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ADVANCES IN APPLIED MECHANICS, Vol. I, 1948

**Survey of Papers on Elasticity
Published in Holland
1940—1946**

by

C. B. BIEZENO

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ADVANCES IN APPLIED MECHANICS

EDITED BY

RICHARD VON MISES

*Harvard University
Cambridge, Massachusetts*

THEODORE VON KÁRMÁN

*California Institute of Technology
Pasadena, California*

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Survey of Papers on Elasticity Published in Holland 1940-1946.

By C. B. BIEZENO

Technical University, Delft, Holland

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The invitation of Professor von Mises to contribute to these "Advances" a survey of the work done in Holland during the war in the field of applied mechanics, in particular in the field of elasticity, has been met by the author with the greatest pleasure. After the sad years in which Holland suffered from the monstrous brutalities to which all western European peoples were subjected by a merciless occupant, it is generally felt as a duty to re-establish as soon as possible the manifold scientific connections by which our country was bound to its allies.

Since he has devoted a great deal of his endeavour to the international exchange of scientific advancement, the author enjoys the possibility of re-establishing contact with his American colleagues and hopes to profit as soon as possible from the enormous amount of knowledge and experience that has been heaped up during the past few years in the United States of America where the war did not stop—but even gave rise to—the creation of different scientific institutions, as for example, the Society for Experimental Stress Analysis.

It will not surprise the reader that—given the prevailing circumstances—Holland cannot claim to have contributed a great deal to the total number of newly issued publications. As a matter of fact in this country only two or three centers of specialized "mechanical" activity do exist: the Technische Hoogeschool (Technical University) at Delft, the recently founded department for experimental research of constructions of the Centrale Organisatie voor Toegepaste Natuurwetenschappelijk Onderzoek at the Hague (Central Institute for Technical Scientific Research), and the Nationaal Luchtvaartlaboratorium (the National Aeronautical laboratory) at Amsterdam. It is mainly the work of these three institutes that will be reviewed here.

Chronological order has been neglected in favor of a methodical treatment, but it must be emphasized that, in consequence of the diversity of the different subjects, coherence of this monograph as a whole could not be reached. Matters are divided according to their theoretical or experimental character, whereas publications in book form have been treated separately. The reader is referred to the index for further information about the topics of research.¹

¹ Of nearly all treatises mentioned in this monograph, reprints are still available, and it will be a pleasure to the author to send them on application to the reader who may be interested in any of them; requests may be directed to him personally at the address: Laboratorium voor Toegepaste Mechanica, Nieuwe Laan 76, Delft, Holland.

I. PUBLICATIONS OF THE LABORATORY FOR APPLIED MECHANICS OF THE
TECHNISCHE HOOGESCHOOL AT DELFTA. *Elastic Stability*

1. *On the Stability of Elastic Equilibrium.* Hitherto the general theories of stability as developed by Bryan, Southwell, Biezeno and Hencky, Trefftz, Marguerre, Biot and others have been restricted to the investigation of neutral equilibrium; they aim particularly at the determination of the stability limit. The phenomena occurring on reaching and possibly on surpassing this limit were left out of account. This restriction as to the extent of the investigations is caused by two circumstances. First of all there must be mentioned the great mathematical difficulties that obstruct the theoretical treatment of elastic behavior after the stability limit is surpassed. Whereas the investigation of states of neutral equilibrium is still possible by means of linear differential equations, the equations describing elastic behavior after the stability limit is surpassed are no longer linear. Moreover engineering has long been satisfied with the knowledge of the stability limit (critical or buckling load). The recognized principle based on considerations of safety was that the load on a structure should always be kept below this limit, so that an investigation of the phenomena occurring above this limit seemed superfluous. However, it has been known for a long time that some structures (*e.g.* flat plates) are capable of sustaining considerably larger loads than the buckling one without exceeding the elastic limit at any point of the structure; in modern engineering, especially in aircraft engineering, where saving on structural weight is of the greatest importance, these higher loads are actually allowed. The theoretical treatment of this problem has been given by Marguerre and Trefftz, among others. On the other hand it has been noted that the experimentally determined buckling loads of some shell structures, *e.g.*, axially compressed thin-walled cylinders, lie considerably below the theoretical stability limit. Moreover the experimental results are widely divergent. (Cf. investigations by Flügge, Donnell, Cox, von Kármán and Tsien.)

From all this it appears that the general theories of stability framed so far do not suffice. They have to be completed in such a way that the divergent behavior of various structures in the case of loads in the neighborhood of the theoretical buckling load can be described as well. The present treatise aims at such an extension.

Chapter 1 gives a brief survey of the theory of elasticity for finite deformations, whereas the general equations of motion are derived by means of Hamilton's principle, using the potential or strain-energy func-

tion to describe the elasticity of the body. The equations of equilibrium are obtained by putting equal to zero all inertia forces.

In Chapter 2 the general theory of stability is dealt with. A precise definition is given by means of the energy criterion. In accordance with Trefftz it leads to two conditions of stability. The first condition is that the first variation of the potential energy is zero for any kinematically possible variation of displacement. The second condition requires that the second variation of the energy cannot be negative for any such possible variation of displacement. The stability at the stability limit is studied at full length.

In the next chapter the states of equilibrium at loads in the neighborhood of the buckling load are investigated. The approximate method used for this purpose naturally gives better results the less the load differs from the critical one. The character of these states actually appears to be governed by the stability at the buckling load. It must be said explicitly that the method exclusively enables to deal with buckling problems with a so-called point of bifurcation; so-called "oilcanning" problems are not taken into account. The most important result presented in Chapter 3 is that with stability of the equilibrium at the critical load neighboring states of equilibrium exist only for larger loads; these states are stable. Therefore larger loads than the buckling can be sustained. With an unstable buckling load, on the contrary, neighboring states of equilibrium do occur at loads smaller than the buckling load; these states are unstable.

The theory treated up to now does not give an explanation of the fact that for some structures the experimental buckling loads are considerably smaller than the theoretical buckling load. To arrive at such an explanation the influence of small deviations of a real structure from the simplified model, designed to represent the structure, is considered in Chapter 4. The most important result that this investigation shows is that with an unstable buckling state of the model the buckling load of the structure may be considerably lower in consequence of very small differences between structure and model. Hence the discrepancy between theoretical and experimental critical loads can be explained purely elastically by assuming small deviations of such a structure from the corresponding model; moreover the great sensibility of the buckling load of the structure for small variations in the magnitude of the deviations explains too the wide divergence of experimental results.

The most interesting example of the application of the theory developed here is the application to the axially compressed thin-walled cylinder; for in this technically important case the great discrepancy between the theoretical and experimental buckling loads has up to now not been

accounted for satisfactorily. To apply the general theory to this problem it is necessary to dispose of the knowledge of the elastic energy of the thin-walled cylindrical shell for finite displacements. With a view to the possibility of application to other shell structures as well, a general theory of thin shells for finite displacements is given in Chapter 5. Before passing to the rather complicated theory of the thin-walled cylinder it seemed advisable to deal first with some simpler applications, in order to elucidate the general theory (Chapter 6). The well-known problem of the elastica, Cox's problem, and the problem of equivalent width of compressed flat rectangular plates are considered in this connection.

Finally, Chapter 7 is concerned with the axially compressed cylindrical shell. Neglect of boundary conditions leads to the same result for the buckling load as known from existing literature. The same neglect leads to the conclusion that equilibrium in the buckling state is unstable. As to the neighboring states of equilibrium at loads in the neighborhood of the critical load, it is found that all existing neighboring states of equilibrium are unstable. The results are compared with those obtained by von Kármán and Tsien, and seem to be more satisfactory, at least for loads in the neighborhood of the critical load. Thereupon the influence of small deviations from the true cylindrical form is discussed. As the investigation is rather complicated the detailed calculations are restricted to one form of deviations. In this case a very marked decrease of buckling load is found already with very small deviations. This result is in striking contrast to that of Donnell and it may be stated indeed that the theory given here supplies an explanation of the large discrepancy between theoretical and experimental critical loads. The wide divergence of experimental results is likewise satisfactorily accounted for by the extreme sensibility of the critical load for small variations in the magnitude of the deviations.

2. *On the Buckling of a Thin-Walled Circular Tube Loaded by Pure Bending.* If a thin-walled circular tube (length l , radius a , thickness h) is loaded by two equal and opposite terminal bending moments M , its cross section alters its circular shape into an oval one, owing to the fact that apart from the normal bending stresses in the cross section of the tube, tangential bending stresses arise in its meridional sections. A closer examination of this fact reveals that the curvature of the "axis" of the tube does not increase proportionally with the loading moment M . The phenomenon has been studied at great length by Brazier. In the present paper another phenomenon is studied, which is characterized by the simultaneous appearance of longitudinal and circumferential waves in the cylindrical shape of the tube. It is assumed that—if unloaded—the tube possesses such initial curvature that under the action of the buckling

moment it is straight and—in cross section—circular and of constant thickness h . The coordinates used are cylinder coordinates r, φ, z ; the corresponding displacements are denoted by u, v, w ; the surface stresses by R, Φ, Z . It is well known that with the load system

$$R = a_{pq} \cos p\varphi \sin \lambda \frac{z}{a}; \quad \Phi = b_{pq} \sin p\varphi \sin \lambda \frac{z}{a}; \quad Z = 0 \quad (1)$$

(where p and q represent arbitrary positive integers and λ stands for $\lambda = \pi qa/l$), the following displacements are compatible:

$$u = u_{pq} \cos p\varphi \sin \lambda \frac{z}{a}; \quad v = v_{pq} \sin p\varphi \sin \lambda \frac{z}{a}; \quad w = w_{pq} \cos p\varphi \cos \lambda \frac{z}{a}. \quad (2)$$

As far as u and v are concerned u_{pq} and v_{pq} are represented by

$$u_{pq} = \alpha_{pq} a_{pq} + \beta_{pq} b_{pq}; \quad v_{pq} = \beta_{pq} a_{pq} + \gamma_{pq} b_{pq} \quad (3)$$

α_{pq}, β_{pq} and γ_{pq} standing for

$$\alpha_{pq} = \left(\frac{T_1}{N} \right)_{pq} \frac{a^2}{B}; \quad \beta_{pq} = - \left(\frac{T_2}{N} \right)_{pq} \frac{a^2}{B}; \quad \gamma_{pq} = \left(\frac{T_3}{N} \right)_{pq} \frac{a^2}{B} \\ \left(B = \frac{Eh}{1 - \nu^2} \right) \quad (4)$$

T_1, T_2, T_3, N themselves standing for

$$T_1 = (1 + k)p^4 + [2\lambda^2 + 2(1 - \nu)\lambda^2 k]p^2 + (1 + 3k)\lambda^4 \\ T_2 = p\{[1 + k(2\lambda^2 + 1)]p^2 + (\nu + 2)\lambda^2 + 2k\lambda^4\} \\ T_3 = kp^2(p^2 - 1)^2 + k \frac{2(2 - \nu)}{1 - \nu} \lambda^2 p^4 + \left[1 + \frac{5 - \nu}{1 - \nu} k\lambda^4 \right. \\ \left. + \frac{2(\nu - \nu^2 - 2)}{1 - \nu} k\lambda^2 + k \right] p^2 + 2(1 + \nu)\lambda^2 + \frac{2}{1 - \nu} k(\lambda^6 - 2\nu\lambda^4) \\ N = kp^8 + k[4\lambda^2 - 2]p^6 + k[6\lambda^4 - 2(4 - \nu)\lambda^2 + 1]p^4 + k[4\lambda^6 - 6\lambda^4 \\ + 2(2 - \nu)\lambda^2]p^2 + (1 - \nu^2)\lambda^4 + k\lambda^8 - 2\nu\lambda^6 k + (4 - 3\nu^2)\lambda^4 k.$$

It can be proved that two systems of so-called "elementary characteristic loads" B , with corresponding "elementary characteristic deformations" D , exist, for which the load components R_{pq}, Φ_{pq} are proportional to the corresponding displacements u_{pq}, v_{pq} , such that

$$\frac{R_{pq}}{u_{pq}} = \frac{\Phi_{pq}}{v_{pq}} = \omega. \quad (6)$$

These systems are defined except for a factor of proportionality, and possess certain properties of orthogonality. The factor of proportionality can be fixed by a suitable condition of standardization.

Returning to the buckling problem under consideration, we refer to the fact (cf., for instance, W. Flügge, *Statik und Dynamik der Schalen*, or C. B. Biezeno und R. Grammel, *Technische Dynamik*) that the differential equations for the *supplementary* displacements u, v, w (which transfer the cylinder from its state of "neutral" equilibrium into an infinitesimally neighboring state of equilibrium) agree with the differential equations for the displacements u, v, w of a cylinder, the loads R, Φ, Z of which are represented by

$$R = -Q \frac{\partial^2 u}{\partial z^2}, \quad \Phi = -Q \frac{\partial^2 v}{\partial z^2}, \quad Z = 0 \quad (7)$$

where $Q = Ma \cos \varphi / \pi a^3$ stands for the normal load per unit of circumferential length in the terminal cross sections of the cylinder, as exerted by the bending moment M .

Let (u, v, w) occurring under the critical load $M = \mu \bar{M}$ ($\bar{M}$ = unit moment) be named the "total" characteristic deformation T . Then T can be decomposed into a (infinite) series of elementary normal deformations D

$$T = \sum_{i=1}^{\infty} d_i D_i \quad (8)$$

If the load system, derived from D_i with the aid of (7) be called $\bar{B}_i$, then the load system belonging to the "total characteristic deformation" T may be written as $\Sigma d_i \bar{B}_i$.

Each separate system $\bar{B}_i$ can be expanded into a (finite or infinite) series of "elementary characteristic loads" B_i , so that the load system corresponding with the "total characteristic deformation" finally can be looked upon as the sum of an infinite number of groups of a (finite or infinite) number of "elementary characteristic loads" B_i . In this reasoning it has been assumed that all deformations and load systems are connected with the *same* parameter λ , characteristic for the "total deformation" under consideration, so that all loads and deformations have the same number of longitudinal waves. All functions T, B, D can therefore be denoted by a single subscript p .

If artificially the tube be given the "elementary characteristic deformation" D_i , then we can, by the aid of (7), formally calculate the corresponding load system (R, Φ, Z) produced by two unit terminal moments $\bar{M}$. As stated before, this load system can be developed in a (finite or

infinite) series of "elementary characteristic" functions B . The coefficient α_{ij} , which in this expansion belongs to the elementary characteristic function B_i , is called the "influence number of the elementary characteristic deformation D_j with respect to the elementary characteristic load B_i ."

This formal definition provides us with an expedient for obtaining a system of homogeneous linear equations for the coefficients d_i in the expansion (8), viz.,

$$d_i = \sum_j \mu d_j \alpha_{ij} \quad (i = 1, 2 \dots) \quad (9)$$

and therefore the problem of the buckling cylinder under bending is formally reduced to the solution of the equation

$$\begin{vmatrix} \alpha_{11} - \frac{1}{\mu} & \alpha_{12} & \dots & \dots \\ \alpha_{21} & \alpha_{22} - \frac{1}{\mu} & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0. \quad (10)$$

It can be proved, that—if only the elementary characteristic functions are suitably numbered— $\alpha_{ij} = \alpha_{ji}$, so that equation (10) is a secular one possessing only real roots.

For the computation of the smallest root μ of this equation recourse is made to a generalization of an iterative method, which by the second author of this treatise has been established in his doctor's thesis, and which consists in deriving from an arbitrary set of quantities ϑ_{1i} another set ϑ_{2i} , defined by

$$\vartheta_{2i} = \sum \alpha_{ij} \vartheta_{1j} \quad \text{etc.} \quad (11)$$

Two consecutive iterations ϑ_{si} and $\vartheta_{s+1,i}$ tend with increasing s to proportionality, the factor of which represents the smallest characteristic number. Its value μ_1 is fairly approximated by the expression given by Koch:

$$\mu_1 = \frac{\sum_i \vartheta_{1i} \vartheta_{2i}}{\sum_i \vartheta_{2i}^2} \quad (12)$$

so that only one iteration needs to be performed. The iteration process relates to a *fixed* value of λ , viz., to a fixed number of longitudinal waves in the buckling deformation. Therefore the iteration process has to be repeated for a great number of values λ if the absolute minimum of the critical buckling moment is required. This seems to be a rather tiresome technique, but means are provided to abbreviate considerably the numerical calculations.

3. *The Generalized Buckling Problem of the Circular Ring.* A circular ring, subjected to a uniform radial pressure q per unit of circumferential length, is apt to buckle under the action of one of the so-called "critical" loads $q = (n^2 - 1)EI/r^3$ (n integer and ≥ 2), EI representing the flexural rigidity of the ring and r its radius. This case of buckling is analogous to the buckling of a straight rod under the action of two compressive forces, as far as the cross sections of both ring and rod are loaded by a normal force of constant magnitude. If the straight rod is loaded by a prescribed system of axial forces, so that the normal force of the cross section varies with its coordinate, proportional increase of the load-system, say to the multiple λ , leads as well to critical buckling loads. The (positive or negative) value of the factor of magnification λ can best be found by a method of iteration. Obviously the analogous problem exists for the circular ring, if only it is subjected to an external load-system such that in every cross section the bending moment M and the shearing force D are zero, whereas the normal force of the section varies with its coordinate. Evidently the first of these conditions will not be fulfilled if the ring is loaded by an arbitrary system of radial and tangential forces, but it can be shown that every system can be split up into two components A and B , the first of which is called the "compressive" system because it is characterized by $M = D = 0$, whereas the second one is called the "bending" system characterized by $N = 0$.

The first system A , if suitably magnified, leads to the generalized buckling problem of the circular ring, which is the object of this treatise, but it is seen at once that an arbitrary load-system, consisting of both components A and B , gives rise to a problem, which again is analogous to a well-known problem, viz. the straight rod subjected to axial thrust and transverse bending loads. Just as with the straight rod the axial forces tend to increase the deflections caused by the transverse loads, the A -load-system of the circular ring will affect the deflections due to the B -system (cf. Section 4).

If the arbitrary load-system (q, t) — q = radial load, t = tangential load per unit of circumferential length—of the ring is expanded into Fourier series

$$q = a_0 + \sum_1^{\infty} a_k \cos k\varphi + \sum_1^{\infty} b_k \sin k\varphi; \quad t = c_0 + \sum_1^{\infty} c_k \cos k\varphi + \sum_1^{\infty} d_k \sin k\varphi \quad (1)$$

the A - and B -components are represented by

$$A \left\{ \begin{aligned} q^* &= a_0 + \frac{1}{2} a_1 \cos \varphi + \sum_2^{\infty} \left[\frac{-1}{k^2 - 1} a_k + \frac{k}{k^2 - 1} d_k \right] \cos k\varphi \\ &\quad + \frac{1}{2} b_1 \sin \varphi + \sum_2^{\infty} \left[-\frac{1}{k^2 - 1} b_k - \frac{k}{k^2 - 1} c_k \right] \sin k\varphi \\ t^* &= \frac{1}{2} b_1 \cos \varphi + \sum_2^{\infty} \left[\frac{k}{k^2 - 1} b_k + \frac{k^2}{k^2 - 1} c_k \right] \cos k\varphi \\ &\quad + \frac{1}{2} a_1 \sin \varphi + \sum_2^{\infty} \left[-\frac{k}{k^2 - 1} a_k + \frac{k^2}{k^2 - 1} d_k \right] \sin k\varphi \end{aligned} \right\} \quad (2)$$

$$B \left\{ \begin{aligned} q^{**} &= \frac{1}{2} a_1 \cos \varphi + \sum_2^{\infty} \left[\frac{k^2}{k^2 - 1} a_k - \frac{k}{k^2 - 1} d_k \right] \cos k\varphi \\ &\quad + \frac{1}{2} b_1 \sin \varphi + \sum_2^{\infty} \left[\frac{k^2}{k^2 - 1} b_k + \frac{k}{k^2 - 1} c_k \right] \sin k\varphi \\ t^{**} &= -\frac{1}{2} b_1 \cos \varphi + \sum_2^{\infty} \left[-\frac{k}{k^2 - 1} b_k - \frac{1}{k^2 - 1} c_k \right] \cos k\varphi \\ &\quad + \frac{1}{2} a_1 \sin \varphi + \sum_2^{\infty} \left[\frac{k}{k^2 + 1} a_k - \frac{1}{k^2 - 1} d_k \right] \sin k\varphi. \end{aligned} \right\} \quad (3)$$

Here only attention will be paid to A -load systems from which it is presumed that the q^* and t^* components *keep their directions* with respect to the element of the ring on which they act. The differential equation of the problem is then given by

$$U'' + U = \frac{Nr^2}{EI} U + C \quad (4)$$

($U = u'' + u$,— u representing the increase of the radius vector r — N = prescribed normal force acting on the cross section, C = constant of integration to be found in the course of the investigation). Both N and U can be expanded into Fourier series

$$\left. \begin{aligned} N &= \lambda N_0 = \lambda \left[A_0 + \sum_{k=1}^{\infty} A_k \cos k\varphi + \sum_{k=1}^{\infty} B_k \sin k\varphi \right] \\ U &= a_0 + \sum_{l=1}^{\infty} a_l \cos l\varphi + \sum_{l=1}^{\infty} b_l \sin l\varphi \end{aligned} \right\} \quad (5)$$

and it can be proved that buckling of the ring can occur only if all multiples of φ in the Fourier series of N have a factor in common.

If the greatest factor in common be called p , all terms a_0 , $a_1 \cos \varphi$, $b_1 \sin \varphi$, $\alpha_{\alpha p \pm 1} \cos (\alpha p \pm 1)\varphi$, and $b_{\alpha p \pm 1} \sin (\alpha p \pm 1)\varphi$, (α representing any positive integer $\neq 0$) have to be excluded from U .

Substitution of N and U in (4) would lead to an infinite system of recurrent relations between the coefficients a_l and b_l , and it would be seen that the system would break up into a number of minor systems, each of which is related to a distinct class of coefficients a_l , b_l , defined by

$$\left. \begin{aligned} q &= 0, 2, 3 \cdots \frac{p-1}{2} \quad (p = \text{odd}) \\ l &= \pm q, \pmod{p} \\ q &= 0, 2, 3 \cdots \frac{p}{2} \quad (p = \text{even}). \end{aligned} \right\} \quad (6)$$

With any prescribed p the buckling problem therefore is split up into $\frac{p-1}{2}$ or $\frac{p}{2}$ cases, in every one of which U is composed exclusively of terms relating to one of the congruences (6).

With respect to the constant C in (4) it must be observed that if the product NU in the right-hand side of the equations happens to miss a constant term after having been written as an ordinary Fourier series, then C must be suppressed; if not so, C serves to annul this term.

The practical solution of the problem is given in an iterative way, the justification of which is given by expressing the problem in terms of an integral equation.

Starting with an arbitrary function $V_1(\varphi)$, containing only cosine and sine terms of suitable multiples of φ , another function $V_2(\varphi)$ is derived, formally defined by the differential equation,

$$V_2'' + V_2 = \frac{N_0 r^2}{EI} V_1 + C$$

and the condition that it does not contain terms of the type $\alpha + \beta \cos \varphi +$

$\gamma \sin \varphi$ (C being again a constant to be determined *a posteriori*). From V_2 another function V_3 is deduced in the same way, etc.

It can be proved that the smallest characteristic value λ_1 is approximated by one of the following formulas:

$$\lambda_1 = \frac{\int_0^{2\pi} N_0 V_{m-1}^2 d\varphi}{\int_0^{2\pi} N_0 V_{m-1} V_m d\varphi}; \quad \lambda_1 = \frac{\int_0^{2\pi} N_0 V_{m-1} V_m d\varphi}{\int_0^{2\pi} N_0 V_m^2 d\varphi};$$

the second of which exceeds in general the first in accuracy. The practical iteration scheme, along which the iteration has to be performed in fact, cannot be described here, but is elucidated at full length in the treatise itself.

The cases $N = \lambda N_0 = \lambda(1 + 2 \cos k\varphi)$ ($k = 2, 3, \dots, 12$) and $N = \lambda(1 + 2 \cos 2\varphi + \cos 4\varphi)$ are worked out numerically. Higher characteristic values can be calculated as well. In all such cases where the normal force N changes its sign, negative characteristic values are to be expected. A negative value of λ interchanges the compressed and the stretched parts of the construction, and a sufficiently great negative λ therefore causes the buckling of the initially stretched parts. As an example it can be stated that with a compressive force $N = \lambda(1 + 4 \cos 2\varphi)$ the second characteristic number proves to be negative $\lambda_2 = -2.1133 \frac{3EI}{r^2}$, whereas the first characteristic number is equal to $\lambda_1 = 0.6275 \frac{3EI}{r^2}$.

4. *The Circular Ring under the Combined Action of Compressive and Bending Loads.* In the previous section the problem treated here has already been announced. The ring is simultaneously subjected to an A - and B -system, but it is supposed that the A -system is small enough to guarantee the elastic stability of the ring. It is obvious that the deflections (and internal stresses) of the ring, due to the *single* action of the B -system, will be affected in a rather complicated way by the simultaneous action of the A -system, which, alone, would produce no deflections at all. The influence of the A -system upon the bending effect produced by the B -system can be expressed in terms of the characteristic numbers λ and the corresponding characteristic functions U of the A -system. To this end the so-called "reduced moment," $-\frac{r^2}{EI} M_B$, belonging to the B -system has to be expanded into a series

$$-\frac{r^2}{EI} M_B = \sum_{k=1}^{\infty} b_k U_k \quad (1)$$

of the characteristic functions U_k of the A -system. The resultant bending moment M , resp. the resultant function U , due to the joint systems A and B , is then represented by

$$-\frac{r^2}{EI} M = U = \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - 1} b_k U_k. \quad (2)$$

The resultant distortion of the ring is governed by the equation

$$u'' + u = U. \quad (3)$$

A numerical application is given for the ring compressed by two diametral forces P .

5. *On the Nonlinear Deflection of a Semicircular Ring, Clamped at Both Ends.* The aim of this paper is to deduce in a theoretical way the force-deflection relation (P, u) of a semicircular ring clamped at both ends and loaded (in its midpoint characterized by $\varphi = 0$) by a radial force P . The treatment of the problem (which finds its origin in a remarkable paper on the buckling of cylindrical shells by Th. von Kármán, L. G. Dunn, and Hue-Shen Tsien) is essentially based upon the methods developed in the two preceding papers. Instead of the semiring clamped at both ends, first a *complete* ring is considered loaded by two pairs of diametral forces P and Q , placed at right angled (P at $\varphi = 0$ and π , Q at $\pi/2$ and $3\pi/2$). For a number of values of the ratio P/Q , viz. $P/Q = 1, 1.05, 1.10$ and 1.2 and their reciprocals, the radial deflections u are calculated at $\varphi = 0$ and $\varphi = \pi/2$. From each of these results (relating to a distinct value of PQ), the absolute value of P can be deduced, for which the deflection u at $\varphi = \pi/2$ and $3\pi/2$ becomes zero, and by substituting the smallest of these values in the corresponding expression of u for $\varphi = 0$, the deflection of the point $\varphi = 0$ can be calculated. In a graph $\frac{P}{u} \frac{r^3}{EI}$ is plotted against u/r and the results are compared with the experimental ones of von Kármán l.c.

6. *On the Nonlinear Deflection of a Semicircular Ring, Clamped at Both Ends.* The method discussed in the preceding paper requires a considerable amount of computation. It therefore seemed desirable to investigate whether the results could be obtained in a simpler (be it approximative) way. Again the problem is attacked by considering a *complete* ring, loaded by two pairs of diametral forces P and Q . The simplification of the problem consists in replacing the rather complicated A -load-system (connected with these forces) by a radial *constant* pressure $(P + Q) : \pi r$ equally distributed along the circumference of the ring, giv-

ing rise to a *constant* normal force $(P + Q): \pi$ in the cross section. The approximative results obtained in this way are quite satisfactory.

7. *Large Distortions of Circular Rings and Straight Rods.* The present paper is closely-connected with the foregoing three papers, and provides the reader with a valuable extension. Indeed the authors of these previous treatises use the differential equation of the slightly curved beam to describe the distortions, an approximation, which can be made only if the difference in slope of the elastic line in its deformed and its undeformed shape is small. So their solution is limited to distortions not too large. Moreover, the distortions, as calculated by their method, appear as an infinite series of terms. As the calculation of the separate terms is rather tedious, one has for practical reasons to confine oneself to a small number of terms. The authors, for instance, use only two terms.

It therefore is of interest to give an exact treatment of the same problem, partly to have the exact results for themselves, partly to provide a check to the approximate results. The method used in this paper is principally based on the ideas used by some authors in the classical period of elasticity and consists in the use of a system of natural coordinates (s, φ) where s is the length of the arc measured from an arbitrary zero point, and φ the angle included by the normal of the distorted ring and a fixed direction. The method is not restricted to closed circular rings: many other problems of large distortions (as for instance the one referred to in section C2) can be solved by the same means. As may be expected, it calls for a knowledge of elliptic integrals. Four particular problems, one dealing with a closed circular ring, one with a semicircular ring, and two with straight beams admitting large deflections, have been numerically calculated, and the results have been plotted in diagrams. Comparisons are made with approximate results of other authors and with own experiments.

8. *On the Buckling and the Lateral Rigidity of Helical Compression Springs.* The buckling of helical springs was first studied by R. Grammel in a paper read before the first International Congress for Applied Mechanics. Later on C. B. Biezeno and J. J. Koch pointed out that the results obtained did not agree with the experiments and that the reason for this discrepancy was to be sought in the neglect of the "shear elasticity" of the spring. Unfortunately a partial misinterpretation of this shear effect leads Biezeno and Koch to an overestimation, which makes itself clearly perceptible in the end results. In this paper the correction is given, and it is proved that now full agreement between theory and experiment exists. The theory is not restricted to the determination of the buckling load of helical springs under compression but can be applied as well to springs subjected to an axial force combined with a bending

moment and a lateral force, as to springs under the combined action of compression and torsion. Apart from the fact that the normal and shear elasticity of the spring have to be brought into account if the latter (as a whole) is expected to behave like a very elastical rod, it must be remembered that the compression of the spring results in an increase of the number of coils per unit of axial length, and therefore leads to a decrease of all coefficients of rigidity as compared with those in the unloaded state.

If n is the number of active coils, D the mean spring diameter, I the linear moment of inertia of the circular wire section, l_0 the length of the unloaded spring, l the length of the loaded spring, E Young's modulus of elasticity, m Poisson's ratio, then the various coefficients of elastic rigidity per unit of length of the unloaded spring are (in succession of bending, shear, compression, and torsion)

$$\frac{1}{\alpha_0} = \frac{n \pi D}{l_0 EI} \frac{2m+1}{2m}; \quad \frac{1}{\beta_0} = \frac{n \pi D^3}{l_0 8EI}; \quad \frac{1}{\gamma_0} = \frac{n \pi D^3}{l_0 EI} \frac{m+1}{4m};$$

$$\frac{1}{\delta_0} = \frac{n \pi D}{l_0 EI} \quad (1)$$

those defined per unit of length of the loaded spring are represented by

$$\alpha = \frac{l}{l_0} \alpha_0; \quad \beta = \frac{l}{l_0} \beta_0; \quad \gamma = \frac{l}{l_0} \gamma_0; \quad \delta = \frac{l}{l_0} \delta_0. \quad (2)$$

The differential equation of the problem is

$$y'' + \frac{P}{\alpha} \left(1 + \frac{P}{\beta} \right) y = 0 \quad (3)$$

(in contrast to the equation

$$y'' + \frac{\beta}{\alpha} \frac{P}{(\beta - P)} y = 0 \quad (4)$$

given by Biezeno and Koch), and it is an easy matter to deduce from this equation the required buckling load, taking due regard to the boundary conditions. The relative compressions ξ [defined by $l = (1 - \xi)l_0$] at which the spring buckles can be plotted against the ratio l_0/D and be compared with experimental results; this leads to the agreement already mentioned. Most interesting is the behavior of a spring the length of which is about 2.7 times its diameter. If the length exceeds this amount by some 5 per cent, the spring buckles under a compression of about 50 per cent, whereas no instability at all occurs if the length is some 5 per cent smaller. Similar results hold for other end conditions of the spring.

With the aim in view to use his results in the construction of so-called "vibration" tables, the author pursues his treatise in examining a spring clamped at one end and loaded at its free end by an axial force P , a lateral force L , and a bending moment M_0 , and calculates the lateral displacement y_l and the slope ψ_l of the free end in terms of these loads. P being prescribed, the lateral force and the moment required to produce a given y_l and ψ_l can be written as

$$L = c_1 y_l - c_2 \psi_l; \quad M_0 = -c_2 y_l + c_3 \psi_l. \quad (5)$$

The coefficients c_1 , c_2 , c_3 , which are of the greatest importance to the designer, can be derived from graphs (for different values of ξ resp. P) as function from l_0/D .

Finally the author considers the buckling of the spring under combined compression (P) and torsion (W), but the treatment of this problem is far more intricate so that no details can be given here. The results are given in graphs, in which for various values of the "angle of twist" curves are drawn the ordinate of which represents the critical relative compression ξ of the spring as function of the ratio l_0/D . Theoretical and experimental data are again in full agreement.

B. Plates and Shells

1. *On the State of Stress in Perforated Plates.* This important monograph deals with the stress distribution occurring in plates perforated by an arbitrary number of circular holes of arbitrary size and arbitrary situation (though the author restricts himself to holes of a fixed radius a).

The starting point of all calculations is the well-known stress function F of Airy, satisfying the equation:

$$\Delta\Delta F = 0 \left(\Delta = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ resp. } \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right). \quad (1)$$

Every solution of this equation—if only satisfying certain conditions of continuity—gives rise to a realizable distribution of stress ($\sigma_y, \sigma_z, \tau_{yz}$) resp. ($\sigma_r, \sigma_\varphi, \tau_{r\varphi}$), represented by

$$\sigma_y = \frac{\partial^2 F}{\partial z^2}; \quad \sigma_z = \frac{\partial^2 F}{\partial y^2}; \quad \tau_{yz} = -\frac{\partial^2 F}{\partial y \partial z} \text{ resp.} \quad (2)$$

$$\sigma_r = \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \varphi^2}; \quad \sigma_\varphi = \frac{\partial^2 F}{\partial r^2}; \quad \tau_{r\varphi} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial F}{\partial r} \right). \quad (3)$$

If we restrict ourselves to stress functions the corresponding displacements of which are zero at infinity and unambiguous throughout the field the most general F useful in our problem is given by

$$F = A_0 F_0 + A_1 F_1 + \bar{A}_1 \bar{F}_1 + \sum_{n=2}^{\infty} (A_n F_n + B_n F_n^* + \bar{A}_n \bar{F}_n + \bar{B}_n \bar{F}_n^*) \quad (4)$$

with

$$\left. \begin{aligned} F_0 &= \ln r, & F_1 &= r^{-1} \cos \varphi, & \bar{F}_1 &= r^{-1} \sin \varphi, & F_n &= r^{-n} \cos n\varphi \\ F_n^* &= r^{-n+2} \cos n\varphi, & \bar{F}_n &= r^{-n} \sin n\varphi, & \bar{F}_n^* &= r^{-n+2} \sin n\varphi \end{aligned} \right\} \quad (n \geq 2). \quad (5)$$

Evidently every function $F_s, F_s^*, \bar{F}_s, \bar{F}_s^*$, gives rise to stresses σ_r, σ_φ and $\tau_{r\varphi}$ in the points of the circle $r = a$ (afterwards to be considered as the boundary of a circular hole I). But it is an easy matter to construct such linear combinations $F_{ss}, F_{\tau s}, \bar{F}_{ss}, \bar{F}_{\tau s}$ of these elementary functions that apart from tangential stresses σ_φ all along the circle $r = a$ only cosinusoidal or sinusoidal radial stresses σ_r , resp. only cosinusoidal or sinusoidal shearing stresses $\tau_{r\varphi}$ occur. It can easily be checked, for instance, that with the stress function

$$F_{ss} = \frac{a^{s+2}}{2(s+1)} F_s - \frac{a^s}{2(s-1)} F_s^* \quad (6)$$

the stresses $\sigma_r = \cos s\varphi$ and $\tau_{r\varphi} = 0$ do correspond in points of the circle $r = a$. In many cases it is recommended that the function F be developed into a series of the arguments $F_{ss}, F_{\tau s}, \bar{F}_{ss}, \bar{F}_{\tau s}$. It takes the form

$$F = C_0 F_{s0} + C_1 (F_{s1} + F_{\tau 1}) + \bar{C}_1 (\bar{F}_{s1} - \bar{F}_{\tau 1}) + \sum_{s=2}^{\infty} (C_s F_{ss} + D_s F_{\tau s} + \bar{C}_s \bar{F}_{ss} + \bar{D}_s \bar{F}_{\tau s}). \quad (7)$$

If now a second circle (radius a) is considered (later on to be regarded as the boundary of a second hole II), a second system of polar coordinates ($r'\varphi'$) may be introduced in the center of this circle. The polar axis coincides with the polar axis of the first system but has the inverse positive direction, whereas the azimuthal positive directions are opposite as well. Under these conditions the stresses σ_r and $\tau_{r\varphi}$ occurring in the points of circle I as produced by the stress functions $F'_{ss}, F'_{\tau s}, \bar{F}'_{ss}, \bar{F}'_{\tau s}$ connected with the center of circle II can be expanded into a Fourier series of the argument φ by the aid of a simple complex transformation.

This being stated we draw our attention to an infinite plate with two circular holes I and II (of equal radius) at infinity subjected to uniform tension perpendicularly directed to the centerline of the holes. If for a moment the plate is thought to be unperforated, the stresses σ_r^0 and $\tau_{r\varphi}^0$ occurring along the boundaries I and II can be calculated and brought into the form

$$-\sigma_r^0 = C_0^0 + \sum_{n=1}^{\infty} C_n^0 \cos n\varphi, \quad -\tau_{r\varphi}^0 = \sum_{n=1}^{\infty} D_n^0 \sin n\varphi. \quad (8)$$

Thereupon a stress function F is introduced into each of the centers I and II, represented by

$$F = C_0 F_{\sigma 0} + C_1 (F_{\sigma 1} + F_{\tau 1}) + \sum_{n=2}^{\infty} (C_n F_{\sigma n} + D_n F_{\tau n}). \quad (9)$$

The stress function F connected with the center I gives rise to stresses along the boundary I of the magnitude

$$\sigma_{r1} = C_0 + \sum_{n=1}^{\infty} C_n \cos n\varphi, \quad \tau_{r\varphi 1} = \sum_{n=1}^{\infty} D_n \sin n\varphi. \quad (10)$$

The stress function F connected with the center II gives rise to stresses along the boundary I, which can be written as

$$\left. \begin{aligned} \sigma_{r2} &= C_0 h_0^0 + \sum_{s=1}^{\infty} (C_s h_0^s + D_s i_0^s) + \sum_{n=1}^{\infty} \left[C_0 h_n^0 + \sum_{s=1}^{\infty} (C_s h_n^s + D_s i_n^s) \right] \cos n\varphi \\ \tau_{r\varphi 2} &= \sum_{n=1}^{\infty} \left[C_0 j_n^0 + \sum_{s=1}^{\infty} (C_s j_n^s + D_s k_n^s) \right] \sin n\varphi. \end{aligned} \right\} \quad (11)$$

The requirement that σ_r and $\tau_{r\varphi}$ shall be zero along the boundary I (and consequently along the boundary II also) leads to the following infinite system of linear equations for the constants C_n and D_n :

$$\left. \begin{aligned} C_n &= C_n^0 - \left[C_0 h_n^0 + \sum_{s=1}^{\infty} (C_s h_n^s + D_s i_n^s) \right] \\ D_n &= D_n^0 - \left[C_0 j_n^0 + \sum_{s=1}^{\infty} (C_s j_n^s + D_s k_n^s) \right] \end{aligned} \right\} \quad (n = 1, 2, \dots) \quad (12)$$

The solution of this system can be found by iteration. By putting

$$\left. \begin{aligned} C_n^i &= - \left[C_0^{(i-1)} h_n^0 + \sum_{s=1}^{\infty} (C_s^{(i-1)} h_n^s + D_s^{(i-1)} i_n^s) \right] \\ D_n^i &= - \left[C_0^{(i-1)} j_n^0 + \sum_{s=1}^{\infty} (C_s^{(i-1)} j_n^s + D_s^{(i-1)} k_n^s) \right] \end{aligned} \right\} \quad (13)$$

C_n and D_n are represented by

$$C_n = \sum_{i=0}^{\infty} C_n^i \quad D_n = \sum_{i=0}^{\infty} D_n^i \quad (14)$$

provided that the convergence of the iteration process be guaranteed. The process itself can be interpreted by the following mechanical direction:

a. Calculate (expressed in Fourier series) the stresses σ_r^0 and $\tau_{r\varphi}^0$ occurring along the boundaries I and II in the unperforated plate.

b. Introduce in the centers I and II (identical) stress functions F_I^0 and F_{II}^0 such that F_I^0 annuls the stresses σ_r^0 and $\tau_{r\varphi}^0$ along the boundary I and that F_{II}^0 annuls the stresses σ_r^0 and $\tau_{r\varphi}^0$ along the boundary II.

c. Calculate the stress systems σ_r^1 $\tau_{r\varphi}^1$ (expressed in Fourier-series) called into existence by F_I^0 and F_{II}^0 along the boundaries of the *non*-corresponding holes.

d. Introduce in the centers I and II new stress functions F^1 , which annul these stresses along their *corresponding* boundaries, etc.

The convergence of the iteration process is brought into relation with the "Eigenwert" problem, defined by the system

$$\mu C_n = - \left[C_0 h_n^0 + \sum_{s=1}^{\infty} (C_s h_n^s + D_s i_n^s) \right]; \mu D_n = - \left[C_0 j_n^0 + \sum_{s=1}^{\infty} (C_s j_n^s + D_s k_n^s) \right] \quad (15)$$

but it is impossible to go into further details.

The exposition given here contains (in principle) all data, required for the treatment of the following cases (all of which in the original treatise are considered in full detail, and calculated numerically):

Three holes (the centers of which are collinear)

An infinite row of holes

Three holes (the centers of which are not collinear)

Two infinite parallel rows of holes (shifted or not).

In the last two paragraphs of the treatise the plate with one or two rows is reconsidered, now under the assumption that each hole is loaded

by a resultant force P , adequately distributed along the boundary of the hole. All forces P are equal and perpendicular to the row (or rows) as may be expected to occur in a riveted joint. The new element in the treatment of these problems consists in the starting function of the iteration. It is shown that the maximum stress is smallest for two nonshifted rows. It is proved that the least maximum stress occurring in a plate with shifted rows exceeds the maximum stress occurring in a plate with one row of holes (the total load of the plate being equal in both cases).

2. *On the State of Stress in Perforated Strips and Plates.* The preceding treatise restricted itself exclusively to infinite plates. In a subsequent series of papers the same author draws attention to the strip of finite width (and infinite length) with one or more rows of holes (pitch = b). The calculations are performed only for a strip with one row of holes, though an extensive program of further investigation was projected, which unfortunately could not be executed; the Jewish author, who lived in Holland, was arrested and marched off to Germany or Poland, which meant his destruction.

The system of coordinates $O(yz)$ used now is rectangular; the y -axis coincides with the center line of the holes. Consequently the stress functions $F_0, F_n, \bar{F}_n, F_n^*, \bar{F}_n^*$ introduced in the preceding treatise are written now as

$$\begin{aligned} F_0 &= Re \ln x, & F_n &= Re x^{-n}, & \bar{F}_n &= -Im x^{-n} & (n \geq 1) \\ & & & & & (x = y + iz) & (1) \\ F_n &= Re \bar{x} x^{-n+1}, & \bar{F}_n^* &= -Im \bar{x} x^{-n+1} & (n \geq 2). \end{aligned}$$

Furthermore stress functions $F_0^k, F_n^k, \bar{F}_n^k, F_n^{*k}, \bar{F}_n^{*k}$ are introduced, identical with the preceding ones, every set related however to a point O' of the y -axis, having the abscissa $y = kb$. Finally these functions are expressed in terms of the coordinates (yz) of the point O , so that, for instance, $F_n^k = Re(x - kb)^{-n}$. The total system of identical stress functions F_n , related to the infinity of centers $y = kb$, can with the aid of this transformation be represented by one single stress function U_n

$$= F_n + \sum_k (F_n^k + F_n^{-k}), \text{ related to the system of coordinates } O(yz).$$

It can readily be understood that the generalized stress function

$$\begin{aligned} F &= A_0 U_0 + A_1 U_1 + \bar{A}_1 \bar{U}_1 + \sum_{n=1}^{\infty} (A_n U_n + B_n U_n^* + \bar{A}_n \bar{U}_n \\ &\quad + \bar{B}_n \bar{U}_n^*) \end{aligned} \quad (2)$$

produces the same set of stresses $\sigma_r, \tau_{r\varphi}$ along the boundaries of *all* holes.

Just in the same way as in the preceding treatise the functions F_s ,

F_s^* , $\bar{F}_s$, $\bar{F}_s^*$ were linearly combined, here the functions U_s , U_s^* , $\bar{U}_s$, $\bar{U}_s^*$ are combined by introducing functions $U_{\sigma s}$, $U_{\tau s}$, $\bar{U}_{\sigma s}$, $\bar{U}_{\tau s}$, the first of which, for instance, is defined by

$$U_{\sigma s} = \frac{\alpha^{s+2}}{2(s+1)} U_s - \frac{\alpha^s}{2(s-1)} U_s^*. \quad (3)$$

Each of these functions produces a radial stress σ_r and a shearing stress $\tau_{r\varphi}$ along the boundary I of all circular holes, on the understanding however that these stresses are composed of a main term and an infinite series of minor terms; the main term corresponds with the stress that would have been produced by the corresponding functions $F_{\sigma s}$, $F_{\tau s}$, $\bar{F}_{\sigma s}$, $\bar{F}_{\tau s}$.

If restriction is made to such functions that satisfy the conditions of symmetry inherent to the strip under tension, the most general stress function that can be brought into play is given by

$$F = C_0 U_{\sigma 0} + \sum_{s=1}^{\infty} (C_{2s} U_{\sigma, 2s} + D_{2s} U_{\tau, 2s}) + \sum_{s=0}^{\infty} (\bar{C}_{2s} \bar{U}_{\sigma, 2s+1} + \bar{D}_{2s+1} \bar{U}_{\tau, 2s+1}). \quad (4)$$

It produces along the boundaries II and III of the strip ($z = \pm c$) stresses σ_z and τ_{yz} , for which a Fourier series (as functions of y , and with the period b) can be developed.

On the other hand it is required to construct a stress function F' having reference to the infinite half-plane such that along the straight boundary of the stress field *prescribed* stresses of the type

$$\sigma_z = \sum_{n=1}^{\infty} C_n' \cos 2\pi n \frac{y}{b} \quad \tau_{yz} = \sum_{n=1}^{\infty} D_n' \sin 2\pi n \frac{y}{b} \quad (5)$$

come into existence.

This function has the form

$$F' = \sum_{s=1}^{\infty} (C_s' F_{\sigma s}' + D_s' F_{\tau s}') \quad (6)$$

with

$$\left. \begin{aligned} F'_{\sigma s} &= \frac{b^2}{4\pi^2 s^2} (s\zeta - 1) e^{s\zeta} \cos s\eta, & F'_{\tau s} &= \frac{b^2}{4\pi^2 s^2} s\zeta e^{s\zeta} \cos s\eta \\ \eta &= \frac{2\pi y}{b}; \zeta = \frac{2\pi z}{b} \end{aligned} \right\} \quad (s \geq 1) \quad (7)$$

It is defined with respect to a system of coordinates, the y -axis of which coincides with the straight boundary of the half plane, the z -axis of which points to the inner region of the field). The stresses occurring along the boundaries I of circular holes, the centers of which have a mutual distance b and a distance c to the boundary can be expanded into a Fourier series of the arguments to φ of the holes.

Now the *unperforated* infinite plate may be considered covered with the contours I of the holes afterward to be introduced, and the boundaries II and III of the strip, afterward to be cut out of the plate; and a state of stress may be introduced as belonging to:

- α . a stress function F_I (4) with respect to the center of one of the holes
- β . a stress function F'_{II} (6) with respect to the boundary II
- γ . a stress function F'_{III} (6) with respect to the boundary III
- δ . a uniform tension p' in the direction y .

Then all means are available to calculate the stresses σ_z and τ_{yz} along the boundaries II and III and σ_r and $\tau_{r\varphi}$ along the boundaries I for each of the cases separately. The requirement that the total stresses along these boundaries be zero provides an infinite system of linear equations for the unknown constants C_{2n} , D_{2n} , C_n' , and D_n' , viz.:

$$\begin{aligned}
 C_{2n} &= C_{2n}^0 - \left[C_0 h_{2n}^0 + \sum_{s=1}^{\infty} (C_{2s} h_{2n}^{2s} + D_{2s} i_{2n}^{2s}) \right. \\
 &\quad \left. + \sum_{s=1}^{\infty} (C_s' p_{2n}^s + D_s' q_{2n}^s) \right] \quad n \geq 0 \\
 D_{2n} &= D_{2n}^0 - \left[C_0 j_{2n}^0 + \sum_{s=1}^{\infty} (C_{2s} j_{2n}^{2s} + D_{2s} k_{2n}^{2s}) + \right. \\
 &\quad \left. + \sum_{s=1}^{\infty} (C_s' r_{2n}^s + D_s' t_{2n}^s) \right] \quad n \geq 1 \\
 C_n' &= C_n'^0 - \left[C_0 h_n'^0 + \sum_{s=1}^{\infty} (C_{2s} h_n'^{2s} + D_{2s} i_n'^{2s}) \right. \\
 &\quad \left. + C_n' p_n' + D_n' q_n' \right] \quad n \geq 1 \\
 D_n' &= D_n'^0 - \left[C_0 j_n'^0 + \sum_{s=1}^{\infty} (C_{2s} j_n'^{2s} + D_{2s} k_n'^{2s}) \right. \\
 &\quad \left. + C_n' r_n' + D_n' t_n' \right] \quad n \geq 1 \quad (8)
 \end{aligned}$$

in which the quantities $h, i, j, k, p, q, r, t, C^0$ etc. are known.

This system is resolved in an iterative way, for which a mechanical interpretation can readily be given.

Three quantities are of outstanding practical importance: (1) The magnitude of the total tensional force $P = 2cp$, to which the strip is subjected; (2) the maximal stress occurring in the strip; (3) the mean elongation of the strip.

The answers to these three questions are:

$$1. \frac{p}{p'} = \frac{P}{2cp'} = 1 + \pi \frac{a^2 C_0 + C_2 - D_2}{bc p'}$$

2. The maximal stress occurs at the boundaries I, and can for different values $\lambda = a/b$ and $\mu = a/c$ be read from Table 1:

TABLE 1
Maximum Stress: p

$\lambda \backslash \mu$	0	0.20	0.40	0.60
0	3.0000			
0.10	2.7677	3.0905	3.7510	
0.20	2.3261	2.7121	3.5210	5.3256*
0.30	1.9956	2.3936	3.0741	4.8310

* Belongs to $\lambda = 0.15$.

$$3. \frac{\Delta b}{b} = \frac{p'}{p} \frac{P}{2Ec}$$

All calculations mentioned before are repeated for the bent strip and all required numerical data are put at the disposal of the reader. The same holds for the problem connected with a row of rivet holes, in the neighborhood of a straight boundary, which is solved under the condition that the boundary stresses of every hole give rise to a resultant force P perpendicular to (and directed toward) the straight boundary, whereas the reaction of these resultant forces consists of a continuous constant tension $p = P/b$ acting at infinity at the "opposite edge" of the infinite plate. The resultant force P is supposed to be exerted by a normal pressure $\sigma_r = (2P/\pi a) \sin \varphi$ acting on the upper half of the circular boundary of the hole; no other stresses on this boundary are present.

It will be clear that only the principal features of the work could be outlined and that a tremendous amount of computation hides itself behind the theoretical deductions.

3. *Some Explicit Formulas of Use in the Calculation of Arbitrary Loaded Thin-Walled Cylinders.*¹

¹ For the notation used here refer to section 2.

An arbitrary load system (R, Φ, Z) acting upon a thin-walled cylinder can be expanded into a double Fourier series:

$$R = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a'_{pq} \cos p\varphi \sin \lambda \frac{z}{a} + \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} b'_{pq} \cos p\varphi \cos \lambda \frac{z}{a} \\ + \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} c'_{pq} \sin p\varphi \sin \lambda \frac{z}{a} + \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} d'_{pq} \sin p\varphi \cos \lambda \frac{z}{a} \quad (1)$$

and two other expansions for Φ and Z , which can be derived from (1) by replacing the coefficients $a'b'c'd'$ by a'', b'', c'', d'' , and a''', b''', c''', d''' , respectively (p and q integers, $\lambda = \pi qa/l$, a = radius, l = length of the cylinder).

It is the aim of this treatise to give *explicit* formulas for *all* internal forces and moments $k_{\varphi\varphi}$, $k_{\varphi z}$, $k_{z\varphi}$, k_{zz} , $m_{\varphi\varphi}$, $m_{\varphi z}$, $m_{z\varphi}$, m_{zz} occurring in the cylinder if it is loaded by any of the elementary loads contained in series (1) or by any of the elementary loads of which Φ and Z are built up. The derivation of the formulas called for a good deal of laborious computation, which, however, needed only to be done once forever, and can be avoided now by others.

4. *The Effective Width of Cylinders, Periodically Stiffened by Circular Rings.* The purpose of this treatise is to provide the reader with a simple rule for the computation of the greatest tangential stress occurring in a thin-walled cylinder which in itself possesses only a slight flexural rigidity against alteration of its circular cross section and which therefore in constant distances ($=2l$) has been stiffened by circular rings. The load system of the cylinder is subject to the condition of periodicity in the axial direction, the period being identical with that of the rings; moreover the load is supposed to be symmetrical with respect to the plane of symmetry of two consecutive rings and to consist of radial and tangential components only. A specialization of this load system presents itself if all loads are concentrated in the planes of the rings, and as a matter of fact in this paper attention is drawn only to such a special system. The justification of this restriction is mainly to be sought in its agreement with actual practice but also in the fact that the stress problem raised by a general load system can always be decomposed into two other problems, one of which refers to a cylinder of length $2l$ loaded in the prescribed way, but clamped at both ends, while the other one relates to the actual cylinder exclusively loaded in the planes of its stiffening rings. The orthodox method of solving this latter stress problem consists in the separate treatment of the rings and the cylinder, the first ones under the

action of unknown radial and tangential load components, the last one under the action of their reaction components and the prescribed load system. The unknown load components can then be found by equalizing the radial and tangential displacements of the corresponding points of the cylindrical shell and the stiffening rings. The authors used this method in solving a certain mineshaft problem and had thereby full opportunity to take note of the laborious computation that it involves. It was this very problem that led them to the approximate method developed in this paper.

If the cylinder, stripped of its stiffening rings, is subject to an arbitrary load system L in each of its former ring planes, it shows—if thin—a high degree of flexibility, characterized by large radial displacements. There exists, however, an infinite class of "characteristic" load systems L_p ($p = 1, 2, \dots$) with radial and tangential components $r_p = B_p' \cos p\varphi$, $t_p = B_p'' \sin p\varphi$ (φ designating the azimuthal coordinate), which, in their planes of application produce no radial displacements u_p but solely tangential displacements v_p , so that in these planes the cylinder seems to lack any flexibility; the amplitudes B_p' and B_p'' of these characteristic components have a well-defined ratio. The circumferential elasticity of the cylinder can be measured by the quotient $v_p:t_p$, which for all points of the circumference appears to be a constant.

We now examine a circular ring of the same radius a and the same thickness h as the cylinder, and of the width $l' = \mu a$. This ring too can be loaded by "characteristic" loads $\bar{r}_p = \bar{B}_p' \cos p\varphi$, $\bar{t}_p = \bar{B}_p'' \sin p\varphi$ (but now uniformly distributed along the width l' , so that the cross sections of this ring remain plane), such that no radial displacements $\bar{u}_p$ occur and only tangential displacements $\bar{v}_p$ are present. Again it can be stated that this ring (which does not bend in its plane) possesses a tangential elasticity to be measured by the quotient $\bar{v}_p:\bar{t}_p$, which for all points of the circumference proves to be constant. The width $\bar{l} = \mu a$ is said to represent the effective width $\bar{l}_p = \mu_p a$ of the cylinder (and the ring is said to be equivalent to the cylinder with respect to its tangential elasticity), if $\bar{v}_p:\bar{t}_p = v_p:t_p$. Consequently the cylinder and its equivalent ring of effective width $\bar{l}_p = \mu_p a$ will have the same tangential displacements if loaded by characteristic loads (r_p, t_p) , $(\bar{r}_p, \bar{t}_p)$ resp. the t -components of which are equal. It can be shown that the equality $t = \bar{t}_p$ involves—with a high degree of approximation—the equality $r_p = \bar{r}_p$, and therefore it can be said that the cylinder and its equivalent ring of effective width $\bar{l}_p = \mu_p a$ behave similarly (with respect to the displacements $u_p (=0)$ and v_p occurring in their midplanes) under the action of equal characteristic loads (r_p, t_p) .

By comparing a section of the stiffened cylinder with an equally

stiffened ring of effective width $\bar{l}_p = \mu_p a$ and loaded by a system $r = A_p' \cos p\varphi$, $t = A_p'' \sin p\varphi$, which differs from the characteristic load system $r_p = B_p' \cos p\varphi$, $t_p = B_p'' \sin p\varphi$ (so that the ratio of the coefficients A_p' and A_p'' differs from the ratio $B_p':B_p''$), it can be proved that both constructions are elastically identical, in so far as the transverse rings are similarly loaded at their joints with the cylindrical shell.

All this taken for granted, it is easily understood that the maximum stress occurring in the cylinder, which in the planes of its stiffening girders is loaded by an *arbitrary* system L , can be found by expanding L into a Fourier series, computing for every component ($r = A_p' \cos p\varphi$, $t = A_p'' \sin p\varphi$) the maximal tangential stress that occurs in the T -shaped ring of corresponding effective width $\bar{l}_p = \mu_p a$, and summing up the results obtained in this way.

It goes without saying that the computation of the coefficients μ_p is a rather tedious matter, in which the theory of cylindrical shells is fully involved. It has been the aim of the authors to do the computation once and for all, and the result of their attempt is laid down in tables and graphs, in which μ_p can be found up to $p = 50$ for four values of $k = h^2/12a^2$, viz. $k = 10^{-4}$, 10^{-5} , 10^{-6} and 10^{-7} . Obviously any particular case occurring in practice is related to another value of k . In such a case the procedure recommended is to calculate (eventually by suitable interpolation) the required effective width corresponding to the four values of $k = 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}$; to represent these values as ordinates in a graph, the abscissa of which indicates $\lg k$, to join by a smooth curve the four points so obtained and to read from this curve the ordinate corresponding to the value of k under consideration.

5. *On Circular Plates, Supported in a Number of Points (≥ 3) Regularly Distributed Along Its Boundary and Rotatory-Symmetrically Loaded.* The problem is governed by the differential equation

$$\Delta\Delta u = Ap; \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}; \quad A = \frac{12(m^2 - 1)}{m^2 E h^3} \quad (1)$$

(u = deflection; p = local load; h = thickness of the plate). The displacement u is decomposed into two components u^* and u^{**} ($u = u^* + u^{**}$) such that:

(1) u^* satisfies the differential equation (1) without satisfying the boundary conditions of the problem.

(2) ($u^* + u^{**}$) satisfies both the differential equation and the boundary conditions.

Hereby one is free to choose the boundary conditions for u^* , which in

the case under consideration is done in such a way that at the boundary $r = R \rightarrow u^*$ and $\sigma_r = 0$.

The differential equation for u^{**} obviously is

$$\Delta \Delta u^{**} = 0. \quad (2)$$

The boundary conditions are: (a) $u^{**} = 0$ in the points of support. (b) $\sigma_r = 0$ at $r = R$; (c) shear force (proportional to $u^* + u^{**}$) zero in all points $r = R$, except in the points of support, where it has a prescribed value, viz. one k^{th} of the total load, if k designs the total number of supports. To cover the most general case of loading, u^* is calculated under the assumption that the plate is loaded over the annular surface $2\pi r_0 dr_0$ only ($0 < r_0 < R$) by a constant pressure p ; it is found that the (infinitesimal) deflection is given by

$$du^* = \frac{3(m^2 - 1)}{m^2} \frac{pr_0 R^2}{Eh^3} \left[\frac{m - 1}{2(m + 1)} \left(\frac{r_0^2}{R^2} - 1 \right) \frac{r^2}{R^2} + \left(\frac{r^2}{R^2} + \frac{r_0^2}{R^2} \right) \ln \frac{r_0}{R} - \frac{(3m + 1)}{2(m + 1)} \left(\frac{r_0^2}{R^2} - 1 \right) \right] dr_0.$$

if $0 \leq r \leq r_0$, and

$$du^* = \frac{3(m^2 - 1)}{m^2} \frac{pr_0 R^2}{Eh^3} \left[\frac{m - 1}{2(m + 1)} \left(\frac{r^2}{R^2} - 1 \right) \frac{r_0^2}{R^2} + \left(\frac{r^2}{R^2} + \frac{r_0^2}{R^2} \right) \ln \frac{r_0}{R} - \frac{3m + 1}{2(m + 1)} \left(\frac{r^2}{R^2} - 1 \right) \right] dr_0$$

if $r_0 \leq r \leq R$.

If the constant boundary shear force corresponding with u^* be denoted by q^* , then u^{**} consistent with u^* can be represented by

$$u^{**} = \frac{q^* A r^3 m}{(3m + 1)(m - 1)} \left[2 \sum_{n=k}^{\infty} \frac{(2n + 1)m + 1}{n^2(n^2 - 1)} + \sum_{n=k}^{\infty} \left\{ - \frac{m(n^2 + 3n + 2) - n^2 + n + 2}{n^2(n^2 - 1)} \left(\frac{r}{R} \right)^n + \frac{m - 1}{n^2 + n} \left(\frac{r}{R} \right)^{n+2} \right\} \cos n\varphi \right]$$

(where the suffix ' added to the Σ sign means that summation takes place over multiples of k only).

Numerical results are given for a plate with three supports loaded in its center by a concentrated force P , with respect to the deflection in the center as well as to the deflection at the boundary. A simple instruction is given to calculate the central deflection of a plate with k supports ($k = 3, 4$, up to 12), subject to an arbitrary rotatory load.

C. Miscellaneous Papers

1. *On Elastically Supported Beams.* The elastically supported beam has been treated by several authors. It has, for instance, been discussed at great length in *Technische Dynamik* by C. B. Biezeno and R. Grammel, where special attention is drawn to iterative methods developed by the first author and J. J. Koch. The points of support of the beam under consideration in this paper may be indicated by $0, 1, 2, \dots, n$; the shear force, the bending moment, and the spring reaction at the support i by D_i , M_i , and P_i (the shear force and the moment at the left of a support by D_{il} , and M_{il} , at the right of a support by D_{ir} and M_{ir}). The quantities D_{0l} , M_{0l} , D_{nr} , M_{nr} are supposed to be known and are denoted by $\bar{D}_{0l}$, $\bar{M}_{0l}$, etc. The deflection of the beam is denoted by y . The method starts with the assumption of arbitrary values for y_0 and y_0' . Then the first reaction of support P_0 can be calculated with the aid of the spring constant k_0 ($P_0 = k_0 y_0$) and consequently D_{1l} and M_{1l} can be statically determined. The deflection and slope y_1 and y_1' of the end point of the beam (01) clamped at (0) under prescribed y_0 and y_0' can easily be calculated by the aid of elementary deflection formulas, and consequently the reaction $P_1 (= k_1 y_1)$ (and if required D_{1r}) are known. From y_1 , y_1' , D_{1r} (and the external loads) M_2 , D_{2l} , y_2 , y_2' can be derived in the same way, and finally D_{nr} and M_{nr} are calculated, which naturally will not agree with the prescribed values $\bar{D}_{nr}$ and $\bar{M}_{nr}$.

The same process is repeated for two other sets of arbitrarily chosen values, y_0^* , $y_0'^*$ and y_0^{**} , $y_0'^{**}$ (not simultaneously zero, under the understanding that all external loads of the beam are put equal to zero, inclusive D_{0l} and M_{0l}). The end result of these computations may be represented by the sets D_{nr}^* , M_n^* and D_{nr}^{**} , M_n^{**} . It is readily understood that the real values $\bar{y}_0$, $\bar{y}_0'$ can be written as

$$\bar{y}_0 = y_0 + t_1 y_0^* + t_2 y_0^{**}, \quad \bar{y}_0' = y_0' + t_1 y_0'^* + y_0'^{**} \quad (1)$$

and that consequently also the following equations must hold:

$$\begin{aligned} \bar{D}_{nr} &= D_{nr} + t_1 D_{nr}^* + t_2 D_{nr}^{**} \\ \bar{M}_{nr} &= M_{nr} + t_1 M_{nr}^* + t_2 M_{nr}^{**} \end{aligned} \quad (2)$$

from which t_1 and t_2 can be calculated. These constants being known, all other required data can easily be found. The practical applicability of the method lies in the fact that a simple schedule is put at the disposal of the reader, which enables him indeed to leave the whole computation to a calculator who may lack all mechanical knowledge.

2. *On a Special Case of Bending.* In a paper, published in the *Ingenieur Archiv* (Vol. XII, 1941), R. Sonntag draws attention to a special

case of bending, which occurs if a highly elastic beam freely supported in two points of prescribed distance l is subjected to a transverse load P , acting in the middle of the span. The beam is supposed to slide freely—and without friction—over its supports so that great deflections are to be expected even under relatively small loads. In an ingenious but rather artificial way the author of the present paper succeeds in deducing the relations between the load P , the corresponding deflection f , and the length of the deflected beam as far as it lies between the supports. The artificiality lies in the linearization of the problem, which—at the expense of a rather laborious control—is proved to hold within well-defined limits. In the Biezeno paper it is shown that the problem admits of a quite natural solution, which for all beams of constant flexural rigidity requires the construction of one single graph, from which Sonntag's conclusions—inclusive the interesting remark about the stability of the beam—can all be derived. The construction admits of some generalizations. First it can be extended without difficulty to symmetrical girders of varying flexural rigidity. But it can as well be extended to cases where friction comes into being during the gliding motion of the girder over its supports. Finally it can be stated that, whereas Sonntag's deductions hold true only as far as the central line of the beam can be approximated by a suitably chosen circular arc, the method proposed in the present paper admits of the investigation of much more complicated deflections. For another (analytical) treatment of the same problem refer to section A7.

3. *Critical Speeds of Rotating Shafts.* It is well known that a rotating shaft, supported either in a statically determined or a statically undetermined way and bearing n centered and concentrated masses $m_1, m_2, \dots, m_n$ (in comparison of which the mass of the shaft itself may be neglected), may whirl at most at n angular speeds. The reciprocal squares of these critical angular speeds are roots of a so-called secular equation of such particular structure that the reality and positiveness of these roots is recognized by the use of well-known mathematical results. But no guarantee is given for the fact—tacitly assumed by all engineers—that these roots are *different* ones. The paper under consideration deals with this inequality and gives the proof of it.

The proof is rather intricate and depends on deep-seated mechanical properties of the shaft, which find their expression in certain relations between the influence coefficients α_{ij} of Maxwell that are connected with the problem. It must be remarked that it is a fundamental feature of the proof that all supports are rigid and that at these supports no zero slope can occur if the shaft is loaded by a single force. Therefore supports—terminal ones eventually excepted—that should have a “clamp-

ing" effect must be excluded. Elastic supports too must be excluded. Examples are given to show that these restrictions are essential.

4. *On the Elastic Behaviour of the So-Called "Bourdon" Pressure Gage.* The Bourdon pressure gage consists of annular tube of an elliptic or oval cross section and of constant thickness, closed by two meridional planes, which, as a result, include an angle lying between 240° and 300° . The major axis of symmetry of the cross section is parallel, the minor axis of symmetry perpendicular to the axis of revolution of the pressure gage. An internal pressure p (as has initially been discovered by mere change) will change the angular distance of the terminal planes, and consequently the amount of this change can be considered as a measure of this pressure. The phenomenon has been studied by H. Lorenz (Zeitsch. Ver. Deutsch. Ingen. 54, 1890) but serious objections must be made against his mode of deduction: on the one hand, the flexural stiffness of the wall is neglected in one part of the calculations, so that there the wall is assumed to act solely as a membrane; on the other hand, in another part of the treatise in which the flexural rigidity of the wall is brought into account, another phenomenon—occurring with initially curved tubes—is neglected. A recent study by G. Schubert accounts for this latter phenomenon but since it touches on the pressure gage only indirectly it does not bring the subject to a conclusion. The present authors meet the objections just mentioned and develop a method that in a rather simple way leads to numerical calculation of both the deformation and the stresses of the gage. It is proved that exceedingly high stresses are caused by relatively low pressures, and that as a rule the limit of proportionality is greatly surpassed by those pressures for which the pressure gage is meant to be designed. It therefore is no longer surprising that manifold ruptures occur and that elastic hysteresis and nonproportionality between pressure and reading are frequently observed.

The calculations are based on the assumption that the displacements and stresses are the same for all meridional cross sections of the gage. It goes without saying that the similarity of all distorted cross sections, which is a consequence of this assumption, would occur only if the gage were closed by two oval flat pistons lacking all elastic stiffness in their planes. It must, however, be emphasized that the disturbance caused by the compressive stiffness of the actual closing ends will be restricted to their immediate neighborhood, and if necessary, may be taken into account by introducing a smaller effective length of the gage.

The crucial point of the problem lies in the calculation of the normal stresses, acting in a meridional plane; the corresponding system of normal forces (which as a whole represents an equilibrium system) obviously has to fulfill the requirement that the meridional section remains plane,

under consideration of the fact that the deformation of the cross section of the gage is influenced by this system. The solution is found in an iterative way.

The situation of the dangerous point of the structure, determined after the theory of rupture of Guest, is found to be in full agreement with the experiment. Directions are given for performing the required calculations for any given form of cross section.

5. *On the Torsion of Prismatic Beams the Cross Section of Which Is Bounded by Two Pairs of Orthogonal Circular Arcs.* The cross section is defined with respect to a rectangular system of coordinates (y, z) , the origin of which is the center of two limiting arcs (radius R); the radius of the two other arcs (having their center on the y -axis and their convex side to the z -axis) is designed by ρ and is unambiguously fixed by the polar angle φ of one of the "angular" points of the cross section.

It is evident that the problem is solved with the aid of the equations

$$\Delta\Phi = -2G\omega, \quad \tau_{xz} = -\frac{\partial\Phi}{\partial y}, \quad \tau_{zy} = \frac{\partial\Phi}{\partial z} \quad (1)$$

and by the use of bipolar coordinates.

Beforehand the required function Φ is split up into two components Φ^* and Φ^{**} defined by

$$\Delta\Phi^* = -2G\omega \quad \Phi_R^* = 0 \quad \Phi_\rho^* \neq 0 \quad (2)$$

$$\Delta\Phi^{**} = 0 \quad \Phi_R^{**} = 0 \quad \Phi_\rho^{**} = -\Phi^* \quad (3)$$

whereby boundary values of Φ , Φ^* , or Φ^{**} in a point of the circles R and ρ are indicated by subscripts R and ρ respectively.

The function Φ^* is chosen as

$$\Phi^* = \frac{G\omega}{2} (R^2 - y^2 - z^2). \quad (4)$$

The bipolar coordinates used henceforth are defined by the complex transformation

$$u = R \operatorname{tgh} \frac{v}{2} \quad (u = y + iz, v = \eta + i\xi) \quad (5)$$

which gives rise to the following set of transformation formulas:

$$y = R \frac{\sinh \eta}{\cosh \eta + \cos \xi} \quad z = R \frac{\sin \xi}{\cosh \eta + \cos \xi} \quad (6)$$

The differential equation (3) is invariant with respect to this transformation.

The transformation itself is such that the limiting arcs of the cross section are covered by the curves $\rho = \pi/2$ and $\eta = \pm \eta_0$, where η_0 is defined by $\rho \sinh \eta_0 = R$. The problem to be solved now is reduced to the determination of a function $\Phi^{**}(\eta\zeta)$, satisfying (3) and the boundary conditions

$$\Phi_R^{**} = 0 \quad \Phi_\rho^{**} = -\Phi_\rho^* = -G\omega R^2 \frac{\cos \zeta}{\cosh \eta_0 + \cos \zeta}. \quad (7)$$

By putting

$$\Phi^{**} = \sum_{n=1}^{\infty} ' a_n \cosh n\eta \cos n\zeta \quad (8)$$

(the accent denoting that summation takes place over odd n 's only) it is proved that

$$a_n = -\frac{4}{\pi} G\omega R^2 \frac{I_n}{\cosh n\eta_0} \quad \text{with} \quad I_n = \int_0^{\frac{\pi}{2}} \frac{\cos \zeta \cos n\zeta}{\cosh \eta_0 + \cos \zeta} d\zeta \quad (9)$$

and therefore

$$\Phi = G\omega R^2 \left[\frac{\cos \zeta}{\cosh \eta + \cos \zeta} - \frac{4}{\pi} \sum_{n=1}^{\infty} ' \frac{I_n}{\cosh n\eta_0} \cosh n\eta \cos n\zeta \right]. \quad (10)$$

The calculation of the maximum stress occurring at the boundary ρ and of the factor of rigidity requires a fair amount of computation, on which we need not enter here.

It is sufficient that full numerical information is given about all required and important practical quantities.

6. *Some Applications of Fourier Integrals with Respect to Elastic Problems.* In this doctor's thesis two different problems are treated without any connection other than the mathematical method by which they are solved. The first one relates to the so-called St. Venant principle (which from a totally different point of view has been elucidated by R. von Mises: see *Bull. American Math. Soc.* **51**, 555-562). The character of the principle involves a certain vagueness and as a matter of fact different versions of it are met in the current literature. One version is to be found in A. E. H. Love's *Treatise on the Mathematical Theory of Elasticity*: "the strains and (we may add) the stresses that are produced in a body by the application to a small part of its surface of a system of forces statically equivalent to zero force and zero couple are of negligible magnitude at distances that are large compared with the linear dimensions of the part." Another way of putting the theorem is the following: If the small part of the body surface mentioned before is loaded alter-

nately by two statically equivalent load systems S_1 and S_2 (such that $S_1 - S_2$ is an equilibrium system), then the greater the distance of a given point P from the loaded surface the less the state of stress and strain occurring in P as produced by S_1 will differ from those produced by S_2 .

O. Zanaboni (*Atti Accad. Lincei*, 1937) gives a remarkable set of theses with respect to the theorem, which, however ingenious their deduction may be, give nevertheless no satisfaction to the engineer, who is far more interested in the numerical question of what difference will occur in the stresses in the neighborhood of the field of application of the two statically equivalent force systems.

Restricting himself to the half plane and the half space, the author of this treatise indicates the narrowest possible limits between which the local stresses in a prescribed point can vary for all possible statically equivalent loads S .

The plane problem is related to a system of rectangular coordinates (x, y, z) , the x -axis of which coincides with the boundary of the stress field, and the y -axis of which points to the inner stress field. The general solution is given by the stress function F :

$$F = \int_0^\infty [(f_1 + ytf_2) \cos xt + (f_3 + ytf_4) \sin xt] e^{-xt} + f_5x + f_6y + f_7] dt \quad (1)$$

in which the f_i stand for unknown functions of t .

The corresponding stresses σ_x , τ_{xy} , σ_y are:

$$\left. \begin{aligned} \sigma_x &= \int_0^\infty \{(f_1 - 2f_2 + ytf_2) \cos xt + (f_3 - 2f_4 + ytf_4) \sin xt\} e^{-xt} t^2 dt \\ \tau_{xy} &= \int_0^\infty \{(f_3 - f_4 + ytf_4) \cos xt + (-f_1 + f_2 - ytf_2) \sin xt\} e^{-xt} t^2 dt \\ \sigma_y &= \int_0^\infty \{(-f_1 - ytf_2) \cos xt + (-f_3 - ytf_4) \sin xt\} e^{-xt} t^2 dt \end{aligned} \right\} \quad (2)$$

so that at the boundary $y = 0$

$$\left. \begin{aligned} \tau_{xy}(x, 0) &= \int_0^\infty \{(f_3 - f_4) \cos xt + (-f_1 + f_2) \sin xt\} t^2 dt \\ \sigma_y(x, 0) &= \int_0^\infty \{-f_1 \cos xt - f_3 \sin xt\} t^2 dt. \end{aligned} \right\} \quad (3)$$

On the other hand the prescribed boundary stresses as far as τ_{yx} and σ_y are concerned can be written as:

$$\left. \begin{aligned} \tau_{xy} = p_x(x) &= \frac{2}{\pi} \int_0^\infty \{P_{x1}(t) \cos xt + P_{x2}(t) \sin xt\} dt \\ \sigma_y = p_y(x) &= \frac{2}{\pi} \int_0^\infty \{P_{y1}(t) \cos xt + P_{y2}(t) \sin xt\} dt \end{aligned} \right\} \quad (4)$$

with

$$\left. \begin{aligned} P_{x1} &= \int_0^\infty p_{x1}(u) \cos tudu, & P_{y1} &= \int_0^\infty p_{y1}(u) \cos tudu \\ P_{x2} &= \int_0^\infty p_{x2}(u) \sin tudu, & P_{y2} &= \int_0^\infty p_{y2}(u) \sin tudu \\ p_{x1}(x) &= \frac{1}{2}\{p_x(x) + p_x(-x)\} & p_{y1}(x) &= \frac{1}{2}\{p_y(x) + p_y(-x)\} \\ p_{x2}(x) &= \frac{1}{2}\{p_x(x) - p_x(-x)\} & p_{y2}(x) &= \frac{1}{2}\{p_y(x) - p_y(-x)\} \end{aligned} \right\} \quad (5)$$

Comparison of (3) and (4) leads to:

$$\begin{aligned} t^2 f_1 &= \frac{2}{\pi} P_{y1}(t); & t^2 f_2 &= \frac{2}{\pi} \{-P_{x2}(t) + P_{y1}(t)\}; & t^2 f_3 &= \frac{2}{\pi} P_{y2}; \\ t^2 f_4 &= \frac{2}{\pi} \{P_{x1}(t) + P_{y2}(t)\}. \end{aligned} \quad (6)$$

Herewith all data are formally at hand for the calculation of the stress field produced by the prescribed boundary stresses τ_{xy} and σ_y .

For the sake of comparison several special cases are treated first: (a) a constant pressure $p_{y1}(x)$ over the interval $-a \leq x \leq a$; (b) the concentrated force P at $x = 0$; (c) a constant shear p_{x1} over the interval $-a \leq x \leq a$; (d) the concentrated shearing force P_x at $x = 0$; (e) the load p_{y2} , constant in the interval $0 \leq x \leq a$, and $-a \leq x \leq 0$, but of different sign; (f) the statically equivalent "double-force" of moment M_z ; (g) the shearing load p_{x2} , constant over the interval $0 \leq x \leq a$ and $-a \leq x \leq 0$, but of different sign; (h) the statically equivalent "double" force at $x = 0$ (clearly not exerting a moment now).

After having finished this preliminary work the author continues to examine the general case of loading in the interval $-a \leq x \leq +a$. To give an idea of his trend of thought the results of his computations may be reproduced for the specialized case for which the loading consists of normal pressures p_{y1d} only, symmetrical with respect to $x = 0$, and subjected to the restriction (indicated by the subscript d) that the system be definitely positive. (To the case in which this restriction is not fulfilled due attention is given later on by the author.) In this case the stress components can be written as:

$$\left. \begin{aligned} \sigma_x &= -\frac{4}{\pi} \frac{x^2 y}{r^4} \int_0^a p_{y1d}(u) F_1 du; & \tau_{xy} &= -\frac{4}{\pi} \frac{xy^2}{r^4} \int_0^a p_{y1d}(u) F_2 du \\ \sigma_y &= -\frac{4}{\pi} \frac{y^3}{r^4} \int_0^a p_{y1d}(u) F_3 du; & r^2 &= x^2 + y^2. \end{aligned} \right\} \quad (6)$$

In these formulas the factors before the integrals represent the stress components of σ_x' , τ_{xy}' , σ_y' produced by a statically equivalent concentrated load P_y at $x = 0$; and therefore the integrals themselves can be

looked upon as the "correction" factors of the stresses under consideration. They converge properly to unity if r increases indefinitely and are represented by:

$$\left. \begin{aligned} F_1(\alpha, \xi) &= \frac{(1 + \alpha^2)^2 \{1 + (\alpha/\xi)^2\} - 4\alpha^2(2 + \alpha^2 - \xi^2)}{\{(1 + \alpha^2)^2 - 4\alpha^2\xi^2\}^2} \\ F_2(\alpha, \xi) &= \frac{(1 - \alpha^2)^2 + 4\alpha^2\xi^2 - 4\alpha^4}{\{(1 + \alpha^2)^2 - 4\alpha^2\xi^2\}^2}; & F_3 &= \frac{(1 + \alpha^2)^2 + 4\alpha^2\xi^2}{\{(1 + \alpha^2)^2 - 4\alpha^2\xi^2\}^2} \\ F_3(\alpha, \xi) &= \frac{(1 + \alpha^2)^2 + 4\alpha^2\xi^2}{\{(1 + \alpha^2)^2 - 4\alpha^2\xi^2\}^2}; & \alpha &= \frac{u}{r}; & \xi &= \frac{x}{r}. \end{aligned} \right\} \quad (7)$$

By introducing new indeterminate functions F_i' , limited by the inequalities:

$$\min F_i \leq F_i' \leq \max F_i,$$

the stresses can be written as

$$\sigma_x = \sigma_x' F_1', \quad \tau_{xy} = \tau_{xy}' F_2', \quad \sigma_y = \sigma_y' F_3' \quad (8)$$

which represent the closest estimation to be made. The results are brought into numerical and graphical form, both for the special case that has served as an example here and for all other possible cases of loading.

The same problem is brought to a successful end for the symmetrically loaded half space (loaded over the surface $r \leq a$) with the aid of Fourier-Bessel integrals of the type

$$f(r) = \int_0^\infty J_n(rt) dt \int_0^\infty f(u) J_n(tu) \cdot u du. \quad (9)$$

It is not possible to go into further details here but it may be emphasized that the solution of the problem requires a considerable amount of analytical skill. (The reader will find the answers and the method of solution of a great number of important integrals at the end of the treatise.)

The second object treated with the same expedients concerns the torsion of prismatic rods, the cross section of which is limited by two circular arcs. All possible modifications (without exception) are considered; and all special cases already treated by other authors are found again as specializations of the general question put forward here, so that the results are checked in several ways. Extensive use is made of conformal transformation, viz:

$$\left. \begin{aligned} x + iy &= c \cotg \frac{1}{2}(u - iv) \\ x &= c \frac{\sinh u}{\cosh u - \cos v} & y &= c \frac{\sin v}{\cosh u - \cos v} \\ \tanh u &= \frac{2cx}{x^2 + y^2 + c^2} & \operatorname{tg} v &= \frac{2cy}{x^2 + y^2 - c^2} \end{aligned} \right\} \quad (9)$$

It goes without saying that the transformation is chosen such that the points of intersection of the two boundary circles coincide with the singular points of the transformation.

All questions that relate to the maximum stress, the torsional rigidity, and the torsional moment are answered in detail but here too mathematical skill is required on the part of the reader if he wishes to follow all computations in detail (Bessel functions, elliptic functions, Γ functions, Ψ functions, conformal transformation, etc. are used). The treatise concludes with some practical applications.

7. *The Convergence of a Specialized Iterative Process in Use in Structural Analysis.* In recent years so-called "relaxation" methods have been developed that have completely altered the treatment of a great number of technically important structures, such as redundant girders, redundant frame-works, trusses of "Vierendeel" construction, etc.

These methods originate from an article by Hardy Cross, Analysis of continuous frames by distributing fixed-end moments, *Proc. Amer. Soc. C.E.* **96**, 1-156 (1932), and have been developed mainly by R. V. Southwell and his disciples (cf. R. V. Southwell, Relaxation methods in engineering science, a treatise on approximate computation. Oxford University Press, New York, 1940).

All of them are iterative methods, the convergence of which—if considered at all—is generally discussed from the energetic point of view. The iterative method, under consideration here, is investigated in a different way. It is restricted to two-dimensional frameworks the joints of which are only liable to rotations, and for the rest are fixed in space. The joints of the framework may be denoted by A_i ($i = 1, 2, \dots, n$); the length of the member joining the points A_i and A_j may be indicated by $l_{ij} = l_{ji}$, and it may be prearranged by definition that $l_{ij} = 0$ if the A_i and A_j are unconnected and that, in agreement with this definition, $l_{ii} = 0$.

If the structure is loaded it will be brought into its "natural" state of stress and all joints A_i will rotate. If the rotations are prevented by suitable moments M_i applied in the joints A_i , the construction will be said to be "locked." It is supposed that the state of stress can easily be calculated for this fully "locked" structure. Then the problem to be solved consists in calculating all stresses that occur if the structure is "unlocked." It is this unlocking of the structure that is effectuated in degrees and in an iterative way. The locking moments, which prevail after the k 'th iteration will be designated by $M_i(k)$. The procedure of iteration having once been fixed, it must be shown that all locking moments $M_i(k)$ ($i = 1, 2, \dots, n$) tend to zero if k tends to infinity. Obviously it must be guaranteed for the sake of practical applicability that this procedure of iteration be such that the change in state of stress, brought about by one single iteration, can easily be calculated.

The iterative method. Let the iteration be carried out k times, so that the locking moments are $M_i(k)$ ($i = 1, 2, \dots, n$) and let the joint A_i alone be unlocked. Then A_i must be loaded with $-M_i(k)$. All joints connected with A_i will be influenced by this unloading in a way that depends upon the distribution of $-M_i(k)$ over all members by which A_i is connected with other joints A_j . This distribution can be described by attributing to every member $A_i A_j$ a certain "elastic weight" α_{ij} . The portion of $-M_i(k)$ for which the member $A_i A_j$ stands, will then be represented by

$$\frac{\alpha_{ij}}{\sum_h \alpha_{ih}} M_i(k) \quad (1)$$

(where the summation over h may be extended over all values $h = 1, 2, \dots, n$, if only α_{ih} is put equal to zero for every h for which $l_{ih} = 0$).

The "extra" locking moment, caused by the unlocking of A_i , depends solely on the elastic properties of the member $A_i A_j$ and can be calculated by introducing the so-called "coefficient of transition" β_{ij} ; it then becomes

$$-\frac{\beta_{ij}\alpha_{ij}}{\sum_h \alpha_{ih}} M_i(k). \quad (2)$$

If the liquidation of the moments $M_i(k)$ is supposed to take place simultaneously for all joints A_i , it is evident that at the point A_i the moment $M_i(k)$ vanished, but is replaced by a moment $M_i(k+1)$ represented by

$$M_i(k+1) = - \sum_j \frac{\beta_{ji}\alpha_{ji}}{\sum_h \alpha_{jh}} M_j(k) \quad (i = 1, 2, \dots, n) \quad (3)$$

The system (3) is the analytical representation of the iterative method proposed in this paper.

By putting

$$N_i(k) = \frac{M_i(k)}{\sum_h \alpha_{ih}} \quad (4)$$

it is reduced to the system:

$$N_i(k+1) \cdot \sum_j \alpha_{ij} = - \sum_j \beta_{ji}\alpha_{ji} N_j(k) \quad (i = 1, 2, \dots, n). \quad (5)$$

The convergence of the iteration defined by these equations depends upon a secular equation, of which all the roots ought to be smaller than 1

(in absolute magnitude). The proof that this condition is fulfilled under all circumstances is given in two distinct ways. The first proof is based on well-known properties of quadratic definite positive polynomials; the second one requires the introduction of so-called reciprocal (or conjugated) influence numbers of Maxwell. The interconnection of both trends of thought leads to remarkable inequalities for the latter numbers.

A simplification of the problem, bearing only on an elastic beam of uniform cross section supported in n distinct points, found its treatment in the Dutch periodical *De Ingenieur* (1946).

8. *A Draft of Nomenclature for Deformations.* At the meeting of the Executive Committee of the International Council of Scientific Unions in 1932 in London Dr. G. E. Hale proposed that there be established a committee for "instruments and methods of research" with the purpose of accumulating data on the use of instruments and methods developed in special branches of science in order to make them known also in other branches. The Committee for International Scientific Relations of the Royal Academy of Sciences at Amsterdam received this proposal with much interest, and since Dr. Hale had also stimulated the formation of national committees that should be related to the international committee, it was considered that in this connection the subject of viscous and plastic deformation seemed particularly well suited to provide a theme that could bring together scientists from the rather different domains of physics, chemistry, and biology. The object of the committee was formulated as follows:

1. to gather information regarding the phenomena of viscous and plastic deformation as they present themselves in various domains of physics, chemistry, technology, and biology.
2. to investigate the relations existing between these phenomena.
3. to make proposals for a nomenclature that should eliminate existing uncertainties in the various denominations.
4. to study the methods used for the measurement of viscosity and of related properties of matter, to interpret the meaning of the results given by various technical instruments, and where possible to indicate instruments that allow an unambiguous measurement of scientifically well-defined quantities.

The first report of the Committee (256 pages) appeared in 1935 and contained six chapters, viz.: I. Mechanical considerations; model systems, phenomenological theories of relaxation and of viscosity. II. Remarks in connection with the experimental investigation of self-restoring substances. III. Viscosity measurements with special reference to their application in colloid chemistry. IV. Viscosity and plasticity from a technical point of view. V. Plasticity of crystalline substances, in par-

ticular of metals. VI. Viscosity effects in the living protoplasm and in muscles.

The second report (287 pages) appeared in 1938 and contains the following six chapters: I. Introductory remarks on recent investigations concerning the structure of liquids. II. Viscosity of liquids in connection with their chemical and physical constitution. III. On the motion of small particles of elongated form, suspended in a viscous liquid. IV. The yield value. V. Recent plastometers. VI. Technical capillary viscometers.

These two elaborate reports, which may already have found their way to the United States of America, cover items 1 and 2 and partly item 4 of the program mentioned before, whereas the treatise mentioned in the heading fulfills requirement 3. In contrast with the two reports this treatise is provisionally written in Dutch, but it is meant to be translated into English as well, in order that it can meet the criticism of countries abroad. Up to now no serious objections have been made by the scientific workers in the field of mechanics in Holland to whose judgment it has been subjected. It is impossible to go into details here, because nearly every one of them would call for some nearer explanation, but it seems to be worth while to draw attention to this important attempt in the furtherance of unity of expression in a field that up to now excels in diversity of expressions. The monograph is subdivided into 5 chapters: I. Fundamental views and definitions. II. Classification of all different possibilities of deformation of a homogeneous material subject to simple shear. III. On the applicability of the nomenclature introduced in the preceding chapter to states of stress differing from pure shear. IV. Complementary definitions. V. Denominations for the distinguishing of different classes of materials.

The treatise concludes with a conveniently arranged table of all possible deformations and their names.

D. *Experimental Work*

1. *The "Reduced" Length of (Cylindrical) Twisted Shafts of Variable Cross Section.* For shafts of variable cross section the customary standard formulas with respect to the maximum stress and the angle of twist (or the torsional rigidity) can be used only if the change in section is not too rapid. In the case of sharp variation of cross section (frequently occurring in practice) not only may high concentrations of stresses occur, but also the torsional rigidity may be perceptibly influenced. It was the aim of the investigation to provide the reader with experimental data that would enable him to calculate this torsional rigidity, by altering the lengths of the different parts of the shaft in such a way that no influence

of fillets needs to be brought into account and no other formula has to be used than $\varphi = Wl/GI_p$ (W = torsional moment, G = shear modulus, I_p = polar moment of inertia). The results, which are united in conveniently arranged graphs, are of great use in the calculation of the torsional frequency of shafts of the type in view.

2. *Some Experimental Data Concerning Flange Couplings.* This article is closely connected with the preceding one. The experiments described here are performed with the aim of comparing the reduced length of a normally constructed flange coupling with that of a "blind" flange the length of which has been reduced according to the directions indicated in Section 1. It appears that the last one has to be increased by about 50 per cent to yield the first one. A very accurate calculation of the reduced length of the coupling, however, is possible if the flanges are only pressed together along a small circular (circumferential) ring of their plane of separation. Another important series of experiments has been undertaken to answer the question whether it is useful or not to insist on fitting bolts and to investigate whether or not the usual number of bolts is necessary. Three series of $10 \cdot 10^6$ cycles of stress variations performed on the model of the coupling demonstrated that fitting of the bolts is entirely superfluous, and that even an absolutely reliable joint is possible with two bolts only. As far as can be seen no "scale" effects are to be feared, so that it may be expected that all conclusions arrived at in this paper are valid for full-scale constructions. It goes without saying that the last word in all these matters is to be spoken by engineering practice. The gain in labor and costs that is to be expected from the proposed alterations in the construction of flange connections make it, however, desirable that such engineering practice be pursued.

3. *Strength and Stiffness of a Twisted Cross Section Weakened by a Deep Key Way with Sharp Corners.* The direct cause of this investigation was given by the desirability of exactly knowing the stress concentrations in the ground of the key way, as occurring in the neighborhood of the fillets and the need for reliable directions for the choice of the radius r of such fillets. The method used was that of the well-known soap-film theory. For all measurements a brass model plate was used, the circular hole of which had a radius $R = 5.0$ cm; the width (b) of the key way amounted to $b = 1.5$ cm. Three series of tests were made corresponding with a depth l of the key way: $l = 0.5R$, $l = R$, and $l = 1.5R$. Every series was performed for a number of different values of r .

It appeared that for any prescribed value of r the greatest stress concentrations occur with $l = R$. With a prescribed value of l the stress concentration naturally decreases with increasing r , but it is noteworthy that for great and small values of l no perceptible improvement occurs if r

surpasses 2 mm; if $l = R$, however, it is recommended that r be enlarged to $r = 5$ mm. Denoting the angle of twist per unit of length by ω , the factor of rigidity by $S_w (= M:\omega; M = \text{torsional moment})$, the "moment of resistance" W_w by $W_w = M:\tau_{\max}$, two characteristic numbers μ and λ , standing for the torsional rigidity and the moment of resistance can be introduced by putting $\mu = S_w:GR^4$ and $\lambda = W_w:R^3$.

For all cross sections under consideration, λ and μ have been calculated from the measured soap films, expressed in terms of the two parameters $\alpha = l:2R$ and $\beta = r:b$. Thanks to the theoretical deductions referred to in Section C6, many experimental results could be checked and reliable extrapolations could be made so that full information is available for the whole field (α, β) as far as it is of practical importance. It must be admitted that all results are restricted to those cases in which $b:R$ has the fixed value 1.5:5.0. Since this ratio represents the value that nearly always occurs in machine practice, it seemed inappropriate to waste a considerable amount of work on other cases.

4. *Experimental Determination of the Stresses Occurring in a Ship Propeller.* The object of this investigation is the same as that taken in view by the American authors E. S. Carmick and H. B. Dodge, who in the *Journal of the Society of Naval Engineers*, **52**, 1940, described the results of "the use of an electrical strain gage in investigating stresses (in a ship propeller)." It has also been treated in a Dutch periodical, *Schip en Werf* (1944-45) by W. H. C. E. Rösingh. The present treatise is restricted to the static load of the propeller as furnished by the water pressure, which of course could be only roughly imitated by a statically equivalent system of concentrated forces. No efforts are made to reproduce stresses as caused by the centrifugal forces. Stresses occurring at the root of the blades are considered only qualitatively, in order to compare them with estimated results obtained by the method of photoelasticity. No attention was given to problems as presented by the coupled bending and torsional vibrations of the blades, which in the future certainly will become of great importance. The measurements were performed with the aid of the Huggenberger extensometer, mounted in a newly designed frame, such that measurements in different directions (as required for the determination of the total state of stress) could easily be executed. This total state of stress was computed for a great number of circumferential points of four cross sections. The analysis of the results obtained led to the result that the maximum normal stress σ occurring in such a cross section can be represented with fair approximation by a formula of the classical form $\sigma = M/W$, in which M represents the bending moment with respect to a well-defined axis of the cross section and W a well-defined geometrical quantity depending on the shape of the cross

section and the position of the axis just mentioned. The maximum stress occurring in the fillet of the propeller can be estimated by using an experimental formula that will be mentioned in Section 5.

5. *Optical Determination of Stress Concentrations in Fillets of Flat Bars of Constant Thickness and Sharp Varying Width.* If the smallest width of the bar is denoted by h , the greatest width by H , and the radius of the fillets by r , the so-called factor of stress concentration α is a function of the two variables h/H and h/r . From the experimental data acquired by the author a formula for α has been deduced that may be considered as accurate for the range $h/H = 0 \dots 1$, and $h/r = 1 \dots 10$, viz.,

$$\alpha = 1 + 0.17 \left(\frac{h}{r} \right)^{0.6} \sqrt{1 - \left(\frac{h}{H} \right)^2}.$$

The factor α relates to the case of bending so that the maximum bending stress can be calculated by aid of the formula:

$$\sigma_{\max} = \alpha M : W.$$

6. *Effect of Change in Section of Rotating Shafts on Fatigue Properties.* The treatise under consideration arose from a question put to the authors by the Dutch Central Bureau of Normalization and bearing on the effect of fillets in rotating shafts of variable cross section on the endurance limit of the bending moment. The German and Swiss rules with respect to this problem diverged largely: on the one hand the radius r of the fillet was made dependent on the smallest (d) of the two diameters only, on the other hand it was made dependent solely on the difference ($D - d$) of both diameters. Renewed examination of the question seemed desirable, and, as was to be expected, the result was that the dynamical concentration factor β proved to be dependent on both quantities d/r and d/D . (If M denotes the limiting moment that can be endured by the shaft under 10^7 revolutions, and M^* the limiting moment that under the same conditions can be endured by a smooth shaft of diameter d , then β is defined by: $\beta = M : M^*$.) The way in which this factor β can be determined in a satisfactorily short time will be described in the next section. The experiments mentioned in this paper lead to the result that the dynamical factor β can be represented in a reliable way by the formula referred to in the preceding section if only h and H be replaced by d and D , so that

$$\beta = 1 + 0.17 \left(\frac{d}{r} \right)^{0.6} \sqrt{1 - \left(\frac{d}{D} \right)^2}.$$

In structural design β will be prescribed, and after having calculated d on the base of strength requirements and D on the base of structural

considerations, the designer will have to determine the radius r of the fillet. For this reason a graph is given in which for different given values of β the ratio d/r is plotted against the ratio d/D .

7. *An Abbreviated Method for the Determination of the Factor β Mentioned in the Preceding Section, for Torsional Stress Cycles as Well as for Bending Ones.* It is well-known that the experimental determination of the factor β , defined in the preceding section takes up much time and is in addition very expensive. It therefore is worth while to seek for an abbreviated method, perhaps lacking to some extent in accuracy, but saving much time and cost.

An effort in this direction is made in this paper; it can be summarized as follows. Starting with a bending moment (of such magnitude that the bending stress is perceptibly smaller than the endurance stress σ_e belonging to 10^7 cycles) the load is regularly increased by 10 per cent after a period in which 10^6 revolutions are made. The experiment is performed not only with the shaft under consideration but also with a smooth shaft the diameter of which is equal to the smaller diameter of the first one. For both shafts the bending moment for which breakdown occurs can be determined to a high degree of approximation, and the factor β can be represented by the ratio β' of these two bending moments. The conclusion to be drawn from laborious experiments, performed in the Laboratory of the Technical University at Delft, is that results of the abbreviated method may be expected that deviate ± 5 per cent from the required value.

8. *A Highly Sensitive Electrical Apparatus for the Magnification, Reproduction, and Measurement of Mechanical Quantities Occurring with Vibratory Phenomena.* The apparatus under consideration is of the "capacity" type, so that the phenomenon to be registered always governs some displacement to be measured by the change of capacity of a condenser. The requirements to be fulfilled are:

(1) The apparatus must not influence the phenomenon to be measured; accordingly the displacement must be held between very narrow limits, and consequently the apparatus must have a great sensitivity.

(2) The phenomenon must be truly reproduced. This requirement can be divided into two other ones:

(a) Rapid and slow-passing phenomena (including static phenomena) must be reproduced with the same sensitivity. (If this requirement is fulfilled the apparatus admits of being statically gaged, which is of great practical importance.)

(b) The amplitude of the reproduction must depend linearly upon the amplitude of the mechanical quantity to be measured.

(c) On account of the manifold problems to be investigated it is

required that it be possible to regulate the sensitivity of the apparatus between wide limits.

In broad lines the apparatus operates as follows: In a so-called detector the quantity to be measured is converted into the relative displacement of the two plates of a condenser which itself is placed in a capacity Wheatstone bridge, subjected to an alternating tension of such high frequency ($1\text{ }M \cdot Hz$) that the period is negligibly small in comparison with the period of the phenomena to be measured. The modulated high-frequency tension in the measuring branch of the bridge is magnified in a high-frequency amplifier and this amplified tension is rectified; the result is a low frequency tension varying in accordance with the modulation. After a tension of constant magnitude is superposed, this low-frequency tension can be led into a dead-tension amplifier, where it is magnified. This magnified tension controls an electron ray tube, by which finally the phenomenon is made visible.

The apparatus satisfies all conditions mentioned before and has already been used with great success in different cases. Magnifications of 200,000 can be reached.

II. PUBLICATIONS OF THE NEDERLANDSCHE CENTRALE ORGANISATIE VOOR TOEGEPAST NATUURWETENSCHAPPELIJK ONDERZOEK (T.N.O.), THE HAGUE—DUTCH CENTRAL INSTITUTE FOR TECHNICAL SCIENTIFIC RESEARCH; VIBRATIONS AND PHOTOELASTICITY

1. *Experimental Determination of the Time Integral of an Impact with the Aid of a Vibrating System.* The theory of measuring the time integral of a mathematical impact S (defined by $S = \lim_{T_s \rightarrow 0} \int_0^{T_s} P dt$, where P tends to infinity in such a way that the integral tends to a finite value if $T_s \rightarrow 0$) with the aid of a simple vibrating system, consisting of an elastically suspended mass, is well known. The theory is equally known in case the vibrating system is subject to viscous damping. This treatise deals with the same problem, but now related to a real physical impact. In the time integral $S = \int_0^{T_s} P dt$, the duration of the impact will now be small though not negligible compared with the period T_s of the free vibration of the undamped system. With respect to P it is supposed that during the impact time it does not change its sign, but no other conditions are to be fulfilled. The vibrating system may have a small viscous damping. The question to be solved is: what error can be made when the formula that holds for the mathematical impact is used for the physical one? The answer given is the following:

When a mathematical impact S acting on an undamped system gives a characteristic indication $x_m = S/\omega M$ (x_m designating the first amplitude

of the undamped vibrating system, ω the natural frequency of the undamped system, and M the vibrating mass), then a physical impact acting on the damped vibrating system gives a characteristic indication

$$x_m = \frac{S}{\omega M} (1 + \alpha - \beta). \quad (1)$$

In this formula α depends on the damping only, whereas β depends on the ratio $T_s:T_e$ and on the functional character of P during the period of impact. If in the (P,t) -graph, the unit of time is represented by T_e , and the unit of force is chosen such that $\int_0^{T_s} P dt = 1$, if furthermore I represents the moment of inertia of the surface covered by the (P,t) -diagram with respect to the line $t = c$, passing through its center of gravity, then $\beta = 2\pi^2 I$. It is readily seen that no greater β can occur, whatever the time dependency of P may be, than

$$\beta_{\max} = \frac{\pi^2}{2} \left(\frac{T_s}{T_e} \right)^2 \quad (2)$$

Therefore the correction β provided by formula (1) is in most cases negligible.

2. The Infinite, Elastically Supported Beam, Subject to an Impact Load.

It is well known that railway wheels locally flattened by abrasion occasionally give rise to such strong impacts that rail rupture occurs. Only a detail of the complex of problems related to this phenomenon is treated here. The elastic support of the beam is considered as obeying Zimmermann's hypothesis (after which the local deflection $\bar{Y}$ is proportional to the local pressure $\bar{p}$, $\bar{Y} = \bar{p}:\bar{\alpha}$). The damping resistance acting on the vibrating beam is supposed to be proportional to the local velocity of the beam (damping coefficient = $\bar{\beta}$). Up to the moment $t = 0$ the beam is unloaded and at that time a concentrated load $\bar{K}(t)$ comes into action at the fixed point $\bar{x} = 0$. The deflection and the bending moment at the point of application are represented by definite integrals

$$Y(0,t) = \frac{2^{-\frac{1}{2}} \sqrt{\pi}}{\Gamma(\frac{3}{4})} \int_0^t K(t-\tau) e^{-\beta \tau} J_{\frac{1}{4}}(\tau) d\tau \quad (1)$$

$$M(0,t) = - \frac{2^{-\frac{1}{2}} \sqrt{\pi}}{\Gamma(\frac{1}{4})} \int_0^t K(t-\tau) e^{-\beta \tau} J_{-\frac{1}{4}}(\tau) d\tau. \quad (2)$$

The quantities occurring in these integrals are defined by EI = coefficient of rigidity of the beam, F = surface of its cross section; γ = specific weight:

$$\alpha^* = \frac{\bar{\alpha}}{EI} \text{ cm}^{-4} \quad a = \sqrt{\frac{gE}{\gamma}} \text{ cm sec}^{-1} \quad 2\beta^* = \bar{\beta} \frac{a}{EI} \sqrt{\frac{I}{F}} \text{ cm}^{-2}$$

$$\epsilon = \sqrt[4]{\alpha^* - \beta^{*2}} \text{ cm}^{-1} \quad (\beta^{*2} < \alpha^*)$$

$\alpha = \frac{\alpha^*}{\epsilon^4}, \beta = \frac{\beta^*}{\epsilon^2} (\alpha - \beta^2 = 1), Y = \epsilon \bar{Y}, x = \epsilon \bar{x}, M = \frac{\bar{M}}{EI\epsilon}$ (M = moment in kg cm), $K = \frac{\bar{K}}{EI\epsilon^2}$ ($\bar{K}$ = loading force in kg), $t = \bar{t}\epsilon^2 \sqrt{\frac{I}{F}}$ (t = time in sec).

Equations (1) and (2) are derived with the aid of Laplace transformations.

Graphs of the functions $e^{-\beta\tau} J_{\frac{1}{2}}(\tau)$ and $e^{-\beta\tau} J_{-\frac{1}{2}}(\tau)$ occurring in these equations are put at the disposal of the reader.

Several specializations of the general case represented by (1) and (2) are treated, viz.:

1. P = constant (up from $t = 0$ to $t = \infty$).
2. P = constant during a short duration t_s of impact (the maxima both of Y and M , in their dependency on β and t_s are studied and graphically represented).

3. Two important theorems, valid for all sorts of impacts are deduced:

(a) The maximum bending moment occurs within or at the end of the interval of impact.

(b) Impacts of the same intensity and of the same character (in as much the one impact can be deduced from the other one by a proportional transformation) yield maximum bending moments inversely proportional to the square roots of the impact times.

If the impact problem is put in another way, such that the force P is the unknown quantity (as may occur if a falling mass comes into contact with the beam) then again the solution is procured by integral equations of the type (1) and (2). They can be solved by using a step by step method indicated by Timoshenko.

3. *The Smallest Characteristic Number of Certain "Eigenwert" Problems.* Many problems in vibration theory and stability lead to the determination of the roots λ of the secular equation

$$\begin{vmatrix} \alpha_{11}m_1 - \frac{1}{\lambda} & \alpha_{12}\sqrt{m_1m_2} & \cdots & \alpha_{1n}\sqrt{m_1m_n} \\ \alpha_{21}\sqrt{m_2m_1} & \alpha_{22}m_2 - \frac{1}{\lambda} & \cdots & \alpha_{2n}\sqrt{m_2m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}\sqrt{m_nm_1} & \cdots & \cdots & \alpha_{nn}m_{nn} - \frac{1}{\lambda} \end{vmatrix} = 0 \quad (1)$$

in which the numbers α_{ij} represent so-called Maxwell numbers, characterized by $\alpha_{ij} = \alpha_{ji}$, $\alpha_{ii} > 0$, and in which $m_i > 0$.

In technical problems the smallest root λ_1 is in general the most important one. Following a trend of thought developed by Graeffe in solving algebraical equations and restricting himself to such problems for which all roots λ_i are different, the author derives a monotonously increasing infinite series of values, the limiting value of which is λ_1 .

In defining

$$\alpha_{ij}^{(p)} = \sum_{k=1}^n \alpha_{ik}^{(p-1)} \alpha_{kj}^{(p-1)} \quad \text{with} \quad \alpha_{ij}^{(0)} = \alpha_{ij} \sqrt{m_i m_j} \quad (2)$$

he proves that

$$\lambda_1 > \frac{1}{\sqrt[2p]{\sum_{i=1}^n \alpha_{ii}^{(p)}}} \quad (p = 1, 2, \dots) \quad \text{and that}$$

$$\lambda_1 = \lim_{p \rightarrow \infty} \frac{1}{\sqrt[2p]{\sum_{i=1}^n \alpha_{ii}^{(p)}}} \quad (3)$$

$p = 0$ and $p = 1$ lead to well-known approximative formulas the first of which is ascribed to Dunkerley (cf., for instance, Biezeno and Grammel, *Technische Dynamik* Chapter III, 13, 25 and Chapter III, 13, 31).

4. *A Comparative Study About the Main Photoelastic Properties of Different Materials* (Rept. IV of the Dept. of Photoelasticity). Though the laboratory for applied mechanics of the Technische Hoogeschool at Delft possessed a photoelastic plant, no use was made of it for industrial purposes. During the war the Institute, mentioned in the heading of this Chapter, founded a separate department for photoelastic research, with the main aim to put its results at the disposal of the industry and other interested groups. The lack of material for models made it impossible up to now to do much work in the field of stress determination. On the other hand relatively much time could be spent on the investigation of the various properties and qualities of different materials in relation with their suitability for optical stress determination. The results of this investigation are gathered in this paper, in which attention is given to polystyrene, celluloid, phenolite. Their mechanical and optical properties are investigated at great length and for one and the same object, the stress distribution of which was known theoretically, the experimental results obtained with every one of the three materials are compared. Much

attention is given to the machining of the materials. All those who have to make their choice with respect to reliability, optical sensitiveness, durations of heat treatment, and so on will find valuable information in this report.

5. *On the Stress Concentration in the Corners of a Cross, the Rectangular Branches of Which Have Equal Width (Rept. V of the Dept. of Photoelasticity)*. The stress concentration mentioned in the title is studied in regard to its dependence on the radius of the fillets of the cross when the latter is subjected to tension in the two perpendicular directions. Naturally the experiments are carried out only under one dimensional tension P . If tensions P and Q are simultaneously applied in two perpendicular directions the required results can be derived from the data obtained by the first mentioned experiment by making use of the law of superposition. The material used was xylonite, because of lack of polystyrene. A very effective method was introduced to obtain the required results in a short time. With the aid of a test strip a tensile stress σ_0 was determined, connected with a sharply perceptible change of color of the strip. This change of color was chosen such that the corresponding tensile stress was not too high (in this case $\sigma_0 = 78 \text{ kg/cm}^2$). The model was charged in such a way that successively in all points in which the stress was to be measured, the change of color occurred. With every such experiment the load was registered and the mean value σ was calculated in such section of the cross as might be expected to show a uniform distribution of stress. The stress concentration factor occurring in the point under consideration evidently was given by the quotient $\sigma_0:\sigma$. If the width of the straight parts of the cross be denoted by $2B$, the radius of the fillets by R , the ratio of the loads P and Q in the two perpendicular cross directions by μ , a graph can be designed in which for distinct values of μ , the maximum stress concentration is represented as function of R/B . As a matter of fact, this graph is put at the disposal of the reader. The stress-concentration factor appears in usual practice to vary between 1.35 and 2.10.

6. *Photoelastic Determination of the Stresses Occurring in a Tunnel Projected between the Banks of the "Noordzee Kanaal" Connecting Amsterdam with the North Sea (Rept. IX)*. No special remarks are to be made. The report is mentioned just to give an idea of the work that is done by the department in question.

7. *Stress Concentrations in Corner Joints (Rept. VIII)*. The data laid down in this report are the result of laborious and lengthy experiments performed on a large series of glass models, and bearing upon the stress concentration occurring in corner joints such as those appearing in innumerable technical constructions. The characteristic dimensions

of such a rectangular corner joint are the width h of the two straight sides, the direction and the length of the inner bevel, and the radii of the fillets constituting the transition from bevel to the normal border. One side of the construction was clamped, whereas the other side was loaded in succession by a bending moment, a normal force, and a shearing force. In each of these cases the stress distribution along the inner boundary of the corner joint was measured and critical examination of the results enabled the author to draw up elementary and simple directions by which henceforth all stresses at the inner boundary of a corner joint can be calculated with great accuracy, whatever the load at the end sections of the joint may be.

III. PUBLICATIONS OF THE NATIONAAL LUCHTVAARTLABORATORIUM AT AMSTERDAM. (AERONAUTICAL RESEARCH LABORATORY)

1. *Stresses of the Second Order in the Skin and Ribs of Wings in Bending.* Wing structures in which the skin in the unloaded condition is not curved in the direction of the tensile stresses are considered. Bending of the wing causes a curvature of the skin in the direction of the tensile stresses; thus the wing ribs are loaded in consequence of the fact, that tensile stresses are changing their inclination; a second-order loading arises. If the structure of the wing ribs is such that their flanges are allowed to deflect in the plane of the rib (truss type and particularly open ribs), the deflection of the flanges by the second-order loads will affect the magnitude of the stresses in the skin; the extensions of the skin are allowed to decrease as the skin is situated now on a smaller radius of curvature. Flexibility of the rib flanges thus changes the distribution of tensile stresses over the width of the skin. Formulas are derived and graphs are given for the stresses in the ribs and in the skin. Usually the decrease of skin stresses is small; it becomes of more importance if the ratio of the height of the rib flange to the unsupported flange length is small, the height of the wing section is small, and the skin (longitudinal stiffeners included) is heavy. Even if the decrease of skin stresses is small, the "second-order" stresses in the ribs may be many times larger than the stresses, caused directly by local airload.

2. *Stress and Strain in Shell Wings with two Spars.* a. *The effect of elastic rib shear on stresses in two-spar wings with stressed skin.*

b. *The stress distribution in wings with two nonparallel spars interconnected by elastically deformable ribs and skin.*

c. *The stress distribution in wings with two nonparallel torsionally rigid spars, interconnected by elastically deformable ribs and skin.*

d. *The distortions of wings with two spars.*

e. *Manual for the stress analysis of shell wings with two spars.*

Preceding Dutch papers on the subject of stress and strain in shell wings with two main spars considered the spars to be deformable in bending and shear; the skin was allowed to shear under shear stresses, whereas the ribs were completely rigid against bending and shear in their plane. Spars and ribs had no stiffness against torque. Furthermore the rib system was a continuous system, thus the behavior of the structure could be described by differential equations holding for the whole of the wing span. Solving these equations by numerical integration the spar deflections were obtained and from these the stresses could be calculated.

Later research made it clear that in certain structures the deformation of the ribs in shear should not be neglected. In order to show the effect of rib shear, paper *a* communicates the stress distribution in 8 wings under continuously distributed and locally applied torque loads, these 8 wings differing only in the shear rigidity of their ribs. The assumption of completely rigid ribs proves to give a good approximation when the shear rigidity of the most stressed ribs surpasses certain well-defined limits.

Introduction of the deformability of the ribs showed that discontinuities in torsional rigidity of the wing shell or in torsional moment necessitates the assumption of "local" ribs apart from the system of continuous ribs. This means that the differential equations, complicated already by the fact that their order increases by two, determine the stress distribution only between consecutive local ribs and that at each of these ribs special boundary conditions have to be satisfied. This mixed system of local and continuous ribs is hardly suitable to practical analysis. With nonparallel spars the difficulties increase still more.

In order to eliminate these difficulties the assumption of continuous ribs is abandoned in paper *b*; the rib system consists of local ribs, 5 to 10 per half wing located at those points along the span, where the ribs of the actual structure will carry high shear loads. In general each rib replaces a group of ribs and its rigidity equals the total rigidity of the group.

The wing scheme is such that it may represent wings with built-up spars, in which the shear load is carried not only by the spar web but for some part too by an auxiliary spar or by the wing nose. In order to account for this the scheme allows the torsional centers of the spars to lie outside the plane through the spar caps. The torsional rigidity of these built-up spars may be accounted for by increasing the shear rigidity of the skin between the spars. This approximative consideration of the torsional rigidity of the spars has been adopted in paper *b*. Structures with built-up spars the torsional rigidity of which is not small compared with the rigidity of the tube between the spars have to be considered more

accurately; paper *c* deals with this problem in assuming torsionally rigid tubes between the ribs in front of the front spar and to the rear of the rear spar.

A more complete description of the scheme of paper *b* is the following.

There are two spars, connected by n parallel ribs per half wing. Between the ribs the spar flanges as well as the lines through the torsional centers of the spars are straight. At the ribs the flanges may change their directions and the torsional centers may shift. There is a skin between the flanges of both spars; it is connected with these flanges and with the ribs. Besides, the skin is supported by a continuous system of stiffeners running parallel to the ribs; these stiffeners are required for kinematical reasons if the skin is not exactly cylindrical.

Finite stiffness is assumed for the spars in bending and shear, for ribs and skin in shear only. The stiffeners are absolutely rigid in their own plane. With respect to moments the vectors of which lie in the planes of the spars, the ribs, the stiffeners and the skin, these structural components have no rigidity. Besides the skin has no rigidity against extension in the direction normal to the ribs.

This structure would be statically determinate if n is unity. The addition of $2(n - 1)$ ribs makes the structure redundant to the $2(n - 1)$ th degree. By splitting the loads up in a symmetrical and an antisymmetrical load system, the structure is with respect to these particular load systems redundant to the $(n - 1)$ th degree. The statically indeterminate quantities X_i are chosen such that they are something like the bending moment in one of the spars at the junction i of the ribs and the spar. By the conditions of equilibrium all stresses can be expressed in these quantities.

Application of Castigliano's theorem of least work yields the conditions of elasticity. They are linear equations in the X_i , containing five of them at five consecutive ribs. All those equations together form a linear recurrent system. They are solved by linear combination of one solution of the complete equations, which has arbitrary starting values X_1 and X_2 , and two solutions of the homogeneous equations with different but arbitrary initial values X_1 and X_2 ; the linear combination is such that the end conditions at the end rib are satisfied. (Compare Section C1 for this trend of thought.)

With completely rigid ribs the "equations of 5 moments" simplify to "equations of 3 moments."

The structure with torsionally stiff spars (paper *c*) contains in each bay of the wing besides the statically indeterminate quantity X , two other statically indeterminate quantities Y_f and Y_r , being the torsional moments in the spar tubes. Each condition of elasticity now contains besides five successive X_i 's four successive unknown quantities Y_{fi} and five Y_{ri} 's.

There are now two other systems of recurrent equations, containing four successive X_i 's and three successive Y_{ji} 's and Y_{ri} 's.

These recurrent systems of equations are solved by combining a solution of the complete equations with arbitrary initial values X_1 , X_2 , Y_{f1} , Y_{r1} and four mutually independent solutions of the homogeneous equations.

When the ribs can be considered to be completely stiff in their own plane the quantities Y can be eliminated; then the system of recurrent equations is formally identical with the system of equations for wings with spars without torsional stiffness.

For particular purposes in addition to the stress distribution the knowledge of the deflections may be needed. Calculating the distortion from the stresses by means of geometrical considerations is very cumbersome with this highly complex structure. Paper *d* gives the solution of this problem by means of a modification of Castigliano's theorem: the principle of virtual forces. According to Castigliano's theorem among all stress distributions satisfying the conditions of equilibrium the actual stress distribution makes $F = A - A_r$ a minimum, A being the elastic energy expressed in the stresses, A_r being the work done by the unknown surface stresses at those parts of the boundary where the deflections are prescribed. According to the rules of the calculus of variations the supplementary conditions of equilibrium $E = 0$ may be accounted for by adding their left-hand sides each multiplied by a factor μ to the function that is to be made a minimum. It can be shown that the multipliers have the meaning of displacements. The method is given a clear mechanical interpretation.

Paper *e* gives instructions with regard to the practical application of the wing analysis given in paper *b*; much attention is given to the problem of how to replace a given wing by a mechanically equivalent wing, which fits into the adopted scheme; formulas are given for the calculation of the required stiffness factors.

3. *Effective Width.* *a. The effective width of flat axially loaded strips of sheet beyond the buckling stress for various edge conditions.* The theory of the effective width given by Marguerre and Trefftz is restricted to loads slightly greater than the buckling load of the sheet. Far beyond the buckling load the wave form assumed by these authors deviates too much from the actual wave form; so the effective width calculated from their formula is too large. An improvement was made by Plantema and Floor (*N.L.L. Report S 262*) who introduced one parameter more into the expression for the assumed wave form; however they too found a finite effective width for $\epsilon_k/\epsilon = 0$ (ϵ being the direct strain of the longitudinal edges of the sheet, ϵ_k = the direct strain causing buckling of the sheet), which is not to be expected. Cox introduced a new type of wave form

and made at the same time some supplementary suppositions concerning the stresses by which however the basis of his investigation becomes mechanically unsound.

In this investigation the main characteristic of the assumed wave form, introduced by Cox, has been adopted; the further method of dealing with the problem however works along the lines given by Marguerre and Trefftz.

For a given strain at the edges the displacements are such that the elastic energy is a minimum. An approximative solution is obtained by assuming displacements that, having at least the character of the wave form to be expected, can be modulated by varying the value of one or more parameters. The parameters introduced by Marguerre and Trefftz are the length and the depth of the waves. Whereas Cox assumes the wave to be of constant depth over some width in the center of the sheet (the edges being curved in two directions), the author of this paper introduces this width as a third parameter.

Various wave forms have been assumed for supported, rigidly clamped and elastically restrained edges each. The effective width proving to be not very much affected by the particular wave form, it seems plausible that the approximation is fairly good. A consequence of the approximation is that the elastic energy as well as the effective width "in the mean" is too large. However, when the approximate effective width is compared with experimental results the experimental values prove to be larger than the theoretical values. The explanation for this rather surprising result is that in an actual sheet, having finite length, the wave length is not free to change gradually in making the energy a minimum as is the case with a sheet of infinite length. This means that the theory is at the conservative side as long as the proportionality limit is not surpassed.

Another remarkable feature of the results is that the effective width of a supported, rigidly and elastically clamped sheet is practically equal for equal ϵ_k/ϵ .

If ϵ_k/ϵ approaches zero the effective width is proportional to $(\epsilon_k/\epsilon)^{1/2}$. This result was obtained by the transformation of Friedrichs, known from the calculus of variations. In theory this method can give a lower limit of the effective width over the *whole* range $0 < \epsilon_k/\epsilon < 1$. In view of the elaborate calculations required these calculations have not been carried out.

b. The effective width of flat sheet supported by longitudinal stiffeners with open section. The longitudinal stiffeners in aircraft-stressed skin construction present some elastical restraint to the edges of the sheet. The amount of this restraint by stiffeners having open section does not

depend merely upon the torsional stiffness of the stiffener. In the first place the torsional moment varies along the stiffener; the variation of warping of the cross section which would occur with a free stiffener is prevented and for this reason its stiffness is increased. In the second place the compression of the stiffener decreases its torsional stability and therewith its torsional stiffness. In the third place the way in which the torsional moment is applied to the stiffener by the sheet causes distortions of the cross section of the stiffener in the plane normal to the axis of the stiffener. This latter fact decreases the amount of restraint.

The first and second circumstances are dealt with in literature. This paper deals with the third effect, which proves to be of major importance. The calculation has been carried out for stiffeners having *L*- and *Z*-shaped sections and the results have been presented in graphs for various ratios of the dimensions of the cross sections.

4. *The Tension Field for Stresses Beyond the Buckling Stress.* The theory of the incomplete tension field in a sheet strip with supported edges, given by Marguerre and Kromm, is reliable for small ratios τ/τ_k (τ_k being the shear stress causing buckling). However the results of this theory for τ_k/τ approaching zero do not coincide with the results given by Wagner's theory for the complete tension field; the additional tensile stresses in S_1 and S_2 the longitudinal and lateral directions being less and the effective rigidity (G') being larger than predicted by Wagner's theory.

Along the same lines as indicated in section 2a this paper gives an approximative solution of the problem. The parameters governing the assumed wave form are, firstly the same as those introduced by Marguerre and Kromm: the wave length and wave depth, the angle between the edges of the sheet, and the nodal line of the waves; and secondly the width in the center of the sheet, over which the depth of the wave is constant. In this way the assumed wave form allows for the tendency of the waves to flatten out from the center towards the edges with increasing stress ratio τ/τ_k .

Sheets with clamped edges as well as sheets with supported edges have been investigated for various wave forms in the double curved-sheet region near the edges.

With equal τ/τ_k the stress ratios S_1/τ and S_2/τ and the effective rigidity are different for supported and clamped edges; the clamped sheet approaches the complete tension field more than does the supported sheet. This contrast to the behavior of the effective width may be explained by saying that the central portions of the tension field, where the influence of the edge conditions has faded out, is very active indeed in supporting the load.

For τ_k/τ approaching zero the deviations from Wagner's results for the complete tension field are presumably proportional with $(\tau_k/\tau)^{\frac{1}{2}}$.

The formula has been evaluated for the supported sheet with various stiffness of the longitudinal and lateral stiffeners under shear load and the results have been laid down in graphs, giving the mean stress ratios S_1/τ and S_2/τ and the effective rigidity G' .

These graphs may be applied too for sheet-stiffener combinations under shear and normal load in longitudinal and lateral direction (and not too large ratios between the normal loads and the shear load).

The solution being an approximative one the energy for given edge displacements will be too high and the actual stresses and rigidity will approach Wagner's results for the complete tension field more closely than do the calculated stresses and rigidity. Again experimental evidence seems to deny this conclusion; the explanation once more will be that the wave length in a sheet of finite length is not free to change gradually as the load increases. Explosive changes of wave number occur, but it may be that this change is postponed by the stability of the preceding wave form. In conclusion it may be stated that the theory for infinite long strips will give a conservative estimate of the actual conditions.

5. *The Effect of the Flexibility of Fuselage Frames on the Stress Distribution in Fuselage Shells.* The stress distribution is influenced to a large extent by the deformations of the frames, especially with large airplanes the frames of which tend to be relatively light.

The stress analysis of fuselages with four longerons can be based on the method developed for wings with two spars and ribs that are flexible in shear (see Section 2). With fuselages having more longerons the application of the analysis developed by Ebner requires extensive simplification of the structure and even then the number of longerons should not exceed twelve. Fuselages with a great many longitudinal stringers will probably be schematized better by assuming longitudinals and frames, the stiffness of which is distributed continuously over the circumference and along the axis respectively. The relations of stress and strain for these orthotropic tubes with circular cross section have been determined. The external loads at the end sections of the homogeneous tube have been expanded in Fourier series; the stresses and the deflections caused by each component of the load have been expressed in these components by explicit formulas. These formulas may serve as the starting point for calculations on the stress distribution of shells, which have a less homogeneous structure along their longitudinal axis and which are loaded along their entire length.

6. *Qualitative Pictures of Stresses in Wings and Fuselages.* The elementary beam theory applied to wings and fuselages gives stresses com-

plying with the equilibrium conditions for the cross section as a whole. However owing to discontinuities in the structure or by local application of external loads the conditions of equilibrium of structural elements and the conditions of compatibility of deformations will not be satisfied by "elementary" stresses. These circumstances give rise to "secondary" stresses, which occasionally may be of primary importance. In many cases the knowledge of the magnitude of these stresses is not needed or may be acquired in a later stage of designing. Then it may be useful to know in which points of the structure the stresses are increased by such effects. A number of cases occurring in wing and fuselage shell structures have been considered qualitatively and the results have been laid down in diagrams of the stress distributions.

7. *General Instability of Longitudinally and Laterally Stiffened Cylindrical Shells under Axial Compression.* Fuselage shells stiffened by stringers and frames may fail at the compression side by "general instability," a type of buckling in which the frames as well as the stringers are distorted. This type of failure is more critical with weaker frames.

In order to simplify the problem the axially loaded cylinder of infinite length has been considered.

If the wave length of the buckled cylinder comprises several stringer spacings tangentially and several frame spacings axially, the action of the stiffeners upon the frames and the action of the frames upon the stiffeners will be practically the same when the system of stringers and frames is replaced by a continuous system of axial and circumferential stiffeners. Thus the structure is transformed into an orthotropic shell. The first part of the investigation deals with this.

Usually the stringers are closely spaced indeed; the frame spacing however can be of the same order of magnitude as the half-wave length. Then it is not to be expected that the orthotropic shell will yield a reliable result. Therefore the second part of the investigation deals with a cylinder stiffened by uniformly distributed longitudinals and by regularly spaced frames.

In both systems the stiffeners are assumed to possess bending and torsional stiffness. Their neutral axis has some eccentricity with respect to the plane of the skin. Stiffeners and frames are interconnected such that the material situated in the line normal to the neutral lines of stiffener and frame remains upon this normal in the distorted position of the neutral lines. The skin is supposed to carry shear stresses only; in so far as normal stresses are taken up by the skin a corresponding effective width of skin is included in the cross section of the stiffening members.

In regard to the orthotropic cylinder the conditions of equilibrium

for an arbitrary deformation adjacent to the unbuckled state have been formulated. These equations have been solved for the cylinder of infinite length. The axial stress at which buckling is possible depends upon the number of waves over the circumference (n) and the wave length in axial direction (l). The complexity of the structure necessitates the introduction of a great number of parameters; they have been defined such that they all represent nondimensional figures of the order of unity; in this way the process of eliminating high-order terms was made clearer and the buckling form, yielding the smallest critical stress, could be determined quite easily.

There are three main groups of buckling forms: (1) short waves axially and long waves circumferentially; (2) long waves axially and intermediate wave length circumferentially; (3) short waves axially and circumferentially. The critical loads for group 1 and 2 could be given in explicit formulas; the minimum critical load in the third group is to be determined implicitly out of a function of n and l . Usually the smallest buckling load falls into the third group.

It may be emphasized that the eccentricity of the stiffeners with respect to the plane of the skin is a factor of major importance for the magnitude of the critical stress in the third group. Internal stiffening lowers the critical stress, sometimes to as much as one third of the value corresponding to stiffening in the plane of the skin. Outside stiffening would increase the critical stress.

The shell with regularly spaced frames represents between two successive frames a particular case of an orthotropic cylinder: the circumferential stiffeners are missing. The conditions of equilibrium of this orthotropic shell are known from the preceding investigation; their integration yields the displacements containing eight integration constants. With each bay of the cylinder eight integration constants are introduced.

The consistency of displacements at each side of a frame and of the deflections of the frame under the loads imposed on it by the skin and stiffeners gives eight equations between the integration constants of successive bays. All these sets of equations constitute a recurrent linear system of equations in the integration constants. The general solution of these equations is given; to any combination of axial load and wave number over the circumference correspond eight longitudinal modes of buckling. For a cylinder composed of m bays the stability determinant of the $8m$ th degree is reduced to a determinant of the 8th degree. Though there are no further essential difficulties the practical evaluation of the solution is highly cumbersome. Usually the cylinder will be so long that the influence of the end conditions will be negligible. Then the

buckling load of the cylinder of infinite length gives a good approximation of the actual buckling load. Therefore the solution is completed for this case. Formulas and graphs are given for the calculation of the buckling load.

It may be stated that the error caused by the uniform distribution of the frames is no more than 2 per cent if the longitudinal half-wave length exceeds 2 frame spacings; the error increases up to 10 per cent if the ratio of half wave length and frame spacing approaches 1.5. For smaller wave lengths the theory of the orthotropic cylinder does not apply and must be replaced by the theory for cylinders with regularly spaced frames.

Column failure of the longitudinals between the frames may occur, when the tensional stiffness of the frames exceeds a particular limit.

8. *The Stability of Sandwich Strips and Plates.* The investigation deals with plates composed of two thin plates and an intermediate thick layer of a much softer material than that of the plates. The soft layer has to have a stabilizing effect on the thin outer plates if the latter are loaded by compressive stresses.

A theoretical investigation of the stability of such compound strips under axial load has been carried out by Gough, Elam, and de Bruyne under the assumptions that the axial load is completely taken by the outer plates and that the extension and compression of the outer plates are negligible compared with those of the inner layer. This latter assumption proves to give faulty results if the wave length of the buckled strip is long compared with the thickness of the plate; for instance, on the basis of the second assumption the ordinary Euler load never will occur.

Therefore the above-mentioned investigation was repeated after dropping the second assumption. The soft layer in this investigation was considered to be a homogeneous isotropic elastic material loaded two-dimensionally; no simplifying assumptions have been incorporated.

The problem of the instability of plates and shells of the sandwich type of construction is more involving, as the soft layer in the buckled condition is loaded three-dimensionally. In order to simplify the mechanism the behavior of this soft layer was schematized. The soft layer was replaced by rigid elements of negligible thickness normal to the outer surface and connected to the outer plates; these members are interconnected by a two-dimensionally isotropic elastic material. In this way all displacements and stresses in the soft layer can be expressed in the displacements of the outer plates.

This scheme proved to give an extremely good approximation of the above-mentioned results of the exact theory for compound strips. So this scheme could be adopted in attacking the problem of the buckling

of plates. The general stability equation of compound plates has been deduced and its solution has been given for the case of the rectangular plate loaded by normal stresses parallel to the edges.

IV. BOOKS

1. *Technische Dynamik* by C. B. Biezeno and R. Grammel. This book on general elasticity appeared just before the outbreak of the war. As far as the author of the present article knows, no American review of the book has appeared. The book has, however, been photo offset in the United States (in a limited number of copies) "in the public interest" on behalf of the Alien Property Custodian. The reviewer is proud indeed that in this way he and his collaborator were given the opportunity to contribute something—as modest as it may be—to the general cause of liberty and freedom that has been so pre-eminently furthered by the American nation.

As to the contents of the book it seems best to quote the English review, written by L. M. Milne-Thomson (*Nature*, Feb. 1941), the more so because this authority reproduces the views and aims of the authors, laid down in the preface of the book, in a perfect way.

"In introducing the student to technical mechanics either by lectures or textbook it is usual to lay stress on those simplified problems which illustrate the principles. The more difficult problems in the form in which they face the mechanical engineer, even if touched upon, do not receive the detailed consideration which their importance not only warrants but demands. The present book is designed to form a continuation of the introductory course and may be described as a treatise on technical applications of elastomechanics for advanced students.

"According to Kirchhoff's definition, mechanics consists of kinematics, or the study of motion and of dynamics, or the study of forces. In this sense dynamics can be divided into statics, the geometry of forces and the theory of equilibrium, and into kinetics, the theory of forces not in equilibrium and the relations between force and motion. For example, an oscillation problem is a dynamical one consisting of a statical part, namely, the investigation of certain elastic properties, and a kinetic part, the calculation of the frequencies. The looser current usage takes dynamics as a synonym of kinetics, and while still describing the oscillation problem as dynamical, neglects the statical part, which is often the more interesting.

"The authors use dynamics in the sense defined by Kirchhoff, and have accordingly investigated both the statical and kinetic parts of a variety of problems concerning rods, shafts, rings, springs, plates, shells and their combinations, keeping in view that any such problem is only

worth solving when it has a practical application, and can only be described as solved when a detailed solution in numbers is forthcoming without recourse to too prohibitive numerical calculations. It must not be inferred that the book is a wearisome list of numerical results. On the contrary, great emphasis has been laid on methods of solution: in fact a chapter of 100 pages is devoted entirely to such methods. In the applications, solutions are not merely outlined but are given in full as the engineer would have to work them, and with the choice of method which the authors consider to be the best.

"The text is divided into four sections: fundamental theory, 225 pages; single parts of mechanisms, 400 pages; the steam turbine, 220 pages; internal combustion engines, 210 pages.

"The first section contains an exposition of the mathematical theory of elasticity as applied to isotropic bodies which obey Hooke's law; indeed phenomena outside the elastic limit are not considered in the sequel. This account of the theory is well and carefully written. It is refreshing to see the authors emphasize the strict limitations of the theorem of unique solution of elastic problems, in that in actuality finite (but small) strains and finite (but not small) stresses represent only approximations to the conditions in which the classical equations are valid. A simple illustration is Euler's strut at the first critical load. Another welcome feature is the inclusion of the proof of de Saint Venant's principal based on that given by Zanaboni (*Atti Lincei*, **25**; 1937). With regard to the general theorems of elasticity it is to be regretted that the authors have not given the vector formulation which makes the physical implications much easier to grasp.

"The remaining three sections consist of applications of elastomechanical theory with special emphasis on the needs of machine construction. Special prominence is given throughout to critical loads and values.

"The authors remark in the preface that the chapters have been arranged so far as possible so that each can be read independently. This has been achieved within limitations, but the text requires close attention on the part of the reader, who is assumed to be proficient in elementary mechanics and to have a fairly extensive knowledge of mathematics.

"The book is a mine of detailed information such as could probably not be found in collected form elsewhere (for example there are 89 pages on the loaded rod) and it should therefore prove to be of great value."

2. *Plasticiteitsleer* by F. K. Th. van Iterson. Starting from the well-known idealized (σ , ϵ) diagram for metals in which σ remains constant after having reached its yield value, and neglecting as usual the elastic

deformations with respect to the much larger plastic ones, the author restricts himself in the first half of his book to the two-dimensional problem. A survey is given of the work done by Prandtl, von Mises, Hencky, Nadai and others and several problems scattered over manifold periodicals are brought together. The other part of the book is devoted to the three-dimensional problem under assumption of the Maxwell-Huber-Hencky hypothesis. In this part the author develops personal ideas on the subject, which are put to the test of his own experiments and the opinions of others. A series of articles on the same subject by the same writer appeared in the *Proceedings of the Royal Academy of Sciences* at Amsterdam: Prof. ir. F. K. Th. van Iterson, Bijdrage tot de Plasticiteitstheorie, *Verslagen Nederland. Akad. Wetensch. Afdel. Natuurkunde*, Vol. LII, No. 1 (1943); Les déformations plastiques près des entailles, Vol. XLV, No. 2 (1942); La Pression du toit sur le charbon près du front, dans les exploitations par tailles chassantes, Vol. XLII, No. 2 (1939); Vol. XLIII, Nos. 2, 3, 4 (1940), Vol. XLIV, Nos. 2, 3 (1941).

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