\omega^T \mathbf{C} + \mathbf{C} \omega = - \sum_{i,j=1}^{3} \omega_{ij} \Lambda_j^2 \left( \mathbf{L}_i \otimes \mathbf{L}_j + \mathbf{L}_j \otimes \mathbf{L}_i \right)
\]

Substituting this into Eq. 3.44, the tensor \(\dot{\mathbf{E}}\) is obtained in terms of its diagonal and off-diagonal components as

\[
\dot{\mathbf{E}} = \sum_{i=1}^{3} \Lambda_i \dot{\Lambda}_i \mathbf{L}_i \otimes \mathbf{L}_i + \frac{1}{2} \left( \sum_{i 
eq j}^{3} \omega_{ij} \left( \Lambda_i^2 - \Lambda_j^2 \right) \mathbf{L}_i \otimes \mathbf{L}_j \right)
\]

The time derivative of \(\mathbf{S}\) can also be cast in a similar format. From Eq. 3.37

\[
\dot{\mathbf{S}} = \sum_{i=1}^{3} \dot{S}_i \mathbf{L}_i \otimes \mathbf{L}_i + \sum_{i=1}^{3} S_i \left( \dot{\mathbf{L}}_i \otimes \mathbf{L}_i + \mathbf{L}_i \otimes \dot{\mathbf{L}}_i \right)
\]

in which the material derivative of the principal stress \(\dot{S}_i\) is

\[
\dot{S}_i = \sum_{j=1}^{3} \frac{\partial S_i}{\partial \Lambda_j^2} \dot{\Lambda}_j^2 = \sum_{j=1}^{3} \frac{\partial S_i}{\partial \Lambda_j^2} \dot{\Lambda}_j
\]

Also, in similarity to Eq. 3.44

\[
\sum_{i=1}^{3} S_i \left( \omega \mathbf{L}_i \otimes \mathbf{L}_i + \mathbf{L}_i \otimes \omega \mathbf{L}_i \right) = - \sum_{i,j=1}^{3} \omega_{ij} S_j \left( \mathbf{L}_i \otimes \mathbf{L}_j + \mathbf{L}_j \otimes \mathbf{L}_i \right)
\]

By means of Eq. 3.48 and Eq. 3.49, \(\dot{\mathbf{S}}\) can then be written as

\* Note: \(\mathbf{L}_i \otimes \omega \mathbf{L}_i = \mathbf{L}_i \left( \omega \mathbf{L}_i \right)^T = \mathbf{L}_i \mathbf{L}_i^T \omega^T = - \mathbf{L}_i \mathbf{L}_i^T \omega = - \mathbf{L}_i \otimes \mathbf{L}_i \omega\)
Hyperelasticity

\[
\dot{\mathbf{S}} = \sum_{i,j=1}^{3} \frac{\partial S_{ij}}{\partial \Lambda_j} \dot{\Lambda}_j \mathbf{L}_i \otimes \mathbf{L}_i + \sum_{i \neq j}^{3} \omega_{ij} (S_i - S_j) \mathbf{L}_i \otimes \mathbf{L}_j \quad 3.50
\]

Examination of the particular forms of \( \dot{\mathbf{E}} \) and \( \dot{\mathbf{S}} \) reveals that both are expressed as a sum of their diagonal and off-diagonal components. By exploiting this particular characteristic, the Lagrangian elasticity tensor \( \mathbf{C} \) for a material whose response is described in terms of the principal stretches is obtained in Appendix 3.2 as

\[
\mathbf{C} = \sum_{i,j=1}^{3} \frac{\partial S_{ij}}{\partial \Lambda_j} \dot{\Lambda}_j \mathbf{L}_i \otimes \mathbf{L}_i \otimes \mathbf{L}_j \otimes \mathbf{L}_j + 2 \sum_{i \neq j}^{3} \omega_{ij} (S_i - S_j) \mathbf{L}_i \otimes \mathbf{L}_j \otimes \mathbf{L}_i \otimes \mathbf{L}_j \quad 3.51
\]

Once \( \mathbf{C} \) is available, the spatial or Eulerian elasticity tensor can be computed via Eq. 2.181.
Expansion of Eq. 3.28 leads to

\[ \frac{\partial}{\partial C} \left[ \frac{\partial \Psi}{\partial \mathbf{I}_1} \right] = \frac{\partial}{\partial C} \left[ \frac{\partial \Psi}{\partial \mathbf{I}_1 \partial \mathbf{I}_1} + \frac{\partial \Psi}{\partial \mathbf{I}_2 \partial \mathbf{C}} + \frac{\partial \Psi}{\partial \mathbf{I}_3 \partial \mathbf{C}} \right] \]

A.3.1.1

First term

\[ \frac{\partial}{\partial C} \left[ \frac{\partial \Psi}{\partial \mathbf{I}_1 \partial \mathbf{I}_1} \right] = \frac{\partial}{\partial \mathbf{I}_1 \partial \mathbf{I}_1} \frac{\partial}{\partial \mathbf{C}} \left[ \frac{\partial \Psi}{\partial \mathbf{I}_1} \right] + \frac{\partial}{\partial \mathbf{I}_1} \frac{\partial}{\partial \mathbf{C}} \left[ \frac{\partial \Psi}{\partial \mathbf{I}_1} \right] \]

A.3.1.2

\[ \frac{\partial}{\partial \mathbf{C}} \left[ \frac{\partial \Psi}{\partial \mathbf{I}_1} \right] = \frac{\partial^2 \Psi}{\partial \mathbf{I}_1 \partial \mathbf{C}} + \frac{\partial^2 \Psi}{\partial \mathbf{I}_2 \partial \mathbf{C}} + \frac{\partial^2 \Psi}{\partial \mathbf{I}_3 \partial \mathbf{C}} \]

A.3.1.3

\[ = \frac{\partial^2 \Psi}{\partial \mathbf{I}_1 \partial \mathbf{I}_1} \mathbf{I} + \frac{\partial^2 \Psi}{\partial \mathbf{I}_1 \partial \mathbf{I}_2} (\mathbf{I} \mathbf{I}_1 \mathbf{I}_1 - \mathbf{C}) + \frac{\partial^2 \Psi}{\partial \mathbf{I}_1 \partial \mathbf{I}_3} \mathbf{I}_3 \mathbf{C}^{-1} \]

Hence

\[ \frac{\partial}{\partial C} \left[ \frac{\partial \Psi}{\partial \mathbf{I}_1 \partial \mathbf{I}_1} \right] = \frac{\partial}{\partial \mathbf{C}} \left[ \frac{\partial \Psi}{\partial \mathbf{I}_1} \right] = 0 \]

A.3.1.4

Second term

\[ \frac{\partial}{\partial C} \left[ \frac{\partial \Psi}{\partial \mathbf{I}_2 \partial \mathbf{I}_2} \right] = \frac{\partial}{\partial \mathbf{C}} \left[ \frac{\partial \Psi}{\partial \mathbf{I}_2} \right] = 0 \]

A.3.1.6
Hyperelasticity

\[
\frac{\partial}{\partial C} \left[ \frac{\partial \psi}{\partial I_2} \right] = \frac{\partial^2 \psi}{\partial I_1 \partial I_2} \frac{\partial I_1}{\partial C} + \frac{\partial^2 \psi}{\partial I_2 \partial I_2} \frac{\partial I_2}{\partial C} + \frac{\partial^2 \psi}{\partial I_2 \partial I_3} \frac{\partial I_3}{\partial C}
\]

\[
= \frac{\partial^2 \psi}{\partial I_1 \partial I_2} I + \frac{\partial^2 \psi}{\partial I_2 \partial I_2} (I_1 I - C) + \frac{\partial^2 \psi}{\partial I_2 \partial I_3} I_3 C^{-1}
\]

\[
\frac{\partial}{\partial C} \left[ \frac{\partial I_2}{\partial C} \right] = \frac{\partial}{\partial C} [I_1 I - C]
\]

\[
= I \frac{\partial I_1}{\partial C} + I_1 \frac{\partial I}{\partial C} - \frac{\partial C}{\partial C}
\]

\[
= I \otimes I - I
\]

Hence

\[
\frac{\partial}{\partial C} \left[ \frac{\partial \psi}{\partial I_2} \frac{\partial I_2}{\partial C} \right] = \frac{\partial I_2}{\partial C} \otimes \left[ \frac{\partial^2 \psi}{\partial I_1 \partial I_2} I + \frac{\partial^2 \psi}{\partial I_2 \partial I_2} (I_1 I - C) + \frac{\partial^2 \psi}{\partial I_2 \partial I_3} I_3 C^{-1} \right] + \frac{\partial \psi}{\partial I_2} [I \otimes I - I]
\]

\[
= (I_1 I - C) \otimes \left[ \frac{\partial^2 \psi}{\partial I_1 \partial I_2} I + \frac{\partial^2 \psi}{\partial I_2 \partial I_2} (I_1 I - C) + \frac{\partial^2 \psi}{\partial I_2 \partial I_3} I_3 C^{-1} \right] + \frac{\partial \psi}{\partial I_2} [I \otimes I - I]
\]

\[
\frac{\partial \psi}{\partial I_2} [I \otimes I - I]
\]

\[
= I_1 \frac{\partial^2 \psi}{\partial I_1 \partial I_2} I \otimes I - I_1 \frac{\partial^2 \psi}{\partial I_1 \partial I_2} C \otimes I + I_2 \frac{\partial^2 \psi}{\partial I_2 \partial I_2} I \otimes I - I_1 \frac{\partial^2 \psi}{\partial I_2 \partial I_2} I \otimes C
\]

\[
- I_1 \frac{\partial^2 \psi}{\partial I_2 \partial I_2} C \otimes I + \frac{\partial^2 \psi}{\partial I_2 \partial I_2} C \otimes C + I_1 I_3 - \frac{\partial^2 \psi}{\partial I_2 \partial I_3} I \otimes C^{-1}
\]

\[
- I_1 \frac{\partial^2 \psi}{\partial I_2 \partial I_3} C \otimes C^{-1} + \frac{\partial \psi}{\partial I_2} I \otimes I - \frac{\partial \psi}{\partial I_2}
\]

\[
= \left( I_1 \frac{\partial^2 \psi}{\partial I_1 \partial I_2} + I_2 \frac{\partial^2 \psi}{\partial I_2 \partial I_2} + \frac{\partial \psi}{\partial I_2} \right) I \otimes I - \left( \frac{\partial^2 \psi}{\partial I_1 \partial I_2} + I_1 \frac{\partial^2 \psi}{\partial I_2 \partial I_2} \right) C \otimes I
\]

\[
- I_1 \frac{\partial^2 \psi}{\partial I_2 \partial I_2} C \otimes I + \frac{\partial^2 \psi}{\partial I_2 \partial I_2} C \otimes C + I_1 I_3 - \frac{\partial^2 \psi}{\partial I_2 \partial I_3} I \otimes C^{-1}
\]

\[
- I_1 \frac{\partial^2 \psi}{\partial I_2 \partial I_3} C \otimes C^{-1} - \frac{\partial \psi}{\partial I_2}
\]

Third term

\[
\frac{\partial}{\partial C} \left[ \frac{\partial \psi}{\partial I_3} \frac{\partial I_3}{\partial C} \right] = \frac{\partial I_3}{\partial C} \otimes \frac{\partial}{\partial C} \left[ \frac{\partial \psi}{\partial I_3} \right] + \frac{\partial \psi}{\partial I_3} \frac{\partial}{\partial C} \left( \frac{\partial I_3}{\partial C} \right)
\]

A.3.1.7

A.3.1.8

A.3.1.9

A.3.1.10
\[
\frac{\partial}{\partial C} \left[ \frac{\partial \Psi}{\partial I_3} \right] = \frac{\partial^2 \Psi}{\partial I_1 \partial I_3} \frac{\partial I_1}{\partial C} + \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} \frac{\partial I_2}{\partial C} + \frac{\partial^2 \Psi}{\partial I_3 \partial I_3} \frac{\partial I_3}{\partial C} \\
= \frac{\partial^2 \Psi}{\partial I_1 \partial I_3} I + \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} (I_1 I - C) + \frac{\partial^2 \Psi}{\partial I_3 \partial I_3} I_3 C^{-1}
\]

\[
\frac{\partial}{\partial C} \left[ \frac{\partial I_3}{\partial C} \right] = \frac{\partial}{\partial C} \left[ I_3 C^{-1} \right] \\
= C^{-1} \frac{\partial I_3}{\partial C} + I_3 \frac{\partial C^{-1}}{\partial C}
\]

A.3.1.11

A.3.1.12

Hence

\[
\frac{\partial}{\partial C} \left[ \frac{\partial \Psi}{\partial I_3} \right] = I_3 C^{-1} \otimes \left[ \frac{\partial^2 \Psi}{\partial I_1 \partial I_3} I + \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} (I_1 I - C) + \frac{\partial^2 \Psi}{\partial I_3 \partial I_3} I_3 C^{-1} \right] + \\
\frac{\partial \Psi}{\partial I_3} \left[ I_3 C^{-1} \otimes C^{-1} - I_3 C^{-1} \otimes C^{-1} \right]
\]

\[
= I_3 \frac{\partial^2 \Psi}{\partial I_1 \partial I_3} C^{-1} \otimes I + I_3 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} C^{-1} \otimes I - I_3 \frac{\partial^2 \Psi}{\partial I_3 \partial I_3} C^{-1} \otimes C
\]

\[
+ I_3 \frac{\partial^2 \Psi}{\partial I_3 \partial I_3} C^{-1} \otimes C^{-1} + I_3 \frac{\partial \Psi}{\partial I_3} C^{-1} \otimes C^{-1} - I_3 \frac{\partial \Psi}{\partial I_3} C^{-1} \otimes C^{-1}
\]

A.3.1.13

\[
= \left( I_3 \frac{\partial^2 \Psi}{\partial I_1 \partial I_3} + I_1 I_3 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} \right) C^{-1} \otimes I + \left( I_3 \frac{\partial^2 \Psi}{\partial I_3 \partial I_3} + I_3 \frac{\partial \Psi}{\partial I_3} \right) C^{-1} \otimes C^{-1}
\]

\[
- I_3 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} C^{-1} \otimes C - I_3 \frac{\partial \Psi}{\partial I_3} C^{-1} \otimes C^{-1}
\]

Finally
\[
\frac{\partial}{\partial C} \left[ \frac{\partial \Psi}{\partial C} \right] = \left[ \frac{\partial^2 \Psi}{\partial I_1 \partial I_1} + I_1 \frac{\partial^2 \Psi}{\partial I_1 \partial I_2} \right] I \otimes I + \frac{\partial^2 \Psi}{\partial I_1 \partial I_2} I \otimes C + \frac{\partial^2 \Psi}{\partial I_1 \partial I_3} I \otimes C^{-1} + \\
\left( I_1 \frac{\partial^2 \Psi}{\partial I_1 \partial I_2} + I_1 \frac{\partial^2 \Psi}{\partial I_2 \partial I_2} + \frac{\partial \Psi}{\partial I_2} \right) I \otimes I - \frac{\partial^2 \Psi}{\partial I_1 \partial I_2} I \otimes C + \frac{\partial^2 \Psi}{\partial I_1 \partial I_3} I \otimes C^{-1} \\
- I_1 \frac{\partial^2 \Psi}{\partial I_2 \partial I_2} I \otimes C + \frac{\partial^2 \Psi}{\partial I_2 \partial I_2} C \otimes C + I_1 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} I \otimes C^{-1} \\
- I_1 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} C \otimes C^{-1} - \frac{\partial \Psi}{\partial I_2} I \\
+ \left( I_1 \frac{\partial^2 \Psi}{\partial I_1 \partial I_3} + I_1 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} \right) C^{-1} \otimes I + \left( I_3 \frac{\partial^2 \Psi}{\partial I_3 \partial I_3} + I_3 \frac{\partial \Psi}{\partial I_3} \right) C^{-1} \otimes C^{-1} \\
- I_3 \frac{\partial \Psi}{\partial I_2} C^{-1} \otimes C - I_3 \frac{\partial \Psi}{\partial I_3} C^{-1} \otimes C^{-1} \\
= \left( \frac{\partial^2 \Psi}{\partial I_1 \partial I_1} + I_1 \frac{\partial^2 \Psi}{\partial I_1 \partial I_2} I_1 \frac{\partial^2 \Psi}{\partial I_1 \partial I_2} + \frac{\partial^2 \Psi}{\partial I_2 \partial I_2} + \frac{\partial \Psi}{\partial I_2} \right) I \otimes I + \\
\left( - \frac{\partial^2 \Psi}{\partial I_1 \partial I_2} - I_1 \frac{\partial^2 \Psi}{\partial I_2 \partial I_2} \right) \left( I \otimes C + C \otimes I \right) + \\
\left( I_3 \frac{\partial^2 \Psi}{\partial I_1 \partial I_3} + I_1 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} \right) \left( I \otimes C^{-1} + C^{-1} \otimes I \right) + \frac{\partial^2 \Psi}{\partial I_2 \partial I_2} C \otimes C + \\
\left( - I_3 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} \right) \left( C \otimes C^{-1} + C^{-1} \otimes C \right) + \\
\left( I_3 \frac{\partial \Psi}{\partial I_3} \right) C^{-1} \otimes C^{-1} + \left( - I_3 \frac{\partial \Psi}{\partial I_3} \right) C^{-1} \otimes C^{-1} + \\
\left( - \frac{\partial \Psi}{\partial I_2} \right) I \\
\right] \tag{A.3.1.14}
\]
The last term of Eq. 3.44 can be simplified as follows

\[ \omega^T C + C \omega = \sum_{i=1}^{3} \left( \dot{L}_i \otimes L_i \right)^T \sum_{j=1}^{3} \Lambda_j^2 L_j \otimes L_j + \sum_{j=1}^{3} \Lambda_j^2 L_j \otimes L_j \sum_{i=1}^{3} \left( \dot{L}_i \otimes L_i \right) \]

\[ = \sum_{i,j=1}^{3} \Lambda_j^2 L_i \dot{L}_i^T L_j L_j^T + \sum_{i,j=1}^{3} \Lambda_j^2 L_j \dot{L}_j^T L_i L_i^T \]

\[ = \sum_{i,j=1}^{3} \Lambda_j^2 L_i \dot{L}_i \otimes L_j + \sum_{i,j=1}^{3} \Lambda_j^2 L_j \dot{L}_j \otimes L_i \]

\[ = \sum_{i,j=1}^{3} \omega_{ij} \Lambda_j^2 L_i \otimes L_j + \sum_{i,j=1}^{3} \omega_{ij} \Lambda_j^2 L_j \otimes L_i \]

\[ = -\sum_{i,j=1}^{3} \omega_{ij} \Lambda_j^2 \left( L_i \otimes L_j + L_j \otimes L_i \right) \]

A diagonal term of \( \hat{E} \) can be computed as

\[ \hat{E}_{ii} = L_i \otimes L_i : \hat{E} \]

\[ = \sum_{j=1}^{3} \Lambda_j \dot{\Lambda}_j L_i \otimes L_i : L_j \otimes L_j + \frac{1}{2} \left( \sum_{i \neq j}^{3} \omega_{ij} \left( \Lambda_i^2 - \Lambda_j^2 \right) L_i \otimes L_j : L_i \otimes L_j \right) \]  \hspace{1cm} \text{(A.3.2.2)}

\[ = \Lambda_i \dot{\Lambda}_i L_i \otimes L_i \]

also, an off-diagonal term of \( \hat{E} \) can be computed as

\[ \hat{E}_{ij} = L_i \otimes L_j : \hat{E} \]

\[ = \sum_{j=1}^{3} \Lambda_j \dot{\Lambda}_j L_i \otimes L_j : L_j \otimes L_j + \frac{1}{2} \left( \sum_{i \neq j}^{3} \omega_{ij} \left( \Lambda_i^2 - \Lambda_j^2 \right) L_i \otimes L_j : L_j \otimes L_j \right) \]  \hspace{1cm} \text{(A.3.2.3)}

\[ = \frac{1}{2} \omega_{ij} \left( \Lambda_i^2 - \Lambda_j^2 \right) L_i \otimes L_j \quad \forall \ i \neq j \]

In explicit form the contraction \( \dot{S} = C : \hat{E} \) is written as
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\[ \dot{\mathbf{S}} = \mathbf{C} : \dot{\mathbf{E}} = \sum_{i,j,k,l,m,n=1}^{3} C_{ijkl} \dot{E}_{mn}(N_i \otimes N_j \otimes N_k \otimes N_l)(N_m \otimes N_n) \]  

A.3.2.4

\[ = \sum_{i,j,k,l=1}^{3} C_{ijkl} \dot{E}_{kl}(N_i \otimes N_j) \]

For a typical component is then

\[ \dot{S}_{ij} = \sum_{i,j,k,l=1}^{3} C_{ijkl} \dot{E}_{kl}(N_i \otimes N_j) = C_{i1j1} \dot{E}_{11} + C_{i1j2} \dot{E}_{12} + \ldots C_{i1j32} \dot{E}_{32} + C_{i1j33} \dot{E}_{33} \]  

A.3.2.5

\[ = \frac{\partial \dot{S}_{ij}}{\partial \dot{E}_{11}} \dot{E}_{11} + \frac{\partial \dot{S}_{ij}}{\partial \dot{E}_{12}} \dot{E}_{12} + \ldots \frac{\partial \dot{S}_{ij}}{\partial \dot{E}_{32}} \dot{E}_{32} + \frac{\partial \dot{S}_{ij}}{\partial \dot{E}_{33}} \dot{E}_{33} \]

For \( i = j \)

\[ \dot{S}_{ii} = C_{i111} \dot{E}_{11} + C_{i112} \dot{E}_{12} + \ldots C_{i132} \dot{E}_{32} + C_{i133} \dot{E}_{33} \]  

A.3.2.6

A comparison of Eq. 2.6 with Eq. 3.50 shows that \( C_{ikkl} = 0 \). This implies that in the contraction \( \dot{S} = \mathbf{C} : \dot{\mathbf{E}} \), only terms of the 4th order tensor \( \mathbf{C} \) involving tensor products with the form \( (N_i \otimes N_j \otimes N_k \otimes N_l) \) participate. Hence the diagonal component of the constitutive matrix can be written as

\[ C_{d} = \sum_{i,j=1}^{3} \frac{\partial S_{ii}}{\partial \Lambda_j} \Lambda_j L_i \otimes L_i \otimes L_j \otimes L_j \]  

A.3.2.7

For \( i \neq j \) a comparison with Eq. 3.50 shows that \( C_{ijkl} = 0 \). This implies that in the contraction \( \dot{S} = \mathbf{C} : \dot{\mathbf{E}} \), only terms of the 4th order tensor \( \mathbf{C} \) involving tensor products with the form \( (N_i \otimes N_j \otimes N_k \otimes N_l)_{k=1}^{x} \) participate. The conditions \( i \neq j \) and \( k = 1 \) also enable the reduction of the contraction operation free indices from 4 to 2 so that a typical tensor product term can be written as \( (N_i \otimes N_j \otimes N_i \otimes N_j) \). Hence the off-diagonal component of the constitutive matrix can be written as

\[ C_{nd} = 2 \sum_{i=1}^{3} \omega_{ij} \left(S_{ij} - S_{ji}\right) L_i \otimes L_i \otimes L_i \otimes L_i \]  

A.3.2.8

Summing the two 4th order tensors, the Lagrangian elasticity tensor \( \mathbf{C} \) for a material whose response is described in terms of the principal stretches is obtained as

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\[ C = C_d + C_{nd} = \sum_{i,j=1}^{3} \frac{\partial S_i}{\partial \Lambda_j} \hat{A}_j L_i \otimes L_i \otimes L_j \otimes L_j \]
\[ + 2 \sum_{\substack{i \neq j \atop i,j=1}}^{3} \omega_{ij} \left(S_i - S_j\right) L_i \otimes L_j \otimes L_i \otimes L_j \]
Hyperelasticity
4.1 Elementary Model of Elastoplastic Response

Elastoplastic materials are typical examples of dissipative materials with history dependent response controlled by a number of internal variables such as the plastic strain and variables controlling isotropic and/or kinematic hardening.

In this Section, an elementary rheological model will be utilised to establish the fundamental aspects of elastoplastic material response. In subsequent Sections these will be extended to the three dimensional case.

Considering first the case of an ideal elastic perfectly plastic model, from Fig. 4.1, after yield, the stress in the spring of the elementary plastic model is

\[ \sigma = E(\varepsilon - \varepsilon_p) = E \varepsilon_e \]  \hspace{1cm} (4.1)

\[ \begin{array}{c}
\sigma \\
\uparrow \\
\varepsilon \\
\varepsilon_e \\
\varepsilon_p
\end{array} \quad \begin{array}{c}
\downarrow \\
\uparrow \\
\uparrow
\end{array} \quad \begin{array}{c}
E \\
\sigma_y \\
\sigma \\
\end{array} \]

Fig. 4.1 Elementary elastoplastic model

Since the stress appears to be a function of the elastic strain, it is natural to postulate the existence of a Helmholtz free energy function of the form

\[ \Psi = \Psi(\varepsilon_e, \ldots) \]  \hspace{1cm} (4.2)

so that (see Chapter 2) at least for the elastic phase of response it is valid that

\[ \sigma = \frac{\partial \Psi}{\partial \varepsilon_e} \]  \hspace{1cm} (4.3)

Considering now the case of a hardening elastoplastic model, an internal parameter \( \xi \) can be introduced to control the post-yield response of the model, Fig. 4.2

\[ \Psi = \Psi(\varepsilon_e, \xi) \]  \hspace{1cm} (4.4)
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\[ \sigma \cdot \dot{\varepsilon} - \dot{\Psi} = \sigma \cdot \dot{\varepsilon} - \frac{\partial \Psi}{\partial \varepsilon_e} \cdot \frac{\partial \varepsilon_e}{\partial t} - \frac{\partial \Psi}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} \]

\[ = \sigma \cdot \dot{\varepsilon} - \frac{\partial \Psi}{\partial \varepsilon_e} \cdot \frac{\partial \varepsilon}{\partial t} + \frac{\partial \Psi}{\partial \varepsilon_e} \cdot \frac{\partial \varepsilon_p}{\partial t} - \frac{\partial \Psi}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} \]

\[ = \left( \sigma - \frac{\partial \Psi}{\partial \varepsilon_e} \right) \cdot \dot{\varepsilon} + \left( \frac{\partial \Psi}{\partial \varepsilon_e} \cdot \dot{\varepsilon}_p - \frac{\partial \Psi}{\partial \xi} \cdot \dot{\xi} \right) \geq 0 \]

which according to the arguments of Coleman and Nol [1963] results to a constitutive relation for the stress

\[ \sigma = \frac{\partial \Psi}{\partial \varepsilon_e} \]

and the inequality

\[ \sigma \cdot \dot{\varepsilon}_p + q \cdot \dot{\xi} \geq 0 \]

with \( q = -\frac{\partial \Psi}{\partial \xi} \).

In anticipation of the generalisation of the model to the three dimensional case, the notion of a domain of admissible stresses in stress space is defined as

\[ \omega = \left\{ \sigma : f(\sigma, q) \leq 0 \right\} \]

in which \( f \) is the yield surface. A schematic of \( f \) in the principal stress space is shown in Fig. 4.3.
The principle of *maximum plastic dissipation* has played a crucial role in the modern mathematical formulation of plasticity, Teman [1985], Simo & Hughes [1998]. It states that for a given set of \((\dot{\varepsilon}_p, \dot{\xi})\), among all possible sets \((\sigma, q)\) satisfying the condition of Eq. 4.8, the actual one is the one which maximizes the argument of inequality 4.7, that is, the one for which

\[
\sigma \in \omega : \max \left(\sigma \cdot \dot{\varepsilon}_p + q \cdot \dot{\xi}\right)
\]

Utilizing the formal notation of Appendix 4.1, Simo [1992] has pointed out that the above two relations can be recast as the following constraint minimization problem

\[
\text{minimize} \quad -\left(\sigma \cdot \dot{\varepsilon}_p + q \cdot \dot{\xi}\right) \\
\text{subject to} \quad f(\sigma, q) \leq 0
\]

According to Appendix 4.4 this is equivalent to

\[
-\dot{\varepsilon}_p + \lambda \frac{\partial f(\sigma, q)}{\partial \sigma} = 0 \\
-\dot{\xi} + \lambda \frac{\partial f(\sigma, q)}{\partial q} = 0 \\
\lambda f(\sigma, q) = 0 \\
\lambda \geq 0
\]

Eq. 4.11 represents the well known notion of *normality* of the plastic strain to the yield surface. The corresponding plasticity models are termed *associative*. The conditions expressed by Eq. 4.11 and Eq. 4.11 are known in optimization theory as the *Kuhn-Tucker* conditions, Luenberger [1984]. In plasticity theory, the Lagrange multiplier \(\lambda\) is known as the *plastic multiplier*. Its computation will be presented in a following section.
4.2 Three Dimensional Elastoplastic Model

In the following a model for simulation of the response of a wide variety of engineering materials ranging from soils to concrete to rocks and to asphalitic materials will be presented. It is based on the theory of rate dependent consistent plasticity. By retaining the fundamental to classical plasticity notions of flow surface, decomposition of strains, hardening and/or softening, the theory has emerged as an attempt to provide a realistic, unified, phenomenological approach for materials exhibiting strain rate dependent inelastic deformations.

4.2.1 Constitutive Framework

In similarity to inviscid plasticity, in the theory of rate dependent consistent plasticity, the plastic rate is defined as:

\[ \dot{\epsilon}_p = \dot{x} \frac{\partial f}{\partial \sigma} \quad 4.12 \]

whith \( \dot{x} \) a constant of proportionality\(^1\), and \( f \) is a response surface associated with a locus of states of stress \( \sigma \) corresponding to a certain magnitude of inelastic response.

Two main phases of material response are distinguished by the formulation, Fig. 4.4:

(a) “hardening”, spanning the range from zero stress to ultimate response and,

(b) “softening”, spanning from ultimate response to response annihilation.

![Diagram showing hardening and softening phases](image)

Fig. 4.4 Schematic of main phases of material model response

The standard Kuhn-Tucker conditions are imposed:

\[ \dot{x} \geq 0 \quad , \quad f \leq 0 \quad , \quad \dot{x} \cdot f = 0 \quad 4.13 \]

Evolution of plastic flow is determined by the consistency condition:

\[ \]

\(^1\) For Perzyna type viscoplasticity \( \dot{x} = \Gamma \cdot \langle \Phi \rangle \) with \( \Gamma \) the material fluidity and \( \langle \Phi \rangle \) an overstress function.
\[
\dot{\sigma} = f(\sigma, \dot{\varepsilon}, T, \kappa) = 0
\]

in which \(\dot{\varepsilon}\) is the deformation rate, \(T\) is the temperature and \(\kappa\) is some measure of hardening/softening.

### 4.2.2 Hardening Response

The model utilizes a single flow surface for hardening response simulation.

#### Flow surface characteristics

The flow surface proposed by Desai [1990], has been chosen. In stress invariant space it is defined as:

\[
f_D = \frac{J_2}{P_a} - F_b \cdot F_c = 0
\]

in which:

\[
F_b = \left[ -\alpha \cdot \left( \frac{I_1 + R}{P_a} \right)^n + \gamma \cdot \left( \frac{I_1 + R}{P_a} \right)^2 \right]
\]

\[
F_c = \left( 1 - \beta \cdot \cos \theta \right)^{-1/2}
\]

\[
\cos \theta = \frac{3\sqrt{3}}{2} \cdot \frac{J_3}{J_2^{3/2}}
\]

\(I_1, J_2\) and \(J_3\) are stress invariants and \(P_a\) is the atmospheric pressure.

Eq. 4.15 represents a closed surface in the \(\left( I_1, \sqrt{I_2}, \theta \right)\) space, Fig. 4.3, alleviating thus the need for definition of additional cap surfaces along the hydrostatic axis. The trace of the surface on the meridian and the deviatoric planes are shown schematically in Fig. 4.5.

![Fig. 4.5 Trace of the Desai surface in the meridian and the deviatoric planes](image-url)
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The trace of the surface on the octahedral plane is determined by parameter $\beta$. For $\beta = 0$ the trace is circular. As $\beta$ increases, the trace progressively becomes triangular.

The hardening response of the material is controlled by parameter $\alpha$. As $\alpha$ decreases, the size of the flow surface increases, Fig. 4.6. By defining $\alpha$ as a decreasing function of plastic strain or plastic work, the characteristics of the hardening response of the material can be simulated.

![Desai surface hardening for decreasing $\alpha$](image)

**Fig. 4.6** Parameter $\alpha$ controls hardening of the Desai surface

$R$ is the triaxial tensile strength of the material. The slope of the ultimate response surface in the $I_1 - \sqrt{J_2}$ plane is controlled by parameter $\gamma$. It is attained for $\alpha = 0$.

![Graph with Desai surface parameters](image)

**Fig. 4.7** Hardening response of Desai surface

Parameter $n$ determines the apex of the surface on the $I_1 - \sqrt{J_2}$ plane. It defines the state of stress beyond which the material begins to dilate, Fig. 4.8.
Fig. 4.8...Parameter $n$ determines the point of material dilation

As shown in Appendix 4.2, by computing the values of the stress invariants at the state of stress at which dilation initiates, the value of $n$ can be determined from the expression

$$
n = \frac{2}{\left(1 - \frac{J_2}{(I_1 + R)^2} \gamma (1 - 3 \cos \theta) \right)^{1/2}} \tag{4.19}
$$

**4.2.3 Response Degradation**

For $\alpha = 0$ the ultimate response of the material is attained. In the model, for deformation levels beyond those corresponding to $\alpha = 0$, two independent mechanisms are activated for controlling the subsequent response.

**$\gamma$ Degradation**

An isotropic measure of response surface degradation has been introduced in the current implementation by means of specifying $\gamma$, after response degradation initiation, as a decaying function of the monotonically increasing post fracture plastic work, the deformation rate $\dot{\gamma}$ and the temperature $T$:

$$
\gamma = \gamma(\dot{\delta}, T, W_{pf}) \tag{4.20}
$$

Only work contributed by compressive stress paths is taken into account. After response degradation initiation, in all subsequent steps, the principal values of the plastic strain vector are computed. The increment of post fracture plastic work is constructed then as

$$
dW_{pf} = \sum_{i=1}^{3} \sigma_i d\varepsilon_i \quad : \quad d\varepsilon_i < 0 \tag{4.21}
$$

in which $\sigma_1$ is the stress component along the principal strain direction $1$. 

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Isotropic softening is meant to simulate, primarily, the overall material response degradation that is observed experimentally as a result of compressive loading. In the model it is complemented by an additional softening mechanism acting on specific material planes as described in the following.

Cracking

In contrast to the diffused nature of compressive damage, in materials like concrete, rocks and asphalts at low temperatures, tensile states of stress can lead to tensile damage whose nature is much more discrete and which tends to concentrate along planes of cracking.

An additional characteristic of these materials is that, for cyclic states of stress in which the material is subjected to alternating stress paths, tensile cracking of the material in one part of the loading cycle does not weaken its compressive strength after stress reversal.

On the other hand, as mentioned earlier, compressive loading does seem to weaken the tensile strength of the material even in the cross direction introducing thus some degree of isotropic softening.

In order to incorporate both of these aspects of material response into the model formulation, an additional, independent of γ softening, softening criterion has been introduced whose main purpose is the simulation of the tensile softening response.

Along the lines of the classical notion of fixed cracking, for states of stress exceeding the magnitude of the flow surface (as determined by the current value of parameter γ ), a plane of cracking is introduced perpendicular to the principal tensile stress direction. On the crack plane, a Hoffman type criterion similar to that utilized by Schellekens [1992] and Scarpas & Blaauwendraad [1992] is specified to control the subsequent softening response:

\[ \sigma^2 + q \left( \tau_s^2 + \tau_t^2 \right) = f_t^2 \left( \delta, T, \kappa \right) \]

in which \( \sigma \) is the normal stress on the crack plane, Fig. 4.9(a), \( \tau_s \) and \( \tau_t \) are the shear stress components, \( f_t \) the uniaxial tensile stress after crack initiation and \( \kappa \) some measure of softening.

Fig. 4.9 (a) Stresses on crack plane, (b) Schematic of Hoffman surface on crack plane
A schematic of the Hoffman surface is shown in Fig. 4.9(b). Up to three orthogonal cracking planes can be introduced at a material point, Fig. 4.10(a). Compatibility of shear stresses along orthogonal crack planes is ensured, Fig. 4.10(b).

![Diagram of Hoffman surface and orthogonal cracking planes]

Fig. 4.10  (a) Orthogonal cracking planes, (b) Shear stresses compatibility on cracking planes

### 4.3 Material Parameter Determination

Evaluation of the model parameters is based on the results of appropriate laboratory investigations. Ideally, results from triaxial tests at different stress paths, rates and temperatures are required if the full simulation capabilities of the model are to be realised. Nevertheless, these are not always easy to obtain. In the following determination of the most significant model parameters will be demonstrated on the basis of uniaxial tests only.

Laboratory tests from uniaxial tests on an asphaltic concrete mix of type 0/16 with 6% bitumen 80/100 will be utilised.

#### 4.3.1 Hardening Response

**Parameters $\beta$, $\gamma$ and $R$**

At ultimate, the hardening parameter $\alpha = 0$. Hence Eq. 4.15 can be written as:

$$J_2 - \gamma \left(1 - \beta \cos \theta \right)^{-1/2} (I_1 + R)^2 = 0$$  \hspace{1cm} 4.23

If uniaxial test results are available only, the influence of Lode angle $\theta$ on the response can not be included. In terms of the Desai surface, this is equivalent to setting $\beta = 0$.

For determination of the values of the parameters $\beta$, $\gamma$ and $R$ on the basis of uniaxial tests, for any given combination of temperature and displacement rate values, the ultimate strength values of compressive and tensile tests were plotted on the $I_1 - \sqrt{J_2}$ space. $R$ was then determined by the intersection of the line connecting these two points with the $I_1$ axis. The
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value of $\gamma$ was determined subsequently by substituting either of the ultimate strength values into Eq. 4.15.

By means of this procedure, for the range of temperatures and deformation rates the following relation was determined:

$$
\gamma(T,\dot{\varepsilon}) = \left( \frac{1/\sqrt{3} \cdot f_{cu}(T,\dot{\varepsilon})}{f_{cu}(T,\dot{\varepsilon}) + 3 \cdot f_{tu}(T,\dot{\varepsilon})} \right)^2
$$

in which $f_{cu}(T,\dot{\varepsilon})$ and $f_{tu}(T,\dot{\varepsilon})$ are the values of the laboratory measured monotonic uniaxial compressive and tensile material strengths.

In CAPA-3D, on the $I_1 - \sqrt{J_2}$ plane, $R = R(T,\dot{\varepsilon})$ is computed as the intersection of the line joining the points $f_{cu}(T,\dot{\varepsilon})$ and $f_{tu}(T,\dot{\varepsilon})$ with the $I_1$ axis.

In case that triaxial test results at constant deformation rate and temperature are available, the values of $I_1$, $J_2$, and $\cos3\theta$ at various states of stress, can be computed. Rearranging the Desai surface equation, a nonlinear equation can be obtained with $\beta$ and $\gamma^2$ as the unknowns:

$$
\begin{bmatrix}
J_2^2 \cos3\theta & (I_1 + R)^4 \\
\gamma^2
\end{bmatrix}
\begin{bmatrix}
\beta \\
\gamma^2
\end{bmatrix} = J_2^2
$$

An overdetermined system of equations can then be set up for the available sets of $I_1$, $J_2$, and $\cos3\theta$ values:

$$
\begin{bmatrix}
A
\end{bmatrix}
\begin{bmatrix}
\beta \\
\gamma^2
\end{bmatrix} = [B]
$$

which can be solved via a least squares technique.

**Dilation parameter $n$**

This constant is associated with the state of stress beyond which the specimen begins to dilate. Its value can be determined from

$$
n = \frac{2}{1 - \frac{J_2}{(I_1 + R)^2} \gamma (1 - \beta \cos3\theta)^{-1/2}}
$$

in which all invariants are computed for the state of stress at initiation of dilation.
**Parameter $\alpha$**

For a given deformation rate $\dot{\delta}$ and temperature $T$, parameter $\alpha$ determines the size of the successive dynamic flow surfaces and can therefore be utilized as a hardening measure. On a physical basis it can be associated with some measure of plastic deformation, plastic work $W_p$, etc.

In Scarpas et al. (1997) the postulate was made:

$$\alpha = \alpha(T, \dot{\delta}, W_p) \quad 4.28$$

For any test, at a given stress level, Eq. 4.15 can be solved in terms of $\alpha$. At the same stress level, $W_p$ can be computed from

$$W_p = \int \sigma : d\varepsilon_p \quad 4.29$$

![Graph](image)

**Fig. 4.11** $\alpha$ variation for all temperatures and strain rates. Eq. 4.31 indicated by solid line

Fig. 4.11 summarizes the experimental results over all tested temperature and displacement rate ranges. The following relationship was found to represent adequately the dependence of $\alpha$ on $W_p$ over the range of test data

$$\alpha = \frac{\alpha_0 - W_{p,\text{lim}} \cdot W_p}{1 + c \cdot W_p} \quad 4.30$$

in which $W_{p,\text{lim}}$ is the experimentally determined plastic work at maximum response and $c$ a degradation constant. Non-linear curve fitting of Eq. 4.30 over the available data sets resulted to

$$\alpha = \frac{0.1 - 0.67 \cdot W_p}{1 + 250 \cdot W_p} \quad 4.31$$

It is compared in Fig. 4.11 with the experimental data.

Currently models based on the work of Erkens et al. (2002), Erkens (2002) and Dunhil (2002) for asphaltic materials and Liu (2003) for sand are also implemented. In these the postulate was made:
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\[ \alpha = \alpha \left( T, \delta, \xi \right) \quad 4.32 \]

in which

\[ \xi = \int \left( \frac{d\varepsilon^p}{d\varepsilon^p} \right)^{1/2} \quad 4.33 \]

Details can be found in the respective references.

Identification of the initial value \( \alpha_0 \) of \( \alpha \) at which initiation of plastic response occurs can be done on the basis of the ratio \( \left( \varepsilon_{\text{accr}} / \varepsilon_{\text{vol}} \right) \) in Fig. 4.8. \( \alpha_0 \) is computed at that state of stress corresponding to the point of divergence of this ratio from linearity.

4.3.2 Response Degradation

Two different sets of model parameters control response degradation. The first is associated with the shrinking size of the Desai flow surface. The second refers to the Hoffman surface.

Parameter \( \gamma \)

During response degradation, for a given deformation rate \( \delta \) and temperature \( T \) parameter \( \gamma \) determines the size of the successive Desai dynamic flow surfaces.

In Scarpas et al. (1997) the degradation of \( \gamma \) was associated with \( W_{\text{pf}} \), the amount of work expended during the post fracture plastic deformation of the material. Some typical curves are shown in Fig. 4.12.

\[ T = 25^\circ \text{C} \]

![Graph](image_url)

Fig. 4.12 Variation of \( \gamma \) with \( W_{\text{pf}} \)

On the basis of the available experimental evidence the following relationship was derived:

\[ \gamma = \left( \gamma_{\text{max}}^2 - k \cdot W_{\text{pf}}^{1.5} \right)^{1/2} \quad 4.34 \]
in which $\gamma_{\text{max}}$ represents the value of $\gamma$ the moment of response degradation initiation and $k$ is a degradation parameter which depends on both, deformation rate and temperature. For $T=25^\circ \text{C}$ non linear fitting of the experimental data resulted to

$$k(T=25^\circ \text{C}) = 0.24 + \frac{0.32}{\delta^2}$$

4.35

Models based on the work of Erkens et al. (2002) and Erkens (2002) for asphaltic materials and Liu (2003) for sand are also implemented. Details can be found in the respective references.

**Cracking**

By assuming the tensile stress direction as the reference direction, coefficient $q$ of the Hoffman surface can be computed as:

$$q = \frac{f_{tu}^2}{\tau_u}$$

4.36

in which the uniaxial tensile strength $f_{tu}$ and the shear strength $\tau_u$ are computed by means of the Desai surface at the moment of crack plane introduction.

Isotropic softening of the Hoffman surface is postulated in the form:

$$f_c = f_{tu} \exp \left( -\frac{c_w}{w_{\text{max}}} \right)$$

4.37

in which $c$ is a degradation constant, $w_{\text{max}}$ is the crack width at stress annihilation and $w$ is the experimentally measured crack opening on the crack plane, Fig. 4.13.

![Fig. 4.13 Uniaxial response of specimen in tension](image-url)
Elastoplasticity

It is important that out of the total measured change in specimen length $\Delta l$, only that part corresponding to the true crack opening $w$ be utilised in Eq. 4.37. As shown in Fig. 4.13, the remaining component $\Delta l_u$ corresponds to the unloading deformation of those parts of the specimen outside the fracture zone. Erkens et al. (2000) have proposed special testing and instrumentation techniques for measurement of the crack opening after crack initiation. Details can be found in Erkens (2002).

In addition to Eq. 4.37, the following expression proposed by Erkens (2002) for asphaltic materials is also implemented in CAPA-3D

$$f_t = f_{tu} \left( \frac{1}{1 + \left( \frac{w}{aw_{max} + b} \right)^c} \right)$$

with

$$w_{max} = \frac{g}{1 + g \delta \exp \left( -f + \frac{h}{T} \right)}$$

in which $a, b, c, f, g$ and $h$ are laboratory determined material parameters and $T$ is the temperature in Kelvin.

4.4 Algorithmic Aspects

Implementation aspects of the specific stress reduction algorithms as they relate to the Desai and Hoffman response surfaces are presented in the following.

4.4.1 Hardening Response

The model considers a yield condition of the form:

$$f_D = \frac{J_2}{P_a} - \left[ -\alpha \left( \frac{I_1 + R}{P_a} \right)^n + \gamma \left( \frac{I_1 + R}{P_a} \right)^2 \right] = 0$$

in which formalistic expressions for $\alpha, \gamma$ and $R$ are determined experimentally as indicated in the previous Section.

Integration of Eq. 4.12 with a backward Euler procedure yields:

$$t + \Delta t \Delta \epsilon^p = \dot{\lambda} \Delta t \frac{\partial f}{\partial \sigma}$$

For the Desai surface:

$$\frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial I_1} I + \frac{\partial f}{\partial \sqrt{I_2}} \frac{1}{2\sqrt{I_2}} \dot{s}$$

84
in which $I = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ \\ \\ \\ \end{pmatrix}$ and $\mathbf{s} = (s_x, s_y, s_z, 2\tau_{xy}, 2\tau_{yz}, 2\tau_{xz})$. Substituting in Eq. 4.42 and defining:

$$
\Delta \varepsilon^p_{v} = \dot{\lambda} \Delta t \left( \frac{\partial f}{\partial \lambda_1} \right)^{t+\Delta t}
$$

$$
\Delta \varepsilon^p_{d} = \dot{\lambda} \Delta t \left( \frac{\partial f}{\partial \sqrt{J_2}} \right)^{t+\Delta t}
$$

the increment of plastic strain $t+\Delta t \Delta \varepsilon^p$ can be expressed in terms of volumetric and deviatoric components as:

$$
t+\Delta t \Delta \varepsilon^p = \Delta \varepsilon^p_{v} I + \frac{1}{2} \Delta \varepsilon^p_{d} \frac{\mathbf{s}}{\sqrt{J_2}}
$$

### Stress update procedure

The implemented stress update procedure is shown in Fig. 4.14. A trial state of stress is defined as:

$$
\sigma^{\text{trial}} = t \sigma + D^e \Delta \varepsilon
$$

![Diagram of stress update procedure](image)

**Fig. 4.14** Stress update procedure

The updated stress state at $t + \Delta t$ is computed by means of a stress correction:

$$
t+\Delta t \sigma = \sigma^{\text{trial}} - D^e \Delta \varepsilon^p
$$

Utilizing Eq. 4.42-4.45 it is shown in Appendix 4.3 that the term $D^e t+\Delta t \Delta \varepsilon^p$ can be expressed in terms of the hydrostatic and deviatoric plastic strain components and the elastic bulk $K$ and shear $G$ moduli. Then, the updated stress state can be written as:
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\[ t^+ \Delta t \sigma^t = \sigma^{\text{trial}} - 3K \Delta \varepsilon^P_v I - G \Delta \varepsilon^P_d \left( \frac{\dot{s}}{\sqrt{J_2}} \right)^{t^+ \Delta t} \]

which indicates the process of stress correction along the hydrostatic and the deviatoric axes. A schematic of this process is shown in Fig. 4.15.

![Schematic of stress correction](image)

**Fig. 4.15 Stress update along the hydrostatic and the deviatoric axes**

By substituting the vector \( t^+ \Delta t \left( \frac{\dot{s}}{\sqrt{J_2}} \right) \) which defines the return direction on the deviatoric plane, with the known vector \( \left( \frac{\dot{s}}{\sqrt{J_2}} \right)^{\text{trial}} \) evaluated at the trial stress state, Aravas (1987), proposed a formulation which reduces the order of the nonlinear system of equations that have to be solved to two only. It is this formulation that has been implemented in CAPA-3D.

Rewriting Eq. 4.46 on the basis of the Aravas postulate, the increment of viscoplastic strain can be expressed by means of two unknowns only, namely \( \Delta \varepsilon^P_v \) and \( \Delta \varepsilon^P_d \):

\[ t^+ \Delta t \Delta \varepsilon^P = \Delta \varepsilon^P_v I + \frac{1}{2} \Delta \varepsilon^P_d \left( \frac{\dot{s}}{\sqrt{J_2}} \right)^{\text{trial}} \]

\( \Delta \varepsilon^P_v \) and \( \Delta \varepsilon^P_d \) can be computed by means of a Newton-Raphson iterative procedure set up at local material level.

**Newton-Raphson methodology**

Eliminating the term \( \dot{\lambda} \Delta t \) between Eq. 4.44 and Eq. 4.45:
$$t+\Delta t\left(\frac{\partial f}{\partial \sqrt{I_2}}\right)\Delta \varepsilon^P_v - t+\Delta t\left(\frac{\partial f}{\partial I_1}\right)\Delta \varepsilon^P_d = 0$$ 4.51

This relationship together with the consistency condition:

$$t+\Delta t f = f(I_1, \sqrt{J_2}, \Delta \varepsilon^P_v, \Delta \varepsilon^P_d, \dot{\varepsilon}, \ldots) = 0$$ 4.52

constitute a system of two nonlinear algebraic equations in the primary variables $\Delta \varepsilon^P_v$ and $\Delta \varepsilon^P_d$. The system can be solved by means of a Newton-Raphson iteration process. Details are included in Appendix 4.4.

Once $\Delta \varepsilon^P_v$ and $\Delta \varepsilon^P_d$ are known, the updated state of stress can be computed from Eq. 4.49.

### 4.5 Response Degradation

As mentioned in Section 4.2.3, both flow surfaces are assumed active during the response degradation phase of material response.

#### 4.5.1 Desai Surface

The algorithm of Section 4.4.1 is applicable without any significant modifications for controlling the softening response of the Desai surface.

#### 4.5.2 Hoffman Surface

On the basis of the implicit stress reduction procedure of Fig. 4.14:

$$\sigma^{trial} = t \sigma + E(t+\Delta t \Delta \varepsilon_n) \rightarrow t+\Delta t \Delta \varepsilon_n = \frac{1}{E}(\sigma^{trial} - t \sigma)$$ 4.53

Also it holds

$$t+\Delta t \sigma = t \sigma + t+\Delta t \Delta \sigma$$ 4.54

$$= t \sigma + E(t+\Delta t \Delta \varepsilon_n - t+\Delta t \Delta \varepsilon^P_n)$$

from which

$$\frac{1}{E}(t+\Delta t \sigma - t \sigma) = t+\Delta t \Delta \varepsilon_n - t+\Delta t \Delta \varepsilon^P_n$$ 4.55

Substituting Eq. 4.53 into Eq. 4.55 after rearrangement it results:

$$t+\Delta t \sigma + E t+\Delta t \Delta \varepsilon^P_n = \sigma^{trial}$$ 4.56

Similarly for the shear stresses on the crack plane:

$$t+\Delta t \tau_s + G t+\Delta t \Delta \varepsilon^P_s = \tau^{trial}_s$$ 4.57

$$t+\Delta t \tau_t + G t+\Delta t \Delta \varepsilon^P_t = \tau^{trial}_t$$ 4.58
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The values of plastic strains on the crack plane can be evaluated as

\[ t + \Delta t \Delta \varepsilon^P_n = \Delta \lambda \left. \frac{\partial \tau_n}{\partial \sigma} \right|_{t + \Delta t} = 2 \Delta \lambda ^{t + \Delta t} \sigma \]

\[ t + \Delta t \Delta \varepsilon^P_s = \Delta \lambda \left. \frac{\partial \tau_s}{\partial \tau_n} \right|_{t + \Delta t} = 2 \Delta \lambda ^{t + \Delta t} \tau_s \]

\[ t + \Delta t \Delta \varepsilon^P_t = \Delta \lambda \left. \frac{\partial \tau_t}{\partial \tau_n} \right|_{t + \Delta t} = 2 \Delta \lambda ^{t + \Delta t} \tau_t \]

Substituting these into Eq. 4.56 – Eq. 4.58 the stresses on the crack plane at time \((t + \Delta t)\) can be expressed in terms of \(\Delta \lambda\)

\[ t + \Delta t \sigma = \left( 1 + 2E \Delta \lambda \right)^{-1} \sigma_{\text{trial}} \]

\[ t + \Delta t \tau_s = \left( 1 + 2G \Delta \lambda \right)^{-1} \tau_{s,\text{trial}} \]

\[ t + \Delta t \tau_t = \left( 1 + 2G \Delta \lambda \right)^{-1} \tau_{t,\text{trial}} \]

The softening function can also be expressed in terms of \(\Delta \lambda\) as follows:

\[ f_{lt + \Delta t} = f_{tu} e^{-(t + \Delta t) \Delta \varepsilon^P_n} = f_{tu} e^{-a \left( t \varepsilon^P_n + \Delta \lambda ^{t + \Delta t} \sigma \right)} \]

\[ = f_{tu} e^{-a \left( t \varepsilon^P_n + 2 \Delta \lambda ^{t + \Delta t} \sigma \right)} \]

Substituting the stresses \((\sigma, \tau_s, \tau_t)\) and the softening function in terms of \(\Delta \lambda\) in Eq. 4.22 a non-linear equation in \(\Delta \lambda\) is obtained. This can be solved by means of a Newton-Raphson iterative procedure, Appendix 4.5.

### 4.6 Large Strains Formulation

From a mathematical point of view, the usual mathematical operations like addition, multiplication etc can only be rigorously applied to tensorial quantities associated with a common configuration. In the context of small strains analysis, the postulate of a common spatial configuration is plausible and, as such, tensor operations like those indicated by Eq. 4.47 and similar are permissible. Nevertheless, the plasticity reduction algorithms presented in this Chapter can also be utilised for large strains analyses by means of a technique proposed by Pinsky et al. [1983].

Recalling that the Piola pull back of the Cauchy stress tensor \(\sigma\) is the second Piola-Kirchhoff stress tensor \(S\), it is legitimate to perform operations between second Piola-Kirchhoff stress tensors corresponding to different configurations in time since they are all defined on the reference configuration.

On the basis of the terminology developed in Chapter 2, it holds

\[ t + \Delta t S - t S = \Delta t ^{t + \Delta t} \dot{S} \]
in which \( \dot{S} \) is to be evaluated at some intermediate configuration with \( 0 \leq \alpha \leq 1 \). Expressing \( S \) via the Piola pull back of the Cauchy stress tensor \( \sigma \) it holds

\[
{t + \Delta t} \left[ J \phi_{\ast}^{-1} [\sigma] \right] - {t} \left[ J \phi_{\ast}^{-1} [\sigma] \right] = \Delta t \left[ \phi_{\ast}^{-1} \left[ J \dot{\sigma} \right] \right]
\]

with \( \phi_{\ast}^{-1} [\sigma] = F^{-1} \sigma (T)^{-T} \).

If the reference configuration is redefined as the configuration at \( t + \Delta t \) then

\[
t + \Delta t F = I ; \quad ^t F = \frac{\partial \left( ^t x \right)}{\partial \left( t + \Delta t x \right)} ; \quad t + \alpha \Delta t F = \frac{\partial \left( t + \alpha \Delta t x \right)}{\partial \left( t + \Delta t x \right)}
\]

and \( t + \Delta t J = 1 \) so that from Eq. 4.63

\[
t + \Delta t \sigma - t J \left[ F^{-1} \right] \left[ \sigma \right] F^{-T} = \Delta t \left( t + \alpha \Delta t \right) J \left[ t + \alpha \Delta t F^{-1} \right] \left( t + \alpha \Delta t \right) F^{-T}
\]

In terms of the pull-back terminology of Chapter 2, the term \( t J \left[ F^{-1} \right] \left[ \sigma \right] F^{-T} \) maps the stress \( \gamma \) to the configuration at \( t + \Delta t \). The same is valid for the Truesdell stress rate.

The relative deformation gradient implied by Eq. 4.64 can be easily evaluated on the basis of the deformation gradients at \( t \) and \( \Delta t \) as

\[
t F^{-1} = \frac{\partial \left( t + \Delta t x \right)}{\partial ^t x} = \frac{\partial \left( t + \Delta t X \right)}{\partial X} \frac{\partial ^t x}{\partial ^t x} = t + \Delta t F_0 \left( t F_0^{-1} \right)
\]

Similarly

\[
t + \alpha \Delta t F^{-1} = t + \Delta t F_0 \left( t + \alpha \Delta t F_0^{-1} \right)
\]

In Section 2.7 it was shown that the Truesdell stress rate is related to the rate of deformation tensor via

\[
t + \alpha \Delta t \dot{\sigma} = t + \alpha \Delta t \dot{c}: \gamma
\]

According to Pinsky et al. [1983] \( t + \alpha \Delta t \gamma \) can be computed as

\[
t + \alpha \Delta t \gamma = \frac{1}{\Delta t} \text{sym} \left[ \left( \left( I - \alpha \right) I + \alpha \left( t F^{-1} \right)^{-1} \left( t F^{-1} - I \right) \right) \right]
\]

and \( t + \alpha \Delta t \gamma \) as in Section 2.7.

* The subscript 0 is utilised to emphasize that evaluation of the deformation gradient is wrt. the reference configuration.
4.7 Utilization

A constitutive model for strain rate sensitive materials has been presented. A methodology has been described which enables the determination of all relevant parameters on the basis of simple laboratory tests.

In its present form the model has been utilized for the study of the dynamic response of pavements subjected to impulse loads Scarpas et al. (1997), Scarpas & Blaauwendraad (1998a), the response of pavements subjected to repetitive traffic loads Erkens et al. (2000), the response of various types of laboratory specimens Erkens et al. (2002a), Erkens et al. (2002b), Dunhil (2002), Molenaar & Liu (2003), Liu et al. (2003b), the response of surfacings of steel bridges Huurman et al. (2002), and the simulation of sand layers in integral pavement constructions Liu et al. (2003a), Zhao et al. (2004).

A non-associative formulation of the model and a nonlinear formulation of the ultimate envelope of the Desai surface can be found in Liu (2004) and Liu et al. (2004a).

Utilization of the model for the simulation of localization phenomena in water saturated sand specimens is shown in Liu (2004), Liu & Scarpas (2004) and Liu et al. (2004b).
Appendix 4.1

Notions of Constrained Minimization

Let $g(x)$ be a scalar function of the $n$-dimensional vector $x$ and $f(x)$ a set of $m$ constraint equations in $x$. In optimization theory it can be shown, Luenberger [1984], that the following problem

$$\begin{align*}
\text{minimize} & \quad g(x) \\
\text{subject to} & \quad f(x) \leq 0
\end{align*} \quad \text{A.4.1.1}$$

can be recast to the solution of

$$\begin{align*}
\nabla g(x) + \nabla f(x) \cdot \lambda &= 0 \\
f(x) \cdot \lambda &= 0 \\
\lambda &\geq 0
\end{align*} \quad \text{A.4.1.2}$$

in which the $m$-dimensional real vector $\lambda$ is termed the vector of Lagrange multipliers.

It is worth noticing that since $\lambda \geq 0$ and $f(x) \leq 0$, Eq. A.4.1.2 is equivalent to the statement that a component of $\lambda$ can be nonzero only if the corresponding constraint is zero and vice versa.

This observation enables the development of a solution procedure for the system of Eq. A.4.1.2 according to which, for any given combination of active constraints, the sign of the corresponding Lagrange multipliers is checked after solving the reduced system of equations. Only those combinations of constraints fulfilling Eq. A.4.1.2 are retained.

In optimization theory, Eq. A.4.1.2 and Eq. A.4.1.2 are known as the Kuhn-Tucker conditions.
Elastoplasticity

Appendix 4.2

The methodology for the derivation of an expression for coefficient $n$ of the Desai surface on the basis of the state of stress is presented in the following.

The volumetric plastic strain is

$$ d\varepsilon^p_{kk} = \sum_{i=1}^{3} d\varepsilon^p_{ii} = d\lambda \sum_{i=1}^{3} \frac{\partial f}{\partial \sigma_{ii}} $$ \hspace{1cm} A.4.2.1

with

$$ \frac{\partial f}{\partial \sigma_{ii}} = \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_{ii}} + \frac{\partial f}{\partial J_2} \frac{\partial J_2}{\partial \sigma_{ii}} + \frac{\partial f}{\partial J_3} \frac{\partial J_3}{\partial \sigma_{ii}} $$ \hspace{1cm} A.4.2.2

It holds

$$ \frac{\partial I_1}{\partial \sigma_{ii}} = 1 \quad ; \quad \frac{\partial J_2}{\partial \sigma_{ii}} = s_{ii} \quad ; \quad \frac{\partial J_3}{\partial \sigma_{ii}} = s_{ik}s_{ki} - \frac{2}{3}J_2 = s_{ik}s_{ki} - \frac{1}{3}s_{mn}s_{nm} $$ \hspace{1cm} A.4.2.3

Substituting successively into Eq. A.4.2.2 and into Eq. A.4.2.1

$$ d\varepsilon^p_{kk} = d\lambda \sum_{i=1}^{3} \frac{\partial f}{\partial \sigma_{ii}} $$

$$ = d\lambda \left[ 3 \frac{\partial f}{\partial I_1} + \frac{\partial f}{\partial J_2} \sum_{i=1}^{3} s_{ii} + \frac{\partial f}{\partial J_3} \sum_{i=1}^{3} \left( s_{ik}s_{ki} - \frac{1}{3}s_{mn}s_{nm} \right) \right] $$ \hspace{1cm} A.4.2.4

$$ = 3d\lambda \frac{\partial f}{\partial I_1} $$

Fig. A.4.2.1 Parameter $n$ controls initiation of volumetric dilation
At the point where the specimen inelastic response changes from contractant to dilatant (point A in Fig. A.4.2.1(a)) \( d\varepsilon_{kk}^P = 0 \) or, equivalently, from Eq. A.4.2.4, \( \frac{\partial f}{\partial I_1} = 0 \). In the \( I_1 - \sqrt{J_2} \) space of Fig. A.4.2.1(b) this condition corresponds to the apex point of the trace of the surface.

From the definition of the Desai surface, it holds

\[
\frac{\partial f}{\partial I_1} = \alpha n \left( \frac{I_1 + R}{P_a} \right)^{n-1} - 2\gamma \left( \frac{I_1 + R}{P_a} \right) \left( 1 - \beta \cos 3\theta \right)^{-1/2} = 0 \quad \text{A.4.2.5}
\]

Considering that the term \( 1 - \beta \cos 3\theta \) can not be zero

\[
\frac{2\gamma}{\alpha n} = \left( \frac{I_1 + R}{P_a} \right)^{n-2} \quad \text{A.4.2.6}
\]

Substituting Eq. A.4.2.6 into Eq. 4.15 an expression for \( n \) can be obtained on the basis of the stress invariants

\[
n = \frac{2}{1 - \frac{J_2}{I_1^2 + K^2} \gamma \left( 1 - \beta \cos 3\theta \right)^{-1/2}} \quad \text{A.4.2.7}
\]
Elastoplasticity

Appendix 4.3

The methodology for the derivation of expressions relating the first stress invariant $I_1$ to the equivalent volumetric plastic strain increment $\Delta \varepsilon^p_v$ and the deviatoric stress $s$ to the equivalent deviatoric plastic strain increment $\Delta \varepsilon^p_d$ is presented in the following.

The stress-strain relation for an elasto-plastic material is:

$$d\varepsilon_{ij} = d\varepsilon^p_{ij} + d\varepsilon^e_{ij} = \left( \frac{dI_1}{9K} \delta_{ij} + \frac{ds_{ij}}{2G} \right) + d\lambda \frac{\partial f}{\partial \sigma_{ij}}$$  \hspace{1cm} \text{A.4.3.1}

Also, the stress increment vector can be expressed as:

$$d\sigma_{ij} = ds_{ij} + \frac{1}{3} dI_1 \delta_{ij}$$  \hspace{1cm} \text{A.4.3.2}

Solving Eq. A.4.3.1 in terms of $ds_{ij}$ and substituting in Eq. A.4.3.2:

$$d\sigma_{ij} = 2G d\varepsilon_{ij} - 2G d\lambda \frac{\partial f}{\partial \sigma_{ij}} + \frac{1}{3} \left( \frac{2G}{9K} \right) dI_1 \delta_{ij}$$  \hspace{1cm} \text{A.4.3.3}

This can be further simplified as follows. From Eq. A.4.3.1, for $i = j$:

$$dI_1 = 3K \left\{ d\varepsilon_{kk} - d\lambda \frac{\partial f}{\partial \sigma_{mn}} \delta_{mn} \right\}$$  \hspace{1cm} \text{A.4.3.4}

Substituting into Eq. A.4.3.3 and employing the identity:

$$d\varepsilon_{ij} = d\varepsilon_{ij} + \frac{1}{3} \varepsilon_{kk} \delta_{ij}$$  \hspace{1cm} \text{A.4.3.5}

after some algebraic manipulation it results:

$$d\sigma_{ij} = 2G d\varepsilon_{ij} + K d\varepsilon_{kk} \delta_{ij} - d\lambda \left\{ \left( K - \frac{2}{3} G \right) \frac{\partial f}{\partial \sigma_{mn}} \delta_{mn} \delta_{ij} + 2G \frac{\partial f}{\partial \sigma_{ij}} \right\}$$  \hspace{1cm} \text{A.4.3.6}

For the chosen form of the hardening surface $f\left( \sigma_{ij} \right) = f \left( I_1, \sqrt{J_2} \right)$, hence:

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_{ij}} + \frac{\partial f}{\partial \sqrt{J_2}} \frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}}$$  \hspace{1cm} \text{A.4.3.7}

$$= \frac{\partial f}{\partial I_1} \delta_{mn} + \frac{1}{2\sqrt{J_2}} \frac{\partial f}{\partial \sqrt{J_2}} \delta_{mn}$$

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so that in Eq. A.4.3.6 the term:

\[ \left( K - \frac{2}{3} G \right) \frac{\partial f}{\partial \sigma_{mn}} \delta_{mn} \delta_{ij} \]

A.4.3.8

can be simplified as:

\[ \left( K - \frac{2}{3} G \right) \left[ \frac{\partial f}{\partial I_1} \delta_{mn} + \frac{1}{2\sqrt{J_2}} \frac{\partial f}{\partial \sqrt{J_2}} \delta_{mn} \right] \delta_{mn} \delta_{ij} = \]

A.4.3.9

\[ 3\left( K - \frac{2}{3} G \right) \frac{\partial f}{\partial I_1} \delta_{ij} \]

Substituting the last of Eq. A.4.3.9 and Eq. A.4.3.7 into Eq. A.4.3.6, the differential increment of stress can be computed in term of an elastic trial stress increment and a stress correction:

\[ d\sigma_{ij} = 2Gd\varepsilon_{ij} + Kd\varepsilon_{kk} \delta_{ij} \]

A.4.3.10

\[ - d\lambda \left[ 3K \frac{\partial f}{\partial I_1} \delta_{ij} + \frac{G}{\sqrt{J_2}} \frac{\partial f}{\partial \sqrt{J_2}} \delta_{ij} \right] \]

Fig. 4.14 portrays the reduction procedure postulated by Eq. A.4.3.10.

In terms of finite stress increments:

\[ \Delta\sigma_{ij}^p = d\lambda \Delta t \left[ 3K \frac{\partial f}{\partial I_1} \delta_{ij} + \frac{G}{\sqrt{J_2}} \frac{\partial f}{\partial \sqrt{J_2}} \delta_{ij} \right] \]

A.4.3.11

As shown in Fig. 4.14, it is also valid that:

\[ \sigma_{ij}^{t+\Delta t} = \sigma_{ij}^{trial} - \Delta\sigma_{ij}^p \]

A.4.3.12

Define:

\[ \Delta\varepsilon_{ij}^p = \Delta\lambda \Delta t \frac{\partial f}{\partial I_1} \]

A.4.3.13

\[ \Delta\varepsilon_{d}^p = \Delta\lambda \Delta t \frac{\partial f}{\partial \sqrt{J_2}} \]

A.4.3.14

so that from Eq. A.4.3.11 and Eq. A.4.3.12:

\[ \sigma_{ij}^{t+\Delta t} = \sigma_{ij}^{trial} - 3K\Delta\varepsilon_{ij}^p \delta_{ij} - \frac{G}{\sqrt{J_2}} \Delta\varepsilon_{d}^p \]

A.4.3.15
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Substituting $t+\Delta t \sigma_{ij}$ and $\sigma_{ij}^{\text{trial}}$ in the above with their corresponding expressions in terms of $I_1$ and $s_{ij}$:

$$\frac{1}{3}(t+\Delta t I_1)\delta_{ij} + t+\Delta t s_{ij} = \frac{1}{3}(I_1^{\text{trial}}) + s_{ij}^{\text{trial}} - 3 K \Delta \varepsilon_v^P \delta_{ij} - \frac{G}{\sqrt{J_2}} t+\Delta t \dot{s}_{ij} \Delta \varepsilon_d^P$$  \hspace{1cm} \text{A.4.3.16}

from which it can be deduced:

$$t+\Delta t I_1 = I_1^{\text{trial}} - 9 K \Delta \varepsilon_v^P$$  \hspace{1cm} \text{A.4.3.17}

and

$$t+\Delta t s_{ij} = s_{ij}^{\text{trial}} - \left(\frac{\dot{s}_{ij}}{\sqrt{J_2}}\right) G \Delta \varepsilon_d^P$$  \hspace{1cm} \text{A.4.3.18}

As mentioned in Section 0, this expression can be recast for stress reduction purposes as:

$$t+\Delta t s_{ij} = s_{ij}^{\text{trial}} - \left(\frac{\dot{s}_{ij}}{\sqrt{J_2}}\right)^{\text{trial}} G \Delta \varepsilon_d^P$$  \hspace{1cm} \text{A.4.3.19}
Appendix 4.4

Local Newton-Raphson Iterative Process

Non-linear system set up

The magnitudes of the volumetric $\Delta \varepsilon_y^P$ and the deviatoric $\Delta \varepsilon_d^P$ equivalent plastic strain increments can be computed on the basis of Newton-Raphson iterative process. In Section 4.4.1 the following system of nonlinear equations was set up:

\[ g_1 = \left( \frac{\partial f}{\partial J_2} \right) \Delta \varepsilon_y^P - \left( \frac{\partial f}{\partial I_1} \right) \Delta \varepsilon_d^P = 0 \]  

\[ A.4.4.1 \]

\[ g_2 = f_D \left( I_1, \sqrt{J_2}, \Delta \varepsilon_y^P, \Delta \varepsilon_d^P, \ldots \right) = 0 \]

\[ A.4.4.2 \]

\[ \sum \frac{\partial g_1}{\partial \varepsilon_i} \frac{\partial \Delta \varepsilon_y^P}{\partial \varepsilon_i} \Delta \varepsilon_y^P + \frac{\partial g_1}{\partial \varepsilon_d} \frac{\partial \Delta \varepsilon_d^P}{\partial \varepsilon_d} \Delta \varepsilon_d^P \]

\[ r_1 = -\left( \frac{\partial f}{\partial J_1} \right) \Delta \varepsilon_y^P + \left( \frac{\partial f}{\partial I_1} \right) \Delta \varepsilon_d^P \]

\[ A.4.4.3 \]

\[ r_2 = -f \left( I_1, \sqrt{J_2}, \Delta \varepsilon_y^P, \Delta \varepsilon_d^P, \ldots \right) \]

\[ A.4.4.4 \]

\[ \begin{bmatrix} \frac{\partial g_1}{\partial \Delta \varepsilon_y^P} & \frac{\partial g_1}{\partial \Delta \varepsilon_d^P} \\ \frac{\partial g_2}{\partial \Delta \varepsilon_y^P} & \frac{\partial g_2}{\partial \Delta \varepsilon_d^P} \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta \varepsilon_y^P}{\partial \varepsilon_i} \\ \frac{\partial \Delta \varepsilon_d^P}{\partial \varepsilon_i} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \]

Analytical expressions for the individual derivatives are presented in the following.
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**Computation of $\partial g_1 / \partial \Delta \varepsilon_v^p$:**

$$
\frac{\partial g_1}{\partial \Delta \varepsilon_v^p} = \frac{\partial}{\partial \Delta \varepsilon_v^p} \left( \frac{\partial f}{\partial \sqrt{J_2}} \Delta \varepsilon_v^p - \frac{\partial f}{\partial I_1} \Delta \varepsilon_d^p \right) \quad \text{A.4.4.5}
$$

$$
= \frac{\partial}{\partial \Delta \varepsilon_v^p} \left( \frac{\partial f}{\partial \sqrt{J_2}} \Delta \varepsilon_v^p \right) - \frac{\partial}{\partial \Delta \varepsilon_v^P} \left( \frac{\partial f}{\partial I_1} \Delta \varepsilon_d^P \right)
$$

For the Desai surface

$$
\frac{\partial f}{\partial I_1} = - \left( \frac{1}{p_a} \right) \left[ n \alpha \left( \frac{I_1 + R}{p_a} \right)^{n-1} + 2 \gamma \left( \frac{I_1 + R}{p_a} \right) \right] \quad \text{A.4.4.6}
$$

and

$$
\frac{\partial f}{\partial \sqrt{J_2}} = \frac{2}{p_a} \sqrt{J_2} \quad \text{A.4.4.7}
$$

so that

$$
\frac{\partial}{\partial \Delta \varepsilon_v^p} \left( \frac{\partial f}{\partial \sqrt{J_2}} \Delta \varepsilon_v^p \right) = \frac{\partial f}{\partial \sqrt{J_2}} + \Delta \varepsilon_v^p \frac{\partial}{\partial \Delta \varepsilon_v^p} \left( \frac{\partial f}{\partial \sqrt{J_2}} \right) \quad \text{A.4.4.8}
$$

$$
= \frac{2}{p_a} \sqrt{J_2} + \frac{1}{p_a} \Delta \varepsilon_v^p \frac{\partial \sqrt{J_2}}{\partial \Delta \varepsilon_v^p}
$$

In Section 0 $^{t+\Delta t}_s$ is expressed as:

$$
^{t+\Delta t}_s = s_{\text{trial}} \left( \frac{8}{\sqrt{J_2}} \right)^{\text{trial}} G \Delta \varepsilon_d^p \quad \text{A.4.4.9}
$$

so that $\frac{\partial}{\partial \Delta \varepsilon_v^p} \left( \sqrt{J_2} \right) = 0$. Hence from Eq. A.4.4.8:

$$
\frac{\partial}{\partial \Delta \varepsilon_v^p} \left( \frac{\partial f}{\partial \sqrt{J_2}} \Delta \varepsilon_v^p \right) = \frac{2}{p_a} \sqrt{J_2} \quad \text{A.4.4.10}
$$

The second term of Eq. A.4.4.5 yields:

$$
\frac{\partial}{\partial \Delta \varepsilon_v^p} \left( \frac{\partial f}{\partial I_1} \Delta \varepsilon_d^p \right) = \Delta \varepsilon_d^p \frac{\partial}{\partial \Delta \varepsilon_v^p} \left( \frac{\partial f}{\partial I_1} \right) \quad \text{A.4.4.11}
$$

Substituting $\partial f / \partial I_1$ for the Desai surface in Eq. A.4.4.11:
Elastoplasticity

\[
\frac{\partial}{\partial \Delta \varepsilon^p_v} \left( \frac{\partial f}{\partial \varepsilon^p_v} \Delta \varepsilon^p_v \right) = -\frac{\Delta \varepsilon^p_d}{p_a} \frac{\partial}{\partial \Delta \varepsilon^p_v} \left[ -n \alpha (I_1 + R)^{n-1} + 2\gamma p_a^{n-2} (I_1 + R) \right]
\]

\text{term A} \quad \text{term B} \quad \text{A.4.4.12}

\text{Computation of term A}

\[\frac{\partial}{\partial \Delta \varepsilon^p_v} \left[ \alpha (I_1 + R)^{n-1} \right] = (I_1 + R)^{n-1} \frac{\partial \alpha}{\partial \varepsilon^p_v} + \alpha \frac{\partial}{\partial \Delta \varepsilon^p_v} \left[ (I_1 + R)^{n-1} \right] \quad \text{A.4.4.13}\]

\text{term A1} \quad \text{term A2}

\text{Computation of term A1}

\[\frac{\partial \alpha}{\partial \Delta \varepsilon^p_v} = \frac{\partial \alpha}{\partial \xi} \frac{\partial \xi}{\partial \Delta \varepsilon^p_v} \frac{\partial \Delta \varepsilon^p_v}{\partial \Delta \varepsilon^p_v} \quad \text{A.4.4.14}\]

\text{term A11} \quad \text{term A12} \quad \text{term A13}

\text{Computation of term A11}

On the basis of the expression for \(\alpha\) from Section 4.3, \(\partial \alpha / \partial \xi\) can be computed as:

\[\frac{\partial \alpha}{\partial \xi} = \alpha_1 \left( \frac{\partial}{\partial \xi} \left( \frac{\xi_{\text{lim}}}{1 + \xi_{\eta_1}} \right) - \frac{\partial}{\partial \xi} \left( \frac{\xi}{1 + \xi_{\eta_1}} \right) \right) \quad \text{A.4.4.15}\]

The partial derivatives are computed as follows:

\[\frac{\partial}{\partial \xi} \left( \frac{\xi_{\text{lim}}}{1 + \xi_{\eta_1}} \right) = \frac{\xi_{\text{lim}}}{(1 + \xi_{\eta_1})^2} (0 - \eta_1 \xi_{\eta_1} - 1) = -\eta_1 \xi_{\text{lim}} \frac{\xi_{\eta_1} - 1}{(1 + \xi_{\eta_1})^2} \quad \text{A.4.4.16}\]

\[\frac{\partial}{\partial \xi} \left( \frac{\xi}{1 + \xi_{\eta_1}} \right) = \frac{1}{(1 + \xi_{\eta_1})^2} \left( \xi_{\eta_1} (1 - \eta_1) + 1 \right) \quad \text{A.4.4.17}\]

Substituting the above expressions into Eq. A.4.4.15:

\[\frac{\partial \alpha}{\partial \xi} = -\frac{\alpha_1}{(1 + \xi_{\eta_1})^2} \left( \xi_{\eta_1}^{-1} \left( \eta_1 \xi_{\text{lim}} + \xi (1 - \eta_1) \right) + 1 \right) \quad \text{A.4.4.18}\]

\text{Computation of term A12}

The increment of equivalent plastic strain is defined as:

\[t + \Delta t \varepsilon = t \varepsilon + \left( \Delta \varepsilon^p : \Delta \varepsilon^p \right)^{\frac{1}{2}} \quad \text{A.4.4.19}\]
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\[ \frac{\partial \xi}{\partial \Delta \varepsilon^p} = \frac{\Delta \varepsilon^p}{(\Delta \varepsilon^p : \Delta \varepsilon^p)^{\frac{1}{2}}} \]

**Computation of term A13**

The increment of the plastic strain vector can be written:

\[ \Delta \varepsilon_{ij}^p = \Delta \varepsilon_{ij}^p \delta_{ij} + \frac{\Delta \varepsilon_d^p}{2\sqrt{J_2}} \hat{s}_{ij} \]

so that

\[ \frac{\partial \Delta \varepsilon_{ij}^p}{\partial \Delta \varepsilon_{ij}^p} = \delta_{ij} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}^T \]

By means of Eq. A.4.4.20 and Eq. A.4.4.22

\[ \frac{\partial \xi}{\partial \Delta \varepsilon^p} \frac{\partial \Delta \varepsilon^p}{\partial \Delta \varepsilon_{ij}^p} = \sum_{i=1}^{3} \frac{\Delta \varepsilon_{ii}^p}{(\Delta \varepsilon^p : \Delta \varepsilon^p)^{\frac{1}{2}}} \]

This concludes computation of term A1 of Eq. A.4.4.13.

**Computation of term A2**

\[ \frac{\partial}{\partial \Delta \varepsilon^p} \left[ \left( I_1 + R \right)^{n-1} \right] = (n-1) \left( I_1 + R \right)^{n-2} \frac{\partial I_1}{\partial \Delta \varepsilon^p} \]

In Appendix 3 it was shown that \( I_1 \) at time \( t + \Delta t \) can be expressed in terms of \( \Delta \varepsilon_v^p \) as

\[ t + \Delta t I_1 = I_1^{\text{trial}} - 9 K \Delta \varepsilon_v^p \]

so that

\[ \frac{\partial (t + \Delta t) I_1}{\partial \Delta \varepsilon_v^p} = -9 K \]

and from Eq. A.4.4.24:

\[ \frac{\partial}{\partial \Delta \varepsilon_v^p} \left[ (I_1 + R)^{n-1} \right] = -9K(n-1)(I_1 + R)^{n-2} \]

**Computation of term B**

\[ \frac{\partial}{\partial \Delta \varepsilon_v} \left[ 2 \gamma (I_1 + R) \right] = 2 \gamma \frac{\partial (I_1 + R)}{\partial \Delta \varepsilon_v} = -18 \gamma K p_{a}^{n-2} \]

This concludes computation of all necessary terms for \( \frac{\partial \varepsilon_1}{\partial \Delta \varepsilon_v^p} \).
Computation of $\frac{\partial g_1}{\partial \Delta \varepsilon_p^d}$:

$$
\frac{\partial g_1}{\partial \Delta \varepsilon_p^d} = \frac{\partial}{\partial \Delta \varepsilon_p^d} \left( \frac{\partial f}{\partial J_2} \Delta \varepsilon_p^d - \frac{\partial f}{\partial I_1} \Delta \varepsilon_p^d \right)
= \frac{\partial}{\partial \Delta \varepsilon_p^d} \left( \frac{\partial f}{\partial J_2} \Delta \varepsilon_p^d \right) - \frac{\partial}{\partial \Delta \varepsilon_p^d} \left( \frac{\partial f}{\partial I_1} \Delta \varepsilon_p^d \right)
$$

term A term B

A.4.4.27

Computation of term A

$$
\frac{\partial}{\partial \Delta \varepsilon_p^d} \left( \frac{\partial f}{\partial J_2} \Delta \varepsilon_p^d \right) = \Delta \varepsilon_p^d \frac{\partial}{\partial \Delta \varepsilon_p^d} \left( \frac{\partial f}{\partial J_2} \right) + \frac{\partial f}{\partial \sqrt{J_2}} 0
$$

A.4.4.28

Substituting $\frac{\partial f}{\partial \sqrt{J_2}}$ for the Desai surface:

$$
\frac{\partial}{\partial \Delta \varepsilon_p^d} \left( \frac{\partial f}{\partial J_2} \right) = 2 \frac{1}{p_a} \frac{\partial \sqrt{J_2}}{\partial \Delta \varepsilon_p^d}
$$

A.4.4.29

Differentiation of the last term of Eq. A.4.4.29 results to:

$$
\frac{\partial \sqrt{J_2}}{\partial \Delta \varepsilon_p^d} = \frac{\partial J_2}{\partial \varepsilon_p^d} \frac{\partial J_2}{\partial s} \frac{\partial s}{\partial \Delta \varepsilon_p^d}
$$

A.4.4.30

From Appendix 3:

$$
\sigma_{ij}^{t+\Delta t} = \sigma_{ij}^{trial} - \frac{\hat{s}_{ij}}{\sqrt{J_2}} \Delta \varepsilon_p^d
$$

A.4.4.31

so that

$$
\frac{\partial \sigma_{ij}^{t+\Delta t}}{\partial \Delta \varepsilon_p^d} = -G \left( \frac{\hat{s}_{ij}}{\sqrt{J_2}} \right)^{trial}
$$

A.4.4.32

Substituting into Eq. A.4.4.30:

$$
\frac{\partial \sqrt{J_2}}{\partial \Delta \varepsilon_p^d} = -G \frac{t+\Delta t}{2} \left( \frac{\hat{s}_{ij}}{\sqrt{J_2}} \right)^{trial}
$$

A.4.4.33

This concludes computation of term A of Eq. A.4.4.27.
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**Computation of term B**

\[
\frac{\partial}{\partial \Delta \varepsilon_d^p} \left( \frac{\partial f}{\partial I_1} \right) = \frac{\partial f}{\partial I_1} + \Delta \varepsilon_d^p \frac{\partial}{\partial \Delta \varepsilon_d^p} \left( \frac{\partial f}{\partial I_1} \right)
\]

A.4.4.34

For the Desai surface \( \partial f / \partial I_1 \) has been computed in Eq. A.4.4.6. Hence:

\[
\frac{\partial}{\partial \Delta \varepsilon_d^p} \left( \frac{\partial f}{\partial I_1} \right) = \left( \frac{1}{\frac{b}{p_a}} \right) \left\{ -n \frac{\partial}{\partial \Delta \varepsilon_d^p} \left[ \alpha(1 + R)^{n-1} \right] + 2\gamma \frac{\partial}{\partial \Delta \varepsilon_d^p} \left( I_1 + R \right) \right\}
\]

A.4.4.35

\[
= \left( \frac{n}{\frac{b}{p_a}} \right) \frac{\partial}{\partial \Delta \varepsilon_d^p} \left[ \alpha(1 + R)^{n-1} \right] + 0
\]

Partial differentiation yields:

\[
\frac{\partial}{\partial \Delta \varepsilon_d^p} \left[ \alpha(1 + R)^{n-1} \right] = (1 + R)^{n-1} \frac{\partial \alpha}{\partial \Delta \varepsilon_d^p} + 0
\]

A.4.4.36

\[
= (1 + R)^{n-1} \frac{\partial \alpha}{\partial \xi} \frac{\partial \varepsilon_d^p}{\partial \Delta \varepsilon_d^p}
\]

On the basis of Eq. A.4.4.21:

\[
\frac{\partial \Delta \varepsilon_d^p}{\partial \Delta \varepsilon_d^p} = \frac{1}{2\sqrt{J_2}} \hat{s}
\]

A.4.4.37

while all other terms of Eq. A.4.4.36 have been computed earlier.

**Computation of \( \partial g_2 / \partial \Delta \varepsilon_v^p \):**

Rewriting the Desai surface function as

\[
f = \left( \frac{\sqrt{J_2}}{p_a^2} \right)^2 - \left[ -\alpha \left( \frac{I_1 + R}{p_a} \right)^n + \gamma \left( \frac{I_1 + R}{p_a} \right)^2 \right]
\]

A.4.4.38

\[
= F_a - F_b
\]

partial differentiation results to
\[ \frac{\partial g_2}{\partial \Delta \varepsilon_v^p} = \frac{\partial f}{\partial \Delta \varepsilon_v^p} \]

\[ = \frac{\partial f}{\partial F_a} \frac{\partial F_a}{\partial \Delta \varepsilon_v^p} + \frac{\partial f}{\partial F_b} \frac{\partial F_b}{\partial \Delta \varepsilon_v^p} \]

\[ = \frac{\partial f}{\partial F_a} \frac{\partial F_a}{\partial I_2 I_2 \partial \Delta \varepsilon_v^p} + \frac{\partial f}{\partial F_b} \left[ \frac{\partial F_b}{\partial I_1} \frac{\partial I_1}{\partial \Delta \varepsilon_v^p} + \frac{\partial F_b}{\partial \alpha} \frac{\partial \alpha}{\partial \Delta \varepsilon_v^p} \right] \]

\[ = 0 + \frac{\partial f}{\partial F_b} \left[ \frac{\partial F_b}{\partial I_1} \frac{\partial I_1}{\partial \Delta \varepsilon_v^p} + \frac{\partial F_b}{\partial \alpha} \frac{\partial \alpha}{\partial \Delta \varepsilon_v^p} \right] \]

in which all terms have been computed earlier.

**Computation of \( \frac{\partial g_2}{\partial \Delta \varepsilon_d^p} \):**

\[ \frac{\partial g_2}{\partial \Delta \varepsilon_d^p} = \frac{\partial f}{\partial \Delta \varepsilon_d^p} \]

\[ = \frac{\partial f}{\partial F_a} \frac{\partial F_a}{\partial \Delta \varepsilon_d^p} + \frac{\partial f}{\partial F_b} \frac{\partial F_b}{\partial \Delta \varepsilon_d^p} \]

\[ = \frac{\partial f}{\partial F_a} \left( \frac{\partial F_a}{\partial I_2} \frac{\partial I_2}{\partial \Delta \varepsilon_d^p} \right) + \frac{\partial f}{\partial F_b} \left( \frac{\partial F_b}{\partial I_1} \frac{\partial I_1}{\partial \Delta \varepsilon_d^p} + \frac{\partial F_b}{\partial \alpha} \frac{\partial \alpha}{\partial \Delta \varepsilon_d^p} \right) \]

\[ = \frac{\partial f}{\partial F_a} \left( \frac{\partial F_a}{\partial I_2} \frac{\partial I_2}{\partial \Delta \varepsilon_d^p} \right) + \frac{\partial f}{\partial F_b} \left( \frac{\partial F_b}{\partial \alpha} \frac{\partial \alpha}{\partial \Delta \varepsilon_d^p} \frac{\partial \varepsilon_d}{\partial \Delta \varepsilon_d^p} \right) \]

in which all terms have been computed earlier.


Appendix 4.5

Response Degradation Stress Update Procedures

$\gamma$ softening

The magnitudes of the volumetric $\Delta \varepsilon^p_v$ and the deviatoric $\Delta \varepsilon^p_d$ equivalent plastic strain increments can be computed on the basis of Newton-Raphson iterative process. In Appendix 4.4 the following system of nonlinear equations was set up:

\[
\begin{bmatrix}
\frac{\partial g_1}{\partial \Delta \varepsilon_{pf}^v} & \frac{\partial g_1}{\partial \Delta \varepsilon_{pf}^d} \\
\frac{\partial g_2}{\partial \Delta \varepsilon_{pf}^v} & \frac{\partial g_2}{\partial \Delta \varepsilon_{pf}^d}
\end{bmatrix}
\begin{bmatrix}
\Delta \varepsilon_{pf}^v \\
\Delta \varepsilon_{pf}^d
\end{bmatrix} = \begin{bmatrix}
r_1 \\
r_2
\end{bmatrix}
\]

A.4.5.1

The same formulation is utilised for stress updates during the $\gamma$ softening phase of material response. Analytical expressions for the individual derivatives are presented in the following.

Computation of $\frac{\partial g_1}{\partial \Delta \varepsilon_{pf}^v}$:

\[
\frac{\partial g_1}{\partial \Delta \varepsilon_{pf}^v} = \frac{\partial}{\partial \Delta \varepsilon_{pf}^v} \left( \frac{\partial f}{\partial J_2} \Delta \varepsilon_{pf}^v - \frac{\partial f}{\partial I_1} \Delta \varepsilon_{pf}^d \right)
\]

A.4.5.2

\[
= \frac{\partial}{\partial \Delta \varepsilon_{pf}^v} \left( \frac{\partial f}{\partial J_2} \Delta \varepsilon_{pf}^v \right) - \frac{\partial}{\partial \Delta \varepsilon_{pf}^v} \left( \frac{\partial f}{\partial I_1} \Delta \varepsilon_{pf}^d \right)
\]

For the Desai surface, during degradation:

\[
\frac{\partial f}{\partial I_1} = -\left( \frac{2}{p_a} \right) \left[ \gamma (I_1 + R) \right]
\]

A.4.5.3

and

\[
\frac{\partial f}{\partial J_2} = \frac{2}{p_a} \sqrt{J_2}
\]

A.4.5.4

Differentiation of Eq. A.4.5.2 by parts yields:
\[
\frac{\partial}{\partial \Delta \varepsilon_{\text{pf}}^v} \left( \frac{\partial f}{\partial \sqrt{J_2}} \Delta \varepsilon_{\text{pf}}^v \right) = \frac{\partial f}{\partial \sqrt{J_2}} + \Delta \varepsilon_{\text{pf}}^v \frac{\partial}{\partial \Delta \varepsilon_{\text{pf}}^v} \left( \frac{\partial f}{\partial \sqrt{J_2}} \right)
\]

\[
= \frac{2}{p_a} \sqrt{J_2} + \frac{2}{p_a} \Delta \varepsilon_{\text{pf}}^v \frac{\partial \sqrt{J_2}}{\partial \Delta \varepsilon_{\text{pf}}^v}
\]

\[
= \frac{2}{p_a} \sqrt{J_2}
\]

The second term of Eq. A.4.5.2 yields:

\[
\frac{\partial}{\partial \Delta \varepsilon_{\text{pf}}^v} \left( \frac{\partial f}{\partial I_1} \Delta \varepsilon_{\text{pf}}^v \right) = \Delta \varepsilon_{\text{pf}}^v \frac{\partial}{\partial \Delta \varepsilon_{\text{pf}}^v} \left( \frac{\partial f}{\partial I_1} \right)
\]

\[
\text{A.4.5.6}
\]

Substituting \( \partial f / \partial I_1 \) for the Desai surface in Eq. A.4.5.6:

\[
\frac{\partial}{\partial \Delta \varepsilon_{\text{pf}}^v} \left( \frac{\partial f}{\partial I_1} \Delta \varepsilon_{\text{pf}}^v \right) = -2 \frac{\Delta \varepsilon_{\text{pf}}^v}{p_a} \frac{\partial}{\partial \Delta \varepsilon_{\text{pf}}^v} \left[ \gamma (I_1 + R) \right]
\]

\[
\text{A.4.5.7}
\]

Partial differentiation yields:

\[
\frac{\partial}{\partial \Delta \varepsilon_{\text{pf}}^v} \left[ \gamma (I_1 + R) \right] = \gamma \frac{\partial}{\partial \Delta \varepsilon_{\text{pf}}^v} (I_1 + R) + (I_1 + R) \frac{\partial \gamma}{\partial \Delta \varepsilon_{\text{pf}}^v}
\]

\[
\text{term A term B}
\]

\[
\text{A.4.5.8}
\]

Term A has been computed in Appendix 4.4. Computation of term B is presented in the following.

Model of Scarpas & Blauwendraad (1998a)

\[
\gamma = \left( \gamma_{\text{max}}^2 - k \left( W_{\text{pf}} \right)^{3/2} \right)^{1/2}
\]

\[
\text{A.4.5.9}
\]

with

\[
W_{\text{pf}} = \int \sigma \cdot d \varepsilon_{\text{pf}}
\]

\[
\text{A.4.5.10}
\]

Then

\[
\frac{\partial \gamma}{\partial \Delta \varepsilon_{\text{pf}}^v} = \frac{\partial \gamma}{\partial W_{\text{pf}}} \frac{\partial W_{\text{pf}}}{\partial \Delta \varepsilon_{\text{pf}}^v} \frac{\partial \Delta \varepsilon_{\text{pf}}^v}{\partial \Delta \varepsilon_{\text{pf}}^v}
\]

\[
\text{A.4.5.11}
\]

\[
\frac{\partial \gamma}{\partial W_{\text{pf}}} = -\frac{3}{4} K \left( \gamma_{\text{max}}^2 - k \left( W_{\text{pf}} \right)^{3/2} \right)^{-1/2} \left( W_{\text{pf}} \right)^{1/2}
\]

\[
\text{A.4.5.12}
\]
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\[ \frac{\partial W^{pf}}{\partial \Delta \varepsilon^{pf}} = \sigma \]  

A.4.5.13

Model of Erkens et al. (2002a)

\[ \gamma = \eta \gamma_f + (1 - \eta) \gamma_r \]  

A.4.5.14

with

\[ \eta = e^{-\kappa \xi^{pf}} \]  

A.4.5.15

Then

\[ \frac{\partial \gamma}{\partial \Delta \varepsilon^{pf}} = \frac{\partial \eta}{\partial \Delta \varepsilon^{pf}} \frac{\partial \xi^{pf}}{\partial \Delta \varepsilon^{pf}} \frac{\partial \Delta \varepsilon^{pf}}{\partial \Delta \varepsilon^{pf}} \]  

A.4.5.16

\[ \frac{\partial \gamma}{\partial \eta} = (\gamma_f - \gamma_r) \]  

A.4.5.17

\[ \frac{\partial \eta}{\partial \xi^{pf}} = -\kappa \xi^{pf} e^{-\kappa \xi^{pf}} \]  

A.4.5.18

All other terms of Eq. A.4.5.16 have been already computed in Appendix 4.4.

**Computation of** \[ \frac{\partial g_1}{\partial \Delta \varepsilon^{pf}} : \]

\[ \frac{\partial g_1}{\partial \Delta \varepsilon^{pf}} = \frac{\partial}{\partial \Delta \varepsilon^{pf}} \left( \frac{\partial f}{\partial I_2} \Delta \varepsilon^p \right) - \frac{\partial f}{\partial I_1} \Delta \varepsilon^{pf} \]  

A.4.5.19

\[ = \frac{\partial}{\partial \Delta \varepsilon^{pf}} \left( \frac{\partial f}{\partial I_2} \Delta \varepsilon^p \right) - \frac{\partial}{\partial \Delta \varepsilon^{pf}} \left( \frac{\partial f}{\partial I_1} \Delta \varepsilon^{pf} \right) \]

\[ \text{term A} \quad \text{term B} \]

Term A has been computed already in Appendix 4.4. Computation of term B is presented in the following.

\[ \frac{\partial}{\partial \Delta \varepsilon^{pf}} \left( \frac{\partial f}{\partial I_1} \Delta \varepsilon^{pf} \right) = \frac{\partial f}{\partial I_1} + \Delta \varepsilon^{pf} \frac{\partial}{\partial \Delta \varepsilon^{pf}} \left( \frac{\partial f}{\partial I_1} \right) \]  

A.4.5.20

For the Desai surface during \( \gamma \) softening \( \frac{\partial f}{\partial I_1} \) has been computed in Eq. A.4.5.3. Hence:
\[
\frac{\partial}{\partial \Delta \varepsilon_{pf}} \left( \frac{\partial f}{\partial I_1} \right) = - \left( \frac{2}{p_a^n} \right) \frac{\partial}{\partial \Delta \varepsilon_{pf}} \left[ \gamma \left( I_1 + R \right) \right] \\
= - \left( \frac{2}{p_a^n} \right) \gamma \left( \frac{\partial}{\partial \Delta \varepsilon_{pf}} \left( I_1 + R \right) + \left( I_1 + R \right) \frac{\partial \gamma}{\partial \Delta \varepsilon_{pf}} \right) \\
= - \left( \frac{2}{p_a^n} \right) \left( I_1 + R \right) \frac{\partial \gamma}{\partial \Delta \varepsilon_{pf}}
\]

Depending on the chosen expression for \( \gamma \) degradation computation of the partial derivative term in Eq. A.4.5.21 is as follows:

**Model of Scarpas & Blauwendraad (1998a)**

\[
\frac{\partial \gamma}{\partial \Delta \varepsilon_{pf}} = \frac{\partial \gamma}{\partial W^{pf}} \frac{\partial W^{pf}}{\partial \Delta \varepsilon_{pf}} \frac{\partial \Delta \varepsilon_{pf}}{\partial \Delta \varepsilon_{pf}}
\]

A.4.5.22

in which all terms have been computed earlier

**Model of Erkens et al. (2002a)**

\[
\frac{\partial \gamma}{\partial \Delta \varepsilon_{pf}} = \frac{\partial \gamma}{\partial \eta} \frac{\partial \eta}{\partial \varepsilon^{pf}} \frac{\partial \varepsilon^{pf}}{\partial \Delta \varepsilon_{pf}} \frac{\partial \Delta \varepsilon_{pf}}{\partial \Delta \varepsilon_{pf}}
\]

A.4.5.23

in which all terms have been computed earlier.

**Computation of \( \partial g_2 / \partial \Delta \varepsilon_{v} \):**

During \( \gamma \) softening the Desai surface function can be expressed as:

\[
f = \left( \sqrt{f_2} \right)^2 = \left[ \gamma \left( \frac{I_1 + R}{p_a} \right) \right]^2
\]

A.4.5.24

\[
f = F_a - F_b
\]

Partial differentiation results to:
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\[ \frac{\partial g_2}{\partial \Delta \varepsilon^p_d} = \frac{\partial f}{\partial \Delta \varepsilon^p_d} \]
\[ = \frac{\partial f}{\partial F_a} \frac{\partial F_a}{\partial \Delta \varepsilon^p_d} + \frac{\partial f}{\partial F_b} \frac{\partial F_b}{\partial \Delta \varepsilon^p_d} \]
\[ = \frac{\partial f}{\partial F_a} \left( \frac{\partial F_a}{\partial \Delta \varepsilon^p_d} \right) + \frac{\partial f}{\partial F_b} \left( \frac{\partial F_b}{\partial \Delta \varepsilon^p_d} \right) \]
\[ \begin{array}{c}
\text{A.4.5.25}
\end{array} \]

\[ = 0 + \frac{\partial f}{\partial F_b} \left( \frac{\partial F_b}{\partial \Delta \varepsilon^p_d} \right) \]

in which all terms have been computed earlier.

**Computation of \( \frac{\partial g_2}{\partial \Delta \varepsilon^p_d} \):**

\[ \frac{\partial g_2}{\partial \Delta \varepsilon^p_d} = \frac{\partial f}{\partial \Delta \varepsilon^p_d} \]
\[ = \frac{\partial f}{\partial F_a} \frac{\partial F_a}{\partial \Delta \varepsilon^p_d} + \frac{\partial f}{\partial F_b} \frac{\partial F_b}{\partial \Delta \varepsilon^p_d} \]
\[ = \frac{\partial f}{\partial F_a} \left( \frac{\partial F_a}{\partial \Delta \varepsilon^p_d} \right) + \frac{\partial f}{\partial F_b} \left( \frac{\partial F_b}{\partial \Delta \varepsilon^p_d} \right) \]
\[ \text{A.4.5.26} \]

in which all terms have been computed earlier.

**Cracking**

As mentioned in Section 4.5.2, substituting the stresses \( \{\sigma, \tau_s, \tau_t\} \) and the softening function in terms of \( \Delta \lambda \) in Eq. 4.22 a non-linear equation in \( \Delta \lambda \) is obtained. A Newton-Raphson iterative procedure can be set up for its solution.

For any given iteration the increment in \( \Delta \lambda \) is computed from:

\[ \Delta \lambda = \frac{f_H(n \Delta \lambda)}{\left( \frac{\partial f_H}{\partial \Delta \lambda} \right)} \]
\[ \text{A.4.5.27} \]

On the basis of Eq. 4.22

\[ \frac{\partial f_H}{\partial \Delta \lambda} = \frac{\partial f_H}{\partial \sigma} \frac{\partial \sigma}{\partial \Delta \lambda} + \frac{\partial f_H}{\partial \tau_s} \frac{\partial \tau_s}{\partial \Delta \lambda} + \frac{\partial f_H}{\partial \tau_t} \frac{\partial \tau_t}{\partial \Delta \lambda} - \frac{\partial f_H}{\partial \xi} \frac{\partial \xi}{\partial \Delta \lambda} \]
\[ \text{A.4.5.28} \]
\[
\frac{\partial \sigma}{\partial \Delta \lambda} = -\frac{2E}{(1+2E\Delta \lambda)^2} \sigma^{\text{trial}}
\]

\[
\frac{\partial \tau_s}{\partial \Delta \lambda} = -\frac{2G}{(1+2G\Delta \lambda)^2} \tau_s^{\text{trial}}
\]

\[
\frac{\partial \tau_t}{\partial \Delta \lambda} = -\frac{2G}{(1+2G\Delta \lambda)^2} \tau_t^{\text{trial}}
\]

\[
\frac{\partial f_t}{\partial \Delta \lambda} = f_{tu} \frac{\partial}{\partial \Delta \lambda} \left( e^{-a \Delta \lambda + \Delta t_{tu}} \right)
\]

\[
= -2af_{tu} f_t \Delta \lambda \sigma
\]

\[
= -2af_{tu} f_t \left( \sigma + \Delta \lambda \frac{\partial \sigma}{\partial \Delta \lambda} \right)
\]
Elastoplasticity
5.1 Elementary Model of Viscoelastic Response

Viscoelastic materials represent a wide range of engineering materials ranging from asphaltic materials to rubber-like materials and to natural and synthetic polymers.

Without any loss of generality, an elementary model of viscoelastic material can be defined as shown in Fig. 5.1. It is also known as a standard linear solid. By judicious choice of the individual material constants, other models can be obtained. For example by setting $E_\infty = \infty$, the well known Maxwell material model is obtained or, by setting $E_M = \infty$ the Voigt-Kelvin material model is obtained.

\begin{equation}
\sigma_M(t) = \eta \dot{\varepsilon}_v(t)
\end{equation}

The stress in the dashpot is defined proportional to the inelastic strain rate $\dot{\varepsilon}_v(t)$

\begin{equation}
\sigma_M(t) = E_M [\varepsilon(t) - \varepsilon_v(t)]
\end{equation}

Also, the stress in the spring of the Maxwell element can be computed as

\begin{equation}
\eta \ddot{\varepsilon}_v(t) = E_M [\varepsilon(t) - \varepsilon_v(t)]
\end{equation}

in which $\varepsilon(t)$ is the total strain. On the basis of Eq. 5.1 and Eq. 5.2

\begin{equation}
\eta \ddot{\varepsilon}_v(t) = E_M [\varepsilon(t) - \varepsilon_v(t)]
\end{equation}

If $\kappa = \eta / E_M$, from Eq. 5.3

\begin{equation}
\ddot{\varepsilon}_v(t) + \frac{\varepsilon_v(t)}{\kappa} = \frac{\varepsilon(t)}{\kappa}
\end{equation}
Viscoelasticity

which is a linear differential equation. By means of Laplace transforms (Appendix 5.1) it results

\[ \varepsilon_v(t) = \varepsilon(t) - \int_0^t e^{-(t-\tau)/\kappa} \ddot{\varepsilon}(\tau) d\tau \]  

5.5

The total stress in the elementary model is

\[ \sigma(t) = E_\infty \varepsilon(t) + E_M [\varepsilon(t) - \varepsilon_v(t)] \]

\[ = \left( E_\infty + E_M \right) \varepsilon(t) - E_M \varepsilon_v(t) \]

\[ = E \varepsilon(t) - E_M \varepsilon_v(t) \]

\[ = E \varepsilon(t) - q(t) \]

5.6

It can be recast in a more frequently encountered format by means of a convolution integral. Rearranging Eq. 5.5 and substituting into Eq. 5.6

\[ \sigma(t) = E_\infty \int_0^t \ddot{\varepsilon}(\tau) d\tau + E_M \int_0^t e^{-(t-\tau)/\kappa} \ddot{\varepsilon}(\tau) d\tau \]

\[ = \int_0^t \left( E_\infty + E_M e^{-(t-\tau)/\kappa} \right) \ddot{\varepsilon}(\tau) d\tau \]

\[ = \int_0^t G(t-\tau) \ddot{\varepsilon}(\tau) d\tau \]

5.7

in which

\[ G(t) = E_\infty + E_M e^{-(t)/\kappa} \]

5.8

is commonly termed the relaxation function, Lubliner [1973], Holzapfel [2000].

5.1.1 Generalized Linear Viscoelastic Model

Generalizing the elementary model of Fig. 5.1 to an arbitrary number on N Maxwell components, Fig. 5.2, the total stress can be expressed as

\[ \sigma(t) = E \varepsilon(t) - \sum_{i=1}^N q_i(t) \]

5.9

with

\[ E = E_\infty + \sum_{i=1}^N E_i \]

5.10

Also, the response of every one of the viscous components it is postulated to be described by

\[ \dot{q}_i(t) + \frac{q_i(t)}{\kappa_i} = \frac{E_i \varepsilon(t)}{\kappa_i} \]

5.11

with \( q_i = E_i \varepsilon_i \).
In similarity to Eq. 5.7 the total stress can be expressed as

\[
\sigma(t) = E_\infty \int_0^t \dot{\varepsilon}(\tau) \, d\tau + \sum_{i=1}^N \left( E_i \int_0^t e^{-\frac{(t-\tau)}{\kappa_i}} \dot{\varepsilon}(\tau) \, d\tau \right)
\]

\[
= \int_0^t \left( E_\infty + \sum_{i=1}^N E_i e^{-\frac{(t-\tau)}{\kappa_i}} \right) \dot{\varepsilon}(\tau) \, d\tau
\]

\[
= \int_0^t G(t-\tau) \dot{\varepsilon}(\tau) \, d\tau
\]

with

\[
G(t) = E_\infty + \sum_{i=1}^N E_i e^{-\frac{t}{\kappa_i}}
\]

5.1.2 Incremental Formulation

On the basis of Eq. 5.12 the current stress is

\[
t+\Delta t \sigma = E_\infty \int_0^{t+\Delta t} \dot{\varepsilon}(\tau) \, d\tau + \sum_{i=1}^N \left( E_i \int_0^{t+\Delta t} e^{-\frac{(t+\Delta t-\tau)}{\kappa_i}} \dot{\varepsilon}(\tau) \, d\tau \right)
\]

The first term of the above can be written as

\[
E_\infty \int_0^{t+\Delta t} \dot{\varepsilon}(\tau) \, d\tau = E_\infty \left[ \int_0^t \dot{\varepsilon}(\tau) \, d\tau + \int_t^{t+\Delta t} \dot{\varepsilon}(\tau) \, d\tau \right]
\]

\[
= E_\infty \left[ t^H + \Delta \varepsilon \right] + E_\infty \int_t^{t+\Delta t} \dot{\varepsilon}(\tau) \, d\tau
\]

\[
= E_\infty (t+\Delta t) H
\]

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For the integral of the second term of Eq. 5.14 it holds

\[
\int_{0}^{t+\Delta t} e^{-\frac{(t+\Delta t-\tau)}{\kappa_i}} \dot{\varepsilon}(\tau) d\tau = \int_{0}^{t} e^{-\frac{(t+\Delta t-\tau)}{\kappa_i}} \dot{\varepsilon}(\tau) d\tau + \int_{t}^{t+\Delta t} e^{-\frac{(t+\Delta t-\tau)}{\kappa_i}} \dot{\varepsilon}(\tau) d\tau
\]

\[
= \int_{0}^{t} e^{-\frac{(t-\tau)}{\kappa_i}} e^{-\frac{\Delta t}{\kappa_i}} \dot{\varepsilon}(\tau) d\tau + \int_{t}^{t+\Delta t} e^{-\frac{(t+\Delta t-\tau)}{\kappa_i}} \dot{\varepsilon}(\tau) d\tau
\]

\[
= e^{-\frac{\Delta t}{\kappa_i}} \int_{0}^{t} e^{-\frac{(t-\tau)}{\kappa_i}} \dot{\varepsilon}(\tau) d\tau + \int_{t}^{t+\Delta t} e^{-\frac{(t+\Delta t-\tau)}{\kappa_i}} \dot{\varepsilon}(\tau) d\tau
\]

By means of the mid-point rule the second term of the above can be integrated as

\[
\int_{t}^{t+\Delta t} e^{-\frac{(t+\Delta t-\tau)}{\kappa_i}} \dot{\varepsilon}(\tau) d\tau \approx e^{-\frac{(t+\Delta t-t)}{\kappa_i}} \dot{\varepsilon}(t) \Delta t
\]

\[
\approx e^{-\frac{\Delta t}{\kappa_i}} \Delta \varepsilon
\]

Also let

\[
t_{h_i} = \int_{0}^{t} e^{-\frac{(t-\tau)}{\kappa_i}} \dot{\varepsilon}(\tau) d\tau
\]

Then substituting in Eq. 5.16

\[
\int_{0}^{t+\Delta t} e^{-\frac{(t+\Delta t-\tau)}{\kappa_i}} \dot{\varepsilon}(\tau) d\tau = e^{-\frac{\Delta t}{\kappa_i}} t_{h_i} + e^{-\frac{\Delta t}{\kappa_i}} \Delta \varepsilon
\]

\[
= t+\Delta t_{h_i}
\]

On the basis of Eq. 5.15 and Eq. 5.19 the total stress can be expressed as, Simo [1987], Simo & Hughes [1998], Holzapfel [2000]

\[
t+\Delta t \sigma = E_\infty t+\Delta t H + \sum_{i=1}^{N} E_i t+\Delta t h_i
\]

in which \(t+\Delta t H\) is defined by Eq. 5.15 and \(t+\Delta t h_i\) by Eq. 5.19.

5.2 Generalized Nonlinear Elastic Formulation

In the previous Section, the assumption of linearity was made on the elastic stress-strain response. In this Section, this assumption will be modified in order to enable more general hyperelastic models to be utilized. This can be achieved by recasting the formulation of the previous Section in such a way that the notion of strain energy function is introduced in the formulation. Then any suitable strain energy function can be utilized to express a linear or nonlinear elastic stress-strain response.

The elastic stored energy in the springs associated with the model of Fig. 5.2 is
\[ \psi(t) = \frac{1}{2} E_\infty \left[ \varepsilon(t) \right]^2 + \frac{1}{2} \sum_{i=1}^{N} E_i \left[ \varepsilon(t) - \varepsilon_{vi}(t) \right]^2 \]  

5.21

or in a notation more appropriate for latter developments in Chapter 6

\[ \psi(t) = \psi_\infty [\varepsilon(t)] + \sum_{i=1}^{N} \psi_{vi} [\varepsilon(t), \varepsilon_{vi}(t)] \]  

5.22

Then, the derivative with respect to strain \( \varepsilon(t) \) is

\[ \frac{\partial \psi(t)}{\partial \varepsilon} = E_\infty \varepsilon(t) + \sum_{i=1}^{N} E_i \left[ \varepsilon(t) - \varepsilon_{vi}(t) \right] \]  

5.23

\[ = E \varepsilon(t) - \sum_{i=1}^{N} q_i(t) \]

in which

\[ E = E_\infty + \sum_{i=1}^{N} E_i \]  

5.24

and

\[ q_i(t) = E_i \varepsilon_{vi}(t) \]  

5.25

From a comparison of Eq. 5.9 and Eq. 5.23 it can be concluded that

\[ \sigma(t) = \frac{\partial \psi(t)}{\partial \varepsilon} \]  

5.26

If a new energy function is defined as

\[ \Psi(t) = \frac{1}{2} E \left[ \varepsilon(t) \right]^2 \rightarrow \frac{\partial \Psi(t)}{\partial \varepsilon} = E \varepsilon(t) \]  

5.27

\[
\begin{align*}
\gamma_i &= E_i / E ; \quad \gamma_\infty = E_\infty / E \\
\gamma_\infty + \sum_{i=1}^{N} \gamma_i &= 1
\end{align*}
\]

5.28

and then Eq. 5.9 and Eq. 5.11 can be expressed as

\[ \sigma(t) = \frac{\partial \Psi(\varepsilon)}{\partial \varepsilon} - \sum_{i=1}^{N} q_i(t) \]  

5.29

\[ \dot{q}_i(t) + \frac{q_i(t)}{\kappa_i} = \frac{\gamma_i}{\kappa_i} \frac{\partial \Psi(\varepsilon)}{\partial \varepsilon} \quad i = 1 \ldots N \]

The solution of Eq. 5.29 is (Appendix 5.2)

\[ ^* \text{ Eq. 5.28 is a direct consequence of Eq. 5.24 and the definitions of Eq. 5.28.} \]
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\[ q_i(t) = \gamma_i \left[ \frac{\partial \Psi(t)}{\partial \epsilon} - \int_0^t e^{-(t-\tau)\kappa_i} \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau \right] \]  

5.30

Summing over N

\[
\sum_{i=1}^N q_i(t) = \sum_{i=1}^N \left[ \gamma_i \frac{\partial \Psi(t)}{\partial \epsilon} - \gamma_i \int_0^t e^{-(t-\tau)\kappa_i} \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau \right]
\]

5.31

\[
= \sum_{i=1}^N \left[ \gamma_i \int_0^t \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau - \gamma_i \int_0^t e^{-(t-\tau)\kappa_i} \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau \right]
\]

However, in view of Eq. 5.28, it is valid

\[
\left\{ \sum_{i=1}^N \gamma_i \right\} \int_0^t \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau = (1-\gamma_\infty) \int_0^t \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau
\]

5.32

Substituting in Eq. 5.31

\[
\sum_{i=1}^N q_i(t) = (1-\gamma_\infty) \int_0^t \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau - \sum_{i=1}^N \gamma_i \int_0^t e^{-(t-\tau)\kappa_i} \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau
\]

5.33

\[
= \frac{\partial \Psi(t)}{\partial \epsilon} - \int_0^t \left[ \gamma_\infty + \sum_{i=1}^N \gamma_i e^{-(t-\tau)\kappa_i} \right] \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau
\]

and hence from Eq. 5.29

\[
\sigma(t) = \gamma_\infty \int_0^t \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau + \int_0^t \left[ \sum_{i=1}^N \gamma_i e^{-(t-\tau)\kappa_i} \right] \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau
\]

5.34

In order to obtain an expression convenient for numerical implementation, the stress at the current time step \( t + \Delta t \) can be rewritten as

\[
t + \Delta t \sigma = \gamma_\infty \int_0^{t+\Delta t} \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau + \int_0^{t+\Delta t} \left[ \sum_{i=1}^N \gamma_i e^{-(t+\Delta t-\tau)\kappa_i} \right] \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau
\]

5.35

By means of the same procedure as in Section 5.1.2 the following recursive formula is obtained for computation of \( t + \Delta t \sigma \)

\[
t + \Delta t \sigma = \gamma_\infty t + \Delta t H + \sum_{i=1}^N \gamma_i \left( t + \Delta t h_i \right)
\]

5.36

\[
t + \Delta t H = t H + \Delta \sigma \quad ; \quad t H = \int_0^t \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau \quad ; \quad \Delta \sigma = \int_0^t \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau
\]

\[
t + \Delta t h_i = e^{-\Delta \psi_k} t h_i + e^{-\Delta \psi_k \tau} \Delta \sigma \quad ; \quad t h_i = \int_0^t e^{-(t-\tau)\kappa_i} \frac{\partial^2 \Psi(\tau)}{\partial \tau \partial \epsilon} \, d\tau
\]
5.3 Three Dimensional Nonlinear Formulation

In this Section the elementary multi-component model of Section 5.2 will be extended to the three dimensional case. In the currently implemented version of the model in CAPA-3D the strain energy function $\Psi$ is defined by

$$\Psi(\varepsilon) = \frac{1}{2} \lambda (\varepsilon : I)^2 + \mu \varepsilon : \varepsilon$$

in which $\lambda$ and $\mu$ are the standard elastic material constants. Any other function can be also implemented.

In similarity to Eq. 5.29, the three dimensional stress tensor can be expressed as

$$\sigma(t) = \frac{\partial \Psi(t)}{\partial \varepsilon} - \sum_{i=1}^{N} q_i(t)$$

The internal variables are postulated to be characterized by rate equations analogous to Eq.5.29

$$\dot{q}_i(t) + \frac{q_i(t)}{\kappa_i} = \frac{\gamma_i}{\kappa_i} \frac{\partial \Psi(\varepsilon)}{\partial \varepsilon}$$

Summing up over N an equation analogous to Eq. 5.33 is obtained

$$\sum_{i=1}^{N} q_i(t) = \frac{\partial \Psi(t)}{\partial \varepsilon} - \int_{0}^{t} \left( \gamma_{\infty} + \sum_{i=1}^{N} \gamma_i e^{-(t-\tau)/\kappa_i} \right) \frac{\partial^2 \Psi(t)}{\partial \tau \partial \varepsilon} d\tau$$

Substituting in Eq. 5.38, the expression for the stress tensor at $t + \Delta t$ is finally obtained as

$$(t + \Delta t) \sigma(t) = \gamma_{\infty} \int_{0}^{t + \Delta t} \frac{\partial^2 \Psi(t)}{\partial \tau \partial \varepsilon} d\tau + \sum_{i=1}^{N} \left( \gamma_i \int_{0}^{t + \Delta t} e^{-(t+\Delta t-\tau)/\kappa_i} \frac{\partial^2 \Psi(t)}{\partial \tau \partial \varepsilon} d\tau \right)$$

By means of the same procedure as in Section 5.1.2 the recursive formulae necessary for computation of $(t + \Delta t) \sigma$ are

$$(t + \Delta t) \sigma = \gamma_{\infty} (t + \Delta t) H + \sum_{i=1}^{N} \gamma_i (t + \Delta t) h_i$$

$$t + \Delta t H = H_{iso} + \Delta \sigma ; \quad t H_{iso} = \int_{0}^{t} \frac{\partial^2 \Psi(t)}{\partial \tau \partial \varepsilon} d\tau$$

$$t + \Delta t h_i = e^{-\Delta \kappa_i} t h_i + e^{-\Delta \gamma \kappa_i} \Delta \sigma ; \quad t h_i = \int_{0}^{t} e^{-(t-\tau)/\kappa_i} \frac{\partial^2 \Psi(t)}{\partial \tau \partial \varepsilon} d\tau$$
5.4 Burger’s Model

So far the assumption has been made that the Maxwell components comprising the constitutive model were acting in parallel. Nevertheless there is a wide range of engineering materials in which other arrangements of the individual components are more appropriate. A typical example are materials whose response is represented by the well known generalized Burger’s model which consists of a number of Kelvin components acting in series together with a dashpot and an elastic spring.

5.4.1 Elementary Model

A schematic of the elementary generalized Burger’s model is shown in Fig. 5.3. In this type of model the stress is transmitted through each element and the strains are additive so that

\[ \varepsilon(t) = \varepsilon_{el}(t) + \varepsilon_{vp}(t) + \sum_{i=1}^{N} \varepsilon_{ve,i}(t) \]  

5.43

where \( \varepsilon, \varepsilon_{el}, \varepsilon_{ve,i} \) and \( \varepsilon_{vp} \) are the total, elastic, viscoelastic and viscoplastic strain components.

![Fig. 5.3 Elementary Burger’s model](image)

The strain in the elastic component when a stress \( \sigma \) is applied to the mechanical device of Fig. 5.3 can be readily calculated as

\[ \varepsilon_{el}(t) = \frac{\sigma(t)}{E_{\infty}} \]  

5.44

in which \( E_{\infty} \) is the modulus of elasticity.

On the basis of the observations made in Section 5.1, the stress in the \( i \)-th viscoelastic component can be computed from

\[ \sigma(t) = \eta_i \dot{\varepsilon}_{ve,i}(t) + E_i \varepsilon_{ve,i}(t) \]  

5.45

Setting \( \kappa_i = \frac{\eta_i}{E_i} \)

\[ \dot{\varepsilon}_{ve,i}(t) + \frac{\varepsilon_{ve,i}(t)}{\kappa_i} = \frac{\sigma(t)}{\eta_i} \]  

5.46

which can be solved by means of a Laplace transform (Appendix 5.3) to obtain

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\[ \varepsilon_{ve,i}(t) = \frac{1}{E_i} \left\{ \sigma(t) - \int_0^t e^{-\frac{(t-\tau)}{\kappa_i}} \dot{\sigma}(\tau) d\tau \right\} \quad 5.47 \]

Finally, the strain increment corresponding to a stress increment \( \Delta \sigma \) is (Appendix 5.3)

\[ \Delta \varepsilon_{ve,i} = t^{+} \Delta t \varepsilon_{ve,i} - t_{-} \varepsilon_{ve,i} \]

\[ = \frac{1}{E_i} \left\{ \left( 1 - e^{\frac{-\Delta t}{2\kappa_i}} \right) \Delta \sigma + \left( 1 - e^{\frac{-\Delta t}{\kappa_i}} \right) t_{h_i} \right\} \quad 5.48 \]

with

\[ t_{h_i} = \int_0^t e^{\frac{-\tau}{\kappa_i}} \dot{\sigma}(\tau) d\tau \quad 5.49 \]

The response of the viscoplastic component is described by

\[ \dot{\varepsilon}_{vp}(t) = \frac{\sigma(t)}{\eta_\infty} \quad 5.50 \]

which can be solved by means of a Laplace transform (Appendix 5.4) to obtain

\[ \varepsilon_{vp}(t) = \frac{1}{\eta_\infty} \left\{ \sigma(t) t - t_{h_\infty} \right\} \quad 5.51 \]

in which

\[ t_{h_\infty} = \int_0^t \tau \dot{\sigma}(\tau) d\tau \quad 5.52 \]

Finally, the strain increment corresponding to a stress increment \( \Delta \sigma \) is

\[ \Delta \varepsilon_{vp} = t^{+} \Delta t \varepsilon_{vp} - t_{-} \varepsilon_{vp} \]

\[ = \frac{\Delta t}{\eta_\infty} \left\{ \sigma(t) + \frac{\Delta \sigma}{2} \right\} \quad 5.53 \]

5.4.2 Three Dimensional Model

In this Section the elementary model of the previous Section is extended to the three dimensional case.

In three dimensions the equivalent incremental form of Eq. 5.44 is

\[ \Delta \varepsilon_{el} = C_{el} \Delta \sigma \quad 5.54 \]
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\[
C_{el} = \frac{1}{E_\infty} \begin{bmatrix}
1 & -\nu_{el} & -\nu_{el} & 0 & 0 & 0 \\
-\nu_{el} & 1 & -\nu_{el} & 0 & 0 & 0 \\
-\nu_{el} & -\nu_{el} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1+\nu_{el}) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1+\nu_{el}) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1+\nu_{el})
\end{bmatrix}
\]

with \( \nu_{el} \) the Poisson ratio of the elastic component.

Similarly, for the viscoelastic components and the viscoplastic component it holds

\[
\Delta \varepsilon_{ve,i} = C_{ve,i} \left[ \begin{bmatrix}
-\Delta t / \eta_i \\
1 - e^{-\Delta t / \eta_i}
\end{bmatrix} \Delta \sigma + \begin{bmatrix}
-\Delta t / \kappa_i \\
1 - e^{-\Delta t / \kappa_i}
\end{bmatrix} \Delta \varepsilon_{vp,i}
\right]
\]

\[
\Delta \varepsilon_{vp} = C_{vp} \left[ \frac{\Delta \sigma}{2} + \Delta \varepsilon_{vp,i} \right] \Delta t
\]

in which \( C_{ve,i} \) and \( C_{vp} \) have the same structure as \( C_{el} \) but with \( E_\infty \) substituted by \( \eta_i \) or \( \eta_\infty \) and \( \nu_{el} \) by \( \nu_i \) or \( \nu_\infty \).

As indicated in Section 5.4.1, for a given increment of stress, strain compatibility requires

\[
\Delta \varepsilon = \Delta \varepsilon_{el} + \Delta \varepsilon_{vp} + \sum_{i=1}^{N} \Delta \varepsilon_{ve,i}
\]

Substituting in the above the appropriate expressions for the strain increments it results

\[
\Delta \varepsilon = C_{el} \Delta \sigma + C_{vp} \left[ \frac{\Delta \sigma}{2} + \Delta \varepsilon_{vp,i} \right] \Delta t
\]

\[
\sum_{i=1}^{N} C_{ve,i} \left[ \begin{bmatrix}
-\Delta t / \eta_i \\
1 - e^{-\Delta t / \eta_i}
\end{bmatrix} \Delta \sigma + \begin{bmatrix}
-\Delta t / \kappa_i \\
1 - e^{-\Delta t / \kappa_i}
\end{bmatrix} \Delta \varepsilon_{vp,i} \right]
\]

which can be further rearranged as

\[
\Delta \sigma = \tilde{C}^{-1} \Delta \tilde{\varepsilon}
\]

with

\[
\tilde{C} = C_{el} + \sum_{i=1}^{N} C_{ve,i} \left[ \begin{bmatrix}
-\Delta t / \eta_i \\
1 - e^{-\Delta t / \eta_i}
\end{bmatrix} \Delta \sigma + \begin{bmatrix}
-\Delta t / \kappa_i \\
1 - e^{-\Delta t / \kappa_i}
\end{bmatrix} \Delta \varepsilon_{vp,i} \right] + C_{vp} \frac{\Delta t}{2}
\]

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and

\[ \Delta \varepsilon = \Delta \varepsilon - \sum_{i=1}^{N} C_{\nu e,i} \left( 1 - e^{-\frac{\Delta t}{\kappa_i}} \right) t^h_i - C_{\nu p} t^\sigma \Delta t \]

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Once \( \Delta \sigma \) is known, the hereditary terms \( t^+\Delta t^h_i \) and \( t^+\Delta t^h_\infty \) can be computed for use in the next time increment.

### 5.5 Utilization

The above material models have been implemented in CAPA-3D and have been utilised extensively for simulation of the response of various structures. Typical applications include the simulation of the response of the mastic constituent of asphaltic materials in micromechanical studies of the response of chip-seals, Huurman et al. [2003a], Milne [2004] and, the simulation of the response of asphaltic mixes under complex stress conditions, Collop et al. [2002], Collop et al. [2003].
Viscoelasticity

Appendix 5.1

The evolution law for the inelastic strain in the dashpot of Fig. 5.1 was determined in Section 5.1 as

\[ q + \frac{q}{\kappa} = \frac{\varepsilon}{\kappa} \quad \text{A.5.1.1} \]

On the basis of Laplace transforms it can be expressed as

\[ sQ(s) - Q_0 + \frac{1}{\kappa} Q(s) = \frac{1}{\kappa} \mathcal{L}[E(s)] \quad \text{A.5.1.2} \]

\[ Q(s) = \frac{Q_0}{s + \frac{1}{\kappa}} + \frac{1}{s + \frac{1}{\kappa}} \mathcal{L}[E(s)] \quad \text{A.5.1.3} \]

Taking the inverse Laplace transform

\[ \mathcal{L}^{-1}[Q(s)] = q_0e^{-\frac{t}{\kappa}} + \int_0^t \varepsilon(t-s) \frac{1}{\kappa} e^{-\frac{t-s}{\kappa}} ds \]

\[ = q_0e^{-\frac{t}{\kappa}} + \int_0^t \varepsilon(s) \frac{1}{\kappa} e^{-\frac{t-s}{\kappa}} ds \]

\[ = q_0e^{-\frac{t}{\kappa}} + \frac{1}{\kappa} \int_0^t \varepsilon(s) \varepsilon(s) \frac{1}{\kappa} e^{-\frac{t-s}{\kappa}} ds \]

\[ = 0 + \frac{1}{\kappa} e^{-\frac{t}{\kappa}} \int_0^t \varepsilon(s) \varepsilon(s) \frac{1}{\kappa} e^{-\frac{t-s}{\kappa}} ds \]

\[ = q(t) \quad \text{A.5.1.4} \]

Integrating by parts \(^1\)

\[ \frac{1}{\kappa} \int_0^t \varepsilon(s) \varepsilon(s) \frac{1}{\kappa} e^{-\frac{t-s}{\kappa}} ds = \varepsilon(s) \frac{e^{-\frac{t}{\kappa}}}{\kappa} \bigg|_0^t - \int_0^t \varepsilon(s) \varepsilon(s) \frac{1}{\kappa} e^{-\frac{t-s}{\kappa}} ds \]

\[ = \varepsilon(t) e^{-\frac{t}{\kappa}} - \varepsilon(0) - \int_0^t \varepsilon(s) \varepsilon(s) \frac{1}{\kappa} e^{-\frac{t-s}{\kappa}} ds \quad \text{A.5.1.5} \]

Substituting in Eq. A.5.1.4

\[ \int_0^t \frac{dv}{ds} ds = u \bigg|_0^t - \int_0^t \frac{du}{ds} ds \]

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\[ q(t) = q_0 e^{-\gamma t} + e^{-\gamma t} \left[ \varepsilon(t) e^{\gamma t} - \varepsilon(0) - \int_0^t e^{\gamma s} \dot{\varepsilon}(s) \, ds \right] \]

\[ = \left[ q_0 - \varepsilon(0) \right] e^{-\gamma t} + \varepsilon(t) - \int_0^t e^{-\gamma s} \dot{\varepsilon}(s) \, ds \]

\[ = \varepsilon(t) - \int_0^t e^{-\gamma (t-s)} \dot{\varepsilon}(s) \, ds \]

A.5.1.6
The response of every one of the viscous components is described by Eq. 5.292

\[ \dot{q}_i(t) + \frac{q_i(t)}{\kappa_i} = \frac{\gamma_i}{\kappa_i} \frac{\partial \Psi(\varepsilon)}{\partial \varepsilon} \quad i = 1 \ldots N \]  

A.5.2.1

Taking the inverse Laplace transform

\[ q(t) = q_0 e^{-\frac{\gamma}{\kappa}} + \gamma \int_0^t \frac{\partial \Psi(t-\tau)}{\partial \varepsilon} \frac{1}{\kappa} e^{-\frac{\tau}{\kappa}} d\tau \]

\[ = q_0 e^{-\frac{\gamma}{\kappa}} + \gamma \int_0^t \frac{\partial \Psi(t)}{\partial \varepsilon} \frac{1}{\kappa} e^{-\frac{t-\tau}{\kappa}} d\tau \]  

A.5.2.2

\[ = q_0 e^{-\frac{\gamma}{\kappa}} + \frac{\gamma}{\kappa} e^{-\frac{\gamma}{\kappa}} \int_0^t \frac{\partial \Psi(t)}{\partial \varepsilon} e^{\frac{\tau}{\kappa}} d\tau \]

Integrating by parts

\[ \frac{1}{\kappa} \int_0^t \frac{\partial \Psi(t)}{\partial \varepsilon} e^{\frac{\tau}{\kappa}} d\tau = \frac{\partial \Psi(t)}{\partial \varepsilon} e^{\frac{\tau}{\kappa}} \bigg|_{\tau=0}^{\tau=t} - \int_0^t e^{\frac{\tau}{\kappa}} \frac{\partial^2 \Psi(\tau)}{\partial t \partial \varepsilon} d\tau \]

A.5.2.3

\[ = \frac{\partial \Psi(t)}{\partial \varepsilon} e^{\frac{\gamma}{\kappa}} - \frac{\partial \Psi(0)}{\partial \varepsilon} - \int_0^t e^{\frac{\tau}{\kappa}} \frac{\partial^2 \Psi(\tau)}{\partial t \partial \varepsilon} d\tau \]

Substituting in A.5.2.2

\[ q(t) = q_0 e^{-\frac{\gamma}{\kappa}} + \gamma e^{-\frac{\gamma}{\kappa}} \left[ \frac{\partial \Psi(t)}{\partial \varepsilon} e^{\frac{\gamma}{\kappa}} - \frac{\partial \Psi(0)}{\partial \varepsilon} - \int_0^t e^{\frac{\tau}{\kappa}} \frac{\partial^2 \Psi(\tau)}{\partial t \partial \varepsilon} d\tau \right] \]

A.5.2.4

\[ = \left[ q_0 - \gamma \frac{\partial \Psi(0)}{\partial \varepsilon} \right] e^{-\frac{\gamma}{\kappa}} + \gamma \left[ \frac{\partial \Psi(t)}{\partial \varepsilon} - \int_0^t e^{-\frac{(t-\tau)}{\kappa}} \frac{\partial^2 \Psi(\tau)}{\partial t \partial \varepsilon} d\tau \right] \]

is the solution of Eq. 5.292
The evolution law for the strain in the viscoelastic component of Fig. 5.3 was determined in Section 5.4 as

\[ \dot{\varepsilon}_i(t) + \frac{\varepsilon_i(t)}{\kappa_i} = \frac{\sigma(t)}{\eta_i} \quad \text{A.5.3.1} \]

with \( \kappa_i = \frac{\eta_i}{E_i} \). On the basis of Laplace transforms the above equation can be expressed as

\[ sE_i(s) - E_i(0) + \frac{1}{\kappa_i} \frac{E_i(s)}{\eta_i} = \frac{1}{s} \Sigma(s) \quad \text{A.5.3.2} \]

and hence

\[ E_i(s) = \frac{E_i(0)}{s + \frac{1}{\kappa_i}} + \frac{1}{s + \frac{1}{\kappa_i}} \Sigma(s) \quad \text{A.5.3.3} \]

Taking the inverse Laplace transform

\[ \varepsilon_i(t) = \mathcal{L}^{-1}\{E_i(s)\} \]

\[ = \varepsilon_i(0)e^{-\frac{\tau}{\kappa_i}} + \int_0^t \sigma(t-\tau) \frac{1}{\eta_i} e^{-\frac{\tau}{\kappa_i}} d\tau \]

\[ = \varepsilon_i(0)e^{-\frac{\tau}{\kappa_i}} + \int_0^t \sigma(\tau) \frac{1}{\eta_i} e^{-\frac{\tau}{\kappa_i}} d\tau \]

\[ = \varepsilon_i(0)e^{-\frac{\tau}{\kappa_i}} + \frac{\kappa_i}{\eta_i} \int_0^t \sigma(\tau) \frac{1}{\kappa_i} e^{-\frac{\tau}{\kappa_i}} d\tau \]

Integrating by parts

\[ \int_0^t \sigma(\tau) \frac{1}{\kappa_i} e^{-\frac{\tau}{\kappa_i}} d\tau = \sigma(\tau) e^{-\frac{\tau}{\kappa_i}} \bigg|_0^t - \int_0^t \frac{1}{\kappa_i} \sigma(\tau) e^{-\frac{\tau}{\kappa_i}} d\tau \]

\[ = \sigma(t)e^{-\frac{\tau}{\kappa_i}} - \sigma(0) - \int_0^t e^{\frac{\tau}{\kappa_i}} \dot{\sigma}(\tau) d\tau \quad \text{A.5.3.5} \]

Substituting in Eq. A.5.3.4
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\[ \varepsilon_i(t) = \varepsilon_i(0) e^{-\frac{t}{\kappa_i}} + \frac{1}{E_i} e^{-\frac{t}{\kappa_i}} \left[ \sigma(t) e^{\frac{t}{\kappa_i}} - \sigma(0) - \int_0^t e^{\frac{\tau}{\kappa_i}} \dot{\sigma}(\tau) d\tau \right] \]

\[ = \left[ \varepsilon_i(0) - \frac{1}{E_i} \sigma(0) \right] e^{-\frac{t}{\kappa_i}} + \frac{1}{E_i} \sigma(t) - \frac{1}{E_i} \int_0^t e^{-\frac{(t-\tau)}{\kappa_i}} \dot{\sigma}(\tau) d\tau \]

\[ = \frac{1}{E_i} \left\{ \sigma(t) - \int_0^t e^{-\frac{(t-\tau)}{\kappa_i}} \dot{\sigma}(\tau) d\tau \right\} \]

By utilizing the notation of Section 5.1.2, the integral term of Eq. A.5.3.6 can be written as

\[ t_h_i = \int_0^t e^{-\frac{(t-\tau)}{\kappa_i}} \dot{\sigma}(\tau) d\tau \]

At time \( t + \Delta t \) the strain is

\[ \varepsilon_i \left( t + \Delta t \right) = \frac{1}{E_i} \left\{ \sigma \left( t + \Delta t \right) - \int_0^{t+\Delta t} e^{-\frac{(t+\Delta t-\tau)}{\kappa_i}} \dot{\sigma}(\tau) d\tau \right\} \]

Similarly, the integral term of Eq. A.5.3.8 can be written according to Section 5.1.2 as

\[ \int_0^{t+\Delta t} e^{-\frac{(t+\Delta t-\tau)}{\kappa_i}} \dot{\sigma}(\tau) d\tau = e^{-\frac{\Delta t}{\kappa_i}} t_h_i + e^{-\frac{\Delta t}{2\kappa_i}} \Delta \sigma \]

so that

\[ \varepsilon_i \left( t + \Delta t \right) = \frac{1}{E_i} \left\{ \sigma \left( t + \Delta t \right) - \varepsilon_i \left( t + \Delta t \right) \right\} \]

The strain increment at \( t + \Delta t \) is

\[ \Delta \varepsilon_i = \varepsilon_i \left( t + \Delta t \right) - \varepsilon_i (t) \]

\[ = \frac{1}{E_i} \left\{ \left( 1 - e^{-\frac{\Delta t}{\kappa_i}} \right) \Delta \sigma + \left( 1 - e^{-\frac{\Delta t}{\kappa_i}} \right) t_h_i \right\} \]
The evolution law for the strain in the viscoplastic component of Fig. 5.3 was determined in Section 5.4 as

\[ \varepsilon_{\infty}(t) = \frac{\sigma(t)}{\eta_{\infty}} \]  

A.5.4.1

On the basis of a Laplace transform it can be expressed as

\[ sE_{\infty}(s) - E_{\infty}(0) = \frac{1}{\eta_{\infty}} \Sigma(s) \]  

A.5.4.2

\[ E_{\infty}(s) = \frac{E_{\infty}(0)}{s} + \frac{1}{s} \frac{\eta_{\infty}}{\eta_{\infty}} \Sigma(s) \]  

A.5.4.3

Taking the inverse Laplace transform

\[ \varepsilon_{\infty}(t) = \mathcal{L}^{-1} \left[ E_{\infty}(s) \right] \]

\[ = \varepsilon_{\infty}(0) + \frac{1}{\eta_{\infty}} \int_{0}^{t} \sigma(\tau) d\tau \]  

A.5.4.4

\[ = \int_{0}^{t} \frac{1}{\eta_{\infty}} \sigma(\tau) d\tau \]

Integrating by parts

\[ \int_{0}^{t} \frac{1}{\eta_{\infty}} \sigma(\tau) d\tau = \frac{1}{\eta_{\infty}} \left[ \sigma(\tau) \tau \right]_{0}^{t} - \int_{0}^{t} \tau \dot{\sigma}(\tau) d\tau \]  

A.5.4.5

\[ = \frac{1}{\eta_{\infty}} \left[ \sigma(t) t - \int_{0}^{t} \tau \dot{\sigma}(\tau) d\tau \right] \]

By utilizing the notation of Section 5.1.2, the integral term of Eq. A.5.4.5 can be written as

\[ t^h_{\infty} = \int_{0}^{t} \tau \dot{\sigma}(\tau) d\tau \]  

A.5.4.6

so that

\[ \varepsilon_{\infty}(t) = \frac{1}{\eta_{\infty}} \left\{ \sigma(t) t - t^h_{\infty} \right\} \]  

A.5.4.7

At time \( t + \Delta t \) the strain is

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\[ \varepsilon_\infty (t + \Delta t) = \frac{1}{\eta_\infty} \left\{ (\sigma(t) + \Delta \sigma)(t + \Delta t) - t + \Delta t \int_0^{t + \Delta t} \tau \dot{\sigma}(\tau) \, d\tau \right\} \text{ A.5.4.8} \]

Utilizing the mid-point integration rule, the integral term of Eq. A.5.4.8 can be written as

\[
\int_0^{t + \Delta t} \tau \dot{\sigma}(\tau) \, d\tau = \int_{t}^{t + \Delta t} \tau \dot{\sigma}(\tau) \, d\tau \\
= \int_{t}^{t + \Delta t} \tau \dot{\sigma}(\tau) \, d\tau \bigg|_{\tau = t + \frac{\Delta t}{2}} \\
= t \varepsilon_\infty + \left( t + \frac{\Delta t}{2} \right) \Delta \sigma \\
= t + \Delta t \varepsilon_\infty \text{ A.5.4.9} 
\]

so that

\[ \varepsilon_\infty (t + \Delta t) = \frac{1}{\eta_\infty} \left\{ (\sigma(t) + \Delta \sigma)(t + \Delta t) - h_\infty - \left( t + \frac{\Delta t}{2} \right) \Delta \sigma \right\} \text{ A.5.4.10} \]
6.1 Introduction

In earlier Chapters constitutive models were presented for the simulation of elastoplastic and viscoelastic materials. Nevertheless, there is a great variety of engineering materials which, depending on strain rate and/or temperature, exhibit response characteristics varying anywhere between the elastoplastic and the viscoelastic limits. Asphalitic concrete, rubbery polymers and certain types of polymeric foams are typical examples.

Constitutive modelling of such types of materials can be implemented by combining the features of purely elastoplastic and purely viscoelastic materials to create a more general category of constitutive models termed \textit{elasto-visco-plastic} in this Chapter. Fig. 6.1 shows a one-dimensional schematic of the envisaged material model consisting of a single elastoplastic constituent in parallel with an arbitrary number of viscoelastic constituents. The actual number of necessary elastoplastic and viscoelastic constituents and their individual components is to be decided on the basis of the available experimental evidence.

Fig. 6.1 Generalized one dimensional elasto-visco-plastic model

Fig. 6.2 indicates schematically the model response at constant temperature. For extremely low strain rates, the elastoplastic constituent dictates inviscid response. With increasing strain rate, the viscous constituent contributes to an increase in material stiffness.
6.2 Multiplicative Decomposition

In Chapter 2, it was shown that a vector $\text{dx}$ in the deformed current configuration is related by means of the deformation gradient tensor $\mathbf{F}$ to its undeformed configuration via the relation

$$\text{dx} = \mathbf{F} \text{dX}$$  \hspace{1cm} 6.1

If it is now assumed that the forces acting on the material element are removed $^1$, the initial reference configuration will only be obtained if the material is elastic. In all other cases, another configuration will be obtained in which the original vector $\text{dX}$ is mapped onto vector $\text{dx}_r$ with the subscript $r$ indicating the residual nature of deformation, Fig. 6.3.

Let $\mathbf{F}_e$ denote the deformation gradient relating the residual deformation configuration to the current configuration. Then, following the logic of Eq. 6.1

$$\text{dx} = \mathbf{F}_e \text{dx}_r$$  \hspace{1cm} 6.2

Similarly, if $\mathbf{F}_r$ denotes the deformation gradient relating the residual deformation configuration to the reference configuration, then it also holds

$$\text{dx}_r = \mathbf{F}_r \text{dX}$$  \hspace{1cm} 6.3

so that

$$\text{dx} = \mathbf{F}_e \mathbf{F}_r \text{dX}$$  \hspace{1cm} 6.4

and therefore

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_r$$  \hspace{1cm} 6.5

---

$^1$ Without the development of residual stresses due to compatibility
Fig. 6.3 Multiplicative decomposition of the deformation gradient

The process represented by Eq. 6.5 is known as the "multiplicative decomposition" of the deformation gradient to a residual deformation component and a component signifying the elastic unloading that the material must undergo from the configuration at time \( t \) to the residual configuration.

The concept of multiplicative decomposition of the deformation gradient provides an elegant tool for description of the three dimensional response of elasto-visco-plastic material models consisting of elastoplastic and viscoelastic components.

Fig. 6.4 Multiplicative decomposition of the deformation gradient of an elasto-visco-plastic material
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As shown schematically in Fig. 6.4, the deformation gradient of a material in which the elastoplastic and the viscoelastic components act in parallel can be decomposed as

\[ \mathbf{F} = \mathbf{F}_\infty \mathbf{F}_p \quad ; \quad \mathbf{F} = \mathbf{F}_e \mathbf{F}_v \quad (6.6) \]

in which \( \mathbf{F}_\infty \) = the elastic component of the deformation gradient of the elastoplastic element

\( \mathbf{F}_p \) = the plastic component of the deformation gradient of the elastoplastic element

\( \mathbf{F}_e \) = the elastic component of the deformation gradient of the viscoelastic element

\( \mathbf{F}_v \) = the viscous component of the deformation gradient of the viscoelastic element

Furthermore the following definition hold:

\[ \mathbf{C}_\infty = \mathbf{F}_\infty^T \mathbf{F}_\infty \quad \quad \mathbf{C}_e = \mathbf{F}_e^T \mathbf{F}_e \]

\[ \mathbf{C}_p = \mathbf{F}_p^T \mathbf{F}_p \quad ; \quad \mathbf{C}_v = \mathbf{F}_v^T \mathbf{F}_v \quad (6.7) \]

therefore

\[ \mathbf{C} = \mathbf{F}^T \mathbf{F} = \mathbf{F}_v^T \mathbf{C}_e \mathbf{F}_v \]

\[ = \mathbf{F}_p^T \mathbf{C}_\infty \mathbf{F}_p \quad (6.8) \]

6.3 Generalized Model Local Dissipation

The Helmholtz free energy function for a three dimensional model equivalent to the generalised model of Fig. Fig. 6.1 can be expressed as

\[ \Psi = \Psi_v (\mathbf{C}_e) + \Psi_p (\mathbf{C}_\infty, \xi_p) \quad (6.9) \]

The Clausius-Planck local dissipation inequality leads to

\[ \mathbf{S} : \frac{1}{2} \dot{\mathbf{C}} - \left[ \frac{\partial \Psi_v}{\partial \mathbf{C}_e} \dot{\mathbf{C}}_e - \frac{\partial \Psi_p}{\partial \mathbf{C}_\infty} \dot{\mathbf{C}}_\infty \right] - \frac{\partial \Psi_p}{\partial \xi_p} \dot{\xi}_p \geq 0 \quad (6.10) \]

As shown in Appendix 6.1 the above inequality can be reformulated as

\[ \left[ \mathbf{S} - 2 \mathbf{F}_v^{-1} \frac{\partial \Psi_v}{\partial \mathbf{C}_e} \mathbf{F}_v^{-T} - 2 \mathbf{F}_p^{-1} \frac{\partial \Psi_p}{\partial \mathbf{C}_\infty} \mathbf{F}_p^{-T} \right] : \frac{1}{2} \dot{\mathbf{C}} \]

\[ + \left[ 2 \mathbf{F}_e \frac{\partial \Psi_v}{\partial \mathbf{C}_e} \mathbf{F}_v^T \mathbf{F}_e^{-T} : \mathbf{F}_e \dot{\mathbf{C}}_e \right] \quad (6.11) \]

\[ + \left[ 2 \mathbf{F}_\infty \frac{\partial \Psi_p}{\partial \mathbf{C}_\infty} \mathbf{F}_\infty^T \mathbf{F}_\infty^{-T} : \mathbf{F}_\infty \dot{\mathbf{C}}_\infty - \frac{\partial \Psi_p}{\partial \xi_p} \dot{\xi}_p \right] \geq 0 \]
By standard arguments, Coleman & Gurtin [1967], on the basis of Eq. 6.11, the stress tensor \( S \) can be additively decomposed into a viscoelastic \( S_v \) and a plastic component \( S_p \).

\[
S = 2F_v^{-1} \frac{\partial \Psi_v}{\partial C_v} F_v^{-T} + 2F_p^{-1} \frac{\partial \Psi_p}{\partial C_p} F_p^{-T}
= S_v + S_p
\]

Also the following inequalities are obtained

\[
2F_e \frac{\partial \Psi_v}{\partial C_e} F_e^T F_e^{-T} : F_e l_e \geq 0
\]

\[
2F_\infty \frac{\partial \Psi_p}{\partial C_\infty} F_\infty^T F_\infty^{-T} : F_\infty l_\infty - \frac{\partial \Psi_p}{\partial \xi_p} \dot{\xi}_p \geq 0
\]

### 6.4 Plastic Component Integration Procedure

As it is shown in Appendix 6.2, inequality 6.14 can be written as

\[
\tau_\infty : \left[ -\frac{1}{2} \mathcal{L}(b_\infty) \right] b_\infty^{-1} + q : \dot{\xi}_p \geq 0
\]

with \( q = -\frac{\partial \Psi_p}{\partial \xi_p} \).

The principle of maximum plastic dissipation mentioned in Chapter 4, states that for a given set of \( \left[ -\frac{1}{2} \mathcal{L}(b_\infty) \right] b_\infty^{-1}, \dot{\xi}_p \), among all possible sets \( (\tau_\infty, q) \) in the stress space domain \( \omega \) satisfying the condition

\[
\omega = \left\{ \tau_\infty : f(\tau_\infty, q) \leq 0 \right\}
\]

the actual one is the one which maximizes the argument of inequality 6.15, that is, the one for which

\[
\tau_\infty \in \omega : \max \left( \tau_\infty : \left[ -\frac{1}{2} \mathcal{L}(b_\infty) \right] b_\infty^{-1} + q : \dot{\xi}_p \right)
\]

Utilizing the formal notation of Appendix 4.1, Simo [1992] has pointed out that the above two relations can be recast as the following constraint minimization problem

\[
\begin{align*}
\text{minimize} & \quad - \left( \tau_\infty : \left[ -\frac{1}{2} \mathcal{L}(b_\infty) \right] b_\infty^{-1} + q : \dot{\xi}_p \right) \\
\text{subject to} & \quad f(\tau_\infty, q) \leq 0
\end{align*}
\]

According to Appendix 4.1, the above minimization statement is equivalent to the following set of plastic evolution equations, Simo [1998]
\[-\frac{1}{2} \mathcal{L}(b_\infty) = \lambda \left( \frac{\partial f}{\partial \tau_\infty} \right) b_\infty \]

\[\dot{\xi} = \lambda \left( \frac{\partial f}{\partial q} \right)\]

\[\lambda \geq 0 \quad ; \quad f(\tau_\infty, q) \leq 0 \quad ; \quad \lambda f(\tau_\infty, q) = 0\]

in which \( \lambda \) is the plastic consistency parameter and \( f(\tau_\infty, q) \) is a flow surface function. The implemented functions are listed in a latter section.

By utilizing the relation \( b_\infty = FC_p^{-1} F^T \) and the definition of the Lie derivative

\[-\frac{1}{2} \mathcal{L}(b_\infty) = -\frac{1}{2} F \left[ \frac{\partial}{\partial t} \left( C_p^{-1} \right) \right] F^T = \lambda \left( \frac{\partial f}{\partial \tau_\infty} \right) b_\infty = \lambda \left( \frac{\partial f}{\partial \tau_\infty} \right) FC_p^{-1} F^T\]

from which

\[\frac{\partial}{\partial t} \left( C_p^{-1} \right) = -2 \lambda F^{-1} \left( \frac{\partial f}{\partial \tau_\infty} \right) FC_p^{-1}\]

On the basis of the terminology of Chapter 2, the vector \( N = F^T \left( \frac{\partial f}{\partial \tau_\infty} \right) F \) represents the pull-back of the vector \( \left( \frac{\partial f}{\partial \tau_\infty} \right) \) to the reference configuration. Introducing \( N \) into Eq. 6.21 it results, Simo [1987a]

\[\frac{\partial}{\partial t} \left( C_p^{-1} \right) = -2 \lambda C^{-1} NC_p^{-1}\]

6.4.1 Stress Reduction Procedure

The procedure that is utilized for reduction of the state of stress to the plastic response surface is presented in the following.

6.4.1.1 Trial Elastic State

If it is temporarily assumed that during the motion in the time interval \([t, t+\Delta t]\) no further plastic deformation takes place, i.e. if it is temporarily set

\[t+\Delta t F_p = t F_p \quad ; \quad t+\Delta t \xi = t \xi \]

then, an approximate elastic deformation gradient can be computed as

\[t+\Delta t F_p = t + \Delta t F_p^{-1} \]

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The corresponding elastic left Cauchy-Green tensor is therefore

\[
^{t+\Delta t}_{\text{trial}} b_\infty = ^{t+\Delta t}_{\text{trial}} F_\infty^{t+\Delta t}_{\text{trial}} F_\infty^T = ^{t+\Delta t}_{\text{trial}} F^{t+\Delta t}_{\text{trial}} F_\infty^{t+\Delta t}_{\text{trial}} F_\infty^T
\]

which is clearly a push forward operation on \(^{t+\Delta t}_{\text{trial}} C_p^{-1}\).

**6.4.1.2 Flow Rule Discretization**

The evolution problem defined by Eq. 6.22 can be solved in the time interval \([t, t+\Delta t]\) to give a first order accurate estimate for \(^{t+\Delta t}_{\text{trial}} C_p^{-1}\)

\[
^{t+\Delta t}_{\text{trial}} C_p^{-1} = \exp \left[-2\Delta t \lambda \left( ^{t+\Delta t}_{\text{trial}} C^{-1} \right)^{t+\Delta t}_{\text{trial}} N \right] ^{t+\Delta t}_{\text{trial}} C_p^{-1}
\]

with

\[
^{t+\Delta t}_{\text{trial}} N = ^{t+\Delta t}_{\text{trial}} F^T \left( \frac{\partial f}{\partial \tau_\infty} \right)^{t+\Delta t}_{\text{trial}} F
\]

Details can be found in Appendix 6.7. Setting \(\Delta t = \Delta \lambda\), on the basis of Eq. 6.25 and Eq. 6.26, the elastic left Cauchy-Green tensor is

\[
^{t+\Delta t}_{\text{trial}} b_\infty = ^{t+\Delta t}_{\text{trial}} F^{t+\Delta t}_{\text{trial}} C_p^{-1}^{t+\Delta t}_{\text{trial}} F_\infty^T
\]

\[
= ^{t+\Delta t}_{\text{trial}} F \exp \left[-2\Delta \lambda \left( ^{t+\Delta t}_{\text{trial}} C^{-1} \right)^{t+\Delta t}_{\text{trial}} N \right] ^{t+\Delta t}_{\text{trial}} F_\infty^{-1}^{t+\Delta t}_{\text{trial}} b_\infty
\]

\[
= \exp \left[-2\Delta \lambda \left( ^{t+\Delta t}_{\text{trial}} F \left( ^{t+\Delta t}_{\text{trial}} C^{-1} \right)^{t+\Delta t}_{\text{trial}} N \right) \right] ^{t+\Delta t}_{\text{trial}} b_\infty
\]

\[
= \exp \left[-2\Delta \lambda \left( \frac{\partial f}{\partial \tau_\infty} \right) \right] ^{t+\Delta t}_{\text{trial}} b_\infty
\]

Multiplying both sides of Eq. 6.28 by \(\exp \left[2\Delta \lambda \left( \frac{\partial f}{\partial \tau_\infty} \right) \right] \)

\[
^{t+\Delta t}_{\text{trial}} b_\infty = \exp \left[2\Delta \lambda \left( \frac{\partial f}{\partial \tau_\infty} \right) \right] ^{t+\Delta t}_{\text{trial}} b_\infty
\]

Because of isotropy

\[
^{t+\Delta t} \left( \frac{\partial f(\tau_\infty, q)}{\partial \tau_\infty} \right) = \sum_{i=1}^{3} ^{t+\Delta t} \left( \frac{\partial f(\tau_{\alpha, i}, \tau_{\alpha, 2}, \tau_{\alpha, 3}, q)}{\partial \tau_{\alpha, i}} \right) l_i \otimes l_i
\]

\[
= 6.30
\]
in which $\tau_{\infty,i}$ indicates the $i$-th principal stress of the stress tensor $\tau_{\infty}$ and $l_i = 1, 2, 3$ are the corresponding principal directions.

In addition $t^{+\Delta t}\mathbf{b}_{\infty}$ can also be expressed as

$$t^{+\Delta t}\mathbf{b}_{\infty} = \sum_{i=1}^{3} t^{+\Delta t} \left( \lambda_{\infty,i}^2 l_i \otimes l_i \right)$$  \hspace{1cm} 6.31

in which $\lambda_{\infty,i}$, $i = 1, 2, 3$ are the elastic principal stretches.

Substituting Eq. 6.30 and Eq. 6.31 into Eq. 6.29 it results

$$t^{+\Delta t}\mathbf{b}_{\infty}^{\text{trial}} = \exp \left[ 2 \Delta \lambda \left( \sum_{i=1}^{3} \frac{\partial f(\tau_{\infty,1}, \tau_{\infty,2}, \tau_{\infty,3}, q)}{\partial \tau_{\infty,i}} \right) l_i \otimes l_i \right] \sum_{i=1}^{3} \left( \lambda_{\infty,i}^2 l_i \otimes l_i \right)$$ \hspace{1cm} 6.32

which can be simplified (Appendix 6.4) as

$$t^{+\Delta t}\mathbf{b}_{\infty}^{\text{trial}} = \exp \left[ 2 \Delta \lambda \sum_{i=1}^{3} \frac{\partial f(\tau_{\infty,1}, \tau_{\infty,2}, \tau_{\infty,3}, q)}{\partial \tau_{\infty,i}} \right] \lambda_{\infty,i}^2 l_i \otimes l_i$$ \hspace{1cm} 6.33

By means of spectral decomposition $t^{+\Delta t}\mathbf{b}_{\infty}^{\text{trial}}$ can be also expressed as

$$t^{+\Delta t}\mathbf{b}_{\infty}^{\text{trial}} = \sum_{i=1}^{3} \lambda_{\infty,i}^2 l_i \otimes l_i$$ \hspace{1cm} 6.34

Comparing Eq. 6.33 and Eq. 6.34 it follows

$$(t^{+\Delta t}_{\text{trial}} \lambda_{\infty,i})^2 = \exp \left[ 2 \Delta \lambda \sum_{i=1}^{3} \frac{\partial f(\tau_{\infty,1}, \tau_{\infty,2}, \tau_{\infty,3}, q)}{\partial \tau_{\infty,i}} \right] \lambda_{\infty,i}^2$$ \hspace{1cm} 6.35

Taking the logarithm of both sides of Eq. 6.35 and denoting the tensor of elastic principal logarithmic strains by $\mathbf{e}_{\infty,i}$

$$t^{+\Delta t}\mathbf{e}_{\infty,i}^{\text{trial}} = t^{+\Delta t}_{\text{trial}} \mathbf{e}_{\infty,i} + \Delta \lambda \left( \frac{\partial f(\tau_{\infty,i}, q)}{\partial \tau_{\infty,i}} \right)$$ \hspace{1cm} 6.36

Also on the basis of Eq. 6.192, a backward Euler integration scheme results to the following algorithmic scheme

$$t^{+\Delta t}\mathbf{e} = t^\mathbf{e} + \Delta \lambda \left( \frac{\partial f}{\partial q} \right)$$ \hspace{1cm} 6.37
6.4.1.3 Return Mapping Procedure

On the basis of the above the following residual equations can be set up

\[ r_{\varepsilon} = t^{+\Delta t} \varepsilon_{\infty,i} - t^{+\Delta t} \varepsilon_{\text{trial,}\infty,i} + t^{+\Delta t} \Delta \lambda \left( \frac{\partial f}{\partial \tau_{\infty,i}} \right) \]

\[ r_{\xi} = -t^{+\Delta t} \xi + t^{+\Delta t} \Delta \lambda \left( \frac{\partial f}{\partial q} \right) \]

\[ r_{f} = t^{+\Delta t} f(\tau_{\infty,i}; q) \]

with \( t^{+\Delta t} \tau_{\infty,i} = \left[ \frac{\partial \psi(e_{\infty,i}+\xi)}{\partial e_{\infty,i}} \right] \); \( t^{+\Delta t} q = -\left[ \frac{\partial \psi(e_{\infty,i}+\xi)}{\partial \xi} \right] \)

6.38

From Eq.6.38_2

\[ \Delta \xi = t^{+\Delta t} \xi - t^{+\Delta t} \xi = t^{+\Delta t} \Delta \lambda \left( \frac{\partial f}{\partial q} \right) \]

6.40

hence

\[ \Delta \lambda = \frac{\Delta \xi}{(\partial f/\partial q)} \]

6.41

Substituting into Eq.6.38_1, after rearrangement

\[ r_{\varepsilon} = \left[ \frac{\partial f}{\partial q} \right] (e_{\infty,i} + \text{trial,}\varepsilon_{\infty,i}) + \Delta \xi \left( \frac{\partial f}{\partial \tau_{\infty,i}} \right) \]

6.42

Consider the Newton iterative scheme

\[ t^{+\Delta t}_{n+1} \mathbf{x} = t^{+\Delta t}_{n} \mathbf{x} + \Delta \mathbf{x} \]

6.43

in which

\[ t^{+\Delta t}_{n} \mathbf{x} = \left[ t^{+\Delta t}_{n} \varepsilon_{\infty,i} \right]^{T} \]

6.44

with initial values

\[ t^{+\Delta t}_{0} \mathbf{x} = \left[ t^{+\Delta t}_{0} \varepsilon_{\text{trial,i}} \right] \]

6.45

Define

\[ \Delta \mathbf{x} = -J^{-1} r \left( t^{+\Delta t}_{n} \mathbf{x} \right) \]

6.46

and the residual vector

\[ r \left( t^{+\Delta t}_{n} \mathbf{x} \right) = \begin{bmatrix} r_{\varepsilon} \\ r_{\xi} \end{bmatrix} \]

6.47

As shown in Appendix 6.3, the Jacobian of the iterative scheme is

137
The above iterative scheme is repeated until a preset tolerance is attained.

### 6.4.2 Characteristics of the Plastic Component

The following yield function is currently implemented in CAPA-3D

\[
f(\mathbf{\tau}_{\infty, i}, q) = \sqrt{\frac{3}{2}} \left\| \mathbf{s}_{\infty, i} \right\| - \left( \mathbf{\tau}_{y_0} - q \right)
\]

with \( \left\| \mathbf{s}_{\infty, i} \right\| = \left( \mathbf{s}_{\infty, i} \cdot \mathbf{s}_{\infty, i} \right)^{\frac{1}{2}} \) and \( \mathbf{s}_{\infty, i} = \text{dev}(\mathbf{\tau}_{\infty, i}) = \left( I - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right) : \mathbf{\tau}_{\infty, i} \)

The specified strain energy function is of the form

\[
\psi_p(\mathbf{\varepsilon}_{\infty, i}, \xi) = \frac{\kappa}{2} \left( \sum_{i=1}^{3} \mathbf{\varepsilon}_{\infty, i} \right)^2 + \mu \left( \sum_{i=1}^{3} \mathbf{\varepsilon}_{\infty, i}^2 \right) +
\]

\[
\left( \mathbf{\tau}_{y_0} - \mathbf{\tau}_{y_\infty} \right) \left[ \xi + \frac{1}{\delta} \left( \exp^{-\delta \xi} - 1 \right) \right]
\]

in which the exponent \( \delta \) controls the rate of hardening, Fig. 6.5

![Fig. 6.5 The influence of \( \delta \) on strain hardening response](image-url)
6.5 Viscoelastic Component Integration Procedure

In similarity to the plastic component, inequality 6.13 is equivalent to

$$\tau_e : \left[ -\frac{1}{2} \mathcal{L}(b_e) \right] b_e^{-1} \geq 0$$

6.52

By observing that condition 6.52 is satisfied for quadratic forms $^1 a : A = a$, Reese & Govindjee [1998] have proposed the evolution law

$$\left[ -\frac{1}{2} \mathcal{L}(b_e) \right] = C_v^{-1} : \tau_e b_e$$

6.53

with

$$C_v^{-1} = \frac{1}{2\eta_D} \left[ 1 - \frac{1}{3} I \otimes I \right] + \frac{1}{9\eta_V} I \otimes I$$

6.54

while $\eta_D$ and $\eta_V$ are the deviatoric and volumetric viscosities which may be deformation dependent

$$\eta_D = \eta_D(b_e) > 0 \quad ; \quad \eta_V = \eta_V(b_e) > 0$$

6.55

By utilizing the relation $b_e = FC_v^{-1} F^T$ and the definition of the Lie derivative

$$-\frac{1}{2} \mathcal{L}(b_e) = -\frac{1}{2} F \left[ \frac{\partial}{\partial t} \left( C_v^{-1} \right) \right] F^T = C_v^{-1} : \tau_e b_e = C_v^{-1} : \tau_e F C_v^{-1} F^T$$

6.56

Eq. 6.53 can be expressed in the reference configuration as

$$\frac{\partial}{\partial t} \left( C_v^{-1} \right) = -2 F^{-1} \left[ C_v^{-1} : \tau_e \right] F C_v^{-1}$$

6.57

6.5.1 Stress Reduction Procedure

The procedure that is utilized for determination of the state of stress of the viscoelastic component is presented in the following.

6.5.1.1 Trial Elastic State

If it is temporarily assumed that during the motion in the time interval $[t, t + \Delta t]$ no further viscous deformation takes place, i.e. if it is temporarily set

$$t^{+\Delta t} \mathbf{F}_v = t \mathbf{F}_v$$

6.58

then, an approximate elastic deformation gradient can be computed as

$$t^{+\Delta t}_{trial} \mathbf{F}_e = t^{+\Delta t} \mathbf{F} \mathbf{F}_v^{-1}$$

6.59

---

$^1$ with $a$ a second order tensor and $A$ a fourth order tensor
The corresponding elastic left Cauchy-Green tensor is therefore

$$
t_{\text{trial}}b_e^{t+\Delta t} = t_{\text{trial}}F_{e}^t t_{\text{trial}}F_{e}^T
= t^{+\Delta t}F^{t}C_v^{-1}^{t+\Delta t}F^T
$$

6.60

### 6.5.1.2 Flow Rule Discretization

The evolution problem defined by Eq. 6.57 can be solved in the time interval $[t, t + \Delta t]$ to give a first order accurate estimate for $t^{+\Delta t}C_v^{-1}$

$$
t^{+\Delta t}C_v^{-1} = t^{+\Delta t}\left[\exp\left[-2\Delta t F^{-1}\left(C_v^{-1} : \tau_e\right)F\right]^{t}C_v^{-1}\right]
$$

6.61

On the basis of Eq. 6.59 and Eq.6.61, the elastic left Cauchy-Green tensor is

$$
t^{+\Delta t}b_e = t^{+\Delta t}\left|FC_v^{-1}F^T\right|
= t^{+\Delta t}F\exp\left[t^{+\Delta t}-2\Delta t F^{-1}\left(C_v^{-1} : \tau_e\right)F\right]^{t}C_v^{-1}^{t+\Delta t}F^T
= t^{+\Delta t}\left|F\exp\left[-2\Delta t F^{-1}\left(C_v^{-1} : \tau_e\right)F\right]F^{-1}\right|_{\text{trial}}b_e
= t^{+\Delta t}\left|\exp\left[-2\Delta t\left(C_v^{-1} : \tau_e\right)\right]\right|_{\text{trial}}b_e
$$

6.62

Multiplying both sides by $\left|\exp\left[2\Delta t\left(C_v^{-1} : \tau_e\right)\right]\right|_{\text{trial}}b_e$

$$
t_{\text{trial}}b_e^{t+\Delta t} = \left|\exp\left[2\Delta t\left(C_v^{-1} : \tau_e\right)\right]\right|b_e
$$

6.63

In similarity to Section 6.4.1.2, a significant reduction in the number of unknowns can be achieved by resorting to a formulation in principal stress space. It holds

$$
C_v^{-1} : \tau_e = \left[\frac{1}{2\eta_p}\left(I - \frac{1}{3}I \otimes I\right) + \frac{1}{9\eta_v}I \otimes I\right] : \tau_e
= \frac{1}{2\eta_p}\text{dev}(\tau_e) + \frac{1}{9\eta_v}\text{tr}(\tau_e)I
= \frac{1}{2\eta_p}\sum_{i=1}^{3}s_{ei}l_i \otimes l_i + \frac{1}{9\eta_v}\sum_{i=1}^{3}\sigma_m l_i \otimes l_i
$$

6.64

$$
= \sum_{i=1}^{3}\left(\frac{1}{2\eta_p}s_{ei} + \frac{1}{9\eta_v}\sigma_m\right)l_i \otimes l_i
$$

in which $s_{ei} = \sigma_{ei} - \sigma_m$ with $\sigma_{ei}$ the i-th principal component of the stress tensor $\sigma_e$ and $l_i = 1, 2, 3$ the corresponding principal directions.
In addition \( t+\Delta t \mathbf{b}_e \) can also be expressed as

\[
\mathbf{b}_e = \sum_{i=1}^{3} (t+\Delta t) \left( \lambda_{e,i}^2 \mathbf{l}_i \otimes \mathbf{l}_i \right)
\]

in which \( \lambda_{\infty,i} \), \( i = 1,2,3 \) are the elastic principal stretches.

Substituting Eq. 6.64 and Eq. 6.65 into Eq. 6.63 it results

\[
t_{\text{trial}}^{t+\Delta t} \mathbf{b}_e = \exp \left[ 2 \Delta t \sum_{i=1}^{3} \left( \frac{1}{2 \eta_0} s_{e,i} + \frac{1}{9 \eta_V} \sigma_m \right) \mathbf{l}_i \otimes \mathbf{l}_i \right] \sum_{i=1}^{3} \left( \lambda_{e,i}^2 \mathbf{l}_i \otimes \mathbf{l}_i \right)
\]

which can be simplified (Appendix 6.4) as

\[
t_{\text{trial}}^{t+\Delta t} \mathbf{b}_e = \sum_{i=1}^{3} \exp \left( 2 \Delta t \left( \frac{1}{2 \eta_0} s_{e,i} + \frac{1}{9 \eta_V} \sigma_m \right) \lambda_{e,i}^2 \right) \mathbf{l}_i \otimes \mathbf{l}_i
\]

By means of spectral decomposition \( t_{\text{trial}}^{t+\Delta t} \mathbf{b}_e \) can be also expressed as

\[
t_{\text{trial}}^{t+\Delta t} \mathbf{b}_e = \sum_{i=1}^{3} \lambda_{e,i}^2 \mathbf{l}_i \otimes \mathbf{l}_i
\]

Comparing Eq. 6.67 and Eq. 6.68 it follows

\[
\left( t_{\text{trial}}^{t+\Delta t} \lambda_{e,i} \right)^2 = \exp \left( 2 \Delta t \left( \frac{1}{2 \eta_0} s_{e,i} + \frac{1}{9 \eta_V} \sigma_m \right) \lambda_{e,i}^2 \right)
\]

Taking the logarithm of both sides of Eq. 6.69 and denoting the elastic principal logarithmic strains by \( \mathbf{e}_{e,i} = \ln(\lambda_{e,i}) \), Eq. 6.69 can be expressed as

\[
t_{\text{trial}}^{t+\Delta t} \mathbf{e}_{e,i} = t_{\text{trial}}^{t+\Delta t} \mathbf{e}_{e,i} - \Delta t \left( \frac{1}{2 \eta_0} s_{e,i} + \frac{1}{9 \eta_V} \sigma_m \right)
\]

6.5.1.3 Return Mapping Procedure

On the basis of the above the following system of residual equations can be set up

\[
r_{e_{e,i}} = t_{\text{trial}}^{t+\Delta t} \mathbf{e}_{e,i} - t_{\text{trial}}^{t+\Delta t} \mathbf{e}_{e,i} + \Delta t \left( \frac{1}{2 \eta_0} s_{e,i} + \frac{1}{9 \eta_V} \sigma_m \right) \quad i = 1,2,3
\]

each of which can be solved iteratively by means of Newton iterative scheme

\[
t_{n+1}^{t+\Delta t} \mathbf{x} = t_n^{t+\Delta t} \mathbf{x} + \Delta \mathbf{x}
\]
Elasto-Visco-Plasticity

\[
\begin{align*}
    t+\Delta t \quad &\begin{pmatrix}
        \frac{t+\Delta t}{n}x &= \begin{pmatrix}
        t+\Delta t \quad &\begin{pmatrix}
            \frac{t+\Delta t}{n}e_{\infty,i} \\
        \end{pmatrix}_n
    \end{pmatrix} \\
    \text{with initial values} \quad &\begin{pmatrix}
        t+\Delta t \quad &\begin{pmatrix}
            \frac{t+\Delta t}{n}x &= \begin{pmatrix}
        t+\Delta t \quad &\begin{pmatrix}
            \text{trial} \quad &\begin{pmatrix}
            e_{\infty,i} \\
        \end{pmatrix}
    \end{pmatrix}_n
    \end{pmatrix}
    \end{pmatrix}
\end{align*}
\]

Define

\[
\Delta x = - \left[ J \left( \begin{pmatrix}
        t+\Delta t \\
        n \end{pmatrix}x \right) \right]^{-1} \begin{pmatrix}
        t+\Delta t \\
        n \end{pmatrix}r \begin{pmatrix}
        t+\Delta t \\
        n \end{pmatrix}x
\]

with the residual vector

\[
\begin{pmatrix}
        t+\Delta t \\
        n \end{pmatrix}r = \begin{pmatrix}
        r_{\infty,i}
    \end{pmatrix}
\]

As shown in Appendix 6.6, the Jacobian of the iterative scheme is

\[
J \left( \begin{pmatrix}
        t+\Delta t \\
        n \end{pmatrix}x \right) = \begin{pmatrix}
        t+\Delta t \\
        n \end{pmatrix} \nabla r =
\begin{pmatrix}
    \frac{\partial \tau_{e,1}}{\partial \varepsilon_{e,1}} & \frac{\partial \tau_{e,1}}{\partial \varepsilon_{e,2}} & \frac{\partial \tau_{e,1}}{\partial \varepsilon_{e,3}} \\
    \frac{\partial \tau_{e,2}}{\partial \varepsilon_{e,1}} & \frac{\partial \tau_{e,2}}{\partial \varepsilon_{e,2}} & \frac{\partial \tau_{e,2}}{\partial \varepsilon_{e,3}} \\
    \frac{\partial \tau_{e,3}}{\partial \varepsilon_{e,1}} & \frac{\partial \tau_{e,3}}{\partial \varepsilon_{e,2}} & \frac{\partial \tau_{e,3}}{\partial \varepsilon_{e,3}} \\
\end{pmatrix}
\]

6.6 Utilization

The contribution to the overall response of the individual components of a model consisting of one viscoelastic and one elastoplastic component (without hardening) is shown in Fig. 6.6. The model is first subjected to a constant displacement rate 1.25*E-04 mm/sec. The characteristics of the components are:

\[
\begin{align*}
    E_\infty &= 0.1 \text{ MPa} & \tau_{\gamma_0} &= 2 \cdot 10^{-5} \text{ MPa} \\
    E_e &= 0.1 \text{ MPa} & \eta_D &= 0.1 \text{ MPa sec} & \eta_V &= 1 \text{ MPa sec}
\end{align*}
\]

![Fig. 6.6 Contribution of model components to overall response](image-url)
Fig. 6.7  The influence of strain rate on model response

The influence of strain rate on the response of the same model is shown in Fig. 6.7. The contribution of the slider can be easily recognized from the figure. Upon unloading (when the plastic component is inactive) the response of the model is typical of that of a linear comparison solid.

Implementation details of this particular model are presented in Appendix 6.5.
Appendix 6.1

Verification of Eq. 6.11 is shown in the following. From Eq. 6.7

\[ \mathbf{C}_\infty = \mathbf{F}_p^{-T} \mathbf{C} \mathbf{F}_p^{-1} \quad \text{hence} \quad \dot{\mathbf{C}}_\infty = \dot{\mathbf{F}}_p^{-T} \mathbf{C} \mathbf{F}_p^{-1} + \mathbf{F}_p^{-T} \dot{\mathbf{C}} \mathbf{F}_p^{-1} + \mathbf{F}_p^{-T} \mathbf{C} \dot{\mathbf{F}}_p^{-1} \]  

A.6.1.1

so that the term \( \frac{\partial \Psi_p}{\partial C_\infty} : \dot{\mathbf{C}}_\infty \) of Eq. 6.10 can be expressed as

\[ \frac{\partial \Psi_p}{\partial C_\infty} : \dot{\mathbf{C}}_\infty = \frac{\partial \Psi_p}{\partial C_\infty} : \left[ \dot{\mathbf{F}}_p^{-T} \mathbf{C} \mathbf{F}_p^{-1} + \mathbf{F}_p^{-T} \dot{\mathbf{C}} \mathbf{F}_p^{-1} + \mathbf{F}_p^{-T} \mathbf{C} \dot{\mathbf{F}}_p^{-1} \right] \]  

A.6.1.2

By means of the double contraction identities \( \mathbf{A} : (\mathbf{B} \mathbf{C}) = (\mathbf{B}^T \mathbf{A}) : \mathbf{C} = (\mathbf{A} \mathbf{C}^T) : \mathbf{B} \) the term \( \frac{\partial \Psi_p}{\partial C_\infty} : \mathbf{F}_p^{-T} \dot{\mathbf{C}} \mathbf{F}_p^{-1} \) can be transformed as follows

\[ \frac{\partial \Psi_p}{\partial C_\infty} : \mathbf{F}_p^{-T} \dot{\mathbf{C}} \mathbf{F}_p^{-1} = \mathbf{F}_p^{-1} \frac{\partial \Psi_p}{\partial C_\infty} : \dot{\mathbf{C}} \mathbf{F}_p^{-1} \]  

A.6.1.3

\[ = \mathbf{F}_p^{-1} \frac{\partial \Psi_p}{\partial C_\infty} : \mathbf{F}_p^{-T} : \dot{\mathbf{C}} \]

By means of the identities \( \dot{\mathbf{F}}_p^{-1} = -\mathbf{F}_p^{-1} \dot{\mathbf{F}}_p^{-1} \) and \( \dot{\mathbf{F}}_p^{-T} = -\mathbf{F}_p^{-T} \dot{\mathbf{F}}_p^T \mathbf{F}_p^{-T} \) the remaining terms of Eq. A.6.1.2 can be also restructured as

\[ \dot{\mathbf{F}}_p^{-T} \mathbf{C} \mathbf{F}_p^{-1} + \mathbf{F}_p^{-T} \dot{\mathbf{C}} \mathbf{F}_p^{-1} = -\left[ \mathbf{F}_p^{-T} \dot{\mathbf{F}}_p^T \mathbf{F}_p^{-T} \mathbf{C} \mathbf{F}_p^{-1} + \mathbf{F}_p^{-T} \mathbf{C} \mathbf{F}_p^{-1} \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} \right] \]  

A.6.1.4

so that
\[
\frac{\partial \Psi_p}{\partial C_\infty} : \hat{F}_p^{-T} C \hat{F}_p^{-1} + \hat{F}_p^{-T} C \hat{F}_p^{-1} = -2 \frac{\partial \Psi_p}{\partial C_\infty} : t_p^T C_\infty + C_\infty t_p
\]
\[
= -2 \frac{\partial \Psi_p}{\partial C_\infty} : C_\infty t_p
\]
\[
= -2 \left[ F_\infty^{-1} F_\infty \right] \frac{\partial \Psi_p}{\partial C_\infty} : C_\infty t_p
\]
\[
= -2 F_\infty \frac{\partial \Psi_p}{\partial C_\infty} : F_\infty^{-T} C_\infty t_p
\]
\[
= -2 F_\infty \frac{\partial \Psi_p}{\partial C_\infty} : F_\infty t_p
\]
\[
= -2 F_\infty \frac{\partial \Psi_p}{\partial C_\infty} : F_\infty^{-T} F_\infty F_\infty^{-T} : F_\infty t_p
\]

Similarly, for the viscous components of Eq. 6.10 it holds
\[
\frac{\partial \Psi_v}{\partial C_e} : \dot{C}_e = \frac{\partial \Psi_v}{\partial C_e} : \left[ \hat{F}_v^{-T} C \hat{F}_v^{-1} + \hat{F}_v^{-T} \dot{C} \hat{F}_v^{-1} + \hat{F}_v^{-T} \hat{F}_v^{-1} \right]
\]
\[
= \frac{\partial \Psi_v}{\partial C_e} : \hat{F}_v^{-T} \dot{C} \hat{F}_v^{-1} - \frac{\partial \Psi_v}{\partial C_e} : \hat{F}_v^{-T} \hat{F}_v^{-1} : \dot{C}
\]
\[
\text{but}
\frac{\partial \Psi_v}{\partial C_e} : \hat{F}_v^{-T} \dot{C} \hat{F}_v^{-1} = F_e^{-1} \frac{\partial \Psi_v}{\partial C_e} : F_e^{-T} : \dot{C}
\]
\[
\text{and}
\frac{\partial \Psi_v}{\partial C_e} : \left[ \hat{F}_v^{-T} C \hat{F}_v^{-1} + \hat{F}_v^{-T} \hat{F}_v^{-1} \right] = -2 F_e \frac{\partial \Psi_v}{\partial C_e} : F_e^{-T} : F_e t_v
\]

Substituting Eq. A.6.1.3, Eq. A.6.1.5, Eq. A.6.1.7 and Eq. A.6.1.8 into Eq. 6.10, Eq. 6.11 is obtained as
\[
\left[ S - 2 F_e^{-1} \frac{\partial \Psi_v}{\partial C_e} : \dot{F}_e - 2 F_p^{-1} \frac{\partial \Psi_p}{\partial C_\infty} : \dot{F}_p \right] : \dot{C}
\]
\[
+ \left[ 2 F_e \frac{\partial \Psi_v}{\partial C_e} : F_e^{-T} : F_e t_v \right]
\]
\[
+ \left[ 2 F_\infty \frac{\partial \Psi_p}{\partial C_\infty} : F_\infty^{-T} : F_\infty t_p - \frac{\partial \Psi_p}{\partial \xi} \dot{\xi} \right]
\geq 0
\]
As shown in Chapter 2, the term \( 2 F_\infty \frac{\partial \Psi_p}{\partial C_\infty} F_p^T \) of Eq. A.6.1.9 can be substituted with \( \tau_\infty \) defined as follows

\[
\tau = J_\sigma = F S F^T = F \left[ 2 F_e^{-1} \frac{\partial \Psi_p}{\partial C_e} F_e^T + 2 F_e^{-1} \frac{\partial \Psi_p}{\partial C_e} F_e^T \right] F^T \\
= 2 F_e \frac{\partial \Psi_p}{\partial C_e} F_e^T + 2 F_\infty \frac{\partial \Psi_p}{\partial C_\infty} F_\infty^T \\
= \tau_e + \tau_\infty
\]

Then from Eq. A.6.1.9

\[
2 F_\infty \frac{\partial \Psi_p}{\partial C_\infty} F_\infty^T : F_\infty l_p = \tau_\infty F_\infty^{-T} : F_\infty l_p \\
= \tau_\infty F_\infty^{-T} : F_\infty l_p F_\infty^{-T} \\
= \tau_\infty F_\infty^{-T} F_\infty^{-1} : F_\infty l_p F_\infty^T \\
= \tau_\infty b_\infty^{-1} : F_\infty l_p F_\infty^T
\]

Furthermore the term \( F_\infty l_p F_\infty^T \) can be split as

\[
F_\infty l_p F_\infty^T = \text{sym} [F_\infty l_p F_\infty^T] + \text{skw} [F_\infty l_p F_\infty^T] \\
= F_\infty \text{sym} [l_p F_\infty^T] + \text{skw} [F_\infty l_p F_\infty^T]
\]

Because

\[
b_\infty = F C_p^{-1} F^T
\]

it holds for the Lie derivative of \( b_\infty \), Simo [1987a]

\[
\mathcal{L}(b_\infty) = F \left[ \frac{\partial}{\partial t} (C_p^{-1}) \right] F^T \\
= -F(C_p^{-1} \dot{C} C_p^{-1}) F^T \\
= -F F_p^{-1} \left[ F_p^{-T} \dot{F}_p^T + \dot{F}_p F_p^{-1} \right] F_p^{-T} F^T \\
= -2 F_\infty \text{sym} [l_p F_\infty^T]
\]

Hence Eq. A.6.2.2 is equivalent to
\[ 2F_\infty \frac{\partial \Psi_p}{\partial C_\infty} F_\infty^T F_\infty^{-T} : F_\infty L_p = \tau_\infty b_\infty^{-1} : \left[ -\frac{1}{2} \mathcal{L}(b_\infty) + \text{skw} \left( F_\infty L_p F_\infty^T \right) \right] \]

\[ = \tau_\infty \left[ -\frac{1}{2} \mathcal{L}(b_\infty) + \text{skw} \left( F_\infty L_p F_\infty^T \right) \right] b_\infty^{-1} \]

\[ = \tau_\infty \left[ -\frac{1}{2} \mathcal{L}(b_\infty) \right] b_\infty^{-1} \quad \text{(A.6.2.6)} \]

where the identity \( b_\infty^{-1} = b_\infty^{-T} \) has been utilized.

Substituting into inequality 6.14 it results

\[ \tau_\infty \left[ -\frac{1}{2} \mathcal{L}(b_\infty) \right] b_\infty^{-1} - \frac{\partial \Psi_p}{\partial \xi} \dot{\xi} \geq 0 \quad \text{(A.6.2.7)} \]
The Jacobian of the iterative scheme defined by Eq. 6.43 is

\[
J^{t+\Delta t} = \begin{vmatrix}
\frac{\partial \tau_{\infty,1}}{\partial \xi} & \frac{\partial \tau_{\infty,1}}{\partial \xi} & \frac{\partial \tau_{\infty,1}}{\partial \xi} \\
\frac{\partial \varepsilon_{\infty,1}}{\partial \xi} & \frac{\partial \varepsilon_{\infty,2}}{\partial \xi} & \frac{\partial \varepsilon_{\infty,3}}{\partial \xi} \\
\frac{\partial \tau_{\infty,2}}{\partial \xi} & \frac{\partial \tau_{\infty,2}}{\partial \xi} & \frac{\partial \tau_{\infty,2}}{\partial \xi} \\
\frac{\partial \varepsilon_{\infty,2}}{\partial \xi} & \frac{\partial \varepsilon_{\infty,3}}{\partial \xi} & \frac{\partial \varepsilon_{\infty,3}}{\partial \xi} \\
\frac{\partial \tau_{\infty,3}}{\partial \xi} & \frac{\partial \tau_{\infty,3}}{\partial \xi} & \frac{\partial \tau_{\infty,3}}{\partial \xi} \\
\frac{\partial \varepsilon_{\infty,3}}{\partial \xi} & \frac{\partial \varepsilon_{\infty,3}}{\partial \xi} & \frac{\partial \varepsilon_{\infty,3}}{\partial \xi}
\end{vmatrix}
\]

The individual terms of the Jacobian can be elaborated as follows:

\[
\frac{\partial f}{\partial \varepsilon_{\infty,j}} = \frac{t+\Delta t}{t+\Delta t} \left( \sum_{k=1}^{3} \left( \frac{\partial f}{\partial \tau_{\infty,k}} \frac{\partial \tau_{\infty,k}}{\partial \varepsilon_{\infty,j}} \right) + \frac{\partial f}{\partial q} \frac{\partial q}{\partial \varepsilon_{\infty,j}} \right)
\]

\[
= \frac{t+\Delta t}{t+\Delta t} \left( \sum_{k=1}^{3} \left( \frac{\partial f}{\partial \tau_{\infty,k}} \frac{\partial ^2 \psi}{\partial \varepsilon_{\infty,j} \partial \varepsilon_{\infty,j}} \right) + \frac{\partial f}{\partial q} \frac{\partial ^2 \psi}{\partial \varepsilon_{\infty,j} \partial \varepsilon_{\infty,j}} \right)
\]

\[
= \frac{t+\Delta t}{t+\Delta t} \left( \sum_{k=1}^{3} \left( \frac{\partial f}{\partial \tau_{\infty,k}} \frac{\partial ^2 \psi}{\partial \varepsilon_{\infty,j} \partial \varepsilon_{\infty,j}} \right) \right)
\]

\[
\frac{\partial f}{\partial \xi} = \frac{t+\Delta t}{t+\Delta t} \left( \sum_{k=1}^{3} \left( \frac{\partial f}{\partial \tau_{\infty,k}} \frac{\partial \tau_{\infty,k}}{\partial \xi} \right) + \frac{\partial f}{\partial q} \frac{\partial q}{\partial \xi} \right)
\]

\[
= \frac{t+\Delta t}{t+\Delta t} \left( \sum_{k=1}^{3} \left( \frac{\partial f}{\partial \tau_{\infty,k}} \frac{\partial ^2 \psi}{\partial \varepsilon_{\infty,j} \partial \xi} \right) + \frac{\partial f}{\partial q} \frac{\partial ^2 \psi}{\partial \xi ^2} \right)
\]

\[
= \frac{t+\Delta t}{t+\Delta t} \left( -\frac{\partial ^2 \psi}{\partial \xi ^2} \right)
\]
\[
\frac{\partial \tau_{\alpha\beta,i}}{\partial \varepsilon_{\alpha\beta}} = \begin{bmatrix}
\frac{\partial f}{\partial q} \frac{\partial}{\partial \varepsilon_{\alpha\beta}} \left( \varepsilon_{\alpha\beta} - \text{trial} \varepsilon_{\alpha\beta} \right) + \left( \varepsilon_{\alpha\beta} - \text{trial} \varepsilon_{\alpha\beta} \right) \frac{\partial}{\partial \psi} \left( \frac{\partial f}{\partial \psi} \right)
+ \Delta \xi \frac{\partial}{\partial \tau_{\alpha\beta,i}} \left( \frac{\partial f}{\partial \psi} \right)
\end{bmatrix}
\]

A.6.3.4

\[
\begin{align*}
\frac{\partial \tau_{\alpha\beta,i}}{\partial \xi} & = \begin{bmatrix}
\frac{\partial f}{\partial q} \frac{\partial}{\partial \xi} \left( \varepsilon_{\alpha\beta} - \text{trial} \varepsilon_{\alpha\beta} \right) + \left( \varepsilon_{\alpha\beta} - \text{trial} \varepsilon_{\alpha\beta} \right) \frac{\partial}{\partial \xi} \left( \frac{\partial f}{\partial \psi} \right)
+ \left( \frac{\partial f}{\partial \tau_{\alpha\beta,i}} \right) \frac{\partial}{\partial \xi} \left( \Delta \xi \right) + \Delta \xi \frac{\partial}{\partial \tau_{\alpha\beta,i}} \left( \frac{\partial f}{\partial \xi} \right)
\end{bmatrix}
\end{align*}
\]

A.6.3.5

\[
\begin{align*}
\frac{\partial \tau_{\alpha\beta,i}}{\partial \xi} & = \begin{bmatrix}
\frac{\partial f}{\partial \tau_{\alpha\beta,i}} + \Delta \xi \frac{\partial}{\partial \tau_{\alpha\beta,i}} \left( \frac{\partial f}{\partial \xi} \right)
\end{bmatrix}
\end{align*}
\]

\[
\frac{\partial \tau_{\alpha\beta,i}}{\partial \xi} = \begin{bmatrix}
\frac{\partial f}{\partial \psi}
\end{bmatrix}
\]

A.6.3.6

\[
\frac{\partial \tau_{\alpha\beta}}{\partial \xi} = \begin{bmatrix}
\frac{\partial f}{\partial \xi}
\end{bmatrix}
\]

A.6.3.7
Verification of Eq. 6.32 is presented in the following.

If $A$ is a square $n \times n$ matrix, it can be shown by means of the Cayley-Hamilton theorem, Bronson 1973, that

$$
\exp(At) = a_{n-1}A^{n-1}t^{n-1} + a_{n-2}A^{n-2}t^{n-2} + \ldots + a_2A^2t^2 + a_1At + a_0I
$$

A.6.4.1

in which $a_0, a_1, \ldots, a_{n-1}$ are functions of $t$ which must be determined on the basis of $A$.

Furthermore if

$$
r(\lambda) = a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_2\lambda^2 + a_1\lambda + a_0
$$

A.6.4.2

then if $\lambda_j$ is an eigenvalue of $At$

$$
\exp(\lambda_j) = r(\lambda_j)
$$

A.6.4.3

while if $\lambda_j$ is an eigenvalue of multiplicity $k$ it holds additionally

$$
\exp(\lambda_j) = \frac{d[r(\lambda)]}{d\lambda} \bigg|_{\lambda=\lambda_j}
$$

A.6.4.4

$$
\exp(\lambda_j) = \frac{d^2[r(\lambda)]}{d\lambda^2} \bigg|_{\lambda=\lambda_j}
$$

A.6.4.5

......

$$
\exp(\lambda_j) = \frac{d^{k-1}[r(\lambda)]}{d\lambda^{k-1}} \bigg|_{\lambda=\lambda_j}
$$

A.6.4.6

The above concepts can be utilized to obtain a pliable form of Eq. 6.32. Considering the $i$-th term of the sum in the exponential term of Eq. 6.32

$$
\exp \left( 2 \Delta\lambda \frac{\partial f}{\partial \tau_i} I \otimes I \right)
$$

and setting $t_i = 2 \Delta\lambda \frac{\partial f}{\partial \tau_i}$ and $A = I \otimes I$ then

$$
\det(At_i - \lambda I) = 0
$$

A.6.4.7
has a root \( \lambda = t_1 \) of multiplicity one and a root \( \lambda = 0 \) of multiplicity two. On the basis of Eq. A.6.4.3 and Eq. A.6.4.4 the following set of equations is obtained

\[
\exp(t_i) = a_2 t_i^2 + a_1 t_i + a_0 \\
\exp(0) = a_0 \\
\exp(0) = a_1
\]

Hence

\[
a_0 = 1 \\
a_1 = 1 \\
a_2 = \frac{\exp(t_i) - t_i - 1}{t_i^2}
\]

Substituting the values of \( a - s \) into Eq. A.6.4.1

\[
\exp(A t_i) = \begin{bmatrix}
\exp(t_i) & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

On the basis of Eq. A.6.4.10 the exponential term of Eq. 6.32 can be written as

\[
\exp \left[ 2 \Delta \lambda \left( \sum_{i=1}^{3} \frac{\partial f}{\partial \tau_i} l_i \otimes l_i \right) \right] = \exp \left[ t_1 l_1 \otimes l_1 + t_2 l_2 \otimes l_2 + t_3 l_3 \otimes l_3 \right]
\]

\[
= \exp(t_1 l_1 \otimes l_1) \exp(t_2 l_2 \otimes l_2) \exp(t_3 l_3 \otimes l_3)
\]

\[
= \begin{bmatrix}
\exp(t_1) & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\exp(t_1) & 0 & 0 \\
0 & \exp(t_2) & 0 \\
0 & 0 & \exp(t_3)
\end{bmatrix}
\]

Therefore, Eq. 6.32 can be further elaborated as
\[
\exp \left[ 2 \Delta \lambda \left( \sum_{i=1}^{3} \frac{\partial f}{\partial \tau_i} l \otimes l \right) \right] \sum_{i=1}^{3} \lambda_{\infty,i}^2 l \otimes l_i \\
= \begin{bmatrix}
\exp(t_1) & 0 & 0 \\
0 & \exp(t_2) & 0 \\
0 & 0 & \exp(t_3)
\end{bmatrix} \sum_{i=1}^{3} \lambda_{\infty,i}^2 l \otimes l_i \\
= \sum_{i=1}^{3} \exp \left( 2 \Delta \lambda \frac{\partial f(\tau_1, \tau_2, \tau_3, q)}{\partial \tau_i} \right) \lambda_{\infty,i}^2 l \otimes l_i
\]
The response surface of the implemented material model is defined by

\[ f(\mathbf{T}_{\infty,i}, q) = \sqrt{\frac{3}{2}} \| \mathbf{s}_{\infty,i} \| - (\mathbf{T}_{\infty,i} - \mathbf{q}) \]  

in which the vector of principal deviatoric Kirchhoff stresses \( \mathbf{s}_{\infty,i} \) is defined by

\[ \mathbf{s}_{\infty,i} = \text{dev}(\mathbf{T}_{\infty,i}) = \left( \mathbf{I} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right) : \mathbf{T}_{\infty,i} \]

\[ = \mathbf{T}_{\infty,i} - \frac{1}{3} \sum_{i=j}^{3} \mathbf{T}_{\infty,j} \]

and

\[ \| \mathbf{s}_{\infty,i} \| = \left( \mathbf{s}_{\infty,i} \cdot \mathbf{s}_{\infty,i} \right)^{\frac{1}{2}} \]

\[ = \left[ \sum_{i=1}^{3} \left( \mathbf{T}_{\infty,i} - \frac{1}{3} \sum_{j=1}^{3} \mathbf{T}_{\infty,j} \right)^{2} \right]^{\frac{1}{2}} \]

so that

\[ \partial_{\mathbf{T}_{\infty,i}} f = \frac{\partial f}{\partial \mathbf{s}_{\infty,1}} \frac{\partial \mathbf{s}_{\infty,1}}{\partial \mathbf{T}_{\infty,i}} + \frac{\partial f}{\partial \mathbf{s}_{\infty,2}} \frac{\partial \mathbf{s}_{\infty,2}}{\partial \mathbf{T}_{\infty,i}} + \frac{\partial f}{\partial \mathbf{s}_{\infty,3}} \frac{\partial \mathbf{s}_{\infty,3}}{\partial \mathbf{T}_{\infty,i}} \]

\[ = \sqrt{\frac{3}{2}} \frac{1}{2} \left( \frac{\mathbf{s}_{\infty,1}^2 + \mathbf{s}_{\infty,2}^2 + \mathbf{s}_{\infty,3}^2}{\frac{2 \mathbf{s}_{\infty,1}^2}{\partial \mathbf{T}_{\infty,i}} + 2 \mathbf{s}_{\infty,2} \frac{\partial \mathbf{s}_{\infty,2}}{\partial \mathbf{T}_{\infty,i}} + 2 \mathbf{s}_{\infty,3} \frac{\partial \mathbf{s}_{\infty,3}}{\partial \mathbf{T}_{\infty,i}}} \right) \]

Considering that

\[ \frac{\partial \mathbf{s}_{\infty,1}}{\partial \mathbf{T}_{\infty,i}} = \delta_{ij} - \frac{1}{3} \]

\[ \partial_{\mathbf{T}_{\infty,i}} f = \sqrt{\frac{3}{2}} \frac{\mathbf{s}_{\infty,i}}{\sqrt{\mathbf{s}_{\infty,1}^2 + \mathbf{s}_{\infty,2}^2 + \mathbf{s}_{\infty,3}^2}} \]
Elasto-Visco-Plasticity

\[
\begin{align*}
\partial^2_{\tau_{\infty,i}} \partial_{\tau_{\infty,j}} f &= \frac{\partial \left( \frac{\partial_{\tau_{\infty,i}} f}{\partial_{\tau_{\infty,j}}} \right)}{\partial_{\tau_{\infty,j}}} \\
&= \frac{\partial}{\partial_{\tau_{\infty,j}}} \left( \sqrt{\frac{3}{2}} \left[ \frac{s_{\infty,i}}{\left( s_{\infty,1}^2 + s_{\infty,2}^2 + s_{\infty,3}^2 \right)^{1/2}} \right] \right) \\
&= \sqrt{\frac{3}{2}} \left[ -\frac{s_{\infty,i} s_{\infty,j}}{\left( s_{\infty,1}^2 + s_{\infty,2}^2 + s_{\infty,3}^2 \right)^{1/2}} + \frac{1}{\left( s_{\infty,1}^2 + s_{\infty,2}^2 + s_{\infty,3}^2 \right)^{1/2}} \frac{\partial s_{\infty,i}}{\partial_{\tau_{\infty,j}}} \right] \\
\partial_{\tau_{\infty,i}} f &= 1 \\
\partial^2_{\tau_{\infty,i}} f &= \frac{\partial (\partial_{\tau_{\infty,i}} f)}{\partial_{\tau_{\infty,i}}} = 0 \\
\partial^2_{\tau_{\infty,i}} f &= \frac{\partial}{\partial_{\tau_{\infty,i}}} \left( \frac{\partial f}{\partial_{\tau_{\infty,j}}} \right) = 0 \tag{A.6.5.7, A.6.5.8, A.6.5.9, A.6.5.10} \end{align*}
\]

The chosen form of the strain energy function is

\[
\psi_p \left( \epsilon_{\infty,i} \xi \right) = \frac{\kappa}{2} \left( \sum_{i=1}^{3} \epsilon_{\infty,i} \right)^2 + \mu \left( \sum_{i=1}^{3} \epsilon_{\infty,i}^2 \right) + \left( \tau_{\infty} - \tau_{y_0} \right) \left( 1 - \frac{1}{\delta} \exp^{-\delta \xi} - 1 \right) \tag{A.6.5.11} \]

so that

\[
\partial_{\epsilon_{\infty,i}} \psi_p = \kappa \sum_{i=1}^{3} \epsilon_{\infty,i} + 2 \mu \epsilon_{\infty,i} \tag{A.6.5.12} \]

\[
\partial_{\epsilon_{\infty,i}}^2 \psi_p = \kappa + 2 \mu \delta_{ij} \tag{A.6.5.13} \]

\[
\partial_{\xi} \psi_p = \left( \tau_{\infty} - \tau_{y_0} \right) \left( 1 - \exp^{-\delta \xi} \right) \tag{A.6.5.14} \]

\[
\partial_{\xi}^2 \psi_p = \delta \left( \tau_{\infty} - \tau_{y_0} \right) \exp^{-\delta \xi} \tag{A.6.5.15} \]

\[
\partial_{\tau_{\infty,i}} \xi \psi_p = \frac{\partial}{\partial_{\tau_{\infty,i}}} \left( \frac{\partial \psi}{\partial \xi} \right) = 0 \tag{A.6.5.16} \]

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The Jacobian of the iterative scheme suggested in Section 6.5.1.3 is

\[
J\left(\frac{t+\Delta t}{n}X\right) = \frac{t+\Delta t}{n} \nabla r = \begin{vmatrix}
\frac{\partial r_{e,1}}{\partial e_{e,1}} & \frac{\partial r_{e,1}}{\partial e_{e,2}} & \frac{\partial r_{e,1}}{\partial e_{e,3}} \\
\frac{\partial e_{e,1}}{\partial e_{e,1}} & \frac{\partial e_{e,2}}{\partial e_{e,2}} & \frac{\partial e_{e,3}}{\partial e_{e,3}} \\
\frac{\partial r_{e,2}}{\partial e_{e,1}} & \frac{\partial r_{e,2}}{\partial e_{e,2}} & \frac{\partial r_{e,2}}{\partial e_{e,3}} \\
\frac{\partial e_{e,2}}{\partial e_{e,1}} & \frac{\partial e_{e,2}}{\partial e_{e,2}} & \frac{\partial e_{e,2}}{\partial e_{e,3}} \\
\frac{\partial r_{e,3}}{\partial e_{e,1}} & \frac{\partial r_{e,3}}{\partial e_{e,2}} & \frac{\partial r_{e,3}}{\partial e_{e,3}} \\
\frac{\partial e_{e,3}}{\partial e_{e,1}} & \frac{\partial e_{e,3}}{\partial e_{e,2}} & \frac{\partial e_{e,3}}{\partial e_{e,3}} 
\end{vmatrix}
\]

A.6.6.1

An individual term is computed as follows.

\[
\frac{\partial r_{e,i}}{\partial e_{e,j}} = \frac{\partial}{\partial e_{e,j}} \left( \frac{t+\Delta t}{n} \epsilon_{e,i}^{\text{trial}} - \frac{t+\Delta t}{n} \epsilon_{e,j}^{\text{t}} \right) + \frac{\partial}{\partial e_{e,j}} \left( \frac{t+\Delta t}{2\eta_D} \left( \tau_{e,i} - \frac{1}{3} \sum_{m=1}^{3} \tau_{e,m} \right) + \frac{\Delta t}{9\eta_V} \left( \frac{1}{3} \sum_{m=1}^{3} \tau_{e,m} \right) \right)
\]

A.6.6.2

\[
= \frac{t+\Delta t}{n} \delta_{ij} + \frac{\Delta t}{2\eta_D} \left( \frac{\partial^2 \Psi_v}{\partial e_{e,j} \partial e_{e,i}} - \frac{1}{3} \sum_{m=1}^{3} \frac{\partial^2 \Psi_v}{\partial e_{e,j} \partial e_{e,m}} \right) + \frac{\Delta t}{9\eta_V} \left( \frac{1}{3} \sum_{m=1}^{3} \frac{\partial^2 \Psi_v}{\partial e_{e,j} \partial e_{e,m}} \right)
\]
The steps necessary for the solution of the evolution problem defined by Eq. 6.22 are presented in the following. Eq. 6.22 which states

$$\frac{d}{dt} \left( C_p^{-1} \right) = \left[ -2 \lambda C^{-1} N \right] C_p^{-1} \tag{A.6.7.1}$$

can be solved in the time interval $[t, t + \Delta t]$ to give a first order accurate estimate for $t + \Delta t C_p^{-1}$.

Define

$$p = 2 \lambda C^{-1} N \tag{A.6.7.2}$$

Define also an integrating factor as

$$I(t, C_p^{-1}) = \exp \left( \int p \, dt \right) = \exp \left( 2 \lambda C^{-1} N t \right) \tag{A.6.7.3}$$

Multiplying Eq. A.6.7.1 by the integrating factor

$$\exp \left( 2 \lambda C^{-1} N t \right) C_p^{-1} + \exp \left( 2 \lambda C^{-1} N t \right) 2 \lambda C^{-1} N C_p^{-1} = 0 \tag{A.6.7.4}$$

or equivalently

$$\frac{d}{dt} \left[ C_p^{-1} \exp \left( 2 \lambda C^{-1} N t \right) \right] = 0 \tag{A.6.7.5}$$

Integrating

$$\int \frac{d}{dt} \left[ C_p^{-1} \exp \left( 2 \lambda C^{-1} N t \right) \right] \, dt = \int 0 \Rightarrow \quad C_p^{-1} \exp \left( 2 \lambda C^{-1} N t \right) = D \quad \Rightarrow \quad C_p^{-1} = D \exp \left( -2 \lambda C^{-1} N t \right) \tag{A.6.7.6}$$

At time $t$ denote $C_p^{-1}(t) = \ ^t C_p^{-1}$. Hence from Eq. A.6.7.6

$$\ ^t C_p^{-1} = D \exp \left( -2 \lambda C^{-1} N t \right) \quad \Rightarrow \quad D = \ ^t C_p^{-1} \exp \left( 2 \lambda C^{-1} N t \right) \tag{A.6.7.7}$$

Substituting into Eq. A.6.7.6, at $t = t + \Delta t$

$$\ ^{t + \Delta t} C_p^{-1} = \ ^t C_p^{-1} \exp \left( 2 \lambda C^{-1} N \Delta t \right) \exp \left[ -2 \lambda C^{-1} N (t + \Delta t) \right] \tag{A.6.7.8}$$
Section II

Finite Elements Technology
Chapter 7

Cubic Finite Element Overview

7.1 Introduction
The 20-noded isoparametric cubic element is particularly suited for the simulation of 3D continua. The element is compatible with all material models presented in earlier Chapters. The element stiffness is computed on the basis of numerical integration over the volume of the element. Various integration schemes can be chosen. Several types of element loads can be applied.

7.2 Element Formulation
Fig. 7.1 shows the local ordering of the nodes and the orientation of the associated local element axes system $\xi = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix}^T$.

![20-noded isoparametric cubic element](image)

Fig. 7.1 20-noded isoparametric cubic element

7.2.1 Element Geometry Interpolation
Following a typical isoparametric formulation, the vector $N$ of interpolation functions for the 20-noded element can be defined as:

$$N = \begin{bmatrix} N_1 & N_2 & \ldots & N_{20} \end{bmatrix}$$  \hspace{1cm} (7.1)$$

with
Cubic Finite Element Overview

\[ N_1 = (1 - \xi_1)(1 - \xi_2)(1 - \xi_3) (\xi_1 - \xi_2 - \xi_3 - 2)/8 \]
\[ N_2 = (1 - \xi_1^2)(1 - \xi_2)(1 - \xi_3)/4 \]
\[ N_3 = (1 + \xi_1)(1 - \xi_2)(1 - \xi_3)(\xi_1 + \xi_2 - \xi_3 - 2)/8 \]
\[ N_4 = (1 + \xi_1)(1 - \xi_2^2)(1 - \xi_3)/4 \]
\[ N_5 = (1 + \xi_1)(1 + \xi_2)(1 - \xi_3)(\xi_1 + \xi_2 - \xi_3 - 2)/8 \]
\[ N_6 = (1 - \xi_1^2)(1 + \xi_2)(1 - \xi_3)/4 \]
\[ N_7 = (1 - \xi_1)(1 + \xi_2)(1 - \xi_3)(\xi_1 + \xi_2 - \xi_3 - 2)/8 \]
\[ N_8 = (1 - \xi_1)(1 - \xi_2^2)(1 - \xi_3)/4 \]
\[ N_9 = (1 - \xi_1)(1 - \xi_2)(1 - \xi_3^2)/4 \]
\[ N_{10} = (1 + \xi_1)(1 - \xi_2)(1 - \xi_3^2)/4 \]
\[ N_{11} = (1 + \xi_1)(1 + \xi_2)(1 - \xi_3^2)/4 \]
\[ N_{12} = (1 - \xi_1)(1 + \xi_2)(1 - \xi_3^2)/4 \]
\[ N_{13} = (1 - \xi_1)(1 - \xi_2)(1 + \xi_3)(\xi_1 - \xi_2 + \xi_3 + 2)/8 \]
\[ N_{14} = (1 - \xi_1^2)(1 - \xi_2)(1 + \xi_3)/4 \]
\[ N_{15} = (1 + \xi_1)(1 - \xi_2)(1 + \xi_3)(\xi_1 + \xi_2 + \xi_3 - 2)/8 \]
\[ N_{16} = (1 + \xi_1)(1 - \xi_2^2)(1 + \xi_3)/4 \]
\[ N_{17} = (1 + \xi_1)(1 + \xi_2)(1 + \xi_3)(\xi_1 + \xi_2 + \xi_3 - 2)/8 \]
\[ N_{18} = (1 - \xi_1^2)(1 + \xi_2)(1 + \xi_3)/4 \]
\[ N_{19} = (1 - \xi_1)(1 + \xi_2)(1 - \xi_3)(\xi_1 + \xi_2 - \xi_3 - 2)/8 \]
\[ N_{20} = (1 - \xi_1)(1 - \xi_2^2)(1 + \xi_3)/4 \]

If the vector of nodal coordinates \( \mathbf{X}_k \) is defined as:

\[ \mathbf{X}_k = \begin{bmatrix} X_{k1} & X_{k2} & X_{k3} \end{bmatrix}^T \]

then the initial configuration of any point within the element can be interpolated in terms of the corresponding nodal coordinates as:

\[ \mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}^T = \sum_{i=1}^{3} \left( \sum_{k=1}^{20} N_k X_{ki} \right) \mathbf{E}_i \]
\[ = \sum_{k=1}^{20} N_k \left( \sum_{i=1}^{3} X_{ki} \mathbf{E}_i \right) = \sum_{k=1}^{20} N_k \mathbf{X}_k \]
Anticipating an isoparametric formulation, at any subsequent time, the current configuration \( \mathbf{x} \) is interpolated in terms of the corresponding current nodal quantities \( a_k \) as:

\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T = \sum_{i=1}^{3} \left( \sum_{k=1}^{20} N_k x_{k1} e_i \right) = \sum_{k=1}^{20} N_k \mathbf{x}_k
\]

7.5

### 7.2.2 Field Variables Interpolation

Similarly, the displacements can be expressed in terms of the current nodal displacements \( \mathbf{u}_k \):

\[
\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}^T = \sum_{i=1}^{3} \left( \sum_{k=1}^{20} N_k u_{k1} \right) e_i = \sum_{k=1}^{20} N_k \mathbf{u}_k
\]

7.6

As shown in Chapter 2, the deformation gradient tensor \( \mathbf{F} \) can be interpolated over an element as

\[
\mathbf{F} = \sum_{i,j=1}^{3} \frac{\partial x_i}{\partial x_j} e_i \otimes e_j = \sum_{i,j=1}^{3} \left[ \frac{\partial}{\partial x_j} \left( \sum_{k=1}^{20} N_k x_{k1} \right) \right] e_i \otimes e_j = \sum_{k=1}^{20} x_k \otimes \frac{\partial N_k}{\partial \mathbf{x}} = \sum_{k=1}^{20} x_k \otimes \nabla_0 N_k
\]

7.7

In order to evaluate the individual Cartesian derivatives \( \nabla_0 N_k \) in Eq. 7.7, the following transformation can be utilized

\[
\left( \frac{\partial N_k}{\partial \xi} \right) = \mathbf{J}^T \nabla_0 N_k
\]

7.8

in which \( \mathbf{J} \) is the coordinate Jacobian matrix. Its computation has been presented in Chapter 2. In CAPA-3D \( \mathbf{J} \) and \( \mathbf{J}^{-T} \) are evaluated at every integration point of the element.

Similarly, the individual terms of the deformation Jacobian \( \mathbf{j} \) defined in Chapter 2 can be computed as

\[
\left( \frac{\partial x_i}{\partial \xi} \right) = \sum_{k=1}^{20} \frac{\partial N_k}{\partial \xi} x_{kj} \quad ; \quad \xi = 1 \ldots 3
\]

7.9

The engineering strain vector for the element is defined as
Cubic Finite Element Overview

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{23} \\
\gamma_{13}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u_1}{\partial X_1} \\
\frac{\partial u_2}{\partial X_2} \\
\frac{\partial u_3}{\partial X_3} \\
\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \\
\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \\
\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}
\end{bmatrix} = L \mathbf{u} = \begin{bmatrix}
\frac{\partial}{\partial X_1} & 0 & 0 \\
0 & \frac{\partial}{\partial X_2} & 0 \\
0 & 0 & \frac{\partial}{\partial X_3} \\
\frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_1} & 0 \\
\frac{\partial}{\partial X_3} & \frac{\partial}{\partial X_2} & 0 \\
\frac{\partial}{\partial X_3} & 0 & \frac{\partial}{\partial X_1}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
\]

with the notation \( \gamma_{ij} = 2 \varepsilon_{ij} \).

Rewriting Eq. 7.6 as

\[
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix} = \begin{bmatrix}
\tilde{N}_1 & \tilde{N}_2 & \ldots & \tilde{N}_{20}
\end{bmatrix} \mathbf{\bar{u}} \tag{7.11}
\]

in which \( \mathbf{\bar{u}} \) is the vector of nodal displacements

\( \mathbf{\bar{u}} = \begin{bmatrix} u_1 & u_2 & \ldots & u_{20} \end{bmatrix}^T \) \tag{7.12}

with

\[
\mathbf{u}_k = \begin{bmatrix} u_{k1} & u_{k2} & u_{k3} \end{bmatrix}^T
\]

\[
\tilde{N}_k = \begin{bmatrix} N_k & 0 & 0 \\
0 & N_k & 0 \\
0 & 0 & N_k
\end{bmatrix}
\] \tag{7.14}

then, after substitution in Eq. 7.10, the familiar expression for the strains as a function of the nodal displacements is obtained as

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{23} \\
\gamma_{13}
\end{bmatrix} = \begin{bmatrix}
B_1 & B_2 & \ldots & B_{20}
\end{bmatrix} \mathbf{\bar{u}} \tag{7.15}
\]
in which the terms $B_k$ have the form

$$
B_k = \begin{vmatrix}
\frac{\partial N_k}{\partial X_1} & 0 & 0 \\
0 & \frac{\partial N_k}{\partial X_2} & 0 \\
0 & 0 & \frac{\partial N_k}{\partial X_3} \\
\frac{\partial N_k}{\partial X_2} & \frac{\partial N_k}{\partial X_1} & 0 \\
0 & \frac{\partial N_k}{\partial X_3} & \frac{\partial N_k}{\partial X_2} \\
\frac{\partial N_k}{\partial X_3} & 0 & \frac{\partial N_k}{\partial X_1}
\end{vmatrix}
$$

7.3 Numerical Integration

A standard $3\times3\times3$ Gauss integration scheme is available. In earlier versions of the system, a 15 points integration scheme proposed by Irons was utilised. Current advances in hardware, hardly justify economizing at integration point level.

The locations of the Gauss integration points wrt. the element local axes system are identified in Fig. 7.2 and are also listed in Table 7.1. It must be remembered that it is the local numbering of the element nodes that determines the direction of the local element axes and hence the locations of the Gauss points.

Fig. 7.2 Gauss points locations in the element local system
### Table 7.1  Gauss points locations in the local element system

<table>
<thead>
<tr>
<th>Gauss Point #</th>
<th>$\xi_1$ - coord.</th>
<th>$\xi_2$ - coord.</th>
<th>$\xi_3$ - coord.</th>
<th>Gauss Point #</th>
<th>$\xi_1$ - coord.</th>
<th>$\xi_2$ - coord.</th>
<th>$\xi_3$ - coord.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.775</td>
<td>-0.775</td>
<td>-0.775</td>
<td>14</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-0.775</td>
<td>-0.775</td>
<td>0</td>
<td>15</td>
<td>0</td>
<td>0</td>
<td>0.775</td>
</tr>
<tr>
<td>3</td>
<td>-0.775</td>
<td>-0.775</td>
<td>0.775</td>
<td>16</td>
<td>0</td>
<td>0.775</td>
<td>-0.775</td>
</tr>
<tr>
<td>4</td>
<td>-0.775</td>
<td>0</td>
<td>-0.775</td>
<td>17</td>
<td>0</td>
<td>0.775</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>-0.775</td>
<td>0</td>
<td>0</td>
<td>18</td>
<td>0</td>
<td>0.775</td>
<td>0.775</td>
</tr>
<tr>
<td>6</td>
<td>-0.775</td>
<td>0</td>
<td>0.775</td>
<td>19</td>
<td>0.775</td>
<td>-0.775</td>
<td>-0.775</td>
</tr>
<tr>
<td>7</td>
<td>-0.775</td>
<td>0.775</td>
<td>-0.775</td>
<td>20</td>
<td>0.775</td>
<td>-0.775</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>-0.775</td>
<td>0.775</td>
<td>0</td>
<td>21</td>
<td>0.775</td>
<td>-0.775</td>
<td>0.775</td>
</tr>
<tr>
<td>9</td>
<td>-0.775</td>
<td>0.775</td>
<td>0.775</td>
<td>22</td>
<td>0.775</td>
<td>0</td>
<td>-0.775</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>-0.775</td>
<td>-0.775</td>
<td>23</td>
<td>0.775</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>-0.775</td>
<td>0</td>
<td>24</td>
<td>0.775</td>
<td>0</td>
<td>0.775</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>-0.775</td>
<td>0.775</td>
<td>25</td>
<td>0.775</td>
<td>0.775</td>
<td>-0.775</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>0</td>
<td>-0.775</td>
<td>26</td>
<td>0.775</td>
<td>0.775</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>27</td>
<td>0.775</td>
<td>0.775</td>
<td>0.775</td>
</tr>
</tbody>
</table>

### 7.4 Constitutive Law

The element functions with all models included in this document and is compatible with all other element types presented in the following Chapters.

### 7.5 Utilization

As mentioned in the Introduction, the cubic element is particularly suited for simulation of the response of linear and nonlinear continua.

Chapter 8

Interface Finite Element Overview

8.1 Introduction

The finite element method is a structural analysis method based on the theory of mechanics of continua. Interface regions constitute discrete discontinuities and as such their modelling within the context of the finite element method requires the use of specially developed elements, elements which because of their formulation are capable of capturing the discontinuous nature of the deformation.

With the exception of highly discretised meshes, the artifice of degenerating an ordinary continuous finite element into an element of very small thickness, renders the element grossly inaccurate and unreliable.

Several authors in the past have proposed various types of interface elements, Ngo & Scordelis [1967], Goodman et al. [1968], Schäfer [1975], Plesha et al. [1989], de Groot et al. [1981], Rots [1988], Schellekens [1992]. Comprehensive reviews can be found in Selvadorai & Boulon [1995] and more recently in Selvadorai & Yu [2005].

In CAPA-3D simulation of interface regions is achieved by means of a 16-noded isoparametric interface element. The element is compatible with the ordinary quadratic elements and can therefore be used in the modelling of layer and/or crack interfaces.

Because of its special formulation, the element can also be used for the simulation of reinforcement and its bond with the surrounding medium.

8.2 Element Formulation

Fig. 8.1 shows the local ordering of the nodes. The thickness of the element in its undeformed configuration can be specified to be very small or even zero.

![16-noded isoparametric interface element](image)

Fig. 8.1 16-noded isoparametric interface element

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8.2.1 Element Geometry Interpolation

By utilizing the element nodal coordinates, a mid-surface is defined as follows. For every pair of nodes, a local thickness vector $\chi_k$ is defined spanning from a node on the bottom surface of the element to the corresponding node on the top surface, Fig. 8.2.

![Diagram showing element geometry interpolation](image)

**Fig. 8.2 Local thickness vectors**

If the vector of nodal coordinates for a node k on the bottom surface of the element is defined as:

$$X_k = \begin{bmatrix} X_{k1} & X_{k2} & X_{k3} \end{bmatrix}^T$$  \hspace{2cm} (8.1)

then, the global coordinates of the middle point the corresponding local thickness vector $t_k$ can be computed as:

$$\begin{bmatrix} mX_{k1} \\ mX_{k2} \\ mX_{k3} \end{bmatrix} = \begin{bmatrix} X_{k1} \\ X_{k2} \\ X_{k3} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} t_{k1} \\ t_{k2} \\ t_{k3} \end{bmatrix} \hspace{2cm} k = 1, ..., 8$$  \hspace{2cm} (8.2)

As shown in Fig. 8.3, at each middle point $k$, a unit local thickness vector $\chi_k$ can be defined as:

$$\chi_{kj} = \frac{t_{kj}}{\|t_k\|} \hspace{2cm} j = 1, ..., 3$$  \hspace{2cm} (8.3)

![Diagram showing local unit thickness vector definition](image)

**Fig. 8.3 Local unit thickness vector definition**
On the element middle surface, Fig. 8.4, two curvilinear coordinate axes \( \xi_1 \) and \( \xi_2 \) are defined and a third linear axis \( \xi_3 \) that is locally perpendicular to the plane defined by \( \xi_1 \) and \( \xi_2 \). All axes span between \([-1, +1]\). The orientation of the axes is determined by the local numbering of the bottom surface nodal points.

![Curvilinear coordinate system](image)

**Fig. 8.4 Curvilinear coordinate system**

The global coordinates of any point within the element can be computed on the basis of the global coordinates of the middle surface defining points and the curvilinear coordinate system

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
\end{bmatrix} = \sum_{k=1}^{8} N_k \begin{bmatrix}
n X_{k1} \\
n X_{k2} \\
n X_{k3} \\
\end{bmatrix} + \sum_{k=1}^{8} N_k \left( \frac{t_k}{2} \right) \xi_3 \begin{bmatrix}
\chi_{k1} \\
\chi_{k2} \\
\chi_{k3} \\
\end{bmatrix}
\]

in which \( N_k \) are the standard 2-D shape functions

\[
\begin{align*}
N_1 &= (1 - \xi_1) (1 - \xi_2) (\xi_1 - \xi_2 - 1) / 4 \\
N_2 &= (1 - \xi_1^2) (1 - \xi_2) / 2 \\
N_3 &= (1 + \xi_1) (1 - \xi_2) (+\xi_1 - \xi_2 - 1) / 4 \\
N_4 &= (1 + \xi_1) (1 - \xi_2^2) / 2 \\
N_5 &= (1 + \xi_1) (1 + \xi_2) (+\xi_1 + \xi_2 - 1) / 4 \\
N_6 &= (1 - \xi_1^2) (1 + \xi_2) / 2 \\
N_7 &= (1 - \xi_1) (1 + \xi_2) (-\xi_1 + \xi_2 - 1) / 4 \\
N_8 &= (1 - \xi_1^2) (1 - \xi_2^2) / 2
\end{align*}
\]
8.2.2 Field Variables Interpolation
The local response of the contact region between two interacting surfaces can be described in terms of the relative motion of two points j and i one on each of the surfaces. As shown in Fig. 8.5, two relative slip displacements and a relative normal displacement can be defined.

![Relative displacement at an interface region](image)

Fig. 8.5 Relative displacement at an interface region

Within the context of the finite element method, it is these relative displacements that are to be simulated by means of an interface element. Only local behavior is meant to be simulated by each individual element. The overall behavior can be simulated by a series of elements placed along the trace of the physical interface.

The element is formulated so as to enable two relative shear displacements and one normal between the top and the bottom faces. Utilizing the same mapping for the global displacements as for the coordinates, the global displacements of two originally coinciding points j and i, one on each face of the element, can be expressed in terms of the nodal displacements of the corresponding element face as:

\[
\begin{pmatrix}
  u^j_1 \\
  u^j_2 \\
  u^j_3 \\
  u^i_1 \\
  u^i_2 \\
  u^i_3 \\
\end{pmatrix} = \begin{pmatrix}
  0 & \bar{N} \\
  \bar{N} & 0 \\
\end{pmatrix} \begin{pmatrix}
  u_{11} \\
  u_{12} \\
  u_{13} \\
  u_{21} \\
  \vdots \\
  u_{15} \\
  u_{16} \\
  u_{17} \\
\end{pmatrix}
\]

8.6

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\[
\tilde{\mathbf{N}} = \begin{bmatrix}
N_1 & 0 & 0 & N_2 & \ldots & 0 & N_8 & 0 & 0 \\
0 & N_1 & 0 & 0 & \ldots & 0 & 0 & N_8 & 0 \\
0 & 0 & N_1 & 0 & \ldots & N_7 & 0 & 0 & N_8
\end{bmatrix}
\]

\[u = \mathbf{N} \ddot{u}\]

\[\nabla_0 N_k = J^{-T} \nabla_{\xi} N_k\]

in which \(J = \sum_{i,n=1}^{3} \frac{\partial X_i}{\partial \xi_n} E_i \otimes E_n\).

On the basis of Eq. 8.4 the individual terms of \(J\) are computed as:

\[
\left( \begin{array}{c}
\frac{\partial X_1}{\partial \xi} \\
\frac{\partial X_2}{\partial \xi} \\
\frac{\partial X_3}{\partial \xi}
\end{array} \right)_{\xi_1, \xi_2} = \sum_{k=1}^{s} \frac{\partial N_k}{\partial \xi} m X_{ki} + \left( \frac{t_k}{2} \right) \xi_3 \sum_{k=1}^{s} \frac{\partial N_k}{\partial \xi} \chi_{ki}
\]

\[i = 1, \ldots 3\]

\[
\left( \begin{array}{c}
\frac{\partial X_1}{\partial \xi_3} \\
\frac{\partial X_2}{\partial \xi_3} \\
0
\end{array} \right) = \left( \frac{t_k}{2} \right) \sum_{k=1}^{s} N_k \chi_{ki}
\]

In CAPA-3D \(J\) is evaluated at every integration point of the element. \(J^{-T}\) is also evaluated and stored at every integration point. It can be shown that the values of \(J\) evaluated at \(\xi_3 = 0\) are independent of the thickness of the interface element.

### 8.4 Constitutive Law

Three mutually orthogonal axes of material anisotropy are defined. For this reason, at each integration point of the element, a local Cartesian coordinate system \(\left( \zeta_i, i = 1, \ldots 3 \right)\) is set up as follows.

- axis \(\zeta_i\) spans along the vector \(v_{\zeta_i}\) tangent to the \(\xi_1\) axis

\[
v_{\zeta_i} = \begin{bmatrix}
\frac{\partial X_1}{\partial \xi_1} \\
\frac{\partial X_2}{\partial \xi_1} \\
\frac{\partial X_3}{\partial \xi_1}
\end{bmatrix}^T
\]

then

\[
\zeta_i = \left[ v_{\zeta_i} \right]
\]
Interface Finite Element Overview

Fig. 8.6 Local Cartesian system

- axis $\zeta_3$ is defined by the cross product of vector $v_{\xi_3}$ and vector $v_{\xi_2}$ tangent to the $\xi_2$ axis

$$v_{\xi_2} = \begin{bmatrix} \frac{\partial X_1}{\partial \xi_2} \\ \frac{\partial X_2}{\partial \xi_2} \\ \frac{\partial X_3}{\partial \xi_2} \end{bmatrix}^T$$

then

$$\zeta_3 = [v_{\xi_3} \times v_{\xi_2}]$$

8.13

- axis $\zeta_2$ is defined by the cross product

$$\zeta_2 = [\zeta_3 \times \zeta_1]$$

8.14

In the above defined local Cartesian coordinate system, the relative displacements between points $j$ and $i$ can be computed from

$$\begin{bmatrix} \Delta u_{\zeta_1} \\ \Delta u_{\zeta_2} \\ \Delta u_{\zeta_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_{\zeta_1}^i \\ u_{\zeta_2}^i \\ u_{\zeta_3}^i \\ u_{\zeta_1}^j \\ u_{\zeta_2}^j \\ u_{\zeta_3}^j \end{bmatrix}$$

8.15

or

$$\Delta u_{\zeta} = L u_{\zeta}$$

8.16
The relation between local interface displacements $u_\zeta$ and the corresponding global displacements is

$$
\begin{bmatrix}
u_{\zeta_1}^j \\
v_{\zeta_2}^i \\
v_{\zeta_3}^j \\
0 \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
\cos(\zeta_1, x_1) & \cos(\zeta_1, x_2) & \cos(\zeta_1, x_3) & 0 & 0 & 0 \\
\cos(\zeta_2, x_1) & \cos(\zeta_2, x_2) & \cos(\zeta_2, x_3) & 0 & 0 & 0 \\
\cos(\zeta_3, x_1) & \cos(\zeta_3, x_2) & \cos(\zeta_3, x_3) & 0 & 0 & 0 \\
0 & 0 & 0 & \cos(\zeta_1, x_1) & \cos(\zeta_1, x_2) & \cos(\zeta_1, x_3) \\
0 & 0 & 0 & \cos(\zeta_2, x_1) & \cos(\zeta_2, x_2) & \cos(\zeta_2, x_3) \\
0 & 0 & 0 & \cos(\zeta_3, x_1) & \cos(\zeta_3, x_2) & \cos(\zeta_3, x_3)
\end{bmatrix}
\begin{bmatrix}
u_1^i \\
u_2^i \\
u_3^i \\
u_1^j \\
u_2^j \\
u_3^j
\end{bmatrix}
$$

8.18

or

$$
\begin{bmatrix}
u_{\zeta_1}^i \\
u_{\zeta_2}^i \\
u_{\zeta_3}^i \\
u_{\zeta_1}^j \\
u_{\zeta_2}^j \\
u_{\zeta_3}^j
\end{bmatrix} =
\begin{bmatrix}
\theta & 0 \\
0 & \theta
\end{bmatrix}
\begin{bmatrix}
u_1^i \\
u_2^i \\
u_3^i \\
u_1^j \\
u_2^j \\
u_3^j
\end{bmatrix}
$$

8.19

or

$$u_\zeta = T u$$

8.20

Substituting successively Eq. 8.8 into Eq.8.20 and the latter into Eq.8.17 it results

$$
\Delta u_\zeta = L T N \ddot{u}
$$

8.21

Once the local relative displacements are known, the local tractions $t_\zeta$ within the element can be computed as

$$
t_\zeta = D_\zeta \Delta u_\zeta$$

8.22

in which $D_\zeta$ is the constitutive matrix of the interface in the local Cartesian system.
Interface Finite Element Overview

\[
\mathbf{D}_\zeta = \begin{bmatrix}
D_{\xi_1\xi_1} & D_{\xi_1\xi_2} & D_{\xi_1\xi_3} \\
D_{\xi_2\xi_1} & D_{\xi_2\xi_2} & D_{\xi_2\xi_3} \\
D_{\xi_3\xi_1} & D_{\xi_3\xi_2} & D_{\xi_3\xi_3}
\end{bmatrix}
\]

8.4.1 Nonlinear Material Response

Three stress components, namely the two in-plane shear stresses and a normal to the interface mid-plane stress are considered, Fig. 8.7.

Earlier, in Chapter 4, the physical and computational advantages of the Desai surface were pointed out. A specially adopted version of the general 3-D Desai surface is utilized for simulation of the nonlinear response of interface regions. Only those aspects, which differ from the general model, will be focused upon.

Fig. 8.7 Stress components on an interface plane

Adaptation of the three dimensional expression of the Desai surface to the case of 3-D material interfaces results to the following definition for the response surface:

\[
f = \left(\frac{\tau_{\xi_1}^2 + \tau_{\xi_2}^2}{p_a^2}\right) - \left[-\alpha \left(\frac{\sigma + R}{p_a}\right)^n + \gamma \left(\frac{\sigma + R}{p_a}\right)^2\right] = 0
\]

A comparison with the corresponding expression in Chapter 4 reveals that due to the discontinuous nature of the interface, the term describing the variation of the surface on the octahedral plane has been dropped.

The hardening response of the model is controlled by parameter \(\alpha\). As \(\alpha\) decreases the size of the yield surface increases.

8.4.2 Parameter Determination

Standard laboratory shear tests can be utilized for evaluation of all necessary material parameters. In order to illustrate the calibration of the model parameters, the case of a "true" shear test (similar to the one developed and currently utilized at the Lab. of Road Engineering.
at TU Delft, de Bondt & Scarpas [1993, 1994]) will be considered. Nevertheless, the methodology can be applied to any other available test.

**Parameter $\gamma$**

At ultimate $\alpha = 0$. From Eq. 8.24 for uni-directional shear loading $\tau_{c2} = 0$, hence

$$\left( \frac{\tau_{c1}}{P_{a}} \right)^2 = \gamma \left( \frac{\sigma + R}{P_{a}} \right)^2$$

8.25

Taking the logarithm of both sides results to

$$2 \ln \left( \frac{\tau_{c1}}{P_{a}} \right) = \ln \gamma + 2 \ln \left( \frac{\sigma + R}{P_{a}} \right)$$

8.26

If the results of a series of tests are plotted in a $\ln \left( \frac{\tau_{c1}}{P_{a}} \right)$ versus $\ln \left( \frac{\sigma + R}{P_{a}} \right)$ space, $\gamma$ can be computed as the intercept of the best-fit line.

**Parameter $\alpha$**

The results of a typical shear stress test of an asphalt concrete interface performed at TU Delft, de Bondt [1999] are shown in Fig. 8.8. After some linear region, nonlinearity sets in and the strength of the material is reached.

![Monotonic Crack Shear Test](image)

**Fig. 8.8** TU Delft interface test results, de Bondt [1999]

Manifestation of an initial linear range in the stress-slip response of the material indicates the existence of an elastic locus of states of stress beyond which nonlinearity sets in. Damage occurs for states of stress outside the boundaries of this surface. In addition, slip values exceeding a limiting value $\varepsilon_{\text{lim}}$ result to response degradation.

The above two experimental facts can be simulated, by the following definition of hardening parameter $\alpha$

$$\alpha = \frac{\alpha}{1 + \varepsilon_{\text{lim}} - \varepsilon}$$

8.27

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in which $\alpha_1$ and $\eta_1$ are material coefficients to be determined experimentally, $\xi$ is defined as

$$\xi = \int \left( \Delta \varepsilon_r^2 + \Delta \varepsilon_\ell^2 + \Delta \varepsilon_\gamma^2 \right)^{1/2}$$

and $\xi_{\text{lim}}$ is the value of total equivalent plastic slip at maximum response.

Assuming that in a unidirectional shear test both interface slip $s_{\text{lim}}$ and dilatancy $w_{\text{lim}}$ have been measured then, at ultimate, from Eq. 8.28

$$\xi_{\text{lim}} = \left( s_{\text{lim}}^2 + w_{\text{lim}}^2 \right)^{1/2}$$

The value of $\tau$ corresponding to initiation of nonlinearity, can be determined from the experimental stress-slip curves. Substituting this value of $\tau$ into Eq. 8.24 and solving for $\alpha$, the value $\alpha_0$ corresponding to the initiation of nonlinearity is determined.

At this stage of response $\xi = 0$ hence from Eq. 8.27

$$\alpha_1 = \frac{\alpha_0}{\xi_{\text{lim}}}$$

The value of the remaining parameter $\eta_1$ can be determined as follows. Taking the logarithm of both sides of Eq. 8.27

$$\ln \alpha = \ln \alpha_1 - \ln \left( 1 + \xi_\ell \right) + \ln \left( \xi_{\text{lim}} - \xi \right)$$

At any stage of a given test all terms except $\eta_1$ are known. Least squares fitting of this equation can be used to determine the value of $\eta_1$.

### 8.5 Algorithmic Aspects

The increment of current plastic strain vector can be expressed as

$$t + \Delta t \Delta \varepsilon^p = \dot{\lambda} \Delta t \left( \frac{\partial g}{\partial \sigma} \right) = \dot{\lambda} \Delta t \begin{bmatrix} \varepsilon^p_{\xi_1} \\ \varepsilon^p_{\xi_2} \\ \varepsilon^p_{\xi_3} \end{bmatrix}$$

For the case of associated plasticity $g = f$ and then on the basis of Eq. 8.24 for the modified Desai surface:
\[
\frac{\partial f}{\partial \sigma} = \left( \begin{array}{c}
\frac{2\tau_{\xi_1}}{\sigma_{\xi_1}} \\
\frac{2\tau_{\xi_2}}{\sigma_{\xi_2}} \\
\alpha n \left( \frac{\sigma + R}{\sigma_{\xi_1}} \right) \frac{1}{\sigma_{\xi_1}} - 2\gamma \left( \frac{\sigma + R}{\sigma_{\xi_2}} \right) \frac{1}{\sigma_{\xi_2}} 
\end{array} \right)
\]

For shear type loading of non-smooth interfaces like cracks, the surface roughness is a controlling factor in the relationship between crack opening and crack slip. The dilatancy predicted by the model is

\[
\frac{t + \Delta t_{\xi_1}}{t + \Delta t_{\xi_1}}^p = \frac{\alpha n \left( \frac{\sigma + R}{\sigma_{\xi_1}} \right) \frac{1}{\sigma_{\xi_1}} - 2\gamma \left( \frac{\sigma + R}{\sigma_{\xi_2}} \right) \frac{1}{\sigma_{\xi_2}}}{\frac{2\tau_{\xi_1}}{\sigma_{\xi_1}} - \frac{\sigma_{\xi_2}^2}{\sigma_{\xi_2}}}
\]

If this differs from the experimentally observed one then it can be adjusted by means of a non-associative formulation. Introducing a new definition for \( g \) in Eq. 8.32

\[
g = \sqrt{\tau_{\xi_1}^2 + \tau_{\xi_2}^2} - \sigma_{\xi_1} \tan \alpha
\]

Then the flow vector is

\[
\frac{\partial g}{\partial \sigma} = \left( \begin{array}{c}
\tau_{\xi_1} / \sqrt{\tau_{\xi_1}^2 + \tau_{\xi_2}^2} \\
\tau_{\xi_2} / \sqrt{\tau_{\xi_1}^2 + \tau_{\xi_2}^2} \\
\tan \alpha
\end{array} \right)
\]

and the dilatancy predicted by the non-associative model is

\[
\frac{t + \Delta t_{\xi_1}}{t + \Delta t_{\xi_1}}^p = \frac{\tan \alpha}{\tau_{\xi_1} / \sqrt{\tau_{\xi_1}^2 + \tau_{\xi_2}^2}}
\]

which for the case of unidirectional shear results to

\[
\frac{t + \Delta t_{\xi_1}}{t + \Delta t_{\xi_1}} = \frac{\tan \alpha}{\tau_{\xi_1} / \sqrt{\tau_{\xi_1}^2 + 0}} = \tan \alpha
\]

Disassociating the model dilatancy from the strength criterion allows implementation of any experimentally observed dilatancy, even a variable one.
8.6 The Reinforcement Component

The reinforcement component of the interface element is assumed to be spanning along a layer perpendicular to the $\zeta_3$ local axis of the element at $\zeta_3 = -1$. A schematic of the resulting element is shown in Fig. 8.9.

![Fig. 8.9 Schematic of the reinforcement component](image)

Only the effects of a reinforcing layer are meant to be simulated by the reinforcing component and not those of a reinforcing bar. In this sense the terminology "smeared reinforcement simulation" is frequently encountered in the literature.

As shown in Fig. 8.10, within its layer, the direction of the reinforcement can make an angle $\phi$ with the local Cartesian system defined in Section 8.4. On the basis of this angle and the directions of the axes of the local Cartesian system, the directional cosines of the reinforcing layer are determined.

![Fig. 8.10 Schematic of reinforcement orientation wrt. the local Cartesian system](image)
At each integration point, the components of strain are defined with respect to the local Cartesian coordinate system. Only three mutually perpendicular strain components are admitted.

\[
\mathbf{e}_\zeta = \begin{bmatrix}
\varepsilon_{\zeta_1} \\
\varepsilon_{\zeta_2} \\
\varepsilon_{\zeta_3}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u_{\zeta_1}}{\partial \zeta_1} \\
\frac{\partial u_{\zeta_2}}{\partial \zeta_2} \\
\frac{\partial u_{\zeta_3}}{\partial \zeta_3}
\end{bmatrix}
\]

The above local derivatives can be obtained from the corresponding global ones by means of the transformation

\[
\begin{bmatrix}
\frac{\partial u_{\zeta_1}}{\partial \zeta_1} & \frac{\partial u_{\zeta_2}}{\partial \zeta_1} & \frac{\partial u_{\zeta_3}}{\partial \zeta_1} \\
\frac{\partial u_{\zeta_1}}{\partial \zeta_2} & \frac{\partial u_{\zeta_2}}{\partial \zeta_2} & \frac{\partial u_{\zeta_3}}{\partial \zeta_2} \\
\frac{\partial u_{\zeta_1}}{\partial \zeta_3} & \frac{\partial u_{\zeta_2}}{\partial \zeta_3} & \frac{\partial u_{\zeta_3}}{\partial \zeta_3}
\end{bmatrix} = [\theta]^T
\begin{bmatrix}
\frac{\partial u_1}{\partial X_1} & \frac{\partial u_2}{\partial X_1} & \frac{\partial u_3}{\partial X_1} \\
\frac{\partial u_1}{\partial X_2} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_3}{\partial X_2} \\
\frac{\partial u_1}{\partial X_3} & \frac{\partial u_2}{\partial X_3} & \frac{\partial u_3}{\partial X_3}
\end{bmatrix} [\theta]
\]

The global derivatives in Eq. 8.40 are computed via the procedure indicated in Section 2.8.2.

Once the local strains are known, the local tractions \( \mathbf{t}_\zeta \) within the reinforcement can be computed as

\[
\mathbf{t}_\zeta = [\mathbf{D}_\zeta] \mathbf{e}_\zeta
\]

in which \( [\mathbf{D}_\zeta] \) is the constitutive matrix of the interface in the local Cartesian system.

\[
[\mathbf{D}_\zeta] = \begin{bmatrix}
D_{\zeta_1} & 0 & 0 \\
0 & D_{\zeta_2} & 0 \\
0 & 0 & D_{\zeta_3}
\end{bmatrix}
\]

\( D_{\zeta_1} \) can be typically associated with the axial stiffness of the reinforcing layer. The other two stiffness coefficients can be set to zero indicating the very flexible nature of typical pavement reinforcing materials in the transverse directions.
8.7 Utilization

Since the early stages of CAPA-3D development, the interface element found extensive utilization in a very wide range of applications. Among others, de Bondt & Scarpas [1993, 1994] and de Bondt [1999] utilized the element for simulation of the response of the crack interface regions in cracked structural members and pavements. Scarpas et al. [1993], Scarpas & de Bondt [1996] and de Bondt [1996] utilized the element for simulation of the reinforcement and its bond with the surrounding material in road and airfield pavements.


On the basis of the formulation of the interface element as presented in the above, a new contact interface element is currently under development in the PhD thesis of Zhao [2006].
Chapter 9

Infinite Boundary Finite Element Overview

9.1 Introduction

In both soil and pavement engineering unbound continua arise in several situations. The traditional “engineering” approach of simply truncating the finite element mesh beyond a “reasonable” distance from the area of interest can introduce indeterminate errors in the solution. Truncation also introduces uncertainties about the actual nature of boundary conditions at the edges of the mesh.

Historically, several approaches have been proposed for solution of the problem, Wood [1976], Silvester & Hsieh [1971], Zienkiewicz et al. [1977], Bettess [1977], Thatcher [1978], Rukos [1978], Kelly et al. [1979], Bettess [1980], Telles & Brebbia [1981], Beer & Meek [1981] and several others. Even though these approaches addressed adequately the issue of the infinite domain, their implementation into the finite element method required the development and utilization of specialized procedures and solution techniques.

The above mentioned implementation difficulties were circumvented when in 1983 Zienkiewicz et al. [1983] proposed a new type of element which is compatible with the typical isoparametric elements utilised in most finite element analyses. The element has become known as the “mapped infinite element” and has since its inception found several applications Marques & Owen [1984], Bettess [1992]. The element is based on the notion of mapping a semi-infinite domain to that of a standard finite element as shown in Fig. 9.1.

![Diagram of mapped infinite element](image)

Fig. 9.1 Mapping of a semi-infinite domain to a finite element space
Infinite Boundary Finite Element Overview

As shown in Fig. 9.2, the essence of the mapping lies in the choice of appropriate functions capable of mapping any physical length PQR (with R lying at infinity) onto an equivalent length spanning from $\xi = -1$ to $\xi = +1$.

![Diagram](image)

**Fig. 9.2** Mapping of the physical coordinates to the local ones

Defining the coordinates of P and Q at distance $a$ and $2a$ from the origin of axes, the desired mapping can be expressed as a linear combination of some functions $M_i(\xi)$ and the physical coordinates $A_i$ of the “finite” points i.e. points P and Q

$$X = \sum_{i=1}^{m} M_i A_i$$  \hspace{1cm} 9.1

in which $m = 2$ is the number of “finite” nodes. Postulating for example

$$M_1 = \frac{2\xi}{1-\xi} ; \quad M_2 = \frac{1+\xi}{1-\xi}$$  \hspace{1cm} 9.2

it can be verified that for $\xi = -1, 0$ and $+1$ the coordinates $a, 2a$ and $\infty$ of points P, Q and R are obtained.

Further insight into the mapping can be obtained by solving Eq. 9.1 in terms of $\xi$. Then

$$\xi = 1 - \frac{2a}{X}$$  \hspace{1cm} 9.3

If it is assumed that a field variable $\phi$ is associated with the points of the line PQR, then a standard isoparametric interpolation will provide the values of $\phi$ as

$$\phi = \sum_{i=1}^{n} N_i \phi_i$$  \hspace{1cm} 9.4

in which $n$ is the total number of interpolation points (i.e. $n = 3$ for the example in Fig. 9.2) and $N_i$ are the standard interpolation functions

$$N_1 = \frac{\xi^2 - \xi}{2} ; \quad N_2 = 1 - \xi^2 ; \quad N_3 = \frac{\xi^2 + \xi}{2}$$  \hspace{1cm} 9.5
Substituting the value of $\xi$ from Eq. 9.3 into Eq. 9.5 and subsequently into Eq. 9.4 it results

$$\phi = c_3 + \left( -\phi_1 + 4\phi_2 - 3\phi_3 \right) \left( \frac{a}{X} \right) + \left( 2\phi_1 - 4\phi_2 + 2\phi_3 \right) \left( \frac{a}{X} \right)^2$$

or

$$\phi = c_3 + c_2 \left( \frac{a}{X} \right) + c_1 \left( \frac{a}{X} \right)^2$$

from which it can be concluded that as $X$ tends to infinity, $\phi$ decays parabolically to the limit value of $\phi_3$.

It is not necessary for $X$ to represent the distance to the origin of the coordinate system. Any other point like $C$ in Fig. 9.1 can be utilised as the origin from which the locations of nodes $P$ and $Q$ are measured. The terminology "pole" has been utilised, Marques & Owen [1984], to indicate such a point.

It is apparent that different choices of the ratio $CP/CQ$ affect the coefficients of the parabolic decay of the field variable $\phi$ denoted by Eq. 9.6. Also, the order of the decay is affected by the order of the interpolation as expressed by Eq. 9.4.

### 9.2 Element Formulation

Extension of the procedure described above to two or three dimensional situations can be accomplished by taking shape function products of the $M_i$ functions with the standard interpolation functions $N_i$. In CAPA-3D a 12 noded infinite finite element has been implemented which is based on the cubic 20 noded element of Chapter 7.

#### 9.2.1 Element Geometry Interpolation

Fig. 9.3 shows the local ordering of the nodes and the orientation of the associated local axes system $\left( \xi_1, \xi_2, \xi_3 \right)$.

![Fig. 9.3 12-noded infinite element](image)
Following a typical isoparametric formulation, the matrix $M$ of interpolation functions for a 12-noded element can be defined as:

$$ M = \begin{bmatrix} M_1 & M_2 & \ldots & M_{12} \end{bmatrix} $$

in which $M_k = M_k(\xi)$ and $\mathbf{\xi} = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix}^T$.

If the vector of nodal coordinates $\mathbf{X}_k$ is defined as:

$$ \mathbf{X}_k = \begin{bmatrix} X_{k1} & X_{k2} & X_{k3} \end{bmatrix}^T $$

then the initial configuration of any point within the element can be interpolated in terms of the corresponding nodal coordinates as:

$$ \mathbf{X} = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}^T = \sum_{k=1}^{12} M_k \mathbf{X}_k $$

The mapping functions are:

$$ M_1 = (1 - \xi_1)(1 - \xi_2)(-\xi_1 - \xi_2 - \xi_3 - 2)/[2(1 - \xi_3)] $$
$$ M_2 = (1 - \xi_1^2)(1 - \xi_2)/(1 - \xi_3) $$
$$ M_3 = (1 + \xi_1)(1 - \xi_2)(+\xi_1 - \xi_2 - \xi_3 - 2)/[2(1 - \xi_3)] $$
$$ M_4 = (1 + \xi_1)(1 - \xi_2^2)/(1 - \xi_3) $$
$$ M_5 = (1 + \xi_1)(1 + \xi_2)(+\xi_1 + \xi_2 - \xi_3 - 2)/[2(1 - \xi_3)] $$
$$ M_6 = (1 - \xi_1^2)(1 + \xi_2)/(1 - \xi_3) $$
$$ M_7 = (1 - \xi_1)(1 + \xi_2)(-\xi_1 + \xi_2 - \xi_3 - 2)/[2(1 - \xi_3)] $$
$$ M_8 = (1 - \xi_1)(1 - \xi_2^2)/(1 - \xi_3) $$
$$ M_9 = (1 - \xi_1)(1 - \xi_2)(1 + \xi_3)/[4(1 - \xi_3)] $$
$$ M_{10} = (1 + \xi_1)(1 - \xi_2)(1 + \xi_3)/[4(1 - \xi_3)] $$
$$ M_{11} = (1 + \xi_1)(1 + \xi_2)(1 + \xi_3)/[4(1 - \xi_3)] $$
$$ M_{12} = (1 - \xi_1)(1 + \xi_2)(1 + \xi_3)/[4(1 - \xi_3)] $$

### 9.2.2 Field Variables Interpolation

The displacements can be expressed in terms of the nodal displacements $\mathbf{u}_k$:

$$ \mathbf{u} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}^T = \sum_{k=1}^{20} N_k \mathbf{u}_k $$

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in which the interpolation functions $N_k$ have been defined in Chapter 7.

The engineering strain vector for the element is defined as

$$
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{23} \\
\gamma_{13}
\end{bmatrix}
= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_1}
= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_2}
= \frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_3}
= \mathbf{L} \mathbf{u}
= \begin{bmatrix}
\frac{\partial}{\partial x_1} & 0 & 0 \\
0 & \frac{\partial}{\partial x_2} & 0 \\
0 & 0 & \frac{\partial}{\partial x_3}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= \mathbf{L} \mathbf{u}
\tag{9.12}
$$

with the notation $\gamma_{ij} = 2\varepsilon_{ij}$.

Rewriting Eq. 9.11 as

$$
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= \begin{bmatrix}
\tilde{N}_1 & \tilde{N}_2 & \ldots & \tilde{N}_{20}
\end{bmatrix}
\tilde{\mathbf{u}}
\tag{9.13}
$$

in which $\mathbf{\tilde{u}}$ is the vector of nodal displacements $\mathbf{\tilde{u}} = \begin{bmatrix} u_1 & u_2 & \ldots & u_{20} \end{bmatrix}^T$

with

$$
\mathbf{u}_k = \begin{bmatrix} u_{k1} & u_{k2} & u_{k3} \end{bmatrix}^T
\tag{9.15}
$$

$$
\mathbf{\tilde{N}}_k = \begin{bmatrix}
N_k & 0 & 0 \\
0 & N_k & 0 \\
0 & 0 & N_k
\end{bmatrix}
\quad k = 1, 2, \ldots, 20
\tag{9.16}
$$

then, after substitution in Eq. 9.12, the familiar expression for the strains as a function of the nodal displacements is obtained as

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\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{23} \\
\gamma_{13}
\end{bmatrix}
= \begin{bmatrix}
B_1 & B_2 & \ldots & B_{20}
\end{bmatrix} \tilde{u} \\
9.17
\]

in which the terms $B_k$ have the form

\[
B_k = \begin{bmatrix}
\frac{\partial N_k}{\partial x_1} & 0 & 0 \\
0 & \frac{\partial N_k}{\partial x_2} & 0 \\
0 & 0 & \frac{\partial N_k}{\partial x_3}
\end{bmatrix}
9.18
\]

In order to evaluate the individual Cartesian derivatives $\nabla_0 N_k$ in Eq. 9.18, the following transformation is utilized

\[
\nabla_0 N_k = J^{-T} \nabla_\xi N_k
9.19
\]

in which $J$ is the coordinate Jacobian matrix:

\[
J = \sum_{i,j=1}^{3} \frac{\partial x_i}{\partial \xi_j} E_j \otimes E_i = \begin{bmatrix}
\frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\
\frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\
\frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3}
\end{bmatrix}
9.20
\]

On the basis of Eq. 9.9 the individual terms of $J$ can be computed as:

\[
\begin{bmatrix}
\frac{\partial x_1}{\partial \xi_j} \\
\frac{\partial x_2}{\partial \xi_j} \\
\frac{\partial x_3}{\partial \xi_j}
\end{bmatrix} = \sum_{k=1}^{12} \frac{\partial M_k}{\partial \xi_j} X_{ki} \quad 9.21
\]
In CAPA-3D $\mathbf{J}$ and $\mathbf{J}^{-\mathbf{T}}$ are evaluated at every integration point of the element. The element stiffness is computed on the basis of numerical integration over the volume of the element. Only a $3 \times 3 \times 3$ integration scheme can be specified (see Chapter 7).

### 9.3 Element Utilization

Because the infinite element implemented in CAPA-3D is based on the 20 noded isoparametric element of Chapter 7, it can simulate a parabolic decay of field variables within its domain (see Eq. 9.6 and Eq. 9.9). For most soil mechanics/pavement engineering applications such a decay is adequate for simulation of the far field conditions.

When positioning the poles, the following issues must be taken into account:

- the poles must be external to the infinite element
- the element sides extending towards infinity, must be parallel or divergent to avoid the overlapping of elements and in order to preserve mapping uniqueness
- material variations in the domain of the element are not possible

Two typical examples of the utilization of infinite elements are shown in the following. In the first example, Fig. 9.4, a slice of a geological profile under (approximate) plain strain conditions is simulated. In the second example, Fig. 9.5, an axisymmetric simulation is presented of a circular load applied on a layered profile.

![Fig. 9.4 3D slice of a geologic profile](image)

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Fig. 9.5 Axisymmetric simulation of a distributed circular load on a layered profile
The local derivatives of the geometry interpolation functions are as follows

\[ \frac{\partial M_1}{\partial \xi_1} = \frac{(1 - \xi_2) (2\xi_1 + \xi_2 + \xi_3 + 1)}{2 (1 - \xi_3)} \]

A.9.1.1

\[ \frac{\partial M_1}{\partial \xi_2} = \frac{(1 - \xi_1) (\xi_1 + 2\xi_2 + \xi_3 + 1)}{2 (1 - \xi_3)} \]

\[ \frac{\partial M_1}{\partial \xi_3} = \frac{(1 - \xi_1) (1 - \xi_2) (-\xi_1 - \xi_2 - 3)}{2 (1 - \xi_3)^2} \]

A.9.1.2

\[ \frac{\partial M_2}{\partial \xi_1} = -2\xi_1 (1 - \xi_2) / (1 - \xi_3) \]

\[ \frac{\partial M_2}{\partial \xi_2} = -(1 - \xi_1^2) / (1 - \xi_3) \]

A.9.1.3

\[ \frac{\partial M_2}{\partial \xi_3} = (1 - \xi_1^2) (1 - \xi_2) / [(1 - \xi_3)^2] \]

\[ \frac{\partial M_3}{\partial \xi_1} = \frac{(1 - \xi_2) (2\xi_1 - \xi_2 - \xi_3 - 1)}{2 (1 - \xi_3)} \]

\[ \frac{\partial M_3}{\partial \xi_2} = \frac{(1 + \xi_1) (-\xi_1 + 2\xi_2 + \xi_3 + 1)}{2 (1 - \xi_3)} \]

\[ \frac{\partial M_3}{\partial \xi_3} = \frac{(1 + \xi_1) (1 - \xi_2) (\xi_1 - \xi_2 - 3)}{2 (1 - \xi_3)^2} \]
\[
\frac{\partial M_4}{\partial \xi_1} = \frac{(1 - \xi_2^2)}{(1 - \xi_3)}
\]

\[
\frac{\partial M_4}{\partial \xi_2} = -2\xi_2 \frac{(1 + \xi_1)}{(1 - \xi_3)}
\]

\[
\frac{\partial M_4}{\partial \xi_3} = \frac{(1 + \xi_1)(1 - \xi_2^2)}{(1 - \xi_3)^2}
\]

\[
\frac{\partial M_5}{\partial \xi_1} = \frac{(1 + \xi_3)(2\xi_1 + \xi_2 - \xi_3 - 1)}{2(1 - \xi_3)}
\]

\[
\frac{\partial M_5}{\partial \xi_2} = \frac{(1 + \xi_1)(\xi_1 + 2\xi_2 - \xi_3 - 1)}{2(1 - \xi_3)}
\]

\[
\frac{\partial M_5}{\partial \xi_3} = \frac{(1 + \xi_1)(1 + \xi_2)(\xi_1 + \xi_2 - 3)}{2(1 - \xi_3)^2}
\]

\[
\frac{\partial M_6}{\partial \xi_1} = -2\xi_1 \frac{(1 + \xi_2)}{(1 - \xi_3)}
\]

\[
\frac{\partial M_6}{\partial \xi_2} = \frac{(1 - \xi_1^2)}{(1 - \xi_3)}
\]

\[
\frac{\partial M_6}{\partial \xi_3} = \frac{(1 - \xi_1^2)(1 + \xi_2)}{(1 - \xi_3)^2}
\]

\[
\frac{\partial M_7}{\partial \xi_1} = \frac{(1 + \xi_2)(2\xi_1 - \xi_2 + \xi_3 + 1)}{2(1 - \xi_3)}
\]

\[
\frac{\partial M_7}{\partial \xi_2} = \frac{(1 - \xi_1)(-\xi_1 + 2\xi_2 - \xi_3 - 1)}{2(1 - \xi_3)}
\]

\[
\frac{\partial M_7}{\partial \xi_3} = \frac{(1 - \xi_1)(1 + \xi_2)(-\xi_1 + \xi_2 - 3)}{2(1 - \xi_3)^2}
\]
\[
\frac{\partial M_8}{\partial \xi_1} = -\frac{(1 - \xi_2^2)}{(1 - \xi_3)}
\]

\[\frac{\partial M_8}{\partial \xi_2} = -2\xi_2 \cdot \frac{(1 - \xi_1)}{(1 - \xi_3)} \quad A.9.1.8
\]

\[
\frac{\partial M_8}{\partial \xi_3} = \frac{(1 - \xi_1) \cdot (1 - \xi_2^2)}{(1 - \xi_3)} \quad \left(1 - \xi_3 \right)^2
\]

\[
\frac{\partial M_9}{\partial \xi_1} = -\frac{(1 - \xi_2) \cdot (\xi_3 + 1)}{4(1 - \xi_3)}
\]

\[
\frac{\partial M_9}{\partial \xi_2} = -\frac{(1 - \xi_1) \cdot (\xi_3 + 1)}{4(1 - \xi_3)} \quad A.9.1.9
\]

\[
\frac{\partial M_9}{\partial \xi_3} = \frac{(1 - \xi_1) (1 - \xi_2)}{2(1 - \xi_3)^2}
\]

\[
\frac{\partial M_{10}}{\partial \xi_1} = \frac{(1 - \xi_2) (\xi_3 + 1)}{4(1 - \xi_3)}
\]

\[
\frac{\partial M_{10}}{\partial \xi_2} = -\frac{(1 + \xi_1) (\xi_3 + 1)}{4(1 - \xi_3)} \quad A.9.1.10
\]

\[
\frac{\partial M_{10}}{\partial \xi_3} = \frac{(1 + \xi_1) (1 - \xi_2)}{2(1 - \xi_3)^2}
\]

\[
\frac{\partial M_{11}}{\partial \xi_1} = \frac{(1 + \xi_2) (1 + \xi_3)}{4(1 - \xi_3)}
\]

\[
\frac{\partial M_{11}}{\partial \xi_2} = \frac{(1 + \xi_1) (1 + \xi_3)}{4(1 - \xi_3)} \quad A.9.1.11
\]

\[
\frac{\partial M_{11}}{\partial \xi_3} = \frac{(1 + \xi_1) (1 + \xi_2)}{2(1 - \xi_3)^2}
\]
\[ \frac{\partial M_{12}}{\partial \xi_1} = -\frac{(1 + \xi_2)(1 + \xi_3)}{4(1 - \xi_3)} \]

\[ \frac{\partial M_{12}}{\partial \xi_2} = \frac{(1 - \xi_1)(1 + \xi_3)}{4(1 - \xi_3)} \]

\[ \frac{\partial M_{12}}{\partial \xi_3} = \frac{(1 - \xi_1)(1 + \xi_2)}{2(1 - \xi_3)^2} \]
Chapter 10

Silent Boundary Finite Element Overview

10.1 Introduction

The finite element solution of problems involving wave phenomena in unbounded media has been an active area of research ever since the establishment of the method. Over the years several finite elements and techniques have been proposed for simulation of the absorbing or non-reflecting conditions at the boundaries of the finite element domain. An extensive literature survey can be found in Bettiess and Bettiess [1991a and 1991b] and more recently in Givoli and Harari [1998].

From these surveys it becomes clear that the majority of the published techniques address steady state wave propagation. Others, even though they can address non-steady state wave propagation, require multistage solution processes and/or nonstandard finite element domain arrangements. Both of these are beyond the capabilities of the average finite element method user and constitute domains of active research rather than standard engineering analysis.

On the basis of these considerations it was decided that the method of the absorbing boundary element as proposed originally by Lysmer and Kuhlemeyer [1969] provided a relatively accurate and yet practical methodology for the finite element handling of non-reflecting boundaries.

The method is based on the idea of developing special finite elements which, when placed at the boundary regions of the finite element domain, will ensure that the energy arriving at the boundary is absorbed, Fig. 10.1.

![Diagram](image)

Fig. 10.1 (a) Wave propagation in a semi-infinite medium
(b) Wave absorption at the boundary of a finite domain
By means of the principle of conservation of momentum it can be shown that in an isotropic medium of density $\rho$, the traction vector at a point along the direction $\zeta_1$ perpendicular to a boundary, Fig. 10.2, due to an incident wave of velocity $\mathbf{v}$ is

$$t_{\zeta_1} = \rho c_p \left( \zeta_1^T \mathbf{v} \right) \zeta_1$$ \hspace{2cm} \text{(10.1)}

Similarly, along the directions $\zeta_2$ and $\zeta_3$ of the locally defined Cartesian system $\left( \zeta_i, i = 1\ldots3 \right)$ it holds

$$t_{\zeta_2} = \rho c_s \left( \zeta_2^T \mathbf{v} \right) \zeta_2$$ \hspace{2cm} \text{(10.2)}

$$t_{\zeta_3} = \rho c_s \left( \zeta_3^T \mathbf{v} \right) \zeta_3$$ \hspace{2cm} \text{(10.3)}

in which $c_p$ is the dilatational wave velocity and $c_s$ is the shear wave velocity.

The concept of the introduction of an absorbing boundary at the far regions of the finite element domain is equivalent to ensuring what is known in wave propagation theory as the \textit{radiation condition}. This condition requires that no energy is transmitted from infinity back to the structure.

\textbf{10.2 Element Formulation}

The element formulation is based on that of the interface element presented in Chapter 8. Fig. 10.3 shows the local ordering of the nodes. The thickness of the element in its undeformed configuration can be specified to be very small or even zero.

As shown schematically in Fig. 10.3, incident waves are assumed to arrive at the top face of the element and dissipated within the element.
10.2.1 Field Variables Interpolation

Only local behavior is meant to be simulated by each individual element. The overall behavior can be simulated by a series of elements placed along the trace of the physical far field boundary. The global displacements of a point on the top face of the element, can be expressed in terms of the nodal displacements of the same face and the standard shape functions defined in Chapter 8 as:

\[
\begin{bmatrix}
    u_1 \\
    u_2 \\
    u_3
\end{bmatrix}
= N^k
\begin{bmatrix}
    u_{91} \\
    u_{92} \\
    u_{93} \\
    u_{101} \\
    \vdots \\
    u_{153} \\
    u_{161} \\
    u_{162} \\
    u_{163}
\end{bmatrix}
\]

10.4

with

\[
N^k =
\begin{bmatrix}
    N_1 & 0 & 0 & N_2 & \ldots & 0 & N_8 & 0 & 0 \\
    0 & N_1 & 0 & 0 & \ldots & 0 & 0 & N_8 & 0 \\
    0 & 0 & N_1 & 0 & \ldots & N_7 & 0 & 0 & N_8
\end{bmatrix}
\]

10.5

or

\[
u = N \ddot{u}
\]

10.6

The velocities can be similarly interpolated as

\[
\dot{u} = N \ddot{v}
\]

10.7
10.3 Constitutive Law

At each integration point of the element, a local Cartesian coordinate system \( (\zeta_i; i=1\ldots3) \) is set up as elaborated in Chapter 8, Fig. 10.4. The relation between the local velocity vector \( \dot{\zeta} \) and the corresponding global velocity vector is:

\[
\begin{bmatrix}
\dot{u}_1 \\
\dot{u}_2 \\
\dot{u}_3
\end{bmatrix} =
\begin{bmatrix}
\cos(\zeta_1, x_1) & \cos(\zeta_1, x_2) & \cos(\zeta_1, x_3) \\
\cos(\zeta_2, x_1) & \cos(\zeta_2, x_2) & \cos(\zeta_2, x_3) \\
\cos(\zeta_3, x_1) & \cos(\zeta_3, x_2) & \cos(\zeta_3, x_3)
\end{bmatrix}
\begin{bmatrix}
\dot{u}_1 \\
\dot{u}_2 \\
\dot{u}_3
\end{bmatrix}
\]

or

\[
\dot{\zeta} = \mathbf{T} \dot{u}
\]

Fig. 10.4 Local Cartesian system

Substituting Eq. 10.6 into Eq.10.9 it results

\[
\dot{\zeta} = \mathbf{T} \mathbf{N} \dot{\mathbf{v}}
\]

\[
= \mathbf{B} \dot{\mathbf{v}}
\]

Once the local velocities are known, the local tractions \( t_{\zeta} \) can be computed as

\[
t_{\zeta} = \rho \cdot \mathbf{c} \cdot \dot{\zeta}
\]

in which

\[
\mathbf{c} =
\begin{bmatrix}
c_p & 0 & 0 \\
0 & c_s & 0 \\
0 & 0 & c_s
\end{bmatrix}
\]
10.4 Virtual Work

The governing equilibrium equation can be obtained from the virtual work principle. Assuming that nodal forces $F$ are the only external actions applied on the element, then for a set of virtual displacements it holds

$$\delta d^T \Delta t F = \int_A \delta u^T \Delta t \epsilon^T dA$$  \hspace{1cm} 10.13

Substituting $\epsilon$ from Eq. 10.11 and $\dot{u}$ from Eq. 10.10

$$\delta d^T \Delta t F = \rho \int_A \delta d^T B^T c B \Delta t \epsilon dA$$

$$= \rho \int_A \delta d^T B^T c B \left( t \epsilon + \epsilon \right) dA$$  \hspace{1cm} 10.14

On the basis of the constant average acceleration scheme $d\ddot{v}$ can be computed as

$$d\ddot{v} = \frac{2}{\Delta t} d\ddot{u}$$  \hspace{1cm} 10.15

Replacing $d\dot{v}$ in Eq. 10.14 and eliminating $\delta d^T$ from both sides it results

$$t^+ \Delta t F = \rho \int_A B^T c B \left( t \ddot{v} + \frac{2}{\Delta t} d\ddot{u} \right) dA$$

$$= \rho \int_A B^T c B dA \epsilon + \frac{2}{\Delta t} \int_A B^T c B dA d\ddot{u}$$  \hspace{1cm} 10.16

which can be rearranged as

$$t^+ \Delta t F - \rho \int_A B^T c B dA \epsilon \dot{v} = \rho \frac{2}{\Delta t} \int_A B^T c B dA d\ddot{u}$$  \hspace{1cm} 10.17

From Eq. 10.17 it becomes apparent that the term on the right hand side constitutes the contribution of the element to the global stiffness matrix of the structure while the integral term on the left hand side represents the internal element actions.

10.5 Utilization

The element has been extensively utilised in dynamic analyses of the response of various structures. Typical examples can be found in the thesis of Duvert [2000] who utilised the element for sensitivity studies of wave propagation is layered media and, in the thesis of Carree [2000] who utilised the element for simulation of the far field conditions in a vulnerability study of various geological profiles in the city of Pereira, Colombia.

It is recommended that the element is utilised in all dynamic analyses in which wave reflections from the boundaries of the finite element mesh may influence the response of the region of interest.
Silent Boundary Finite Element Overview
Summary & Conclusions

Current pavement design methodologies are based primarily on empirical rules. Relying on experience is however not acceptable anymore because of the rapidly changing conditions of the international road network.

Because these are far beyond our experience, one needs to address pavement design by applying proper design and evaluation models, which are based on a sound engineering approach towards material behaviour. For this reason, interest in mechanics based approaches for road engineering design has recently grown considerably, both nationally and internationally.

The current advent of powerful computational hardware systems, will make mechanics based tools like finite element software accessible to a wider research and engineering audience.

The adoption of finite element based tools will facilitate the way for a new generation of pavement design techniques based on rational mechanics principles. A typical example of this change in design philosophy is the recent NCHRP 1-37A [2004] report on future pavement design, in which the finite element method in combination with advanced material constitutive models and characterisation techniques constitute the backbone of the whole design process.

In response to the above changes in national and international pavement design philosophy, the finite element system CAPA-3D has been developed as a computational platform for the static/dynamic analysis of very large scale three dimensional pavement and soil engineering models across a broad range of hardware platforms.

The system consists of a mesh generation facility with pre- and post processing capability, the main finite element code and an analysis data post-processing facility. All these facilities are integrated in a Windows based user interface.

A library of generic constitutive models suitable for pavement and soil engineering applications is included. The architecture of the system allows the combination of these for the development of more complex constitutive models.

Also, a library of various specialised finite element types necessary for efficient analyses of pavement and soil engineering problems is included.

By accounting for the idiosyncrasies of pavement material response and by enabling the visualisation of the internal distributions of stresses and strains in the body of a pavement, the finite element method constitutes a valuable tool in understanding the mechanisms and the processes leading to pavement deterioration. As such it will make possible the adoption of the new generation of “mechanistic” pavement design techniques.

A. Scarpas
References


References


References


References


References


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Samenvatting

De huidige wegontwerp methoden zijn voornamelijk gebaseerd op empirische regels. Helaas is het rekenen op opgedane ervaringen in het verleden niet langer meer voldoende, gezien de snel veranderende omstandigheden van de internationale infrastructuur.

Omdat deze omstandigheden niet eens in de buurt komen van onze ervaringen, moet men nu voor het ontwerpen van wegen uitgaan van toepasselijke ontwerp- en evaluatiemodellen, gebaseerd op solide ingenieurs benaderingen van materiaalgedrag. Om deze reden is de interesse, zowel nationaal als internationaal, in op mechanica gebaseerde methoden voor wegontwerp de laatste tijd sterk toegenomen.

De huidige opkomst van sterke computer systemen zal de op mechanica gebaseerde software, zoals eindige elementenmethode pakketten, toegankelijk maken voor een breed publiek van onderzoekers en ingenieurs.


In reactie op de bovenstaande veranderingen in de nationale en internationale wegontwerpfilosofie, is het eindige elementen systeem CAPA-3D ontworpen als zijdde een computer platform voor de statische/dynamische analyse van grootschalige drie dimensioele weg- en grondmechanica ingenieursmodellen, verspreid over een grote variatie van hardware platforms.

Het systeem bestaat uit een mesh generator voorzien met pre- en post processing mogelijkheden, een hoofd programma met de eindige elementen code en een voorziening voor data-analyse post-processing. Al deze voorzieningen zijn geïntegreerd binnen een Windows gebaseerde gebruikers interface.

Een uitgebreide bibliotheek van algemene materiaalmodellen, geschikt voor weg- en grondmechanica ingieurstoepassingen, is geïntegreerd binnen het systeem. De opzet van het systeem staat een combinatie van deze modellen toe voor de ontwikkeling van complexere materiaalmodellen. Er is tevens een bibliotheek aanwezig van verschillende speciale eindige elementen types die nodig zijn voor een efficiënte analyse van weg- en grondmechanica ingenieursproblemen.

Door rekening te houden met de eigenaardigheden van het gedrag van de verschillende wegmaterialen en door het in staat stellen van de visualisatie van de inwendige verspreiding van de spanningen en rekken in het weglichaam, vormt de eindige elementen methode een waardevolle techniek voor het verkrijgen van inzicht in de mechanismen en processen die uiteindelijk leiden tot de verslechtering van wegen. In deze hoedanigheid zal deze techniek het algemene gebruik van een nieuwe generatie mechanische wegontwerp technieken mogelijk maken.

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<table>
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