ARTIFICIAL BOUNDARY CONDITIONS FOR NONLINEAR DISCRETE PROBLEMS IN GAS DYNAMICS

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Abstract. The previous theory of artificial boundary conditions in gas dynamics has been elaborated mostly for the Euler equations. However, in practice, such boundary treatments are applied to discrete methods to solution of gasdynamic problems. In this study, based on exact nonreflecting boundary conditions for the 1D linearized Euler equations, we proceed to 2D external viscous subsonic flows. The implementation of boundary conditions passes a number of stages. For linearized formulations—the 2D Euler equations and the 1D inviscid central-difference scheme—a Fourier analysis is applied to develop and examine boundary conditions. A theoretical investigation of saw-tooth spurious waves in nonlinear finite-difference schemes is also performed. Owing to this analysis, the appropriate discrete form for outflow boundary conditions has been found. Properties of boundary conditions are tested on direct simulation of 2D viscous flows.

1 INTRODUCTION

For numerical simulation of external gasflows, e.g., flows about rigid bodies, finite computational domains are used having artificial boundaries. This implies formulation of some artificial boundary conditions at these boundaries. In computations of subsonic and, especially, low Mach-number flows, inaccurate boundary treatments lead to considerable distortions of the flow pattern and sometimes to instability of the problem solution.

From the point of view of mathematical physics, artificial boundary conditions must provide the unique solution which coincides with the solution of the unbounded domain formulation, for a class of problems described by the given governing equations. In the field of continuum mechanics and electrodynamics, conditions on artificial (nonphysical) boundaries should not cause reflection of waves inside the domain. Therefore, such conditions are known as nonreflecting boundary conditions (NRBC).

The development of nonreflecting boundary conditions for various systems of equations has a nearly 30-year history. Most of the results are concerned with linear differential equations and are based on a normal mode analysis (the Fourier–Laplace transform) or some
other expansions. In such cases, NRBC’s are algebraic or differential equations, specified at each point of the boundary (local conditions) or, otherwise, nonlocal pseudodifferential equations. The latter are, by many authors, replaced with their local approximations.

In the course of simulation of gasdynamic flows, artificial boundary conditions are used along with discrete methods to solution of nonlinear problems. Such descriptions are still unreachable for rigorous analysis. In gas dynamics, the basic theoretically substantiated results for NRBC’s are the following. In the case of the linearized Euler equations, exact, for a single spatial dimension, and approximate, for two dimensions, artificial boundary conditions (e.g., see [1]). In the nonlinear case, the characteristic boundary conditions [2] for the 1D Euler equations. The latter remain the dominating approach to boundary treatments for nonlinear models of gas dynamics in general.

For many applied problems, the methods mentioned above meet difficulties which are connected with the following phenomena listed here by the decreasing order of importance,

- nonlinearity,
- discretization,
- multidimensionality, and
- viscosity (dissipation).

In nonlinear formulations, nonlocal boundary conditions cannot usually be expressed. However, a simple procedure can be used to transform certain classes of linear differential NRBC’s into a nonlinear form. Discrete (finite-difference and finite-element) schemes
typically deal with additional spurious waves. An extensive analysis and development of linear discrete NRBC’s was carried out in [3, 4]. The first consideration of properties of boundary conditions for nonlinear schemes was made in [5].

In this study, the procedure of constructing artificial boundary conditions for discrete nonlinear 2D applied problems passes a number of stages which are, in simplified form, depicted in Fig. 1. We start with the revision of NRBC’s for the 1D linearized Euler equations. Since these conditions are to be used for more general flow descriptions, we should deal with sets of boundary conditions excessive for the Euler equations. Then, for each gasdynamic model (including also the 2D Euler equations not indicated in Fig. 1), versions of 1D NRBC’s are selected and implemented without change or modified, based on physical reasoning or mathematical substantiation.

Let us describe this in detail.

2 BOUNDARY CONDITIONS FOR 1D LINEAR EULER EQUATIONS

Consider a two-dimensional flow of an inviscid gas depending on time \( t \) and a single spatial coordinate \( x \). Such flow is described by the following system of Euler equations

\[
\frac{\partial V}{\partial t} + C_x \frac{\partial V}{\partial x} = 0 ,
\]

where

\[
V = (\rho \ u \ v \ p)^T
\]

is the vector of primitive variables, \( \rho \) is density, \( u \) and \( v \) are velocity components, \( p = \rho c^2/\gamma \) is pressure, \( c \) is sound speed, and \( \gamma \) is the ratio of specific heats. The matrix \( C_x \) has the form

\[
C_x = \begin{pmatrix}
    u & \rho & 0 & 0 \\
    0 & u & 0 & 1/\rho \\
    0 & 0 & u & 0 \\
    0 & \rho c^2 & 0 & u
\end{pmatrix}.
\]

Equation (2.1), linearized with respect to perturbations of the primitive variables

\[
U = (\rho' \ u' \ v' \ p')^T,
\]

looks like

\[
\frac{\partial U}{\partial t} + C_x \frac{\partial U}{\partial x} = 0 .
\]

Here, the matrix \( C_x \) is expressed through unperturbed values of gasdynamic variables which are assumed to be constant.

In this study, we will examine reflection of waves by the left-hand (inflow) and the right-hand (outflow) domain boundaries in the case of a subsonic flow \((0 < u < c)\). For simplicity, both inflow and outflow boundary conditions are supposed to be set at \( x = 0 \),
because each boundary is to be dealt independently. Let the boundary condition be linear, have the general form
\[ \mathcal{L}U \bigg|_{x=0} = 0, \quad (2.4) \]
and consist of \( n \leq 4 \) scalar equations.

Consider solutions to the 1D Euler equations (2.3) in the normal mode form
\[ U = \hat{U} \exp\{i\omega t - ikx\}, \quad \omega > 0, \quad k \in \mathbb{C}. \quad (2.5) \]
For a fixed frequency \( \omega \), the general solution is a linear combination
\[ U = \sum_{j=1}^{4} a_j \hat{U}_j \exp\{i\omega t - ik_j x\} \quad (2.6) \]
with arbitrary coefficients (amplitudes) \( a_j \), where the four modes are characterized by the following wavenumbers \( k_j = k_j(\omega) \) and eigenvectors \( \hat{U}_j \):
\[ k_1 = \frac{\omega}{c+u}, \quad \hat{U}_1 = (\rho \ c \ 0 \ \rho c^2)^T, \]
\[ k_2 = -\frac{\omega}{c-u}, \quad \hat{U}_2 = (\rho \ -c \ 0 \ \rho c^2)^T, \]
\[ k_3 = \omega/u, \quad \hat{U}_3 = (\rho \ 0 \ 0 \ 0)^T, \]
\[ k_4 = k_3 = \omega/u, \quad \hat{U}_4 = (0 \ 0 \ c \ 0)^T. \quad (2.7) \]

Modes (2.7) describe, consequently, the rightward and leftward-propagating acoustic waves, the entropy wave, and the vorticity wave. Considering the expressions \( \omega = \omega(k) \) following (2.7), one determines the directions of wave propagation [1] by the signs of their group velocities
\[ c_j = \frac{d\omega}{dk} \bigg|_{k=k_j} = \frac{\omega}{k_j}, \quad j = 1, \ldots, 4. \]
So, in (2.7), the left acoustic mode \((k_2)\) propagates leftward and all the others do rightward.

Consider boundary condition (2.4). The Fourier–Laplace transform (image) of the linear operator \( \mathcal{L} \) applied to function (2.5) is an \( n \times 4 \) matrix \( \hat{L}(k, \omega) \). The specification of (2.4) makes the coefficients \( a_j \) satisfy the system of linear algebraic equations
\[ V a = 0, \quad \text{where} \quad a = (a_1 \ a_2 \ a_3 \ a_4)^T, \quad (2.8) \]
\[ V = \begin{pmatrix} \hat{V}_1 & \hat{V}_2 & \hat{V}_3 & \hat{V}_4 \end{pmatrix}, \quad \hat{V}_j = \hat{L}(k_j, \omega) \hat{U}_j, \quad j = 1, \ldots, 4. \]
Here, \( V \) is \( n \times 4 \) matrix.

Let us discuss the requirements to boundary conditions. Principally, they must result in a well-posed problem for governing equations (2.3). Besides, we expect the absence of incoming waves. These statements can be expressed in terms of wave amplitudes \( a_j \).
For any hyperbolic system having form (2.3), denote the set of outgoing mode indices by $\mathcal{J}$. The requirement of well-posedness [6] yields the consequence [4] that, for prescribed outgoing wave amplitudes $a_j$, $j \in \mathcal{J}$, there exist at most a single set of incoming wave amplitudes $a_j$, $j \notin \mathcal{J}$. This imposes a condition on columns of the matrix $V$ from (2.8), namely,

$$\hat{L}(k_j, \omega) \hat{U}_j \text{ are linearly independent, } j \notin \mathcal{J}, \forall \omega.$$  

(2.9)

Boundary condition (2.4) be nonreflecting, if for incoming modes, the coefficients $a_j = 0$, $j \notin \mathcal{J}$, as coefficients for outgoing modes are arbitrary. This is satisfied when, in addition to (2.9),

$$\hat{L}(k_j, \omega) \hat{U}_j = 0, \ j \in \mathcal{J}, \forall \omega.$$  

(2.10)

For the Euler equations characterized by modes (2.7), a right-hand subsonic outflow ($\mathcal{J} = \{1, 3, 4\}$) specifies the following expression for criterion (2.9), (2.10) of NRBC:

$$\hat{L}(k_2, \omega) \hat{U}_2 \neq 0, \ \hat{L}(k_1, \omega) \hat{U}_1 = \hat{L}(k_3, \omega) \hat{U}_3 = \hat{L}(k_3, \omega) \hat{U}_4 = 0.$$  

(2.11)

At a left-hand subsonic inflow ($\mathcal{J} = \{2\}$), criterion (2.9), (2.10) becomes

$$\text{rank } V = 3, \ \hat{L}(k_2, \omega) \hat{U}_2 = 0.$$  

(2.12)

In the case of 1D linearized Euler equations, it is simple to obtain systems of algebraic or differential equations matching requirements (2.11) or (2.12). After work [1], this problem may be considered as resolved completely. However, for applied purposes, we should construct a wide variety of NRBC’s and thereafter make selection of their appropriate representatives. Besides, the situation of partially reflective boundary conditions will be taken into account.

The basic classes of the 1D Euler NRBC’s and methods to derivation of their new versions are stated in [4]. We will mention the following examples.

Start with the algebraic characteristic boundary conditions [1]. For modes (2.7), the left-hand subsonic inflow condition is the system

$$\rho c u' + p' = 0, \ c^2 \rho' - p' = 0, \ v' = 0,$$  

(2.13)

and at the right-hand subsonic outflow, the equation

$$\rho c u' - p' = 0$$  

(2.14)

is specified.

The differential radiation boundary conditions [4] are, at the left boundary,

$$\frac{\partial U}{\partial t} - (c-u) \frac{\partial U}{\partial x} = 0,$$  

(2.15)
and at the right boundary,
\[
\left( \frac{\partial}{\partial t} + (c+u) \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) U \equiv \frac{\partial^2 U}{\partial t^2} + (c+u) \frac{\partial^2 U}{\partial x \partial t} + u (c+u) \frac{\partial^2 U}{\partial x^2} = 0. \tag{2.16}
\]

Indicate two examples of so-called combined boundary conditions for inflow and outflow. Coupling inflow conditions (2.13) with the pressure equation from (2.15), one obtains
\[
\begin{align*}
\rho c u' + p' &= 0, \quad c^2 \rho' - p' = 0, \quad v' = 0, \quad \partial p'/\partial t - (c-u) \partial p'/\partial x = 0. \tag{2.17}
\end{align*}
\]

An outflow NRBC is the set
\[
\begin{align*}
\rho'/\partial t + u \rho'/\partial x + c^{-1} \partial p'/\partial x &= 0, \quad \partial u'/\partial t + (c+u) \partial u'/\partial x = 0, \\
\partial v'/\partial t + u \partial v'/\partial x &= 0, \quad \partial p'/\partial t + (c+u) \partial p'/\partial x = 0. \tag{2.18}
\end{align*}
\]

Note that boundary conditions (2.15)–(2.18) are composed of 4 equations each. These boundary treatments are intended to be incorporated into algorithms based on three-point stencil spatial approximations to governing equations. For subsonic Euler equations (2.3), the number of boundary equations is always excessive, but such systems degenerate resulting in well-posed mathematical problems.

3 BOUNDARY CONDITIONS FOR 2D LINEAR EULER EQUATIONS

The linearized Euler equations in the 2D case take the form
\[
\frac{\partial U}{\partial t} + C_x \frac{\partial U}{\partial x} + C_y \frac{\partial U}{\partial y} = 0, \tag{3.1}
\]
where \( C_x \) is the same as in (2.2) and
\[
C_y = \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & 1/\rho \\ 0 & 0 & \rho c^2 & v \end{pmatrix}.
\]

Let artificial boundaries form a rectangle. A flow is assumed to be subsonic, i.e.,
\[
u^2 + v^2 < c^2, \quad u > 0.
\]

Consider boundary conditions at the left-hand inflow and the right-hand outflow only, so that expression (2.4) holds again. Similarly treated are lateral (top and bottom) boundaries which can be inflow and outflow as well.

Normal-mode solutions
\[
U = \tilde{U} \exp\{i \omega t - ikx - ily\}, \quad \omega > 0, \quad k, l \in \mathbb{C}, \tag{3.2}
\]
to system (3.1) for fixed frequency $\omega$ and $y$-wavenumber $\ell$ are represented as

$$U = \sum_{j=1}^{4} a_j \hat{U}_j \exp\{i\omega t - ik_j x - i\ell y\},$$

(3.3)

where $k_j = k_j(\omega, \ell)$ and $\hat{U}_j = \hat{U}_j(\ell/\omega)$ are indicated in [1, 4]. The modes have the same physical meaning as in the 1D case (2.7), and their group velocities in $x$-direction

$$c_{g,j} = \frac{\partial \omega}{\partial k} \bigg|_{k=k_j}, \quad j = 1, \ldots, 4,$$

for $\omega = \omega(k, \ell)$, preserve the sign at any $y$-wavenumbers $\ell$.

Boundary condition (2.4), where the image of operator $\mathcal{L}$ in terms of (3.2) is denoted by $\hat{\mathcal{L}}(k, \ell, \omega)$, results in the following system for the amplitudes,

$$Va = 0, \quad \text{where} \quad a = (a_1 \ a_2 \ a_3 \ a_4)^T,$$

$$V = (\hat{V}_1 \ \hat{V}_2 \ \hat{V}_3 \ \hat{V}_4), \quad \hat{V}_j = \hat{\mathcal{L}}(k_j, \ell, \omega) \hat{U}_j, \quad j = 1, \ldots, 4.$$

(3.4)

Requirements to boundary conditions consist of the following. To make the problem well-posed by excluding an infinite variety of incoming mode coefficients $a_j$, $j \notin \mathcal{J}$, we must ensure that, in (3.4),

$$\hat{V}_j \text{ be linearly independent}, \quad j \notin \mathcal{J}, \ \forall \omega, \ell.$$

(3.5)

Nonreflecting boundary conditions for the 2D Euler equations are always nonlocal, as the description of acoustic waves in system (3.1) can be reduced to the 2D wave equation. For the latter, only approximate algebraic or differential boundary conditions may be specified [7].

On the other hand, available are boundary conditions being exactly nonreflecting for entropy and vorticity waves. This means inflow amplitudes of entropy and vorticity modes be $a_3 = a_4 = 0$. At the outflow, $a_3$ and $a_4$ are arbitrary and the left acoustic-mode amplitude $a_2$ is expressed through $a_1$ only. In terms of the columns of (3.4), this means, at a subsonic outflow,

$$\hat{V}_2 \neq 0, \quad \hat{V}_3 = \hat{V}_4 = 0, \quad \hat{V}_1 = \alpha \hat{V}_2,$$

(3.6)

where $\alpha = \alpha(\ell, \omega)$, and at a subsonic inflow,

$$\text{rank } V = 3, \quad \hat{V}_2 = \alpha \hat{V}_1.$$

(3.7)

In the case of an excessive number of boundary equations, the requirements should be less strong. At the outflow,

$$\hat{V}_2 \neq 0, \quad \hat{V}_3 = \hat{V}_4 = 0,$$

(3.8)

and at the inflow,

$$\text{rank } V = 4.$$

(3.9)
2D approximate NRBC’s are constructed as corrections to 1D nonreflecting conditions. Two examples are the modified inflow system (2.17),
\[
\rho c u' + p' = 0, \quad c^2 \rho' - p' = 0, \quad \partial u'/\partial y - \partial v'/\partial x = 0,
\]
\[
\partial p'/\partial t - (c-u) \partial p'/\partial x + v \partial p'/\partial y = 0,
\]
and the analogue of outflow condition (2.18),
\[
\frac{\partial \rho'}{\partial t} + u \frac{\partial \rho'}{\partial x} + v \frac{\partial \rho'}{\partial y} + \frac{1}{c} \frac{\partial p'}{\partial x} = 0, \quad \frac{\partial u'}{\partial t} + (c+u) \frac{\partial u'}{\partial x} + v \frac{\partial u'}{\partial y} + c \frac{\partial v'}{\partial y} = 0,
\]
\[
\frac{\partial v'}{\partial t} + u \frac{\partial v'}{\partial x} + v \frac{\partial v'}{\partial y} + \frac{1}{\rho} \frac{\partial p'}{\partial y} = 0, \quad \frac{\partial p'}{\partial t} + (c+u) \frac{\partial p'}{\partial x} + v \frac{\partial p'}{\partial y} = 0.
\]

Boundary operators from (3.10) and (3.11) satisfy (3.9) and (3.8), respectively.

In the general case, sets of 4 equations do not match strict criteria (3.6) or (3.7). Such formulations result in overdetermined problems for 2D Euler equations (3.1). However, this situation can be resolved in practice.

As well as the two-dimensional linearized Euler equations, systems of equations describing viscous gas dynamics, such as the Navier–Stokes equations and the quasi-gasdynamic system [8], admit only approximate NRBC’s even in the 1D case. It should be noted that these differential models yield (in linearized form) expansions over 7 or 8 modes rather than 4 (see [4, 9]), and they require a greater number of boundary equations than the Euler equations do. Therefore, sets of four boundary equations produce wave reflections in the form of modes additional with respect to inviscid models. The same is true for computational gasdynamic algorithms.

4 BOUNDARY CONDITIONS FOR NONLINEAR PROBLEMS

For nonlinear systems of equations, linear boundary conditions stated above should be reformulated in terms of the original physical variables. The procedure is analogous to the transition from a linear hyperbolic system back to its quasilinear origin. First consider a case when such transformation of boundary equations is obvious.

4.1 1D Euler equations

For linearized 1D Euler equations (2.3), a number of boundary operators proposed in §2 contain only first derivatives with respect to time and space,
\[
\mathcal{L}U = L_t \frac{\partial U}{\partial t} + L_x \frac{\partial U}{\partial x},
\]
with any matrices $L_t$ and $L_x$ depending on background values $V$. To treat quasilinear 1D Euler equations (2.1) being expressed through the variables $V = \bar{V} + U$, let us construct an operator and the correspondent boundary condition
\[
\mathcal{L}[V] V \equiv L_t(V) \frac{\partial V}{\partial t} + L_x(V) \frac{\partial V}{\partial x} = 0.
\]
Here, background values are replaced with local ones in coefficients and differentiated are
basic variables themselves.

In this way, we obtain exact NRBC’s for the nonlinear Euler equations in the 1D case.
It is proved in [5] that, if operator (4.1) is nonreflecting in sense (2.9)–(2.10), then, with its
quasilinear counterpart, equation (4.2) be nonreflecting boundary condition for nonlinear
equations (2.1).

In the general case of gasdynamic model, when a linear boundary operator contains
only first derivatives, a similar procedure can be applied to it, although, without strict
substantiation. Note that even for the 2D Euler equations, boundary conditions nonre-
reflecting for entropy and vorticity waves do lose this property after the transition to the
nonlinear case.

Extension of the class of boundary conditions meets some obsta-
cles. Algebraic equations cannot generally be represented in a nonlinear form. Boundary conditions with
second or higher order derivatives, contrarily, admit various nonlinear transformations.
In such circumstances, we will hold the principle that any nonlinear boundary condition,
after its linearization and dropping \( y \)-derivatives, be nonreflecting for the 1D linear Euler
equations (2.3).

4.2 Two-dimensional problems

In the case of two dimensions and viscosity, 1D NRBC’s cease to be exact. Moreover,
the error introduced is difficult to estimate. Some types of boundary conditions exhibit
bad behavior that cannot be predicted by a linear analysis. Thus, linear artificial bound-
ary conditions should be selected and then implemented directly or subjected further
modifications based on physical reasoning.

Consider typical subsonic viscous flows about bodies. The restricted domain is assumed
to be a rectangle.

At the left-hand inflow boundary, the free-stream values of en-
tropy, the right Riemann
invariant, and vorticity are specified (for any magnitude \( \varphi \), denote its free-stream value
by \( \varphi_\infty \)), together with the radiation condition for pressure,

\[
\frac{p}{\rho^\gamma} = \frac{p_\infty}{\rho_\infty^\gamma}, \quad u + \frac{2}{\gamma - 1} c = u_\infty + \frac{2}{\gamma - 1} c_\infty, \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \quad \frac{\partial p}{\partial t} - (c - u) \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} = 0. \tag{4.3}
\]

Linearization of equations (4.3) results in 2D condition (3.10) nonreflecting for entropy
and vorticity waves.

At the right-hand outflow, no algebraic equation may be set since for the 1D nonlinear
Euler equations this always leads to waves reflection. Meanwhile, when purely differential
boundary conditions are specified, a gradual drift (wandering) is observed in the gas-
dynamic parameters that frequently causes instability, especially, for low Mach numbers
\( u \ll c \).
In practice of viscous flow simulation, small algebraic terms are commonly added to differential outlet boundary equations [10]. Such terms may be argued as a tool to accelerate the convergence to a steady-state solution [11], as a 2D correction to the waves far-field behavior [12], or as allowing for the asymptotics [13] of a viscous laminar wake flow [5]. In most instances, the boundary conditions obtained become partially reflecting in the 1D linear case.

For low Mach-number flows, the following outlet set is the most suitable,

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + \frac{1}{c} \frac{\partial p}{\partial x} &= 0, \\
\frac{\partial u}{\partial t} + (c+u) \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \alpha |u_0 - u| (u_0 - u), \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} &= 0, \\
\frac{\partial p}{\partial t} + (c+u) \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} &= 0.
\end{align*}
\]

(4.4)

The right-hand side of the 2nd equation in (4.4) contains a term which will be referred to as the square relaxation. Here, \(u_0\) is a prescribed (reference) value. One may choose \(u_0 = u_\infty\) or a function \(u_0 = u_0(y)\). Magnitude \(\alpha\) is constant or variable. System (4.4), except its second equation, is the quasilinear form of 2D boundary condition (3.11). With the inclusion of this equation, linearization to (4.4) only in the 1D case yields NRBC (2.18).

Lateral boundaries have a feature that a flow can be directed both outside and inside the domain varying in space and time. The way is performing a permanent monitoring if the boundary point be inflow or outflow. This arrives in a kind of simulated active flow control. Otherwise, one may set some robust boundary condition acceptable in the both cases.

Ask a question. Can an outflow boundary condition be used at an inflow and can an inflow condition be applied at an outflow? Consider this situation for the 1D linear inviscid case (§2) assuming, by connotation, the boundary to be left-hand instead of top or bottom.

In the view of well-posedness requirement (2.9), at an inflow (see (2.12)) the property

\[\hat{V}_1, \hat{V}_3, \hat{V}_4 \text{ be linearly independent} \quad (4.5)\]

must be fulfilled. However, NRBC criterion (2.10) for a left-hand outflow (rearranged (2.11)) reads

\[\hat{V}_3 = \hat{V}_4 = 0. \quad (4.6)\]

This contradicts (4.5). Thus, any outflow condition should not be specified at an inflow.

If the boundary is outflow then (2.11) turns to

\[\hat{V}_1 \neq 0. \quad (4.7)\]

This equality also follows linear independence of the columns in inflow NRBC criterion (2.12). So inflow conditions are allowed at an outflow.
The same statements remain valid for 2D Euler linear boundary conditions satisfying (3.6) and (3.7) respectively, even in more general cases of flow descriptions. Note that inequalities (4.5) and (4.7) are primarily to be fulfilled for any normal-mode parameters, whereas equality (4.6) must never take place rather than cease to be an identity like in (2.10).

In the practice of modeling external flows, the lateral boundaries are much less sensitive to wave reflections than, for example, the outlet is. It is suitable to take a 1D inflow condition, say, (2.15), and rewrite it for the nonlinear case and in correspondence with the boundary orientation:

\[
\frac{\partial V}{\partial t} + (c+v) \frac{\partial V}{\partial y} = 0, \quad \text{where} \quad V = (\rho u v p)^T
\]  

(4.8)

(for the upper boundary).

5 DISCRETE BOUNDARY CONDITIONS

As mentioned above, finite-difference and finite-element approximations to gasdynamic systems of equations possess additional nonphysical modes and require specific approaches to the formulation of boundary conditions. For simplicity, we consider models being discrete in the single spatial coordinate and differential in time. The spatial grid is assumed to be uniform,

\[ \{x_l : l = 0, \ldots, N, \quad x_{l+1} - x_l \equiv h\}. \]

In this study, we will concentrate at scalar finite-difference equations that exhibit spurious waves. By these examples, most of the problems of numerical boundary conditions can be explained and partially resolved. A more detailed analysis has been performed in [4, 5].

5.1 Linear central-difference scheme

The linear transport equation

\[
\frac{\partial u}{\partial t} + c_0 \frac{\partial u}{\partial x} = 0
\]  

(5.1)

is approximated by the central-difference scheme

\[
du_l/dt + c_0 \left( u_{l+1} - u_{l-1} \right)/(2h) = 0.
\]  

(5.2)

Scheme (5.2) is based on a three-point stencil and needs a boundary condition on the right-hand end of the domain.

The general normal-mode solution to (5.2) is conveniently represented in the form

\[
u(x_l, t) = \exp\{i\omega t - ikx_l\} + R(-1)^l \exp\{i\omega t + ikx_l\},
\]

11
where $R$ to be called the reflection coefficient,

$$
  k = (1/h) \arcsin r, \quad r = \omega h/c_0.
$$

Thus, for moderate frequencies $\omega$, a solution is decomposed into an analogue of the physical wave and a spurious spatial saw-tooth oscillation.

A right-hand boundary condition must suppress the spurious wave, i.e., minimize the reflection coefficient $R$. As a boundary condition, consider the second-order approximation to equation (5.1) by the use of a three-point upwind difference

$$
du_N/dt + c_0(D_x^h u)_N = 0, \quad (5.3)
$$

where

$$
(D_x^h \varphi)_l \equiv (3\varphi_l - 4\varphi_{l-1} + \varphi_{l-2})/(2h). \quad (5.4)
$$

Setting (5.3), (5.4) determines the following reflection coefficient (see [5]),

$$
R = ir^3/8 + O(r^4). \quad (5.5)
$$

In [5], a number of other types of a right-hand boundary condition for scheme (5.2) is investigated and correspondent reflection coefficients are obtained.

Application of the central-difference approximation (5.2) to 1D Euler equations (2.3) reads

$$
\frac{dU_l}{dt} + C_x \frac{U_{l+1} - U_{l-1}}{2h} = 0. \quad (5.6)
$$

The general normal-mode solution to (5.6) is

$$
U = \sum_{j=1}^{4} \hat{U}_j [a_{j1} \exp\{i\omega t - ik'_j x_l\} + (-1)^ja_{j2} \exp\{i\omega t + ik'_j x_l\}]. \quad (5.7)
$$

The eigenvectors $\hat{U}_j$ are the same as in the continuous case (see (2.7)), and the wavenumbers

$$
k'_j = (1/h) \arcsin(k_j h), \quad j = 1, \ldots, 4,
$$

are associated with the magnitudes $k_j$ from (2.7).

Correct boundary conditions are constructed based on continuous NRBC’s of class (4.1), with the replacement of spatial derivatives by finite differences

$$
L_t \frac{dU_N}{dt} + L_x (D_x^h U)_N = 0, \quad (5.8)
$$

where operator (5.4) can be successfully applied at the right-hand boundary and its mirror counterpart can on the left.
5.2 Nonlinear central-difference scheme

In the nonlinear case, Fourier analysis is inapplicable. However, an approximate analysis of saw-tooth parasite waves can be performed. The following technique was proposed in [5]. A solution to 1D nonlinear finite-difference scheme $V(x_l, t)$, where $l$ is the node number, is decomposed to the background and oscillatory values

$$V = \overline{V}(x_l, t) + (-1)^l u^h(x_l, t). \tag{5.9}$$

The following hypotheses are done,

1. the functions $\overline{V}(x, t)$ and $u^h(x, t)$ be smooth with respect to $x, t \in \mathbb{R}$ and gradually varying,

2. in linear approximation, $O(U^h)^2 = 0$.

These assumptions permit one to find interrelations among behavior of the spurious oscillation $U^h$ and the background parameters $\overline{V}$ that follow the combination of governing equations with some boundary conditions.

As in [5], consider the simplest nonlinear analogue of equation (5.1)—the quasilinear transport equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0. \tag{5.10}$$

It is assumed that $u > u_\ast > 0$ and that the solution does not undergo a gradient catastrophe. We choose a conservative central-difference approximation for (5.10) analogous to (5.2),

$$\frac{du_l}{dt} + \frac{1}{2} \frac{u^2_{l+1} - u^2_{l-1}}{2h} = 0. \tag{5.11}$$

Represent a solution to (5.11) in form (5.9),

$$u = \bar{u}(x_l, t) + (-1)^l a(x_l, t). \tag{5.12}$$

Irrespective of a boundary condition, substituting function (5.12) into equation (5.11) results, owing to hypothesis (1), the two equalities at the same time:

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \frac{h^2}{6} \frac{\partial^3 \bar{u}}{\partial x^3} + \frac{h^2}{2} \frac{\partial \bar{u}}{\partial x} \frac{\partial^2 \bar{u}}{\partial x^2} = O(h^4), \tag{5.13}$$

$$\frac{\partial a}{\partial t} - a \frac{\partial \bar{u}}{\partial x} - \bar{u} \frac{\partial a}{\partial x} = O \left( h^4 + ah^2 + a^2 \right). \tag{5.14}$$

Now, we consider a boundary condition set at $x = x_N$. An approximation to nonlinear equation (5.11) of type (5.3), (5.4) gives the boundary condition

$$\frac{du_N}{dt} + u_N (D_x^h u)_N = 0,$$
By this equation, using formulas (5.13) and (5.14), we obtain the following asymptotics for the amplitude of the oscillation from (5.12) at the boundary point,

\[ a_N = (-1)^N \left[ \frac{h^3}{8} \frac{\partial^3 \bar{u}}{\partial x^3} + \frac{h^3}{8\bar{u}} \frac{\partial \bar{u}}{\partial x} \frac{\partial^2 \bar{u}}{\partial x^2} \right] + O(h^4). \]  

Expression (5.15) of the spurious oscillation amplitude exhibits some similarity with the reflection coefficient \( R \) from (5.5) that was obtained for linear equation (5.2). If we define the correspondence

\[ (-ir)^q \leftrightarrow h^q \frac{\partial^q \bar{u}}{\partial x^q} \]

then the magnitude \( R \) coincide with the first term on the right-hand side of (5.15). However, in the nonlinear case, an additional term of the same order as the basic term appears. In [5], a number of pairs of analogous boundary conditions for the linear and the quasi-linear transport equations is considered, and both common features and distinctions of operators characteristics are indicated for the two cases.

### 5.3 General system of equations

A detailed analysis of more complex problems—such as discrete models of external viscous flows—would be complicated considerably. However, a useful information can be extracted even from the situation when a boundary equation is specified but the system of governing equations is unknown. The only assumption is the decomposability of the solution into smooth and saw-tooth components.

![Figure 2: A saw-like function and its behavior near the right-hand boundary](image)

For example, consider a relaxation condition for velocity at the right-hand boundary

\[ \frac{\partial u}{\partial x} = \beta (u_\infty - u). \]  

Let it be approximated by the use of upwind difference (5.4),

\[ (D^h_x u)_N = \beta (u_\infty - u_N). \]
Velocity $u$ is decomposed into background and oscillatory components (5.12) and the sum is substituted into (5.17). We obtain the expression

$$\frac{\partial \bar{u}}{\partial x} = \beta (u^*_\infty - \bar{u}), \quad \text{where} \quad u^*_\infty \approx u_\infty - (-1)^N \frac{4a}{\beta h}. \quad (5.18)$$

This implies that velocity relaxes not to $u_\infty$, as occurred in differential case (5.16), but rather to the value $u^*_\infty$. The solution behavior is outlined in Fig. 2. The deviation of the asymptotic velocity $u^*_\infty$ from the free-stream value $u_\infty$ is proportional to inverse of $\beta$. Note that the result obtained is not affected by the dependence of the functions involved on $t$ and $y$ (if any).

If, in boundary condition (5.16), apply, instead of operator (5.17), a shifted central difference, in the form

$$\left( u_N - u_{N-2} \right) / (2h) = \beta (u_\infty - u_N), \quad (5.19)$$

then, the background velocity equation becomes

$$\bar{u} + \frac{1}{\beta} \frac{\partial \bar{u}}{\partial x} \approx u_\infty - (-1)^N \left[ a + \frac{1}{\beta} \frac{\partial a}{\partial x} \right].$$

Here, the correction to $u_\infty$ is less in order of magnitude than that in (5.18).

The qualitative and quantitative validity of the formulas presented is confirmed by the computation of test 2D problems in [5].

For the simulation of viscous subsonic flows, of high importance is an appropriate discrete representation for an outlet condition. In set of equations (4.4), at low Mach numbers the use of expanded differences (5.19) in the $x$-velocity equation

$$(\partial u / \partial t)_N + (c+u)_N (u_N - u_{N-2}) / (2h) + v_N (\partial u / \partial y)_N = \alpha |u_0 - u_N| (u_0 - u_N) \quad (5.20)$$

is preferable, but the rest of the derivatives $\partial \varphi / \partial x$ should be replaced by operator (5.4) (the quasilinear form of (5.8)). Note that the latter approximation is desirable for the term $\partial p / \partial x$, since pressure exhibits strong decreasing of oscillation amplitude near the boundaries. Although, this phenomenon has no explanation yet.

6 IMPLEMENTATION TO APPLIED PROBLEMS

The NRBC technique described in this study was applied to direct numerical simulation of two viscous flows—about a plate and in a channel with two strongly heated sidewall sites. In all the computations, we used the quasi-gasdynamic system, where first spatial derivatives were approximated by centered differences as in (5.11), and time derivatives were represented in the form of scheme with a weight of 1/2 implemented by explicit iteration (see [8]). Numerical boundary treatments applied for both problems combine common principles with distinctions in their adaptation.
The flow about a plate has the free-stream Mach number $M = u_\infty/c_\infty = 0.01$ and the typical Reynolds number $Re \sim 10^3$. The computational domain is the rectangle $\{ -20 \leq x \leq 93, 0 \leq y \leq 8 \}$ located above the plate $\{ 0 \leq x \leq 47, y = 0 \}$ in view of the flow symmetry. The plate is held at the constant temperature $T_w = 1.4 T_\infty$. The computations were performed on a rectangular nonuniform $181 \times 35$ grid being condensed toward the plate.

Initial conditions, boundary conditions on the rigid body, and the symmetry conditions at bottom in front of and behind the plate were borrowed from [4, 5]. Calculations were made until steady state. Consider conditions at the artificial boundaries.

At the left boundary, we set inlet condition (4.3) with the $x$-derivatives approximated by the three-point biased differences as (5.4) for the opposite direction. Here and in the rest of boundary equations involving time derivatives, the symmetric two-level treatment was applied to $x$-derivatives and the explicit form of $y$-derivatives was taken.

At the right-hand boundary, outlet condition (4.4) has been implemented. The source term in the square relaxation equation (5.20) for velocity contains the free-stream velocity $u_\infty$ as the reference value and is taken implicitly:

$$
\frac{u_{N-1}^{n+1} - u_{N-1}^n}{\Delta t} + (c+u)_{N-1/2}^n \frac{u_{N-1}^{n+1} - u_{N-2}^{n+1} + u_{N-2}^n - u_{N-2}^n}{4 \Delta x} + v_{N-1/2}^n \frac{u_{N,+}^n - u_{N,-}^n}{2 \Delta y} = \alpha \left| u_\infty - u_{N,+}^n \right| (u_\infty - u_{N,+}^{n+1}).
$$

Here, the notation of grid objects is typical, whereas values $u_{N,+}$ and $u_{N,-}$ refer to upper
and lower neighbors of a current node. The constant parameter $\alpha$ was chosen experimentally.

The upper boundary condition approximates 1D inflow set (4.8) by the pattern described above, with the change from $x$ to $y$. More precisely, in the equation for velocity $v$, an expanded difference

$$v_l,N_y - v_l,N_y - 2 \Delta y$$

is used, whereas in the equations for $\varphi = \rho, u, p$, an operator $D_h y \varphi$ analogous to (5.4) is applied.

Fig. 3 demonstrates lines of horizontal velocity $u$. The plate is marked with a bold line. The uppermost portion of the domain is not shown because of the absence of levels to be displayed. The flow is satisfactorily similar to the natural solution supposed. Some improvement of the results [5, 9] was achieved and convergence to a steady state was accelerated owing to the application of the boundary condition being transparent for acoustic waves outgoing through the upper boundary.

![Figure 4: Sketch of the channel flow](image)

In the second problem, we consider the flow between two plates with a small high-temperature spot in the middle of each plate. The flow geometry is shown in Fig. 4. The free-stream Mach number is $M = 0.02$, and the Reynolds number is $Re = 2300$. The heating elements maintain temperature $T_w = 4T_\infty$. The difficulty in the simulation of such a flow is associated with low velocities, high temperature gradients along with thermal waves, and considerable difference in scales of the construction parts.

The computational domain is a rectangle, the sides of which are the lower plate, the middle between the plates (the problem is symmetric), and two vertical lines in front of and behind the channel, namely, $\{x = -13\}$ and $\{x = 36\}$. In this formulation, the flow may be classified as external. A rectangular $301 \times 35$ mesh was used. The mesh in $y$ is condensed toward the plate. The $x$-mesh is maximally fine over the heating spots and gradually becomes sparser varying in size essentially along the domain. For more details of the problem setting, see [5]. A steady-state solution was obtained.
Figure 5: Isotherms ($T$) in the channel; $\min T = 0.8616T_\infty$, $\max T = 4T_\infty$

Figure 6: The isolines of velocity $u$ in the channel
Nonreflecting boundary conditions were specified at the left and right boundaries. The inlet condition is the same as in the previous case. The outlet conditions differ from the single plate formulation by a variable value of \( u_0 \) in the equation for \( u \),

\[
\frac{u_{N-1}^{n+1} - u_{N-1}^n}{\Delta t} + (c+u)_{N-1/2}^n u_{N}^{n+1} - u_{N-2}^n + u_N^n - u_{N-2}^n + v_{N-1/2}^n u_{N,+}^n - u_{N,-}^n
\]

\[
= \alpha \left| u_0(y) - u_{N}^{n+1} \right| \left( u_0(y) - u_{N}^{n+1} \right),
\]

with an empirical constant \( \alpha \) and the function \( u_0(y) \) defined by a formula consistent with the theory [13] for a laminar flooded stream.

Figures 5 and 6 show lines of constant temperature and \( x \)-velocity. The whole computational domain is displayed in Fig. 6. The temperature distribution looks quite reasonable, whereas velocity undergoes considerable oscillations. However, within the channel, where the flow is of practical interest, these oscillations are very weak.

7 CONCLUSIONS

- Artificial boundary conditions for applied gasdynamic problems are based on NRBC’s for the 1D linearized Euler equations and on 2D boundary conditions non-reflecting for entropy and vorticity waves.

- High-accuracy approximate NRBC’s for linear gasdynamic models are not useful in the simulation of nonlinear problems.

- For the simulation of external flows in rectangular domains, the most important and most difficult is an appropriate setting of the outlet boundary condition. This may involve a special technique.

- The discrete representation of the boundary conditions (especially, at the outlet) plays essential role and it can be chosen by means of a mathematical instrument.

REFERENCES


