Stellingen behorende bij het proefschrift:

**Collapse Behavior of Imperfect Sandwich Cylindrical Shells**

door Erwan Karyadi

1. In het algemeen kan de klassieke aanpak voor het analyseren van sandwich-constructies, die gebaseerd is op de eerste-orde afschuif-vervormingstheorie, niet gebruikt worden voor het voorspellen van de bezwijkeigenschappen van sandwich-constructies met een in de dwarsrichting (dikte richting) zachte kern.

2. Door de samenstelling van drie verschillende lagen, de buitenste twee lagen en de kern, zijn de bezijkvormen van een cilinder schaal van sandwich materiaal ingewikkelder dan de bezijkvormen van eenzelfde schaal die bestaat uit conventionele materiaal (isotroop of anisotroop materiaal).

3. De termen *symmetrische* of *antisymmetrische* “wrinkling“ passen niet in het kader van de niet-lineaire oplossingen waarbij de lokale invloeden (de randvoorwaarden, locatie van de belastingen, enzovoort) in acht worden genomen.

4. Deze lokale invloeden genoemd bij 3 veroorzaken hoge spanningspieken in de verbindingslagen huid-kern-huid die bij cilinderschalen leiden tot vervroegd bezwijken in de locale breukvormen van het kernmateriaal of het loslaten van de lijm-verbindingen tussen de lagen.

5. In de praktijk gebruikte sandwich cilinderschalen knikken bij een hogere belasting en zijn in het algemeen minder imperfectiegevoelig dan cilinderschalen die opgebouwd zijn uit hetzelfde huidmateriaal zonder kern.

6. Gelet op het oorspronkelijk betekenis van het woord “Sandwich” (zie Webster’s Encyclopedie of the English Language):

   Two thin slices of bread, having between them meat, cheese, etc., ........, [after John Montagu, Fourth Earl of Sandwich, 1718-1792, who is said to have originated it in order to eat without leaving the gaming table]

   is dit woord meer een geschikt term voor een snel-kookboek dan voor een technische wetenschappelijk proefschrift.

7. Sport is de beste manier om verlies te leren accepteren.

8. Het succes voor het behoud van het tropische regenwoud wordt niet in de eerste plaats bepaald door een goed beheer maar eerder door de activiteiten van de mensen die wonen in de omringende gebieden.

9. Het gebruik van *eco*-labels in de handel wordt eerder door *eco*-nomische motieven bevorderd dan door de wens milieuvriendelijke produkten te categoriseren.
10. Het vele gebruik van het toetsenbord van een computer bevordert een slecht handschrift.

11. Bergtoppen zullen er altijd wel bestaan. De kunst is er bovenop te komen. *(vrij naar Hervey Voge)*.

12. In Indonesië is het (auto-)rijgedrag geen graadmeter voor de tolerantie en bescheidenheid van de bevolking als geheel.

13. Het langdurig leven tussen twee verschillende culturen kan leiden tot een beter inzicht in de waarden van beiden.
Collapse Behavior of Imperfect Sandwich Cylindrical Shells

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door

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Summary

The collapse behavior of cylindrical shells which are built from sandwich material depends on the boundary conditions at the shell edges and initial geometric imperfections. It is also influenced strongly by the property of the core of sandwich layers. The core, which acts as an elastic foundation of the two adjacent faces, disposes the shells to fail in several local and overall instability or failure modes.

In the past years, instability behavior of sandwich structures with traditional metallic honeycomb cores has been investigated by many researches [1, 40] using a classical approach. This approach, which assumed “transversely” stiff formulation of the core, becomes dubious since the arrival of modern cores in the last two decades. These modern cores are made of light materials which generally have very low rigidity and characteristics resembling those of traditional cores. However, due to their very low rigidity, they are also transversely flexible. This leads to difficulties in achieving an accurate model for analyzing the instability behavior of sandwich constructions. A new approach which encompasses “transversely” flexible cores is derived. Both classical and new approaches are presented in this thesis.

In the first part of the thesis, a classical theory for sandwich cylinders is described that takes into account the effects of boundary conditions and initial geometric imperfections. A single-layer sandwich modeling approach based on first-order shear deformation theory, is used to derive the governing equations and boundary conditions. As a result, transverse shear stresses of the core are constant through the thickness and vanish on the free surface of the faces. The model described is valid only for sandwich cylinders with a thin metallic honeycomb core and identical isotropic thin faces that may be loaded by axial compression. The buckling problem formulated for perfect shells and for shells with axisymmetric imperfections is solved using bifurcation analysis and for shells with asymmetric imperfections using Koiter’s initial postbuckling theory.

A new approach is presented in the second part of the thesis. A high-order theory for sandwich cylinders is utilized using three-layer sandwich modeling. The governing equations and the boundary conditions of each individual layer, i.e. the faces and the core, are derived separately whereas continuity requirements of the displacement and internal stress fields are imposed at the interfaces. The boundary conditions are presented both in the local and global formulations. Nonlinear distributions of transverse shear and normal stresses of the core through the thickness are obtained. The displacement fields in the core are also nonlinear. A distinct imperfection model of the faces is considered. The model described is suitable for
sandwich cylinders with a thick "transversely" soft foam core or nonmetallic honeycomb-like core and laminated faces under axial compression or external pressure load. Nonlinear equilibrium equations are solved using continuation procedures based on path following technique. Separate equations are formulated for the bifurcation analysis of axisymmetric prebuckling states.

In both parts of the thesis, Fourier decomposition and finite difference method are used to reduce the governing partial differential equations into algebraic matrix equations. Bifurcation solutions for shells from axisymmetric primary path are found using the inverse power method.

Several buckling problems of perfect and imperfect shells are analyzed in order to check the validity of the current formulations. For special cases, the results of the two approaches described are compared with results from literature or with those obtained from finite element program. The problems of bifurcation buckling, edge effects, prediction of (core) material failures, local and overall instability modes, and imperfection sensitivities are considered in the present investigation.
Samenvatting

De bezwijkeigenschappen van cilindrische schalen, gebouwd met sandwich materialen, zijn afhankelijk van de randvoorwaarden en de initiële vormonzuiverheden. Ook de eigenschappen van het material van de kern hebben een grote invloed op deze bezwijkeigenschappen. De kern, die functioneert als een elastische fundatie van de boven- en onderhuid, is er de oorzaak van dat de schalen verschillende locale en overall instabiliteits-of breukvormen vertonen.

In het verleden zijn, door verschillende onderzoekers [1, 40], de instabiliteits-eigenschappen van sandwich-constructies met een traditionele metalen honingraat kern bestudeerd door middel van een klassieke aanpak. Deze aanpak, waarbij werd aangenomen dat de kern stijf is geformuleerd in dwarsrichting, is minder betrouwbaar gebleken met de aankomst van de moderne kernen in de laatste twee decaden. De moderne kernen zijn vervaardigd uit licht materialen, die in het algemeen een lagere stijfheid bezitten en waarvan de andere kenmerken lijken op die van de traditionele kernen. Door die lage stijfheid zijn ze dan ook flexibel in de dwarsrichting. Dit leidt tot problemen bij het verkrijgen van een nauwkeurig model voor het analyseren van de instabiliteits-eigenschappen van sandwich-constructies. Een nieuwe aanpak, waarbij rekening gehouden wordt met deze flexibiliteit van de kern in de dwarsrichting, is ontwikkeld. Beide methoden van aanpak worden in dit proefschrift gepresenteerd.

In het eerste deel van dit proefschrift wordt een klassieke aanpak voor sandwich cilindrische schalen geformuleerd waarmee de effecten van de randvoorwaarden en de initiële vormonzuiverheden kunnen worden bestudeerd. Voor een enkel laags sandwich-model wordt, voor het afleiden van de vergelijkingen en de randvoorwaarden, een aanpak gebruikt welke is gebaseerd op de eerste-orde afschuif-vervormingstheorie. Hieruit volgt dat de afschuifspanningen uit het vlak van de kern constant zijn over de dikte en dat ze verdwenen zijn aan het vrije oppervlak van de boven- en onderhuid. Het bovengenoemde model is alleen geschikt voor sandwich cilindrische schalen met een dunne metaal honingraat kern en met identieke isotrope dunne buitenste huiden die belast kunnen worden door axiale compressie. Het knikprobleem geformuleerd voor perfecte schalen of schalen met axiaalsymmetrische vormonzuiverheden is opgelost door middel van een bifurcatie analyse en voor schalen met asymmetrische vormonzuiverheden door middel van initiële na-knik theorie van Koiter.

Een nieuwe aanpak wordt gepresenteerd in het tweede deel van dit proefschrift, door middel van drie-lagen sandwich-model, waar een hoge-orde theorie voor sand-
Samenvatting

wich cilindrische schalen wordt verkregen. De vergelijkingen en de randvoorwaarden van elke individuele laag, de twee huidplaten en de kern, worden apart afgeleid waarbij de continuïteits-eisen voor de verplaatsing en de interne spanningen aan de tussen lagen worden opgelegd. De randvoorwaarden worden in zowel de locale als de globale formulering gepresenteerd. Een niet-lineaire verdeling van de afschuiﬁn- en de normaal-spanningen over de dikte van de kern wordt daarbij verkregen. Hierbij zijn ook de verplaatsing-velden in de kern niet-lineair. Een apart model voor de vormonzuiverheden van de buitenste twee lagen wordt beschouwd. Het bovengenoemde model is geschikt voor, op axiale compressie of uitwendige belaste, sandwich cilindrische schalen met een dikke kern van een schuim of van een niet-metaal honingraat kern en composiet boven- en onderhuid. De niet-lineaire vergelijkingen worden opgelost door middel van de continuïteits procedures die gebaseerd zijn op padvolg-methode. Het probleem van bifurcatie van de axiaalsymmetrische grondtoestanden wordt afzonderlijk geformuleerd.

In beide delen van het proefschrift worden de Fourier decompositie en de eindige differentie methode gebruikt voor het reduceren van het stelsel partiële differentiaal vergelijkingen tot een stelsel algebraïsche vergelijkingen. De oplossing van het probleem van bifurcaties van axiaalsymmetrische grondtoestanden wordt opgelost door middel van de inverse power methode.

Om de juistheid van de huidige formulering te controleren worden verschillende knikproblemen van cilindrische schalen met en zonder initiële vormonzuiverheden geanalyseerd. Voor bijzondere gevallen, worden de resultaten uit de klassieke- en nieuwe aanpak vergeleken met resultaten uit de literatuur of met resultaten verkregen met behulp van de eindige elementen methode. Verschillende aspecten zoals de bifurcatie knikproblemen, de rand-effecten, het voorspellen van de breuk van het kernmaterial, de locale en overall instabiliteits-vormen en de gevoeligheid voor vormonzuiverheden worden nader bekeken in het huidige onderzoek.
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<th>Description</th>
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<tr>
<td>A, D</td>
<td>in-plane and bending stiffnesses</td>
</tr>
<tr>
<td>a, b</td>
<td>first and second initial postbuckling coefficients</td>
</tr>
<tr>
<td>c</td>
<td>$\sqrt{1 - v^2}$</td>
</tr>
<tr>
<td>$E_c$</td>
<td>elasticity modulus of the core</td>
</tr>
<tr>
<td>$E_f$</td>
<td>elasticity modulus of the faces</td>
</tr>
<tr>
<td>$\bar{E}$</td>
<td>reduced elasticity modulus of the face ($\bar{E} = E_f \frac{1}{R}$) in Part I</td>
</tr>
<tr>
<td>$e_z$</td>
<td>nondimensional elasticity modulus of the core ($e_z = E_c/E_f$)</td>
</tr>
<tr>
<td>$G_{x}, G_{y}$</td>
<td>transverse shear modulus w.r.t. $xz$ and $yz$-axes, resp.</td>
</tr>
<tr>
<td>$G_{th}$</td>
<td>transverse shear modulus in circumferential direction</td>
</tr>
<tr>
<td>$g_{x}, g_{y}$</td>
<td>nondimensional shear modulus ($g_x = G_x/\bar{E}$, $g_y = G_y/\bar{E}$) in Part I or ($g_x = G_x/E_f$, $g_y = G_y/E_f$) in Part II</td>
</tr>
<tr>
<td>h</td>
<td>distance between the mid-surfaces of the top and bottom face</td>
</tr>
<tr>
<td>$h_i$</td>
<td>distances between subsequent gridpoints</td>
</tr>
<tr>
<td>i</td>
<td>integer number of axial half waves of imperfection</td>
</tr>
<tr>
<td>L, R</td>
<td>length and radius of the cylinder</td>
</tr>
<tr>
<td>$M_{x}, M_{y}, M_{xy}$</td>
<td>moment resultants</td>
</tr>
<tr>
<td>$N_{x}, N_{y}, N_{xy}$</td>
<td>normal stress resultants</td>
</tr>
<tr>
<td>N</td>
<td>number of finite difference gridpoints</td>
</tr>
<tr>
<td>n</td>
<td>number of half waves in the circumferential direction</td>
</tr>
<tr>
<td>$n_{x}, n_{xy}, n_{x}, m_{x}, m_{xy}$</td>
<td>nondimensional basic stress variables</td>
</tr>
<tr>
<td>P, S, M, Q</td>
<td>applied axial load, torsion, bending moment, transverse shear load, resp.</td>
</tr>
<tr>
<td>$p_e$</td>
<td>external pressure</td>
</tr>
<tr>
<td>$\bar{p}_e$</td>
<td>nondimensional external pressure ($\bar{p}_e = p_e c(R^3/E_f t^2)$)</td>
</tr>
<tr>
<td>$Q_{x}, Q_{y}$</td>
<td>shearing stress resultants</td>
</tr>
<tr>
<td>$R_1, R_2$</td>
<td>radius of the bottom face-core and top face-core interface, resp.</td>
</tr>
<tr>
<td>$R^c, R^t, R^b$</td>
<td>radius of the core, top face and bottom face, resp.</td>
</tr>
<tr>
<td>r</td>
<td>nondimensional radial coordinate of the core ($r = R/R^c$)</td>
</tr>
<tr>
<td>$r_1, r_2$</td>
<td>nondimensional radius of the interfaces ($r_1 = R_1/R^c, r_2 = R_2/R^c$)</td>
</tr>
<tr>
<td>$t^c, t^t, t^b$</td>
<td>thickness of the core, top face and bottom face, resp.</td>
</tr>
<tr>
<td>$\bar{t}$</td>
<td>reduced thickness ($\bar{t} = \sqrt{t^t t^b/(1 - v^2)}$) in Part I</td>
</tr>
<tr>
<td>t</td>
<td>thickness of the faces ($t = t^t = t^b$)</td>
</tr>
<tr>
<td>U, V, W</td>
<td>shell displacements of the reference surface in ($x, y, z$) direction</td>
</tr>
<tr>
<td>u, v, w</td>
<td>nondimensional basic displacement variables ($u = U/h$, $v = V/h$, $w = W/h$) in Part I or ($u = U/t$, $v = V/t$, $w = W/t$) in Part II</td>
</tr>
<tr>
<td>$\bar{W}$</td>
<td>initial radial imperfection (positive inward)</td>
</tr>
<tr>
<td>$\bar{w}$</td>
<td>nondimensional initial radial imperfection</td>
</tr>
</tbody>
</table>
List of Symbols

x, y, z  axial, circumferential, and radial coordinates of the reference surface
\( \bar{x}, \bar{y} \)  nondimensional coordinates of the reference surface (\( \bar{x} = x/R \), \( \bar{y} = y/R \))
B  boundary condition matrices
\( b_0, b_N \)  right-hand side boundary condition vectors
\( f^D, f^F \)  vectors contain the displacement and stress basic variables
Y  vectors with basic shell variables
\( \alpha, \beta \)  imperfection form factors
\( \beta_x, \beta_y \)  rotations around the x and y-axis, respectively
\( \epsilon_x, \epsilon_y, \gamma_{xy} \)  in-plane normal and shear strains
\( \epsilon_z, \gamma_{xz}, \gamma_{yz} \)  out-of plane normal and shear strains
\( \eta \)  path parameter
\( \lambda \)  special nondimensional applied axial load (\( \lambda = \tilde{P}/(2hE) \)) in Part I
\( \lambda_a \)  nondimensional applied axial load (\( \lambda_a = \tilde{P}(cR^c)/(E_r t^2) \)) in Part II
\( \lambda_e \)  estimated eigenvalue
\( \Lambda \)  generalized nondimensional variable load factor
\( \Lambda_c \)  \( \Lambda \) evaluated at the bifurcation load
\( \Lambda_s \)  \( \Lambda \) evaluated at the limit load
\( \mu \)  eigenvalue
\( \mu_0 \)  eigenvalue shift
\( \bar{\mu} \)  amplitude of axisymmetric imperfection
\( \nu \)  Poisson’s ratio of the faces
\( \xi, \eta \)  perturbation parameter
\( \rho_s \)  amplitude of the initial imperfection
\( \sigma_x, \sigma_y, \tau_{xy} \)  in-plane stresses
\( \tau_{xz}, \tau_{yz} \)  transverse shear stresses (in Part I)
\( \tau_{x0}, \tau_{y0}, \sigma_{T} \)  transverse shear and normal stresses (in Part II)
\( \chi, \psi \)  nondimensional basic rotation variables

( )^G  variables related to global point at the rigid stiffener
( )^0, ( )^1, ( )^2  quantities belonging to prebuckling, buckling and postbuckling state, respectively
( )^C, ( )^T, ( )^b  quantities belonging to core, top face and bottom face, resp.
( )  quantities with z-dependence
( )'  modified stress resultant quantities
( )'  differentiation with respect to load factor
( )_x, ( )_y  differentiation with respect to x and y coordinate
Circular cylindrical shells are commonly used in aerospace vehicles, for instance as aircraft fuselages, the body of launch vehicles, missiles, and satellite components. Although these aerospace structures are usually fabricated from metals, advanced composites are also gaining widespread usage because of their higher strength/weight and stiffness/weight ratios. Prior to the advent of composites, the development of sandwich construction emerged because, by a proper choice of materials and geometry, a significant stiffness/weight improvement can be achieved [40]. For this particular reason, sandwich materials will take an increasingly important role in the future usage of aerospace materials.

A sandwich could be defined as a layered construction, which consists of three layers, formed by bonding two thin faces to a comparatively thick core. The faces, the outermost layer components of a sandwich construction, are generally thin and of high density, which provide practically all of the over-all bending and in-plane extensional rigidity to the sandwich. It is the task of the faces to carry the tensile and compressive stresses that occur in the structure due to tensile, compressive and bending loads and to provide for the in-plane shear stresses due to shear loads and torsion. The core serves to position the faces at locations removed from the neutral axis, provides all of the transverse shear rigidity of the sandwich and stabilizes the faces against local buckling and transmits shear between them. Thus the task of the core in sandwich constructions is to take care of shear forces that work on the structure, and to support and maintain the proper distance between the faces.

1.1. Theory Background

The face sheets are generally thin in comparison with the total thickness of sandwich wall. In general, modeling the faces using the two-dimensional classical laminate theory based on the Kirchhoff-Love assumptions is sufficiently accurate. Early applications of sandwich structures have been made of metallic honeycomb core in which the transverse normal stress in the modeling approach can be
neglected since the core is "transversely" (in its thickness direction) stiff. However, the core, with usually low density, can have a low transverse shear modulus. It becomes obvious that modeling the core using the classical laminate theory is inadequate and will result in a very stiff formulation, i.e. a non shear-deformable core. Therefore, it is necessary to model the core using another approach that includes transverse shear effects.

In the literature, several models for laminate plates and shells that include the transverse shear effects are available. Basically, these modeling approaches can roughly be divided in two categories depending upon the variation of the displacement components through the thickness.

The first category of modeling approach is based on a first-order shear deformation theory. This theory implies the Reissner-Bollé-Mindlin assumption [47], namely a cross section, normal to the mid-surface of the un-deformed state remains flat but not necessarily normal in the deformed state. Note that the approaches of Reissner and Whitney [64] are different in the way for the specification of the shear correction factor. This correction factor has to be used to adjust the transverse shear stiffness following the assumption of uniform shear stress through the thickness. This factor was evaluated by comparison with an exact elasticity solution. Recently, Noor in [14,15] improved the prediction of this correction factor using a predictor-corrector method. All of the approaches described here will result in a linear variation of in-plane displacement components through the thickness of the laminates.

The second category of modeling approach is based on the higher-order theory. It should be cautioned that the terminology high order or higher-order theories, as used herein, refers to the level of truncation of terms in a power series expansion for displacements rather than to the order of the final system of differential equations. In the past years, two higher-order theories have been developed for laminate plates and shells depending upon the transverse shear and normal stresses assumptions. The first theory is the higher-order theory of Lo-Christensen-Wu in [36]. This theory does not neglect the transverse normal stress. Further, the transverse shear stresses are assumed constant through the thickness which implies that they do not satisfy the stress-free boundary conditions. The second one is the refined higher-order theory of Reddy in [44] that satisfies the transverse shear stress-free boundary conditions and accounts for shear rotations and parabolic variation of the transverse shear stresses. This particular choice eliminates the need for the specification of the shear correction factors. However, Reddy's theory neglects the transverse normal stress. Both theories mentioned result in a nonlinear variation of the displacement components through the thickness.
If the core is considered as a continuum, from the analysis point of view, the sandwich panels are no different from hybrid laminated structures. Therefore, the aforementioned modeling approaches for laminate plates and shells can directly be applied for sandwich structures. Noor in [15] has performed extensive numerical studies for sandwich panels and shells using these two category of modeling approaches.

Knowing the characteristic of the core that provides mainly the transverse shear rigidity, an assumption can be made that the core is in an anti-plane state of stresses (carries no in-plane stresses). This assumption implies that the transverse shear stresses are constant through the thickness of the core. With this additional knowledge, a modeling approach based on the first-order shear deformation theory is proposed by Tennyson in [55]. In his model, the transverse shear stresses distribution across the sandwich wall is assumed to satisfy the continuity requirement at the interfaces between the faces and core, and vanishes on free surfaces. As a result of this approach one obtains a linear distribution of the in-plane displacement components of the core through the thickness, as illustrated in figure 1-1.a. The faces are modeled using the classical laminate theory. Note that in this model one still neglects the transverse normal stress of the core.

![Diagram](image)

**Figure 1-1. Deformation pattern through the thickness of the sandwich shell:**
(a) First-order shear deformation theory, (b) High-order theory

Most of the buckling analyses performed on sandwich cylinders [2, 12, 35, 46, 52, 54, 60] have considered perfect shells. Tennyson in [55], as far as it is known to the author, is the first to deal with imperfect shells. However, only cylinder shells with axisymmetric imperfections are studied here. The cylinder is considered to
have classical simply supported boundary conditions or is considered long, thereby making the boundary conditions unimportant and a membrane prebuckling analysis is considered. The extension of modeling approach of Tennyson will be used in the first part of this thesis and is called the classical theory for sandwich structures.

It must be emphasized that the classical approach of Tennyson in modeling sandwich structures treated the faces and the core as one single layer. Actually, this is an extension of application of the first order shear deformation theory of conventional material (isotropic or anisotropic material) for analyzing the behavior of sandwich structures. In fact, a sandwich modeling is done by adjusting the in-plane and bending stiffnesses and including transverse shear stiffness properties of conventional material to those appropriate for sandwich material properties.

Today’s application of sandwich structures is usually made of a “transversely” flexible core such as foam and Aramid honeycomb cores. This modern core is more attractive in use rather than the traditional metal honeycomb core because of its advantages in terms of weight, manufacturing processes and resources. For this core, the transverse normal stress, which is important to describe the behavior of sandwich configurations in its thickness direction, cannot be neglected. Therefore a new formulation is introduced to model the behavior of sandwich shells more accurately.

An entirely different modeling approach for sandwich cylindrical shells, without neglecting the transverse normal stress of the core, is proposed in the second part of this thesis inspired by the approach of Bartelds [10] and the more recent works of Frostig and Baruch et.al. in [20-25]. This more general formulation of sandwich model is established by use of higher-order theory of “three-layer sandwich model” consisting of two face layers and one core layer in the middle. The basic idea of this approach is quite different from the previously discussed approaches in the sense that the sandwich layers are no longer considered as one single layer and then analyzing their behavior using the earlier mentioned first-order or high-order theories. The key element of this new approach is the treatment of each part of sandwich layers as an individual layer and considering the continuity requirement of deformation and stress distribution between the layers. As a result of this theory, the displacement components of the core can vary non-linearly through its thickness, as displayed in figure 1-1.b. A short historical review of this new approach might be given as follows.

Hemp in [28] has introduced this kind of model to investigate the buckling loads for perfect sandwich panels loaded by bi-axial in-plane compression. The formulation used for the face layers is based on a classical plate theory. For the core layer, a three dimensional elasticity theory is utilized with an additional assumption
that the in-plane extensional and shear stiffness properties are negligible with respect to the other (transverse) stiffness properties. However, separate formulation is established for the solution of symmetric or antisymmetric deformations of the isotropic faces with respect to the mid-surface of sandwich panels. Hemp's work was extended by Pearce and Webber in [41] for orthotropic face plates and later by Webber and Stewart in [62] for anisotropic face plates. All of the works mentioned dealt with an isotropic or orthotropic core and the obtained governing equations were derived via equilibrium of force and moments and the boundary conditions were not consistently accounted for.

Another application of this three-layers sandwich model has been successively implemented by Baruch and Frostig in [20] to study the bending behavior of perfect sandwich panels with a soft-core under partially distributed or concentrated loads normal to the mid-surface of the panels. This study is a logical extension of their earlier studies dealing with sandwich beams in [21-25]. They derived the governing equations and the boundary conditions consistently using the minimum of total potential energy principle for the faces and core as a separate medium. It is important to note their statement related to the high-order effects of the core displacement fields which are an outcome from this three-layers model. They stated that the nonlinear pattern of the displacements of the core are results of the proposed theory rather than consisting of a priori assumed deflection shapes. This is contrary to several modeling approaches for sandwich panels and shells summarized by Noor in [15] which were based on a priori assumptions of the nonlinear (cubic or quadratic) through-the-thickness displacement fields of separate layers of the sandwich model. Recently, inspired by the work of Baruch and Frostig, Thomsen in [56] succeeded to use this three-layers sandwich model to analyze sandwich panels with "through-the-thickness" type of inserts which are used whenever severe loads were introduced into the panels. Recently, Frostig in [27] categorized this kind of three-layers modeling of sandwich constructions as a "closed form high-order theory" of sandwich plates and shells.

For sandwich cylindrical shells, the work of Bartelds in [10] back in 1967, as far as it is known to the author, is the first and the only one which used this kind of sandwich modeling and derived both governing equations and boundary conditions within the consistent framework of a complete variational treatment. His so-called "unified theory" included also the effects of transverse shear and normal strain of the core. At almost the same time, Wempner in [63] used the same kind of modeling in a more complex manner, i.e., by use of the nonlinear kinematic relations of the core which is in contradiction with the assumed linear state of anti-plane stress of the core. Moreover, in Wempner's work the boundary conditions were developed independently of the governing equations.
Local and Global Instability Modes

There are significant differences in approaching the sandwich layers model between the work of Bartelds [10] and the more recent work of Frostig in [27]. In his study Bartelds considers the composed sandwich layer as a thin shell. He assumed that the thickness of the core is small in compared to the radius and the thin shell assumption is thus valid for the whole sandwich configuration. The treatment of boundary conditions is not further investigated and only the “classical” simple support is used.

On the contrary, Frostig considers the sandwich curved panels in more general way. The core is considered as a thick layer and the thin shell assumption is only imposed for the faces part of the sandwich panel. The boundary conditions are presented either in the local or global formulation, both follows from a variational derivation of the equilibrium equations. In the local formulation, the boundary conditions for the faces and the core are not necessary equal for the same section of the edges of sandwich panel since they are not related to each other. In the global formulation, the boundary conditions of the whole sandwich layer are defined at a certain global point along the height of the sandwich panel by use of a rigid stiffener which forces a connection between the boundary conditions for the faces and the core and to the global point as well. The same general approach as done by Frostig will be used in the second part of this thesis and is called the higher-order theory for sandwich structures.

1.2. Local and Global Instability Modes

In general, the structural instability of sandwich constructions can be categorized in the following type of modes. These various possibilities are described in [53] and summarized in figure 1-2.

The first overall instability mode is shear crimping (A). This mode is often referred to as a local mode of failure but is actually a special form of general instability for which the buckle wavelength is very short due to a low transverse shear modulus for the core. This phenomenon occurs quite suddenly and usually causes the core to fail in shear, however, it may also cause a shear failure in the core to face bond. The second one is the general instability mode (B). This mode involves overall bending of the composite wall coupled with transverse shear deformations. Usually, transverse normal strains do not play a significant role in this behavior. The wavelength associated with general instability are normally considerably larger than those encountered in local instability modes as inter-cellular buckling or face dimpling (C) and face wrinkling (D).
Figure 1-2. Sandwich instability and failure modes

In sandwich modeling point of view, these two overall instability modes can be obtained when one is analyzing the sandwich behavior by use of the classical theory mentioned, in which only transverse shear deformations of the core are considered.

The face dimpling (C) is a localized mode of instability which occurs only when the core is not continuous (such as those of a honeycomb core). In the regions directly above core cells, the faces buckle in plate-like fashion with the cell walls acting as edge supports. The progressive growth of these buckles can eventually precipitate the buckling mode identified below as face wrinkling. Face wrinkling (D) is a localized mode of instability which manifest itself in the form of short wavelengths in the faces, is not confined to individual cells of cellular-type cores, and involves the transverse normal deformation of the core material. As shown in figure 1-2, one must consider the possible occurrence of wrinkles which may be either symmetric or antisymmetric with respect to the mid-surface of the original un-deformed sandwich. However, when localized effects such as edge effects, inserts, introduced concentrated loads, etc. are considered, the faces can wrinkle in a general form where they will deform no longer either symmetric or antisymmetric to each other. As shown in figure 1-2, the final failure from wrinkling will usually result either in crushing of the core (E), tensile rupture of the core (F), or tensile rupture of the core to face bond (G). However, if proper care is exercised in the selection of the
adhesive system, one can reasonably assume that the tensile bond strength will exceed both the tensile and compressive strengths of the core proper.

Essentially, we need information of the normal and shear stress distributions in the interfaces between the core and the adjacent face to investigate this kind of local failure modes. Therefore, an investigation into such phenomena of local failure modes as well as its interaction with global failure modes requires a higher-order theory formulation which takes into account both transverse shear and normal deformation of the core.

1.3. Research Objectives

The main objective of the present research is to develop a numerical tool for the analysis to general collapse behavior of sandwich cylindrical shell with or without initial geometric imperfections under various edge conditions. Basically the newly developed code will be separated in two parts, both are written for the numerical solution of classical and higher-order theory part of this thesis.

The first part of this thesis, which is dealing with the classical theory, can be considered as a preliminary study to understand the buckling behavior of sandwich cylinders. This work is an extension of several earlier works, which are based on a classical approach for sandwich cylinder, in the sense that the current analysis includes the effect of nonlinear solution of the prebuckling state, satisfies rigorously the boundary conditions and accounted for general imperfections. The formulation in this part of the thesis is only capable to study overall instability modes, i.e. general buckling and shear crimping mode.

It becomes obvious that the whole phenomena of instability modes that occur in sandwich structures is not yet well understood due limitation of this classical approach. Therefore, the second part of this thesis will cover a new study of the local instability modes phenomena and their interaction with overall instability modes which usually occur in sandwich structures, using the new higher-order theory approach.

In this approach, the boundary conditions for the faces and core can be defined either in the global or local formulation. This opens new possibilities to investigate the effect of using various kind of boundary conditions of sandwich constructions which is also the objective of the current research. In reality, engineering designers will face also this kind of possibilities in modeling the boundary conditions for sandwich constructions. Hence, the new formulation of edge conditions can assist them to define proper boundary conditions for their particular design purposes.
It is known that in real structures there will always be initial geometric imperfections at the faces of sandwich layers. Therefore, imperfection sensitivity studies are also the main goal of this research. The new study with distinct imperfection models for the faces is carried out in the second part of the thesis. The numerical tools developed in the present research can be used to set up the development of "Improved Shell Design Criteria", see [7], for buckling sensitive of sandwich cylindrical shells, by carrying out a systematic study of their buckling and postbuckling behavior.

1.4. Overview of the Thesis

We can now outline the contents of this thesis which is mainly divided in two parts.

In Part I we deal with a classical approach of sandwich modeling based on first order shear deformation theory. Chapter 2 contains a detailed description of the analytical formulation for sandwich cylindrical shells with initial geometrical imperfections using a single-layer sandwich modeling technique. The formulation as described in this chapter is valid only for sandwich cylinder with 'transversely' stiff honeycomb core and identical isotropic faces. It is capable to analyze shells loaded in axial compression.

In chapter 3, the numerical procedures use to solve the governing equations of imperfect sandwich cylinder are described. Two numerical analysis are discussed namely bifurcation and initial postbuckling analysis. Procedures of numerical solution of the governing partial differential equations are based on Fourier decompositions and the finite difference method.

Numerical examples are given in chapter 4. In the first section we give the results for perfect sandwich cylinder under variation of core properties, shell geometry and boundary conditions. In the last two sections imperfection sensitivities of sandwich cylinders under a variation of transverse shear stiffnesses of the core are studied.

In Part II we deal with a new approach of sandwich modeling based on a higher-order theory. Chapter 5 contains a detailed description of the analytical formulation for sandwich cylindrical shell with initial geometrical imperfections using a three-layer sandwich modeling technique. The formulation as described in this chapter is valid for sandwich shells with "transversely" soft foam or honeycomb thick cores and laminated faces with the same thickness. Using the variational principle, the set of equilibrium equations and the natural boundary conditions are derived. Closed form solutions for the equilibrium equations of the core are sought by considering the continuity requirement of deformation and stress distribution between the sandwich layers. The resulting compatibility equations com-
plete the definition of the governing equations. The governing partial differential equations are then reduced into ordinary differential equations by use of Fourier decompositions and the finite difference method.

In chapter 6, the numerical procedures use to solve the governing ordinary differential equations are described. First, we discuss the continuation procedure for the solution of nonlinear equilibrium states. In the second section of this chapter we will describe the procedure for finding the lowest eigenvalue (i.e. the critical bifurcation point on a fundamental path) with the corresponding eigenmode by use of the standard inverse iteration method.

In chapter 7 numerical examples are given. In the first section we review the results from "unified theory" of Bartels which uses a membrane assumption of prebuckling state, thin core consideration and classical boundary conditions. In the next two section we give the results for nonlinear prebuckling responses and bifurcation buckling loads using both local and global boundary conditions under axial compression and external pressure. Comparison is made for "transversely" stiff core and global boundary conditions with the results obtained from the classical approach in Part I. In section 7.3, the influence of the variation of core property on the buckling behavior and comparison with the several classical formulation of local instability modes are presented. In section 7.4, we investigate the collapse behavior of sandwich cylinder with practically used "soft" honeycomb and foam core material. Since we know the allowable stresses of the core, we can make a prediction of material failure under increasing the applied load in section 7.5. In section 7.6, the buckling behavior and core material failure of cylindrical shell with laminated faces under external pressure are investigated. In the last section, the collapse load of imperfect sandwich cylinder is studied. First, a comparison is made for a "transversely" stiff core case with identical faces imperfection mode, with the results obtained in Part I of this thesis and the Finite Element Method MSC/NASTRAN. The investigation is followed by considering distinct imperfection models for the top and bottom faces of sandwich cylinders with a "soft" core under axial compression and external pressure.

In the last chapter we present some discussions, general conclusions and recommendations following the results of the current research.
Part I : Classical Theory
In this chapter, the governing equations of axially compressed cylindrical sandwich shells are derived based on the first-order shear deformation theory. The faces of sandwich layers are built from an isotropic shell and the core is modeled as an equivalent homogeneous orthotropic continuum with the principal material directions coinciding with the coordinate directions. In addition, the sandwich cylinders can have both axisymmetric or asymmetric initial geometric imperfections. This provides the current problem with two different possibilities of collapse for axially loaded cylindrical shells, namely a bifurcation buckling for perfect shells and shells with axisymmetric imperfection, and a stability problem at a limit point for shells with asymmetric imperfections.

The shallow sandwich cylindrical shell equations utilized here are based on imperfect thin sandwich shells. The equilibrium equation and boundary conditions for the faces and core are consistently derived in the framework of the minimum of total potential energy principle using a global cylindrical coordinate system. The final set of governing equations are composed of 10 partial differential equations, first order in the axial coordinate, and they contain 10 unknown stress or displacement variables of the mid-surface of sandwich layers. The 5 boundary conditions for each of the shell edges can be specified in terms of either a stress or displacement variable.

2.1. Definition of the Problem

The sandwich cylindrical shell geometry and the coordinate system used are shown in figure 2-1. In this figure, R and L denote the radius of the mid-surface and the length of the cylindrical shells, respectively. $t^t, t^c, t^b$ denote the thickness of the top face, core and bottom face, respectively. Let the reference surface be the mid-surface of the geometrically perfect cylinder, see figure 2-1, as defined by...
Basic Shell Equations

\[ h^t = \frac{ht^b}{t^t + t^b} \quad h^b = \frac{ht^t}{t^t + t^b} \quad (2-1) \]

the separation between the mid-surface of the top and bottom faces is

\[ h = t^c + \frac{t^t + t^b}{2} \quad (2-2) \]

From equation (2-1), if the faces have the same thickness \((t = t^t = t^b)\), then the distance between respectively the mid-surface of the top face and bottom face to the mid-surface of the core is equal \((h^t = h^b)\). Thus, the sandwich wall is symmetric with respect to the shells reference surface.

![Diagram showing the Sandwich Cylindrical Shells](image)

**Figure 2-1. Geometry and coordinate system of a sandwich cylindrical shells**

The global coordinate system \((x,y,z)\) is measured with respect to the reference surface of the whole sandwich wall in the axial, circumferential and radial directions, respectively. The in-plane displacements in axial and circumferential direction U, V and the radial displacement (positive inward) W of a point on the shell reference surface are attached to this coordinate system. \(\mu\) denotes the amplitude of initial imperfections. The external loading on the shell considered in this study is only the edge load in the axial direction.

**2.2. Basic Shell Equations**

In section 1.1, two different modeling techniques used in the construction of two dimensional theories for sandwich shells are discussed, namely a single-layer and three-layer sandwich model. The first technique is a global approximation model and will be used in this section. Here, the sandwich layers are replaced by an
equivalent single-layer anisotropic shell, including transverse shear deformation. Global displacement and strain approximations in the thickness direction are introduced and attached at the global coordinate system \((x,y,z)\).

The faces are built from an isotropic shell and the core layer has orthotropic properties. Note that the contribution of in-plane shear and extensional stiffness to the stiffness properties of the equivalent single-layer sandwich model is the outcome from the stiffness properties of the faces. On the other hand, the contribution of transverse shear stiffness is the outcome from the stiffness properties of the core. In addition, the model can have an initial geometric imperfection that is assumed to be stress-free and its amplitude is kept small with respect to the other (read shell thickness) geometric properties. The model is based on small strains-moderate rotations shell theory.

*Face and core considerations*

The equivalent single-layer sandwich shell model, consisting of the faces and the core, is treated as a thin shell. The nonlinear Kármán-Donnell strain-displacement relations for shallow shells, including the initial imperfection term \(\bar{W}\), are used. In addition, the rotations about the normal to the mid-surface are neglected with respect to rotations about directions tangent to the mid-surface. According to this theory, the mid-surface nonlinear in-plane strain-displacement relations are

\[
\begin{align*}
\varepsilon_x &= U_{rx} + \frac{1}{2}(W_{rx}^2 + 2W_{rx}\bar{W}_{rx}) \\
\varepsilon_y &= V_{ry} - \frac{W}{R} + \frac{1}{2}(W_{ry}^2 + 2W_{ry}\bar{W}_{ry}) \\
\gamma_{xy} &= V_{rx} + U_{ry} + W_{rx}W_{ry} + W_{rx}\bar{W}_{ry} + W_{ry}\bar{W}_{rx}
\end{align*}
\]  
\[(2-3)\]

and the out of plane mid-surface strain-displacement relations, with an additional assumption that the transverse normal stiffness is infinite, are defined as

\[
\begin{align*}
\gamma_{xz} &= W_{rx} + \beta_x \\
\gamma_{yz} &= W_{ry} + \beta_y \\
\varepsilon_z &= 0
\end{align*}
\]  
\[(2-6)\]

where \(\beta_x, \beta_y\) denote for the components of change of slope (rotation) of the normal to undeformed mid-surface. Further, the changes of curvature are defined as

\[
\begin{align*}
\kappa_x &= \beta_{x rx} \\
\kappa_y &= \beta_{y ry} \\
\kappa_{xy} &= \beta_{x ry} + \beta_{y rx}
\end{align*}
\]  
\[(2-7)\]

From the definition of the transverse shear and normal strain in equation (2-6), it follows that normal to the undeformed shell mid-surface remains straight and in-extensional during deformation, but no longer normal to the deformed shell mid-surface. Here, \(\gamma_{xz}\) and \(\gamma_{yz}\) are respectively the angles in the \(xz\) and \(yz\)-plane between normal to the undeformed and normal to deformed shell mid-surface.
Basic Shell Equations

Thus, this assumption satisfies the first-order shear deformation theory for thin shells and does not constitute the Kirchhoff-Love hypothesis for classical laminate theory which assumes that $\gamma_{xz}$ and $\gamma_{yz}$ are equal to zero.

The first-order shear deformation theory implies that the displacement components $\tilde{U}$ and $\tilde{V}$ are assumed to vary linearly through the thickness of the shell and the displacement component $\tilde{W}$ is assumed to be constant. This implies that the relations between the displacement components $\tilde{U}$, $\tilde{V}$ and $\tilde{W}$ of the deformed mid-surface and the displacement components $\tilde{U}$, $\tilde{V}$ and $\tilde{W}$ of a point in the shell can be expressed as

$$
\tilde{U} = U + z\beta_x \\
\tilde{V} = V + z\beta_y \\
\tilde{W} = W
$$

(2-8)

where $z$ denotes the radial distance from the reference surface. Consequently, the strain-displacement relations through the thickness of the whole sandwich shells can be approximated as

$$
\tilde{\epsilon}_x = \epsilon_x + zK_x \\
\tilde{\epsilon}_y = \epsilon_y + zK_y \\
\tilde{\gamma}_{xy} = \gamma_{xy} + zK_{xy}
$$

(2-9)

$$
\tilde{\gamma}_{xz} = \gamma_{xz} \\
\tilde{\gamma}_{yz} = \gamma_{yz} \\
\tilde{\epsilon}_z = 0
$$

(2-10)

Note that, it can be seen from equation (2-10) that the transverse shear strains are constant through the thickness.

Constitutive equations

The in-plane stress-strain relations for isotropic faces are assumed to be in a plane stress state and defined as follows

$$
\begin{bmatrix}
\tilde{\sigma}_x \\
\tilde{\sigma}_y \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{21} & C_{22} & 0 \\
0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\tilde{\epsilon}_x \\
\tilde{\epsilon}_y \\
\tilde{\gamma}_{xy}
\end{bmatrix}
$$

(2-11)

where

$$
C_{11} = C_{22} = \frac{E_f}{(1-v^2)} \\
C_{12} = C_{21} = vC_{11} \\
C_{66} = C_{11} \left( \frac{1-v}{2} \right)
$$

and $E_f$ is the modulus of elasticity of the faces.

Furthermore, the transverse shear stresses in the faces are equal to the transverse shear stresses in the core at the interfaces, and vanish on the free surfaces. The distribution of the transverse shear stresses across the faces is assumed to be linear as illustrated in figure 2-2. Here, only the distribution through the thick-
ness in xz-plane is portrayed. Note that the distribution of the transverse shear stress through the thickness in yz-plane can be displayed in a similar way.

\[
\begin{align*}
\tau_{xz}^t &= \frac{1}{2} \left( \frac{z + h^1}{t} \right) \tau_{xz}^c \\
\tau_{xz}^c &= \tau_{xz}^b \\
\tau_{xz}^b &= \frac{1}{2} \left( \frac{z + h^b}{t^b} \right) \tau_{xz}
\end{align*}
\]

**Figure 2-2. Transverse shear stresses distribution across the thickness of sandwich wall in xz-plane**

Assuming that the core resists only transverse shear and does not carry any in-plane stresses, which implies that \(\sigma_x = \sigma_y = \tau_{xy} = 0\), then the stress-strain relations for the core are

\[
\begin{bmatrix}
\tau_{xz} \\
\tau_{yz}
\end{bmatrix} =
\begin{bmatrix}
G_x & 0 \\
0 & G_y
\end{bmatrix}
\begin{bmatrix}
\tilde{\sigma}_{xz} \\
\tilde{\sigma}_{yz}
\end{bmatrix}
\]  \hspace{1cm} (2-12)

where \(G_x\) and \(G_y\) are the transverse shear modulus in xz and yz-plane, respectively.

In [18] Filon uses the term anti-plane to describe the state of stress in equation (2-12). An anti-plane core is an idealized core in which the modulus of elasticity in-planes parallel with the faces is zero but the shear modulus in-planes perpendicular to the faces is finite. In addition, the transverse normal stress \(\tilde{\sigma}_z\) is equal to zero. By this definition the in-plane stiffness is zero and the anti-plane core makes no contribution to the bending stiffness of the shell. One may also conclude here that because there is no in-plane stresses acting in the core, the transverse shear stresses cannot vary with \(z\) and thus are constant through the thickness of the core.

An equivalent system of stress and moment resultants is considered to be acting at the mid-surface of an element of the shell, as shown figure 2-3. The stress and moment intensities are related to the internal stresses by the following definitions
Basic Shell Equations

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix}
= \left[ \begin{array}{c}
h^b + \frac{t^b}{2} \\
-\frac{h^t - t^t}{2} \\
-\frac{h^t + t^t}{2}
\end{array} \right]
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
dz
\]

\[
\begin{bmatrix}
Q_x \\
Q_y \\
Q_{xy}
\end{bmatrix}
= \left[ \begin{array}{c}
h^b + \frac{t^b}{2} \\
-\frac{h^t - t^t}{2} \\
-\frac{h^t + t^t}{2}
\end{array} \right]
\begin{bmatrix}
\tau_{xz} \\
\tau_{yz}
\end{bmatrix}
dz
\tag{2-13}
\]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
= \left[ \begin{array}{c}
h^b + \frac{t^b}{2} \\
-\frac{h^t - t^t}{2} \\
-\frac{h^t + t^t}{2}
\end{array} \right]
\begin{bmatrix}
\ddot{\sigma}_x \\
\ddot{\sigma}_y \\
\ddot{\tau}_{xy}
\end{bmatrix}
zdz
\tag{2-14}
\]

where the integrations of equations (2-13) and (2-14) are taken across the whole sandwich wall. \(h^b\) and \(t^b\) are the distances of the mid-surface of the top face and bottom face, respectively, from the shell mid-surface, as depicted in figure 2-1.

\[\text{Stress resultants} \quad \text{Moment resultants}\]

Figure 2-3. The definition of stress-and moment resultants

By use of equations (2-9) and (2-10), the stress-strain relations for the faces in equation (2-11) and the core in equation (2-12) can be written in terms of the strain and curvature of the mid-surface. Introducing the resulting relations into equations (2-13) and (2-14), followed by carrying out the indicated integrations, gives the following constitutive equations in terms of the mid-surface strains and rotations

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix}
= \begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{12} & A_{22} & 0 \\
0 & 0 & A_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
= \begin{bmatrix}
D_{11} & D_{12} & 0 \\
D_{12} & D_{22} & 0 \\
0 & 0 & D_{66}
\end{bmatrix}
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\tag{2-15}
\]

\[
\begin{bmatrix}
Q_x \\
Q_y \\
Q_{xy}
\end{bmatrix}
= \begin{bmatrix}
S_{77} & 0 \\
0 & S_{88}
\end{bmatrix}
\begin{bmatrix}
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix}
\tag{2-16}
\]

18
where

\[
[A_{11}, A_{12}, A_{22}, A_{66}] = \left[ \frac{A}{1 - \nu^2}, \frac{A\nu}{(1 - \nu^2)^2}, \frac{A}{(1 - \nu^2)^2}, \frac{A}{2(1 + \nu)} \right]
\]

\[
[D_{11}, D_{12}, D_{22}, D_{66}] = \left[ D, \nu D, D, \frac{D(1 - \nu)}{2} \right]
\]

\[
[S_{77}, S_{88}] = [G_x h, G_y h]
\]

It should be mentioned here that with the definition of the mid-surface as given in equation (2-1), the coupling between in-plane strains and the rotation of the normal disappears. Furthermore, the in-plane stiffness A and bending stiffness D parameters are given by

\[
A = E_t (t^t + t^b) \quad D = \frac{E_t}{(1 - \nu^2)} \left\{ \frac{h^2 t^t t^b}{(t^t + t^b)} + \frac{1}{12} [(t^t)^3 + (t^b)^3] \right\}
\] (2-17)

### 2.3. Equilibrium Equations and Boundary Conditions

The basic approach followed in this study is to establish the equilibrium equations and consistent boundary conditions using the minimum of total potential energy principle with the aid of variational procedures. This principle states that a body is in equilibrium if and only if the virtual strain energy stored in the cylinder due to deformations equals the virtual work done by the external loads for every kinematically admissible displacement field. Applications of the minimum total potential energy principle requires that the first variation of the sum of the strain energy and potential of the applied loads of the face layers and the core is equal to zero. This implies

\[
\delta(U + V) = 0
\] (2-18)

where the elastic strain energy U, including the transverse shear deformation effect, and the potential energy of the applied axial compression load V are defined as

\[
U = \frac{1}{2} \int \int \int_{0}^{2\pi} \int_{-h}^{h} (\sigma_x \ddot{e}_x + \sigma_y \ddot{e}_y + \tau_{xy} \ddot{\gamma}_{xy} + \tau_{xz} \ddot{\gamma}_{xz} + \tau_{yz} \ddot{\gamma}_{yz})(1 - \frac{Z}{R})dx dy dz
\] (2-19)

\[
V = - \int \int (\hat{P}U_{nx})dx dy
\] (2-20)
Nondimensional Governing Equations

The actual variation is carried out in Appendix A.1. The resulting equilibrium equations, and boundary conditions are given in equations (A.5) to (A.9) and equations (A.10) to (A.14), respectively. The equilibrium equations can be rewritten, if the modified shear force $\tilde{Q}_x$ is introduced (see Appendix A.1), as follows

$$\begin{align*}
N_{x'y'} &= -N_{x'y''} \quad (2-21) \\
N_{xy'y'} &= -N_{y'y''} \quad (2-22) \\
\tilde{Q}_{x'y'} &= -\frac{N_y}{R}Q_{y'y''} - \{N_y(W_{x'y'} + \overline{W}_{y'y'}) + N_{xy}(W_{x'y'} + \overline{W}_{y'y'})\}_{y''} \quad (2-23) \\
M_{x'y'} &= \tilde{Q}_x - N_x(W_{x'y'} + \overline{W}_{x'y'}) - N_{xy}(W_{x'y'} + \overline{W}_{y'y'}) - M_{xy'y'} \quad (2-24) \\
M_{xy'y'} &= Q_y - M_{y'y''} \quad (2-25)
\end{align*}$$

and the boundary conditions at the edge of the cylinder, are such that one can prescribe either

$$\begin{align*}
N_x & \quad \text{or} \quad U \quad (2-26) \\
N_{xy} & \quad \text{or} \quad V \quad (2-27) \\
\tilde{Q}_x & \quad \text{or} \quad W \quad (2-28) \\
M_x & \quad \text{or} \quad \beta_x \quad (2-29) \\
M_{xy} & \quad \text{or} \quad \beta_y \quad (2-30)
\end{align*}$$

Two groups of variables are distinguished, namely basic and secondary variables. The basic variables are the unknowns that can be prescribed at the edges of the shell. From equations (2-26) to (2-30) one can see that these 10 variables are $(N_x, N_{xy}, \tilde{Q}_x, M_x, M_{xy}, U, V, W, \beta_x, \beta_y)$. The remaining three stresses $(N_y, M_y, Q_y)$ are the secondary variables and they can be eliminated from the equilibrium equations using the partially inverted constitutive equations derived below.

### 2.4. Nondimensional Governing Equations

The five equilibrium equations in (2-21) to (2-25) derived above are first order differential equations in the x-coordinate. There are in total five more unknowns in the set of 5 equations, so that an additional set of equations must be formulated in order to complete the definition of the governing equations that can be solved. These equations are the partially inverted constitutive equations derived from equations (2-15) and (2-16).
**Chapter 2**

**Partially inverted constitutive equations**

The constitutive equations given in (2-15) and (2-16) can be modified in such a way that they are partially inverted. Detailed derivations are given in Appendix A.2. They can be expressed as follows

\[
\varepsilon_x = \frac{h}{R} L_1(a) \quad N_y = 2\bar{E}hL_5(a) \tag{2-31}
\]

\[
\kappa_x = \frac{h}{R^2} L_2(a) \quad M_y = 2\bar{E}hRL_6(a) \tag{2-32}
\]

\[
\gamma_{xy} = \frac{h}{R} L_3(a) \quad Q_y = 2\bar{E}hL_7(a) \tag{2-33}
\]

\[
\gamma_{xz} = \frac{h}{R} L_4(a) \quad \kappa_{xy} = \frac{h}{R^2} L_7(a) \tag{2-34}
\]

where the linear operators \(L_i(a)\) are defined Appendix A.2.

The left-side relations of equations (2-31) to (2-33) and equation (2-34), together with the kinematic relations in equations (2-3), (2-5), (2-6) and (2-7), give the following 5 additional differential equations for the basic displacement variables

\[
U_{ix} + \frac{1}{2}(W_{ix}^2 + 2W_{ix}xW_{ix}x) = \frac{h}{R} L_1(a) \tag{2-35}
\]

\[
V_{ix} + U_{iy} + W_{ix}W_{iy} + W_{ix}xW_{iy} + W_{iy}W_{ix} = \frac{h}{R} L_3(a) \tag{2-36}
\]

\[
W_{ix} + \beta_x = \frac{h}{R} L_4(a) \tag{2-37}
\]

\[
\beta_{x'x} = \frac{h}{R^2} L_2(a) \tag{2-38}
\]

\[
\beta_{x'y} + \beta_{y'x} = \frac{h}{R^2} L_7(a) \tag{2-39}
\]

Furthermore, to eliminate the secondary variables from the equilibrium equations these variables are expressed in the basic variables using the remaining partially inverted constitutive equations, i.e., the right-side relations in equations (2-31) to (2-33).

Introducing the secondary variables into the equilibrium equations in (2-21) to (2-25) one finally obtains the following set of governing partial differential equations,
first order in the axial coordinate, which are entirely expressed in the basic variables

\[
N_{x'y} = -N_{xy'y} \quad (2-40)
\]

\[
N_{xy'y} = -2\bar{E}hL_5(a_y) \quad (2-41)
\]

\[
\dot{Q}_{x,x} = \frac{2\bar{E}h}{R}L_5(a) - 2\bar{E}hL_8(a_y)
\]

\[
- \{2\bar{E}hL_5(a)(W_{xy} + \overline{W}_{xy}) + N_{xy}(W_{xx} + \overline{W}_{xx})\},_y \quad (2-42)
\]

\[
M_{x'y} = \dot{Q}_x - N_x(W_{x'y} + \overline{W}_{x'y}) - N_{xy}(W_{yy} + \overline{W}_{yy}) - M_{xy'y} \quad (2-43)
\]

\[
M_{xy'y} = 2\bar{E}hL_8(a) - 2\bar{E}hR\Omega_4(a_y) \quad (2-44)
\]

\[
U_x = -\frac{1}{2}(W_{xx}^2 + 2W_{x'y}W_{y'y}) + \frac{h}{R}L_1(a) \quad (2-45)
\]

\[
V_x = -U_{y'y} - W_{x'y}W_{yy} - W_{xy}\overline{W}_{yy} - W_{yy}\overline{W}_{xy} + \frac{h}{R}L_3(a) \quad (2-46)
\]

\[
W_x = -\beta_x + \frac{h}{R}L_4(a) \quad (2-47)
\]

\[
\beta_{x'y} = \frac{h}{R^2}L_2(a) \quad (2-48)
\]

\[
\beta_{y'y} = -\beta_{x'y} + \frac{h}{R^2}L_7(a) \quad (2-49)
\]

The governing partial differential equations and the boundary conditions are written in nondimensional form with the following nondimensional quantities. The stress and displacement basic variables and the imperfection \( \overline{W} \) are made nondimensional as follows

\[
\begin{align*}
\begin{bmatrix}
    u \\
    v \\
    w \\
    \chi \\
    \eta \\
    \dot{w}
\end{bmatrix} &= \frac{1}{h} R^2 \begin{bmatrix}
    U \\
    V \\
    W \\
    R\beta_x \\
    R\beta_y \\
    \dot{W}
\end{bmatrix} \\
\begin{bmatrix}
    n_x \\
    n_{xy} \\
    q_x \\
    m_x \\
    m_{xy}
\end{bmatrix} &= \frac{1}{\sigma_0 h} \begin{bmatrix}
    N_x \\
    N_{xy} \\
    \dot{Q}_x \\
    M_x/R \\
    M_{xy}/R
\end{bmatrix}
\end{align*}
\]  

(2-50)

where \( \sigma_0 \) is the reference stress defined as
\[
\alpha_0 = 2E = \frac{2E_t t}{R} \quad \dot{t} = \sqrt{\frac{t t^b}{\gamma(1 - v^2)}}
\]

Next, the axial and circumferential coordinates and the differentiations of the basic variables with respect to these coordinates are written in nondimensional form as follows

\[
\bar{x} = \frac{x}{R} \quad \bar{y} = \frac{y}{R} \quad (\cdot)_x = \frac{1}{R}(\cdot)_\bar{x} \quad (\cdot)_y = \frac{1}{R}(\cdot)_\bar{y} \quad (2-51)
\]

Introduction of the nondimensional basic variables and imperfection from equation (2-50), the nondimensional axial and circumferential coordinates, and the nondimensional basic variables differentiated with respect to these nondimensional coordinates from equation (2-51) into the governing equations in (2-40) to (2-49) and the boundary conditions given by equations (2-26) to (2-30) yields the following set of nondimensional governing partial differential equations expressed in a short vector notation

\[
Y_{,x} = f(\bar{x}, \bar{y}, \bar{w}, Y, Y_{,x}, Y_{,y}, Y_{,xy}, Y_{,yy}, \ldots) \quad (2-52)
\]

where \( f \) denotes the linear function and its axial first derivative and circumferential derivatives. Note that this function contains also the imperfection term \( \bar{w} \).

The 10x1 vector \( Y \) of the nondimensional basic variables is

\[
Y = \{u, v, w, \chi, \psi, n_x, n_{xy}, q_x, m_x, m_{xy}\} \quad (2-53)
\]

and the nondimensional boundary conditions

\[
B(Y)_{x=0} = b_0(\lambda) \quad B(Y)_{x=L/R} = b_N(\lambda) \quad (2-54)
\]

where \( B(Y) \) are 5x10 matrices. The right hand side vectors \( b_0 \) and \( b_N \) depend linearly on the axial load factor \( \lambda \), since only the axial compression case is considered in this study. The full definition of the nondimensional governing partial differential equations and boundary conditions are given in Appendix A.3.
Nondimensional Governing Equations
The governing equations for sandwich cylindrical shell under axial compression with initial imperfections have been derived in the previous chapter. The main goal of analysis of the stability behavior of imperfect sandwich cylindrical shells is to determine the critical loads, i.e. bifurcation buckling and limit loads under variation of shell geometry, material property, boundary conditions, amplitude and shape of imperfections as well.

Koiter in [32] derived a buckling analysis of cylindrical shells with axisymmetric imperfections within the framework of bifurcation analysis. This kind of analysis will be used in the present study to investigate the effect of axisymmetric imperfection on buckling of sandwich cylindrical shells. The second analysis of Koiter in [33] is more general and will be used to investigate the effect of imperfections of asymmetric shape on the shell buckling behavior. This well-known initial postbuckling theory of Koiter has been used by many investigators [6] in the past years. Numerous results of application of this theory to various kinds of shell stability problems have been published. Most of the results, however, dealt with shell structures made of conventional materials, i.e. isotropic or anisotropic material. This theory will be derived in the form which is suitable for sandwich cylindrical shells. Both bifurcation and initial postbuckling analysis described here are based on perturbation methods.
3.1. Bifurcation Analysis

As it has already been stated, the effect of small initial imperfections can be investigated within the framework of bifurcation analysis. In contrast to the general Koiter's theory [33] which is valid only in an asymptotic sense, this method is valid for finite imperfections.

Because both the shell itself and the applied load are axisymmetric, shell configurations along the equilibrium path also are axisymmetric. The prebuckling path of axisymmetric deformation can be shown to be of the general form of the path displayed in figure 3-1 for a cylinder with axisymmetric imperfections. It is shown from the figure that, under increasing load the displacement amplitude of the axisymmetric form will grow non-linearly [19] until the bifurcation point is reached, where there is an intersection between the equilibrium configuration on the primary and secondary path. The objective of the analysis in this section is to locate the first bifurcation point along the (axisymmetric) imperfect equilibrium path by use of the so-called adjacent equilibrium criterion.

The work of Tennyson in [55] inspired the author to extend the analysis with the use of a nonlinear prebuckling analysis and satisfying rigorously the boundary conditions at edges of cylinder. This offers the possibility to study the effect of various boundary conditions on the buckling behavior of sandwich cylinder under axial compression.

![Figure 3-1. Bifurcation from prebuckling path of perfect or axisymmetric imperfect shells](image)

3.1.1. Adjacent-Equilibrium Criterion

The set of nondimensional partial differential equations governing the boundary value problem in equations (2-52) and (2-54) are rewritten as follows
\[
D_{\bar{x}} = f^D(Y, \bar{x}, \theta, \bar{w}) \quad (3-1)
\]
\[
F_{\bar{x}} = f^F(Y, \bar{x}, \theta, \bar{w}) \quad (3-2)
\]
\[
B(Y)_{\bar{x} = 0} = b_0(\lambda) \quad B(Y)_{\bar{x} = L/R} = b_N(\lambda) \quad (3-3)
\]

where the vector \( Y \) of the nondimensional basic stress and displacement variables is as follows

\[
Y^T = [D \ F] \quad (3-4)
\]
\[
D = [u, v, w, \chi, \psi]^T \quad F = [n_x, n_{xy}, q_x, m_x, m_{xy}]^T \quad (3-5)
\]

and \( \bar{w} \) denotes the given axisymmetric initial imperfection.

From figure 3-1 it is seen that the primary path can intersect another path (secondary path) at a certain point along its solution path. This intersection point is called a bifurcation point and the corresponding load \( \lambda_c \) is known as the bifurcation buckling load.

The so-called adjacent equilibrium criterion can be used to derive the governing equations for this particular point. Once an equilibrium configuration has been obtained on the primary path, the existence of an adjacent equilibrium configuration under the same applied load is examined. Linearized equations derived from the nonlinear one by means of perturbation technique are used to determine the existence of the adjacent equilibrium configuration. Thus, to investigate the possible existence of adjacent equilibrium configurations, we give a small perturbation \( Y^{(1)}(\lambda) \) to the equilibrium configuration on the primary path \( Y^{(0)}(\lambda) \) as follows

\[
Y \rightarrow Y^{(0)} + Y^{(1)} \quad (3-6)
\]

Adjacent equilibrium configurations exist if these expressions satisfy the nonlinear equilibrium equations and a non-trivial solution of a small perturbation can be found. The lowest load, at which this criterion is satisfied, is called the critical buckling load.

Introducing the expressions (3-6) into the governing equations (3-1) to (3-3) yields a set of ordinary and partial differential equations for the prebuckling and buckling state, respectively. They can be written in vector notations as follows

**Axisymmetric prebuckling state:**

\[
D_{\bar{x}}^{(0)} = f_0^D(Y^{(0)}, \bar{x}, \bar{w}, \lambda) \quad (3-7)
\]
Bifurcation Analysis

\begin{align}
\mathbf{F}^{(0)}_{\mathbf{x}} &= \mathbf{f}^{F}_{0}(\mathbf{Y}^{(0)}, \mathbf{x}, \mathbf{w}, \lambda) \\
\mathbf{B}(\mathbf{Y}^{(0)})_{\mathbf{x} = 0} &= \mathbf{b}_{0}(\lambda) \quad \mathbf{B}(\mathbf{Y}^{(0)})_{\mathbf{x} = L/R} = \mathbf{b}_{N}(\lambda)
\end{align}

(3-8) \quad (3-9)

where \( \mathbf{Y}^{(0)} = \{ w^{(0)}, \chi^{(0)}, q^{(0)}, m^{(0)} \}^{T} \) and \( n^{(0)}_{x} = -\lambda \)

Buckling state:

\begin{align}
\mathbf{D}^{(1)}_{\mathbf{x}} &= \mathbf{f}^{D}_{1}(\mathbf{Y}^{(1)}, \mathbf{Y}^{(1)}, \mathbf{x}, \mathbf{\theta}, \mathbf{\bar{w}}) \\
\mathbf{F}^{(1)}_{\mathbf{x}} &= \mathbf{f}^{F}_{1}(\mathbf{Y}^{(1)}, \mathbf{Y}^{(1)}, \mathbf{x}, \mathbf{\theta}, \mathbf{\bar{w}}) \\
\mathbf{B}(\mathbf{Y}^{(1)})_{\mathbf{x} = 0} &= 0 \quad \mathbf{B}(\mathbf{Y}^{(1)})_{\mathbf{x} = L/R} = 0
\end{align}

(3-10) \quad (3-11) \quad (3-12)

As it was already stated, the prebuckling state is axisymmetric which implies that all derivatives of the unknowns of the prebuckling state with respect to the circumferential coordinate are equal to zero. The complete definition of equations (3-7) to (3-9) and equations (3-10) to (3-12) governing the prebuckling state and buckling state, respectively, with their appropriate boundary conditions is given in Appendix A.4.

3.1.2. Solution of the Prebuckling Problem

It is obviously not possible to obtain an analytical solution of equations (3-7) to (3-9). Therefore these equations will be solved numerically. A finite difference formulation is introduced to reduce the obtained prebuckling ordinary differential equations into linear algebraic equations. For a given value of \( \lambda \), the solution of the prebuckling problem and the calculation of the derivative of the prebuckling problem with respect to the normalized axial compression load factor \( \lambda \) can be computed directly by the solution of algebraic matrix equations. It should be mentioned that the solution of the derivative of the prebuckling problem is needed for solving the buckling problem in the next section.

Finite difference discretization

After substitution of the axisymmetric prebuckling solution, see in Appendix A.4, the ordinary differential prebuckling equations can be written in vector notations as follows

\begin{align}
\mathbf{D}^{0}_{\mathbf{x}} + \mathbf{f}^{D}(\mathbf{Y}^{0}, \mathbf{Y}^{0}, \mathbf{w}, \mathbf{\bar{w}}, \lambda) &= 0 \\
\mathbf{F}^{0}_{\mathbf{x}} + \mathbf{f}^{F}(\mathbf{Y}^{0}, \mathbf{Y}^{0}, \mathbf{\bar{w}}, \mathbf{\bar{w}}, \lambda) &= 0
\end{align}

(3-13) \quad (3-14)
\[ B(Y_i^0)_{x=0} = b_0(\lambda) \quad B(Y_i^0)_{x=L/R} = b_N(\lambda) \]  

(3-15)

The vector functions \( f_D \) and \( f_F \) used in equations (3-13) and (3-14) and the boundary conditions in equation (3-15) are defined in Appendix A.4.

The first order ordinary differential equations in equations (3-13) and (3-14) will be written in finite difference expressions by using an \( O(h_i^2) \) accurate central finite difference formulation. The subdivision of the \( x \)-axis into finite difference grid points is shown in figure 3-2 where \( h_i \) denotes the distance between two successive gridpoints.

---

**Figure 3-2. Finite difference grid with uniform spacing**

The first order differential and function values of the prebuckling unknowns at a mid-point between two successive gridpoints can be written in terms of unknowns at the gridpoints as follows

\[ Y_i^0 \left|_{x=i+\frac{1}{2}} = \frac{1}{2}(Y_{i+1}^0 + Y_i^0) + O(h_i^2) \right. \]  

(3-16)

\[ Y_i^0 \left|_{x=i+\frac{1}{2}} = \frac{1}{h_i}(Y_{i+1}^0 - Y_i^0) + O(h_i^2) \right. \]  

(3-17)

Substitution of the expression in equations (3-16) and (3-17) into the prebuckling equations in (3-13) and (3-14), neglecting the terms of order \( O(h_i^2) \) and higher, yields the following finite difference formulation for the prebuckling solution

\[ -D_i^{00} + D_i^{01} f^D(\frac{Y_i^0}{i+\frac{1}{2}}, \frac{Y_i^0}{i+\frac{1}{2}}, \frac{\overline{w}_i}{i+\frac{1}{2}}, \frac{\overline{w}_i}{i+\frac{1}{2}}, \lambda) = f^D_{i+1}(Y_{i+1}^0, Y_{i+1}^0, \overline{w}_{i+1}, \overline{w}_{i+1}, \lambda) = 0 \]

\[ -F_i^{00} + F_i^{01} f^F(\frac{Y_i^0}{i+\frac{1}{2}}, \frac{Y_i^0}{i+\frac{1}{2}}, \frac{\overline{w}_i}{i+\frac{1}{2}}, \frac{\overline{w}_i}{i+\frac{1}{2}}, \lambda) = f^F_{i+1}(Y_{i+1}^0, Y_{i+1}^0, \overline{w}_{i+1}, \overline{w}_{i+1}, \lambda) = 0 \]

for \( i = 0, ..., N-1 \)  

(3-18)

and differentiation with respect to the load factor \( \lambda \) gives the finite difference formulation for the derivative of the prebuckling solution.
Bifurcation Analysis

\[
\begin{align*}
\mathbf{f}^D_{i + \frac{1}{2}}(\mathbf{Y}_i^0, \mathbf{Y}_{i+1}^0, \lambda) + \mathbf{f}^D_{i + \frac{1}{2}}(\mathbf{Y}_i^0, \mathbf{Y}_{i+1}^0, \overline{w}_i, \overline{w}_{i+1}) &= 0 \\
\mathbf{f}^F_{i + \frac{1}{2}}(\mathbf{Y}_i^0, \mathbf{Y}_{i+1}^0, \lambda) + \mathbf{f}^F_{i + \frac{1}{2}}(\mathbf{Y}_i^0, \mathbf{Y}_{i+1}^0, \overline{w}_i, \overline{w}_{i+1}) &= 0
\end{align*}
\]

for \( i = 0, \ldots, N - 1 \) (3-19)

where \( (\cdot)' = \frac{d(\cdot)}{d\lambda} \) denotes the differentiation with respect to load factor.

This two set of linear algebraic equations can be written in block-tridiagonal form as follows

\[
\begin{bmatrix}
\mathbf{D}_0 & \mathbf{U}_0 & & & \\
\mathbf{L}_1 & \mathbf{D}_1 & \mathbf{U}_1 & & \\
& \ddots & \ddots & \ddots & \\
& & \mathbf{L}_{N-1} & \mathbf{D}_{N-1} & \mathbf{U}_{N-1} \\
& & & \mathbf{L}_N & \mathbf{D}_N
\end{bmatrix}
\begin{bmatrix}
\mathbf{Y}_0 \\
\mathbf{Y}_1 \\
\vdots \\
\mathbf{Y}_{N-1} \\
\mathbf{Y}_N
\end{bmatrix}
= \begin{bmatrix}
\mathbf{f}_0 \\
\mathbf{f}_1 \\
\vdots \\
\mathbf{f}_{N-1} \\
\mathbf{f}_N
\end{bmatrix}
\]

where for the prebuckling solution one has

\[
\mathbf{D}_0 = \begin{bmatrix}
\mathbf{B}_0^0 \\
\mathbf{f}_{i(0)}^F \\
\mathbf{f}_{\frac{1}{2}(0)}^D
\end{bmatrix}, \quad \mathbf{D}_N = \begin{bmatrix}
\mathbf{f}_{N-\frac{1}{2}(i=N)}^D \\
\mathbf{B}^0_N \\
0
\end{bmatrix}, \quad \mathbf{L}_i = \begin{bmatrix}
\mathbf{f}_{i-\frac{1}{2}(i)}^D \\
0
\end{bmatrix} \text{ for } i = 1, \ldots, N
\]

\[
\mathbf{D}_i = \begin{bmatrix}
\mathbf{f}_{i-\frac{1}{2}(i+1)}^D \\
\mathbf{f}_{i+\frac{1}{2}(i)}^F
\end{bmatrix} \text{ for } i = 1, \ldots, N - 1; \quad \mathbf{U}_i = \begin{bmatrix}
0 \\
\mathbf{f}_{i+\frac{1}{2}(i+1)}^F
\end{bmatrix} \text{ for } i = 0, \ldots, N - 1
\]

\[
\mathbf{Y}_i = \mathbf{Y}_i^0, \quad \mathbf{f}_i = \mathbf{0} \quad \text{(3-21)}
\]

and for the derivative of the prebuckling solution with respect to \( \lambda \) one obtains

\[
\mathbf{Y}_i = \mathbf{Y}_i^0
\]

\[
\mathbf{f}_0 = \begin{bmatrix}
\mathbf{0} \\
-\mathbf{f}_{\frac{1}{2}}^F \\
-\mathbf{f}_{\frac{1}{2}}^D
\end{bmatrix}, \quad \mathbf{f}_i = \begin{bmatrix}
-\mathbf{f}_{i-\frac{1}{2}}^D \\
\mathbf{0} \\
-\mathbf{f}_{i+\frac{1}{2}}^F
\end{bmatrix} \text{ for } i = 1, \ldots, N - 1; \quad \mathbf{f}_N = \begin{bmatrix}
-\mathbf{f}_{N-\frac{1}{2}}^D \\
\mathbf{0}
\end{bmatrix} \quad \text{(3-22)}
\]
where the sub-matrices $L_i$, $D_i$ and $U_i$ are the same as given in equation (3-21) for the prebuckling solution. This two algebraic matrix equations can be solved by using the parallel Potter's method discussed in Appendix C.1.

### 3.1.3. Solution of the Buckling Problem

The partial differential equations for the buckling state in equations (3-10) and (3-12) are reduced to a set of ordinary differential equations in the axial coordinate by introducing the following Fourier decompositions for the buckling state of the non-dimensional basic variables

$$\Phi^{(1)}(\bar{x}, \theta) = \Phi_n^{(1)}(\bar{x}) \cos(n\theta) \quad \Psi^{(1)}(\bar{x}, \theta) = \Psi_n^{(1)}(\bar{x}) \sin(n\theta)$$  \hspace{1cm} (3-23)

where

$$\Phi_n^{(1)}(\bar{x}, \theta) = \{ n_{xy}^{(1)}, q_x^{(1)}, m_x^{(1)}, u^{(1)}, w^{(1)}, \chi^{(1)} \}$$  \hspace{1cm} (3-24)

$$\Psi_n^{(1)}(\bar{x}, \theta) = \{ n_{xy}^{(1)}, m_{xy}^{(1)}, v^{(1)}, \psi^{(1)} \}$$  \hspace{1cm} (3-25)

Note that $n$ is the number of circumferential full waves and the basic variables $\Phi_n^{(1)}(\bar{x})$ and $\Psi_n^{(1)}(\bar{x})$ are functions of the axial coordinate only.

Substitution of the axisymmetric prebuckling solution and Fourier series for the buckling state of the basic variables from equation (3-23) into the partial differential equations (3-10) to (3-12) of the buckling state, regrouping and equating coefficients of like trigonometric terms results in a set of 10 ordinary differential equations which can be represented in vector notations as follows

$$D^1_{,x} + \hat{f}^D(Y^1, Y^1_{,x}) = \hat{f}^{D*}(Y^0, Y^0_{,x}, Y^1_{,x}, Y^1_{,x}, \bar{w}, \bar{x}, \lambda)$$  \hspace{1cm} (3-26)

$$F^1_{,x} + \hat{f}^F(Y^1, Y^1_{,x}) = \hat{f}^{F*}(Y^0, Y^0_{,x}, Y^1_{,x}, Y^1_{,x}, \bar{w}, \bar{x}, \lambda)$$  \hspace{1cm} (3-27)

$$B(Y^1)_{\bar{x}=0} = 0 \quad B(Y^1)_{\bar{x}=1/R} = 0$$  \hspace{1cm} (3-28)

Note that the coefficients of the buckling state function $\hat{f}^{D*}$, $\hat{f}^{F*}$ in equations (3-26) and (3-27) also depend on the solution of the prebuckling state unknowns $Y^0$. The vector functions $\hat{f}^D$, $\hat{f}^F$, $\hat{f}^{D*}$, $\hat{f}^{F*}$ and the appropriate boundary conditions are defined in Appendix A.4.

**Linearized buckling equations**

Because of the nonlinear dependence of the prebuckling state on the load factor $\lambda$, in general, it is necessary to approach the critical eigenvalue (for a given number of circumferential full waves $n$) by the solution of a sequence of linearized eigenvalue problems. These equations are obtained by restricting the search for eigen-
values to a sufficiently small neighborhood of $\lambda = \lambda_k$ so that in this neighborhood the prebuckling solution has a linear dependence on $\lambda$. This linearization of prebuckling state is illustrated in figure 3.1. Once the calculation of the prebuckling solution at load level $\lambda = \lambda_k$ and its derivative with respect to the load factor has been established, see in section 3.1.2, the buckling load $\lambda_e$ can be estimated by linearization of prebuckling solutions along the last known point $\lambda = \lambda_k$ as follows

$$Y^0(\lambda_e) = Y^0(\lambda_k) + \Delta \lambda_e Y^0(\lambda_e)$$  \hspace{1cm} (3-29)

where $\Delta \lambda_e = (\lambda_c - \lambda_k)$. Substituting this expression into equations (3-26) and (3-27) of the buckling state yields the following linearized eigenvalue problem

$$D^i_x + \hat{f}^D(Y^i) - \hat{f}^{D*}(Y^0, Y^1, \bar{w}, \lambda_e) = \Delta \lambda_c \hat{f}^{D*}(Y^0, Y^1)$$  \hspace{1cm} (3-30)

$$F^i_x + \hat{f}^F(Y^i) - \hat{f}^{F*}(Y^0, Y^1, \bar{w}, \lambda_e) = \Delta \lambda_c \hat{f}^{F*}(Y^0, Y^1)$$  \hspace{1cm} (3-31)

For the sake of clarity, the notations $Y^0_x$, $Y^1_x$ and $Y^0_\lambda$, $Y^1_\lambda$ in vector functions $f^D$, $f^F$, etc., denoting the derivative of the prebuckling and buckling unknowns with respect to the axial coordinate $x$ are dropped out. The system of differential equations (3-30) and (3-31) are linear homogeneous ordinary differential equations with variable coefficients containing the terms of prebuckling solution at the same load level.

Next, introducing an eigenvalue shift $\mu_0$ such that $\Delta \lambda_e = \mu_0 + \mu$ in equations (3-30) and (3-31), the previous eigenvalue problem can be reformulated as

$$D^i_x + \hat{f}^D(Y^i) - \hat{f}^{D*}(Y^0, Y^1, \bar{w}, \lambda_e) - \mu_0 \hat{f}^{D*}(Y^0, Y^1) = \mu \hat{f}^{D*}(Y^0, Y^1)$$  \hspace{1cm} (3-32)

$$F^i_x + \hat{f}^F(Y^i) - \hat{f}^{F*}(Y^0, Y^1, \bar{w}, \lambda_e) - \mu_0 \hat{f}^{F*}(Y^0, Y^1) = \mu \hat{f}^{F*}(Y^0, Y^1)$$  \hspace{1cm} (3-33)

The eigenvalue problem formulated in equations (3-32) and (3-33) with eigenvalue $\mu$ has the same eigenvectors as the original eigenvalue problem formulated in equations (3-30) and (3-31). These ordinary differential equations can be reduced to an algebraic eigenvalue problem by the same finite difference scheme used earlier in section 3.1.2. The eigenvalue problem in equations (3-32) and (3-33) is then solved numerically using the standard inverse iteration method [11] to compute the lowest eigenvalue $\mu$ and its corresponding eigenmode which is the buckling mode. For a given eigenvalue shift $\mu_0$, a first estimate for the critical value of loading parameter $\lambda_e$ can be directly computed. If this approximation is not satisfactory, a new estimate should be constructed relative to a point on the prebuckling path which is close to the critical point. Such a point can be computed using the following solution
\[ Y^{(0)}(\lambda_{k+1}) = Y^{(0)}(\lambda_k) + \varphi \Delta \lambda_e Y^{(0)}(\lambda_e) \] (3-34)

where \( \varphi \) is a positive real number less than one. This makes it possible to keep the stepsize small and to ensure that the critical point is approached from below. After the new point on the primary path has been obtained, the process for computing the critical eigenvalue can be repeated.

3.2. Initial Postbuckling Analysis

The differential equations for asymmetric imperfections are easily formulated, but they are inherently nonlinear. The initial postbuckling theory presented by Koiter [33] is applicable to any small imperfections, and it avoids the need for the direct solution of the nonlinear equations.

![Figure 3-3. Equilibrium paths for perfect and imperfect cylindrical shell](image)

Koiter has shown that the imperfection sensitivity of shell structures is closely related to their initial postbuckling behavior. This theory is exact in the asymptotic sense, i.e., exact at the bifurcation point itself and a close approximation for postbuckling configurations near the bifurcation point.

In general, buckling may occur at a limit point for asymmetric imperfections, as illustrated in figure 3-3. The influence of imperfections depends on the shape of the postbuckling equilibrium path for the corresponding "perfect" shell. The equilibrium path of the perfect shell represents the upper bound to which the equilibrium paths of imperfect shells with decreasing imperfection converge. The central problem appears now to be the behavior of the perfect shell. For thin shells, the bifurcation point that occurs before the limit point of the prebuckling path is unstable.
This means that the shell with imperfection will buckle at a load below the value determined by the bifurcation point.

To investigate the behavior of perfect sandwich shells the imperfection terms \( \bar{w} \) are set equal to zero in the governing equations of sandwich cylindrical shells with axisymmetric imperfection, i.e., equations (3-1) to (3-3) in section 3.1.1. Setting \( \bar{w} = 0 \) and substituting an assumed perturbation expansion for the basic stress and displacement variables, which is valid in the neighborhood of the bifurcation point, one obtains a set of ordinary differential equations for the axisymmetric prebuckling state and partial differential equations for the buckling and postbuckling states of perfect shells.

For the solution of the prebuckling and buckling states, the numerical approaches of this states with an axisymmetric imperfection, as given in section 3.1.2 and section 3.1.3, can be used directly by setting imperfection terms \( \bar{w} \) equal to zero. In the following section 3.2.1 the solution of the postbuckling problem is given. Finally, after the solution of the postbuckling problem has been obtained, the postbuckling coefficients \((a, b)\), the imperfection form factors \((\alpha, \beta)\) and the limit load \(\lambda_s\) will be evaluated in section 3.2.2.

3.2.1. Prebuckling, Buckling and Postbuckling Equations

The nondimensional governing equations for perfect sandwich shells are obtained by substituting \( \bar{w} = 0 \) in equations (3-1) to (3-3). The resulting first order partial differential equations in the axial coordinate are

\[
\begin{align*}
D_{\bar{x}} &= \mathbf{f}^D(Y, \bar{x}, \theta) \quad (3-35) \\
F_{\bar{x}} &= \mathbf{f}^F(Y, \bar{x}, \theta) \quad (3-36) \\
\mathbf{B}(Y)_{\bar{x} = 0} &= b_0(\lambda) \\
\mathbf{B}(Y)_{\bar{x} = L/R} &= b_N(\lambda) \quad (3-37)
\end{align*}
\]

where the vector \( Y \) of the nondimensional basic stress and displacement variables is given in equations (3-4) and (3-5). Again, the complete definition of these equations and the appropriate nondimensional boundary conditions is given in Appendix A.3.

An asymptotic perturbation expansion of the solution, valid in the neighborhood of a bifurcation point is introduced as follows

\[
Y = Y^{(0)} + \xi Y^{(1)} + \xi^2 Y^{(2)} + ... 
\quad (3-38)
\]

where the vector \( Y \) of the basic stress and displacement variables are defined in equations (3-4) and (3-5) and \( \xi \) is the perturbation parameter.
A formal substitution of this perturbation expansion into the nonlinear governing equations in (3.35) and (3.36) and the boundary conditions in equation (3.37) generates a sequence of equations for the functions appearing in the expansions. Ordering by powers of $\xi$ yields the following set of ordinary and partial differential equations and the appropriate set of boundary conditions for the zero-order, first-order and second-order state.

**Zero-order (prebuckling) state** with coefficients of $\xi^0$

$$D_{\bar{x}}^{(0)} = f_0^D(Y^{(0)}, \bar{x})$$

$$F_{\bar{x}}^{(0)} = f_0^F(Y^{(0)}, \bar{x})$$

$$B(Y^{(0)})_{\bar{x}=0} = b_0(\lambda) \quad B(Y^{(0)})_{\bar{x}=L/R} = b_N(\lambda)$$

**First-order (buckling) state** with coefficients of $\xi$

$$D_{\bar{x}}^{(1)} = f_1^D(Y^{(0)}, Y^{(1)}, \bar{x}, \theta)$$

$$F_{\bar{x}}^{(1)} = f_1^F(Y^{(0)}, Y^{(1)}, \bar{x}, \theta)$$

$$B(Y^{(1)})_{\bar{x}=0} = 0 \quad B(Y^{(1)})_{\bar{x}=L/R} = 0$$

**Second-order (postbuckling) state** with coefficients of $\xi^2$

$$D_{\bar{x}}^{(2)} = f_2^D(Y^{(0)}, Y^{(1)}, Y^{(2)}, \bar{x}, \theta)$$

$$F_{\bar{x}}^{(2)} = f_2^F(Y^{(0)}, Y^{(1)}, Y^{(2)}, \bar{x}, \theta)$$

$$B(Y^{(2)})_{\bar{x}=0} = 0 \quad B(Y^{(2)})_{\bar{x}=L/R} = 0$$

The complete definition of equations (3.45) and (3.46) governing the postbuckling state and the appropriate boundary conditions of equation (3.47) is given in Appendix A.4.

Again, the prebuckling state is axisymmetric which yields that all derivatives of the prebuckling state unknowns with respect to the circumferential coordinate are equal to zero. The partial differential equations obtained for the buckling and postbuckling states above still depend on both the $\bar{x}$ and $\theta$ coordinates. For the buckling state, the dependence on the $\theta$-coordinate is eliminated in section 3.1.3 by introducing Fourier decompositions for the basic variables in such a way that the buckling partial differential equations are solved exactly in the $\theta$-direction.
In the same manner, the partial differential equations of the postbuckling state can be solved exactly in the $\theta$-direction by introducing the following Fourier series for the basic variables of the postbuckling state

$$
\Phi^{(2)}(\tilde{x}, \theta) = \Phi_0^{(2)}(\tilde{x}) + \Phi_n^{(2)}(\tilde{x}) \cos(2n\theta) \quad (3-48)
$$

$$
\Psi^{(2)}(\tilde{x}, \theta) = \Psi_0^{(2)}(\tilde{x}) + \Psi_n^{(2)}(\tilde{x}) \sin(2n\theta) \quad (3-49)
$$

where

$$
\Phi^{(2)}(\tilde{x}, \theta) = \{ n_x^{(2)}, q_x^{(2)}, m_x^{(2)}, u^{(2)}, w^{(2)}, \chi^{(2)} \} \quad (3-50)
$$

$$
\Psi^{(2)}(\tilde{x}, \theta) = \{ n_{xy}^{(2)}, m_{xy}^{(2)}, v^{(2)}, \psi^{(2)} \} \quad (3-51)
$$

Notice that $n$ is the number of circumferential full waves and the basic stress and displacement variables $\Phi_0^{(2)}(\tilde{x}), \Phi_n^{(2)}(\tilde{x}), \Psi_0^{(2)}(\tilde{x})$ and $\Psi_n^{(2)}(\tilde{x})$ are functions of the axial coordinate only.

Substitution of the axisymmetric prebuckling solution and the Fourier series for the basic variables of the buckling state as given in section 3.1.2 and section 3.1.3, respectively, and the Fourier series for the basic variables in the postbuckling state from equations (3-48) and (3-49) into the partial differential equations (3-45) to (3-47) of the second-order state, regrouping and equating coefficients of like trigonometric terms results in the following set of 20 ordinary differential equations of the postbuckling state. These equations are lengthy and they can be represented in vector notation as follows

$$
D^{(2)}_{\tilde{x}} x + f^{D}(Y, Y, Y^{(2)}, Y^{(2)})_{\tilde{x}} = f^{D\theta}(Y, Y, Y, Y) \quad (3-52)
$$

$$
F^{(2)}_{\tilde{x}} x + f^{F}(Y, Y, Y^{(2)}, Y^{(2)})_{\tilde{x}} = f^{F\theta}(Y, Y, Y, Y) \quad (3-53)
$$

$$
B(Y^{(2)})_{\tilde{x} = 0} = 0 \quad B(Y^{(2)})_{\tilde{x} = L/R} = 0 \quad (3-54)
$$

where

$$
D^{(2)} = \{ u, v, w, \chi, \psi, u, v, w, \chi, \psi \}^T
$$

$$
F^{(2)} = \{ n_x, n_{xy}, q_x, m_x, n_{xy}, n_x, n_{xy}, m_x, m_{xy} \}^T
$$

and the vector functions $f^{D}, f^{D\theta}, f^{F}$ and $f^{F\theta}$ are defined in Appendix A.4.

The equations (3-52) and (3-53) form a set of 20 first-order linear inhomogeneous ordinary differential equations with variable coefficients. These coefficients are depending on the prebuckling solutions and the right-hand side terms of the equa-
tions are depending only on the buckling modes. They form together with the appropriate boundary conditions a response problem that can be solved numerically. These ordinary differential equations are reduced to algebraic vector equations using the same finite difference scheme used earlier in section 3.1.2. Once the calculation of the prebuckling solution and buckling solution at bifurcation load level $\lambda = \lambda_c$ has been calculated, the postbuckling solution at this load level can also be solved.

The solution of the prebuckling, buckling and postbuckling state is needed for the calculation of postbuckling coefficients $(a, b)$ and imperfection form factors $(\alpha, \beta)$ discussed in the next section. Note that the imperfection sensitivity study of sandwich shells presented here only considers a single-mode of imperfection.

### 3.2.2. Postbuckling Coefficients and Imperfection Form Factors

The imperfection sensitivity of shell structures is closely related to their initial postbuckling behavior. For perfect shell, one is interested in the variation of $\Lambda(\xi)$ with $\xi$ in the vicinity of the bifurcation point $\Lambda = \Lambda_c$, where $\Lambda$ is the loading parameter and $\xi$ is the amplitude of the buckling mode normalized with the thickness parameter $h$. If the shell structure possesses a unique buckling mode associated with the lowest buckling load, the following asymptotic expansion for its buckling and postbuckling behavior is valid near the bifurcation point $\Lambda_c$,

$$\frac{\Lambda}{\Lambda_c} = 1 + a\xi + b\xi^2 + \ldots \quad (3-55)$$

Three cases can be distinguished for their buckling and postbuckling behavior. In case 1 (unstable skew symmetric) where $a = 0$, then postbuckling equilibrium path is approximately an inclined straight line, as illustrated in figure 3-4(a). In case 2 (unstable symmetric) where $a = 0$ and $b < 0$, then the path has the parabolic shape shown in figure 3-4(b). In case 3 (stable symmetric) where $a = 0$ and $b > 0$, then the path has the parabolic shape shown in figure 3-4(c).

As can be seen from those figures the shape of the postbuckling equilibrium path plays a central role in determining the influence of the initial imperfections. Cases 1 and 2 characterize structures that are sensitive to initial imperfections whereas case 3 depicts the postbuckling behavior of structures that are insensitive to initial imperfections.

Thus, in order to describe the behavior of imperfect shell structures, one is interested in the variation of $\Lambda(\xi, \tilde{\xi})$ with $\xi$ for an imperfect shell $(\tilde{\xi} \neq 0)$ in the vicinity of the bifurcation point $\Lambda = \Lambda_c$ given by the following asymptotic expansion (see also figure 3-5).
\[(\Lambda - \Lambda_c)\tilde{\xi} = a\Lambda_c\tilde{\xi} + b\Lambda_c\tilde{\xi}^2 + \ldots - \alpha\Lambda_c\tilde{\xi} - \beta(\Lambda - \Lambda_c)\tilde{\xi}^2 + O(\tilde{\xi}^3) \quad (3-56)\]

so that the expressions used for "perfect" shells in equation (3-55) are still useful. Here, a small initial stress free imperfection \(\bar{W} = \bar{\xi}\bar{W}\) will be assumed, where \(\bar{W}\) represents the shape of the initial imperfection and \(\bar{\xi}\) is the imperfection amplitude. Notice that if the initial imperfection is assumed to be affine to the buckling mode then \(\bar{W} = W^{(1)}\).

Knowing only whether the structure is imperfection-sensitive or not is not enough. The ultimate aim of imperfection sensitivity analysis is to determine the maximum load carrying capacity. As can be seen from figure 3-5, if the limit point is close enough to the bifurcation point, the relation between the limit load \(\Lambda_s\) of the imperfect structure and the bifurcation load \(\Lambda_c\) of the perfect structure can be found from equation (3-56) by determining the maximum of \(\Lambda(\bar{\xi})\).

It is known that for many practical applications where a unique buckling mode is associated with the lowest buckling load and the initial postbuckling behavior is symmetric. The first postbuckling coefficient \(a\) is identically equal to zero. As it has been discussed before, in that case the sign of \(b\) (cases 2 or 3) determines whether the structure can still carry load after bifurcation. For \(a = 0\), one obtains after some algebraic manipulation the following modified Koiter's formula

\[(1 - \rho_s)^{3/2} = (3/2)\sqrt{-3}\alpha^2 b[1 - (\beta/\alpha)(1 - \rho_s)]\|\bar{\xi}\| \quad (3-57)\]

where \(\rho_s = \Lambda_s/\Lambda_c = \lambda_s/\lambda_c\) since one considers only the axial compression load case.
Once the solution of the prebuckling, buckling and postbuckling problem for perfect shells has been obtained in sections 3.1.2 to 3.2.1, the initial postbuckling coefficients $a$, $b$ and the imperfection form factors $\alpha$, $\beta$ can also be calculated. The formulas for the postbuckling coefficients and for the first and second imperfection form factors for anisotropic cylindrical shells can be found in [6], while the derivation of those for sandwich cylindrical shells with isotropic faces and an orthotropic core is carried out in the same manner. The final expressions for the postbuckling coefficients $a$, $b$ and the imperfection form factors $\alpha$, $\beta$ are given in Appendix A.4.

Finally, with the help of equation (3-57) one can estimate the knockdown factor with which one has to multiply the buckling load prediction of the perfect shell in order to arrive at the safe allowable load level of the real (imperfect) structure. Such an estimate can be computed if besides the postbuckling coefficients and the imperfection form factors one also knows the size of the amplitude of the imperfection $\xi$. 

**Figure 3-5. Equilibrium paths for perfect and imperfect structures**
Initial Postbuckling Analysis
In this chapter numerical computations are performed to determine the collapse load of perfect and imperfect sandwich cylindrical shells using the newly developed computer program SFOSDT (Sandwich First Order Shear Deformation Theory). This program is written in FORTRAN 77. As mentioned before, one can distinguish the collapse analysis of shells by two types, namely, a bifurcation analysis for perfect shells and shells with axisymmetric imperfections, and a limit point analysis for shells with asymmetric imperfections. Only axial compression load is considered in this study.

First, in section 4.1 an investigation of the buckling behavior of perfect sandwich shells is carried out by performing a bifurcation analysis. The calculation of the critical buckling loads and the corresponding modes is performed and the results are compared to those obtained in [55] for the same problem. The effects of using different prebuckling states (membrane or nonlinear), core stiffness properties, shell geometries and boundary conditions are investigated.

Next, the influence of axisymmetric imperfections on the critical buckling loads of sandwich shells is investigated in section 4.2 via bifurcation analysis. Of interest is here the calculation of knockdown factors for sandwich cylindrical shells under various core shear stiffness configurations and different shapes of initial geometric imperfections. Two kinds of axisymmetric imperfection forms are studied, namely, trigonometric imperfections and imperfections which are assumed to be affine to the corresponding mode of the critical buckling load of perfect shells.

Finally, in section 4.3 the imperfection sensitivity of sandwich shells with asymmetric imperfections is studied by performing initial postbuckling analysis. Only imperfections which are assumed to be affine to the corresponding mode of the critical buckling load of perfect shells are considered. Under various core stiffness configurations, the postbuckling coefficient $b$, imperfection form factors $\alpha$ and $\beta$, and limit loads $\lambda_s$ are computed successively in order to present knockdown factor plots.
4.1. Perfect Sandwich Shell Behavior

In this section, we investigate the buckling behavior of sandwich shells with perfect isotropic faces loaded by axial compression. Unless otherwise stated, the geometries and material properties of the shell investigated are shown in Table 4-1 and using simply supported SS-3β boundary conditions at the edges of the shell, as given in Table A-1 in Appendix A.4. The shell under consideration consists of two identical isotropic faces. The nondimensional shear stiffness of the core $g_x$ and $g_y$ are chosen appropriately.

<table>
<thead>
<tr>
<th>Table 4-1. Sandwich shell geometries and material properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry or property</td>
</tr>
<tr>
<td>R (radius), inch</td>
</tr>
<tr>
<td>L (length), inch</td>
</tr>
<tr>
<td>$t^t$ (top face thickness), inch</td>
</tr>
<tr>
<td>$t^b$ (bottom face thickness), inch</td>
</tr>
<tr>
<td>$t^c$ (core thickness), inch</td>
</tr>
<tr>
<td>h (distance between faces), inch</td>
</tr>
<tr>
<td>$E_f$ (modulus of elasticity), psi</td>
</tr>
<tr>
<td>v (Poisson’s ratio)</td>
</tr>
</tbody>
</table>

4.1.1. Variation of Core Properties

Table 4-2 displays the results of computation of the critical buckling loads as function of core transverse shear stiffness coefficients $g_x$ and $g_y$ for an isotropic core $\phi = g_x / g_y = 1$ and orthotropic cores $\phi = 5$ and $\phi = 10$. Note that the eigenvalues in the table are listed in normalized form $\lambda = N_x^{crit} / N_x^{norm}$, where $N_x^{norm} = 2hE = -4193$ lbf/inch. The bracketed number denote the critical number of circumferential full waves $n$. The results of calculations are also displayed in Figure 4-1 specifically using different prebuckling solutions. The critical buckling loads for an isotropic core $\phi = 1$ and orthotropic cores $\phi = 5$ and $\phi = 10$ as functions of the core shear flexibility coefficients $\chi_x = 1 / g_x$ are shown.

Table 4-2 reveals that the critical buckling loads obtained using nonlinear prebuckling analysis are smaller than membrane prebuckling analysis counterparts. A significant reduction in the buckling strength is observed when the core shear stiffness $g_x$ decreases and the ratios $\phi = g_x / g_y$ increases.
For the same problem, although it is not presented here, the computed buckling load for membrane prebuckling case are slightly higher than those of [55]. The reason for this discrepancy is because the different of the bending stiffness of the faces. Tennyson in [55] neglects the faces own bending stiffness about their mid-surfaces. This assumption is valid only for sufficiently thin face sheets with respect to the core thickness. Therefore the faces can be treated as membranes which implies that the flexural stiffness about their mid-surfaces is zero. The present study does not impose this assumption which implies that the thickness of the faces is not necessarily much smaller than the thickness of the core. However, the ratio of radius to thickness of the total sandwich layers falls within the range of validity of thin shell assumptions.

**Table 4-2. The critical buckling loads for perfect sandwich cylinders with isotropic and orthotropic cores as function of core shear stiffness $g_x$.**

<table>
<thead>
<tr>
<th>$g_x$</th>
<th>$\chi_x$</th>
<th>$\phi = 1$</th>
<th>$\phi = 5$</th>
<th>$\phi = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Membrane</td>
<td>Nonlinear</td>
<td>Membrane</td>
</tr>
<tr>
<td>10$^9$</td>
<td>0</td>
<td>1.0108 (2)</td>
<td>0.8211 (4)</td>
<td>1.0107 (2)</td>
</tr>
<tr>
<td>5.00</td>
<td>0.2</td>
<td>0.9194 (2)</td>
<td>0.7489 (5)</td>
<td>0.8672 (6)</td>
</tr>
<tr>
<td>1.25</td>
<td>0.4</td>
<td>0.6020 (0)</td>
<td>0.5781 (5)</td>
<td>0.5727 (9)</td>
</tr>
<tr>
<td>1.00</td>
<td>1.0</td>
<td>0.5000 (0)</td>
<td>0.4978 (5)</td>
<td>0.5000 (0)</td>
</tr>
<tr>
<td>0.83</td>
<td>1.2</td>
<td>0.4167 (0)</td>
<td>0.4167 (0)</td>
<td>0.4167 (0)</td>
</tr>
<tr>
<td>0.50</td>
<td>2.0</td>
<td>0.2500 (0)</td>
<td>0.2500 (0)</td>
<td>0.2500 (0)</td>
</tr>
</tbody>
</table>

One can observe from the brackets in the table that by decreasing the core shear stiffness coefficients $g_x$, buckling is suddenly changed from asymmetric form to axisymmetric form ($n = 0$). The corresponding modes are characterized by axial wavy patterns with a short wavelength. We encounter this kind of mode especially when the core shear stiffness coefficient $g_x$ reaches the value around 1.0 or smaller which are typical configurations of moderately flexible or weak core, see in [55]. This type of instability mode of sandwich structures is also referred to some references as shear crimping mode, see for instance in [53, 55].

To understand the phenomenon of shear crimping, one must keep in mind that this instability mode is simply a limiting case of general instability. The equations for predicting shear crimping emerge from general instability theory when the analytical treatment extends into the region of low shear modulus for the core. In the fol-
Perfect Sandwich Shell Behavior

Lowing the theoretical derivation of [55] yields the result that when the two faces are of the same material, shear crimping will occur in axially compressed sandwich cylinders whenever

\[ \chi_x = \frac{1}{g_x} \geq 1 \]  

(4-1)

where

\[ \lambda_{sc} = \frac{1}{2\chi_x} = \frac{g_x}{2} \]  

(4-2)

For sandwich cylinders with orthotropic cores with \( \phi = 5 \) and \( \phi = 10 \), the plot in figure 4-1 shows that it is possible to have a buckling load lower than the shear crimping load. This phenomenon is also found by Tennyson in [55]. For a constant \( g_x \), the transverse shear stiffness of the core in circumferential direction \( g_y \) decreases when the ratio \( \phi \) increases. Thus, the core provides a low resistance against deformation in the circumferential direction which causes general buckling instability in an asymmetric mode prior to shear crimping instability. Returning to table 4-2 one can see that, before a sudden change to shear crimping modes (n = 0), the number of fullwaves \( n \) in the circumferential direction of general buckling mode increases with decreasing the shear stiffness \( g_y \) which indicates the shear flexibility of the core in this direction.

![Figure 4-1. Buckling load of sandwich cylinders as a function of the core shear flexibility (1/gx) for varying core shear stiffness ratio (ϕ)](image)

\( \chi_x = 1/g_x \)
However, for \( \chi_x \geq 1.4 \) the critical buckling loads for all different values of \( \phi \) converge to the same critical values. The corresponding mode is axisymmetric and characterized by axial wavy patterns which cover the entire cylinder length. As one can observe in table 4-1, there is no difference found between the critical buckling loads obtained with a membrane or nonlinear prebuckling analysis. This might be the reason that in some references [1, 55], this shear crimping mode is referred as a local instability mode. It depends only on the core property and not on the shell geometry.

Practically, sandwich shells with a weak core are never built. When the core shear stiffness coefficient is approaching zero, one can imagine this as an extreme case of two parallel faces with an empty space in between.

4.1.2. Effect of Different Length to Radius Ratios

In figure 4-2 the effect of length to radius ratios on the buckling strengths of sandwich shells is given. The critical buckling loads are plotted as a function of the core shear ratio \( \phi \) for varying core shear flexibility coefficients \( \chi_x = 1/g_x \) for two different lengths of shells, namely, a short length of shells \( (L/R = 1) \) and a moderately long one \( (L/R = 10) \). For both lengths the computations of the critical buckling load are performed using a nonlinear prebuckling analysis. Three different core shear ratios are used, namely, \( \phi = 1 \), \( \phi = 5 \) and \( \phi = 10 \).

![Figure 4-2](image)

**Figure 4-2.** Buckling load of sandwich cylinders as a function of the core shear flexibility \( 1/g_x \) for varying core shear stiffness ratio \( \phi \) using a different length to radius ratio \( L/R \) (nonlinear prebuckling)
Perfect Sandwich Shell Behavior

It can be seen from the figure that a significant difference appears in the critical buckling loads obtained with two different lengths for the values of $\chi_x$ between 0.0 and 0.6. The differences will reduce gradually when the core shear flexibility coefficient $\chi_x$ increases. The critical loads will converge to the same value for $\chi_x \geq 1.4$ and the corresponding instability modes will be characterized by shear crimping mode.

**Figure 4-3.** Prebuckling shapes of isotropic core sandwich shell with $g_x = 5.0$ (left) and $g_x = 1.25$ (right) for $L/R=10$ (only half length depicted)

Notice that the critical values obtained for $L/R = 10$ in figure 4-2 are closer to those for $L/R = 1$ with membrane prebuckling solutions, as depicted in figure 4-1, than the counterpart values for $L/R = 1$. This results can be justified when we look at the prebuckling shapes of sandwich shells with an isotropic core $\phi = 1$ and with $g_x = 5.0$ and $g_x = 1.25$ for $L/R = 10$ as displayed in figure 4-3. The edge

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effects, which obviously are present for the case with \( L/R = 1 \) as displayed in figure 4-4, are diminished and restricted only to the area close to the ends. Further away from the ends towards the midpoint of the cylinder, the prebuckling radial displacements behave more or less as a membrane.

Decreasing the core shear stiffness coefficients to \( g_x = 0.83 \) (\( \chi_x = 1.2 \)), the magnitude of prebuckling solutions start to grow and the shape is characterized by a large number of axial half waves in the area close to the ends. As mentioned before, the shell will buckle suddenly into shear crimping mode and both the critical loads for \( L/R = 10 \) and \( L/R = 1 \) using nonlinear prebuckling solutions approach the values obtained for the membrane prebuckling cases.

### 4.1.3. Effect of Different Boundary Conditions

It is generally known that the effect of boundary conditions is important for the buckling behavior of shell structures. The sandwich cylindrical shell investigated has an isotropic core with the shear stiffness coefficient \( g_x = g_y = 5.0 \) and it is a relatively short one with the length to radius \( L/R = 1 \) where one can expect significant effects of using different boundary conditions.

| Table 4-3. Effect of various boundary conditions on the buckling loads for sandwich cylinders with \( \phi = 1 \), \( g_x = 5.0 \) and \( L/R = 1 \) |
|---|---|---|---|---|---|
| B.C. | \( \lambda_c \) Membrane | \( \lambda_c \) Nonlinear | B.C. | \( \lambda_c \) Membrane | \( \lambda_c \) Nonlinear |
| SS1-\( \beta \) | 0.92054 (3,S) | 0.77971 (5,S) | C1-\( \beta \) | 0.99248 (4,A) | 0.88825 (5,S) |
| SS1-m | 0.91517 (3,S) | 0.77893 (5,S) | C1-m | 0.99247 (4,A) | 0.88824 (5,S) |
| SS2-\( \beta \) | 0.46530 (1,S) | 0.47454 (1,S) | C2-\( \beta \) | 0.93854 (4,A) | 0.88790 (5,S) |
| SS2-m | 0.46360 (1,S) | 0.47272 (1,S) | C2-m | 0.93855 (4,A) | 0.88790 (5,S) |
| SS3-\( \beta \) | 0.91936 (3,S) | 0.74888 (5,S) | C3-\( \beta \) | 0.96623 (4,S) | 0.86520 (5,S) |
| SS3-m | 0.91417 (3,S) | 0.74860 (5,S) | C3-m | 0.96616 (4,S) | 0.86520 (5,S) |
| SS4-\( \beta \) | 0.46517 (1,S) | 0.46742 (1,S) | C4-\( \beta \) | 0.92178 (3,A) | 0.86465 (5,S) |
| SS4-m | 0.46347 (1,S) | 0.46570 (1,S) | C4-m | 0.92114 (3,A) | 0.86464 (5,S) |

Note: (S) - \( w^0 \) symmetric and (A) - \( w^1 \) antisymmetric

Table 4-3 gives the results of computations of the critical buckling load using different boundary conditions. Both membrane and nonlinear prebuckling solutions are used. Numbers in the brackets denote the number of circumferential fullwaves
Shell with Axisymmetric Imperfections

n of the corresponding mode. Since the computations are carried out only with the half length of the shells, one must use symmetric conditions \( w^0 \) symmetric and \( w^1 \) symmetric or antisymmetric) at the middle of the shell length, as given in table A-1 in Appendix A.4. Notice that the letters in the brackets indicate whether the axial variation of the buckling mode is symmetric (S) or anti-symmetric (A) with respect to \( x/L = 0.5 \). The definition of the 16 different sets of boundary conditions used is given in table A-1 in Appendix A.4.

It can be seen from the table that the critical buckling load is strongly reduced for simply supported boundary conditions in which the circumferential displacement \( v \) is free (\( N_{xy} = 0 \)) at the shells edges. The buckling mode is characterized by a small number of circumferential full waves \( n = 1 \). This kind of results has also been obtained by Yamaki in [65] for cylinders with isotropic material. However, boundary conditions in which \( v \) is free never occur in practical applications. The effect of clamping \( \beta_x = 0 \) is clearly present, giving increases of the buckling load for both calculations using membrane and nonlinear prebuckling solutions.

In general the buckling behavior of sandwich cylinders under various boundary conditions are similar to those obtained in the case of conventional material cylinders, see for instance [65]. In facts, this result is not surprising since the modeling technique in this part of the thesis is an extension of first order shear deformation theory of thin shells applicable to sandwich configurations.

The boundary condition is treated in the same manner as for conventional (isotropic or anisotropic) material. This means that at the same section of the edges one cannot make a distinction between the boundary conditions for the individual layers, i.e., the faces and the core. The displacement and stress unknown variables of the faces and the core are attached to one global coordinate system at the mid-surface of the whole sandwich configuration. Therefore, at the edges, one can only prescribe the displacement or stress unknowns of the faces at the mid-surface of the sandwich layers. This form of boundary conditions are called the global boundary conditions of sandwich structures.

4.2. Shell with Axisymmetric Imperfections

Although there are numerous factors involved in the study of imperfection sensitivity such as different boundary conditions, uncertainties of geometry, loading, material property, etc., we will concentrate our present study of imperfection sensitivity of sandwich cylindrical shells only under variation of transverse shear stiffness \( g_x \) and \( g_y \) of the core.

From an engineering design viewpoint, in order to know the imperfection sensitivity behavior of sandwich cylindrical shells, it is very useful to plot a knockdown factor corresponding to the minimum buckling load \( \lambda_s \) as a function of the dimen-
sionless imperfection parameter. The knockdown factor is defined as \( \rho = \frac{\lambda_s}{\lambda_c} \) where \( \lambda_c \) is the perfect shells buckling load, which is calculated in the previous section 4.1. Moreover, the normalized imperfection parameter is defined as \( \bar{\mu} = \mu / h \) where \( h \) denotes for the distance between the mid-surface of the faces.

Two types of initial axisymmetric imperfections will be used for the investigation, namely, trigonometric axisymmetric imperfections and imperfections which are assumed to be affine to the buckling mode of the perfect shell. The trigonometric imperfection shape is given as follows

\[
\bar{w} = \bar{\mu} \cos \frac{i \pi x}{L}
\]  

(4-3)

where the number of imperfection half waves used is four \( (i = 4) \). The second imperfection shape considered is affine to the lowest buckling mode

\[
\bar{W} = \bar{\mu} w^1
\]

(4-4)

where \( w^1 \) is the normalized buckling mode of the perfect shells. In both cases, the values of the normalized amplitude of imperfection \( \bar{\mu} \) are increased incrementally from 0.0 to 1.0.

First, we investigate the imperfection sensitivity of the same cylindrical shells as previously used with the trigonometric (modal) initial axisymmetric imperfections as given in equation (4-3). This type of imperfection is also used by Tennyson in [55]. However, in his investigation, the number of imperfection half waves \( i = 2m \) is corresponding to the critical number of axial half waves \( m \) of the buckling mode obtained. In fact, he minimized the critical buckling loads with respect to the number of imperfection half waves in order to obtain the critical one.

In the present investigation with SPOSĐT, the imperfection half wave number is taken as a fixed value for varying values of the amplitude \( \bar{\mu} \) and no longer related to the critical buckling mode obtained. For a given imperfection half waves number, the buckling mode will converge automatically to that mode which belongs to the critical buckling load. Moreover, the shape of the mode can have an arbitrary form in the axial coordinate direction. Since a finite difference formulation is used, we have only to concern with the number of gridpoints used for the calculation to obtain a converged solution.

Figure 4-5 illustrates the imperfection sensitivity of sandwich shells with an isotropic core with modal half wave cosine axisymmetric imperfections using a membrane and nonlinear prebuckling solution. The left-hand side of figure 4-5 shows the modal cosine axisymmetric imperfection sensitivity for the case of using membrane prebuckling solutions. Three different core stiffness coefficients, namely, \( g_x = 1.25 \), \( g_x = 5.0 \) and a non-shear deformable core \( g_x = 10^9 \) are used in the study.
Note that the critical buckling loads of a non-shear deformable core can be also calculated using the so-called “equivalent” isotropic cylinder, having the same ratio of in-plane to bending stiffness (A/D) as the sandwich shell, see [55]. Hence, one can simply analyze the sandwich cylinder by using conventional isotropic material with the “equivalent” thickness computed from the relation mentioned.

In general it can be seen from the figure that increasing the imperfection amplitude $\bar{\mu}$ decreases the buckling strength. Increasing the value of $g_x$ from 1.25 to 5.0 the shell is more sensitive to the given imperfections. A qualitatively good agreement is found with the results of Tennyson.

On the contrary, when one switches the analysis to using nonlinear prebuckling solutions, the knockdown factor will decrease for lower core stiffness coefficients. This is illustrated in figure 4-5 (right) for the core stiffness coefficients $g_x = 1.25$, $g_x = 5.0$ and $g_x = 10^9$. As a consequence, for a given imperfection form, one cannot use the results of isotropic cylinder solution as discussed earlier as a lower bound curve of the solutions obtained, as it has been proposed by Tennyson in [55]. However, it can be seen that with the increase of the imperfection amplitude $\bar{\mu}$, the buckling strength will also decrease.

**Figure 4-5.** Modal cosine ($i = 4$) axisymmetric imperfection sensitivities of isotropic core sandwich cylinders using membrane prebuckling (left) and nonlinear prebuckling (right)

Figure 4-6 displays the imperfection sensitivity of sandwich shells with isotropic (left) and orthotropic (right) cores with axisymmetric imperfections, which are assumed to be affine to the (critical) buckling mode of perfect shells. The imperfection sensitivity of sandwich shells with an isotropic core $\phi = 1$ and three different core stiffness coefficients namely $g_x = 1.25$, $g_x = 5.0$ and a non shear deformable
core $g_x = 10^9$ reveals that by increasing the imperfection amplitude $\bar{u}$ the buckling strength decreases. Again, the knockdown factor will decrease for a lower core stiffness coefficient $g_x$, similar to those obtained with modal imperfections.

The right-hand side figure 4-6 gives the imperfection sensitivity for sandwich shells with $g_x = 5.0$ and three core shear stiffness ratios $\phi = 1$, $\phi = 5$ and $\phi = 10$. Again, here we may conclude that increasing the imperfection amplitude $\bar{u}$ and the core shear ratio $\phi$ decreases the buckling strength. This also indicates that a sandwich cylinder with orthotropic cores is more sensitive to initial imperfections.

From the cases considered, as one can observe by comparing figure 4-5 (right) and figure 4-6 (left), it is possible that shells with modal axisymmetric imperfections are more imperfection sensitive than shells with imperfections which are assumed to be affine to the buckling mode.

![Graphs showing imperfection sensitivity](image)

**Figure 4-6.** Axisymmetric imperfection sensitivity of isotropic core (left) and orthotropic core with $g_x = 5.0$ (right) for sandwich cylinders with imperfections which are affine to the buckling modes using nonlinear prebuckling solution

### 4.3. Shell with Asymmetric Imperfections

In this section, a numerical investigation is carried out to determine the collapse load of sandwich shells with initial asymmetric imperfections using the newly developed computer program SFOSDT which is based on the Koiter's initial postbuckling theory. First, the postbuckling coefficient $b$ and the imperfection form factors $\alpha$ and $\beta$ are computed respectively. By knowing the size of the amplitude $\bar{\xi}$ of the imperfection, the limit load can be computed with the use of equation (3-57) in chapter 3. All the following calculations are performed with nonlinear prebuckling solutions.
In the following, the imperfections sensitivity computations are limited to a few cases of interest, based on the assumption that the shapes of the initial imperfections are affine to the corresponding buckling modes and only the imperfection sensitivity of the lowest buckling load is calculated. Again, of interest is here to study the imperfection sensitivities only with the variation of the core shear stiffness coefficients \( g_x \) and \( g_y \).

First, the influence of core shear stiffness coefficients on the imperfection sensitivity of sandwich shells with an isotropic core \((\phi = g_x/g_y = 1)\) is studied. In figure 4-7 the computed postbuckling coefficients \( b \) for varying values of \( g_x \) are depicted. It is seen that the magnitude of the postbuckling coefficient \( b \) decreases gradually for the values of \( g_x \) from \( g_x = 10.0 \) to \( g_x = 2.15 \). For the core shear stiffness values smaller than \( g_x = 2.15 \) the magnitude of the postbuckling coefficient \( b \) changes abruptly. Decreasing \( g_x \) smaller than 1.0 into the shear crimping domain, then it is not possible to compute the postbuckling coefficient \( b \) since the critical buckling loads found are axisymmetric. This is caused by the singularity of the postbuckling problem for \( n = 0 \), in which the assumption made that the buckling and postbuckling mode must be orthogonal in some appropriate sense is no longer satisfied, see also [6].

![Figure 4-7. Effect of core shear stiffness variables on the critical buckling load and postbuckling coefficient b of isotropic core sandwich cylinders (\( \phi = 1 \))]
are less sensitive to asymmetric imperfections. Note that for all values of $g_x$, the corresponding buckling modes have 5 full waves in the circumferential direction.

### Table 4-4. The buckling loads, postbuckling coefficient and imperfection form factors of isotropic core sandwich cylinders ($\phi = 1$)

<table>
<thead>
<tr>
<th>$g_x$</th>
<th>$n$</th>
<th>$\lambda_c$</th>
<th>$b$</th>
<th>$\alpha$</th>
<th>$b = ba^2$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.0</td>
<td>5</td>
<td>0.781957</td>
<td>-0.134696</td>
<td>0.348922</td>
<td>-0.012175</td>
<td>-0.133415</td>
</tr>
<tr>
<td>5.0</td>
<td>5</td>
<td>0.748884</td>
<td>-0.152919</td>
<td>0.360520</td>
<td>-0.019875</td>
<td>-0.132140</td>
</tr>
<tr>
<td>2.5</td>
<td>5</td>
<td>0.702547</td>
<td>-0.215053</td>
<td>0.392819</td>
<td>-0.033184</td>
<td>-0.011004</td>
</tr>
<tr>
<td>2.0</td>
<td>5</td>
<td>0.690776</td>
<td>-0.205981</td>
<td>0.395341</td>
<td>-0.032194</td>
<td>1.379410</td>
</tr>
<tr>
<td>1.25</td>
<td>5</td>
<td>0.578047</td>
<td>-0.170889</td>
<td>0.189449</td>
<td>-0.006133</td>
<td>4.142424</td>
</tr>
</tbody>
</table>

It is interesting to see the mode shapes in which the fluctuation and decrease of the postbuckling coefficient $b$ occurs between the values of $g_x$ around 5.0 to 1.25. The corresponding mode shapes for $g_x = 5.0$ and $g_x = 1.25$ are depicted figure 4-8 and for $g_x = 2.5$ and $g_x = 2.0$ are depicted in figure 4-9, respectively. Looking at the buckling and postbuckling modes for $g_x = 2.5$ and $g_x = 5.0$ we can observe that the shapes are quite similar. Although they are not given here, it can be revealed that the modes found for the values of $g_x > 2.15$ are more or less similar to these modes. Therefore, the computed postbuckling coefficient $b$ and imperfection form factors $\alpha$ and $\beta$ are only change gradually. Looking now at the mode shapes for $g_x = 2.0$ and $g_x = 1.25$, it appears that the buckling and postbuckling modes very much differ from those for $g_x > 2.15$. This might explain the fluctuation of the computed postbuckling coefficient $b$ and imperfection form factors $\alpha$ and $\beta$ in table 4-4.

In figure 4-10 the knockdown factor plots are given for the variation of the amplitude $\xi$ from 0 to 1.0 of the affine imperfections. The imperfection sensitivity curves are displayed for isotropic cores with a variation of core shear stiffness values $g_x$. One can see in the left figure that apparently the imperfection sensitivity increases as $g_x$ decreases. However, from the right figure, it is seen that for $g_x$ smaller than 2.5, since the imperfection sensitivity coefficient $b$ does increase, the shells become less sensitive to the affine imperfections.
Figure 4-8. Prebuckling, buckling and postbuckling shapes of isotropic core sandwich shell with $g_x = 5.0$ (a,b,c) and $g_x = 1.25$ (d,e,f) using nonlinear prebuckling solutions.
Figure 4-9. Prebuckling, buckling and postbuckling shapes of isotropic core sandwich shells with $g_x = 2.5$ (a,b,c) and $g_x = 2.0$ (d,e,f) using nonlinear prebuckling solutions.
Figure 4-10. Asymmetric imperfection sensitivity of isotropic core sandwich cylinders with varying core shear stiffness values $g_x$

Further, we investigate the imperfection sensitivity of sandwich cylinders with an orthotropic core for a constant value of $g_x$, namely $g_x = 5.0$ and $g_x = 1.25$, respectively. The effect of varying the core shear stiffness ratio $\phi$ on the computed imperfection sensitivity coefficient $b$ are shown in table 4-5.

Table 4-5. The buckling loads and imperfection sensitivity coefficients of orthotropic core sandwich cylinders

<table>
<thead>
<tr>
<th>$\phi = \frac{g_x}{g_y}$</th>
<th>Orthotropic core $g_x = 5.0$</th>
<th>Orthotropic core $g_x = 1.25$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>$\lambda_c$</td>
<td>$b = b_0^2$</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>0.748884</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>0.740442</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>0.732674</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>0.725500</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0.718851</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>0.712667</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>0.706902</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>0.701507</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>0.696451</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>0.691689</td>
</tr>
</tbody>
</table>
It is apparent from table 4-5 that, increasing the ratio $\phi$ or decreasing the value of $g_y$, increases the magnitude of the second imperfection sensitivity coefficient $\tilde{b}$. Notice that sharp changes in the coefficient $b$ in the case with $g_x = 1.25$ always occur where there is a change in the critical circumferential wave number $n$, see also [8].

The imperfection sensitivity of these orthotropic core sandwich cylinders is illustrated in the left and right side of figure 4-11 for $g_x = 5.0$ and $g_x = 1.25$, respectively. In general it can be seen that the imperfection sensitivity increases as $\phi$ increases. However, it should be noted that for the value of $g_x = 1.25$ the predicted imperfection sensitivity appears reasonable only for $\tilde{\xi} < 0.2$. If we recall that Koiter’s imperfection sensitivity theory is asymptotically exact, i.e. it yields accurate predictions for sufficiently small imperfection $\tilde{\xi}$ then for the values of the imperfection amplitude higher than $\tilde{\xi} = 0.2$, the predicted very high imperfection sensitivity appears to be unrealistic.

Moreover, one should be aware that the amplitude $\tilde{\xi}$ of imperfections is related to the distance $h$ between the mid-surface of the top and bottom face. In the current study, the distance is four times higher than the thickness of the individual faces. In real sandwich structures, the amplitude of imperfections, which is usually present in the face sheets during the material process, is much smaller than the total thickness of sandwich layers. Therefore, it is also important before discussing the imperfections sensitivity results, that one also has an idea of the size of $\tilde{\xi}$ that is likely to occur in sandwich structures.

Figure 4-11. Asymmetric imperfection sensitivity of orthotropic core sandwich cylinders with varying core shear ratio $\phi = g_x / g_y$ and with $g_x = 5.0$ (left) and $g_x = 1.25$ (right)
Shell with Asymmetric Imperfections
Part II : Higher-Order Theory
In part I of this thesis, the classical formulation of first-order shear deformation for sandwich shells with a single-layer sandwich shell model is discussed. This theory is used by many researchers [2, 12, 35, 46, 52, 54, 55, 60] to predict the collapse load of sandwich cylindrical shells loaded by axial compression. A good prediction is obtained for the collapse loads of sandwich cylinder with "transversely" stiff core. The failure mechanism of the sandwich occurs, as it is expected, in the overall instability mode (general buckling or shear crimping).

In today's application of sandwich structures, however, the cores are also made of foam or low-strength Aramid or Nomex honeycomb which have generally very low rigidity and they can have a low transverse normal stiffness. This modern "soft" core is compressible and localized effects such as at the place of concentrated loading and boundary conditions can produce on early sudden core failure below the expected classical instability load level. The use of the classical theory to predict the buckling load for this type of core is no longer adequate. Therefore, a more general formulation of sandwich model emerged based on a higher-order theory with a three-layer sandwich modeling technique, [10, 27].

The model comprises two face layers and one core layer in the middle with their own formulation of coordinate systems, stress and displacement fields, geometry and stiffness properties. In contrast to the higher-order theory proposed in [15] for sandwich structures which also included the transverse normal stress, no a priori assumptions are made on the deformation fields through the thickness of the core in the present theory.

In this work, the equilibrium equations and natural boundary conditions for the faces and for the core are consistently derived in the framework of the minimum of total potential energy principle. In addition, the interface continuity conditions between the core and the faces are imposed. The obtained boundary conditions can be defined as global or local formulation. The displacement fields through the thickness of the core are analytically derived and described as explicit functions of
the thickness coordinate. Two types of core are considered namely foam and honeycomb-like cores and both are thick. By use of the interfaces continuity requirements, they can be expressed in the faces displacement components and the internal stresses at the interfaces. Each of these unknown variables only depends on the surface coordinates. These closed-form solutions will result in the two compatibility equations which complete the definition of the governing equations.

5.1. Definition of the Problem

The cylindrical shell geometry and the coordinate system used are shown in figure 5-1. Note that, to expose the sandwich layers section, the thickness of the cylinder is drawn much thicker compared to its length and radius. Here, \( R^t \), \( R^c \), \( R^b \) and \( t^t \), \( t^c \), \( t^b \) denote the radius and thickness of the top face, core and bottom face, respectively. \( L \) is the length of the cylindrical shells.

The faces rectangular coordinate system \( x^t, y^t, z^t \) and \( x^b, y^b, z^b \) is measured with respect to the faces mid-surface in the axial, circumferential and radial directions, respectively. The core cylindrical coordinate system \( x^c, \theta, R \) is measured with respect to the middle point of the cylinder in the axial, circumferential and radial directions, respectively.

For the faces, the in-plane displacements in axial and circumferential direction of a point on the faces reference surface are denoted \( U^t \), \( V^t \) and \( U^b \), \( V^b \), whereas the radial displacement is denoted as \( W^t \) and \( W^b \) with positive inward. For the core, they are given as \( \bar{U}^c \), \( \bar{V}^c \) and \( \bar{W}^c \) (positive outward) where the superscript \( \sim \) denotes the \( R \)-dependence of the quantities. The displacement fields are attached
to the appropriate coordinate systems. In addition, the face layers can have an initial geometric imperfection that is assumed to be stress-free and its amplitude is kept small with respect to the other (read face thickness) geometric properties.

It is tacitly assumed that the face layers and core are perfectly bonded at the interfaces. The face layers are thin and may have isotropic or composite laminated properties. In this study, both face layers are assumed to have an equal thickness and elastic material properties. They are placed symmetric with respect to the mid-surface of the sandwich wall.

The thick core, which actually acts as an elastic foundation for the face layers, and may have three dimensional elastic properties. However, an assumption is made that in-plane shear and extensional stiffness of the core are negligible in comparison to those of the faces. The core is thus transversely flexible in its thickness direction and very flexible, with negligible or even null rigidity in the in-plane direction.

Figure 5-2. External loads and internal stress and moment resultants on layers of sandwich shells
5.2. Basic Shell Equations and Continuity Requirements

In the development of a three-layer sandwich cylindrical shell model, the transverse flexibility of the core material, on the contrary to the earlier formulation using the classical theory, is accounted for.

The shell model, as depicted in figure 5-2, consists of two face layers which are held apart by a core. The internal stresses working in the core are the two transverse shear stresses $\tau_{xr}^c$, $\tau_{or}^c$ and the transverse normal stress $\sigma_{tr}^c$. In the following, the modeling of the faces and the core and the continuity conditions between this two models as well, which guarantee the continuity of deformation of the composed sandwich model, will be discussed separately.

5.2.1. Face Considerations

It has been stated that the face layers are treated as thin shells. The nonlinear Sanders and Koiter strain-displacement relations for non-shallow shells, including the initial imperfection term $\bar{W}$ are used. In addition, the rotations about the normal to the mid-surface are neglected with respect to rotations about directions tangent to the mid-surface. In the notations below, to distinguish the quantities belonging to the two faces, the superscript $j$ is introduced denoting the letters t and b for the top and bottom face layers respectively.

The mid-surface nonlinear in-plane strain-displacement relations corresponding to this theory are

$$\varepsilon_x^j = U_{,x}^j + \frac{1}{2}(\beta_x^j)^2 + 2\beta_x^j\tilde{\beta}_x^j) \tag{5-1}$$

$$\varepsilon_y^j = V_{,y}^j - \frac{W^j}{R^j} + \frac{1}{2}(\beta_y^j)^2 + 2\beta_y^j\tilde{\beta}_y^j) \tag{5-2}$$

$$\gamma_{xy}^j = U_{,y}^j + U_{,x}^j + \beta_x^j\tilde{\beta}_y^j + \beta_y^j\tilde{\beta}_x^j + \beta_x^j\tilde{\beta}_x^j \tag{5-3}$$

where $\beta_x^j, \beta_y^j$ denote flexural rotations and $\tilde{\beta}_x^j, \tilde{\beta}_y^j$ are initial rotations due to the initial imperfection $\bar{W}^j$. The flexural rotations and changes of curvature are defined as

$$\beta_x^j = -W_{,x}^j \quad \beta_y^j = -W_{,y}^j - \frac{V^j}{R^j} \tag{5-4}$$

$$\kappa_x^j = \beta_{x,xx}^j \quad \kappa_y^j = \beta_{y,yy}^j \quad \kappa_x^j = \beta_{x,yx}^j + \beta_{y,xx}^j \tag{5-5}$$

The off-midsurface in-plane strain-displacement relations are approximated as
\[ \begin{align*}
\varepsilon_x^j &= \varepsilon_x^j + z^j \kappa_x^j & \varepsilon_y^j &= \varepsilon_y^j + z^j \kappa_y^j & \gamma_{xy}^j &= \gamma_{xy}^j + z^j \kappa_{xy}^j \\
\end{align*} \]

(5-6)

where \( z^j \) denotes the radial distance from the reference surface. In the scope of the classical two-dimensional Kirchhoff-Love type shell theory for thin shells, i.e., it is assumed that normal to the undeformed mid-surfaces of the face layers remains straight, normal and in-extensional during deformation, the transverse shearing strains and normal strains are assumed to be zero which implies that

\[ \begin{align*}
\gamma_{xz}^j &= W_x^j + \beta_x^j = 0 & \gamma_{yz}^j &= W_y^j + \frac{V_y^j}{R} + \beta_y^j = 0 & \varepsilon_z^j &= 0 \\
\end{align*} \]

(5-7)

The following anisotropic laminate constitutive equations are used for the face layers. Detailed derivation is given in Appendix B.1 and the results are

\[ \begin{bmatrix}
N_x^j \\
N_y^j \\
N_{xy}^j \\
M_x^j \\
M_y^j \\
M_{xy}^j
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x^j \\
\varepsilon_y^j \\
\varepsilon_{xy}^j \\
\kappa_x^j \\
\kappa_y^j \\
\kappa_{xy}^j
\end{bmatrix} \]

(5-8)

where the \( A, B \) and \( D \) matrices are defined in Appendix B.1.

### 5.2.2. Core Considerations

When dealing with a curved sandwich panel built from a soft core, it is necessary to distinguish between analysis where one is using a core made of foam (which usually has constant properties through the thickness) or a cellular structure. Therefore, two types of core will be considered separately in the present model. The first type is foam cores which are uniformly distributed between the faces and of a high density. The second one is honeycomb type cores which have a cellular structure and are usually of a lower density. In the second type, both traditional cores (metallic honeycomb) and modern cores (Aramid and Nomex) can be considered and they are distinct to each other only by the range of their stiffness properties.

For sandwich plates, the definition of the stiffness properties \( G_x, G_y, \) and \( E_r \) of both types of cores is constant through the thickness of the core layer. Foam cores are made of an isotropic material and can be characterized by use of two elastic constants \( (G_x = G_y = G_c, E_r = E_c) \). Honeycomb cores, due to their "cellular" micro structure, are orthotropic material and have to be characterized by three elastic
constants ($\tilde{G}_x = G_x$, $\tilde{G}_0 = G_0$, $\tilde{E}_r = E_c$). Note that, in both types the stiffness properties of the core in-plane direction (in the direction of its plane) are always neglected because their small rigidity as compared to those of the faces.

(a) Foam-type core  
(b) Honeycomb-type core

**Figure 5-3. Sandwich element with two types of core material**

On the other hand, for curved sandwich structures such as sandwich cylindrical shells, the core layer can have different stiffness properties through its thickness, as illustrated in figure 5-3. For honeycomb cores, the density of the region near the top interface is considerably lower than the bottom region ($\rho_t < \rho_b$). The honeycomb is placed in a radial way and one can assume that the density changes as $R_c/R$. However, for foam cores the density are more and less constant through the thickness ($\rho_t = \rho_b$), i.e. the foam is placed uniformly between the faces.

Consequently, the stiffness properties for foam type cores can appropriately be designated as

$$\tilde{G}_x(R) = G_x \quad \tilde{G}_0(R) = G_0 \quad \tilde{E}_r(R) = E_c$$  \hspace{1cm} (5-9)$$

where the transverse shear modulus $G_x = G_0 = G_c$. The isotropic shear modulus $G_c$ is equal to $E_c/2(1+v_c)$ and $v_c$ denotes for the Poisson’s ratio of core material.

The honeycomb core stiffness properties of sandwich cylinders should be defined as

$$\tilde{G}_x(R) = \frac{G_x R_c}{R} \quad \tilde{G}_0(R) = \frac{G_0 R_c}{R} \quad \tilde{E}_r(R) = \frac{E_c R_c}{R}$$  \hspace{1cm} (5-10)$$

where $G_x$, $G_0$ and $E_c$ are the values of these quantities at $R = R_c$. Again, it should be mentioned here that no thin shell assumption is made for the core layer.

For honeycomb-type cores, the principal directions of materials are designated as the L (longitudinal) and W (warp) directions as displayed in figure 5-3. In the sandwich cylinder model of the present study, the assumption is made that the material
principal directions coincide with the in-plane cylinder coordinates \((x, y)\) which implies that the transverse shear modulus \(G_x = G_L\) and \(G_\theta = G_W\).

The assumption that the in-plane shear and extensional stiffnesses of the core are negligible with respect to the transverse shear and extensional stiffness quantities corresponds to a state of stress in which \(\sigma_x = \sigma_\theta = \tau_{x\theta} = 0\). Consistent with this initial simplification, therefore, it is justifiable to linearize the relevant strain-displacement relations for the core, see [10]. Furthermore, the rotation about the normal to the \(x-y\) surface is considered to be much smaller than the rotation about tangents to the same surface, consistent with the assumption made for the face layers. The linearized strain-displacement relations are

\[
\begin{align*}
\dot{\gamma}^c_{x\tau} &= \dot{U}^c_{\tau} + \dot{W}^c_{,x} \\
\dot{\gamma}^c_{\theta r} &= \dot{V}^c_{\tau} + \frac{1}{R}(-\dot{V}^c + \dot{W}^c_{,\eta}) \\
\dot{\varepsilon}^c_r &= \dot{W}^c_{,r}
\end{align*}
\]  

(5-11)

where \(R\) is the radial coordinate of the core measured from the middle point of the cylinder and superscripts ~ denote the \(R\)-dependence of the quantities.

On the basis of this core model, the relevant constitutive equations for the core are

\[
\begin{align*}
\dot{\gamma}^c_{x\tau} &= \dot{G}_x(R)\dot{\gamma}^c_{x\tau} \\
\dot{\gamma}^c_{\theta r} &= \dot{G}_\theta(R)\dot{\gamma}^c_{\theta r} \\
\dot{\varepsilon}^c_r &= \dot{E}_r(R)\dot{\varepsilon}^c_r
\end{align*}
\]  

(5-12)

5.2.3. Interface Continuity Considerations

In section 5.3, the equilibrium equations and boundary conditions for the faces and the core will be derived separately. Therefore, the components of the face and core displacements must satisfy the following conditions in order to ensure continuity of deformation in the sandwich layers.

\[
\begin{align*}
\dot{U}^c(\bar{x}, \theta, r=r_2) &= U^t - \frac{t}{2R^c}(W^t_{,\bar{x}}) \\
\dot{U}^c(\bar{x}, \theta, r=r_1) &= U^b + \frac{t}{2R^b}(W^b_{,\bar{x}}) \\
\dot{V}^c(\bar{x}, \theta, r=r_2) &= V^t - \frac{t}{2R^t}(W^t_{,\theta}) \\
\dot{V}^c(\bar{x}, \theta, r=r_1) &= V^b + \frac{t}{2R^b}(W^b_{,\theta}) \\
\dot{W}^c(\bar{x}, \theta, r=r_2) &= -W^t \\
\dot{W}^c(\bar{x}, \theta, r=r_1) &= -W^b
\end{align*}
\]  

(5-13)

Note that \(r_2 = \frac{R_2}{R^c}\) and \(r_1 = \frac{R_1}{R^c}\), as shown in figure 5-2, denote the nondimensional outer and inner radius of the core, respectively. Furthermore, the nondimensional axial and radial coordinates \(\bar{x} = x^c/R^c\), \(r = R/R^c\) are introduced.

It will be demonstrated in section 5.3 that the satisfaction of these conditions, following integration of the core strain-displacement equations, results in the two interface compatibility equations and that they define, together with the equilib-
rium equations of the faces derived below, the set of equations governing the behavior of the current sandwich shell model.

5.3. Equilibrium Equations and Boundary Conditions

The basic approach followed in this study is to establish the equilibrium equations and consistent boundary conditions using the minimum of total potential energy principle with the aid of variational procedures. This principle states that a body is in equilibrium if and only if the virtual strain energy stored in the cylinder due to deformations equals the virtual work done by the external loads for every kinematically admissible displacement field. Applications of the minimum total potential energy principle requires that the first variation of the sum of the strain energy and potential of the applied loads of the thin face layers and the thick core is equal to zero. This implies (the superscript j is introduced denoting the letters t and b for the top and bottom face, respectively)

$$\delta (\Pi_t^j + \Pi_c) = \delta (U_t^j + V_t^j) + \delta (U_c + V_c) = 0$$  \hspace{1cm} (5-16)

The strain energy of the top face (j = t) and bottom face (j = b) is defined as

$$U_t^j = \frac{1}{2} \int_{-\frac{t'}{2}}^{\frac{t'}{2}} \int_0^{2\pi} \int_0^{R_t^j} \{ \tilde{\sigma}_{xx} \tilde{\varepsilon}_x^j + \tilde{\sigma}_{yy} \tilde{\varepsilon}_y^j + \tilde{\sigma}_{zz} \tilde{\varepsilon}_z^j \} (1 - \frac{z^j}{R_t^j}) dx^j dy^j dz^j$$  

$$+ \int_{-\frac{t'}{2}}^{\frac{t'}{2}} \int_0^{2\pi} \int_0^{R_t^j} \{ \tilde{\tau}_{xy} \tilde{\gamma}_{xy}^j + \tilde{\tau}_{yz} \tilde{\gamma}_{yz}^j + \tilde{\tau}_{xz} \tilde{\gamma}_{xz}^j \} (1 - \frac{z^j}{R_t^j}) dx^j dy^j dz^j$$  \hspace{1cm} (5-17)

Note that the terms of the order $z^j/R_t^j$ will be neglected in the integrand of the strain energy of the faces in (5-17) due to the thin shell assumption made for the faces.

The strain energy of the core is

$$U_c = \frac{1}{2} \int_{-R_c^j}^{R_c^j} \int_0^{2\pi} \int_0^{\frac{R^j}{2}} \{ \tilde{\tau}_{xx}^c \tilde{\gamma}_{xx}^c + \tilde{\tau}_{yy}^c \tilde{\gamma}_{yy}^c + \tilde{\tau}_{zz}^c \tilde{\gamma}_{zz}^c \} R dx^c d\theta dR$$  \hspace{1cm} (5-18)

Again, $R_2$ and $R_1$ denote the outer and inner radius of the core, respectively and $R$ is the radial distance from the middle point of the cylinder.
For the axial line load $\tilde{P}^j$, the torsion line load $\tilde{S}^j$, the transverse load $\tilde{Q}^j$, the bending moment $\tilde{M}^j$ and the normal pressure $p_e^j$, as depicted in figure 5-2, the potential of applied loads for the faces is

$$V^j_c = \left[\begin{array}{c}
\int_0^{2\pi R'} \{ \tilde{P}^j U^j_x \big|_{x=0} - \tilde{S}^j V^j_x \big|_{x=L} + \tilde{Q}^j W^j_x \big|_{x=0} + \tilde{M}^j \beta_x^j \big|_{x=0} \} \, dx \bigg|_{x=0}^{2\pi R'} \\
+ \int_0^{2\pi R'L} \{ \pm p_e^j W^j \} \, dx \bigg|_{x=0}^{2\pi R'}
\end{array}\right]$$

(5-19)

where the upper sign applies to the top face while the lower sign is for the bottom face.

Furthermore, the core carries no external load which follows that the potential of the applied load $V_c = 0$.

The first variation of the strain energy density expressions for linearly elastic thin faces, consistent with the Kirchhoff-Love assumptions, is defined as

$$\delta U^j_c = \int \int \left[ \tilde{\epsilon}_x^j \delta \epsilon_x^j + \tilde{\epsilon}_y^j \delta \epsilon_y^j + \tilde{\epsilon}_z^j \delta \epsilon_z^j \right] (1 - \frac{z^j}{R^j}) \, dx \, dy \, dz$$

(5-20)

Introducing into equation (5-20) the expressions for the off-midsurface strains in terms of the mid-surface strains and curvatures, as given in equations (5-6) and (5-7) and keeping in the mind that the terms of the order $z^j/R^j$ as compared to one are neglected in the integrand of equation (5-20) yields

$$\delta U^j_c = \int \int \left[ N_x^j \delta \epsilon_x^j + N_y^j \delta \epsilon_y^j + N_{xy}^j \delta \epsilon_{xy} \right] \, dx \, dy$$

(5-21)

$$+ \int \int \left[ M_x^j \delta \kappa_x^j + M_y^j \delta \kappa_y^j + M_{xy}^j \delta \kappa_{xy} + Q_x^j \delta \gamma_x^j \right] \, dx \, dy$$

where the stress- and moment resultants, acting at the mid-surface of the faces are defined as
Three-layer Sandwich Model

\[
\begin{align*}
\begin{bmatrix}
N_x^j \\
N_y^j \\
N_{xy}^j \\
Q_x^j
\end{bmatrix} &= \int \begin{bmatrix}
\tau_{xy}^j \\
\tau_{xz}^j
\end{bmatrix} dz^j \\
\begin{bmatrix}
M_x^j \\
M_y^j \\
M_{xy}^j
\end{bmatrix} &= \int \begin{bmatrix}
\sigma_{xy}^j \\
\sigma_{xz}^j
\end{bmatrix} z dz^j
\end{align*}
\] (5-22)

Note that \( N_{xy}^j = N_{yx}^j \) and \( M_{xy}^j = M_{yx}^j \), following the neglecting of the terms of order \( z^{1/R} \) in the definition of the stress-and moment resultant. In figure 5-2 the sign convention of stress and moment resultant of the faces are given.

It should be mentioned that the expression for the off-midsurface strains \( \tilde{\gamma}_{yz}^j = 0 \) and \( \tilde{\varepsilon}_{z}^j = 0 \) are substituted directly in the definition of the first variation of the strain energy in equation (5-20). The expression for the off-midsurface strain \( \tilde{\gamma}_{xz}^j = 0 \) is not yet introduced but is later on added.

From equation (5-19) the first variation of the potential of applied loads for the faces is

\[
\delta V_i^j = -\int_0^{2\pi R^j} \left\{ P_i^j \delta U_i^j \bigg|_{x=0}^{x=L} + S_i^j \delta V_i^j \bigg|_{x=0}^{x=L} + Q_i^j \delta W_i^j \bigg|_{x=0}^{x=L} + M_i^j \delta \beta_i^j \bigg|_{x=0}^{x=L} \right\} dy^j
\] (5-23)

where again the upper sign applies to the top face while the lower sign is for the bottom face.

From equation (5-11), at any point in the core the strain-displacement relations are

\[
\tilde{\gamma}_{xr} = \frac{1}{R^c} (\tilde{U}^c_{rr} + \tilde{W}^c_{r\bar{x}}) \quad \tilde{\gamma}_{0r} = \frac{1}{R^c} (\tilde{V}^c_{rr} - \frac{\tilde{V}^c_{r\bar{r}}}{r}) \quad \tilde{\varepsilon}_r = \frac{1}{R^c} \tilde{W}^c_{r\bar{r}}
\] (5-24)

where the nondimensional axial and radial coordinates \( \bar{x} = x^c / R^c, \bar{r} = r / R^c \) are introduced. Substituting these expressions into equation (5-18), the first variation of the strain energy function of the core is defined as

\[
\delta U_c = \int \int \int_{0 \bar{r}, 0} \left\{ (\tau_{xrr}^c + \frac{\tau_{xrr}^c}{r}) \delta U^c + (\tau_{0rr}^c + 2 \frac{\tau_{0rr}^c}{r}) \delta V^c \right\}
\]
The first variation of the strain energy $\delta U_j^i$, defined in equation (5-21), can be expressed in terms of the variation of mid-surface displacements and rotations by substituting the nonlinear mid-surface strain-displacement relations in equations (5-1) to (5-3) and the changes of curvature in equation (5-5). Next one substitutes into equation (5-16) the resulting expressions together with the expressions for the first variation of the potential of applied loads $\delta V_j^i$ in equation (5-23) and the strain energy for the core $\delta U_c$ in equation (5-25). Further, by use of the interfaces continuity conditions equations (5-13) to (5-15), integrating the obtained equations by parts and some algebraic manipulation, the equilibrium equations for the faces are

\[
N_{x}^{j} + N_{y}^{j} \frac{M_{xy}^{j}}{R} \beta_{x}^{j} - \frac{N_{xy}^{j}}{R} (\beta_{x}^{j} + \beta_{y}^{j}) + \frac{1}{R} (N_{x}^{j} \beta_{x}^{j} + N_{y}^{j} \beta_{y}^{j}) = 0
\]

\[
N_{y}^{j} + N_{x}^{j} \frac{M_{xy}^{j}}{R} \beta_{y}^{j} - \frac{N_{xy}^{j}}{R} (\beta_{x}^{j} + \beta_{y}^{j}) + \frac{1}{R} (N_{x}^{j} \beta_{x}^{j} + N_{y}^{j} \beta_{y}^{j}) = 0
\]

\[
(Q_{x}^{j} + M_{xy}^{j} \beta_{x}^{j}) \beta_{x}^{j} + M_{y}^{j} \beta_{y}^{j} + \frac{N_{y}^{j}}{R} (\beta_{x}^{j} + \beta_{y}^{j}) - \frac{N_{xy}^{j}}{R} (\beta_{x}^{j} + \beta_{y}^{j}) = 0
\]

\[
(Q_{x}^{j} + M_{xy}^{j} \beta_{x}^{j}) \beta_{x}^{j} + M_{y}^{j} \beta_{y}^{j} + \frac{N_{y}^{j}}{R} (\beta_{x}^{j} + \beta_{y}^{j}) - \frac{N_{xy}^{j}}{R} (\beta_{x}^{j} + \beta_{y}^{j}) = 0
\]

where for the top face $j = t$, $k = 2$ and for the bottom face $j = b$, $k = 1$. The equilibrium equations for the core are

\[
\tau_{rx}^{c} \frac{r}{R} = 0
\]
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\[ \tau_{\theta r r}^c + 2 \frac{\tau_{\theta r r}^c}{r} = 0 \]  
(5-31)

\[ \alpha_{rr}^c + \frac{\alpha_{rr}^c}{r} + \tau_{xr x}^c + \frac{\tau_{\theta r \theta}}{r} = 0 \]  
(5-32)

and the natural boundary conditions are:

\[ \begin{align*}
&\int_0^{2\pi} \left( (N_x^j - \tilde{P}^j) \delta U^j + (N_{xy}^j - \frac{M_{xy}^j}{R^j} - \tilde{S}^j) \delta V^j + (Q_x^j + M_{xy'y'}^j - \tilde{Q}^j) \delta W^j \right. \\
&\left. \left. + (M_x^j - \tilde{M}^j) \delta \beta_x^j \right|_{x=0}^{x=L} R^j d\theta + \int_0^{2\pi} \int_0^{r_c} \tau_{xr}^c R^j d\theta dr \right|_{x=0}^{x=L} = 0 \\
&+ R^j d\theta \right|_{x=0}^{x=L} = 0 \\
\end{align*} \]  
(5-33)

Local and global boundary conditions

The local boundary conditions at the edge of the cylinder can be derived directly from equation (5-33). Thus, at the edges of the cylinder at \( x = 0 \) and \( x = L \) one can prescribe either

\[ N_x^j = \tilde{P}^j \quad \text{or} \quad U^j = \tilde{U}^j \]  
(5-34)

\[ N_{xy}^j = N_{xy}^j - \frac{M_{xy}^j}{R^j} = \tilde{S}^j \quad \text{or} \quad V^j = \tilde{V}^j \]  
(5-35)

\[ Q_x^j = Q_x^j + M_{xy'y'}^j = \tilde{Q}^j \quad \text{or} \quad W^j = \tilde{W}^j \]  
(5-36)

\[ M_x^j = \tilde{M}^j \quad \text{or} \quad \beta_x^j = \tilde{\beta}_x^j \]  
(5-37)

\[ \tau_{xr}^c = 0 \quad \text{or} \quad \tilde{W}^c = 0 \]  
(5-38)

where the notations \( \tilde{N}_{xy}^j \) and \( \tilde{Q}_x^j \) are introduced to denote the modified shear forces. \( \tilde{P}^j \), \( \tilde{S}^j \), \( \tilde{Q}^j \) and \( \tilde{M}^j \) are respectively axial, torsional, transverse and bending external loads applied at the faces, \( \tilde{U}^j \), \( \tilde{V}^j \), \( \tilde{W}^j \) and \( \tilde{\beta}_x^j \) are prescribed deformations and rotation of the faces.
In [27] Frostig mentioned that the global boundary conditions have to be implemented since in most practical applications sandwich panels have rigid inserts (stiffeners, membranes) at their edges. As displayed in figure 5-4, these extraneous rigid members force boundary conditions for the faces and the core to be related to each other and to the global boundary conditions (at point G) as well. The relations between the displacements and rotations of the faces and global point G of the rigid stiffener are as follows

\[
U^j = U^G - \frac{(R^L - R^B)}{2} W^G_{,x'}
\]  
\[
V^j = \frac{R^j}{R^G} V^G + \frac{(R^j - R^G)}{R^G} W^G_{,y'}
\]  
\[
W^j = W^G
\]  
\[
W^j_{,x'} = W^G_{,x'}
\]

After formal substitution of these expressions into equation (5-33) and integrating, the global boundary conditions at point G for the sandwich cylinder with rigid inserts at their edges can be written as follows

\[
W^t = W^b
\]
\[ \beta_x^t = \beta_x^b \]  
\[ \beta_x^t = \frac{U^t - U^b}{(R^t - R^b)} \]  
\[ W_{t,\theta}^t = \frac{V^t R^b - V^b R^t}{(R^t - R^b)} \]  
\[ N_x^t R^t + N_x^b R^b = P^G R^G \quad \text{or} \quad \frac{U^t + U^b}{2} = \tilde{U}^G \]  
\[ \tilde{N}_{xy}^t R^t + \tilde{N}_{xy}^b R^b = S^G R^G \quad \text{or} \quad \frac{V^t (R^G - R^b) - V^b (R^t - R^G)}{(R^t - R^b)} = \tilde{V}^G \]  
\[ \tilde{Q}_x^t R^t + \tilde{Q}_x^b R^b = Q^G R^G \quad \text{or} \quad W^t = \tilde{W}^G \]  
\[ M_x^t R^t + M_x^b R^b + N_x^t (R^t - R^G) - N_x^b (R^G - R^b) = \tilde{M}^G R^G \quad \text{or} \quad \frac{U^t - U^b}{(R^t - R^b)} = \tilde{\beta}_x^G \]  
\[ \tilde{\tau}_{xr}^c = 0 \quad \text{or} \quad \tilde{W}^c = 0 \]  

where \( \tilde{P}^G, \tilde{S}^G, \tilde{Q}^G \) and \( \tilde{M}^G \) are respectively axial, torsional, transverse and bending external loads applied at the supports of the shell, \( U^G, V^G, W^G \) and \( \beta^G_x \) are prescribed deformations and rotation of the edge stiffener. In general, one can prescribe the global boundary conditions (5-43) to (5-51) at any point \( G \) along the height of the rigid stiffener. Note that for similar top and bottom faces the sandwich shells configuration is symmetric. Thus, if one prescribes the boundary conditions at the shell mid-surface then one has \( R^G = R^C \).

### 5.3.1. Core Stresses and Displacement Fields

The equilibrium equations for the core can be integrated with respect to the radial coordinate \( r \). Integrating equations (5-30) to (5-32), the following expressions for the core stress components are obtained

\[ \tau_{xr}^c(r) = \frac{\tau_{xr}}{r} \quad \tilde{\tau}_{\theta r}^c(r) = \frac{\tau_{\theta r}}{r^2} \quad \tilde{\sigma}_r^c(r) = \frac{\tau_{\theta r,\theta}}{r^2} - \tau_{xr,\tilde{x}} \]  

where the core shear stress components \( \tau_{xr}^c, \tilde{\tau}_{\theta r}^c \) and the constant \( A_1 = A_1(\tilde{x}, \theta) \) are independent of the coordinate \( r \), and thus they are a function of the in-plane coordinates \( \tilde{x} \) and \( \theta \) only. Here, it should be noted that the shear stress distribu-
tion across the thickness of the core model is completely determined by the equilibrium equations.

Two types of core are considered depending upon the definition of the stiffness properties as they are given in equation (5-9) or (5-10). The first case is dealing with the foam-type core and the second case is classified as the honeycomb-type core. They will be discussed separately below.

_Foam-type core_

For foam cores, see equation (5-9), the elastic constants are assumed constant through the core thickness which results in the following expressions

\[
\tilde{G}_x(r) = G_x \quad \tilde{G}_\theta(r) = G_{\theta} \quad \tilde{E}_r(r) = E_c
\]  
(5-53)

The core constitutive equations in equation (5-12), using the definition of the stiffness functions in equation (5-53), are combined with the kinematic relations in equation (5-11) giving the following relations

\[
\tilde{\tau}_{xr}^c(r) = \frac{G_x r}{R^c}(\tilde{U}_x^c \gamma_r + \tilde{W}_{s_x}^c) 
\]  
(5-54)

\[
\tilde{\tau}_{r\theta}^c(r) = \frac{G_{\theta} r^2}{R^c}(\tilde{V}_\theta^c - \frac{\tilde{V}_r^c}{r} + \frac{\tilde{W}_{s_{\theta}}^c}{r}) 
\]  
(5-55)

\[
\tilde{\sigma}_r^c(r) = \frac{E_c}{R^c}(\tilde{W}_{s_r}^c) 
\]  
(5-56)

The variation of the core displacement components \( \tilde{U}^c, \tilde{V}^c \) and \( \tilde{W}^c \) with respect to the \( r \)-coordinate can be derived by use of equation (5-52) and equations (5-54) to (5-56). This means that the core displacement components will be described as explicit functions of \( r \).

Substituting the expression for the core stress \( \tilde{\sigma}_r^c \) from equation (5-52) into equation (5-56) and by integrating the result with respect to the coordinate \( r \) yields the variation of the radial displacement component \( \tilde{W}^c \) through the thickness coordinate

\[
\tilde{W}^c = \frac{R^c}{E_c}\left\{\frac{\tau_{r\theta}^c}{r} - r \tau_{xr}^c + A_1(\tilde{x}, 0) \ln(r) + A_2(\tilde{x}, 0)\right\} 
\]  
(5-57)

The variation of the axial displacement component \( \tilde{U}^c \) through the thickness coordinate can be derived by substituting the shear stress component \( \tilde{\tau}_{xr}^c \) from equa-
tion (5-52) and $\hat{W}^c$ from equation (5-57) into equation (5-54) and by integrating the resulting expression with respect to the coordinate $r$. This yields

$$\hat{U}^c = \frac{R_c \ln(r) \tau_{xr}}{G_x} - \frac{R_c}{E_c} \left\{ -\tau_{r\theta} \frac{\ln(r)}{2} \tau_{x\theta} + A_{1, x}(r \ln(r) - r) + A_{2, x} r + A_3(\tilde{x}, \theta) \right\}$$  \tag{5-58}

Similar to the derivation of $\hat{U}^c$, the variation of the circumferential displacement component $\hat{V}^c$ can be obtained by substituting the expression for $\tau_{r\theta}^c$ and $\hat{W}^c$ into equation (5-55). The result is

$$\hat{V}^c = \frac{R_c \tau_{xr}}{2G_y r} + \frac{R_c}{E_c} \left\{ -\tau_{r\theta} \frac{\ln(r)}{2} + A_{1, \theta}(\ln(r) + 1) + A_{2, \theta} + r A_4(\tilde{x}, \theta) \right\}$$  \tag{5-59}

From the equations (5-57), (5-58) and (5-59), it can be seen that the radial, axial and circumferential deformations of the core behave non-linearly through its thickness coordinate. These are higher-order effects and must be considered when transversely flexible cores are investigated.

First, by use of the continuity requirement of the radial displacement $\hat{W}^c$ in equation (5-15) and the expression in equation (5-57), one can obtain the functions $A_1(\tilde{x}, \theta)$ and $A_2(\tilde{x}, \theta)$. Substituting these functions into equation (5-57) yields the following expressions for the radial displacement of the core

$$\hat{W}^c = \frac{1}{\ln(r_2/r_1)} \left\{ \ln(r_1) W^t - \ln(r_2) W^b - \ln(r)(W^t - W^b) \right\} + \frac{R_c}{E_c \ln(r_2/r_1)} \left\{ (r_2 - r_1) \ln(r) - r \ln(r_2/r_1) + r_1 \ln(r_2) - r_2 \ln(r_1) \right\} \tau_{xr}^c$$  \tag{5-60}

$$+ \frac{R_c}{E_c \ln(r_2/r_1)} \left\{ \frac{1}{r} \ln(r_2/r_1) + \left( \frac{r_2^2}{r_1} - 1 \right) \ln(r) + \frac{\ln(r_2)}{r_2} - \frac{\ln(r_1)}{r_1} \right\} \tau_{r\theta}^c$$

The functions $A_1(\tilde{x}, \theta)$ and $A_2(\tilde{x}, \theta)$ can also be substituted into the expressions for $\hat{U}^c$ and $\hat{V}^c$ in equations (5-58) and (5-59). By use of the continuity conditions in equations (5-13) and (5-14), the expressions of the in-plane displacement fields in (5-58) and (5-59) will result in two compatibility equations namely
\[
\begin{align*}
U^t - U^b + \left\{ -\frac{t^t}{2R^c} + C_1 \right\} W^t_{,x} + \left\{ -\frac{t^b}{2R^c} + C_2 \right\} W^b_{,x} - \frac{R^c}{G_x} C_3 \tau_{xr}^c \\
+ \frac{R^c}{E_c} C_4 \tau_{0r,0x}^c + \frac{R^c}{E_c} C_5 \tau_{x'r,xx}^c = 0 
\end{align*}
\]
\[(5-61)\]

\[
\begin{align*}
V^t r_1 - V^b r_2 + \left\{ -\frac{t^t}{2R^t} + C_6 \right\} W^t_{,0} + \left\{ -\frac{t^b}{2R^b} + C_7 \right\} W^b_{,0} - \frac{R^c}{G_0} C_8 \tau_{0r}^c \\
+ \frac{R^c}{E_c} C_9 \tau_{0r,00}^c + \frac{R^c}{E_c} C_{10} \tau_{x'r,00}^c = 0 
\end{align*}
\]
\[(5-62)\]

Note that these compatibility equations are expressed in terms of the components of the face displacement and the core shear stresses \(\tau_{xr}^c\) and \(\tau_{0r}^c\) only. Each of these unknown variables depends on the surface coordinates \(\bar{x}\) and \(\theta\). The coefficients \(C_1\) to \(C_{10}\) used in equations (5-61) and (5-62) are given in Appendix B.3.

Next, by use of the expression for the core stress components in equation (5-52) and the expression for \(A_1(\bar{x}, \theta)\), the shear and normal stresses at the interfaces between the top face-core and the bottom face-core are become

\[
\begin{align*}
\tilde{\tau}_{xr}(r=r_2) &= \frac{\tau_{xr}}{r_2} \\
\tilde{\tau}_{0r}(r=r_2) &= \frac{\tau_{0r}}{(r_2)^3} 
\end{align*}
\]
\[(5-63)\]

\[
\begin{align*}
\tilde{\sigma}_r(r=r_2) &= C_1^t \tau_{0r,00}^c + C_2^t \tau_{x'r,\bar{x}}^c + \frac{E_c C_3^t}{R^c} (W^t - W^b) \\
\tilde{\tau}_{xr}(r=r_1) &= \frac{\tau_{xr}}{r_1} \\
\tilde{\tau}_{0r}(r=r_1) &= \frac{\tau_{0r}}{(r_1)^2} 
\end{align*}
\]
\[(5-64)\]

where the coefficients \(C_1^t\) to \(C_3^t\) and \(C_1^b\) to \(C_3^b\) are given in Appendix B.3.

Finally, introducing the following modified shear resultants and the average radial displacement for the core

\[
\begin{align*}
\tilde{Q}_x^c &= -C_3 R^c \tau_{xr}^c \\
\tilde{Q}_0^c &= t^c \tau_{0r}^c \\
W^c &= \frac{R^c}{E_c} \{ C_4 W^t + C_2 W^b + \frac{R^c}{E_c} (C_4 \tau_{0r,00}^c + C_5 \tau_{x'r,\bar{x}}^c) \} 
\end{align*}
\]
\[(5-65)\]
into equations (5-63) and (5-64), i.e. the normal stress $\dot{\sigma}_r^{c}(r=r_k)$ working on the top face-core interface (k = 2) and bottom face-core interface (k = 1), one obtains

$$\dot{\sigma}_r^{c}(r=r_k) = \frac{E_c}{R^c}(D_2^tW^t + D_3^bW^b - D_4^t\frac{t^c}{R^c}W^c) + D_4^t\dot{\Theta}_{r,\theta}$$  \hspace{1cm} (5-66)$$

where the coefficients $D_1^t$ to $D_4^t$ are also given in Appendix B.3.

**Honeycomb-type core**

For the case of honeycomb-type core, the stiffness functions of the core are assumed to vary linearly through the thickness as follows

$$\dot{\mathcal{G}}_x(r) = \frac{G_x}{r} \quad \dot{\mathcal{G}}_\theta(r) = \frac{G_\theta}{r} \quad \dot{\mathcal{E}}_r(r) = \frac{E_c}{r}$$  \hspace{1cm} (5-67)$$

Similar to the foam-type core, the core constitutive equations in equation (5-12), using the definition of the stiffness function in equation (5-67), are combined with the kinematic relations in equation (5-11) to give the following relations

$$\tau_x^{c}(r) = \frac{G_x}{R^c}(\dot{U}^{c}_r + \dot{W}^{c}_x)$$  \hspace{1cm} (5-68)$$

$$\tau_\theta^{c}(r) = \frac{G_\theta r}{R^c}(\dot{V}^{c}_r - \frac{\dot{V}^{c}}{r} + \frac{\dot{W}^{c}_\theta}{r})$$  \hspace{1cm} (5-69)$$

$$\dot{\sigma}_r^{c}(r) = \frac{E_c}{R^c} (\dot{W}^{c}_r)$$  \hspace{1cm} (5-70)$$

All the derivations are performed in the same way as for the foam-type core case. The compatibility equations, the shear and normal stresses are similar to those given in equations (5-61) and (5-62), equations (5-63), (5-64) and (5-66), except that the coefficients $C_1^t$ to $C_{10}^t$ used in equations (5-61) and (5-62) and the coefficients $C_1^t$ to $C_3^t$ and $C_1^b$ to $C_3^b$ used in equations (5-63) and (5-64) are different. They are also given in Appendix B.3.

**5.3.2. Equilibrium Equations**

Finally, the equilibrium equations can be rewritten after formal substitution of the expressions for the transverse shear stresses and normal stresses of the core in equations (5-63), (5-64), (5-66) and the modified shear resultant and average displacement in equation (5-65). The final set of eight equilibrium equations are (for the top face: $j = t$, $k = 2$, for the bottom face: $j = b$, $k = 1$)
\[ N_i^{ij} - \hat{N}_{xy}^{ij} + \frac{M_{xy}^{ij}}{R^j} + (-1)^k \frac{\hat{Q}_x^c}{C_3 r_k R^c} = 0 \]  
(5-71)

\[ N_y^{ij} + \hat{N}_{xy}^{ij} - \frac{M_{xy}^{ij}}{R^j} + \frac{N_y^{ij}}{R^j} (\beta_y^j + \tilde{\beta}_y^j) \]
\[ + \frac{1}{R^j} (\hat{N}_{xy} + \frac{M_{xy}^{ij}}{R^j}) (\beta_x^j + \tilde{\beta}_x^j) - (-1)^k \frac{\hat{Q}_0^c}{t^c (r_k)^2} = 0 \]  
(5-72)

\[ \hat{Q}_x^{ij} + M_{xy}^{ij} + \frac{N_y^{ij}}{R^j} - N_y^{ij} (\beta_y^j + \tilde{\beta}_y^j), y^j, + (\hat{N}_{xy} + \frac{M_{xy}^{ij}}{R^j} (\beta_x^j + \tilde{\beta}_x^j)) y^j, \]
\[ + (-1)^k \left\{ \frac{E^c}{R^c} (D_2^c W_t + D_3^c W_b - D_4^c t^c W_c) + \frac{D_1^j}{t^c} \hat{Q}_{0,0}^c + p^j \right\} - \frac{t^j \hat{Q}_0^c y^j}{2 t^c (r_k)^2} = 0 \]  
(5-73)

\[ M_{xy}^{ij} + 2 M_{xy}^{ij} - \hat{Q}_x^{ij} - N_x^{ij} (\beta_x^j + \tilde{\beta}_x^j) - (\hat{N}_{xy} + \frac{M_{xy}^{ij}}{R^j} (\beta_y^j + \tilde{\beta}_y^j)) + \frac{t^j \hat{Q}_0^c}{2 C_3 r_k R^c} = 0 \]  
(5-74)

and the resulting compatibility equations, after introducing the modified shear resultants and average displacement in equation (5-65) into equations (5-61) and (5-62), are

\[ W^c x^s - \frac{R^c}{t^c} (U^t - U^b) + \frac{t}{2 t^c} (W_t + W_b), x^s - \frac{R^c}{t^c} \hat{Q}_x^c = 0 \]  
(5-75)

\[ \dot{Q}_x^c - \frac{C_3 E^c}{C_5} (C_1 W_t + C_2 W_b + \frac{t^c}{R^c} W_c) - \frac{C_3 C_4 R^c}{C_5 t^c} \hat{Q}_{0,0}^c = 0 \]  
(5-76)

\[ C_8 \hat{Q}_0^c - \frac{D_1 G^c}{E^c} Q_{0,0}^c - \frac{G^c t^c}{R^c} (V_t r_1 - V_b r_2) \]
\[ - \frac{G^c t^c}{R^c} \{(D_2 - \frac{t^t}{2 R^t}) W_t^c + (D_3 - \frac{t^b}{2 R^b}) W_b^c - D_4 \frac{t^c}{R^c} W_c^c \} = 0 \]  
(5-77)

where the coefficients D_1 to D_4 are defined in Appendix B.3.

The boundary conditions for the face layers and the core are given in equations (5-34) to (5-38) for local formulation and in equations (5-43) to (5-51) for global formulation. Two groups of variables are distinguished, namely basic and secondary variables. The basic variables are the unknowns that can be prescribed at the edges of
the shell. From equations (5-34) to (5-51) one can see that these 18 variables are \( N^j_x, \tilde{N}^j_{xy}, \tilde{Q}^j_x, M^j_x, U^j_j, V^j_j, W^j_j, \beta^j_x \) for \( j = t, b \) and \( (\tilde{Q}^c_x, W^c) \).

The secondary variables are the six face stresses \( (N^j_y, M^j_y, M^j_{xy}) \) for \( j = t, b \) and the core shear stress \( \tilde{Q}^c_0 \). Note that equation (5-77) can be used to express the core shear stress \( \tilde{Q}^c_0 \) in the basic variables. The elimination of the secondary variable \( \tilde{Q}^c_0 \) from the governing equations will be performed after Fourier decomposition in section 5.4.

### 5.3.3. Partially Inverted Constitutive Equations

The eight equilibrium equations for the faces in (5-71) to (5-74) derived above are first order differential equations in the x-coordinate. There are in total 8 more unknowns in the set of 8 equations, so that an additional set of equations must be formulated in order to complete the definition of the governing equations that can be solved. These equations are the partially inverted constitutive equations of the faces derived from equation (5-8) and the condition \( \dot{\gamma}^j_{xz} = 0 \) from the kinematic relations in equation (5-7), which completes the introduction of the Kirchhoff-Love assumption for the faces.

The constitutive equations for the faces given in equation (5-8) can be modified in such a way that they are partially inverted. Detailed derivations are given in Appendix B.2. By use of the linear operators \( L^j_i \) defined in equation (B.12), they can be expressed as follows

\[
\begin{align*}
\varepsilon^j_x &= \frac{t}{cR^j}L^j_1(a) & N^j_y &= \frac{E_I t^2}{cR^j}L^j_2(a) \quad (5-78) \\
\kappa^j_x &= \frac{2}{R^j}L^j_2(a) & M^j_y &= \frac{E_I t^3}{2c^2 R^j}L^j_5(a) \quad (5-79) \\
\gamma^j_{xy} &= \frac{t}{cR^j}L^j_3(a) & M^j_{xy} &= \frac{E_I t^3}{2c^2 R^j}L^j_6(a) \quad (5-80)
\end{align*}
\]

The left-side relations of equations (5-78) to (5-80), together with the kinematic relations in equations (5-1), (5-3) and (5-5), and the expression for \( \dot{\gamma}^j_{xz} = 0 \) in equation (5-7) give the following 8 additional differential equations for the basic displacement variables

\[
U^j_{j',x'} + \frac{1}{2}(\beta^2 + 2\beta_x \beta_x^j) - \frac{t}{cR^j}L^j_1(a) = 0 \quad (5-81)
\]
\[ V_{y}^{j} + U_{y}^{j} + (\beta_{y} \beta_{y})^{j} + (\beta_{y} \bar{\beta}_{y})^{j} + (\beta_{x} \bar{\beta}_{x})^{j} - \frac{t}{cR} L_{3}(\alpha)^{j} = 0 \]  
(5-82)

\[ W_{y}^{j} + \beta_{x}^{j} = 0 \]  
(5-83)

\[ \beta_{x}^{j} - \frac{2}{R} L_{2}(\alpha)^{j} = 0 \]  
(5-84)

where \( j \) stands either for \( t \) (top) or \( b \) (bottom). To eliminate the secondary variables \( N_{y}^{j}, M_{y}^{j} \) and \( M_{xy}^{j} \) from the equilibrium equations in (5-71) to (5-74) these variables are expressed in the basic variables using the right-side relations of equations (5-78) to (5-80).

### 5.3.4. Nondimensional Governing Equations

The governing equations of the current problem, consisting of the equilibrium equations in (5-71) to (5-74), the compatibility equations in (5-75) to (5-77), and the additional equations in (5-81) to (5-84) can be written in the following short vector notations

\[ \mathbf{D}_{\chi}^{j} + f_{1}(\mathbf{D}, \mathbf{F}, \bar{\omega}, 0) + f_{1}^{nl}(\mathbf{D}, \mathbf{F}, 0) = 0 \]  
(5-85)

\[ \mathbf{F}_{\chi}^{j} + f_{2}(\mathbf{D}, \mathbf{F}, \bar{\omega}, 0) + f_{2}^{nl}(\mathbf{D}, \mathbf{F}, \bar{\omega}, \Lambda, 0) = 0 \]  
(5-86)

where \( \mathbf{D} \) and \( \mathbf{F} \) denote the vector of nondimensional displacement and stress variables

\[ \mathbf{D}^{T} = \{ u^{j}, v^{j}, w^{j}, \chi^{j}, w^{c} \} \quad \mathbf{F}^{T} = \{ n_{x}^{j}, n_{xy}^{j}, q_{x}, m_{x}^{j}, q_{c}, q_{\Omega}^{c} \} \]  
(5-87)

and \( f_{i}, f_{1}^{nl} \) for \( i = 1, 2 \) denote the linear and nonlinear functions in \( \mathbf{D} \) and \( \mathbf{F} \) and its derivatives with respect to the circumferential coordinate \( \theta \). The full definition of the nondimensional governing equations (5-85) and (5-86), nondimensional boundary conditions, as well as the definition of nondimensional variables are given in Appendix B.4.

### 5.4. Reduction to Ordinary Differential Equations

Arbocz in [7] considers three types of solution methods to solve the two dimensional field problems of shell structures. Based on the complexities in the numerical analysis of the solution method, the solution method is categorized into three hierarchical level computational modules.

In the first level computational modules, the problem is directly reduced into algebraic equations by using double trigonometric Fourier expansions. Only a simple
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type of boundary condition can be satisfied in this analytical solution. Although this method can provide an insight of the problem very quickly, it is sometimes not very accurate.

In the second level computational modules, the problem is first reduced to one dimensional problem by using single trigonometric Fourier expansions. Thereafter a numerical method (discretization) is applied to solve the remaining problem. This method can satisfy arbitrary boundary condition at the edges of the coordinate axis along which the discretization is applied.

The third level computational modules, the most general and expensive one, is dealing with a direct discretization of the field problem into two dimensional approximation, such as finite difference or finite element approach. The advantage of this method is that the general boundary conditions can be satisfied at all boundaries regardless of the form of the field boundaries.

From the viewpoint of the present problem, the second level computational modules is chosen. The boundaries are only found at the edges of the cylinder (one dimensional boundary) and in the circumferential direction the continuity condition is to be fulfilled. Any variable of the problem described in equations (5-85) and (5-86) can be expanded into trigonometric Fourier series in terms of the circumferential coordinate \( \theta \). The trigonometric functions fulfil the periodicity condition in \( 2\pi \), which satisfies the continuity conditions in the circumferential direction.

The nonlinear partial differential equations (5-85) and (5-86) are reduced to a set of ordinary differential equations by expanding the dependent variables in trigonometric Fourier series as follows

\[
\begin{align*}
\begin{bmatrix}
u^j(x^j, \theta) \\
v^j(x^j, \theta) \\
w^j(x^j, \theta) \\
\chi^j(x^j, \theta)
\end{bmatrix}
&= \frac{1}{2}
\begin{bmatrix}
u_{c0}^j(x^j) \\
v_{c0}^j(x^j) \\
w_{c0}^j(x^j) \\
\chi_{c0}^j(x^j)
\end{bmatrix}
+ \sum_{n=1}^{N}
\begin{bmatrix}
u_n^j(x^j) \\
v_n^j(x^j) \\
w_n^j(x^j) \\
\chi_n^j(x^j)
\end{bmatrix}
\begin{bmatrix}
\cos(n \theta) \\
\sin(n \theta)
\end{bmatrix} \\
\begin{bmatrix}
\begin{bmatrix}
u_{x0}^j(x^j) \\
v_{x0}^j(x^j) \\
q_{x0}^j(x^j) \\
m_{x0}^j(x^j)
\end{bmatrix}
&= \frac{1}{2}
\begin{bmatrix}
u_{x0}^j(x^j) \\
v_{x0}^j(x^j) \\
q_{x0}^j(x^j) \\
m_{x0}^j(x^j)
\end{bmatrix}
+ \sum_{n=1}^{N}
\begin{bmatrix}
u_{xn}^j(x^j) \\
v_{xn}^j(x^j) \\
q_{xn}^j(x^j) \\
m_{xn}^j(x^j)
\end{bmatrix}
\begin{bmatrix}
\cos(n \theta) \\
\sin(n \theta)
\end{bmatrix}
\end{align*}
\]
Inserting the Fourier series into the governing equations in (5-85) and (5-86), applying Galerkin’s method to eliminate the θ-dependence and eliminating the secondary variable \( q_0 \) (by use of the expression from equation (B.30) in Appendix B.4), yields a set of 18 + 36N ordinary first order nonlinear differential equations. These equations are very lengthy and therefore they are presented here in a compact form by introducing the following vector notations,

\[
\begin{bmatrix}
  \mathbf{w}_c^c(\mathbf{x}_c^c, \theta) \\
  \mathbf{q}_x^c(\mathbf{x}_c^c, \theta) \\
  \mathbf{q}_0^c(\mathbf{x}_c^c, \theta)
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
  \mathbf{w}_0^c(\mathbf{x}_c^c) \\
  \mathbf{q}_0^c(\mathbf{x}_c^c)
\end{bmatrix} + \sum_{n = 1}^{N} \begin{bmatrix}
  \mathbf{w}_n^c(\mathbf{x}_c^c) \\
  \mathbf{q}_n^c(\mathbf{x}_c^c)
\end{bmatrix} \cos(n\theta) + \begin{bmatrix}
  \mathbf{w}_n^c(\mathbf{x}_c^c) \\
  \mathbf{q}_n^c(\mathbf{x}_c^c)
\end{bmatrix} \sin(n\theta)
\]

(5-90)

Then the resulting equations can be rewritten as follows

\[
\begin{align*}
\mathbf{D}^T \mathbf{X}^k + \mathbf{L}_{11}(\mathbf{D}, \mathbf{\bar{w}}) + \mathbf{L}_{12}(\mathbf{F}) &= \mathbf{L}_{11}^n(\mathbf{D}, \mathbf{F}) \\
\mathbf{F}^T \mathbf{X}^k + \mathbf{L}_{21}(\mathbf{D}, \mathbf{\bar{w}}) + \mathbf{L}_{22}(\mathbf{F}, \mathbf{\bar{w}}, \Lambda) &= \mathbf{L}_{22}^n(\mathbf{D}, \mathbf{F}, \mathbf{\bar{w}})
\end{align*}
\]

(5-91)  
(5-92)

where \( \mathbf{L}_{ij} \) and \( \mathbf{L}_{ij}^n \) are respectively linear and nonlinear vector operators. Their components are given in Appendix B.5. Note that here the superscript \( k = t, c, b \) denotes the top face, core and bottom face nondimensional axial coordinate, respectively. \( \mathbf{\bar{w}} \) indicates of the existence of imperfection terms in the operators and \( \Lambda \) denotes the variable loading parameter.

The boundary conditions are treated in the same manner. After expanding the basic variables and the given loads (or given displacements) in Fourier series and eliminating the θ-dependence using Galerkin’s method, two sets of \( 9 + 18N \) algebraic equations, respectively at \( \mathbf{x}_c^k = 0 \) and \( \mathbf{x}_c^k = L/R_c^k \), are obtained. In compact form, the boundary conditions can be written as follows

\[
\mathbf{B}(\mathbf{D}, \mathbf{F}, \Lambda)_{\mathbf{x}_c^k = 0} = 0 \quad \mathbf{B}(\mathbf{D}, \mathbf{F}, \Lambda)_{\mathbf{x}_c^k = L/R_c^k} = 0
\]

(5-93)
where $\Lambda$ denotes a load parameter that determines the intensity of the applied loads.
6

Numerical Procedures

The analysis of the stability behavior of shell structures requires the solution of nonlinear equilibrium equations under a variety of load intensity, imperfection amplitudes, boundary conditions, material property, etc. Of interest is here the location and characterization of the critical loads in these solution sets, i.e. limit and bifurcation points. In section 6.1 we will discuss the solution of nonlinear equilibrium equations and the computation of a limit point of the solution path. Once the solution of the equilibrium equations is obtained, bifurcation points are sought by the adjacent equilibrium criterion discussed in section 6.2.

6.1. Computation of Nonlinear Equilibrium Equations

It is obviously not possible to obtain an analytical solution for the nonlinear governing equations (5-91) to (5-93). This section is devoted to a numerical procedure for calculating the fundamental or primary equilibrium solution path which is needed for a computation of the critical points. Once the equilibrium solution path has been established, the existence of the first critical point (limit point) can also be determined.

Two basic approaches to the solution of the limit point problem for small imperfections structures are available today, namely Koiter's initial postbuckling method and the continuation method. In chapter 3 the first approach has been discussed to the solution of this problem and successfully implemented in the computer code. Since the development of this approach is based on power series expansion techniques, it only yields an approximate solution of the equilibrium locally around the pre determined solution point of equilibrium at which the series are evaluated.

In contrast to the first method, the continuation methods, which try to solve the equations step by step, produce results that are no longer confined to a particular region of solution space. The key difference between this and Koiter's approach is the use of iteration methods. In the beginning many of the procedures developed were hampered by an inappropriate choice of the continuation parameter which is resulted in an awkward behavior in the neighborhood of limit points. These difficulties were removed when Riks in [48] suggested to use adaptive continuation
parameters. Since then the continuation technique is looked at as a most versatile
technique for the solution of the aforementioned problem.

6.1.1. Continuation Procedure

The two points boundary value problem in equations (5-91) and (5-92) is specified
by \( 9 + 18N \) nonlinear ordinary differential equations and the boundary conditions
in (5-93) with \( 9 + 18N + 1 \) unknowns \( (Y^t = \{D^t, F^t\} \subseteq R^{9+18N} \) and \( \Lambda \)). Thus, in order
to solve the problem completely one extra condition is needed. For example one can
choose straightforward the load factor \( \Lambda \) as a prescribed variable.

For a given value of \( \Lambda \), these equations can admit a number solutions \( Y^t \) because
of their nonlinear behavior. However, we will be focussed only on the solutions \( Y^t \)
which correspond to a continuous deformation of the structure from the undeformed state. The equilibrium solution of the un-deformed state is usually
known close to the origin of the load-displacement space and a solution path
through this reference state is called a primary or fundamental equilibrium solution path. Since there is no analytical solution for this boundary value equations,
they will be solved numerically in sequel.

As is well known, iteration methods for the solution of nonlinear equations require
a starting approximation which is “close” to the solution to be determined. This
because most iteration methods (e.g. Newton’s method) are only locally convergent
[39, 43]. A “good” starting approximation for the solution of a still unknown point
can be constructed by using information that belongs to the initially known solution
point and keeping the distance between the known and the still unknown point within certain bounds. In the same way, the newly obtained solution point
can be used to construct the starting configuration for the next point to be computed. By continuing this process a desired number of solution points along the path can be obtained which will give a discrete image of the solution path. It
becomes obvious that the key to the success of this iteration process depends on the
quality of a starting approximation. In the following we will first tackle this crucial subject.

Two well known procedures are illustrated in figures 6-1.a and 6-1.b for constructing a starting approximation. In the first case, the load parameter \( \lambda \) is used as the
prescribed variable (denoted by path parameter \( \eta \) ) which is then held constant at
its value in the iteration as a constraint parameter. This procedure is better known
as an incremental load procedure. In the second case, the displacement parameter
\( \delta \) is taken to fulfill this role and the procedure is then called an incremental dis-
placement procedure.
Figure 6-1. Load incrementing procedure (a), displacement incrementing procedure (b), arc-length procedure (c)

It is seen from the figure that the incremental load procedure and incremental displacement procedure cannot pass through the limit point A and the turning point B, respectively. These two critical points are defined as follows

\[
\lambda' = \frac{d\lambda}{ds} = 0 \quad \text{for} \quad \eta = \eta^A \tag{6-1}
\]

\[
\delta' = \frac{d\delta}{ds} = 0 \quad \text{for} \quad \eta = \eta^B \tag{6-2}
\]

where \( s \) denotes the arc length of the solution path. Riks has explained in [49] why both procedures may break down in the neighborhood of the critical points, in terms of the quality of the intersection of the constraint surfaces with the solution path. In the same paper he introduced a so-called arc-length procedure in which an approximation to the arc-length is used as the continuation parameter, see figure 6-1.c. This constraint parameter can handle effectively the previously mentioned difficulties and will be applied in the current problem.

Let a known configuration and its tangent to the solution curve at the known point \( \eta_k \) be defined as follows

\[
\mathbf{Y}^l(\eta_k) = \{ \mathbf{D}^l(\eta_k), \mathbf{F}^l(\eta_k) \} \quad \Lambda(\eta_k) \tag{6-3}
\]

\[
\mathbf{Y}(\eta_k) = \frac{d}{d\eta}(\mathbf{Y}(\eta_k)) \quad \Lambda'(\eta_k) = \frac{d}{d\eta}(\Lambda(\eta_k)) \tag{6-4}
\]

The constraint equations is then specified as follows
\[ Y^i(\eta_k)\{ Y(\eta) - Y(\eta_k) \} + \Lambda'(\eta_k)\{ \Lambda(\eta) - \Lambda(\eta_k) \} = (\eta_k - \eta) \] (6-5)

This equation defines a surface which is normal to the tangent in equation (6-4) and its distance to the known point in equation (6-3) is \( \eta - \eta_k \). It will be intersected by the equilibrium curve if the distance \( \eta - \eta_k \) is kept small enough, even in the vicinity of a turning point as displayed in figure 6-1.c. This means that \( \eta \), introduced by equation (6-5), serves as an approximation of the arc length (s) of the solution curve. The constraint equation (6-5) completes the equilibrium equations and the boundary conditions in equation (5-91) to (5-93) for a computation of the primary path and an evaluation of the critical point (e.g. limit load) along the path considered.

*Newton’s method*

In general we need iteration methods to compute a particular point \( \{ Y^i(\eta), \Lambda(\eta) \} \) for some value of the path parameter \( \eta \) due to the nonlinear behavior of equations (5-91) and (5-92). The method of Newton [39, 43] is used for this purpose. Suppose that we have a configuration \( \{ Y^i(\eta), \Lambda(\eta) \} \) in the neighborhood of the exact solution. In Newton’s method the improved approximation \( \{ Y^i(\eta), \Lambda(\eta) \} \) can be expressed as follows

\[ Y^i(\eta) = Y(\eta) + \Delta Y(\eta) \] (6-6)

\[ \Lambda(\eta) = \Lambda(\eta) + \Delta \Lambda(\eta) \] (6-7)

where \( \{ \Delta Y^i(\eta), \Delta \Lambda(\eta) \} \) are the corrections of the original approximation \( \{ Y^i(\eta), \Lambda(\eta) \} \).

Introducing the improved approximations from equations (6-6) and (6-7) into the equilibrium equations and the boundary conditions in equations (5-91) to (5-93) and the constraint equation in (6-5) and retaining only linear terms of the corrections yields,

\[ \Delta D_{\cdot i}(\eta) + d\Delta D(\eta) + c\Delta F(\eta) - d^F(Y(\eta))\Delta D(\eta) = R^D(Y(\eta)) \]

\[ \Delta F_{\cdot i}(\eta) + b\Delta D(\eta) + a\Delta F(\eta) + e\Delta \Lambda(\eta) - b^D(Y(\eta))\Delta D(\eta) \]

\[ - a^F(Y(\eta))\Delta F(\eta) = R^F(Y(\eta)) \] (6-8)

\[ B_{b0}\Delta Y(\eta) + b_{b0}\Delta \Lambda(\eta) = -B(Y(\eta), \Lambda(\eta))_{\xi^b = 0} \]

\[ B_{N}\Delta Y(\eta) + b_{N}\Delta \Lambda(\eta) = -B(Y(\eta), \Lambda(\eta))_{\xi^b = L - R^b} \]

and
\[ Y'(\eta_k) \Delta Y(\eta) + \Lambda'(\eta_k) \Delta \Lambda(\eta) = R(Y(\eta), \Lambda(\eta)) \]  

(6-9)

where \( \eta_k \) denote the path parameter of the previously known configuration and

\[
B_0 = \begin{bmatrix} \frac{\partial B}{\partial D} & \frac{\partial B}{\partial F} \end{bmatrix}_{\eta_k = 0} \\
\frac{\partial B}{\partial D} & \frac{\partial B}{\partial F} \end{bmatrix}_{\eta_k = 1 - R_k} \\
\frac{\partial B}{\partial D} & \frac{\partial B}{\partial F} \end{bmatrix}_{\eta_k = 1 - R_k}
\]

\[
b_0 = \left\{ \frac{\partial B}{\partial \Lambda} \right\}_{\eta_k = 0} \\
b_k = \left\{ \frac{\partial B}{\partial \Lambda} \right\}_{\eta_k = 1 - R_k}
\]

\[a = \frac{\partial L_{22}}{\partial F}, \quad b = \frac{\partial L_{21}}{\partial D}, \quad c = \frac{\partial L_{12}}{\partial F}, \quad d = \frac{\partial L_{11}}{\partial D}, \quad e = \frac{\partial L_{22}}{\partial \Lambda}
\]

\[R(Y(\eta), \Lambda(\eta)) = -Y'(\eta_k)\{Y(\eta) - Y(\eta_k)\} - \Lambda'(\eta_k)\{\Lambda(\eta) - \Lambda(\eta_k)\} + (\eta - \eta_k)
\]

\[R^D(Y(\eta)) = -D_{\eta^k}(\eta) - L_{11}(D(\eta)) - L_{12}(F(\eta)) + L_{11}(Y(\eta))
\]

\[R^F(Y(\eta)) = -F_{\eta^k}(\eta) - L_{21}(D(\eta)) - L_{22}(F(\eta)) + L_{22}(Y(\eta))\]  

(6-10)

The iteration process in equations (6-6) to (6-9) is repeated successively until convergence is obtained, that means the improved approximations \( \{Y^m(\eta), \Lambda^m(\eta)\} \) approach the exact solution sufficiently close. In section 6.1.3 we will pay more attention to the convergence criterion of this process.

**Constructing a starting approximation**

As it was previously mentioned, this method requires a starting approximation \( \{Y'(\eta), \Lambda(\eta)\} \) in sufficiently close neighborhood of the exact solution. To obtain a good starting configuration, one can apply one-step Euler's method

\[Y(\eta) = Y(\eta_k) + (\eta - \eta_k)Y'(\eta_k)\]  

(6-11)

\[\Lambda(\eta) = \Lambda(\eta_k) + (\eta - \eta_k)\Lambda'(\eta_k)\]  

(6-12)

This procedure requires the computation of the tangent to the solution curve at each known point \( \eta_k \) of the path. The equations that govern the tangents \( \{Y'(\eta_k),\Lambda'(\eta_k)\} \) are obtained by differentiating equations (5-91) to (5-93) and equation (6-5) with respect to the path parameter \( \eta \). This yields

\[D'_{\eta^k}(\eta_k) + dD'(\eta_k) + cF'(\eta_k) - dF'(Y(\eta_k))D'(\eta_k) = 0\]

\[F'_{\eta^k}(\eta_k) + bD'(\eta_k) + aF'(\eta_k) + e\Lambda'(\eta_k)\]
\[ -b^D(\mathbf{Y}(\eta_k))\mathbf{D}'(\eta_k) - a^F(\mathbf{Y}(\eta_k))\mathbf{F}'(\eta_k) = 0 \quad (6-13) \]

\[ \mathbf{B}_0\mathbf{Y}'(\eta_k) + \mathbf{b}_0\Lambda'(\eta_k) = 0 \]

\[ \mathbf{B}_N\mathbf{Y}'(\eta_k) + \mathbf{b}_N\Lambda'(\eta_k) = 0 \]

and

\[ \mathbf{Y}'(\eta_{k-1})\mathbf{Y}'(\eta_k) + \Lambda'(\eta_{k-1})\Lambda'(\eta_k) = 1 \quad (6-14) \]

It is obvious that the tangent of the starting point must be determined initially (i.e. \{\mathbf{Y}'(\eta_{k-1}), \Lambda'(\eta_{k-1})\}). For instance the reference configuration in the undeformed state \{\mathbf{Y}(0), \Lambda(0)\} = \{0,0\}. In that case, to obtain the tangent at the initial point usually the load parameter is taken as the path parameter \(\eta\). This means that the constraint equation in (6-14) becomes \(\Lambda'(0) = 1\).

After constructing a starting approximation with equations (6-11) and (6-12), one can compute the improved approximations \{\mathbf{Y}'(\eta), \Lambda'(\eta)\} in the Newton iteration process. Equations (6-11) and equation (6-12) can be solved for the tangent at the new point \(\eta\), which is required to reconstruct the constraint equation for the new solution point and to make a starting approximation that can be used to compute the exact solution at \(\eta\).

### 6.1.2. Finite Difference Discretization

The first order ordinary differential equations in the axial coordinates \(x^i, x^c, x^b\), i.e. equations (6-8) and (6-9) for the Newton’s iteration method and equations (6-13) and (6-14) for the tangents of the solution point, will be written in finite difference expression by using an \(O(h_i^2)\) accuracy central finite difference formulation.

The subdivision of the \(x^i, x^c, x^b\) -axis of the top face, the core and the bottom face coordinates, respectively, into finite difference gridpoints is displayed in figure 6-2. Note that the distance \(h_i\) is the interval between two subsequence gridpoints and can be arbitrarily chosen. In some cases, in order to achieve a converged solution, we need a variable distance gridpoints scheme in which a more fine distribution of gridpoints is applied to adjust a certain interval of the scheme. The first order differentials and the function values of the faces and core variables at a midpoint between two subsequent gridpoints can be written in terms of the variables at the gridpoints as follows

\[ \mathbf{Y}_{i+\frac{1}{2}} \approx \frac{R^c_i}{h_i+1} (\mathbf{Y}_{i+1} - \mathbf{Y}_{i}) + O(h_i^2) \]
\[ \Delta Y_{i+\frac{1}{2},k} = \frac{R^c}{h_{i+1}R} (\Delta Y_{i+1} - \Delta Y_i) + O(\tilde{h}_i^2) \]

\[ Y_{i+\frac{1}{2}} = \frac{1}{2} (Y_{i+1} + Y_i) + O(\tilde{h}_i^2) \]

\[ \Delta Y_{i+\frac{1}{2}} = \frac{1}{2} (\Delta Y_{i+1} + \Delta Y_i) + O(\tilde{h}_i^2) \]  \hspace{1cm} (6-15)

and

\[ Y_{i+\frac{1}{2},b} = \frac{R^c}{h_{i+1}R} (Y_{i+1} - Y_i) + O(\tilde{h}_i^2) \]

\[ Y_{i+\frac{1}{2}} = \frac{1}{2} (Y_{i+1} + Y_i) + O(\tilde{h}_i^2) \]  \hspace{1cm} (6-16)

where superscripts \( k = t, c, b \) denote for the quantities of the top face, the core and the bottom face, respectively, and \( \tilde{h}_i = \frac{h_i}{R^c} \) is the nondimensional distance between two gridpoints at the mid-surface of the core.

**Figure 6-2. Finite difference grid of in the axial direction \( x \)**

After formal substitution of the expression from equation (6-15) into the Newton's corrector equations in equations (6-8) and (6-9), neglecting the terms of order \( O(\tilde{h}_i^2) \) and higher, yields a set of linear algebraic equations which can be written in block-tridiagonal form as follows
where

\[ \begin{bmatrix} D_0 & U_0 & -g_0 \\ L_1 & D_1 & U_1 & -g_1 \\ \vdots & \vdots & \vdots & \vdots \\ L_{N-1} & D_{N-1} & U_{N-1} & -g_{N-1} \\ L_N & D_N & -g_N \end{bmatrix} \begin{bmatrix} \Delta Y(\eta)_0 \\ \Delta Y(\eta)_1 \\ \vdots \\ \Delta Y(\eta)_{N-1} \\ \Delta Y(\eta)_N \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix} \]

(6-17)
\[ g_i = \begin{bmatrix} 0 \\ e \end{bmatrix} \quad \quad f_i = \begin{bmatrix} R^D(Y(\eta))_{i-1/2} \\ R^D(Y(\eta))_{i+1/2} \end{bmatrix} \quad ; \quad i = 1, \ldots, N-1 \]

\[ g_N = \begin{bmatrix} 0 \\ b_N \end{bmatrix} \quad \quad f_N = \begin{bmatrix} R^D(Y(\eta))_{N-1/2} \\ -B(Y(\eta), \Lambda(\eta))_N \end{bmatrix} \]

(6-18)

and

\[ n_i = Y'(\eta_k) \quad ; \quad i = 0, \ldots, N-1 \]

\[ n = \Lambda'(\eta_k) \]

Similarly, after substitution of the expressions from equation (6-16) into equations (6-13) and (6-14) for the tangents of a starting approximation and neglecting the terms of order \( O(h^2) \) and higher yields the following set of linear algebraic equations in block-tridiagonal form

\[
\begin{bmatrix}
D_0 & U_0 & & & & & \\
L_1 & D_1 & U_1 & & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & L_{N-1} & D_{N-1} & U_{N-1} & -g_0 \\
& & & & L_N & D_N & -g_N \\
 n_0 & n_1 & \ldots & n_{N-1} & n_N & n &
\end{bmatrix}
\begin{bmatrix}
y'_{(\eta_k)}_0 \\
y'_{(\eta_k)}_1 \\
\vdots \\
y'_{(\eta_k)}_{N-1} \\
y'_{(\eta_k)}_N \\
\Lambda'_{(\eta_k)}
\end{bmatrix} =
\begin{bmatrix}
o \\
o \\
\vdots \\
o \\
o \\
1
\end{bmatrix}
\]

(6-19)

where sub-matrices \( D_i, L_i \) and \( U_i \) are the same as given in equation (6-18), but instead of the functions of the known approximations, they are functions of the known solution \( \{ Y'(\eta_k), \Lambda(\eta_k) \} \). Furthermore the matrices \( n_i \) and the scalar \( n \) are functions of the tangent of the previously known point \( \{ Y'(\eta_{k-1}), \Lambda'(\eta_{k-1}) \} \).

### 6.1.3. General Computational Strategy

In this section the computational procedure discussed in the previous sections is summarized. For the sake of simplicity, the subscripts denoting the gridpoint number are dropped and all vectors are now considered as full vectors containing sub-vectors corresponding to all gridpoints in the finite difference scheme (e.g. a vector \( Y'(\eta) \) represents actually the vector \( \{ Y'(\eta)_0, Y'(\eta)_1, \ldots, Y'(\eta)_N \} \)).

As it was mentioned earlier in section 6.1.1, the solutions along the path for specified value of the path parameter cannot be obtained in an arbitrary way. The solutions must be computed step by step along the solution curve. In each step, information that belongs to the solution points previously computed is used to con-
struct a suitable starting approximation for the iteration cycle. Thus the basic feature of a continuation method is the computation of a set of points:

\[ \{ \mathbf{Y}(\eta), \Lambda(\eta) \} \quad \text{for} \quad \eta : \eta_k < \eta_{k+1} < \eta_{k+2} < \ldots < \eta_K \]  

(6-20)

Suppose that of the equilibrium path to be computed, one point is known in advance, e.g. \( \{ \mathbf{Y}(\eta_k), \Lambda(\eta_k) \} \), together with its tangent to the solution curve \( \{ \mathbf{Y}(\eta_k), \Lambda'(\eta_k) \} \). Then the general computational algorithm is organized as follows

**Step 1.** Construct starting approximation (predictor):

The first approximation of the next solution point at \( \eta = \eta_{k+1} \) is computed by applying one step Euler's method (see equations (6-11) and (6-12))

\[ \mathbf{Y}^{(1)}(\eta_{k+1}) = \mathbf{Y}(\eta_k) + (\eta_{k+1} - \eta_k) \mathbf{Y}'(\eta_k) \]

(6-21)

\[ \Lambda^{(1)}(\eta_{k+1}) = \Lambda(\eta_k) + (\eta_{k+1} - \eta_k) \Lambda'(\eta_k) \]

(6-22)

**Step 2.** Improve approximation by iteration method (corrector):

The approximate solution is improved successively by Newton's iteration as follows (see equations (6-8) and (6-9))

\[ \mathbf{K}(\mathbf{Y}^{(j)}(\eta_{k+1}), \Lambda^{(j)}(\eta_{k+1})) \Delta \mathbf{Y}^{(j)}(\eta_{k+1}) - \mathbf{g}(\mathbf{Y}^{(j)}(\eta_{k+1}), \Lambda^{(j)}(\eta_{k+1})) \Delta \Lambda^{(j)}(\eta_{k+1}) = \mathbf{f}(\mathbf{Y}^{(j)}(\eta_{k+1}), \Lambda^{(j)}(\eta_{k+1})) \]

(6-23)

\[ \mathbf{n}^T(\mathbf{Y}^{(j)}(\eta_k) \Delta \mathbf{Y}^{(j)}(\eta_{k+1}) + n(\Lambda^{(j)}(\eta_k) \Delta \Lambda^{(j)}(\eta_{k+1}) = \mathbf{R}(\mathbf{Y}^{(j)}(\eta_{k+1}), \Lambda^{(j)}(\eta_{k+1})) \]

(6-24)

where the coefficient matrix \( \mathbf{K} \) is the block-tridiagonal matrix in equation (6-17) and the short notations \( \mathbf{Y}^{(j)}(\eta_{k+1}) = \mathbf{Y}^{(j)}(\eta_{k+1}), \Lambda^{(j)}(\eta_{k+1}) = \Lambda^{(j)}(\eta_{k+1}) \), etc. are used. This matrix equations can be solved by using the parallel Potter's method discussed in Appendix C.1. Once the corrections are obtained, the improved solutions can be calculated as follows (see equations (6-6) and (6-7))

\[ \mathbf{Y}^{(j+1)}(\eta_{k+1}) = \mathbf{Y}^{(j)}(\eta_{k+1}) + \Delta \mathbf{Y}^{(j)}(\eta_{k+1}) \]

(6-25)

\[ \Lambda^{(j+1)}(\eta_{k+1}) = \Lambda^{(j)}(\eta_{k+1}) + \Delta \Lambda^{(j)}(\eta_{k+1}) \]

(6-26)

This iteration process is repeated successively for \( j = 1, 2, \ldots, J \), until the approximate solution \( \{ \mathbf{Y}^{J}(\eta_{k+1}), \Lambda^{J}(\eta_{k+1}) \} \) converges to a solution \( \{ \mathbf{Y}^{*}(\eta_{k+1}), \Lambda^{*}(\eta_{k+1}) \} \) which approaches the exact solution with sufficient accuracy. Note that if the coefficient matrix \( \mathbf{K} \) of the first iteration step \( (j = 1) \) is used for the next iteration steps \( (j > 1) \) then the method of iteration is better known as the modified Newton's iteration. The itera-
tive improvements of each solution point are terminated after the following conditions are satisfied

\[
\left\| \mathbf{f}^i(\mathbf{Y}_{k+1}, \Lambda_{k+1}^{(j)}, R(\mathbf{Y}_{k+1}, \Lambda_{k+1}^{(j)})) \right\| \leq \varepsilon_R \tag{6-27}
\]

\[
\left\| \Delta \mathbf{Y}_{k+1}^{(j)}, \Delta \Lambda_{k+1}^{(j)} \right\| \leq \varepsilon_C \tag{6-28}
\]

where \(\varepsilon_R\) and \(\varepsilon_C\) are small, preassigned, positive real numbers denoting the accuracy of the iteration.

**Step 3.** Compute the tangent in the new point:

To calculate the next point of the path, the tangent at the new point can be solved as follows (see equations (6-13) and (6-14))

\[
\mathbf{K}(\mathbf{Y}_{k+1}, \Lambda_{k+1}) \mathbf{Y}_{k+1} - \mathbf{g}(\mathbf{Y}_{k+1}, \Lambda_{k+1}) \Lambda_{k+1}' = 0 \tag{6-29}
\]

\[
\mathbf{n}^t(\mathbf{Y}_{k+1}) \mathbf{Y}_{k+1} + \eta(\Lambda_{k}) \Lambda_{k+1}' = 1 \tag{6-30}
\]

Again, this matrix equations can be solved by using the parallel Potter's method discussed in Appendix C.1.

**Step 4.** Check existence of critical point (limit point) along the solution path:

To inspect the existence of a limit load along the solution path, the tangent of the new point calculated in Step 3 should be checked as follows (see equation (6-1))

\[
\frac{d\lambda}{d\eta_{k+1}} < 0 \tag{6-31}
\]

where \(\lambda\) denotes the loading intensity. If this condition is detected during the computation then a restart procedure should be performed with a smaller stepsize \(\eta - \eta_{k+1}\) until the following criterion is fulfilled

\[
\frac{d\lambda}{d\eta_{k+1}} \leq \varepsilon_{l}. \tag{6-32}
\]

where \(\varepsilon_{l}\) is a small positive number of desired accuracy. It should be noted that in order to perform a restart procedure, in each iteration step, the calculated points along the solution curve have to be saved. One can restart the computation from an arbitrarily chosen previous known point by repeating steps 1 through step 3.

For the computation of the next point, step 1 through step 4 are then repeated.
6.2. Bifurcation Buckling Solution

From figure 6-3 it is seen that the primary equilibrium path can intersect another path (secondary path) at a certain point along its solution path. This intersection point is called a bifurcation point and the corresponding load $\Lambda_c$ is known as the bifurcation buckling load.

![Diagram of bifurcation buckling solution]

**Figure 6-3. Linearized prebuckling state at load level $\Lambda = \Lambda_k$**

The so-called adjacent equilibrium criterion can be used to derive the governing equations for this particular point. Once an equilibrium configuration has been obtained on the primary path by method of solution described in section 6.1, the existence of an adjacent equilibrium configuration under the same applied load is examined.

Linearized equations derived from the nonlinear one by means of perturbation technique are used to determine the existence of the adjacent equilibrium configuration. Thus, to investigate the possible existence of adjacent equilibrium configurations, we give a small perturbation $Y^{(1)}(\eta_k) = \{D^{(1)}(\eta_k), F^{(1)}(\eta_k)\}$ to the equilibrium configuration on the primary path $Y^{(0)}(\eta_k) = \{D^{(0)}(\eta_k), F^{(0)}(\eta_k)\}$ as follows

$$D(\eta_k) \rightarrow D^{(0)}(\eta_k) + D^{(1)}(\eta_k) \quad (6-33)$$

$$F(\eta_k) \rightarrow F^{(0)}(\eta_k) + F^{(1)}(\eta_k) \quad (6-34)$$
Adjacent equilibrium configurations exist if these expressions satisfy the nonlinear equilibrium equations and a non-trivial solution of a small perturbation can be found. The lowest load, at which this criterion is satisfied, is called the critical buckling load.

Introducing the expressions (6-33) and (6-34) into the nonlinear governing equations for a perfect cylinder, i.e. by setting imperfection terms in equations (5-91) and (5-92) to zero, yields a set of equations which are nonlinear in the small perturbations. Since the perturbations are sufficiently small, the nonlinear terms in the small perturbations can be neglected. This results in linear homogeneous equations in the perturbation terms with variable coefficients consisting of the terms of the equilibrium configuration along the primary path.

This homogeneous buckling equations can be solved using the same approach as is used for the solution of the equilibrium equations in the previous section 5.4. The linear partial differential equations are reduced to ordinary differential equations by the use of Fourier series expansion. To excite a bifurcation buckling in asymmetric form, it is necessary to include a circumferential wave number \(n\) in the series for the perturbation which indicates the buckling mode. Consistent to the notation used in the previous section 6.1, the linearized buckling equations can be written as follows

\[
\begin{align*}
D_k^{(1)} \delta \chi_k + dD_k^{(1)} + cF_k^{(1)} - d^F (Y_k^{(0)})D_k^{(1)} &= 0 \\
F_k^{(1)} \delta \chi_k + bD_k^{(1)} + aF_k^{(1)} - [b_1^D (Y_k^{(0)}) + b_2^D (Y_k^{(0)}, Y_k^{(0)})]D_k^{(1)} - a^F (Y_k^{(0)})F_k^{(1)} &= 0
\end{align*}
\]

\[
B_0 Y_k^{(1)} = 0 \quad B_N Y_k^{(1)} = 0 \quad (6-35)
\]

where

\[
B_0 = \left[ \frac{\partial B}{\partial D} \frac{\partial B}{\partial F} \right]_{\chi_k = 0} \quad B_N = \left[ \frac{\partial B}{\partial D} \frac{\partial B}{\partial F} \right]_{\chi_k = 1/R_k}
\]

\[
a = \frac{\partial L_{22}}{\partial F} \quad b = \frac{\partial L_{21}}{\partial D} \quad c = \frac{\partial L_{12}}{\partial F} \quad d = \frac{\partial L_{11}}{\partial D}
\]

\[
a^F (Y_k^{(0)}) = \frac{\partial L_{21}^{nl}}{\partial F} (Y_k^{(0)}) \quad d^F (Y_k^{(0)}) = \frac{\partial L_{11}^{nl}}{\partial D} (Y_k^{(0)})
\]

\[
b_1^D (Y_k^{(0)}) + b_2^D (Y_k^{(0)}, Y_k^{(0)}) = \frac{\partial L_{21}^{nl}}{\partial D} (Y_k^{(0)}) \quad (6-36)
\]
For sake of simplicity, the short notations \( \mathbf{Y}_k^{(0)} = \mathbf{Y}^{(0)}(\eta_k), \mathbf{Y}_k^{(1)} = \mathbf{Y}^{(1)}(\eta_k), \) etc. are used.

Suppose that the solution of discrete points along the primary path prior to the critical buckling load, also referred to as the prebuckling solutions, have been computed already using the continuation method in the previous section. Because of the nonlinear dependence of the prebuckling solutions on the load factor \( \Lambda(\eta), \) it is necessary to approach the critical load by solving a sequence of linearized buckling problems. Thus, to estimate the buckling load \( \Lambda_c = \Lambda(\eta_c) \), the prebuckling solution is linearized about the last known point \( \eta_k \) as shown in figure 6-3

\[
\mathbf{Y}^{(0)}(\eta_c) = \mathbf{Y}^{(0)}(\eta_k) + \Delta \eta_c \mathbf{Y}^{(0)}(\eta_k) \tag{6-37}
\]

\[
\Lambda^{(0)}(\eta_c) = \Lambda(\eta_k) + \Delta \eta_c \Lambda'(\eta_k) \tag{6-38}
\]

Inserting equations (6-37) and (6-38) into the linearized buckling equations in (6-35) and (6-36) yields the following eigenvalue buckling problem with \( \Delta \eta_c \) as an eigenvalue.

\[
\begin{align*}
\mathbf{D}_k^{(1)} + \mathbf{d}^1 \mathbf{D}_k^{(1)} + \mathbf{c}^1 \mathbf{F}_k^{(1)} &= \Delta \eta_c \mathbf{d}^F \mathbf{D}_k^{(1)} \\
\mathbf{F}_k^{(1)} + \mathbf{b}^1 \mathbf{D}_k^{(1)} + \mathbf{a}^1 \mathbf{F}_k^{(1)} &= \Delta \eta_c \mathbf{b}^D \mathbf{D}_k^{(1)} + \Delta \eta_c^2 \mathbf{b}^D \mathbf{D}_k^{(1)} + \Delta \eta_c \mathbf{a}^F \mathbf{F}_k^{(1)} \\
\mathbf{B}_0 \mathbf{Y}_k^{(1)} &= 0 \\
\mathbf{B}_N \mathbf{Y}_k^{(1)} &= 0
\end{align*}
\tag{6-39}
\]

where

\[
\begin{align*}
\mathbf{a}^1 &= \mathbf{a} - \mathbf{a}^F(\mathbf{Y}_k^{(0)}) \\
\mathbf{b}^1 &= \mathbf{b} - \mathbf{b}^F(\mathbf{Y}_k^{(0)}) \\
\mathbf{c}^1 &= \mathbf{c} \\
\mathbf{a}^F &= \mathbf{a}^F(\mathbf{Y}_k^{(0)}) \\
\mathbf{d}^F &= \mathbf{d}^F(\mathbf{Y}_k^{(0)}) \\
\mathbf{b}^D &= \mathbf{b}_1^D(\mathbf{Y}_k^{(0)}) + \mathbf{b}_2^D(\mathbf{Y}_k^{(0)}, \mathbf{Y}_k^{(0)}) + \mathbf{b}_2^D(\mathbf{Y}_k^{(0)}, \mathbf{Y}_k^{(0)}) \\
\mathbf{b}^nD &= \mathbf{b}_2^D(\mathbf{Y}_k^{(0)}, \mathbf{Y}_k^{(0)})
\end{align*}
\]

After introducing an additional equation \( \mathbf{E}_k^{(1)} = \Delta \eta_c \mathbf{D}_k^{(1)} \) and an eigenvalue shift \( \mu_0 \) such that \( \Delta \eta_c = \mu_0 + \mu \), the quadratic eigenvalue problem \( \Delta \eta_c \) in equation (6-39) can be written as the following eigenvalue problem with \( \mu \) as an eigenvalue

\[
\begin{align*}
\mathbf{D}_k^{(1)} + \mathbf{d}^1 - \mu_0 \mathbf{d}^F \mathbf{D}_k^{(1)} + \mathbf{c}^1 \mathbf{F}_k^{(1)} &= \mu \mathbf{d}^F \mathbf{D}_k^{(1)} \\
\mathbf{F}_k^{(1)} + \mathbf{b}^1 - \mu_0 \mathbf{b}^D - \mu_0^2 \mathbf{b}^nD \mathbf{D}_k^{(1)} + \mathbf{a}^1 - \mu_0 \mathbf{a}^F \mathbf{F}_k^{(1)}
\end{align*}
\]

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\[ E_k^{(1)} - \mu_0 D_k^{(1)} = \mu D_k^{(1)} \quad (6-40) \]

\[ B_0 Y_k^{(1)} = 0 \quad B_N Y_k^{(1)} = 0 \]

This ordinary differential eigenvalue equations can be reduced to an algebraic eigenvalue problem using the same finite difference scheme used earlier in section 6.1.2. The eigenvalue problem equation (6-40) is then solved numerically using the standard inverse iteration method [11] to compute the lowest eigenvalue \( \mu \) and its corresponding eigenmode which is the critical buckling mode. For a given eigenvalue shift \( \mu_0 \), the eigenvalue \( \Delta \eta_c \) is directly computed and a first estimate for the critical value of the loading parameter \( \Lambda(\eta_c) \), i.e. the critical buckling load, can be calculated from equation (6-38). If this approximation is not satisfactory, a new estimate should be constructed relative to a point on the primary path which is close to the critical point. Such a point can be computed using the continuation method in which the initial predictor of equations (6-11) and (6-12) is modified to

\[ Y^{(0)}(\eta) = Y^{(0)}(\eta_k) + \varphi \Delta \eta_c Y^{(0)}(\eta_k) \quad (6-41) \]

\[ \Lambda^{(0)}(\eta) = \Lambda(\eta_k) + \varphi \Delta \eta_c \Lambda'(\eta_k) \quad (6-42) \]

where \( \varphi \) is a positive real number less than one. This makes it possible to keep the step size small and to ensure that the critical point is approached from below. After the new point on the primary path has been obtained, the process for computing the critical eigenvalue can be repeated. The eigenvalue iteration process along the primary path is terminated when the following condition is satisfied (see figure 6-3)

\[ \left| \frac{\Lambda_c - \Lambda_k}{\Lambda_k} \right| \leq \varepsilon \quad (6-43) \]

where \( \varepsilon \) is a small, preassigned, positive real number denoting the convergence tolerance.
Bifurcation Buckling Solution
Numerical Results

The computer program SHOT (Sandwich Higher-Order Theory) has been written to implement the theory and numerical analysis as described in the previous chapters. This computer program is written in FORTRAN 77. In this chapter, numerical results obtained with this program are given. Where possible, the results are compared with results known from literature and obtained from program SFOSSDT in Part I and the well known finite element program MSC/NASTRAN [4]. This chapter is divided into seven sections. In the first section a review of Bartelds Unified theory [10] is given and comparison is made with the current program using a membrane prebuckling analysis. In the next two sections the results of nonlinear prebuckling and bifurcation buckling analysis for isotropic honeycomb core sandwich cylinders under axial compression and external pressure are given. Both global and local boundary conditions of sandwich layers at the edges are used. In the fourth section variation of core property is considered. In the fifth section the buckling and material failure behavior of soft honeycomb (Nomex) and foam (Rohacell) core are investigated. The same study is carried out in the sixth section for a sandwich cylinder with composite faces material. Finally, the collapse behavior of imperfect sandwich cylinders is investigated in the last section.

7.1. Membrane Prebuckling Analysis

In [10] Bartelds made a simplification in his unified theory by assuming that the total shell thickness is negligible with respect to the principal radii of curvature of the shells. The core is then treated as a thin layer and its thickness is assumed small in comparison with the shell radius, from which it follows that the core thickness to shell radius ratio is neglected as compared to unity. As a consequence, the difference in the radii \((R^t, R^b, R^c)\) of the two face layers and the core is neglected and his theory is only valid for sandwich cylinders with a thin core. The consideration of the whole sandwich configuration, in particularly the core, as a thin shell together with the assumption of the uniform (membrane) prebuckling deformation lead to a great simplification in the buckling analysis. In addition, he also considered the boundary conditions in the way which is usually done in the case of a thin (isotropic or laminated) shell.
Membrane Prebuckling Analysis

The solution is basically divided into three parts, by assuming three possibilities of instability modes, namely two short-wave instability of wrinkling (symmetric and antisymmetric) and one long-wave instability (general buckling). In general, this is not true since there is an interaction possible between these modes, especially when localized effects such as boundary conditions and loading location are involved. The theory and derivations used to obtain the solutions are not given here and the interested reader should consult [10]. In the end of Appendix B.3 the simplification of the current analysis, which leads to the solution of Bartelds, is shortly discussed and also implemented in the code.

The preliminary study in this section has the purpose to reproduce Bartelds’ results and to obtain some improvements of his results by use of exact (thick) core consideration. The current code SHOT, which is primarily developed to solve the buckling problem using a rigorous nonlinear prebuckling solution, is also capable to solve the buckling problem using a membrane prebuckling analysis. Both options of the solution, with thin and exact core considerations are presented.

Global insight in the failure possibilities of thin sandwich cylinders can be quickly obtained using this approach by performing various parameter studies of both shell geometries and material (core and faces) properties. However, the simplification of this theory by assuming a thin sandwich configuration, membrane prebuckling state and simplification of boundary conditions excluded the possibilities of considering the earlier mentioned localized effects, which will be discussed in section 7.2.

| Table 7-1. Geometry and material properties of the sandwich cylinders investigated |
|---------------------------------|------------------|------------------|
| Geometry:                      |                  |                  |
| L (length) = 30 inch           | t_f (face thickness) = 0.010125 inch |
| R_c (radius of the core) = 18 inch | t_c (core thickness) = 0.405 inch |
| Identical isotropic top and bottom face with: |                  |                  |
| E_f (in-plane modulus of elasticity) = 1.0 \times 10^7 psi | ν (Poisson’s ratio) = 0.3 |
| 4 isotropic honeycomb cores with: |                  |                  |
| E_c (transverse modulus of elasticity) = 1.0 \times 10^2, 2.0 \times 10^5, 2.0 \times 10^4 and 1.0 \times 10^2 psi |
| G_x = G_y = G_c (transverse shear modulus) = E_c / 5 |

The geometry and the properties of the isotropic honeycomb core sandwich cylinders investigated are given in table 7-1. Honeycomb cores with four different
values of transverse modulus of elasticity of the core \( E_c \) are considered, each has an transverse shear modulus \( G_z \) equal to \( E_c/5 \) which is typical for a honeycomb core. The core has an isotropic property and the principal directions of the honeycomb core materials are placed so as to coincide with the reference axis of the cylindrical shell. This implies that \( G_x = G_y = G_z \). For \( E_c = 2.0 \times 10^5 \) psi, the core is identical to the Cunningham-Jacobsen cylinder used in [10]. The top and bottom faces are built from isotropic material and they are of the same thickness.

First, the critical buckling loads are calculated for a moderately thin core sandwich cylinder with \( R^c = 18 \) inch \( (R^c/t^c = 44.44) \) using both thin and exact core considerations. It can be expected that for this case the results from both core considerations are more and less comparable. The results obtained with a thin core assumption should agree with those obtained in [10].

Table 7-2 gives the comparison of the critical bifurcation buckling load results. The classical buckling load \( N_x^{\text{class}} \) found by Bartelds is used for the normalization of the critical values \( \lambda_a^{\text{crit}} \). The third column of the table gives the results with thin core consideration and shows that they are exactly the same as those obtained in [10]. The fourth column gives the critical values using exact core consideration and they are more or less equal to the results using thin core consideration.

### Table 7-2. The critical buckling loads of a thin core sandwich cylinder \((R^c/t^c = 44.4)\) with the variation of modulus of elasticity \( E_c \)

<table>
<thead>
<tr>
<th>( E_c ) (psi)</th>
<th>( N_x^{\text{class}} ) (lbf/inch)</th>
<th>( \lambda_a^{\text{crit}} ) (SHOT)</th>
<th>Buckling mode</th>
<th>( l_m = \frac{L}{t_e^{c_m}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0 \times 10^7</td>
<td>-0.490056.10^4</td>
<td>1.000 (0)</td>
<td>G (m=5)</td>
<td>14.81</td>
</tr>
<tr>
<td>2.0 \times 10^5</td>
<td>-0.387630.10^4</td>
<td>1.000 (0)</td>
<td>S.W (m=305)</td>
<td>0.24</td>
</tr>
<tr>
<td>2.0 \times 10^4</td>
<td>-0.122754.10^4</td>
<td>1.000 (0)</td>
<td>S.W (m=172)</td>
<td>0.43</td>
</tr>
<tr>
<td>1.0 \times 10^2</td>
<td>-0.770505.10^2</td>
<td>1.000 (0)</td>
<td>S.C (m=41)</td>
<td>1.81</td>
</tr>
</tbody>
</table>

G: General Buckling S.W: Symmetric Wrinkling S.C: Shear Crimping

Brackets in the third and fourth columns denote for the number of circumferential fullwaves \( n \). Notice that in all cases \( n = 0 \), that is we have the axisymmetric buckling state.

In the fifth column of the table the appropriate buckling modes are given for both core considerations. Notice that the same buckling mode (m denotes the number of
axial halfwaves) is obtained for all cases. In the last column the ratio of the axial halfwave length to the core thickness \( l_m = L/(t^c) \) are given for both results. Regarding the unified theory of Bartels, this examples give a good illustration of the possibility of buckling of sandwich construction in both local (symmetric wrinkling) and overall (general buckling, shear crimping) mode. Here, the wrinkling failure mode is defined as whether \( l_m \leq 1 \) or in other words, the halfwave length in the axial direction of the buckling mode is of the order of smaller than the thickness of the core \( t^c \).

As it has already been discussed in Part I, decreasing the shear modulus \( G_c \) will cause the sandwich shell to fail in shear crimping mode. The shear crimping criterion for the sandwich cylinder with \( E_f = 1.0 \times 10^7 \) psi, calculated with equation (4-2), reveals that shear crimping will occur for \( G_c \leq 6.04 \times 10^3 \) psi. This explains the shear crimping failure mode for \( G_c = 20 \) psi and \( E_c = 100 \) psi, in which the faces

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**Figure 7-1.** Displacements and resultants of the faces and stresses at the interfaces for sandwich cylinder with \( E_c = 2.0 \times 10^5 \) psi.

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deform in antisymmetric mode with many axial halfwaves but the wavelength is still greater than the core thickness.

Note that this mode occurs in the entire cylinder length with the same amplitude. However, the approximate formula for shear crimping mentioned does not depend on the value of $E_c$ and only gives a good approximation for all the cases studied in Part I, i.e. for a core with stiff $E_c$. This maybe is the reason that the buckling mode for $G_c = 400$ psi and $E_c = 2.0 \times 10^4$ psi still occurs in symmetric wrinkling rather than in shear crimping mode.

In figure 7-1, membrane prebuckling responses for $E_c = 2.0 \times 10^5$ psi at the critical buckling load level are displayed for both core considerations. It can be seen that when using exact core consideration, radial displacements of the top and the bottom face are no longer equal. The distribution of axial stress resultants of both faces along the axial coordinate are also different. However, since buckling occurs at almost the same load level in both core considerations, the total axial resultant for both cases are equal.

The radial stresses at the interfaces between the core and the adjacent faces are zero in the case of thin core consideration. Important to note that in the case of exact core consideration they are not zero and constant through the axial direction. In both cases, however, the shear resultants in the faces and shear stresses at the interfaces are zero due to membrane assumptions.

*Figure 7-2. Symmetrical wrinkling mode (buckling) for sandwich cylinder with $E_c = 2.0 \times 10^5$ psi*
Membrane Prebuckling Analysis

The corresponding buckling modes of the top and bottom faces are displayed in figure 7-2 for $E_c = 2.0 \times 10^5$ psi for 1/4 length of the cylinder and show that the amplitude of the symmetrical wrinkling mode is constant along the axial direction x.

**Table 7-3. The critical buckling loads of a thick core sandwich cylinder ($R^c / t^c = 22.2$) with the variation of modulus of elasticity $E_c$**

<table>
<thead>
<tr>
<th>$E_c$</th>
<th>$N^c_x$ (lbf/inch)</th>
<th>$\lambda^c_{a_{(n,m)}}$</th>
<th>$l_m$</th>
<th>$\lambda^c_{a_{(n,m)}}$</th>
<th>$l_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.0 \times 10^7$</td>
<td>$-0.977067 \times 10^4$</td>
<td>1.000 (2,6)</td>
<td>12.35</td>
<td>0.917 (3,1)</td>
<td>74.07</td>
</tr>
<tr>
<td>$2.0 \times 10^5$</td>
<td>$-0.387809 \times 10^4$</td>
<td>1.000 (0,305)</td>
<td>0.24</td>
<td>0.999 (0,305)</td>
<td>0.24</td>
</tr>
<tr>
<td>$2.0 \times 10^4$</td>
<td>$-0.123322 \times 10^4$</td>
<td>1.000 (0,172)</td>
<td>0.43</td>
<td>0.999 (0,172)</td>
<td>0.43</td>
</tr>
<tr>
<td>$1.0 \times 10^2$</td>
<td>$-0.145629 \times 10^3$</td>
<td>1.000 (0,58)</td>
<td>1.28</td>
<td>0.996 (0,58)</td>
<td>1.28</td>
</tr>
</tbody>
</table>

Note: $l_m = L/(t^m)$

![Diagram](image)

(a) General buckling ($n = 3$)  
(E$_c$ = $1.0 \times 10^7$ psi)

(b) General buckling ($n = 0$)  
(E$_c$ = 100 psi)

**Figure 7-3. Instability modes (buckling) of sandwich cylinders with high and low values of modulus of elasticity $E_c$**

Next, the critical buckling loads with the variation of $E_c$ are calculated for a thick core sandwich cylinder with $R^c = 9$ inch ($R^c / t^c = 22.22$) using both core considerations and the results are given in table 7-3. It can be seen that the differences of the results obtained using both thin and exact core considerations increase when buckling occurs in overall (antisymmetric) mode. For $E_c = 1.0 \times 10^7$ psi, the corresponding buckling modes have also different modes, as can be seen from the bracket values.

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In figure 7-3, the buckling modes of the top and bottom faces for $E_c = 1.0 \times 10^7$ psi and $E_c = 100$ psi are displayed. They are antisymmetric with respect to the mid-height of the sandwich layers. For $E_c = 2.0 \times 10^4$ psi and $2.0 \times 10^5$ psi, the critical loads (symmetric wrinkling) found are only a fraction higher than those found for $R^c = 18.0$ inch with the same modes. This indicated that, for the same faces and core, symmetric wrinkling is hardly influenced by the shell geometry (radius of the cylinder).

**Remarks on the results of membrane prebuckling analysis**

- One can observe from the numerical examples presented here that an error is occurred in the modeling approach using thin core consideration when the core is thick ($R^c/t^c << 50$), especially in the case of the overall instability mode. The exact core formulation should be used in the modeling of the core which can guarantee that also for a thick core one will obtain a good solution.

- When local instability with symmetric wrinkling occurs, the critical load is almost constant despite the variation of the radius of cylinder. On the other hand, overall instability with general buckling mode is sensitive to the change of the cylinder geometry.

- In the analysis and numerical study presented localized effects were neglected and a membrane prebuckling state was assumed. The amplitude of critical buckling modes, either local or overall instability mode, is constant along the cylinder length which indicates the absence of the localized effects. Interaction between overall and local instability, mostly triggered by these effects, is not considered in this study. In general, the failure of sandwich structures will occur in the mixed form of overall and local mode. Wrinkling instabilities of the faces are no longer symmetric or antisymmetric and sometimes occur only in a limited area near the edges. In the next section, this phenomena will be extensively investigated by the use of nonlinear analysis.

**7.2. Nonlinear Prebuckling Response**

A more realistic way to describe the sandwich shell behavior under increasing applied load is obtained when full nonlinear solution is performed and the effects of boundary conditions are taken into account. For this purpose, we investigate the response of the cylinder with the increase of the axial compression and external pressure. Unless otherwise stated, the core of the sandwich cylinder considered has a modulus of elasticity $E_c = 2.0 \times 10^4$ psi and a shear modulus $G_c = 4.0 \times 10^3$ psi within the range of practically used honeycomb core, as can be found in several handbook of materials [29, 38]. The same shell geometry and faces properties will be used as
given in the previous table 7.1. Two hypothetical weak core sandwich cylinders with $E_c = 100$ psi will be also investigated. Both global and local boundary conditions will be used in order to investigate the effects of boundary conditions on the nonlinear response of sandwich cylinders under two applied loads mentioned.

**Global Boundary conditions**

The investigation is started by using global simply supported SS3G and clamped C4G boundary conditions at the edges of the cylinder. Physically, as it has been mentioned earlier in section 5 this kind of boundary conditions means that one prescribes either displacement or stress variables of a point (say point G) at the mid-height of the rigid stiffeners which have been attached to the sandwich layers section at both edges of the cylinder. This global boundary conditions are given in equations (5-43) to (5-51) in chapter 5 and the definition of simply supported SS3G and clamped C4G boundary conditions are given in table B-1 in Appendix B.6.

**Axial compression**

The first case considered is a sandwich cylinder with $E_c = 2.0 \times 10^4$ psi and $G_c = 4.0 \times 10^3$ psi under axial compression. The applied load is introduced at the mid-height of the rigid stiffeners and transmitted through the top face and the bottom face into the entire cylinder. Beside looking closely at the deformation pattern and stresses distributions in the faces, we will investigate the radial and shear stress distribution along the interfaces between the core and the adjacent faces by increasing the applied load.

Figures 7-4 and 7-5 display the distribution of displacements and stress resultants in the faces as function of the axial coordinate at $\lambda_a = 0.3$ ($N_x = -372.5$ lbf/inch) for simply supported SS3G (left-hand) and clamped C4G (right-hand) boundary conditions. The distributions of shear and radial stresses at the interfaces along the axial coordinate $x$ are displayed in figure 7-6. All other quantities of the unknown variables that are not displayed here, i.e. $V^t, V^b, N_{xy}^t, N_{xy}^b$ and $\tau_c^t, \tau_c^b$ are equal to zero.

The radial displacements $W^t$ and $W^b$ for the global boundary conditions SS3G and C4G are depicted in figure 7-4.a. On the contrary to the results obtained with membrane prebuckling analysis in the previous section, the edge effects are now clearly present. Note that at this load level the radial displacements of both faces ($W^t, W^b$) are more or less identical. At this load level, the effect of the transverse compressibility of the core in its thickness direction is not yet observable.

Axial displacements $U^t$, $U^b$ are depicted in figure 7-4.b and they are of one order greater than the radial displacements which can be expected since the cylinder is loaded mainly in the axial direction. For the C4G case, due to the condition of zero
rotation of the rigid stiffener at the supports, the axial displacements of both faces are identical at the edges. However, there are not the same through the cylinder length. On the other hand, the axial displacements in the SS3G case are not identical due to the rotation of the rigid stiffener about its mid-point (point G).

Despite that there is no rotation of the faces for the most part of the cylinder, see figure 7-4.c, the rotations $\beta_x^t, \beta_x^b$ of the faces are not zero at the supports. Even for the clamped C4G case the rotation of the individual face is not zero. This is because the clamped global boundary conditions define zero rotation of the mid-point of the stiffener section, $\beta_x^G = (U^t - U^b)/2 = 0$.

These edge effects give also a significant difference of the axial stress resultants distributions of both faces in the SS3G and C4G cases, as shown in figure 7-5.a. At the supports $N_x^t = -185.1$ lbf/inch and $N_x^b = -187.4$ lbf/inch in the SS3G case and $N_x^t = -161.4$ lbf/inch and $N_x^b = -211.7$ lbf/inch in the C4g case. As it is expected, the sum of $N_x^t(R^t / R^c)$ and $N_x^b(R^b / R^c)$ is equal to the applied load $N_x^G = -372.5$ lbf/inch. This global axial resultant is almost constant along the length of the cylinder in both cases.

For the SS3G case, the global bending moment resultant $M_x^G$ is zero at the supports. This global bending moment $M_x^G$ can be computed directly from the boundary equations in (5-50) in chapter 5. It is important to note here that the local bending moment resultants of the top and bottom faces $M_x^t$ and $M_x^b$ are not zero, see figure 7-5.b. In each individual face, the bending moment is respectively $M_x^t = 0.203$ lbf and $M_x^b = 0.204$ lbf. For C4 case, both global bending moment resultant and individual face bending moments are not zero ($M_x^G = 9.8$ lbf, $M_x^t = 0.118$ lbf and $M_x^b = 0.122$ lbf).

The shear resultants in the faces $Q_x^t, Q_x^b$ and the shear stresses at the interfaces between the core and the adjacent face $\tau_{xr}^t, \tau_{xr}^b$, as depicted in figures 7-5.c and 7-6.a, respectively, are antisymmetric with respect to the mid-length of the cylinder. One can see that in the edge region both $Q_x^t$ (or $Q_x^b$) and $\tau_{xr}^t$ (or $\tau_{xr}^b$) undergo sharp change. Outside the edge region both quantities decrease to zero at the shell mid-length ($x = L/2$), whereby $Q_x^t$ decreases noticeably faster than $\tau_{xr}^t$. This indicates the high shear distribution inside the core. It is also clearly demonstrates a shear redistribution between the sandwich layers. The faces endure shear only nearby the supports while the core resists shear along the entire cylinder.
Figure 7-4. Displacements and rotations of the faces at $\lambda_a = 0.3$ ($N_x = -372.5$ lbf/inch) using SS3G (left) and C4g (right) boundary conditions
Figure 7-5. Stress resultants of the faces at $\lambda_a = 0.3$ ($N_x = -372.5$ lbf/inch) using SS3G (left) and C4g (right) boundary conditions.
Nonlinear Prebuckling Response

**Figure 7-6.** Stresses at the interfaces between the core and the adjacent faces at \( \lambda_n = 0.3 \) (\( N_x = -372.5 \text{ lbf/inch} \)) using SS3G (left) and C4G (right) b.c.

Note that the distribution of shear stresses at the interfaces between the core and the adjacent faces \( \tau_{xy}^t, \tau_{xy}^b \) are almost identical but they are not the same. Especially nearby the edges they have different absolute maximum values. Both in the SS3G and C4G cases, they have maximum absolute peak values encountered at the vicinity of the edges. For the SS3G case, \( |\tau_{xy}^t| = 16.67 \text{ psi} \) and \( |\tau_{xy}^b| = 17.05 \text{ psi} \) and for the C4G case they are significantly higher, \( \tau_{xy}^t = 22.69 \text{ psi} \) and \( \tau_{xy}^b = 23.20 \text{ psi} \). Remarkably they are found at a distance smaller than the order of the core thickness, namely at \( x = 0.03 \text{ inch} \) from both side of the edges.

The radial stresses \( \sigma_r^t, \sigma_r^b \) as functions of the axial coordinate, as depicted in figure 7-6.b are symmetric with respect to the mid-length of the cylinder. The behavior of the radial stresses near the edges in the SS3G and C4G cases, not mentioning their magnitudes, is quite identical. Compressive and tensile peak values are found at the top face-core and core-bottom face interfaces respectively, in the vicinity of the edges. The maximum values found in the SS3G case are \( \sigma_r^t = -34.87 \).
psi and $\sigma_r^b = 35.56$ psi and in the C4G case they are $\sigma_r^t = -19.41$ psi and $\sigma_r^b = 19.76$ psi. In both cases, because the loading of the top and bottom faces is almost the same, the magnitude of the maximum radial stresses are quite similar. Far from the edges, the average radial stresses are close to zero and they are constant along the axial coordinate.

**Prebuckling responses with increasing applied load**

The relation between the applied load and the end-shortening of both faces are depicted in figure 7-7 for simply supported SS3G. Because of the rotation of the rigid stiffener, the end-shortening of both faces are not the same. The bottom face experiences more shortening than the top face, as it can also be observed from the axial displacement distribution of the faces in figure 7-4.b at $\lambda_a = 0.3$. The normalized axial end-shortening $\Delta^i$ is defined as $\epsilon^i / \epsilon^{\text{class}}$ where $\epsilon^i = (U^i_{x=0} - U^i_{x=L}) / L$ is the difference between the face axial displacement of both ends normalized to the cylinder length. The superscript $j = t, b$ denotes the top and bottom face quantities, respectively. The classical axial strain $\epsilon^{\text{class}}$ is defined as $N_x^{\text{class}} / (E_t t)$ where $N_x^{\text{class}}$ is the critical buckling load obtained for membrane prebuckling case.

The limit load found for this perfect sandwich cylinder is slightly lower than the load level as the bifurcation buckling load using membrane prebuckling solution in section 7.1. The limit point of the solution path is determined first by performing a sequence of calculating solution points until a solution point is found where the derivative of $\Lambda$ with respect to the path parameter $\eta$ becomes negative. Then, the limit point is calculated by restarting the process from a slightly lower load level than the last calculated point with a smaller step-size until the solution is found where $d\Lambda / d\eta \leq 1.0^{-5}$. However, a buckling load in axisymmetric form ($n = 0$) is encountered at about $\lambda_a = 0.969$, just before approaching the limit load. In figure 7-8 the radial displacements along the axial coordinate are displayed for various load levels below the buckling load level. Due to symmetry only half of the radial displacements of the cylinder are displayed.

The radial displacements of both faces, which grow with increasing load up to load level $\lambda_a = 0.9$, are more or less of the same form. At $\lambda_a = 0.969$, axial wrinkling at the edge region is initiated and its magnitude grows rapidly as the load level approaches the limit load. This kind of deformation pattern is characteristic of the deformation of sandwich cylinders with a low core elastic modulus $E_c$ and moderate shear modulus $G_c$ under axial compression load and using global boundary conditions.
Nonlinear Prebuckling Response

Figure 7-7. Nondimensional axial load parameter $\lambda_a$ as function of the normalized axial end-shortening $\Delta^t, \Delta^b$ (SS3G b.c., axial compression)

Figure 7-8. Radial displacements of the top face and the bottom face at four different load levels (SS3G b.c., axial compression)
Radial stresses and out of plane shear stresses at the interfaces

It has been mentioned in chapter 5 that with the present model the core is not expose to any external loads. The applied load is carried by the faces from both edges through the cylinder surfaces. However, internal stresses that is radial, axial and circumferential stresses, respectively, will exist at the interfaces between the faces and the core.

In figure 1-2 of chapter 1, the possible instability and failure modes of sandwich constructions are summarized. Several failure modes are involved with the local failure of the core and the bond. To predict this material failures, one should know the distribution of the internal stresses working along the interface surfaces. Therefore, at each load step during the computation, the maximum stresses and the associated place of occurrence along the surface of the interfaces will be searched for.

In table 7-4 the maximum stresses along the surfaces coordinate with increasing load factor are given for SS3G boundary conditions. For illustration, stresses only at five different load levels are given here. Note that the circumferential interface stresses $\tau_{\theta r}^1$, $\tau_{\theta r}^b$ are zero, which can be expected since the sandwich cylinder has isotropic faces and is loaded by axial compression. Typical for this case is that the radial displacements of the faces, nearby the edges, show localized edge wrinkling pattern at load level close to the critical buckling load. This usually coincides with the oscillation of the radial stresses distribution at the same region. The growth of the amplitude and wavy form is very sensitive to the increase of the applied load which will cause the maximum stresses. The value can easily be changed either from a maximum compressive to tension stress or vice versa. Accurate prediction of the maximum stresses should be done with a small incremental load.

At $\lambda_a = 0.969$, the maximum compressive radial stress $(\sigma_r^b)_{\max} = -522.3$ psi at the interface between bottom face and core is rather high. One has to be concerned about the possibility of core material failure (core crushing) when comparing this radial stress to the allowable compressive strength of Nomex honeycomb core with $\sigma_{r \text{ tens}} = 300$ psi obtained from [37] which has more or less the same property.

On the other hand, the absolute maximum shear stress $\tau_{\theta x}^1$, $\tau_{\theta x}^b$ at both interfaces are quite similar and they increase more or less proportional to the applied load. Even at $\lambda_a = 0.969$, the maximum stresses are still below the allowable strength for Nomex honeycomb core, namely $|\tau_{\theta x}^L| = 160$ psi.
Nonlinear Prebuckling Response

Table 7-4. Maximum stresses at the interfaces between the core and the adjacent faces for increasing axial compression load with $E_c = 2.0 \times 10^4$ psi, $E_c / G_c = 5.0$ (SS3G b.c.)

<table>
<thead>
<tr>
<th>$\lambda_a$ (lbf/inch)</th>
<th>$N_x$</th>
<th>Radial stresses (psi) (at x/L = 0.002)</th>
<th>Shear stresses (psi) (at x/L = 0.006)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$(\sigma_r^a)_{max}$</td>
<td>$(\sigma_r^b)_{max}$</td>
</tr>
<tr>
<td>0.3</td>
<td>-372.5</td>
<td>-34.9</td>
<td>35.6</td>
</tr>
<tr>
<td>0.5</td>
<td>-613.8</td>
<td>-66.6</td>
<td>68.1</td>
</tr>
<tr>
<td>0.7</td>
<td>-859.3</td>
<td>-103.9</td>
<td>106.5</td>
</tr>
<tr>
<td>0.9</td>
<td>-1104.8</td>
<td>-171.2</td>
<td>178.4</td>
</tr>
<tr>
<td>0.969</td>
<td>-1189.5</td>
<td>-216.9</td>
<td>-522.3</td>
</tr>
</tbody>
</table>

Hypothetical weak core

Driven by the knowledge of the advantage of the current modeling approach for sandwich configurations, two hypothetical cases are considered to show how the faces can undergo different deformation patterns under increasing applied load. In the first case a sandwich cylinder is modeled by use of an isotropic honeycomb core with a very low transverse and shear rigidity ($E_c = 100$ psi and $G_c = 20$ psi). Actually, one can imagine this extreme case as two faces (read cylinders) which are separated by an empty space in between and loaded separately by some ratio of axial compression. The second case is for the same $E_c$ but with a higher core shear rigidity ($E_c = 100$ psi and $G_c = 4.0 \times 10^4$ psi). Here, an interaction occurs between the top face and the bottom face.

Figure 7-9 display the nonlinear radial displacement response of both faces with SS3G boundary conditions at two load levels below the critical buckling load for the first hypothetical case. More details of the predicted buckling calculations will be presented in the next section 7.3. Note that due to the symmetry and axisymmetric deformation of the prebuckling responses only 1/8 of the cylinder segment is shown at the left-side surface plots.
Figure 7-9. Radial displacements of the faces for \( E_c = 100 \text{ psi} \) and \( G_c = 20 \text{ psi} \)
Figure 7-10. Radial displacements of the faces for $E_c = 100$ psi and $G_c = 4 \times 10^4$ psi
Increasing the applied load up to $\lambda_a = 0.876$, the maximum amplitude of radial displacement near the support has started to grow. At this load level, axial wavy displacements at both faces occur for the first time and their magnitudes are growing as the load level approaches $\lambda_a = 0.983$. The axial wavy patterns of the top and bottom faces have a similar form and their magnitudes will grow in the same direction. In fact, both faces will deform antisymmetrically with respect to the deformed state mid-surface of the sandwich layers.

Beyond this load levels up to the predicted buckling load, the magnitude of wavy patterns grow rapidly. However, the axial half-wave length found is of one order greater than the thickness of the core. This kind of mode is typical for sandwich shells with a low transverse shear modulus of the core. Shear crimping has occurred here because the core cannot resist shear along the entire cylinder. Due to the direction of the applied load, the transverse compressibility of the core along the cylinder length is not clearly seen.

On the other hand, the transverse compressibility of the core will be observed when the shear modulus is increased to $G_c = 4.0 \times 10^4$ psi. Figure 7-10 displays the non-linear radial displacement response of both faces for SS3G case at two load levels below the critical buckling load. Again, due to the symmetry and axisymmetric deformation of the prebuckling responses only 1/8 of the cylinder segment is shown at the left-side surface plots.

At $\lambda_a = 0.983$, the axial wavy displacements at both faces near the support are detected for the first time and their magnitudes are growing as the load level approaches $\lambda_a = 0.998$. Contrary to the results obtained in the previous case, the axial wavy patterns of the top and bottom face have a different form and their magnitudes will grow in the opposite direction. Both faces will deform almost symmetrically with respect to the deformed state mid-surface of the sandwich layers.

**External pressure**

The previous sandwich cylinder with $E_c = 2.0 \times 10^5$ psi and $G_c = 4.0 \times 10^4$ psi is now subjected to external pressure $p_c = 33.9$ psi at the top face directed towards to the mid-point of the cylinder. Frostig in [27] investigated sandwich curved panels under such external uniform loading. The load is applied on the top face of the panel and nonlinear responses are presented using similar global simply supported and clamped boundary conditions. This offers the opportunity to make qualitative comparisons. In figures 7-11 to 7-13 the distributions of displacements, resultants and stresses of the response as function of the axial coordinate $x$ are presented. Because of the symmetry, only half of the responses are displayed.

Radial and axial displacements of the top and bottom faces along the axial coordinate are displayed in figure 7-11.a-b. The radial displacement of the top face $W^1$ in
the SS3G and C4G cases are slightly higher than the appropriate displacements of the bottom face \( W^b \), which indicates the transverse compressibility of the core. The same core behavior is also mentioned in [27]. The magnitude of radial displacements \( W^t, W^b \) increases from zero at the edges up to \( W^t_{\text{max}} = 0.0566 \) inch and \( W^b_{\text{max}} = 0.0563 \) inch at about \( x = 0.2 \) inch the SS3G case and \( W^t_{\text{max}} = 0.0511 \) inch and \( W^b_{\text{max}} = 0.0508 \) inch at about \( x = 0.25 \) inch the C4G case. As expected, the axial displacements \( U^t, U^b \) are of one order smaller than \( W^t, W^b \) and their behavior not the same for different boundary conditions. In general, qualitatively the same trend of deformation patterns of the sandwich faces are obtained as in [27], when similar boundary conditions are used.

The distribution of rotations \( \beta^t_x, \beta^b_x \) of the faces as function of the axial coordinate, see in figure 7-11.c, corresponds to the gradient of the changes of the radial displacements. Away from the edges towards the mid-length of the cylinder, the rotations decrease to zero, which indicates that the radial displacements are constant. Again, in the C4G case the rotation of the stiffener \( \beta^G_x \) at the edges is zero.

In the SS3G case, see figure 7-12.a, the axial stress resultant of the top and bottom faces at the supports is respectively \( N^t_x = 5.47 \) lbf/inch and \( N^b_x = -5.60 \) lbf/inch. The axial stress resultant of the top face \( N^t_x \) and bottom face \( N^b_x \) are not identical because there is a difference in the radius of curvatures of the top and bottom faces at the same section along the axial coordinate of the cylinder. However, the global axial resultants \( N^G_x \) of the rigid stiffener at the supports are zero, satisfying the definition of simply supported boundary conditions. The appropriate values at the supports in the C4G case are \( N^t_x = 40.7 \) lbf/inch and \( N^b_x = -206.6 \) lbf/inch, respectively. The bottom face experiences a significant compression at the supports and the global axial resultant \( N^G_x = -163.1 \) lbf/inch is rather high.

The distributions of the bending moment and shear resultants along the axial coordinate \( x \) are presented in figure 7-12.b-c. In the SS3G case, the computed global bending moment \( M^G_x \) is zero at the supports. In each individual face, the bending moment is not zero, i.e. \( M^t_x = -1.2 \) lbf and \( M^b_x = -1.1 \) lbf. In the C4 case, both the computed global bending moment resultant and the individual face bending moments are not zero at the supports \( (M^G_x = 50.9 \) lbf, \( M^t_x = -0.73 \) lbf and \( M^b_x = -0.51 \) lbf). Close to the edges, substantial changes of the shear resultants in the faces occur in the SS3G and C4G cases. One can see in figure 7-12.c that from the edge at \( x = 0 \) to a distance \( x = 0.03 \) inch the absolute value of the shear resultants in the faces decreases to zero. This kind of phenomenon has also been obtained by Frostig in [27]. Far from the edges the shear resultants decrease to zero and they are antisymmetric with respect to the mid-length of cylinder.
Figure 7.11. Displacements and rotations of the faces at $p_e = 0.01$ ($p_e = 33.9$ psi) using SS3G (left) and C4g (right) boundary conditions.
Figure 7-12. Stress resultants of the faces at $p_e = 0.01$ ($p_e = 33.9$ psi) using SS3G (left) and C4g (right) boundary conditions.
Figure 7-13. Stresses at the interfaces between the core and the adjacent faces at 
$p_c = 0.01$ ($p_c = 33.9$ psi) using SS3G (left) and C4g (right) boundary conditions

The characteristic of overall distribution of radial and shear stresses at the top-
core and bottom core interfaces, as displayed in figure 7-13, reflects a typical dis-
btribution as also obtained in [27] where sudden major changes of the magnitude of 
stresses occur in the vicinity of the supports. The shear stresses change abruptly 
from their maxima nearby the edges to zero at the mid-point of cylinder. At the 
same time, major changes of radial stresses occur, which reduce their maximum 
values to almost constant values far from the edges.

The distribution of shear stresses at the interfaces between the core and top face 
$\tau_{xt}^t$ and the core and bottom face $\tau_{xt}^b$ along the axial coordinate, as shown in 
figure 7-13.a are almost identical but not the same. They are antisymmetric with 
respect to the mid-length of the cylinder. Both in the SS3G and C4G cases, they
have maximum absolute peak values at the vicinity of the edges. The absolute maximum values of the top face and bottom face found in the SS3G case are $|\tau_x^t| = 78.3$ psi and $|\tau_x^b| = 80.1$ psi, respectively and the appropriate values in the C4G case are slightly higher, namely $|\tau_x^t| = 107.0$ psi and $|\tau_x^b| = 109.4$ psi. These maximum values are found at about $x = 0.15$ inch from both edges.

The radial stresses $\sigma_r^t$, $\sigma_r^b$ as functions of the axial coordinate are depicted in figure 7-13.b. The behavior of the radial stresses, except their magnitudes, is quite identical for the SS3G and C4G cases. Looking closely at the distributions of these stresses nearby the edges, the maximum values are found at about $x = 0.03$ inch from the edges. Both in the SS3G and C4G cases, a maximum tensile stress is attained in the top face while the bottom face has a maximum compressive stress. In the SS3G case they are $\sigma_r^t = 154.8$ psi and $\sigma_r^b = -164.3$ psi, respectively. In the C4G case they are $\sigma_r^t = 83.7$ psi and $\sigma_r^b = -97.2$ psi, respectively. Due to uniform loading at the top face of the cylinder, the compressive stress develops mainly in the core along the entire cylinder with their average values equal to -17.1 psi and -17.0 psi for the SS3G and C4G case, respectively.

**Table 7-5. Maximum stresses at the interfaces between the core and the adjacent faces for external pressure with $E_c = 2.0 \times 10^4$ psi, $E_c / G_c = 5.0$ (SS3G and C4G b.c.)**

<table>
<thead>
<tr>
<th>$p_e$ (psi)</th>
<th>$p_e$ (psi)</th>
<th>Radial stresses (psi) (at x/L)</th>
<th>Shear stresses (psi) (at x/L)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>($\sigma_r^t$)$_{max}$</td>
<td>($\sigma_r^b$)$_{max}$</td>
</tr>
<tr>
<td>SS3G - SS3G:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>33.9</td>
<td>154.8 (0.001)</td>
<td>-164.3 (0.001)</td>
</tr>
<tr>
<td>0.0125</td>
<td>42.6</td>
<td>194.3 (0.001)</td>
<td>-206.1 (0.001)</td>
</tr>
<tr>
<td>C4G - C4G:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>33.9</td>
<td>83.7 (0.001)</td>
<td>-97.2 (0.001)</td>
</tr>
<tr>
<td>0.0146</td>
<td>49.5</td>
<td>123.2 (0.002)</td>
<td>-150.3 (0.002)</td>
</tr>
</tbody>
</table>

Table 7-5 shows the maximum values of radial and shear stresses at top and bottom face-core interfaces found along the axial coordinate $x$ when the applied load increases. Both in the SS3G and C4G cases, the first row gives the maximum stresses at $p_e = 33.9$ psi and the second row displays those at load level slightly below the predicted buckling load. In each row, bracketed numbers denote the place along the axial coordinate $x$ where the maximum values are detected.
The maximum radial stresses will increase proportional to applied load and they occur almost at the same place. On the contrary to the results obtained in the axial compression case, the maximum values do not change in sign. Even at the buckling load level, the values are still below the allowable strength of Nomex honeycomb core with $\sigma_{r}^{\text{comp}} = 370$ psi and $\sigma_{r}^{\text{tens}} = 300$ psi. This implies that premature material failure of the core, namely core crushing and tensile rupture of core proper will not occur below the buckling load level. However, to avoid the damage of the structure the maximum radial stresses should be less than the allowable glue bondage strength between the top face-core and bottom face-core interfaces.

The maximum absolute shear stresses $\tau_{x \theta}^t$, $\tau_{x \theta}^b$ at both interfaces are rather high in comparison with those obtained in the axial compression case. Especially in the C4G case, comparing the maximum values of both faces found at $p_e = 49.5$ psi with the allowable strength of Nomex honeycomb core, i.e. $|\tau_{x \theta}^t| = 160$ psi, one has to be concerned about the possibilities of core material failure due to shear.

*Local boundary conditions*

Though the boundary conditions of sandwich layers can be modeled by the global boundary conditions defined in the previous section, in real sandwich structures it is possible that the edge conditions are defined at the component level of sandwich layers, i.e. the face and core, separately.

As it has been derived in equations (5-34) to (5-38) in chapter 5, different boundary conditions for the sandwich faces and core can be defined at the edges of the cylinder. In this study, simply supported SS3 boundary conditions ($N_x^t = 0, V_x^t = 0, W_x^t = 0, M_x^t = 0$) are used for the top face at both edges while the bottom face is free which means that all displacement and rotation components are not fixed. The core is also not supported which leads to the condition that the transverse shear stress $\tau_{x \theta}^c = 0$ at the edges. For axial compression load and external pressure, the definition of the current boundary conditions, denotes as SS3-free, are given in table B-3 in Appendix B.6. Note that the notation SS3-free is used to denote the top face is supported with SS3 condition at both edges while the bottom face is free.

*Axial compression*

In the first case, the responses characteristic of sandwich cylinder with $E_c = 2.0 \times 10^4$ psi and $G_c = 4.0 \times 10^4$ psi is already given under axial compression load ($\lambda_{c} = 0.3$) for SS3G global boundary conditions. For the same problem, the local SS3-free boundary conditions will be used. Of interest is here the difference in the behavior of the response quantities close to the edges and the possibilities of core material failure under increasing applied load.
Nonlinear Prebuckling Response

In figure 7-14.a the radial and axial displacement distributions along the axial coordinate $x$ are displayed. Note that due to symmetry at $x=L/2$ only half of the responses are shown. Comparing with the results for the SS3G case in figure 7-4, both the magnitudes and the distribution of the displacements are more or less equivalent. However, at the zone nearby the edge, the difference between the radial displacements $W^l$ and $W^b$ in the two boundary condition cases is noticeable. For the SS3-free case, the displacement $W^b = -0.0022$ inch while the displacement $W^l$ is equal to zero at the edges (see zoom-in on this area).

The rotations $\beta_x^l, \beta_x^b$ at the edges zone are quite different compared with those for the SS3G case in figure 7-4.b (left).

The distributions of the axial stress resultants $N_x^l, N_x^b$ along the axial coordinate in figure 7-14.b (right) show also similar form as those in the SS3G case. The difference in magnitude is caused by the slightly different applied load. Again, the global axial stress resultant defined as $N_x^l(R^l/R^c) + N_x^b(R^b/R^c)$ is equal to the applied load $N_x^G = -379.2$ lbf/inch.

Close to the edges, as shown in figure 7-14.c, the bending moment and shear resultants distributions are quite different compared with the appropriate distributions in the SS3G case. The bending moments $M_x^l, M_x^b$ of both faces as well as the shear resultant $Q_x^b$ are now equal to zero at edges.

The maximum absolute shear stresses at the top face-core and bottom face-core interfaces are $|\tau_x^l| = 22.0$ psi and $|\tau_x^b| = 22.5$ psi, respectively. Similarly as in the SS3G case, the shear stresses change substantially along the axial coordinate $x$, as shown in figure 7-15 (left). They have their maxima at about $x = 0.03$ inch from both edges and at the mid-length of the cylinder the shear stresses at the interfaces are equal to zero.

There is an essential difference of the distribution of the radial stresses of both faces $\sigma_r^l, \sigma_r^b$, as shown in figure 7-15 (right), where the maximum values found at edges are much greater than the appropriate values in the SS3G case. This indicates the higher possibility of damage of sandwich structures due to core material failure or debonding of the glue at relatively low load levels when dealing with this kind of local boundary conditions.
Figure 7-14. Displacements and stress resultants of the faces and the core at $\lambda_a = 0.3$ ($N_x = -379.2 \text{ lbf/inch}$) using SS3-free boundary conditions
**Nonlinear Prebuckling Response**

**Figure 7-15.** Stresses at the interfaces between the core and the adjacent faces at $\lambda_a = 0.3$ ($N_x = -379.2 \text{ lbf/inch}$) using SS3-free boundary conditions

**Increasing the applied load**

The relation between the applied axial compression load and the end-shortening of both faces using SS3-free boundary conditions at both edges is displayed in figure 7-16. The same definition of the normalized axial end-shortening $\Delta^I$ and the load factor $\lambda_a$ is used as for the previous SS3G case.

Similar to the SS3G case, the bottom face experiences more axial deformation than the top face with the increase of the load. It is important to mention the significant decrease obtained in the predicted limit load using these local boundary conditions. The limit load in this case is found at $\lambda_a = 0.512$. Thus, the load reduces to about 52% of the limit load in the previous SS3G case. No bifurcation buckling point will be found when performing a bifurcation buckling analysis in the next section.

The radial displacements of the faces $W^t$, $W^b$ along the axial coordinate are displayed in figure 7-17 for 3 different load levels, namely $\lambda_a = 0.3$, $\lambda_a = 0.5$ and at the limit load. The radial displacements are displayed only for half of the cylinder length due to symmetry.
Figure 7-16. Nondimensional axial load parameter $\lambda_a$ as function of the normalized axial end-shortening $\Delta^t, \Delta^b$ (S33-free b.c., axial compression)

Figure 7-17. Radial displacements of the top face and the bottom face at three different load levels (SS3-free b.c., axial compression)
Nonlinear Prebuckling Response

One can see that the magnitude of displacements increase, especially at the edges zones, when increasing the applied load. The radial displacement $W^t$ of the top face is zero at the edges, corresponding to the definition of SS3 boundary conditions. On the other hand, the radial displacement $W^b$ of the bottom face grows when the applied load increases. As expected, the bottom face is free to move as a consequence of implying free-end condition in this part of sandwich layers. Together with the growth of magnitude of $W^t$, $W^b$, a wavy pattern is initiated in the edge zones where magnitudes will grow in the opposite direction. However, contrary to the results obtained in the SS3G case, this wavy pattern is not a wrinkling mode, because the wavelength of the pattern is much greater than the order of thickness of the core.

Table 7-6 gives the maximum radial and shear stresses of the interfaces between the core and the adjacent faces along the axial coordinate x with increasing load factor. As it can be observed from the table, the maximum radial stresses at the limit load are very high. On the contrary to the results obtained for global boundary conditions in the previous section, these values obviously exceed the allowable compressive and tensile strength of Nomex honeycomb core. Hence, one should be aware of the possibilities of core material failure or debonding of the interfaces when dealing with local boundary conditions such as this SS3-free case, in which one of the faces and the core are not supported.

**Table 7-6. Maximum stresses at the interfaces between the core and the adjacent faces for axial compression load with $E_c = 2.0 \times 10^4$ psi, $E_c / G_c = 5.0$ (SS3-free b.c.)**

| $\lambda_a$ (lbf/inch) | $N_x$ | Radial stresses (psi) (at x/L) | Shear stresses (psi) (at x/L) | $|\tau_{tx}^t|_\text{max}$ | $|\tau_{tx}^b|_\text{max}$ |
|------------------------|-------|-------------------------------|-------------------------------|--------------------------|--------------------------|
| SS3-free               |       |                               |                               |                          |                          |
| 0.3                    | -368.3| -272.0 (0.0)                  | 61.0 (0.0)                    | 21.9 (0.003)             | 22.4 (0.003)             |
| 0.5                    | -613.8| -1500.0 (0.0)                 | -1605.4 (0.0)                | 32.3 (0.001)             | 33.0 (0.001)             |
| 0.512                  | -628.6| -8144.1 (0.0)                 | -13787 (0.0)                 | 285.6 (0.002)            | 292.1 (0.002)            |
External pressure

For local SS3-free boundary conditions, an investigation of the nonlinear responses behavior of the same sandwich cylinders will be performed under external pressure. The normal pressure is exerted at the top face of the sandwich layers towards the centre line of the cylinder. Figures 7-18 and 7-19 display the distributions of displacements, stresses and stress resultants in the sandwich layers at $p_c = 33.9$ psi.

In figure 7-18.a the radial and axial displacement distributions along axial coordinate $x$ are displayed. Comparing with the results for the SS3G case in figure 7-11, both the magnitudes and the distribution of the displacements are more or less equivalent. Except at the zone nearby the edges (see zoom-in plot on this area), where the difference between displacements $W^t$ and $W^b$ of the faces is noticeable. For the SS3-free case, the radial displacement $W^b = 0.008$ inch while the displacement $W^t$ is equal to zero at the edges. This implies the free-end condition of the bottom face. The rotations $\beta^t_x, \beta^b_x$ at the edge zone, see figure 7-18.b (left), are quite different with those for the SS3G case in figure 7-11.

The distribution of axial stress resultants $N^t_x, N^b_x$ in figure 7-18.b (right) is quite similar to that in the SS3G case. Note that at the edges the axial stress resultants $N^t_x, N^b_x$ are zero which satisfies both the simply supported SS3 boundary condition of the top face and the free-end condition of the bottom face. Close to the edges, as displayed in figure 7-18.c, the distributions of the bending moment $M^t_x, M^b_x$ and shear resultants $Q^t_x, Q^b_x$ in the faces are quite different compared with the appropriate distributions in the SS3G case. The bending moments $M^t_x, M^b_x$ of both faces as well as the shear resultant $Q^b_x$ are now equal to zero at edges.

The distributions of shear and radial stresses at the interfaces between the core and the adjacent faces are displayed in figure 7-19. The maximum absolute shear stresses at the top face-core and bottom face-core interfaces are $\left|\tau_{xr}^t\right| = 85.7$ psi and $\left|\tau_{xr}^b\right| = 87.6$ psi, respectively. Similar to those obtained in the SS3G case, the shear stresses change substantially along the axial coordinate $x$. The maximum peak values of the shear stresses are found at about $x = 0.09$ inch from both edges. At the mid-length of the cylinder the shear stresses of the interfaces are equal to zero.

Essential difference of the distributions of the radial stresses of both faces is shown in figure 7-19, where the maximum values found at the edges are much greater, and of an opposite sign, than the appropriate values in the SS3G case. The top face has a maximum tensile value of $\sigma_r^t = 974.8$ psi and the bottom face has a maximum compression value of $\sigma_r^b = -194.1$ psi. Again, this indicates the possibility of damage of the sandwich structure due to core material failure or debonding of the glue at this load level.
Figure 7-18. Displacements and stress resultants of the faces at $\bar{P}_e = 0.01$ ($p_e = 33.9$ psi) using SS3-free boundary conditions
Figure 7-19. Stresses at the interfaces between the core and the adjacent faces at $\bar{p}_e = 0.01$ ($p_e = 33.9$ psi) using SS3-free boundary conditions

7.3. Bifurcation Buckling and Limit Load

In chapter 6 bifurcation buckling equations are derived and the bifurcation buckling load can be solved for iteratively by finding the smallest eigenvalue via inverse iteration method. These eigenvalue equations depend on the cylinder geometry (radius and length), the geometry and the mechanical properties of the sandwich layers, and the type of boundary conditions used.

In particular, a special attention is given here to the effects of the modulus of elasticity $E_c$ and the shear modulus $G_c$. The effects of $E_c$ on the critical loads will be investigated for four different isotropic honeycomb cores namely for a “stiff” core with $E_c = 1.0 \times 10^7$ psi, for two “soft” cores with $2.0 \times 10^5$ psi and $2.0 \times 10^4$ psi used in the practice and for a hypothetical weak core with 100 psi, respectively. For all the cases considered for the transverse shear modulus $G_c = E_c/5$ is used.

A rigorous prebuckling solution is used for the buckling analysis which includes the effects of prebuckling deformations caused by the edge constraints. This nonlinear prebuckling responses have already been investigated in section 7.2. The results based on a membrane prebuckling analysis, listed in section 7.1, are also used for comparison.

It is also known from numerous investigations [65, 30] that the prediction of the critical buckling loads for a cylinder using conventional (isotropic or composite) material under various external loads depends on the type of boundary conditions defined at both cylinder edges. In sandwich structures, however, only a few references have studied the effects of boundary conditions on the collapse behavior of
sandwich cylindrical shells. Reference [55] dealing with sandwich cylinders uses classical boundary conditions, in which the sandwich layers are treated as one single layer and the boundary conditions applied for the faces and core are always identical and mostly simply supported. The possibilities of each part of the sandwich layers to have different boundary conditions at the edges are not considered.

For buckling problem, the newly developed program SHOT can deal with various kind of global or local conditions. Therefore, the following investigation of the effects of boundary conditions will be separated in two parts. First, the global boundary conditions will be used either defined as simply supported SS3G or clamped C4G at both cylinder edges. Under the same applied loads (axial compression and external pressure), the investigation will be continued by using four different local boundary conditions namely SS3-free, C3-free, free-SS3 and free-C4 at the edges.

The eigenvalue problem described above, as a result of introducing Fourier decomposition of the unknown variables, also depends on the circumferential full wave parameter n. The critical buckling load is then sought by sequences of smallest eigenvalue calculations for a certain number of circumferential wave n until the minimum smallest eigenvalue is obtained. In most of the cases, the discretization in the axial direction, as a result of the finite difference formulation, are started using 200 gridpoints with equidistant grids interval scheme and only for half of cylinder length. However, if convergence is not yet reached, i.e. by increasing the number of gridpoints the difference of the calculated eigenvalue with the previous calculation is still greater than a certain desired accuracy then the discretization scheme must be adjusted. For example, in the problem with axial compression load using global boundary conditions, the gridpoints should be increased to 500 points for half length of cylinder in order to obtain a converged solution.

Alternatively, for efficiency of computation, in particular when dealing with local boundary conditions, a variable distance grids interval scheme of discretization (see chapter 6) can be used in which more gridpoints are applied nearby the edge area where usually localized wavy patterns occur. Experiences have shown, such as in [30, 31], that discretizations in axial direction with at least 5 gridpoints for one axial half-wave of buckling mode are sufficient.

Global boundary conditions

The global SS3G and C4G boundary conditions for the nonlinear prebuckling analysis have already been defined in table B-1 in Appendix B.6. For buckling analysis, the definition of global simply supported SS3G and clamped C4G boundary conditions are given in table B-2.
Axial compression

The results of calculation of the critical buckling load are given in Table 7-7. Note that the eigenvalues are listed in normalized form \( \lambda_a = \frac{N_{x}^{\text{crit}}}{N_{x}^{\text{norm}}} \), where \( N_{x}^{\text{norm}} \) is the appropriate buckling load obtained using a membrane prebuckling analysis in section 7.1. Bracketed values denote the number of circumferential full-waves of the buckling mode of both faces, S stands for symmetry and A for antisymmetry at \( x = L/2 \) of the buckling mode.

Except for the hypothetical weak core cases, the eigenvalues displayed are smaller than one, which indicates the critical buckling loads using nonlinear prebuckling solutions are smaller than those using membrane prebuckling solutions. As expected, lower predicted buckling loads are obtained when decreasing the modulus of elasticity \( E_c \). Notice that the results computed using C4G boundary conditions are higher than those for the same case using SS3G boundary conditions.

It should be mentioned that the calculation for \( E_c = 1.0 \times 10^7 \) psi is performed using 200 gridpoints for only half of the cylinder length. Here, because one is dealing with the stability problem, both the symmetric and the antisymmetric boundary conditions at \( x = L/2 \) should be examined. In the other cases, for converged solutions, the number of gridpoints was increased to 500 points and the calculation was performed using half of the cylinder length.

<table>
<thead>
<tr>
<th>( E_c ) (psi)</th>
<th>( N_{x}^{\text{norm}} ) (lbf/inch)</th>
<th>( \lambda_a )</th>
<th>( N_{x} )</th>
<th>( \lambda_a )</th>
<th>( N_{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0 \times 10^7</td>
<td>-0.490056.10^4</td>
<td>0.8529 (4,S)</td>
<td>-0.4180.10^4</td>
<td>0.9320 (4,S)</td>
<td>-0.4567.10^4</td>
</tr>
<tr>
<td>2.0 \times 10^5</td>
<td>-0.387630.10^4</td>
<td>0.9150 (0,S)</td>
<td>-0.3547.10^4</td>
<td>0.9718 (0,A)</td>
<td>-0.3767.10^4</td>
</tr>
<tr>
<td>2.0 \times 10^4</td>
<td>-0.122754.10^4</td>
<td>0.9702 (0,A)</td>
<td>-0.1191.10^4</td>
<td>0.9993 (0,A)</td>
<td>-0.1227.10^4</td>
</tr>
<tr>
<td>1.0 \times 10^2</td>
<td>-0.770505.10^2</td>
<td>1.0000 (0,S)</td>
<td>-0.7705.10^2</td>
<td>1.0000 (0,A)</td>
<td>-0.7705.10^2</td>
</tr>
</tbody>
</table>

For the "stiff" core (\( E_c = 1.0 \times 10^7 \) psi, \( G_c = 2.0 \times 10^6 \) psi), the critical buckling load using a membrane prebuckling analysis and simply supported boundary condition is equal to \( N_x = -4900.6 \) lbf/inch and the corresponding mode is axisymmetric (\( n = 0 \)). Looking at Table 7-7, the critical buckling load is now found to be \( N_x = -4180 \) lbf/
inch for SS3G boundary conditions and \( N_x = -4567 \ \text{lbf/inch} \) for C4G boundary conditions. The corresponding buckling modes, as shown in figure 7-20, are symmetric at \( x = L/2 \) and have 4 full waves in the circumferential direction. Thus, it can be seen that when one switches to an analysis using a rigorous prebuckling solution where the effect of edge constraint is accounted for, the critical buckling load will turn out to be asymmetric. Note that due to the high value of \( E_c \), the transverse compressibility of the core is absent with the result that the modes displayed for the top and bottom faces are identical.

![Diagram](image)

**Figure 7-20.** Prebuckling shapes at buckling load level (axial compression) and buckling modes for transversely "stiff" core calculated with SHOT and SFOSD program using simply supported (left) and clamped (right) b.c.

Comparison is also made with the results obtained for the transversely stiff core formulation in part I of this thesis (SFOSDT) with the same and using a rigorous
Chapter 7

The critical buckling load is found to be \( N_x = -4151 \) lbf/inch for SS3 boundary conditions and \( N_x = -4626 \) lbf/inch for C4 boundary conditions using 200 gridpoints for only half of the cylinder length. A good agreement is found for both results. The difference of the critical values is 0.7% in the simply supported case and 1.3% in the clamped case. The prebuckling solution at the buckling load level and the corresponding buckling mode are shown in figure 7-20. For both SS3 and C4 boundary conditions the buckling modes have the same number of full waves in the circumferential direction \((n = 4)\) as those of SHOT results.

There are three reasons for the difference between the lowest buckling load obtained with the current code SHOT and SFOSDT. The first one is clearly the difference in the modeling of the core. SFOSDT assumed that the modulus of elasticity of the core \( E_c \) is very stiff \((E_c \rightarrow \infty)\) while SHOT can vary this value depending upon the core property used. In this case it is taken equal to the modulus of elasticity of the face \( E_f = 1.0 \times 10^7 \) psi. The second one is that in SFOSDT a thin shell assumption is introduced for the core and the faces while in SHOT the assumption is only made for the faces. The core is solved exactly. Recalling the results of using two core considerations in section 7.1, one can expect the differences especially here, the buckling mode is in asymmetric (overall) mode. The third one is that the difference occurs in the modeling of the faces. In program SFOSDT the faces are modeled based on Donnell's shell equations while in SHOT the non-shallow Sander-Koiter shell equations are used.

As one can see in table 7-7, in the case with \( E_c = 2.0 \times 10^5 \) psi the critical values are found to be \( N_x = -3547 \) lbf/inch for SS3G boundary conditions and \( N_x = -3767 \) lbf/inch for C4G boundary conditions. In the case with \( E_c = 2.0 \times 10^4 \) psi a significant decrease of the critical values is observed and the values are found to be \( N_x = -1191 \) lbf/inch for SS3G boundary conditions and \( N_x = -1227 \) lbf/inch for C4G boundary conditions. Thus, one obtains an about 66% reduction in the buckling load. For all the cases, the corresponding buckling modes of both faces are axisymmetric \((n = 0)\) and they are symmetric (S) or antisymmetric (A) at \( x = L/2 \) as denoted in the table.

For the SS3G case, the prebuckling shape at the critical buckling load and the corresponding buckling modes of both faces for \( E_c = 2.0 \times 10^5 \) psi and \( E_c = 2.0 \times 10^4 \) psi are displayed in figure 7-21. A close-up plot of the buckling modes is given by displaying the modes only over part of the cylinder from the edge at \( x = 0 \) to \( x = 0.15L \). Beyond this area the modes are more or less constant and decrease smoothly to zero at \( x = L/2 \). The transverse compressibility of the core is clearly present in both modes which is demonstrated by the axial wavy patterns of the faces. At the same place of the axial coordinate \( x \), the top face mode will have local maxima while the bottom face mode has local minima or otherwise. One might consider this wavy pattern as localized wrinkling modes because the axial wavelengths of the mode is of the order or smaller than the thickness of the core.
Bifurcation Buckling and Limit Load

On the contrary to the symmetric or antisymmetric wrinkling mode found in previous section 7.1, which covered in the entire cylinder length, this localized wrinkling modes occur only nearby the edges and are triggered by the edge effects. The modes are neither symmetric nor antisymmetric with respect to the shell mid-surface. Note that to obtain the converged solutions, the discretizations are using 500 points of equidistant grids scheme for half of the cylinder length.

![Graph showing prebuckling shape and buckling mode](image)

**Figure 7-21.** Prebuckling shapes at buckling load level (axial compression) and buckling modes for $E_c = 2.0 \times 10^5$ psi (left) and $E_c = 2.0 \times 10^4$ psi (right), SS3G b.c.

The critical buckling load for the hypothetical weak core case ($E_c = 100$ psi) is found equal to the results obtained using membrane prebuckling in section 7.1. This result can be expected since one deals with an extreme low transverse shear mod-
ulus of the core ($G_c = 200$ psi). Thus, the shear crimping mode here occurs overall in the cylinder in which the core undergoes large shear deformation.

**External pressure**

Table 7-8 displays the results of calculations of the critical buckling loads of the same cylinders loaded by external pressure using global simply supported SS3G and clamped C4G boundary conditions. Note that the eigenvalues in the table are listed in normalized form $p_e^* = p_e^\text{crit} / p_e^\text{norm}$, where $p_e^\text{norm} = E_f(t_c^2 + 2t)/(c(R_c)^2) = 3378.0$ psi.

Similar to the axial compression case, the reduction of the predicted buckling loads are observed when decreasing the modulus of elasticity $E_c$. Notice that the results computed using C4G boundary conditions are higher than those for the same case using SS3G boundary conditions.

**Table 7-8. The critical buckling loads (external pressure) for different core property with $E_c / G_c = 5$ using global boundary conditions**

<table>
<thead>
<tr>
<th>$E_c$ (psi)</th>
<th>SS3G</th>
<th>C4G</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_e^{\text{crit}}$</td>
<td>$p_e^{\text{crit}}$ (psi)</td>
<td>$p_e^{\text{crit}}$ (psi)</td>
</tr>
<tr>
<td>$1.0 \times 10^7$</td>
<td>$0.179447 \times 10^{-1}$ (5)</td>
<td>$0.606177 \times 10^2$</td>
</tr>
<tr>
<td>$2.0 \times 10^5$</td>
<td>$0.175885 \times 10^{-1}$ (5)</td>
<td>$0.594142 \times 10^2$</td>
</tr>
<tr>
<td>$2.0 \times 10^4$</td>
<td>$0.125960 \times 10^{-1}$ (5)</td>
<td>$0.425495 \times 10^2$</td>
</tr>
<tr>
<td>$1.0 \times 10^2$</td>
<td>$0.166568 \times 10^{-3}$ (14)</td>
<td>$0.562669$</td>
</tr>
</tbody>
</table>

Some remarks have been made concerning the reduction of the critical buckling values when decreasing $E_c$ from $E_c = 2.0 \times 10^5$ psi to $E_c = 2.0 \times 10^4$ psi in the axial compression case. For the external pressure case, the reduction of the buckling loads with decreasing $E_c$ is less significant than in axial compression case. In the SS3G case, the critical load is reduced from $p_e = 59.4$ psi to $p_e = 42.5$ psi and in the C4G case, from $p_e = 71.6$ psi to $p_e = 49.5$ psi. Thus it is an about 30% of reduction.

For the C4G case, the radial prebuckling displacement shapes at the critical buckling load are displayed in the left-side of figure 7-22 for $E_c = 1.0 \times 10^7$ psi, $E_c = 2.0 \times 10^5$ psi and $E_c = 2.0 \times 10^4$ psi and in the right-side of figure 7-22 for $E_c = 100$ psi. Looking at the left figure, it is interesting to note that the transverse compressibility of the
core becomes more visible when the value of $E_c$ decreases. The right figure for very low $E_c$ clearly demonstrates the compressibility of the core.

On purpose, to get more insight on the asymmetric buckling modes obtained, the surface plots of buckling mode are presented for $E_c = 1.0 \times 10^7$ psi, $E_c = 2.0 \times 10^4$ psi and $E_c = 100$ psi in figure 7-23. Note that full-length buckling mode plots are displayed. Because the buckling modes for top face and bottom face are quite identical, only one of the face modes is depicted here. The mode for $E_c = 2.0 \times 10^5$ psi, which also has 5 full-waves in the circumferential direction is qualitatively identical to the buckling mode for $E_c = 1.0 \times 10^7$ psi.

It can be seen that the number of full-waves in circumferential direction (n) of the buckling mode increases when shear modulus $G_c$ decreases. This obviously indicates that the core provides a small resistance against transverse shear deformation in the circumferential direction.

**Figure 7-22.** Prebuckling shapes at buckling load level (external pressure) for $E_c = 2.0 \times 10^4$, $2.0 \times 10^5$, $1.0 \times 10^7$ psi (left) and $E_c = 100$ psi (right), SS3G b.c.
Figure 7.23. Buckling modes (external pressure) for $E_c = 1.0 \times 10^7$ psi, $E_c = 2.0 \times 10^4$ psi and $E_c = 100$ psi using SS3G boundary conditions
Bifurcation Buckling and Limit Load

Local boundary conditions

It is interesting to investigate the buckling behavior of the same cylinder loaded by the two applied loads using various local boundary conditions. The sandwich cylinder investigated has the modulus of elasticity $E_c = 2.0 \cdot 10^4$ psi and shear modulus $G_c = 4.0 \cdot 10^3$ psi. The definition of the local boundary conditions SS3-free, C4-free, free-SS3, free-C4 for prebuckling and buckling analysis are defined in tables B-3 and B-4 in Appendix B.6. Note that, for instance, the notation SS3-free is used to denote the top face is supported with SS3 condition at both edges while the bottom face is free.

Axial compression

Table 7-9 shows the results of calculation of the critical loads for various set of local boundary conditions for $E_c = 2.0 \cdot 10^4$ psi and $E_c/G_c = 5$. Note that in the table the limit loads are listed in normalized form $\lambda_a^\text{lim} = N_x^\text{crit} / N_x^\text{norm}$, where $N_x^\text{norm} = -1227.54$ lbf/inch is the critical buckling load obtained with a membrane prebuckling analysis.

For all the cases, no bifurcation in an asymmetric mode is encountered below the limit load. When performing the axisymmetric buckling analysis ($n = 0$), the corresponding mode obtained is symmetric at $x = L/2$. The calculated critical buckling load is identical with the limit load obtained from a limit point analysis described earlier in section 7.2.

Table 7-9. The predicted limit loads (axial compression) of sandwich cylinders with $E_c = 2.0 \cdot 10^4$ psi, $E_c / G_c = 5$ using various local boundary conditions

<table>
<thead>
<tr>
<th>Boundary Conditions</th>
<th>$\lambda_a^\text{lim}$</th>
<th>$N_x^\text{lim}$ (lbf/inch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS3-free</td>
<td>0.512044</td>
<td>-0.628554.10^3</td>
</tr>
<tr>
<td>C4-free</td>
<td>0.536542</td>
<td>-0.658626.10^3</td>
</tr>
<tr>
<td>free-SS3</td>
<td>0.506664</td>
<td>-0.621949.10^3</td>
</tr>
<tr>
<td>free-C4</td>
<td>0.651994</td>
<td>-0.800353.10^3</td>
</tr>
</tbody>
</table>

It can be seen from the table that if one uses clamped C4 boundary conditions for one of the faces, the limit loads will be higher than when using simply supported SS3 boundary conditions. Furthermore, the limit load will be higher when the top
face is supported with SS3 condition at the edges \(N_x = -628.6 \text{ lbf/inch}\) than supporting the bottom face with the same condition \(N_x = -621.9 \text{ lbf/inch}\). On the other hand in the C4 case, the limit load will be higher when supporting the bottom face \(N_x = -800.4 \text{ lbf/inch}\) than supporting the top face \(N_x = -658.6 \text{ lbf/inch}\).

Returning now to the results of the calculations of the critical loads of the same core using global boundary conditions SS3G and C4G in table 7-7, which are found to be \(N_x = -1190.3 \text{ lbf/inch}\) for SS3G boundary conditions and \(N_x = -1226.4 \text{ lbf/inch}\) for C4G boundary condition, one can see that the change of the boundary conditions from global to local will result in a significant reduction of the critical loads (about more than 60%).

**Figure 7.24.** Radial displacements of both faces at limit load levels (axial compression) using different local boundary conditions
The radial displacements $W^t$ and $W^b$ of both faces at the limit load are displayed in figure 7-24 for four different local boundary conditions. Only half of the cylinder length are depicted due to symmetry. To get more insight on the behavior of the distribution of $W^t$ and $W^b$ close to the support, in each figure, a zoom-in plot nearby the edge is included.

Comparing with the prebuckling deformations for the same core using global SS3G boundary conditions, as displayed in figure 7-21 (left), one can see here that the localized wrinkling patterns with a very small wave-length nearby the edge zone disappeared. This is because, at the supports, the core and one of the faces are now free to deform. The axial wavy pattern, which is initiated and will grow at the edges zone because of the fixation of all parts of sandwich layers, is released to the unsupported core and face. These parts of sandwich layers will undergo more deformations than the one which is supported. The magnitudes of deformation of the core and the two adjacent faces, especially with the one which is supported, become significantly different. As a result, high radial and shear stress concentrations will occur at the interfaces between the top face - core and the bottom face - core.

Looking at table 7-10, the maximum radial stresses of the interfaces in the SS3-free case, detected at the edge $x = 0$, are both compressive and very high. This should be expected when one looks at the close-up plot in figure 7-24. Recalling that the radial displacement of the faces is measured from the mid-surface of each face and that the radius of the top face is greater than that for the bottom face, one can see that the deformation of the top face and bottom face is growing towards each other. At the supports, the distance between the top face and the bottom face becomes minimum. As a result, the core will be compressed which is indicated by the negative radial stress values at the interfaces.

**Table 7-10. Maximum stresses at the interfaces between the core and the adjacent faces at the limit loads using local boundary conditions for axial compression case**

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>$\lambda^a_{\text{lim}}$</th>
<th>Radial stresses (psi) (at $x/L = 0.0$)</th>
<th>Shear stresses (psi) (at $x/L = 0.002$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$(\sigma^t_r)_{\text{max}}$</td>
<td>$(\sigma^b_r)_{\text{max}}$</td>
</tr>
<tr>
<td>SS3-free</td>
<td>0.512</td>
<td>-8144</td>
<td>-13787</td>
</tr>
<tr>
<td>C4-free</td>
<td>0.537</td>
<td>-710</td>
<td>2501</td>
</tr>
<tr>
<td>free-ss3</td>
<td>0.507</td>
<td>4863</td>
<td>2924</td>
</tr>
<tr>
<td>free-c4</td>
<td>0.652</td>
<td>-21398</td>
<td>5736</td>
</tr>
</tbody>
</table>
On the other hand, the radial stresses at both interfaces found in the free-SS3 case are high tensile stresses. The radial displacements of the top face and the bottom face are now growing away from each other, which certainly is going to separate the core and the adjacent faces (debonding process). At the supports, the distance between the top face and the bottom face becomes maximum. The same reasoning can also be given for the C4-free and free-C4 cases.

**External pressure**

Table 7-11 gives the results of calculations of the critical buckling loads of the sandwich cylinder ($E_c = 2.0 \times 10^4$ psi and $E_c/G_c = 5$) and loaded by external pressure using four different local boundary conditions at both edges. Note that, for instance, the notation SS3-free is used to denote the top face is supported with SS3 condition at both edges while the bottom face is free.

**Table 7-11.** The critical (external pressure) buckling loads of sandwich cylinders with $E_c = 2.0 \times 10^4$ psi and $E_c/G_c = 5$ using various local boundary conditions

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>$P_{e_{bif}}$</th>
<th>$P_{e_{bif}}$ (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS3-free</td>
<td>$0.120528 \times 10^{-1}$ (5,S)</td>
<td>$0.407147 \times 10^2$</td>
</tr>
<tr>
<td>C4-free</td>
<td>$0.144369 \times 10^{-1}$ (6,S)</td>
<td>$0.487683 \times 10^2$</td>
</tr>
<tr>
<td>free-SS3</td>
<td>$0.125473 \times 10^{-1}$ (5,S)</td>
<td>$0.423850 \times 10^2$</td>
</tr>
<tr>
<td>free-C4</td>
<td>$0.133762 \times 10^{-1}$ (6,S)</td>
<td>$0.451852 \times 10^2$</td>
</tr>
</tbody>
</table>

It can be seen that if one uses clamped C4 boundary conditions for one of the faces, the buckling loads will be higher than when using simply supported SS3 boundary conditions. Furthermore, the buckling load will be higher when the bottom face is supported with SS3 condition ($p_e = 42.385$ psi) than applying the same condition to the top face ($p_e = 40.715$ psi). On the other hand in the C4 case, the buckling load will be higher when supporting the top face ($p_e = 48.768$ psi) than when supporting the bottom face ($p_e = 45.185$ psi).

Returning now to the results of the calculation of the buckling loads of the same core using global SS3G and C4G boundary conditions listed in table 7-8. They are found to be $p_e = 42.550$ psi for the SS3G case and $p_e = 49.489$ psi for the C4G case. Thus one can see that the change of boundary conditions at both edges in the external pressure case only gives a slight reduction in the buckling loads (about less than 10%).
Bifurcation Buckling and Limit Load

The prebuckling shape at the critical buckling load and the buckling modes are plotted in figures 7-25 and 7-26. In each figure, the zoom-in plot is also included for the cylinder segment close to the edge zone.

![Graphs showing prebuckling shapes and buckling modes](image)

**Figure 7-25.** Prebuckling shapes at buckling load levels (external pressure) and buckling modes of sandwich cylinders with $E_c = 2.0 \times 10^4$ psi and $E_c / G_c = 5$ using SS3-free (left) and free-SS3 (right) boundary conditions.

Looking closely at the prebuckling shape and the buckling mode for the SS3-free case in figure 7-25 (left) and for the C4-free case in figure 7-26 (left), one can see that the radial displacement and mode of the top face are zero at the edge while the appropriate values of the bottom face are not zero. Removed from the edges, the prebuckling deformation and the buckling mode behave qualitatively identical to...
those for global boundary conditions case as displayed in figure 7-22 (left) for $E_c = 2.0 \times 10^4$ psi.

![Graphs showing radial displacement W versus length (x/L)](image)

**Figure 7-26.** Prebuckling shapes at buckling load level (external pressure) and buckling modes of sandwich cylinders with $E_c = 2.0 \times 10^4$ psi and $E_c / G_c = 5$ using C4-free (left) and free-C4 (right) boundary conditions.

The prebuckling shape and the buckling mode for the free-SS3 case in figure 7-25 (right) and for the free-C4 case figure 7-26 (right) are quite similar, too. However, looking closely at the zoom-in plot of the edges zone, the radial displacement and mode of the bottom face becomes zero at $x = 0$ while the appropriate values for the top face are not zero, indicating the free-end condition.
Variation of the Core Property

The clamping effects, which are indicated by a smaller slope of the changes of radial displacement and mode, are obviously present when one of the faces is supported using C4 conditions.

Remarks on the effect of boundary conditions

- In general, one can conclude that decreasing the core property $E_c$ and $G_c$ will result in reduction of the critical loads. When dealing with local boundary conditions under axial compression, a significant reduction of the critical load is obtained (about 60% reduction) compared with those cases where global boundary conditions are used. The buckling modes in the global boundary conditions case are characterized by a localized edge wrinkling mode. Typical for local boundary conditions, the radial deformations of both faces have an axial wavy pattern limited to the edge zones. The radial deformations of the top face and the bottom face have a maximum or minimum distance to each other found at the both edges which results in a high radial and shear stress concentration at the interfaces between the core and the adjacent faces.

- Less reduction of the buckling load is observed in the external pressure case. The reduction of the buckling loads in the local boundary conditions case is about 10% of those in the global boundary conditions case. The corresponding buckling modes are quite similar, both in asymmetric mode with the same number of full-waves in the circumferential direction. However, close to the edges they clearly have a different distribution in the axial direction.

7.4. Variation of the Core Property

As it has been stated by several researchers [1, 40] in the fields of sandwich structures, the buckling loads are dependent on the geometry and the mechanical properties of the faces and the core. The buckling load also depends heavily on the modulus of elasticity of the core in the vertical direction $E_c$, see in [26], and the transverse shear moduli of the core $G_x$ and $G_y$ [16].

Referring to the previous table 7-7, it is shown that decreasing the values of modulus of elasticity $E_c$ and shear modulus $G_c$ will result in reduction of the critical buckling loads with the change of the corresponding buckling modes from overall to local mode.

For “stiff” core, $E_c = 1.0 \times 10^7$ psi, one obtains an asymmetric buckling mode ($n = 4$) of both faces which is comparable to the general buckling (overall) mode mentioned in section 7.1 using a membrane prebuckling analysis. The two practical cores used, $E_c = 2.0 \times 10^4$ psi and $E_c = 2.0 \times 10^5$ psi, have an axisymmetric buckling mode ($n$
= 0) and a localized edge wrinkling mode. At the same place along axial coordinate, the wrinkling modes of the top face and the bottom face have opposite maxima which is comparable with the symmetric wrinkling in the membrane solution. The hypothetical weak core, \( E_c = 100 \text{ psi} \), has also axisymmetric mode \((n = 0)\) but the mode of both faces have wrinkling pattern everywhere. Here, the buckling mode is comparable with the shear crimping (antisymmetric) in the membrane solution.

This indicates one important aspect in designing sandwich cylinder configurations. The choice of core material must be made by knowing the effect of variation of core property on the buckling behavior of sandwich constructions, which requires some parametric studies.

For cylinders under axial compression, this kind of studies has already been done by several investigators [55]. However, most of them only deal with the parametric study of shear modulus \( G_c \). Note that chapter 4 in Part I also deals with this kind of investigation. In this section, on purpose, the study is focussed on the variation of modulus of elasticity \( E_c \), as one of the new outcomes of the current sandwich modeling approach, and shear modulus \( G_c \) with the ratio \( E_c/G_c \) within the range of practically used honeycomb core.

All the investigations in this section will be carried out for the same shell geometry and faces property using global SS3G boundary conditions. First, the modulus of elasticity \( E_c \) is varied keeping the same ratio \( E_c/G_c \). Next, both \( E_c \) and \( E_c/G_c \) are varied. These two cases mentioned deal with isotropic core sandwich cylinders. Finally, parametric studies are performed for orthotropic honeycomb core cases.

**Variation of \( E_c \) (isotropic honeycomb and foam core)**

As a first step in the study of the effects of variation of \( E_c \) the critical buckling loads for isotropic honeycomb core are calculated. The results will be compared with several classical results such as in [1, 37, 59]. In order to make a comparison with classical values, the orthotropic property of honeycomb core material is replaced by equivalent isotropic property \( (G_c = (G_x + G_y)/2) \). In the following calculations the ratio of modulus of elasticity \( E_c \) to shear modulus \( G_c \) is fixed, that is \( E_c/G_c = 5.0 \).

Figure 7-27 shows the critical load as function of the ratio of modulus of elasticity of the core to the faces \( E_c/E_f \), respectively. The buckling load is normalized with the same factor as used in Chapter 4 in part I, \( N_{x\text{norm}} = 2E_f(t + t^c)t/cR^2 = 4896 \text{ lbf/inch} \). Hence, this normalization factor does not depend on the value of \( E_c \) and remains constant with the variation of \( E_c \). Note that both in the horizontal and vertical axis of the figure a logarithm scale is used.

From the figure it is seen that decreasing the value of \( E_c \) the critical buckling load obtained with a rigorous nonlinear prebuckling analysis (see the points labeled
Honeycomb SHOT) will also reduce. Points (a-d) in the figure denote the four cases calculated in the previous section 7.3. The corresponding buckling modes will change from asymmetric mode \((n = 4)\) to axisymmetric mode \((n = 0)\) at \(E_c/E_f = 0.02\), denoted by point b. Between \(E_c/E_f = 0.02\) to 0.001, the decrease of the critical loads is more or less constant. The corresponding axisymmetric mode of the faces shows localized edge wrinkling, see also section 7.3.

Since \(E_c/G_c = 5.0\) is fixed, this means that also the shear modulus \(G_c\) is very small for \(E_c/E_f < 0.001\), which implies that the core provides no longer shear rigidity. The axisymmetric mode will change to the form in which the transverse shear deformation of core becomes dominant (shear crimping mode).

Two separate solutions of Bartelds' unified theory [10] are also depicted by two lines in figure 7-27. The first line gives the buckling loads resulting from the antisymmetric mode solution. The second one is that obtained from the symmetric mode solution. The critical buckling loads resulting from this two solutions will be the lowest one of the two computed values.

**Figure 7-27.** Normalized axial compression load as function of the ratio \(E_c/E_f\) calculated with the current program and several classical theories
At about $E_c/E_f = 0.02$, the intersection point between the two solutions indicates the change of the buckling mode from overall (general buckling) mode to local (symmetric wrinkling) mode. It is interesting to note that the point where the slope of the antisymmetric solution curve changes abruptly (point e) is actually the point where the mode changes from overall mode to antisymmetric wrinkling mode. However, this point occurs at a load higher than the calculated load for symmetric solution with the same $E_c/E_f$. At another point, i.e. the intersection point between the two solutions at about $E_c/E_f = 0.001$, the mode will change from symmetrical wrinkling to antisymmetric wrinkling. The intersection points of two membrane solutions mentioned determine the instability mode areas: overall mode, symmetrical wrinkling and antisymmetric wrinkling, as displayed in the figure.

Comparing these solutions with results obtained by SHOT using a rigorous nonlinear prebuckling analysis, one can see that the difference between the two values becomes visible in the overall mode area. In the wrinkling mode areas, symmetric or antisymmetric, the two results almost coincide. Nevertheless, the critical loads obtained with SHOT are slightly lower than the membrane ones.

The simplest approach for symmetrical wrinkling in [59] neglects the shear effects of the core and treats the isotropic faces as beams on an elastic foundation. The spring constant is obtained by considering the mid-plane of the core as a symmetric-axis and treating the core material as an elastic spring. This classical symmetrical wrinkling load is given by van Zelst [59] as

$$N_{x}^{\text{wrench}} = 0.82 t \sqrt{\frac{E_f E_c t}{N(1 - \nu^2)t^2}}$$  \hspace{1cm} \text{(7-1)}$$

The line (3) depicted in figure 7-27 shows that the critical symmetrical wrinkling load is reduced as the core property $E_c$ decreases. This line coincides with the symmetric mode line of Bartelds for about $E_c/E_f > 10^{-4}$. Thus, in the symmetric wrinkling mode area a good agreement is found between this two solutions.

Another instability mode in sandwich construction, as displayed in figure 1-2 in chapter 1 is face dimpling mode. This phenomenon, which precipitates wrinkling of the faces, can happen only in a core which has cellular structure, i.e. honeycomb-like core. The face is buckled or dimpled into the spaces between core walls. For isotropic material, uni-axial compression and only one halfwave of cell mode in both directions, the classical dimpling load formula for one of the faces, see [59] yields

$$N_{x}^{\text{dimpl}} = \frac{\pi^2}{2} \frac{E_t t^3}{s^2 \sqrt{3(1 - \nu^2)}}$$  \hspace{1cm} \text{(7-2)}$$
Variation of the Core Property

Here $s$ denotes the core cell size. From a handbook of material [29], typical for honeycomb material with $E_c/G_c = 5.0$, one obtains the core cellsize $0.1562 < s < 0.1875$ inch. As one can see in figure 7-27, the range within which face dimpling will occur for this particular case is given. Of interest is here that the load levels of face dimpling are slightly below the buckling load of transition point (point b) from overall to symmetric wrinkling mode, which indicates that dimpling mode indeed triggers the face to wrinkle when increasing the applied load.

In chapter 5, another type of core is formulated namely a foam-type core. Foam type core is typically built-up from isotropic material. Thus, the shear modulus of foam core can be defined as $G_c = E_c/(2(1+v))$. For all calculations performed use $\nu = 0.3$. Though the practical range of foam type core is $0.001 < E_c/E_f < 0.01$ (see Frostig [26] and Marshall [37]), the critical buckling load as function of the modulus $E_c$ are given at the same $E_c/E_f$ range.

One can see from figure 7-27 that inside the practical area, i.e. in the symmetrical wrinkling mode area, the points for foam type core almost coincides with the points and the lines for honeycomb core with $E_c/G_c = 5.0$. This indicates that in the symmetrical wrinkling mode area, the buckling behavior becomes more dependent on the property of the faces than the core. Since the same faces are used here, one can expect the similarity of the behavior for different core types.

Four points in figure 7-27 at $E_c/E_f = 0.002$ labeled "Local B.C.", denote the computed critical buckling loads using various local boundary conditions (SS3-free, C4free, free-SS3, free-C4) performed in previous section 7.3. One can see that for the same core property, when local boundary condition is involved, the critical loads are reduced.

Variation of $E_c$ and $G_c$ (isotropic honeycomb core)

In real sandwich configurations, however, the core can have a different ratio $E_c/G_c$. For isotropic honeycomb core material, the ratio $E_c/G_c$ is in the range $4.1 < E_c/G_c < 8.2$ (see [26]). Again, it should be noted here that the orthotropic property of honeycomb core material is still under isotropic core assumption ($G_c = (G_x + G_y)/2$).

Figure 7-28 shows the critical buckling load as function of the ratio of modulus of elasticity of the core to the faces $E_c/E_f$, respectively, for different ratios of $E_c/G_c$. The critical load is normalized with the same factor as used in Chapter 4 in part I, $N_{x}^{\text{norm}} = 2E_f(t + t^c)t/cR^c = -4896$ lbf/inch. Calculations are carried out for values in the range $1 < E_c/G_c < 10$.

In general, decreasing the ratio $E_c/E_f$ the critical loads for all values of $E_c/G_c$, as shown by the various lines in the figure, will also reduce. Especially in the range
1.0 \times 10^{-5} < \frac{E_c}{E_f} < 0.001$, the reduction of the critical load is greater for higher values of $\frac{E_c}{G_c}$. Here, for the same $E_c$ the shear modulus $G_c$ becomes smaller. Recalling that in this range the critical loads also depend heavily on the shear property $G_c$, one can justify this result.

For $\frac{E_c}{E_f} > 0.1$ the lines become more or less identical which indicates that the critical buckling loads no longer depend on the value of $G_c$. The corresponding buckling modes will be obtained as overall mode. Here, because $G_c$ is also increased the core can provide sufficient shear rigidity and transfer the shear stresses between the two faces of the sandwich layer. At the same time, because of high values of $E_c$, the core becomes stiff in its thickness direction. The faces and the core are then behave more or less as one layer of conventional material with the equivalent appropriate thickness and membrane, shear and bending stiffnesses.

![Graph](image)

**Figure 7-28.** Normalized axial compression load as function of the ratio $\frac{E_c}{E_f}$ with different ratios $\frac{E_c}{G_c}$

**Orthotropic Honeycomb core**

Until now, the characterization of the honeycomb core material is done by use of only two core properties $E_c$ and $G_c$. Due to its “cellular” micro-structure, the core has to be characterized by three elastic constants $E_c$, $G_x$ and $G_y$. Here, $G_x$ and
$G_y$ denote for the transverse shear modulus of the core in the axial and circumferential direction, respectively. It has been assumed in chapter 5 that the principal directions of honeycomb core materials also coincided with the in-plane cylinder coordinates (x-y). Thus $G_x$ is equal to the core shear modulus in the longitudinal direction $G_L$ and $G_y$ equal to the core shear modulus in the transverse direction $G_W$.

Figure 7-29 shows the critical load as function of the ratio $E_c/G_c$ for different ratios $\phi = G_x/G_y$. Here, the shear modulus $G_c$ is equal to the average of two orthotropic shear moduli, $(G_x + G_y)/2$. The modulus of elasticity of the core is constant with $E_c = 2.0 \times 10^5$ lbf/inch. Again, the critical loads are normalized with the factor $N_x^{\text{norm}} = 4896$ lbf/inch. 

![Graph showing normalized critical load vs. ratio $E_c/(G_x/G_y)$]

**Figure 7-29.** Normalized axial compression load as function of the ratio $2E_c/(G_x/G_y)$ with different ratios $\phi = G_x/G_y$ under isotropic core assumption

As it expected, the error of assuming the orthotropic core property to be an equivalent isotropic property $G_c$ by averaging $G_x$ and $G_y$ becomes greater when the average shear modulus $G_c$ decreases and the ratio $\phi$ increases. The prediction of buckling loads under this isotropic core assumption becomes too conservative. It should be noted that in the case $\phi = 1$ one can replace the orthotropic property $G_x$, $G_y$ exactly with the isotropic property $G_c$. This can happen, for instance, when the core has a cubical micro cellular structure, see handbook of material [37]. However,
a common honeycomb core will have a hexagonal micro cellular structure, which usually has a ratio of $\phi = G_x/G_y = G_L/G_W$ of approximately 1.5 - 1.8, see [56].

The reduction of the critical buckling loads under isotropic core assumption is expected by knowing that the critical loads in the case of axial compression, as mentioned in the previous section 7.3, depend on the value of $G_x$. The corresponding buckling mode obtained is characterized by axisymmetric wrinkling mode. The critical buckling loads will be reduced when the value of $G_x$ decreases, which occurs if one replaces $G_x$ to $G_c$ for $\phi > 1$ under isotropic core assumption.

It is interesting now to investigate the influence of using (isotropic or orthotropic) honeycomb core or foam core on the buckling behavior of cylinders with the same face properties and shell geometry. Table 7-12 gives the result of the various calculation of the critical loads and the corresponding buckling modes under different core property with the same modulus $E_c = 2.0 \times 10^5$ psi.

**Table 7-12. Comparison of collapse loads of sandwich cylinders using foam and honeycomb type cores for the same $E_c$ and faces property under global SS3G B.C.**

<table>
<thead>
<tr>
<th>Foam $(E_c/E_f = 0.02$ and $v = 0.3)$</th>
<th>Isotropic Honeycomb $E_c/G_c = 5.0$</th>
<th>Orthotropic Honeycomb $(\phi = 1.5$ and $(G_x + G_y)/2 = G_c)$</th>
<th>Isotropic Honeycomb (Membrane)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_x^{crit}$ (lbf/inch)</td>
<td>-3602.03</td>
<td>-3544.28</td>
<td>-3567.71</td>
</tr>
<tr>
<td>Instability mode localized edge wrinkling $(n = 0, A)$</td>
<td>localized edge wrinkling $(n = 0, A)$</td>
<td>localized edge wrinkling $(n = 0, A)$</td>
<td>symmetric wrinkling $(n = 0, m = 305)$</td>
</tr>
</tbody>
</table>

From the results listed in table 7-12, as expected, one can see that the results obtained using a rigorous prebuckling analysis with three different core configurations give always smaller buckling loads than the membrane solution. It is also shown that the critical load in both honeycomb core cases is smaller than in the foam core case, which can be expected since the shear modulus in the foam core case is higher. Again, one can see that the result in the orthotropic honeycomb core case is higher than in the isotropic honeycomb core case.

All of the corresponding buckling modes are axisymmetric $(n = 0)$ and the top face mode and bottom face have opposite local maximum amplitudes at the same place along the axial coordinate. The corresponding symmetric wrinkling mode obtained in the membrane solution has 305 halfwaves in the axial direction $(m = 305)$. 

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Counting out the number of axial halfwaves of the modes obtained for the other three different results, they vary from 300 to 305 halfwaves in the axial direction. Note that, on the contrary to the one for membrane solution, the amplitude of the mode is not the same over the entire cylinder. The maximum amplitude is encountered close to the edges and denoted here as localized edges wrinkling.

Finally, in figure 7-30, the shear modulus $G_x$ is varied for four different values of $\phi = G_x/G_y$, i.e., $\phi = 1.0, 1.5, 5.0, 10.0$. The plots give the critical buckling load for two values of elastic modulus $E_c = 2.0 \times 10^5$ psi and $1.0 \times 10^7$ psi. In order to make a comparison with the results obtained in Chapter 4 in Part I, the horizontal axis is given as $\chi = (E_c t)/(G_x c R^5)$. The same normalization factor is used along the vertical axis, namely with $N_x^{\text{norm}} = 4896$ lbf/inch.

![Figure 7-30. Normalized axial compression load as function of $\chi$ for different $\phi = G_x/G_y$ ratios](image)

For $E_c = 1.0 \times 10^7$ psi one can see that the lines denoting the critical loads for different values of $\phi$ do not coincide at the same $E_c$ and $G_x$ for $\chi$ smaller than 1.5. This means that for isotropic or orthotropic cores the critical loads are not the same. Qualitatively the same trend here is obtained as that displayed in figure 4-1 in chapter 4. The reason for this phenomena is because the instability modes obtained are general buckling in asymmetric modes. These modes will also depend on the
value of $G_Y$. Thus, for higher ratio $\phi = G_X/G_Y$, i.e. lower value of $G_Y$, will give smaller critical buckling loads because the core provides less shear rigidity in the circumferential direction. Beyond $\chi = 1.5$, the results for all the cases are more or less identical and the modes are changed into shear crimping mode.

On the other hand, for $E_c = 2.0 \times 10^6$ psi, if one varies the value of $\chi$ the critical loads are more or less the same for different values of $\phi$. This is because the corresponding buckling modes found for this case are always axisymmetric wrinkling modes ($n = 0$). Here, the difference in shear modulus $G_Y$ gives hardly influence to the computed critical loads.

**Remarks on variation of the core properties**

- When one is dealing with a moderate or weak core sandwich cylinder under axial compression and using SS3G boundary conditions, in which the buckling mode is characterized with axial wavy wrinkling patterns (symmetric wrinkling), the buckling behavior of sandwich cylinder becomes more dependent on the property of the faces than the core. In this case, the classical theories discussed also give a good prediction on the critical buckling loads of the cylinder.

- Knowing the buckling behavior of sandwich cylinders under variation of core property for a given shell geometry, face property, loading and boundary conditions, by presenting such plots as in figure 7-27 to figure 7-30, the designer can choose the appropriate core property. Thus the predicted critical load and the corresponding buckling mode can quickly be determined.

- However, the collapse behavior of sandwich cylinders described above excluded the aspect of possible material failures. General collapse behavior for sandwich constructions, especially when localized effects occur due to the boundary conditions, loading conditions, etc., should also consider this kind of phenomena by inspecting the stress distribution between the sandwich layers. Another aspect that one should also consider is the effects of faces imperfections. In the last three sections, we will deal with this kind of problems.

### 7.5. Core Material Failures

For practically used core types, two cases are investigated, namely Nomex honeycomb Hexcel HRH10 core and Rohacell 1101G foam core [37]. Both cores are lightweight and commonly used in aerospace applications. The core properties, the allowable compressive and tensile strength, as well as their shear strength are given in table 7-13. Note that in the following calculations for the Nomex core case, one uses orthotropic honeycomb core material model without isotropic core
assumption. Again, it will be assumed that the principal directions of core material coincide with the in-plane cylinder coordinates (x-y). Thus it follows that the shear moduli \( G_x = G_L \) and \( G_y = G_W \), respectively. The shell geometry and the faces properties are identical to those applied in section 7.1.

In the first stage, global boundary conditions SS3G and C4G at both edges are used for the failure predictions. However, the boundary conditions at the supports of sandwich cylinders, as pointed out earlier in section 7.3, can also be defined as local boundary conditions SS3-free or C4-free. In this case, the edges of the outside surface of the cylinder (the top face of sandwich layers) are attached to the end ring while the edges of the inside surface of the cylinder (the bottom face of sandwich layers) and the core are free.

By increasing the applied load, the radial and shear stresses at the interfaces between the top face - core and bottom face - core are computed directly from the solution of the nonlinear equilibrium equations. The maximum value of the interfaces stresses along the surface coordinate is searched and compared with the allowable corresponding stress for Nomex and Rohacell core.

| Table 7.13. Typical mechanical properties of honeycomb and foam core [37] |
|---|---|---|---|---|---|---|
| Hexcel HRH10-3/16-3.0(2) Honey (Nomex/Phenolic) | | | | | | |
| \( E_c \) (psi) | \( G_L \) (psi) | \( G_W \) (psi) | \( \sigma_z \) comp (psi) | \( \sigma_z \) tension (psi) | \( \tau_{xr} \) L (psi) | \( \tau_{xr} \) W (psi) |
| 2.0.10^4 | 5.8.10^3 | 3.5.10^3 | 370 | 300 | 160 | 90 |
| Rohacell IG 1101G | | | | | | |
| \( E_c \) (psi) | \( G_c \) (psi) | \( v \) | \( \sigma_z \) comp (psi) | \( \sigma_z \) tension (psi) | \( \tau_{xr} \) (psi) |
| 2.27.10^4 | 8.25.10^3 | 0.375 | 427 | 498 | 341 |

The results are given in tables 7-14 and 7-15 for axial compression and in tables 7-16 and 7-17 for external pressure, using global boundary conditions SS3G and C4G and local boundary conditions SS3-free and C4-free at both edges of cylinder. Note that the critical values in the tables are listed in normalized form

\[
\lambda_x^{\text{crit}} = \frac{N_x^{\text{crit}}}{N_x^{\text{norm}}} \quad \text{and} \quad p_e^{\text{crit}} = \frac{p_e^{\text{crit}}}{p_e^{\text{norm}}}
\]

where the normalization factor \( N_x^{\text{norm}} \) listed in the tables is the result obtained using a membrane prebuckling analysis and \( p_e^{\text{norm}} = \frac{E_f(t^c + 2t)^2}{(c(R^c)^2)} = 3378.02 \text{ psi} \), respectively.
Table 7-14. The critical buckling loads (axial compression) and material failures prediction of Nomex (honeycomb-type) core using different boundary conditions

| Boundary conditions | $\lambda_a^{\text{crit}}$  
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(N_x^{\text{norm}} = -1227.5 \text{ lbf/inch})$</td>
</tr>
<tr>
<td>SS3G</td>
<td>0.98512 (0)</td>
</tr>
<tr>
<td></td>
<td>$(\sigma_r)^{\text{max}}$</td>
</tr>
<tr>
<td></td>
<td>$(\sigma_r)^{\text{allow}}$</td>
</tr>
<tr>
<td></td>
<td>$(\lambda_a \times L)$</td>
</tr>
</tbody>
</table>
|                     | 1.050  
|                     | 0.98072  
|                     | 0.05  
|                     | 0.05 |
| C4G                 | 0.99924 (0)                  |
|                     | $(\sigma_r)^{\text{max}}$   |
|                     | $(\sigma_r)^{\text{allow}}$ |
|                     | $(\lambda_a \times L)$      |
|                     | 1.003  
|                     | 0.99840  
|                     | 0.057  
|                     | 0.018 |
| SS3-free            | 0.51537 (0)                  |
|                     | $(\sigma_r)^{\text{max}}$   |
|                     | $(\sigma_r)^{\text{allow}}$ |
|                     | $(\lambda_a \times L)$      |
|                     | -1.157  
|                     | 0.47708  
|                     | 0.0  
|                     | 0.002 |
| C4-free             | 0.56096 (0)                  |
|                     | $(\sigma_r)^{\text{max}}$   |
|                     | $(\sigma_r)^{\text{allow}}$ |
|                     | $(\lambda_a \times L)$      |
|                     | -1.142  
|                     | 0.51553  
|                     | 0.0  
|                     | 0.002 |

Table 7-15. The critical buckling loads (axial compression) and material failures prediction of Rohacell (foam-type) core using different boundary conditions

| Boundary conditions | $\lambda_a^{\text{crit}}$  
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(N_x^{\text{norm}} = -1307.5 \text{ lbf/inch})$</td>
</tr>
<tr>
<td>SS3G</td>
<td>0.98757 (0)</td>
</tr>
<tr>
<td></td>
<td>$(\sigma_r)^{\text{max}}$</td>
</tr>
<tr>
<td></td>
<td>$(\sigma_r)^{\text{allow}}$</td>
</tr>
<tr>
<td></td>
<td>$(\lambda_a \times L)$</td>
</tr>
</tbody>
</table>
|                     | -1.006  
|                     | 0.98854  
|                     | 0.06  
|                     | 0.07 |
| C4G                 | 0.99902 (0)                  |
|                     | $(\sigma_r)^{\text{max}}$   |
|                     | $(\sigma_r)^{\text{allow}}$ |
|                     | $(\lambda_a \times L)$      |
|                     | -1.117  
|                     | 0.99884  
|                     | 0.008  
|                     | 0.02 |
| SS3-free            | 0.51550 (0)                  |
|                     | $(\sigma_r)^{\text{max}}$   |
|                     | $(\sigma_r)^{\text{allow}}$ |
|                     | $(\lambda_a \times L)$      |
|                     | -1.182  
|                     | 0.47427  
|                     | 0.0  
|                     | 0.002 |
| C4-free             | 0.57274 (0)                  |
|                     | $(\sigma_r)^{\text{max}}$   |
|                     | $(\sigma_r)^{\text{allow}}$ |
|                     | $(\lambda_a \times L)$      |
|                     | -1.017  
|                     | 0.56674  
|                     | 0.0  
|                     | 0.002 |

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Table 7.16. The critical buckling loads (external pressure) and material failures prediction of Nomex (honeycomb-type) core using different boundary conditions

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>( \frac{p_{c}}{p_{e}} )</th>
<th>( \frac{\sigma_{t}^{a}}{\sigma_{t}^{b}} )</th>
<th>( \frac{\sigma_{r}^{a}}{\sigma_{r}^{b}} )</th>
<th>( \frac{\tau_{x}^{a}}{\tau_{x}^{b}} )</th>
<th>( \frac{\tau_{x}^{b}}{\tau_{x}^{a}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS3G</td>
<td>0.012310 (5)</td>
<td>0.7127</td>
<td>-0.6020</td>
<td>0.6625</td>
<td>0.6776</td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td>0.001</td>
<td>0.005</td>
<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>C4G</td>
<td>0.014285 (6)</td>
<td>0.3797</td>
<td>-0.3262</td>
<td>1.0059</td>
<td>1.0016</td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td>0.002</td>
<td>0.005</td>
<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>SS3-free</td>
<td>0.011731 (5)</td>
<td>1.0144</td>
<td>-0.5531</td>
<td>0.6824</td>
<td>0.6979</td>
</tr>
<tr>
<td></td>
<td>0.0030</td>
<td>0.0</td>
<td>0.003</td>
<td>0.003</td>
<td>0.003</td>
</tr>
<tr>
<td>C4-free</td>
<td>0.014097 (6)</td>
<td>1.0065</td>
<td>-0.0784</td>
<td>0.6247</td>
<td>0.6389</td>
</tr>
<tr>
<td></td>
<td>0.0094</td>
<td>0.001</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
</tr>
</tbody>
</table>

Table 7.17. The critical buckling loads (external pressure) and material failures prediction of Rohacell (foam-type) core using different boundary conditions

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>( \frac{p_{c}}{p_{e}} )</th>
<th>( \frac{\sigma_{t}^{a}}{\sigma_{t}^{b}} )</th>
<th>( \frac{\sigma_{r}^{a}}{\sigma_{r}^{b}} )</th>
<th>( \frac{\tau_{x}^{a}}{\tau_{x}^{b}} )</th>
<th>( \frac{\tau_{x}^{b}}{\tau_{x}^{a}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS3G</td>
<td>0.014702 (5)</td>
<td>0.5969</td>
<td>-0.7186</td>
<td>0.3987</td>
<td>0.4078</td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td>0.001</td>
<td>0.005</td>
<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>C4G</td>
<td>0.018084 (6)</td>
<td>0.2743</td>
<td>-0.3778</td>
<td>0.7358</td>
<td>0.7526</td>
</tr>
<tr>
<td></td>
<td>0.001</td>
<td>0.001</td>
<td>0.005</td>
<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>SS3-free</td>
<td>0.014154 (5)</td>
<td>1.0059</td>
<td>-0.6176</td>
<td>0.4135</td>
<td>0.4229</td>
</tr>
<tr>
<td></td>
<td>0.0045</td>
<td>0.0</td>
<td>0.003</td>
<td>0.003</td>
<td>0.003</td>
</tr>
<tr>
<td>C4-free</td>
<td>0.017966 (6)</td>
<td>1.0005</td>
<td>-0.1020</td>
<td>0.4129</td>
<td>0.4223</td>
</tr>
<tr>
<td></td>
<td>0.0135</td>
<td>0.001</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
</tr>
</tbody>
</table>
For the axial compression load cases listed in tables 7-14 and 7-15, the predicted critical loads using local boundary conditions (SS3-free and C4-free) dropped to about 50-60% of the critical values found using global simply supported (SS3G) or clamped (C4G). The reduction of the critical loads is mainly caused by the unsupported (free end) and thus unloaded bottom face of the cylinder at both edges. In the case of external pressure loading in tables 7-16 and 7-17, despite using different boundary conditions, the predicted critical loads are almost the same. In both loading cases, in general, the predicted critical loads obtained for Rohacell core are higher than for Nomex core following the slightly higher value of modulus of elasticity $E_c$ and shear modulus ($G_{c,Rohacell} > G_{L,Nomex}$).

The core material failures prediction, given in the last four columns of each tables, detects whether prematurely core crushing, tensile ruptures or shear crimping can occur at load levels below the predicted critical loads. In this study the maximum stress criterion which belongs to the category of independent failure criteria [45] will be used. It must be stressed that the failure criterion is only used to check whether allowables are exceeded. Core crushing will occur when the compressive radial stress is greater than the core compressive strength. Similarly, shear crimping is more likely to happen when the shear stresses in the interfaces exceed the allowable core shear strength.

Depending on the glue bondage allowables between the faces and core, tensile ruptures of sandwich layers can occur in two modes. The first one is tensile rupture of core proper. This case probably will happen when the bond between the core and the faces is strong. The radial stresses in the core exceed the tensile strength of the core material. The second case is tensile rupture of the bond itself which will occur when the bond between the core and faces is weak.

Frostig mentioned in [27] that the failure of sandwich structures occurs more often due to the last two cases. To avoid debonding the core from the adjacent faces nearby the supports, inserts, geometrical discontinuity, etc., the computed maximum radial stresses in the interfaces should be less than the glue bondage allowable between the core and the faces. This is also a reason to call radial stresses in the interfaces between the core and the faces as peeling stresses. However, quantitatively predictions in this study assumed that the bonding between the core and the faces is sufficiently strong, so that only the case of tensile rupture of core proper will be taken into consideration. Nevertheless, knowing the strength of bonding material, it is easy to detect whether debonding of the sandwich layers will be initiated by comparing the maximum tensile radial stresses to the tensile strength of the bond.

Note that prediction of the material failures in this study is only a macro prediction which excludes the detail of core cellular structure by considering the core as a
Core Material Failures

Homogenous continuum. A more accurate micro study of these material failures are beyond the scope of the current investigation. Knowing the detailed core structure, a closer analysis and prediction can be performed by looking only at a small area of, e.g., honeycomb core cells. This structure is then subjected by external loads which correspond to the values of maximum radial and shear stresses detected in the present global analysis.

If the absolute value of the ratio of the maximum stresses to the allowable stresses is greater than one then core material failures might happen before the critical buckling load level is reached. The corresponding load levels and their location along the coordinate axis are given in the tables.

In general, there is no significant different in the core material failure behavior of the Nomex and Rohacell type cores. Returning to table 7-13 one can see that Rohacell core has a greater modulus of elasticity and shear modulus than the corresponding values for the Nomex core. This explains that the material failures for the Rohacell core always happen at a higher load level than those obtained for the Nomex core. It is also indicated the smaller failure ratios found for Rohacell core at the critical loads. Unless otherwise stated, the following explanations are given only for Nomex core cases.

Axial compression

The first material failures detected for axial compression in tables 7-14 and 7-15, which are denoted by the underlining of the values in the tables, is primarily caused by exceeding the allowable radial stresses (core crushing or tensile rupture of core proper). It is interesting to note that in the SS3G and C4G cases the core material failure due to core crushing or tensile rupture occurred at slightly below the critical loads. Prediction of failure of the core due to shear crimping is found only in the case of local boundary conditions SS3-free and C4-free. However, these values are not critical because they are found at load levels higher than the failure loads due to core crushing or tensile rupture. Even for Rohacell core with C4-free boundary conditions, the failure ratio for the shear stress is still smaller than the one at the critical load, which certainly does not occur in the Nomex core case. Again, this indicates that Rohacell foam core has a rather high shear modulus \( G_c \) in comparison with that for the Nomex core.

The corresponding deformation patterns (radial displacements) at the critical loads are depicted in figures 7-31 and 7-32 for Nomex core using various boundary conditions. The radial displacements are displayed in real measurements, which means that both the thickness and radius of the faces and core are accounted for in the deformation. The radial displacement of the core displayed is the average value of radial displacement of the core through the thickness as given in equation (5-65)
in chapter 5. Note that only half the cylinder is depicted and symmetry conditions are implied in the mid-length of cylinder.

Figure 7-31 shows the radial displacements at the critical load for the SS3G case and the SS3-free case. Circles in the figure denote the location where core material failures are first detected. In the SS3G case, it is seen that localized edge wrinkling of the faces occurs. At the bottom face - core interface, see zoom-in figure on the right side, tensile rupture or debonding has probably happened since at this place the distance between the bottom face to the core becomes maximum (denoted by circle 1). Thus the bottom face is moved away from the core and the maximum radial stress exceeds the allowable value \((\sigma_r^{b})_{\text{max}} > (\sigma_r)_{\text{allow}}\).

**Figure 7-31. Radial deformations of sandwich layers (Nomex core) under axial compression using SS3G and SS3-free boundary conditions**

Looking now at the deformations for the SS3-free case, one can observe that the specified boundary conditions are satisfied at the edge at \(x = 0\). Here, the bottom face and the core clearly move from their un-deformed mid-surfaces while the top face is fixed. At this place, the distance between the top face and the core becomes minimum (denoted by circle 2) which indicates that the core is crushed by the top
face and the maximum stress \( \sigma_{r}^{\text{l}} \)_{max} exceeds \( \sigma_{r} \)_{allow}. One can conclude here that core crushing failure is most likely to happen.

The same kinds of reasoning can be also given for the C4G and C4-free case. The radial deformations of these cases are displayed in figure 7-32. It should be mentioned that, on the contrary to the results obtained in the simply supported cases described earlier, the first failure detected in the C4G case is most likely to happen due to tensile rupture close to the top face (denoted by circle 3) while in the C4-free case core crushing occurs close to the bottom face at the edge (denoted by circle 4).

\[
\begin{align*}
&\text{C4G} \\
&\text{C4-free}
\end{align*}
\]

**Figure 7-32.** Radial deformations of sandwich layers (Nomex core) under axial compression using C4G and C4-free boundary conditions

Both radial displacements for the SS3G and C4G cases at the critical loads, as shown in figures 7-31 and 7-32, are dealing with wrinkling mode, in which the growing of the mode is initiated from the areas close to the edges. Looking closely at the radial stress distribution at this stage of loading, as depicted in figure 7-33 for the SS3G case, an oscillation of the radial stresses of both interfaces occurs. The maximum value found along the axial coordinate x can easily change the sign. Here, the difference between the maximum compressive and tensile stress found
for the bottom face-core interface is very small which causes that the core can fail either in tensile rupture or core crushing.

![Figure 7-33. Radial stresses at the interfaces between the core and the adjacent faces at the critical load level (axial compression) for Nomex core using SS3G b.c. along $0 < x < 7.5$ inch](image)

**External pressure**

For external pressure, as shown in tables 7-16 and 7-17 the core material failures caused by exceeding the allowable value of the radial stress at the interfaces only occur in the SS3-free and C4-free cases for both core types.

The radial stress distribution of the interfaces between the core and the adjacent faces along the axial coordinate $x$ are depicted in figure 7-34.a using SS3G and C4G boundary conditions at the critical load level. The same trend is also found in the recent study of Frostig [27] for a curved panel which is subjected to uniform loading at the upper face. The peak tensile stress is found at the top face-core interface close to the support. Far from the edges the radial stresses at both interfaces become negative and almost constant.

The maximum tensile stresses found here do not exceed the allowable tensile strength of the core (Nomex). However, depending on the glue bondage allowable, debonding process of the interfaces can be happened here. In the C4G case, a premature material failure due to shear is detected at $x = 0.15$ inch. The gradient of the shear stresses at the interfaces increases suddenly at this area, see zoom-in plot of distribution of $\tau_{xr}^t$ and $\tau_{xr}^b$ in figure 7-35, which exceeds the allowable
shear strength of the core (Nomex). Note that the shear stresses at both interfaces are not the same. Looking now at table 7-17, for Rohacell core the failure ratios at the critical loads become smaller than one (0.7358 and 0.7526, respectively) which excludes the possibility of failure of the core due to shear.

![Graphs showing radial stresses at interfaces](image)

**Figure 7-34.** Radial stresses at the interfaces at the critical loads (external pressure) for Nomex core using different boundary conditions

For the SS3-free and C4-free cases, the distribution of the radial stresses at the interfaces along the length coordinate are also depicted in figure 7-34.b at load levels about 26% and 64% of the critical loads for respectively SS3-free and C4-free boundary conditions (see table 7-16). In both cases, the maximum tensile values are found at the top face - core interface close to the edges and they clearly exceed the allowable strength. Thus, tensile rupture of core proper would probably happen at this interface nearby both cylinder edges. At the bottom face - core interface, the maximum peak value is also found at the edges. However, the peak value found is smaller than the allowable strength.
Figure 7-35. Shear stresses at the interfaces at the critical loads (external pressure) for Nomex core using C4G boundary conditions

The radial displacements at the critical load for the SS3G case and the SS3-free case are displayed in the top of figure 7-36. Again circles in the figure denote the location where core material failure is first detected. In the SS3G case, the possibility of material failure is absent.

Looking closely at the deformation of the top face, core and bottom face, one can see that they deform almost identically and the distances to each other are more or less constant everywhere. On the other hand in the SS3-free case, it is seen now that because the core and bottom faces are free to move the distance between the top face and the core becomes maximum at the edge x = 0 (denoted by circle 1). Here, peeling of the top face from the core can cause tensile rupture of the core proper when the bond is strong. One can also justify the above results by looking at the distribution of the stresses at the interfaces along the axial coordinate, as discussed earlier.

In the bottom of figure 7-36 the radial displacements at the critical load for the C4G case and the C4-free case are displayed. Circles 2 and 3 in the figure denote the location where the core material failure is first detected. In the C4G case, as mentioned earlier, core material failure due to shear is detected at x = 0.15 inch (denoted by circle 2). To justify the results, one should return to figure 7-35 and look at the shear stress distribution at the interfaces. In the C4-free case, it can be seen that the distance between the top face and the core becomes maximum at the edge x = 0 (denoted by circle 3). Thus, as mentioned earlier, tensile rupture of core proper would probably happen here for strong bonding.
Figure 7-36. Radial deformations of sandwich layers (Nomex core) under external pressure using SS3G, SS3-free (top) and C4G, C4-free (bottom) b.c.
Remarks on core material failures

Finally, some remarks can be given regarding the collapse behavior of the two cores used in practice due to both the critical loads found and the possibilities of material failures:

- In the axial compression case the use of local boundary conditions (SS3-free and C4-free) will reduce the critical loads significantly, about 50-60% of that for the global SS3G and C4G boundary conditions. In the external pressure case the reduction is only about 10%.

- Material failures due to core crushing and tensile rupture of core proper occur almost at the critical loads in axial compression case using global SS3G and C4G boundary conditions. In the SS3-free and C4-free cases, the material failures are encountered far below the critical buckling loads. Material failures due to shear occur in all local boundary cases, except for Rohacell core using C4-free boundary conditions, which indicates that this foam core provides high shear rigidity. Because both faces are exposed to axial compression load, material failures can occur either at the top face-core or the core-bottom face interface.

- In the external pressure case, tensile rupture of the core proper can only occur in the SS3-free and C4-free cases. Since the uniform loading is applied at the top face, the material failure is detected at the interface between the top face and the core. Using clamped C4G boundary conditions, the maximum shear stresses at the interfaces found are rather high and for Nomex core they exceed the allowable value below the critical buckling load.

- In general, the material failures described above occur at the edge zones. This indicates the localized effects of boundary conditions and loading location. Especially in axial compression case, one can observe a strong combination between localized effect of boundary conditions and that due to introducing the applied load at the edges. This results in premature material failures below the critical buckling load in all the cases.

- Finally, one should be aware about the debonding process of the interfaces. Especially when a peak tensile radial or peeling stresses occur nearby the edges. These stresses may exceed the stress allowable of the interfaces and might initiate debonding which usually leads to a premature failure in the form of debonding of one of the faces from the core.

7.6. Laminated Faces

It is interesting to investigate collapse behavior of the same sandwich cylinder of table 7-1 in section 7.1, by replacing the isotropic face sheets with laminated faces.
The sandwich layers are built from two identical Khot's glass epoxy (-40°/40°/0°) unsymmetrical laminated sheets, see [9], with a Rohacell core (foam type) in the middle. The material property of the faces used are: \( E_{11} = 7.5 \times 10^6 \) psi, \( E_{22} = 3.5 \times 10^6 \) psi, \( G = 1.25 \times 10^6 \) psi, \( v_{12} = 0.025 \) and \( t^\text{tot} = 0.036 \) inch. The material property of Rohacell core is already given in table 7-13 in section 7.5. The cylinder is loaded by external pressure at the top face pointing towards the mid-point of the cylinder. The global SS3G boundary conditions are applied at the cylinder edges.

By performing bifurcation analysis the critical buckling load is found to be \( p_{c}^\text{bif} = 0.02453 \) (3140.6) \( = 77.04 \) psi. The corresponding buckling mode is asymmetric with five fullwaves in the circumferential direction. In figure 7-37 the radial prebuckling deformation at the buckling load level and the buckling mode are displayed. Compared with the previously obtained results using isotropic faces \( (p_{c}^\text{bif} = 49.66 \) psi), see table 7-17 in section 7.5, the buckling load is increased up to 50%.

As a comparison, in the case of conventional anisotropic Khot's cylindrical shells with more or less the same ratio \( L/R = 2 \) and loaded by external pressure, see [9], or in other words if we consider only one of the faces, the critical buckling load calculated with a rigorous prebuckling analysis is found to be \( p_{c}^\text{bif} = 5.199 \) psi. This example illustrates how a cylinder, which is built from sandwich material can carry much higher load than the same cylinder made of conventional material. In this case one obtained a more than 10 times higher buckling load. Tactily, the bending stiffness of the case with sandwich material is found higher.

**Figure 7-37.** Prebuckling shapes at buckling load level (external pressure) and buckling modes for laminated face sandwich cylinder with a Rohacell core
**Figure 7-38.** Shear resultants in the laminated faces and shear stresses at the interfaces of a Rohacell core sandwich cylinder at $p_e = 77.04$ psi

Of interest is here also the distribution of response quantities in the circumferential direction that is displacements $V^t, V^b$ and resultants $Q^t_y, Q^b_y$ of the faces and shear stresses $\tau^t_{0r}, \tau^b_{0r}$ at the interfaces. Contrary to the previous results using isotropic faces, they are not zero. The distribution of shear stresses $\tau_{0r}^t, \tau_{0r}^b$ along the axial coordinate $x$ are depicted in the left side of figure 7-38.

**Figure 7-39.** Stresses at the interfaces of a Rohacell core sandwich cylinder with laminated faces using SS3G boundary conditions at $p_e = 77.04$ psi

The distribution along the axial coordinate $x$ of the shear resultants $Q^t_x, Q^b_x$ of the faces is shown in the right side of figure 7-38 and the shear stresses $\tau_{xr}^t, \tau_{xr}^b$ and radial stresses $\sigma^t_r, \sigma^b_r$ at the interfaces are shown in figure 7-39. Again, they show
localized effects which are characterized by shear concentrations nearby the edges and peak radial stresses in almost the same region.

Another aspect of interest is the failure prediction of the core material. The failure prediction for isotropic faces has been discussed in section 7.5 and for the current laminated faces the results are given in table 7-18. Similar to isotropic faces, it can be seen from the table that no material failure predicted in the core when increasing the applied pressure up to the critical buckling load. Depending on the strength of the bonding, the maximum radial stress, which is detected at the interface between the core and the top face in the vicinity of the edge, indicated the possibility of debonding of the face from the core.

It should be mentioned that the shear stresses in the circumferential direction at the interface are very small in comparison with the shear stresses the axial direction, but they are not zero. However, it is very unlikely that they can lead to material failure of the core.

### Table 7-18. The critical buckling loads (external pressure) and material failures prediction of a Rohacell core sandwich cylinder with laminated faces

<table>
<thead>
<tr>
<th>$\bar{p}_{e}^{\text{bif}}$</th>
<th>$\left(\sigma_{r}^{l}\right)_{\max}$</th>
<th>$\left(\sigma_{r}^{b}\right)_{\max}$</th>
<th>$\left(\tau_{x}\right)_{\max}$</th>
<th>$\left(\tau_{x}^{b}\right)_{\max}$</th>
<th>$\left(\tau_{x}^{l}\right)_{\max}$</th>
<th>$\left(\tau_{r}^{l}\right)_{\max}$</th>
<th>$\left(\tau_{r}^{b}\right)_{\max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\left(\bar{p}_{e,x/L}^{\text{norm}} = 3378.0 \text{ psi}\right)$</td>
<td>$\left(\bar{p}_{e,x/L}^{\text{allow}}\right)$</td>
<td>$\left(\bar{p}_{e,x/L}^{\text{allow}}\right)$</td>
<td>$\left(\bar{p}_{e,x/L}^{\text{allow}}\right)$</td>
<td>$\left(\bar{p}_{e,x/L}^{\text{allow}}\right)$</td>
<td>$\left(\bar{p}_{e,x/L}^{\text{allow}}\right)$</td>
<td>$\left(\bar{p}_{e,x/L}^{\text{allow}}\right)$</td>
<td>$\left(\bar{p}_{e,x/L}^{\text{allow}}\right)$</td>
</tr>
<tr>
<td>0.024530 (5)</td>
<td>0.7092</td>
<td>-0.8632</td>
<td>0.5683</td>
<td>0.5812</td>
<td>0.0002</td>
<td>0.0002</td>
<td></td>
</tr>
<tr>
<td>$\bar{p}_{e,x/L}^{\text{bif}}$</td>
<td>$\bar{p}_{e,x/L}^{\text{bif}}$</td>
<td>$\bar{p}_{e,x/L}^{\text{bif}}$</td>
<td>$\bar{p}_{e,x/L}^{\text{bif}}$</td>
<td>$\bar{p}_{e,x/L}^{\text{bif}}$</td>
<td>$\bar{p}_{e,x/L}^{\text{bif}}$</td>
<td>$\bar{p}_{e,x/L}^{\text{bif}}$</td>
<td>$\bar{p}_{e,x/L}^{\text{bif}}$</td>
</tr>
<tr>
<td>0.0025</td>
<td>0.0025</td>
<td>0.0125</td>
<td>0.0125</td>
<td>0.005</td>
<td>0.005</td>
<td>$\bar{p}_{e,x/L}^{\text{bif}}$</td>
<td>$\bar{p}_{e,x/L}^{\text{bif}}$</td>
</tr>
</tbody>
</table>

### 7.7. Collapse of Imperfect Sandwich Shells

The imperfection sensitivity of the axially compressed isotropic or anisotropic cylindrical shells is well known and studied by several investigators [6, 12, 32] in the past years. As it is commonly known, the presence of geometric imperfections can greatly reduce the buckling load predicted for a shell of perfect geometry made of conventional material. On the other hand, there are only few references concerning sandwich cylindrical shells. Tennyson [55] has studied the imperfections sensitivity of sandwich cylindrical shells under axial compression using the classical approach discussed in chapter 2 in Part I of this thesis. In the model of imperfect shells, the whole sandwich configuration is associated with one layer with a certain
stress-free geometric imperfection introduced at the mid-surface of the sandwich layers. There is no distinction made in the initial imperfections and their amplitudes between the different faces.

In real sandwich structures, however, the faces can have different initial imperfections. Frostig in [26] shows that sandwich panels under in-plane external compressive loads with symmetric or unsymmetrical imperfection model with respect to the mid-height of the panels can have different response characteristics, i.e. deformation, stress and internal stress resultants at various load levels. The theory in chapter 5 implemented in the current program SHOT offers the possibility to model different imperfections for the top face and the bottom face. Initial imperfections can be modeled both in the shape affine to the buckling mode or in trigonometric form.

In the first stage of the study, the imperfection sensitivity of a “transversely” stiff core sandwich cylinder is investigated by using the current program SHOT and their results will be compared with those obtained using the classical approach SFOSDT, as presented in chapter 4 in Part I, and the finite element program MSC/NASTRAN. The choice of using this package is simply because, at this moment, it is the only package that offers the possibility of using the available shell elements for modeling aluminum honeycomb core sandwich shells in an efficient way. In order to make a comparison possible, in program SHOT the modulus of elasticity of the core $E_c$ is set equal to the in-plane modulus of elasticity of the faces $E_F$ (to simulate transversely ‘stiff’ honeycomb core) and setting the imperfections of both faces identical.

The second case dealt with a Nomex honeycomb core sandwich cylinder where both edges are locally supported (SS3-free). For the perfect cylinder, this case has been extensively studied in the previous section 7.5. Initial imperfections are imposed in the top face and bottom face with different amplitudes and form. In order to know their general collapse behavior of this cylinder, an investigation to imperfection sensitivity with the variation of amplitudes of imperfections as well as examination of the possibilities of material failure will be performed under two different external loads.

Only single-mode imperfection sensitivity studies are performed, which means the imperfection model of the faces has only one mode either axisymmetric or asymmetric.

**Transversely stiff core**

The same geometry and face property of sandwich cylinder is used as in section 7.1 with a high value of $E_c$ and shear modulus $G_c = 4.0 \times 10^4$ psi. This honeycomb core is more or less representing an Aluminum honeycomb core.

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Collapse of Imperfect Sandwich Shells

As pointed out earlier in chapter 2 in Part I, the imperfection model of sandwich cylinders in program SFOSDT is imposed at the mid-surface of the whole sandwich configuration. In order to investigate the imperfection sensitivity of shells the initial postbuckling theory of Koiter [33] (asymptotic method) is employed. For more detail of the theory one should return to chapter 2.

The finite element program MSC/NASTRAN is also used for validation of the results. This package offers the possibility to create efficient honeycomb core shells using the standard shell element (QUAD4) with PSHELL card, see more details in [3, 4, 5]. The transverse shear stiffness of the shell element can be adjusted such that it is equivalent to the appropriate stiffness for honeycomb core. However, it should be kept in mind that the kind of sandwich modeling here used is restricted only to sandwich shells with a high value of E_c (transversely stiff core), which is the same type as the core considered in classical approach discussed in Part I.

To calculate the bifurcation buckling load of perfect shells, the nonlinear buckling analysis in SOL106 is performed using two sequential analysis runs, a nonlinear statics analysis to establish the prebuckling state and the approximate instability load, followed by a second run to calculate the eigenvalue and eigenvector (buckling load and mode). For more details of the procedure one can see [51].

In the case of imperfect shells, in the second run the solution is restarted using the arc-length method in SOL106 to find the limit point and seeking the postbuckling behavior. Of course, the two runs here involve geometrically imperfect cylinders. Because the axial, circumferential and radial coordinates can be modeled, both axisymmetric and asymmetric imperfections can be accounted for.

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>SHOT ((E_c=1.0\times10^7))</th>
<th>SFOSDT</th>
<th>MSC/NASTRAN ((SOL106, 40\times20) elements)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N_x^{\text{crit}}) (lbf/inch)</td>
<td>-3872.0 (4,S)</td>
<td>-3867.8 (4,S)</td>
<td>-3866.6 (4,S)</td>
</tr>
</tbody>
</table>

Table 7-19. The critical buckling loads of transversely stiff core with \(G_c = 4.0 \times 10^4\) psi using several computer programs

Table 7-19 shows the results of the computed critical buckling load using SHOT, SFOSDT and MSC/NASTRAN. In all the cases, a nonlinear prebuckling analysis is performed prior to buckling which satisfies rigorously the simply supported boundary conditions displayed in the table. A good agreement in the predicted buckling

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loads has been obtained between the three solutions with the maximum deviation of less than 0.1%.

As denoted by the bracketed values, the corresponding buckling modes are asymmetric with four fullwaves in the circumferential direction and symmetric at $x = L/2$. Figure 7-40 shows the prebuckling solution at the bifurcation load level and the corresponding buckling mode obtained from SHOT and SFOSDT program. One should refer to section 7.3 for the reason of the difference between the result of both programs.

![Graphs showing prebuckling and buckling modes.]

**Figure 7-40.** Prebuckling shapes at buckling load level (axial compression) and buckling modes for transversely 'stiff' core with $G_c = 4.0 \times 10^4$ psi calculated using SHOT and SFOSDT program

First, the imperfection sensitivity of the lowest buckling load of the shells is investigated by using the SHOT and SFOSDT programs. The initial imperfections of the top and the bottom faces are assumed to be affine to the lowest buckling mode depicted in figure 7-40.b. Thus

\[
\bar{W}^j = \bar{\xi}^j h W_{c4}^1 \cos(4\theta) \quad j = t, b, \quad \bar{\xi}^j = \bar{\xi}
\]

(7-3)

where $W_{c4}^1$ and $W_{c4}^b$ are the axial shape of the buckling mode of the top face and bottom face, respectively and their imperfection amplitudes $\bar{\xi}^t = \bar{\xi}^b = \bar{\xi}$ are normalized by the total thickness of the sandwich layer $h = t^t + 2t$.

Table 7-20 displays the ratio between the computed limit load and the bifurcation buckling load for various imperfection amplitudes $\bar{\xi}$. Excellent agreement between the two results has been demonstrated in the table, with less than 1% difference,
for sufficiently small amplitudes ($\xi < 0.01$). One important aspect to note here is that the decrease of buckling load is less visible when the amplitude of imperfections is of the order of the thickness of the faces $t$. For this shell with $h/t = 40$, the amplitudes $\xi < 0.01$ are smaller than $t$. The reduction of the ratios is obviously seen when the amplitude is increased to the order of the total thickness $h$ between $\xi = 0.05$ and $\xi = 0.5$ or about 2.0 and 20 times $t$, respectively. A reduction of the buckling load by about 60% is obtained for an imperfection amplitude of 0.5 times the total thickness $h$.

**Table 7-20.** Ratio between the axial compression at the limit point and the bifurcation point of the perfect shell ($P_L/P_{bif}$) for a honeycomb core sandwich shell

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$P_L/P_{bif}$ \text{(E}_c=1.0\times10^7$ psi)</th>
<th>SFOSDT \text{(}\beta=-0.11467, \alpha = 0.35267, \beta = 0.34941\text{)}</th>
<th>diff. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0001</td>
<td>0.998</td>
<td>0.999</td>
<td>0.1</td>
</tr>
<tr>
<td>0.001</td>
<td>0.993</td>
<td>0.995</td>
<td>0.2</td>
</tr>
<tr>
<td>0.01</td>
<td>0.973</td>
<td>0.978</td>
<td>0.5</td>
</tr>
<tr>
<td>0.05</td>
<td>0.919</td>
<td>0.934</td>
<td>1.6</td>
</tr>
<tr>
<td>0.10</td>
<td>0.867</td>
<td>0.893</td>
<td>2.9</td>
</tr>
<tr>
<td>0.50</td>
<td>0.614</td>
<td>0.643</td>
<td>4.5</td>
</tr>
</tbody>
</table>

The difference between the two results becomes visible when the amplitude is increased from $\xi = 0.05$ (1%) up to $\xi = 0.5$ (4.5%). The reason of the deviation might be found by looking at the theoretical background of both programs. In the SFOSDT program, the limit load is calculated based on Koiter’s imperfection sensitivity theory, see chapter 3, which is asymptotically exact and yields accurate predictions only for sufficiency small amplitude $\xi$. On the other hand in the SHOT program the limit load is calculated directly by following the imperfect prebuckling path. As mentioned earlier in section 7.3, the limit point is searched along the imperfect path using a continuation technique based on the path following method. Mathematically speaking, by considering equation (3-56) in chapter 3, SHOT program does not neglect the terms of order $O(\xi^3)$ as is done in the SFOSDT program.
Besides the asymmetric imperfection affine to the buckling mode one might want to know the imperfection sensitivity of the lowest buckling load to the following trigonometric imperfection

\[ \bar{W}^j = \tilde{\xi}^j h \sin \left( \frac{\pi x}{L} \right) \cos (4\theta) \quad j = t, b, \tilde{\xi}^j = \bar{\xi} \] (7-4)

where the imperfection amplitudes \( \tilde{\xi}^t = \tilde{\xi}^b = \bar{\xi} \) are normalized by the total thickness of the sandwich layer \( h = t^c + 2t \).

Comparison between the programs SHOT and MSC/NASTRAN is shown in table 7-21. A reasonable agreement between the two results has been demonstrated with less than 5% difference.

**Table 7-21.** Ratio between the axial compression at the limit point and the bifurcation point of the perfect shell \( P_L/P_{bif} \) for a honeycomb core sandwich shell

<table>
<thead>
<tr>
<th>( \xi )</th>
<th>( \rho = P_L/P_{bif} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SHOT ( E_c = 1.0 \times 10^7 ) psi</td>
</tr>
<tr>
<td>0.01</td>
<td>0.983</td>
</tr>
<tr>
<td>0.05</td>
<td>0.947</td>
</tr>
<tr>
<td>0.10</td>
<td>0.935</td>
</tr>
<tr>
<td>0.50</td>
<td>0.622</td>
</tr>
</tbody>
</table>

Comparing the results of both imperfection models in tables 7-20 and 7-21, the first imperfection model (affine) predicts that the imperfect shell will buckle at a lower load level than its trigonometric model counterpart. The difference in the results might be explained by looking at the buckling mode shapes in the axial direction in figure 7-40, which shows a quite different shape then the trigonometric initial imperfection model specified by equation (7-4).

**Nomex core**

Next, we investigate the same cylinder with the core replaced by a transversely "soft" Nomex honeycomb core. The effect of imperfections on the lowest buckling load is investigated with distinct imperfection models for the faces. Furthermore,
two load cases are considered, namely axial compression load and external pressure, both using local SS3-free boundary conditions at both cylinder edges.

In the previous study, the amplitude of imperfections is increased until a certain size of the order of the total thickness of sandwich layers. Considering the thickness of the faces is very small in comparison to the thickness of the core $h/t = 40$ and the imperfections are only imposed on the faces, one might conclude that such sizeable imperfections are very unlikely to be present under normal circumstances. Therefore, it is more realistic to vary the amplitude size in the range of value that is of the order of the thickness of the faces.

Returning to table 7-17 in section 7.5, one can see that the critical buckling load under axial compression is found to be $N_x = 0.51537 \cdot (-1227.5) = -632.6$ lbf/inch. The corresponding buckling mode of both faces is axisymmetric and they are displayed in figure 7-41. The right figure 7-41 give the zoom-in plot for $0 < x/L < 0.05$. Note that the modes are symmetric with respect to the mid-length of the cylinder at $x = L/2$.

![Figure 7-41. Buckling mode for Nomex core sandwich cylinder under axial compression using SS3-free boundary conditions](image)

The imperfection sensitivity of the lowest buckling load of the shells is investigated using SHOT program. The initial imperfections of the top and the bottom faces are assumed to be affine to the lowest axisymmetric buckling mode of the top and bottom faces, thus

$$\bar{W}^j = \bar{z}_j h W_{c0}^1 \quad j = t, b$$

(7-5)
where $W_{e_0}^1$ and $W_{e_0}^1$ are the axial shape of the buckling mode of the top face and bottom face, respectively and their imperfection amplitudes $\xi_t$ and $\xi_b$ are normalized by the total thickness of the sandwich layer $h$.

Table 7-22 shows the ratio between the computed limit loads and the bifurcation load with the variation of the imperfection amplitude $\xi$. The first column displays the results for the same amplitude at the top face and the bottom face ($\xi_t = \xi_b = \xi$). The other columns exhibit the results when one of the faces has a lower amplitude or when one amplitude is null (perfect). In general, it can be seen that the ratios will decrease when the amplitudes $\xi$ decrease. More significant reduction of the buckling load is observed when the amplitude of the supported top face $\xi_t$ is greater than the one of the unsupported or free bottom face $\xi_b$.

The results might be more clear when one is looking at the knockdown plot in figure 7-42. The figure displays the imperfection sensitivity for this particular case, when a variation of the amplitude size of the top face and the bottom face imperfections is made.

Notice that as one can see from figures 7-43 and 7-44, for increasing the amplitude of imperfection $\xi$, the limit points are found at a lower load level along the imperfect path. In figure 7-43, both faces have the same amplitude of imperfection $\xi$, while in the left-side of figure 7-44 the top face is perfect and in the right-side of the figure the bottom face is perfect.

<table>
<thead>
<tr>
<th>$\xi_t$</th>
<th>$\xi_t = \xi_b = \xi$</th>
<th>$\xi_t = \xi$, $\xi_b = \xi/2$</th>
<th>$\xi_t = 0$, $\xi_b = \xi$</th>
<th>$\xi_t = \xi/2$, $\xi_b = \xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0001</td>
<td>0.9999</td>
<td>0.9999</td>
<td>0.9999</td>
<td>0.9999</td>
</tr>
<tr>
<td>0.0010</td>
<td>0.9989</td>
<td>0.9993</td>
<td>0.9991</td>
<td>0.9989</td>
</tr>
<tr>
<td>0.0025</td>
<td>0.9973</td>
<td>0.9983</td>
<td>0.9978</td>
<td>0.9989</td>
</tr>
<tr>
<td>0.0100</td>
<td>0.9904</td>
<td>0.9939</td>
<td>0.9919</td>
<td>0.9960</td>
</tr>
<tr>
<td>0.0250</td>
<td>0.9787</td>
<td>0.9859</td>
<td>0.9822</td>
<td>0.9907</td>
</tr>
<tr>
<td>0.1000</td>
<td>0.9320</td>
<td>0.9537</td>
<td>0.9422</td>
<td>0.9708</td>
</tr>
</tbody>
</table>
Figure 7-42. Imperfection sensitivity for Nomex core sandwich shells under axial compression for different amplitude of (affine) imperfections of top and bottom face.

Figure 7-43. Nondimensional axial compression versus the face end-shortening for sandwich cylindrical shells with two imperfect faces ($\xi^t = \xi^b = \xi$)
Figure 7-44. Nondimensional axial compression versus the face end-shortening for sandwich cylindrical shells with one imperfect face $\xi_t = 0$, $\xi^b = \xi$ (left) and $\xi^t = \xi$, $\xi^b = 0$ (right)

Material failure of imperfect shells

Table 7-23 shows the results of predicted material failure for imperfect Nomex core cylinders for five different amplitudes of the top face and the bottom face imperfections. The maximum imperfection amplitude used is $\xi = 0.01$. In the first row, the previously obtained results for perfect shell in section 7.5 are displayed. In all cases it revealed that the first material failure of the sandwich layers will occur at the interface between the top face and the core at the edges of the cylinder. At this place, the maximum compressive radial stress $o_r^t$ exceeds the allowable stress, indicated by the absolute failure ratio greater than one.

Looking closely at table 7-23 one can observe that for increasing imperfection amplitudes material failure is likely to occur at a lower load.

While failure ratios due to radial stresses are sensitive to the presence of imperfections, the counterparts failure ratios due to shear stress $\tau_{xr}^t$, $\tau_{xr}^b$ are more or less constant (between 0.50 - 0.51). Tacitly, this can be expected since one imposed imperfections to the faces only in the radial direction.
### Table 7-23. Material failure predictions of Nomex core imperfect sandwich cylinders under axial compression load using SS3-free b.c.

<table>
<thead>
<tr>
<th>$\xi^{t}$</th>
<th>$\xi^{b}$</th>
<th>$(\sigma_{T}^{t})_{\text{max}}$</th>
<th>$(\sigma_{T}^{b})_{\text{max}}$</th>
<th>$(\tau_{SR}^{t})_{\text{max}}$</th>
<th>$(\tau_{SR}^{b})_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$x/L = 0.0$</td>
<td>$x/L = 0.0$</td>
<td>$x/L = 0.002$</td>
<td>$x/L = 0.002$</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>-1.018</td>
<td>-1.157</td>
<td>1.002</td>
<td>1.006</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.36452)</td>
<td>(0.47708)</td>
<td>(0.51353)</td>
<td>(0.51345)</td>
</tr>
<tr>
<td>0.00</td>
<td>0.01</td>
<td>-1.020</td>
<td>-1.053</td>
<td>1.009</td>
<td>1.004</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.36421)</td>
<td>(0.39722)</td>
<td>(0.50747)</td>
<td>(0.50767)</td>
</tr>
<tr>
<td>0.005</td>
<td>0.01</td>
<td>-1.001</td>
<td>-1.015</td>
<td>1.006</td>
<td>1.002</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.30838)</td>
<td>(0.38565)</td>
<td>(0.50497)</td>
<td>(0.50519)</td>
</tr>
<tr>
<td>0.01</td>
<td>0.00</td>
<td>-1.024</td>
<td>-1.136</td>
<td>1.015</td>
<td>1.006</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.26624)</td>
<td>(0.44537)</td>
<td>(0.50837)</td>
<td>(0.50850)</td>
</tr>
<tr>
<td>0.01</td>
<td>0.005</td>
<td>-1.007</td>
<td>-1.070</td>
<td>1.010</td>
<td>1.006</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.26621)</td>
<td>(0.41430)</td>
<td>(0.50575)</td>
<td>(0.50592)</td>
</tr>
</tbody>
</table>

The last case explored is the imperfect cylinder with Nomex core under external pressure using the same SS3-free boundary conditions at both edges. As discussed earlier in section 7.3, the critical buckling load is found at $p_c = 0.014154 \ (3378.0) = 47.81\text{psi}$ with an associated asymmetric mode at both faces $(n = 5)$. The corresponding buckling mode, which will be used as the form of imperfection model, is displayed in figure 7-45. To get more insight on the distribution of the buckling mode close to the support, in each figure, a zoom-in plot nearby the edge is included.

The imperfection sensitivity of the lowest buckling load of the shells with the initial imperfections of the top and the bottom face are assumed to be affine to the lowest buckling mode, thus

$$W_{j}^{i} = \xi^{j} hW_{c5}^{1} \cos(5\theta) \quad j = t, b, \xi^{j} = \xi$$  \hspace{1cm} (7-6)

where $W_{c5}^{1}$ and $W_{c5}^{1}$ are the axial shape of the buckling mode of the top face and bottom face, respectively and their imperfection amplitudes $\xi^{t} = \xi^{b} = \xi$ are normalized by the total thickness of the sandwich layer $h$. The number of circumferential fullwaves taken in the Fourier terms of the stress and displacement variables in equations (5-88) to (5-90) are $n = 0, 5$ and 10. When the shell with the imperfection model as given by equation (7-6) is loaded in compression or external
pressure, then practice has shown that these terms are sufficient to describe the response up to the limit point.

Figure 7-45. Buckling mode for Nomex core sandwich cylinder under external pressure using local SS3-free boundary conditions

In table 7-24 the ratio between the computed limit load and bifurcation load for various imperfection amplitudes $\xi$ is displayed. The influence of the imperfection on the buckling load is less important than in the case of axial compression. Notice that as one can see from figure 7-46, no limit point is found along the imperfect path for the amplitude of imperfection $\xi = 0.5$.

Table 7-24. Normalized limit load (external pressure) for Nomex core sandwich cylinder using SS3-free b.c. with various asymmetric imperfections of the faces

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\bar{P}<em>{c_l} / \bar{P}</em>{c_x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi^t = \xi^b = \xi$</td>
<td>$\xi^t = \xi^b = 0$</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.01170</td>
</tr>
<tr>
<td>0.0100</td>
<td>0.01132</td>
</tr>
<tr>
<td>0.1000</td>
<td>0.01006</td>
</tr>
<tr>
<td>0.5000</td>
<td>x</td>
</tr>
</tbody>
</table>

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Figure 7-46. Nondimensional pressure versus the radial displacements at the mid-length of the cylinder using SS3-free b.c. with two imperfect faces.
The central goal of this research has been to develop a new approach that can be used to study the collapse behavior of sandwich cylindrical shells with initial geometric imperfections and taking into consideration the effect of boundary conditions. The classical approach, which neglects transverse flexibility of the core, has also been presented to improve the existing works in the literatures for sandwich cylinders. However, this approach can only be used to investigate the buckling behavior of sandwich cylindrical shells in overall instability mode, i.e. general buckling and shear crimping mode. In contrast to the classical approaches, a new approach has been derived which considers the transverse normal deformation of the core. Both overall and local instability modes of sandwich constructions can be analyzed with this approach. Local instability and failure modes have been investigated in the present study including wrinkling, core crushing, face dimpling and tensile rupture of the core proper.

Within the scope of the study of collapse behavior of sandwich cylinders, extensive investigations have been carried out to determine the bifurcation buckling, limit and predicted (core) material failure load of perfect and imperfect shells.

Theoretical Formulation:

The classical theory has been developed which is based on a single-layer modeling technique. The model accounts first-order shear deformations of the core. The stress and deformation of the individual sandwich layers has been formulated in one global coordinate system. The present model is suitable for:

- Bifurcation analysis of sandwich cylindrical shells with a “transversely” stiff cores such as a metallic honeycomb core under axial compression load. The core is thin and can have orthotropic properties. Only identical isotropic faces can be modeled.
- Prescribing only the boundary conditions for the faces and the core at the mid-surface of sandwich layers.
Discussions and Conclusions

- Analysis of the collapse behavior of sandwich cylindrical shells with an identical single-mode imperfection of the faces. This imperfection can have axisymmetric or asymmetric form.

A higher-order theory has been introduced by using a three-layer modeling technique. The stress and deformation fields of each of the faces and the core are formulated in their own local coordinate system taking into account both transverse shear and normal deformations of the core. The model formulated is suitable for:

- Nonlinear response and bifurcation analysis of sandwich cylindrical shells with a modern core, which is "transversely" soft, such as a foam or non-metallic honeycomb core, under axial compression and external pressure load. The faces of sandwich shells are built from (thin) composite faces with the same thickness. The core can be modeled either from an orthotropic honeycomb core or (isotropic) foam core, which is not necessarily thin.
- Global and local formulation of boundary conditions.
- Analysis of the collapse behavior of sandwich cylinders with different single-mode imperfections of the faces. These imperfections can have axisymmetric or asymmetric form.

Numerical Results:

Several instability modes, typical for sandwich structures, have been investigated. Their behavior is strongly influenced by the core properties and boundary conditions. Therefore, a numerical study has been carried out for a number of problems considering these effects. The presence of imperfections on both faces has also been investigated. From the results of the sandwich shells analyzed, the following important conclusions may be drawn:

Perfect shells

- For "transversely" stiff cores, in the case of orthotropic core it is possible to have the critical loads of general buckling mode lower than those for shear crimping mode. The transition from general buckling to shear crimping mode is due to a reduction in shear stiffness of the core. This shear crimping mode is initiated from the edge zones.
- Exact core consideration should be used when the ratio between the radius and thickness of the core becomes, say, smaller than 50.
- In the case of axially compressed shells the instability modes heavily depend on the modulus of elasticity and shear modulus of the core. They also depend on the boundary conditions formulated. For a low modulus of elasticity of the core, wrinkling of the faces has been detected nearby the edges where the growing of the magnitude is in the opposite direction, indicative of the low transverse flexibility of the core. For a low transverse shear modulus of the core, wrinkling of
the faces grows in the same direction. The edge effects cause wrinkling modes which are neither in symmetric nor antisymmetric form.

- Analyzing modern cores is more complex since local and global instability modes are involved. Therefore, a higher-order formulation has been developed. However, it has been shown that for axially compressed cylinders using global simply support boundary conditions, several classical formulations of local instabilities such as symmetrical wrinkling and shear crimping, can also give a good prediction for the buckling load.

**Boundary conditions**

- Local boundary conditions where only one of the faces is supported (a situation that sometimes is unavoidable in designing the supports of sandwich structures) reduce the load carrying capacity of the shells. A significant reduction in buckling strength has been found under axial compression load. However, in the external pressure case only a slight reduction of the critical load was observed.

- Large peak values of the radial and shear stress distributions have been found at the interfaces between the core and the adjacent faces close to the supports in all the cases. This indicates the possible damage of sandwich shells due to material failure nearby the supports.

**Imperfect shells**

- For “transversely” stiff cores, in the general buckling mode domain, the imperfection sensitivity increases when the shear stiffness of the isotropic core decreases. Therefore, a non-shear deformable core or “equivalent” isotropic case cannot be used as a lower bound curve to determine the knockdown factor, as proposed by Tennyson. In the shear crimping mode domain the shells become less sensitive to the geometric imperfections.

- Cylinders with orthotropic cores, with a lower transverse shear stiffness in the circumferential direction than its counterpart in the axial direction, become more sensitive to imperfections because of the delay of transition from general to shear crimping mode.

- The collapse behavior of sandwich cylinders with “transversely” stiff core has been studied using both classical and higher-order theory and the finite element program MSC/NASTRAN. In this case, the agreement between the three results was found to be good.

- The computations presented have demonstrated the well known fact that the buckling load reduction due to initial imperfections depends on the amplitude and the shape of initial imperfections. In general, however, imperfect sandwich shells with the amplitude of imperfections of the order of the thickness of the faces, which is more likely to occur in reality, are less imperfection sensitive than usually is found in the case of shells made of conventional material.
Discussions and Conclusions

• In the case of local boundary conditions under axial compression, where the faces can have different imperfection models, it has been shown that the supported face is more sensitive to the initial imperfections than the unsupported one.

Recommendations for Further Research:

• More extensive studies should be performed for sandwich shells with composite faces since they are commonly used in today’s sandwich applications. The model should be extended for cases where the shells can have dissimilar faces.
• The external applied load can easily be extended using other external loads, such as internal pressure, torsion, bending moment and transverse shear load. The general formulation of Fourier decomposition in the current program, that can include more Fourier terms to describe the response of cylindrical shells, offers this possibility.
• The information of stress distributions along the surface coordinates at the interfaces between the core and the adjacent faces obtained for a certain load level from the current (global) response analysis can be used to perform a more detailed analysis of the material failure behavior of the core.
• Due to limitation of time, the study of collapse behavior of imperfect sandwich shells only focussed on the calculation of limit loads using a single-mode imperfection model. An investigation of the postbuckling behavior, where an extension with a multi-mode imperfection model is included, can easily be made with the program developed.
• Until now, there is no capable sandwich shell element available in either commercial or educative finite element programs to describe the general behavior of sandwich constructions which includes the vertically flexibility of the core. Therefore, the author feels the urgent need for the development of such elements, considering the increase of application of sandwich construction with a “soft” core in today’s structures.
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A.1. Variational Derivation of Equilibrium Equations

The first variation of the strain energy density expressions for linearly elastic thin shells, based on the first order shear deformation assumptions, is defined as

\[ \delta U = \int_{V} \left\{ \delta \varepsilon_{x} \delta \varepsilon_{x} + \delta \varepsilon_{y} \delta \varepsilon_{y} + \delta \gamma_{xy} \delta \gamma_{xy} + \delta \gamma_{xz} \delta \gamma_{xz} + \delta \gamma_{yz} \delta \gamma_{yz} \right\} (1 - \frac{z}{R}) dV \]  

(A.1)

where V denotes the volume of the whole sandwich shell and dV denotes the volume of a differential segment.

Introducing into equation (A.1) the expressions for the off-midsurface strains in terms of the mid-surface strains and curvatures, as given in equations (2-9) and (2-10) and keeping in the mind that the terms of the order \( z/R \) are neglected in the integrand of equation (A.1) yields

\[ \delta U = \int_{A} \left\{ N_{x} \delta \varepsilon_{x} + N_{y} \delta \varepsilon_{y} + N_{xy} \delta \gamma_{xy} + M_{x} \delta \kappa_{x} + M_{y} \delta \kappa_{y} + M_{xy} \delta \kappa_{xy} + Q_{x} \delta \gamma_{xz} + Q_{y} \delta \gamma_{yz} \right\} dA \]

(A.2)

where A denotes the reference surface and the stress- and moment resultants are defined in equations (2-13) and (2-14). Note that \( N_{xy} = N_{yx} \) and \( M_{xy} = M_{yx} \), following the neglecting of the terms of order \( z/R \) in the definition of the stress-and moment resultants.

For the edge load \( \tilde{P} \) the first variation of the potential of applied loads is

\[ \delta V = -\int_{A} \left. \left( \tilde{P} \delta U \right) \right|_{x=L}^{x=0} dy \]

(A.3)

where \( \partial A \) denotes boundary curve of domain A.

Application of the minimum total potential energy principle requires that the first variation of the total potential energy is equal to zero.
\[ \delta \Pi = \delta U + \delta V = 0 \]  

(A.4)

The first variation of the strain energy \( \delta U \), defined in equation (A.2), can be expressed in terms of the variation of mid-surface displacements and rotations by substituting the nonlinear mid-surface strain-displacement relations in equations (2-3) to (2-6) and the changes of curvature in equation (2-7). Substituting the resulting expressions together with the expressions for the first variation of the potential of applied load in equation (A.3) into equation (A.4) and integrating the obtained equations by parts yields the equations of equilibrium and the natural boundary conditions. The equilibrium equations are

\[ N_{x'y} + N_{y'x} = 0 \]  

(A.5)

\[ N_{y'y} + N_{x'x} = 0 \]  

(A.6)

\[ Q_{x'} + Q_{y'} + (N_x(W_{x'} + \overline{W}_{x'})_{,x} + N_y(W_{y'} + \overline{W}_{y'})_{,y} + \]
\[ + (N_y(W_{y'} + \overline{W}_{y'})_{,x} + N_x(W_{x'} + \overline{W}_{x'})_{,y} = 0 \]  

(A.7)

\[ Q_x - M_{x'y'} - M_{x'y} = 0 \]  

(A.8)

\[ Q_y - M_{y'x'} - M_{y'x} = 0 \]  

(A.9)

and the boundary conditions at the edge of the cylinder are such that one can prescribe either

\[ N_x \quad \text{or} \quad U \]  

(A.10)

\[ N_{xy} \quad \text{or} \quad V \]  

(A.11)

\[ \dot{Q}_x = Q_x + N_x(W_{,x} + \overline{W}_{,x}) + N_{xy}(W_{,y} + \overline{W}_{,y}) \quad \text{or} \quad W \]  

(A.12)

\[ M_x \quad \text{or} \quad \beta_x \]  

(A.13)

\[ M_{xy} \quad \text{or} \quad \beta_y \]  

(A.14)

where the notation \( \dot{Q}_x \) is introduced to denote the modified shear force. Equations (A.5) and (A.6) are the in-plane equilibrium equations. Equations (A.7) to (A.9) are the out-of-plane equilibrium equations.

Note that, to define the free edge conditions in the current first order shear theory formulation, the modified shear force \( \dot{Q}_x \), the bending moment \( M_x \) and the twisting moment \( M_{xy} \) are prescribed equal to zero. In the classical shell theory, see for instance [57], this three conditions are reduced to two conditions by replacing the boundary conditions dealing with shear force and twisting moment by one bound-
ary condition. Introducing the Kirchoff-Love assumption \( \gamma_{yz} = 0 \), it follows from the strain-displacement relation in equation (2-6) that \( \beta_y = -W_{y} \). By substituting this expression in the strain energy expression in equation (A.2) and working out the resulting variational expressions, the boundary conditions regarding to the modified shear force \( \hat{Q}_x \) and twisting moment \( M_{xy} \) are coupled and they are expressed by the modified shear force \( \hat{Q}_x = \hat{Q}_x + M_{xy,y} \).

### A.2. Partially Inverted Constitutive Equations

In order to eliminate the secondary variables out the equilibrium equations, as described in chapter 2, the secondary variables must be expressed in terms of basic variables. To achieve these relations, the constitutive equations in (2-15) and (2-16) will be reformulated in this section. The remaining relations are used to introduce the additional equations, as will be discussed in section 2.4, to complete the definition of governing equations of the present theory. First the following nondimensional parameters are introduced

\[
\bar{A}_{ij} = \frac{1}{E_t} A_{ij} \quad \bar{D}_{ij} = \frac{1}{E_r h^2} D_{ij} \quad \bar{S}_{ij} = \frac{1}{E_t} S_{ij} \quad (A.15)
\]

in which the quantities \( E_t \) and \( v \) are arbitrarily chosen reference values as used for normalization in section 2.4. Furthermore, the following reduced thickness is introduced

\[
\hat{t} = \frac{t^t b}{\sqrt{1 - v^2}}
\]

The constitutive equations in (2-15) and (2-16) are written nondimensional by using equation (A.15) as follows

\[
\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} \bar{E} & \bar{F} \\ \bar{F}^t & \bar{G} \end{bmatrix} \begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} \quad (A.16)
\]

where

\[
\hat{a} = \begin{bmatrix} N_y, M_y, M_{xy}, Q_y \end{bmatrix}^T \quad \hat{b} = \begin{bmatrix} N_x, M_x, N_{xy}, Q_x \end{bmatrix}^T \quad \hat{c} = \begin{bmatrix} E_t \hat{t} \bar{e}_y, E_r h^2 \bar{t} \bar{k}_y, E_r h^2 \bar{t} \bar{k}_{xy}, E_r \hat{t} \bar{\gamma}_{yz} \end{bmatrix}^T \\
\hat{d} = \begin{bmatrix} E_t \hat{t} \bar{e}_x, E_r \bar{h} h^2 \bar{t} \bar{k}_x, E_r \hat{t} \bar{\gamma}_{xy}, E_r \hat{t} \bar{\gamma}_{xz} \end{bmatrix}^T
\]

and

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\[
[\bar{E}] = \begin{bmatrix}
\bar{A}_{22} & 0 & 0 & 0 \\
0 & \bar{D}_{22} & 0 & 0 \\
0 & 0 & \bar{D}_{66} & 0 \\
0 & 0 & 0 & \bar{S}_{88}
\end{bmatrix},
[\bar{F}] = \begin{bmatrix}
\bar{A}_{12} & 0 & 0 & 0 \\
0 & \bar{D}_{12} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
[\bar{G}] = \begin{bmatrix}
\bar{A}_{11} & 0 & 0 & 0 \\
0 & \bar{D}_{11} & 0 & 0 \\
0 & 0 & \bar{A}_{66} & 0 \\
0 & 0 & 0 & \bar{S}_{77}
\end{bmatrix}
\]

Note that the matrices \([\bar{E}], [\bar{F}]\) and \([\bar{G}]\) are nondimensional and symmetric. Partial inverting of equation (A.16) yields

\[
\begin{bmatrix}
\dot{d} \\
\dot{a}
\end{bmatrix} = \begin{bmatrix}
\bar{G}^* & \bar{F}^* \\
-\bar{F}^*^t & \bar{E}^*
\end{bmatrix} \begin{bmatrix}
b \\
\dot{c}
\end{bmatrix} \tag{A.17}
\]

where

\[
[\bar{G}^*] = [\bar{G}]^{-1}, \quad [\bar{F}^*] = -[\bar{G}]^{-1}[\bar{F}]^t, \quad [\bar{E}^*] = [\bar{E}] - [\bar{F}][\bar{G}]^{-1}[\bar{F}]^t
\]

Note that the matrices \([\bar{G}^*], [\bar{F}^*]\) and \([\bar{E}^*]\) are symmetric. The partially inverted constitutive equations are then rewritten as follows

\[
\begin{align*}
\frac{R}{h} \epsilon_x &= L_1(a) & \frac{R}{h} \epsilon_{xy} &= L_3(a) & N_y &= 2\bar{E}hL_5(a) & \frac{R^2}{h} \kappa_{xy} &= L_7(a) \\
\frac{R^2}{h} \kappa_x &= L_2(a) & \frac{R}{h} \kappa_{xz} &= L_4(a) & M_y &= 2\bar{E}hRL_6(a) & Q_y &= 2\bar{E}hL_8(a)
\end{align*} \tag{A.18}
\]

where \(\bar{E} = E_f l/R\) and linear operators \(L_i(a)\) are defined as follows

\[
\begin{align*}
L_1(a) &= 2\bar{G}_{11}^* a_1 + \bar{F}_{11}^* a_5 & L_4(a) &= 2\bar{G}_{77}^* a_4 & L_7(a) &= \frac{2R^2}{h^2} (\bar{E}_{66}^*)^{-1} a_7 \\
L_2(a) &= \frac{2R^2}{h^2} \bar{G}_{22}^* a_2 + \bar{F}_{22}^* a_6 & L_5(a) &= -\bar{F}_{11}^* a_1 + \frac{1}{2} \bar{E}_{11}^* a_5 \\
L_3(a) &= 2\bar{G}_{66}^* a_3 & L_6(a) &= -\bar{F}_{22}^* a_2 + \frac{h^2}{2R^2} \bar{E}_{22}^* a_6 & L_8(a) &= \frac{1}{2} \bar{E}_{88}^* a_8
\end{align*} \tag{A.19}
\]

and the operand vector \(a\) is

\[
a = \begin{bmatrix}
\frac{1}{2\bar{E}h} N_x, \frac{1}{2\bar{E}hR} M_x, \frac{1}{2\bar{E}h} N_{xy}, \frac{1}{2\bar{E}h} (\bar{Q}_x - N_x(W_x + \bar{W}_x) - N_{xy}(W_y + \bar{W}_y)) \end{bmatrix}^T
\]

\[
\begin{pmatrix}
\frac{R}{h} \epsilon_{xy}, \frac{R^2}{h} \kappa_{xy}, \frac{1}{2\bar{E}hR} M_{xy}, \frac{R}{h} \gamma_{yz}
\end{pmatrix}^T \tag{A.20}
\]
A.3. Nondimensional Governing Equations

The set of nondimensional governing partial differential equations, first order in the axial coordinate are

\[
\begin{align*}
    u_{,x} &= L_1(a) - \frac{h}{2R}(w_{,x}^2 + 2w_{,x}\tilde{w}_{,x}) \\
    v_{,x} &= -u_{,y} - \frac{h}{R}(w_{,x}w_{,y} + w_{,x}\tilde{w}_{,y} + \tilde{w}_{,x}w_{,y}) + L_3(a) \\
    w_{,\tilde{x}} &= -\chi + L_4(a) \\
    \chi_{,\tilde{x}} &= L_2(a) \\
    \psi_{,\tilde{x}} &= -\chi_{,y} + L_7(a) \\
    n_{x,\tilde{x}} &= -n_{xy,y} \\
    n_{x,\tilde{y}} &= -n_{,,y} \\
    q_{x,\tilde{x}} &= -L_8(a,\tilde{y}) - L_5(a) - \{L_5(a)\frac{h}{R}(w_{,y} + \tilde{w}_{,y})\}_{,y} - \{n_{xy}\frac{h}{R}(w_{,x} + \tilde{w}_{,x})\}_{,\tilde{y}} \\
    m_{x,\tilde{x}} &= -m_{xy,y} + q_{,x} - \frac{h}{R}n_{x}(w_{,x} + \tilde{w}_{,x}) - \frac{h}{R}n_{xy}(w_{,y} + \tilde{w}_{,y}) \\
    m_{x,\tilde{y}} &= L_8(a) - L_6(a,\tilde{y})
\end{align*}
\]

where the nondimensional vector \( a = a_1 + a_2 \) with

\[
\begin{align*}
    a_1 &= \begin{bmatrix} n_x, m_x, n_{xy}, q_x - \frac{h}{R}n_x(w_{,x} + \tilde{w}_{,x}) - \frac{h}{R}n_{xy}(w_{,y} + \tilde{w}_{,y}) \end{bmatrix}^T \\
    a_2 &= \begin{bmatrix} (v_{,y} - w) + \frac{h}{2R}(w_{,y}^2 + 2w_{,y}w_{,y}) , \psi_{,y} , m_{xy} , (w_{,y} + \psi) \end{bmatrix}^T
\end{align*}
\]

For perfect shells all the terms involving \( \tilde{w} \) in the above governing equations are disappear. Finally, the consistent boundary conditions at the edge of the shell \( \tilde{x} = 0 \) and \( \tilde{x} = L/R \) are such that one can prescribe either

\[
\begin{align*}
    n_x & \quad \text{or} \quad u \\
    n_{xy} & \quad \text{or} \quad v \\
    q_x & \quad \text{or} \quad w
\end{align*}
\]
A.4. Equations Governing Prebuckling, Buckling and Postbuckling States

The nondimensional constants $C_i$ used in the governing equations for prebuckling, buckling and postbuckling states are defined as follows

\[
\begin{align*}
C_1 &= v \\
C_6 &= \frac{v}{r_h} \\
C_9 &= \frac{2v}{g_y} \\
C_{13} &= \frac{2r_h}{g_x} \\
C_2 &= \frac{r_0}{2l^2_h} \\
C_6 &= \frac{r_0}{4l^2_h} \\
C_{10} &= \frac{(1-v^2)l^2_r}{g_y r_h (r_0^{-1} + r_1^{-1})} \\
C_{14} &= \frac{2}{g_x} \\
C_3 &= \frac{r_0}{2l^2_h} \\
C_7 &= \frac{1}{r_h} \\
C_{11} &= \frac{2(1-v^2)l^2_r}{r_0} \\
C_{15} &= \frac{2r_h}{(r_0^{-1} + r_1^{-1})l^2_r} \\
C_4 &= \frac{(1-v^2)l^2_r}{2r_h (r_0^{-1} + r_1^{-1})^{-1}} \\
C_8 &= \frac{2}{g_y} \\
C_{12} &= \frac{4(1+v)l^2_h}{r_0} \\
C_{16} &= \frac{4r_h(1-v)^{-1}}{(r_0^{-1} + r_1^{-1})^{-1}l^2_r}
\end{align*}
\]

where the nondimensional geometric parameters $l_r$, $l_h$, $r_h$ and the nondimensional material property parameters $r_0$, $r_1$, $g_x$, $g_y$ are defined as follows

\[
\begin{align*}
l_r &= \frac{L}{R} \\
l_h &= \frac{L}{h} \\
r_h &= \frac{R}{h} \\
g_x &= \frac{G_x}{E} \\
g_y &= \frac{G_y}{E} \\
r_0 &= \frac{(L^2 + l^2)R}{12h} \\
r_1 &= \frac{12t^2 h^2 L^2}{R \left\{ (t^1)^2 + (t^2)^2 \right\}}
\end{align*}
\]

Axisymmetric prebuckling state:

The ordinary differential equations for the axisymmetric prebuckling state of imperfect cylindrical shells in equations (3-7) and (3-8) are defined as follows

\[
\begin{align*}
q_{x'x'}^{(0)} &= C_1 \lambda + C_2 w^{(0)} \\
m_{x'x'}^{(0)} &= q_{x'}^{(0)} + C_7 \lambda (w_{x'}^{(0)} + w_{,x}^{(0)}) \\
w_{,x}^{(0)} &= -C_{13} q_{x}^{(0)} + C_{14} \lambda (w_{x'}^{(0)} + w_{,x}^{(0)}) - \chi^{(0)} \\
\chi_{,x}^{(0)} &= -C_{15} m_{x}^{(0)}
\end{align*}
\]
where $\lambda = \bar{P} / (2hE)$ denotes the normalized axial compression load.

Note that the shell behavior is nonlinear because of the presence of the term $\lambda w_\bar{x}^0$ in equations (A.40) and (A.41). However, for any fixed value of the axial load $\lambda$, the differential equations (A.39) to (A.42) are a set of first order linear ordinary differential equations with constant coefficients, which implies that they can directly be solved (see also in [12]).

After substitution of the following axisymmetric prebuckling solution

$$q_x^{(0)} = q_x^0(\bar{x}) \quad w^{(0)} = w_v + w^0(\bar{x}) \quad \chi^{(0)} = \chi^0(\bar{x})$$

(A.43)

where $w_v = -(C_1/C_2)\lambda$, one obtains a set of first order ordinary linear differential equations in (3-13) and (3-14). The vector functions $f^D$ and $f^F$ used in these equations are defined as follows

$$f^D = [f_1^D f_2^D] \quad f^F = [f_1^F f_2^F]$$

(A.45)

where

$$f_1^D = -C_2 w^0 \quad f_2^D = -q_x^0 - C_7 \lambda w_{x\bar{x}}^0 - C_7 \lambda w_{\bar{x}}$$

$$f_1^F = C_{13} q_x^0 - C_{14} \lambda w_{x\bar{x}}^0 + \chi^0 - C_{14} \lambda \bar{w}_{\bar{x}} \quad f_2^F = C_{15} m_x^0$$

It should be noted that for perfect shells all terms involving $\bar{w}$ in the above equations for the prebuckling state are disappeared.

The prebuckling boundary conditions at $\bar{x} = 0$ and $\bar{x} = L/R$ in equation (3-15) become

$$q_x^0 = 0 \quad \text{or} \quad w^0 = -w_v \quad \chi^0 = 0$$

(A.46)

(A.47)

Buckling state:

The partial differential equations for the buckling state of imperfect shells, following equations (3-10) and (3-11) in section 3.1.1, are as follows

$$n_{x1x}^{(1)} = -n_{x1y}^{(1)}$$

$$n_{xy1\bar{x}}^{(1)} = -C_1 n_{x1y}^{(1)} - C_2 (v_{x1y}^{(1)} - w_{xy}^{(1)})$$

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\[ q_{x'}^{(1)} = -C_1 m_{xy}^{(1)} - C_4 \psi^{(1)} - m_{xy}^{(1)} - C_1 n_x^{(1)} - C_2 (v_{y'} - w^{(1)}) + 
 - C n_x^{(0)} w_{x'y'}^{(1)} + C_3 w_{x'y'}^{(1)} - C_2 n_x^{(1)} (w_{x'} + \bar{w}_{x'}) \]
\[ m_{x'y'}^{(1)} = q_{x'}^{(1)} - m_{xy}^{(1)} - C_1 n_x^{(0)} w_{x'y'}^{(1)} + n_x^{(0)} w_{x'y'}^{(1)} + n_x^{(1)} (w_{x'}^{(0)} + \bar{w}_{x'}) \]
\[ m_{xy}^{(1)} = \{-C_9 m_{xy}^{(1)} + C_7 (w_{y'}^{(1)} + \psi^{(1)}) - C_10 \psi^{(1)} \} \]
\[ u_{x'}^{(1)} = -C_1 n_x^{(1)} - C_2 (v_{y'} - w^{(1)}) - C_7 (w_{x'}^{(1)} + \bar{w}_{x'}) \]
\[ v_{x'}^{(1)} = -C_2 n_x^{(1)} - u_{y'}^{(1)} - C_7 w_{x'y'}^{(1)} (w_{x'} + \bar{w}_{x'}) \]
\[ w_{x'}^{(1)} = -C_3 q_{x'}^{(1)} - C_4 n_x^{(0)} w_{x'}^{(1)} + n_x^{(0)} w_{x'}^{(1)} - C_14 n_x^{(1)} (w_{x'}^{(0)} + \bar{w}_{x'}) - \chi_{1}^{(1)} \]
\[ \chi_{x'}^{(1)} = -C_5 n_x^{(1)} - C_6 \psi_{y'}^{(1)} \]
\[ \psi_{x'}^{(1)} = -C_7 n_x^{(1)} - C_{15} w_{x'}^{(1)} \]

where the nondimensional constants \( C_i \) are defined in equation (A.38).

The vector functions \( \mathbf{f}^D \), \( \mathbf{f}^F \), \( \mathbf{f}^{D*} \) and \( \mathbf{f}^{F*} \) for the buckling problem in equations (3-26) and (3-27) in section 3.1.3 are defined as follows:

\[ \mathbf{f}^D = [\hat{f}_1^D \hat{f}_2^D \hat{f}_3^D \hat{f}_4^D \hat{f}_5^D] \quad \mathbf{f}^{D*} = [\hat{f}_1^{D*} \hat{f}_2^{D*} \hat{f}_3^{D*} \hat{f}_4^{D*} \hat{f}_5^{D*}] \quad \mathbf{f}^F = [\hat{f}_1^F \hat{f}_2^F \hat{f}_3^F \hat{f}_4^F \hat{f}_5^F] \quad \mathbf{f}^{F*} = [\hat{f}_1^{F*} \hat{f}_2^{F*} \hat{f}_3^{F*} \hat{f}_4^{F*} \hat{f}_5^{F*}] \]
where the nondimensional constants $\hat{C}_i$ are defined as follows

\[
\begin{align*}
\hat{C}_1 &= n \\
\hat{C}_2 &= -C_1 n \\
\hat{C}_3 &= -C_2 n^2 \\
\hat{C}_4 &= C_2 n \\
\hat{C}_5 &= C_1 \\
\hat{C}_6 &= -C_1 n^2 \\
\hat{C}_7 &= -C_2 \\
\hat{C}_8 &= -C_4 n^3 \\
\hat{C}_9 &= -C_5 n^2 \\
\hat{C}_{10} &= C_3 n^2 \\
\hat{C}_{11} &= C_7 n \\
\hat{C}_{12} &= C_7 \\
\hat{C}_{13} &= C_8 \\
\hat{C}_{14} &= -C_9 n \\
\hat{C}_{15} &= -(C_{10} n^2 + C_7) \\
\hat{C}_{16} &= C_11 \\
\hat{C}_{17} &= C_{12} \\
\hat{C}_{18} &= C_{13} \\
\hat{C}_{19} &= C_{14} \\
\hat{C}_{20} &= -C_{14} n
\end{align*}
\]

It can be seen that the nondimensional constants $\hat{C}_i$ are expressed entirely in the nondimensional constants $C_i$ and the number of circumferential full waves $n$. For perfect shells all terms involving $w$ in the above equations for the buckling state are discarded.

By substituting the Fourier series for the basic variables in the buckling state defined in section 3.1.3, the boundary conditions become

\[
\begin{align*}
n_x^1 &= 0 & \quad \text{or} & & u^1 &= 0 \quad \text{(A.51)} \\
n_{xy}^1 &= 0 & \quad \text{or} & & v^1 &= 0 \quad \text{(A.52)} \\
q_x^1 &= 0 & \quad \text{or} & & w^1 &= 0 \quad \text{(A.53)} \\
m_x^1 &= 0 & \quad \text{or} & & \chi^1 &= 0 \quad \text{(A.54)} \\
m_{xy}^1 &= 0 & \quad \text{or} & & \psi^1 &= 0 \quad \text{(A.55)}
\end{align*}
\]

**Postbuckling state:**

The partial differential equations for the postbuckling state of perfect shells, following from equations (3-45) and (3-46) in section 3.2.1, are as follows

\[
\begin{align*}
n_{xx}^{(2)} &= -n_{xyy}^{(2)} \\
n_{xxy}^{(2)} &= -C_1 n_{xy}^{(2)} - C_2 (v_{yy}^{(2)} - w_{y}^{(2)}) - C_3 w_{y}^{(1)} w_{yy}^{(1)} \\
q_{x}^{(2)} &= -C_1 m_{xyy}^{(2)} - C_4 \psi_{yyy}^{(2)} - m_{xxyy}^{(2)} - C_1 n_{x}^{(2)} - C_2 (v_{yy}^{(2)} - w_{y}^{(2)}) + \\
&\quad - C_5 n_{x}^{(0)} w_{xyy}^{(2)} + C_3 w_{y}^{(0)} w_{yy}^{(2)} - C_7 w_{x}^{(0)} n_{xxyy}^{(2)} + \\
&\quad - \frac{1}{2} C_3 (w_{x}^{(1)})^2 - C_5 n_{x}^{(1)} w_{xyy}^{(1)} - C_3 v_{yy}^{(1)} w_{yy}^{(1)} + C_3 w_{y}^{(1)} w_{yy}^{(1)} + \\
&\quad - C_7 w_{x}^{(1)} n_{xxyy}^{(1)} - C_7 n_{x}^{(1)} w_{xyy}^{(1)} + C_7 n_{xy}^{(1)} w_{x}^{(1)}
\end{align*}
\]

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\[ m_{x,x}^{(2)} = d_{x}^{(2)} - m_{x,y}^{(2)} - C_{7}n_{x}^{(0)}w_{x}^{(2)} - C_{7}w_{x}^{(0)}n_{x}^{(2)} - C_{7}n_{xy}^{(0)}w_{y}^{(2)} + \]
\[ -C_{7}n_{x}^{(1)}w_{x}^{(1)} - C_{7}n_{xy}^{(1)}w_{y}^{(2)} \]
\[ m_{x,y}^{(2)} = \{ -C_{9}m_{x,y}^{(2)} + C_{7}(w_{y}^{(2)} + \psi^{(2)}) - C_{10}\psi_{yy}^{(2)} \}C_{8}^{1} \]
\[ u_{x}^{(2)} = -C_{11}n_{x}^{(2)} - C_{1}(v_{y}^{(2)} - w^{(2)}) - C_{7}w_{x}^{(0)}w_{x}^{(2)} - \frac{1}{2}C_{7}(w_{x}^{(1)})^{2} - \frac{1}{2}C_{5}(w_{y}^{(1)})^{2} \]
\[ v_{x}^{(2)} = -C_{12}n_{xy}^{(2)} - u_{y}^{(2)} - C_{7}w_{x}^{(0)}w_{x}^{(1)} - C_{7}w_{y}^{(0)}w_{y}^{(1)} \]
\[ w_{x}^{(2)} = -C_{13}q_{x}^{(2)} - \chi^{(2)} - C_{14}n_{x}^{(0)}w_{x}^{(2)} - C_{14}n_{x}^{(2)}w_{x}^{(0)} - C_{14}n_{xy}^{(0)}w_{y}^{(2)} \]
\[ -C_{14}n_{x}^{(1)}w_{x}^{(1)} - C_{14}b_{xy}^{(1)}w_{y}^{(1)} \]
\[ x_{x}^{(2)} = -C_{15}m_{x}^{(2)} - C_{1}\psi_{y}^{(2)} \]
\[ \psi_{x}^{(2)} = -C_{16}m_{xy}^{(2)} - \chi_{y}^{(2)} \]

The vector functions \( \ddot{f}^{D}, \ddot{f}^{F}, \ddot{f}^{D*} \) and \( \ddot{f}^{F*} \) for the postbuckling problem in equations (3.52) and (3.53) in section 3.2.1 are defined as follows

\[ \ddot{f}^{D} = [\ddot{f}_{1}^{D} \ddot{f}_{2}^{D} \ddot{f}_{3}^{D} \ddot{f}_{4}^{D} \ddot{f}_{5}^{D} \ddot{f}_{6}^{D} \ddot{f}_{7}^{D} \ddot{f}_{8}^{D} \ddot{f}_{9}^{D} \ddot{f}_{10}^{D}] \quad (A.57) \]
\[ \ddot{f}^{D*} = [\ddot{f}_{1}^{D*} \ddot{f}_{2}^{D*} \ddot{f}_{3}^{D*} \ddot{f}_{4}^{D*} \ddot{f}_{5}^{D*} \ddot{f}_{6}^{D*} \ddot{f}_{7}^{D*} \ddot{f}_{8}^{D*} \ddot{f}_{9}^{D*} \ddot{f}_{10}^{D*}] \quad (A.58) \]
\[ \ddot{f}^{F} = [\ddot{f}_{1}^{F} \ddot{f}_{2}^{F} \ddot{f}_{3}^{F} \ddot{f}_{4}^{F} \ddot{f}_{5}^{F} \ddot{f}_{6}^{F} \ddot{f}_{7}^{F} \ddot{f}_{8}^{F} \ddot{f}_{9}^{F} \ddot{f}_{10}^{F}] \quad (A.59) \]
\[ \ddot{f}^{F*} = [\ddot{f}_{1}^{F*} \ddot{f}_{2}^{F*} \ddot{f}_{3}^{F*} \ddot{f}_{4}^{F*} \ddot{f}_{5}^{F*} \ddot{f}_{6}^{F*} \ddot{f}_{7}^{F*} \ddot{f}_{8}^{F*} \ddot{f}_{9}^{F*} \ddot{f}_{10}^{F*}] \quad (A.60) \]

with

\[ \ddot{f}_{1}^{D} = D_{16}n_{x}^{a} - D_{5}w^{(0)}_{,x}w^{(a)}_{,x} \quad \ddot{f}_{2}^{D} = D_{17}n_{xy}^{a} \]
\[ \ddot{f}_{3}^{D} = D_{18}q_{x}^{a} + \chi^{a} + D_{19}w^{(0)}_{,x}n_{x}^{a} + D_{19}n_{x}^{0}w^{(a)}_{,x} \quad \ddot{f}_{4}^{D} = D_{21}m_{x}^{a} \]
\[ \ddot{f}_{5}^{D} = D_{22}m_{x}^{a} \quad \ddot{f}_{6}^{D} = D_{16}n_{x}^{a} - D_{2}v^{(0)} - D_{5}w^{(0)}_{,x}w^{(a)}_{,x} \]
\[ \ddot{f}_{7}^{D} = D_{17}n_{xy}^{a} - D_{1}u^{a} - D_{11}w^{(0)}_{,x}w^{(a)}_{,x} \quad \ddot{f}_{8}^{D} = D_{18}q_{x}^{a} + \chi^{a} + D_{19}w^{(0)}_{,x}n_{x}^{a} + D_{19}n_{x}^{0}w^{(a)}_{,x} \]
\[ \ddot{f}_{9}^{D} = D_{21}m_{x}^{a} - D_{2}\psi^{(a)} \quad \ddot{f}_{10}^{D} = D_{22}m_{xy}^{a} - D_{1}\chi^{a} \]
\[ \ddot{f}_{2}^{D*} = \ddot{f}_{4}^{D*} = \ddot{f}_{5}^{D*} = \ddot{f}_{9}^{D*} = \ddot{f}_{10}^{D*} = 0 \quad \ddot{f}_{1}^{D*} = -\frac{1}{2}D_{25}(w^{1})^{2} + \frac{1}{2}D_{28}(w^{1}_{,x})^{2} \]

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\[ f_3^D = D_{29} n_x^1 w^1, x - D_{30} n_{xy}^1 w^1 \]
\[ f_7^D = D_{26} w^1 w^1, x \]
\[ f_6^D = \frac{1}{2} D_{25} (w^1)^2 + \frac{1}{2} D_{28} (w^1, x)^2 \]
\[ f_9^D = D_{29} n_x^1 w^1, x + D_{30} n_{xy}^1 w^1 \]

\[ f_1^F = f_2^F = 0 \]
\[ f_3^F = D_5 n_x^a + D_7 w^a \]
\[ f_4^F = -q_x^a + D_{12} w^0, x n_x^a + D_{12} n_x^0 w^a, x \]
\[ f_5^F = -(D_{12}/D_{13}) q^a \]
\[ f_6^F = D_1 n_{xy}^0 \]
\[ f_7^F = D_2 n_x^0 + D_3 v^a + D_4 w^a \]
\[ f_8^F = D_5 n_x^0 + D_6 m_x^0 + D_1 m_{xy}^0 + D_4 v^0 + D_7 w^0 + D_8 q^0 \]
\[ + D_9 n_x^0 w^0 + D_{10} w^0, x n_{xy}^0 \]
\[ f_9^F = -q_x^0 + D_1 m_{xy}^0 + D_{12} w^0, x n_x^0 + D_{12} n_x^0 w^0, x \]
\[ f_{10}^F = (D_{14} m_x^0 + D_{11} w^0 + D_{15} q^0)/D_{13} \]
\[ f_{10}^D = f_2^D = f_5^D = f_6^D = f_10^D = 0 \]
\[ f_4^* = D_{28} n_x^1 w^1, x + D_{26} n_{xy}^1 w^1 \]
\[ f_3^* = D_{25} n_x^1 w^1 - D_{26} n_{xy}^1 w^1 - D_{24} v^1 w^1 + 3 D_{27} (w^1)^2 \]
\[ f_7^* = D_{24} (w^1)^2 \]
\[ f_8^* = D_{25} n_x^1 w^1 - 2 D_{26} w^1, x n_{xy}^1 + D_{26} n_{xy}^1 w^1 - D_{24} v^1 w^1 + D_{27} (w^1)^2 \]
\[ f_9^* = D_{25} n_x^1 w^1, x - D_{26} n_{xy}^1 w^1 \]

where the nondimensional constants \( D_i \) are defined as follows:

\[ D_1 = 2n \quad D_9 = -4C_5 n^2 \quad D_{17} = C_{12} \quad D_{25} = \frac{1}{2} C_5 n^2 \]
\[ D_2 = -2C_1 n \quad D_{10} = 4C_3 n^2 \quad D_{18} = C_{13} \quad D_{26} = \frac{1}{2} C_7 n \]
\[ D_3 = -4C_2 n^2 \quad D_{11} = 2C_7 n \quad D_{19} = C_{14} \quad D_{27} = \frac{1}{4} C_3 n^2 \]
\[ D_4 = 2C_2 n \quad D_{12} = C_7 \quad D_{20} = -2C_{14} n \quad D_{28} = -\frac{1}{2} C_7 \]
\[ D_5 = C_1 \quad D_{13} = C_8 \quad D_{21} = C_{15} \quad D_{29} = -\frac{1}{2} C_{14} \]
\[ D_6 = -4C_1 n^2 \quad D_{14} = -2C_9 n \quad D_{22} = C_{16} \quad D_{30} = -\frac{1}{2} C_{14} n \]
\[ D_7 = -C_2 \quad D_{15} = -(4C_{10} n^2 + C_7) \quad D_{23} = C_3 n^3 \]
\[ D_8 = -8C_4n^3 \quad D_{16} = C_{11} \quad D_{24} = -\frac{1}{2}C_3n^3 \]

Again, the nondimensional constants \( D_i \) are expressed entirely in the nondimensional constants \( C_i \) and the number of circumferential full waves \( n \).

By substituting the Fourier series for the basic variables in the postbuckling state defined in section 3.2.1, the boundary conditions become

\[ n_x^\alpha = n_x^\beta = 0 \quad \text{or} \quad u^\alpha = u^\beta = 0 \]
\[ n_{xy}^\alpha = n_{xy}^\beta = 0 \quad \text{or} \quad v^\alpha = v^\beta = 0 \]
\[ q_x^\alpha = q_x^\beta = 0 \quad \text{or} \quad w^\alpha = w^\beta = 0 \quad (A.61) \]
\[ m_x^\alpha = m_x^\beta = 0 \quad \text{or} \quad \chi^\alpha = \chi^\beta = 0 \]
\[ m_{xy}^\alpha = m_{xy}^\beta = 0 \quad \text{or} \quad \psi^\alpha = \psi^\beta = 0 \]

**Postbuckling coefficients and imperfection form factors**

The formulas for the first and second postbuckling coefficients \( a \) and \( b \) and the first and second imperfection form factors \( \alpha \) and \( \beta \) have been derived by Budiansky and Hutchinson in [13] using a highly abbreviated notation. The derivation for these expressions is not given here and they can directly be used for the present problem of sandwich cylindrical shells

\[ a = 0 \quad b = -\Pi_b/\Lambda_c \hat{\Lambda} \quad (A.62) \]
\[ \alpha = \Pi_\alpha/\Lambda_c \hat{\Lambda} \quad \beta = \{\Pi_{\beta_1} + \Pi_{\beta_2} - \alpha\Lambda_c[\Pi_{\beta_3} + \frac{1}{2}(\Pi_{\beta_4} + \Pi_{\beta_5})]\}/\hat{\Lambda} \quad (A.63) \]

where

\[ \hat{\Lambda} = 2h^3E\pi \int_0^{L/R} \left\{2\left[n_x^1w_{,x}^1 - n(n_{xy}^1w_{,x}^1)\right]w_{,x}^0 + n_x^0(w_{,x}^1)^2 \right. \]
\[ \quad + C_1n^2n_x^0(w^1)^2 - C_2n^2w_{,x}^0(w^1)^2 \}
dx \]
\[ \Pi_b = 2h^3E\pi \int_0^{L/R} \left\{2n_x^1w_{,x}^1w_{,x}^1 + n_x^1w_{,x}^1w_{,x}^1 + 2C_1n^2n_x^1w_{,x}^1w_{,x}^1 \right. \]
\[ \quad + 2C_2n^2(nv^1 - w^1)v^1w^0 + -2n(n_{xy}^1w_{,x}^1w_{,x}^1) - 2(n_{xy}^1w_{,x}^1w_{,x}^1) \]
\[ \quad + n(n_{xy}^1w_{,x}^1w_{,x}^1) + n_x^0(w_{,x}^1)^2 + \frac{1}{2}n_x^1(w_{,x}^1)^2 + C_1n^2n_x^1(w^1)^2 \]

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\[-\frac{1}{2} C_1 n_x^2 n_x^0 (w^1_x)^2 - C_2 n_x^2 w_x (w^1_x)^2 - C_2 n_x^2 (n_x^\beta - \frac{1}{2} w_x^\beta) (w^1_x)^2 + \frac{3}{8} C_3 n_x^4 (w^1_x)^4 - n_{xx} n_x^0 w^1_x w_{xx} \] 

\[ \Pi_{\alpha} = 2h^3 E\pi \int_0^{L/R} \left\{ (n_x^1 w^1_x - n_x n_{xx} w^1_x) w^0_{x} - n_x^0 \{(w^1_x)^2 + C_1 n_x^2 (w^1_x)^2 \} \right\} \, dx \]

\[ \Pi_{\beta_1} = 2h^3 E\pi \int_0^{L/R} \left\{ (n_x^1 w^1_x - n_x n_{xx} w^1_x) w^0_{x} - n_x^0 \{(w^1_x)^2 + C_1 n_x^2 (w^1_x)^2 \} \right\} \, dx \] 

\[ \Pi_{\beta_2} = 2h^3 E\pi \int_0^{L/R} \left\{ \frac{C_3}{1 - C_1} w^0_x w^0_{xx} (w^1_x)^2 + \frac{C_3 n_x^2}{2(1 + C_1)} w^0_{xx} w^0_x (w^1_x)^2 \right\} \, dx \]

\[ \Pi_{\beta_3} = 2h^3 E\pi \int_0^{L/R} \left\{ (n_x^1 w^1_x - n_x n_{xx} w^1_x) w^0_{x} \right\} \, dx \]

\[ \Pi_{\beta_4} = 2h^3 E\pi \int_0^{L/R} \left\{ n_x^0 \{(w^1_x)^2 + C_1 n_x^2 (w^1_x)^2 \} - C_2 n_x^2 w^0 (w^1_x)^2 \right\} \, dx \]

\[ \Pi_{\beta_5} = 2h^3 E\pi \int_0^{L/R} \left\{ \frac{C_3}{1 - C_1} (w^0_{xx})^2 (w^1_x)^2 + \frac{C_3 n_x^2}{2(1 + C_1)} (w^0_{xx})^2 (w^1_x)^2 \right\} \, dx \]

To evaluate the above integrals the trapezoidal rule of the numerical integration scheme referred to in [34] is used here. Once the integrals of equation (A.64) are computed then the postbuckling coefficient \( b \) and the imperfection form factors \( \alpha \) and \( \beta \) can be obtained by substituting the computed integrals in equations (A.62) to (A.63).

**Definition of Boundary conditions**

In the following table A-1 the definition of 16 sets of the boundary conditions at the edges of the shell (at \( \bar{x} = 0 \) and \( \bar{x} = L/R \)) and two sets of symmetry conditions at the mid-length of the cylinder, at \( \bar{x} = L/(2R) \), used in chapter 4 are given.

Note that the notation SS1-\( \beta \) is used to denote that in the first-order shear deformation theory also the rotation \( \psi = \beta_y \) can be prescribed. Further, the notation SS1-m is introduced to indicate that the twisting moment \( m_{xy} \) is prescribed.
### Table A-1. Definition of various boundary conditions

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>$n_x$</th>
<th>$n_{xy}$</th>
<th>$q_x$</th>
<th>$m_x$</th>
<th>$m_{xy}$</th>
<th>$u$</th>
<th>$v$</th>
<th>$w$</th>
<th>$\chi$</th>
<th>$\psi$</th>
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<tr>
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<td>1</td>
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<tr>
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<td>0</td>
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<td>1</td>
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<td>1</td>
</tr>
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<td>0</td>
</tr>
<tr>
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<tr>
<td>C4-β</td>
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<tr>
<td>C4-m</td>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note: 0 - variable is free and 1 - variable is prescribed
APPENDIX B

Derivation of Higher-Order Theory Equations

B.1. Constitutive Equations of the Faces Layers

![Diagram of anisotropic face layer of sandwich cylindrical shells]

**Figure B-1. Anisotropic face layer of sandwich cylindrical shells**

One of the face layers of sandwich wall of the cylindrical shell, as depicted in figure B-1, is a layered anisotropic shell consisting of NL individual layers with different material properties. Using the given sign convention, with W positive inward, the numbering of the layers begins at the outer surface. Notice that the angle of rotation $\theta_k$ ($k = 1, 2, ..., NL$) of the individual layers is defined with respect to the x-axis of the shell. The face reference surface coincides with the mid-surface of the laminate. If the position of the $k^{th}$ lamina is defined by $z_{k-1} < z < z_k$, then the total thickness of the laminate is
Constitutive Equations of the Faces Layers

\[ h^j = \sum_{k=1}^{NL} (z_k - z_{k-1}) \]  \hspace{1cm} (B.1)

where NL denotes the number of lamina.

The classical thin laminate theory is used to derive the constitutive equations for a layered composite shell. No effects of plasticity of the layers are taken into account. For a thin face (say, R/t > 50), one can assume that each lamina may be considered as a homogeneous orthotropic medium in a plane stress state. Thus the stress-strain relations for the \( k \)th lamina in the lamina principal axes (1, 2) can be written as

\[
\begin{bmatrix}
\dot{\sigma}_1^j \\
\dot{\sigma}_2^j \\
\dot{\tau}_{12}^j
\end{bmatrix}_k =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}_k
\begin{bmatrix}
\dot{\varepsilon}_1^j \\
\dot{\varepsilon}_2^j \\
\dot{\gamma}_{12}^j
\end{bmatrix}_k
\]  \hspace{1cm} (B.2)

where with Hooke's law for an orthotropic material

\[
Q_{11} = \frac{E_{11}}{(1 - \nu_{12}\nu_{21})}, \quad Q_{22} = \frac{E_{22}}{(1 - \nu_{12}\nu_{21})}, \quad Q_{12} = \frac{\nu_{21}E_{11}}{(1 - \nu_{12}\nu_{21})} = \frac{\nu_{12}E_{22}}{(1 - \nu_{12}\nu_{21})}, \quad Q_{66} = G_{12}
\]

The superscript \( j \) in equation (B.2) is used to distinguish the top and bottom face laminate quantities. Thus \( j \) can assume the value \( t \), for the top layer, or \( b \) for the bottom layer. The elastic material properties of top and bottom face laminates are the same thus it follows that both laminates have an equivalent elastic material matrix \( Q_{ij} \) for corresponding lamina.

Normally, the lamina principal axes (1, 2) do not coincide with the reference axes of the shell wall (x, y). Transformation to the shell wall reference axes (x, y) gives

\[
\begin{bmatrix}
\ddot{\sigma}_x^j \\
\ddot{\sigma}_y^j \\
\ddot{\tau}_{xy}^j
\end{bmatrix}_k =
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}_k
\begin{bmatrix}
\ddot{\varepsilon}_x^j \\
\ddot{\varepsilon}_y^j \\
\ddot{\gamma}_{xy}^j
\end{bmatrix}_k
\]  \hspace{1cm} (B.3)

where

\[
\bar{Q}_{11} = Q_{11} C^4 + 2(Q_{12} + 2Q_{66})C^2S^2 + Q_{22}S^4
\]

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\[ \bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})C^2S^2 + Q_{66}(C^4 + S^4) \]
\[ Q_{12} = (Q_{11} + Q_{22} - 4Q_{66})C^2S^2 + Q_{12}(C^4 + S^4) \]
\[ \bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})C^3S + (Q_{12} - Q_{22} + 2Q_{66})CS^3 \]
\[ Q_{22} = Q_{11}S^4 + 2(Q_{12} + 2Q_{66})C^2S^2 + 2Q_{22}C^4 \]
\[ \bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})CS^3 + (Q_{12} - Q_{22} + 2Q_{66})C^3S \]

\[ C = \cos(\theta_k), \quad S = \sin(\theta_k) \]

The stress and moment resultant acting at the face mid-surfaces are defined in section 5.3. For a layered composite shell they can be rewritten as follows

\[
\begin{bmatrix}
N^j_x \\
N^j_y \\
N^j_{xy}
\end{bmatrix} = \sum_{k=1}^{NL} \int_{z_k}^{z_{k+1}} \begin{bmatrix}
\dot{\phi}_x^j \\
\dot{\phi}_y^j \\
\tau_{xy}^j
\end{bmatrix} \, dz
\]
\[
\begin{bmatrix}
M^j_x \\
M^j_y \\
M^j_{xy}
\end{bmatrix} = \sum_{k=1}^{NL} \int_{z_k}^{z_{k+1}} \begin{bmatrix}
\ddot{\phi}_x^j \\
\ddot{\phi}_y^j \\
\ddot{\tau}_{xy}^j
\end{bmatrix} \, dz
\]

Recalling the Kirchhoff-Love assumption for a thin shell, the strain at any layer can be written in terms of the strain and curvature of the mid-surface as

\[
\begin{bmatrix}
\epsilon_x^j \\
\epsilon_y^j \\
\gamma_{xy}^j
\end{bmatrix} = \begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{bmatrix} + z \begin{bmatrix}
\kappa_x^j \\
\kappa_y^j \\
\kappa_{xy}^j
\end{bmatrix}
\]

Substituting these expressions into equation (B.3) and introducing the resulting relations into equation (B.4), followed by carrying out the indicated integrations, gives the following constitutive equations

\[
\begin{bmatrix}
N^j_x \\
N^j_y \\
N^j_{xy} \\
M^j_x \\
M^j_y \\
M^j_{xy}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix} \begin{bmatrix}
\epsilon_x^j \\
\epsilon_y^j \\
\gamma_{xy}^j \\
\kappa_x^j \\
\kappa_y^j \\
\kappa_{xy}^j
\end{bmatrix}
\]

where the A, B and D matrices are defined as
\[ A_{ij} = \sum_{k=1}^{NL} (\bar{Q}_{ij})_k (z_k - z_{k-1}) \quad B_{ij} = \frac{1}{2} \sum_{k=1}^{NL} (\bar{Q}_{ij})_k (z_k^2 - z_{k-1}^2) \]

\[ D_{ij} = \frac{1}{3} \sum_{k=1}^{NL} (\bar{Q}_{ij})_k (z_k^3 - z_{k-1}^3) \]

**B.2. Partially Inverted Constitutive Equations**

In order to eliminate the secondary variables out the equilibrium equations, as described in chapter 5, they must be expressed in terms of basic variables. To achieve these relations, the constitutive equations in (B.6) will be reformulated in this paragraph. The remaining relations are used to introduce the additional equations, as it is discussed in section 5.3, which complete the definition of governing equations of the present theory. First, the following nondimensional parameters are introduced

\[
\bar{A}_{ij} = \frac{1}{E_r t} A_{ij} \quad \bar{B}_{ij} = \frac{2c}{E_r t^2} B_{ij} \quad \bar{D}_{ij} = \frac{4c^2}{E_r t^3} D_{ij} \quad c = \sqrt{3(1-v^2)} \quad (B.7)
\]

in which the quantities \( E_r \) and \( v \) are arbitrarily chosen reference values as used for normalization.

The constitutive equations in (B.6) are written nondimensional by using equation (B.7) as follows

\[
\begin{bmatrix}
\hat{\bar{a}} \\
\hat{\bar{b}}
\end{bmatrix} =
\begin{bmatrix}
\bar{E} & \bar{F}^T \\
\bar{F}^T & \bar{G}
\end{bmatrix}
\begin{bmatrix}
\hat{\bar{c}} \\
\hat{\bar{d}}
\end{bmatrix} \quad (B.8)
\]

where

\[
\hat{\bar{a}} = \begin{bmatrix}
t^j_{ij} N^i_y, M^i_y, M^i_{xy}
\end{bmatrix}^T \quad \hat{\bar{c}} = \begin{bmatrix}
\frac{E_r(t^j)^2}{2c} e_j, \frac{E_r(t^j)^3}{4c^2} \kappa^j_y, \frac{E_r(t^j)^3}{4c^2} (\kappa^j_{xy} + \gamma^j_{xy})
\end{bmatrix}^T
\]

\[
\hat{\bar{b}} = \begin{bmatrix}
t^j_{ij} N^i_x, M^i_x, \frac{t^j}{2c} (N^j_{xy} - \frac{1}{R^j} M^j_{xy})
\end{bmatrix}^T \quad \hat{\bar{d}} = \begin{bmatrix}
\frac{E_r(t^j)^2}{2c} e^n_j, \frac{E_r(t^j)^3}{4c^2} \kappa^n_j, \frac{E_r(t^j)^2}{2c} \gamma^n_{xy}
\end{bmatrix}^T
\]

and

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\[
[\tilde{E}] = \begin{bmatrix}
A_{22} & \bar{B}_{22} & \bar{B}_{26} \\
\bar{B}_{22} & D_{22} & D_{26} \\
\bar{B}_{26} & D_{26} & \bar{D}_{66}
\end{bmatrix}
\quad [\tilde{F}] = \begin{bmatrix}
A_{12} & \bar{B}_{12} & (\bar{A}_{26} - \frac{t^j}{2cR^j}\bar{B}_{26}) \\
B_{12} & D_{12} & (B_{26} - \frac{t^j}{2cR^j}\bar{D}_{26}) \\
B_{16} & D_{16} & (\bar{B}_{66} - \frac{t^j}{2cR^j}\bar{D}_{66})
\end{bmatrix}
\]

\[
[\tilde{G}] = \begin{bmatrix}
\bar{A}_{11} & \bar{B}_{11} & (\bar{A}_{16} - \frac{t^j}{2cR^j}\bar{B}_{16}) \\
\bar{B}_{11} & \bar{D}_{11} & (\bar{B}_{16} - \frac{t^j}{2cR^j}\bar{D}_{16}) \\
(\bar{A}_{16} - \frac{t^j}{2cR^j}\bar{B}_{16})(\bar{B}_{16} - \frac{t^j}{2cR^j}\bar{D}_{16})(\bar{A}_{66} - \frac{t^j}{cR^j}\bar{B}_{66} - (\frac{t^j}{2cR^j})^2\bar{D}_{66})
\end{bmatrix}
\]

Note that the matrices \([\tilde{E}], [\tilde{F}]\) and \([\tilde{G}]\) are nondimensional and the matrices \([\tilde{E}]\) and \([\tilde{G}]\) are symmetric. Partial inverting of equation (B.8) yields

\[
\begin{bmatrix}
d \\ \dot{a}
\end{bmatrix} = \begin{bmatrix}
\tilde{G}^* & \tilde{F}^* \\
-F^* & E^*
\end{bmatrix} \begin{bmatrix}
b \\ \dot{c}
\end{bmatrix}
\quad (B.9)
\]

where

\[
[\tilde{G}^*] = [\tilde{G}]^{-1} \quad [\tilde{F}^*] = -(\tilde{G})^{-1}[\tilde{F}]^t \quad [E^*] = [E] - [F][G]^{-1}[F]^t
\]

Note that the matrices \([\tilde{G}^*]\) and \([E^*]\) are symmetric. Recalling the definition for the basic variables in chapter 5, yields the following stress basic variable

\[
N_{xy}^j = \frac{1}{R^j}M_{xy}^j = \hat{N}_{xy}^j
\quad (B.10)
\]

the partially inverted constitutive equations are the rewritten as follows

\[
\begin{align*}
\frac{R^j}{t^j} & \frac{\gamma_x}{x} = \frac{1}{c}L_1^j(a) \\
R^j & \kappa_x^j = 2L_2^j(a) \\
\frac{R^j}{t^j} & \gamma_{xy} = \frac{1}{c}L_3^j(a)
\end{align*}
\]

\[
\begin{align*}
M_{xy}^j & = \frac{E_f(t^j)^3}{2c^2R^j}L_5^j(a) \\
N_{xy}^j & = \frac{E_f(t^j)^2}{cR^j}L_4^j(a) \\
M_{xy}^j & = \frac{E_f(t^j)^3}{2c^2R^j}L_6^j(a)
\end{align*}
\quad (B.11)
\]

where linear operators \(L_i^j(a)\) are defined as follows
\[ \begin{align*}
L^j_1(a) &= \bar{G}^{0}_{11}a_1 + \frac{t^j}{2R^j}G^{0}_{12}a_2 + \bar{G}^{0}_{16}a_3 + c\bar{F}^{0}_{11}a_4 + \frac{1}{2}\bar{F}^{0}_{12}a_5 + \frac{1}{2}\bar{F}^{0}_{16}a_6 \\
L^j_2(a) &= \bar{G}^{0}_{12}a_1 + \frac{t^j}{2R^j}\bar{G}^{0}_{22}a_2 + \bar{G}^{0}_{26}a_3 + c\bar{F}^{0}_{21}a_4 + \frac{1}{2}\bar{F}^{0}_{22}a_5 + \frac{1}{2}\bar{F}^{0}_{26}a_6 \\
L^j_3(a) &= \bar{G}^{0}_{16}a_1 + \frac{t^j}{2R^j}\bar{G}^{0}_{26}a_2 + \bar{G}^{0}_{66}a_3 + c\bar{F}^{0}_{61}a_4 + \frac{1}{2}\bar{F}^{0}_{62}a_5 + \frac{1}{2}\bar{F}^{0}_{66}a_6 \\
L^j_4(a) &= -\bar{F}^{0}_{11}a_1 - \frac{t^j}{2R^j}\bar{F}^{0}_{21}a_2 - \bar{F}^{0}_{61}a_3 + c\bar{E}^{0}_{11}a_4 + \frac{1}{2}\bar{E}^{0}_{12}a_5 + \frac{1}{2}\bar{E}^{0}_{16}a_6 \\
L^j_5(a) &= -\bar{F}^{0}_{12}a_1 - \frac{t^j}{2R^j}\bar{F}^{0}_{22}a_2 - \bar{F}^{0}_{62}a_3 + c\bar{E}^{0}_{12}a_4 + \frac{1}{2}\bar{E}^{0}_{22}a_5 + \frac{1}{2}\bar{E}^{0}_{26}a_6 \\
L^j_6(a) &= -\bar{F}^{0}_{16}a_1 - \frac{t^j}{2R^j}\bar{F}^{0}_{26}a_2 - \bar{F}^{0}_{66}a_3 + c\bar{E}^{0}_{16}a_4 + \frac{1}{2}\bar{E}^{0}_{26}a_5 + \frac{1}{2}\bar{E}^{0}_{66}a_6
\end{align*} \]

and the operand vector \( a \) is

\[ a = \begin{bmatrix}
\frac{cR^j}{E_f(t^j)^2}N^j_x, & \frac{4c^2(R^j)^2}{E_f(t^j)^4}M^j_x, & \frac{cR^j}{E_f(t^j)^2}N^j_{xy}, & \frac{R^j}{t^j}t^j_y, & R^j\kappa^j_y, & R^j(k^j_{xy} + \gamma^j_{xy})/R^j
\end{bmatrix}^T \]  \hspace{1cm} (B.13)

### B.3. Coefficients Used for Core Equilibrium Equations

The coefficients \( D_1 \) to \( D_4 \) are

\[ \begin{align*}
D_1 &= C_9 - \frac{C_4C_{10}}{C_5} \\
D_2 &= C_6 - \frac{C_1C_{10}}{C_5} \\
D_3 &= C_7 - \frac{C_2C_{10}}{C_5} \\
D_4 &= \frac{C_{10}}{C_5}
\end{align*} \]  \hspace{1cm} (B.14)

where the coefficients \( D^j_1 \) to \( D^j_4 \) are given as

\[ \begin{align*}
D^j_1 &= C^j_1 - \frac{C_4}{C_5}C^j_2 \\
D^j_2 &= C^j_3 - \frac{C_1}{C_5}C^j_2 \\
D^j_3 &= -C^j_3 - \frac{C_2}{C_5}C^j_2 \\
D^j_4 &= \frac{C^j_2}{C_5}
\end{align*} \]  \hspace{1cm} (B.15)

**Foam-type core**

The coefficients \( C_1 \) to \( C_{10} \) used through the equations (5-61) and (5-62) are given as follows.
\[ C_1 = C_7 = -r_2 + \frac{(r_2 - r_1)}{\ln(r_2/r_1)} \]
\[ C_2 = C_6 = r_1 - \frac{(r_2 - r_1)}{\ln(r_2/r_1)} \]
\[ C_3 = \ln(r_2/r_1) \]
\[ C_4 = -\ln(r_2/r_1) + \frac{(r_1 - r_2)}{\ln(r_2/r_1)}(1/r_2 - 1/r_1) \]
\[ C_5 = \frac{(r_2^2 - r_1^2)}{2} - \frac{(r_2 - r_1)^2}{\ln(r_2/r_1)} \]
\[ C_8 = \frac{1}{2}(r_2/r_1 - r_1/r_2) \]
\[ C_9 = \frac{(r_2^2 - r_1^2)}{2r_1r_2} - \frac{(r_2 - r_1)^2}{r_1r_2\ln(r_2/r_1)} \]
\[ C_{10} = -r_1r_2\ln(r_2/r_1) + \frac{(r_1 - r_2)^2}{\ln(r_2/r_1)} \]

The coefficients \( C_1^t \) to \( C_3^t \) and \( C_1^b \) to \( C_3^b \) used through the equations (5-63) and (5-64) are given as follows:

\[ C_1^t = 1 - \frac{(1/r_2 - 1/r_1)}{r_2\ln(r_2/r_1)} \]
\[ C_2^t = -1 + \frac{(r_2 - r_1)}{r_2\ln(r_2/r_1)} \]
\[ C_3^t = -\frac{1}{r_2\ln(r_2/r_1)} \]
\[ C_1^b = 1 + \frac{(1/r_2 - 1/r_1)}{r_1\ln(r_2/r_1)} \]
\[ C_2^b = -1 + \frac{(r_2 - r_1)}{r_1\ln(r_2/r_1)} \]
\[ C_3^b = -\frac{1}{r_1\ln(r_2/r_1)} \]

**Honeycomb-type core**

The coefficients \( C_1 \) to \( C_{10} \) in equations (5-61) and (5-62) and the coefficients \( C_1^t \) to \( C_3^t \) and \( C_1^b \) to \( C_3^b \) in equations (5-63) and (5-64) for honeycomb type cores are:

\[ C_1 = C_2 = \frac{r_1 - r_2}{2} \]
\[ C_3 = C_8 = r_2 - r_1 \]
\[ C_4 = r_2 - r_1 + \frac{(r_1 + r_2)}{2}\ln(r_2/r_1) \]
\[ C_5 = \frac{(r_2 - r_1)^2}{12} \]
\[ C_6 = r_1 - \frac{r_1r_2}{(r_2 - r_1)}\ln(r_2/r_1) \]
\[ C_7 = -r_2 + \frac{r_1r_2}{(r_2 - r_1)}\ln(r_2/r_1) \]
\[ C_9 = r_2 - r_1 - \frac{r_1r_2}{(r_2 - r_1)}\ln(r_2/r_1)^2 \]
\[ C_{10} = -r_1r_2(r_2 - r_1) + \frac{r_1r_2(r_2^2 - r_1^2)\ln(r_2/r_1)}{2(r_2 - r_1)} \]

and

\[ C_1^t = 1 - \frac{\ln(r_2/r_1)}{r_2(r_2 - r_1)} \]
\[ C_2^t = -1 + \frac{(r_2^2 - r_1^2)}{2r_2(r_2 - r_1)} \]
\[ C_3^t = -\frac{1}{r_2(r_2 - r_1)} \]
\[ C_1^b = 1 - \frac{\ln(r_2/r_1)}{r_1(r_2 - r_1)} \]
\[ C_2^b = -1 + \frac{(r_2^2 - r_1^2)}{2r_1(r_2 - r_1)} \]
\[ C_3^b = -\frac{1}{r_1(r_2 - r_1)} \]
The coefficients $D_1$ to $D_4$ and the coefficients $D_1^b$ to $D_4^b$ are equal to those for foam cores.

**Thin core - Bartels approximation solution**

If the coefficients $C_1$ to $C_{10}$, $C_1^t$ to $C_3^t$ and $C_1^b$ to $C_3^b$ are approximated by a truncated series expansion of the terms of order $t^c/R^c$ and higher then the coefficients are reduced to

\[
C_1 = C_2 = C_6 = C_7 = C_1^t = C_2^t = C_1^b = C_2^b = -t^c/(2R^c)
\]

\[
C_3 = C_8 = t^c/R^c \quad C_3^t = C_3^b = -R^c/t^c \quad C_4 = C_5 = C_9 = C_{10} = \frac{1}{12}(t^c/R^c)^3
\]

The compatibility equations and the transverse and normal stresses derived in section 5.3 with these coefficients for a consideration of the core with $t^c/R^c \ll 1$ (thin cores) are those of Bartels in [10]. This approximation solution of the core is only yields for thin cores sandwich configurations and in the present study is used for reference purposes.

**B.4. Nondimensional Governing Equations**

Introducing the secondary variables into the equilibrium equations one finally obtains the following set of governing partial differential equations, first order in the axial coordinate, which are entirely expressed in the basic variables (for the top face: $j = t$, $k = 2$, for the bottom face: $j = b$, $k = 1$)

\[
u_{x_1}^j = \frac{1}{c}L_1(a)^j - \frac{R_j}{2t}(\chi^j)^2 - 2\frac{t}{R_j}\chi^j\bar{w}_{j_1}^{j_2}
\]

\[
u_{x_2}^j = -u_{v_1}^j - \frac{R_j}{t}(\chi^j\bar{w}_{j_1}^{j_2}) + (\psi^j\bar{w}_{j_1}^{j_2}) + \frac{1}{c}L_3(a)^j
\]

\[
u_{x_3}^j = \frac{R_j}{t}\chi
\]

\[
u_{x_4}^j = 2L_2(a)^j
\]

\[
u_{x_5}^j = \frac{R_x^c}{t}(u^t - u^b) + \frac{R_x^c}{2t^c}(\chi^t + \chi^b) + \frac{t}{ct^c}\bar{q}_x
\]

\[
u_{x_6}^j = -\frac{t^k}{2cR_j}L_6(a_{j_6})^j - (-1)^k\frac{(R_j)^2}{C_3(R_x^c)^2r_k}\bar{q}_x
\]

\[
u_{x_7}^j = -L_4(a_{j_4})^j + \frac{t}{2cR_j}L_5(a_{j_5})^j - L_4(a)^j(\psi^j - \frac{t}{R_j}\bar{w}_{j_1}^{j_2})
\]

\[
-\frac{t^k}{2cR_j}L_6(a)^j(\chi^j - \frac{t}{R_j}\bar{w}_{j_1}^{j_2}) + (-1)^k\frac{(R_j)^2}{R_x^c(R_x^c)^2r_k}\bar{q}_x
\]
\[ q_{x,x}^j = -\frac{t}{2cR^j}L_5(a_{,\theta\theta})^j - L_4(a)^j + \{L_4(a)^j(\psi^j - \frac{t}{R^j}\bar{w}_{,\theta}^j),_{\theta} \]
\[ + \{n_{x,y}^j(\chi^j - \frac{t}{R^j}\bar{w}_{,\chi}^j),_{\theta} \} + \frac{t}{2cR^j}\{L_6(a)^j(\chi^j - \frac{t}{R^j}\bar{w}_{,\chi}^j),_{\theta} \} \]  
\[ - (-1)^k \frac{e^c(R^j)^2}{R^c t} (D_2 w^t + D_3 w^b - D_4 t^c w^c)) \]
\[ - (-1)^k \frac{(R^j)^2}{R^c t} \frac{(R^j)^2}{D_1^j q_{0,0}^c + \bar{p}_{0}^c} \]
\[ m_{x,x}^j = \frac{1}{c}L_6(a_{,\theta})^j + \frac{R^j}{t} q_{0,0}^c + \frac{R^j}{t} n_{x,y}^j(\chi^j - \frac{t}{R^j}\bar{w}_{,\chi}^j) + \frac{R^j}{t} n_{x,y}^j(\psi^j - \frac{t}{R^j}\bar{w}_{,\theta}^j) \]
\[ + \frac{1}{2c}L_6(a)^j(\psi^j - \frac{t}{R^j}\bar{w}_{,\theta}^j) - \frac{(R^j)^2}{2(R^c)^2 C_3 r_k} q_{x,x}^c \]
\[ q_{x,x}^c = \frac{C_3 cR^j e_z}{C_5 t} \{ C_1 w^t + C_2 w^b + \frac{t^c}{R^c w^c} \} + \frac{C_3 C_4 R^c}{C_5 t^c} q_{0,0}^c \]  
\[ \text{and} \]
\[ C_8 q_{0,0}^c - D_1 q_{0,0}^c = \frac{g_0 c t^c}{t} \{ r_1 v^t - r_2 v^b \} \]
\[ + \frac{g_0 c t^c}{t} \{ (D_2 - \frac{tr_1}{2R^t}) w_{,\theta}^t + (D_3 - \frac{tr_2}{2R^b}) w_{,\theta}^b - \frac{D_4 t^c}{R^c w_{,\theta}^c} \} \]

The following nondimensional stress and displacement variables and the imperfection \( \bar{W}^j \) are introduced,

for the faces \((j = t, b)\):

\[ \begin{bmatrix}
 u^j \\
 v^j \\
 w^j \\
 \bar{w}^j
\end{bmatrix} = \frac{1}{t} \begin{bmatrix}
 U^j \\
 V^j \\
 W^j \\
 \bar{W}^j
\end{bmatrix} \quad \chi^j = \begin{bmatrix}
 n_x^j \\
 n_{x,y}^j \\
 q_x^j \\
 q_{0,0}^j
\end{bmatrix} = \frac{c R^j}{E t^2} \begin{bmatrix}
 N_x^j \\
 N_{x,y}^j \\
 \bar{N}_x^j \\
 \bar{N}_{x,y}^j \\
 \bar{Q}_x^j \\
 \bar{Q}_{x,y}^j \\
 \bar{Q}_{0,0}^j
\end{bmatrix} \]  

for the core:

\[ \begin{bmatrix}
 w^c \\
 \bar{w}^c
\end{bmatrix} = \frac{1}{t} \begin{bmatrix}
 W^c \\
 \bar{W}^c
\end{bmatrix} \quad \begin{bmatrix}
 q_x^c \\
 q_{0,0}^c
\end{bmatrix} = \frac{c R^c}{E t^2} \begin{bmatrix}
 \bar{Q}_x^c \\
 \bar{Q}_{0,0}^c
\end{bmatrix} \]
where the thickness parameter \( t = t^t = t^b \) is used because of the assumption made on the present model that the faces are identical. The following nondimensional normal pressure, either for the top face (external pressure) or for the bottom face (internal pressure), is introduced

\[
\tilde{p}_c^j = \frac{c(R^j)^2}{E_t t^2} (p_c^j) \tag{B.33}
\]

Further, a short notation

\[
\psi^j = -\frac{t}{R^j} (w^j_y + v^j) \tag{B.34}
\]

and the following nondimensional stiffness parameters are used

\[
\begin{align*}
\gamma_x &= \frac{G_x}{E} & \gamma_y &= \frac{G_y}{E} & \gamma_z &= \frac{E_c}{E}
\end{align*} \tag{B.35}
\]

and the nondimensional vector \( a^j \)

\[
[a^j] = \begin{bmatrix}
\gamma_x & \frac{n_x^j}{t} & \frac{4cR^j}{t} m_x^j \\
\gamma_y & n_y^j & (v^j_y - w^j) + \frac{R^j}{2t} (\psi^j_{y0}^2 - 2 \frac{t}{R^j} \overline{w}^j_{y0} \psi^j) \\
\gamma_z & \psi^j_{z0} & (\frac{t}{R^j} u^j_{z0} + 2 \chi^j_{z0}) + (\psi^j_{z0} \chi^j + \frac{t}{R^j} \overline{w}^j_{z0} \psi^j - \frac{t}{R^j} \overline{w}^j_{z0} \chi^j)
\end{bmatrix} \tag{B.36}
\]

Finally, the consistent boundary conditions at the edge of the shell \( x = 0 \) and \( x = L \) are such that one can prescribe either

For the faces:

\[
\begin{align*}
\gamma_x & \quad \text{or} \quad u^j \tag{B.37} \\
\gamma_y & \quad \text{or} \quad v^j \tag{B.38} \\
\gamma_z & \quad \text{or} \quad w^j \tag{B.39} \\
\gamma_{xy} & \quad \text{or} \quad \chi^j \tag{B.40}
\end{align*}
\]
For the core:

\[ q_x^c \quad \text{or} \quad \tilde{w}^c \quad (B.41) \]

The above governing equations and boundary conditions are sufficient to determine the bending, buckling and postbuckling behavior of sandwich cylindrical shells under prescribed axial compression and boundary conditions.

**B.5. Definition of the Operators in the Governing Equations**

For convenience, one rewrite the operators used in the partially inverted constitutive relation in (B.12) as follows

\[ L_i(a)^j = L_i(f)^j + L_i(d)^j + \frac{R^j}{2t} (g_{14}^j \psi^2 - 2 \frac{t}{R^j} \tilde{w}_{y} \psi)^j \]

\[ + g_{16}^j \psi \chi - \frac{t}{R^j} \tilde{w}_{z} \psi - \frac{t}{R^j} \tilde{w}_{z} \chi)^j \quad (B.42) \]

where \( i = 1, ..., 6 \) and \( j = t, b \) denote for the quantities of top and bottom face respectively, and

\[ a^j = \left[ n_x^j, m_x^j, n_{xy}^j, e_y^j, k_y^j, e_{xy}^j \right] \quad t^j = \left[ n_x^j, m_x^j, n_{xy}^j, 0, 0, 0 \right] \]

\[ d^j = \left[ 0, 0, 0, (v^j_y - w^j), -\frac{t}{R^j}(w^j_{yy} + v^j_{y}), (\frac{t}{R^j}u^j_y + 2 \chi^j_y) \right] \]

\[ g_{14}^j = cF_{11} \quad g_{44}^j = cE_{11} \quad g_{16}^j = \frac{1}{2} F_{16} \quad g_{46}^j = \frac{1}{2} E_{16} \]

\[ g_{24}^j = cF_{21} \quad g_{54}^j = cE_{12} \quad g_{26}^j = \frac{1}{2} F_{26} \quad g_{56}^j = \frac{1}{2} E_{26} \]

\[ g_{34}^j = cF_{61} \quad g_{64}^j = cE_{16} \quad g_{36}^j = \frac{1}{2} F_{66} \quad g_{66}^j = \frac{1}{2} E_{66} \]

Further, the nonlinear terms in the Fourier series representation leads to products of Fourier series. Let three arbitrary Fourier series are given as

\[ A(\bar{y}) = \frac{1}{2} A_{co} + \sum_{n=1}^{N} \left\{ A_{cn} \cos(n\bar{y}) + A_{sn} \sin(n\bar{y}) \right\} \]

\[ B(\bar{y}) = \frac{1}{2} B_{co} + \sum_{n=1}^{N} \left\{ B_{cn} \cos(n\bar{y}) + B_{sn} \sin(n\bar{y}) \right\} \]

\[ C(\bar{y}) = \frac{1}{2} C_{co} + \sum_{n=1}^{N} \left\{ C_{cn} \cos(n\bar{y}) + C_{sn} \sin(n\bar{y}) \right\} \]

A quadratic product can be expressed as
Definition of the Operators in the Governing Equations

\[ A(\bar{y}) B(\bar{y}) = \frac{1}{2} L^Q_{co}(A, B) + \sum_{m=1}^{2N} \left\{ L^Q_{cm}(A, B) \cos(m \bar{y}) + L^Q_{sm}(A, B) \sin(m \bar{y}) \right\} \]

where

\[ L^Q_{co}(A, B) = \frac{1}{2} (B_{co} A_{co}) + \sum_{n=1}^{N} (B_{cn} A_{cn} + B_{sn} A_{sn}) \]

\[ L^Q_{cm}(A, B) = \frac{1}{2} (B_{co} A_{cm}) + \frac{1}{2} \sum_{n=1}^{N} (B_{cn}(A_{c(n+m)} + A_{c(n-m)}) + B_{sn}(A_{s(n+m)} + A_{s(n-m)})) \]

\[ L^Q_{sm}(A, B) = \frac{1}{2} (B_{co} A_{sm}) + \frac{1}{2} \sum_{n=1}^{N} (B_{cn}(A_{s(n+m)} - A_{s(n-m)}) - B_{sn}(A_{c(n+m)} - A_{c(n-m)})) \]

A cubic product can be expressed as

\[ A(\bar{y}) B(\bar{y}) C(\bar{y}) = \frac{1}{2} L^C_{co}(A, B, C) + \sum_{m=1}^{3N} \left\{ L^C_{cm}(A, B, C) \cos(m \bar{y}) + L^C_{sm}(A, B, C) \sin(m \bar{y}) \right\} \]

where

\[ L^C_{co}(A, B, C) = \frac{1}{4} C_{co} B_{co} A_{co} + \frac{1}{2} C_{co} \sum_{n=1}^{N} (B_{cn} A_{cn} + B_{sn} A_{sn}) + \frac{1}{2} B_{co} \sum_{n=1}^{N} (C_{cn} A_{cn} + C_{sn} A_{sn}) \]

\[ + \frac{1}{2} \sum_{n=1}^{N} \sum_{k=1}^{N} \left\{ B_{ck}(A_{c(k+n)} + A_{c(k-n)}) + B_{sk}(A_{s(k+n)} + A_{s(k-n)}) \right\} \]

\[ + \frac{1}{2} \sum_{n=1}^{N} \sum_{k=1}^{N} \left\{ B_{ck}(A_{s(k+n)} - A_{s(k-n)}) - B_{sk}(A_{c(k+n)} - A_{c(k-n)}) \right\} \]

\[ L^C_{cm}(A, B, C) = \frac{1}{4} C_{co} B_{co} A_{cm} \]

\[ + \frac{1}{4} \sum_{k=1}^{N} \left\{ B_{ck}(A_{c(k+m)} + A_{c(k-m)}) + B_{sk}(A_{s(k+m)} + A_{s(k-m)}) \right\} \]

\[ + \frac{1}{4} \sum_{n=1}^{N} \sum_{k=1}^{N} \left\{ B_{ck}(A_{c(n+m)} + A_{c(n-m)}) + C_{sn}(A_{s(n+m)} + A_{s(n-m)}) \right\} \]

\[ + \frac{1}{4} \sum_{n=1}^{N} \sum_{k=1}^{N} \left\{ B_{ck}(A_{c(k+n+m)} + A_{c(k+n-m)} + A_{c(k-n+m)} + A_{c(k-n-m)}) \right\} \]

\[ + \frac{1}{4} \sum_{n=1}^{N} \sum_{k=1}^{N} \left\{ B_{sk}(A_{s(k+n+m)} + A_{s(k+n-m)} + A_{s(k-n+m)} + A_{s(k-n-m)}) \right\} \]

\[ + \frac{1}{4} \sum_{n=1}^{N} \sum_{k=1}^{N} \left\{ C_{sk}(A_{s(k+n+m)} + A_{s(k+n-m)} - A_{s(k-n+m)} - A_{s(k-n-m)}) \right\} \]

\[ L^C_{sm}(A, B, C) = \frac{1}{4} C_{co} B_{co} A_{sm} \]

\[ + \frac{1}{4} \sum_{k=1}^{N} \left\{ B_{ck}(A_{s(k+m)} - A_{s(k-m)}) - B_{sk}(A_{c(k+m)} - A_{c(k-m)}) \right\} \]

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\begin{align*}
+ \frac{1}{4} B_{co} \sum_{n=1}^{N} \{ C_{cn}(A_{s(n+m)} - A_{s(n-m)}) - C_{sn}(A_{c(n+m)} - A_{c(n-m)}) \} \\
+ \frac{1}{4} \sum_{n=1}^{N} C_{cn} \sum_{k=1}^{N} \{ B_{ck}(A_{s(k+n+m)} - A_{s(k+n-m)} + A_{s(k-n+m)} - A_{s(k-n-m)}) \\
- B_{sk}(A_{c(k+n+m)} - A_{c(k+n-m)} + A_{c(k-n+m)} - A_{c(k-n-m)}) \} \\
- \frac{1}{4} \sum_{n=1}^{N} C_{sn} \sum_{k=1}^{N} \{ B_{ck}(A_{c(k+n+m)} - A_{c(k+n-m)} - A_{s(k+n-m)} + A_{s(k-n-m)}) \\
+ B_{sk}(A_{s(k+n+m)} - A_{s(k+n-m)} - A_{s(k-n+m)} + A_{s(k-n-m)}) \} \\
\end{align*}

Note that for the case of negative subscript

\[ A_{c(-n)} = A_{cn} \quad A_{s(-n)} = -A_{sn} \]

Using the notations as defined above the linear and nonlinear operators in the governing equations in equations (5-91) and (5-92) can be written as follows

\[ L_{11,1}(D) = -\frac{1}{c} L_{1}(d_{co}) t - L^{Q}_{co}(\overline{w}_{sx}, \chi)^t + \frac{b_{14}}{c} L^{Q}_{co}(\overline{w}_{sy}, \psi)^t + \frac{b_{16}}{cR} \{ L^{Q}_{co}(\overline{w}_{sx}, \psi)^t + L^{Q}_{co}(\overline{w}_{sy}, \chi)^t \} \]

\[ L_{11,2}(D) = -\frac{1}{c} L_{\beta}(d_{co}) t + \frac{b_{14}}{c} L^{Q}_{co}(\overline{w}_{sy}, \psi)^t - (1 - \frac{b_{26}}{cR}) \{ L^{Q}_{co}(\overline{w}_{sx}, \psi)^t + L^{Q}_{co}(\overline{w}_{sy}, \chi)^t \} \]

\[ L_{11,3}(D) = \frac{R_{t}^t}{L_{co}} \]

\[ L_{11,4}(D) = -2L_{2}(d_{co}) t + 2\frac{b_{14}}{c} L^{Q}_{co}(\overline{w}_{sy}, \psi)^t + 2\frac{b_{26}}{R} \{ L^{Q}_{co}(\overline{w}_{sx}, \psi)^t + L^{Q}_{co}(\overline{w}_{sy}, \chi)^t \} \]

\[ L_{11,5}(D) = -\frac{R_{t}^t}{c} u_{co} + \frac{R_{t}^b}{c} v_{co} - \frac{R_{t}^c}{2t} \chi_{co} - \frac{R_{t}^e}{2t} \chi_{co} \]

\[ L_{11,6}(D) = -\frac{1}{c} L_{1}(d_{co}) b - L^{Q}_{co}(\overline{w}_{sx}, \chi)^b + \frac{b_{14}}{c} L^{Q}_{co}(\overline{w}_{sy}, \psi)^b + \frac{b_{16}}{cR} \{ L^{Q}_{co}(\overline{w}_{sx}, \psi)^b + L^{Q}_{co}(\overline{w}_{sy}, \chi)^b \} \]

\[ L_{11,7}(D) = -\frac{1}{c} L_{3}(d_{co}) b + \frac{1}{c} L_{24} L^{Q}_{co}(\overline{w}_{sy}, \psi)^b - (1 - \frac{b_{26}}{cR}) \{ L^{Q}_{co}(\overline{w}_{sx}, \psi)^b + L^{Q}_{co}(\overline{w}_{sy}, \chi)^b \} \]

\[ L_{11,8}(D) = \frac{R_{b}^b}{L_{co}} \]

\[ L_{11,9}(D) = -2L_{2}(d_{co}) b + 2\frac{b_{14}}{c} L^{Q}_{co}(\overline{w}_{sy}, \psi)^b + 2\frac{b_{26}}{R} \{ L^{Q}_{co}(\overline{w}_{sx}, \psi)^b + L^{Q}_{co}(\overline{w}_{sy}, \chi)^b \} \]

\[ L_{11,90,1}(D) = -\frac{1}{c} L_{1}(d_{cn}) t - L^{Q}_{cn}(\overline{w}_{sx}, \chi)^t + \frac{b_{14}}{c} L^{Q}_{cn}(\overline{w}_{sy}, \psi)^t \]

\[ + \frac{b_{16}}{cR} \{ L^{Q}_{cn}(\overline{w}_{sx}, \psi)^t + L^{Q}_{cn}(\overline{w}_{sy}, \chi)^t \} \]

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\[ L_{11g_{s+2}}(\mathbf{D}) = -\frac{1}{c} L_{1c}(\mathbf{d}_{cn})^t - nu_{cn}^t + \frac{g_{34}}{c} L_{sn}(\mathbf{w}_{sy}, \psi)^t - \left(1 - \frac{tg_{36}^t}{cR^t}\right) \{ L_{sn}(\mathbf{w}_{sx}, \psi)^t + L_{cn}(\mathbf{w}_{sy}, \psi)^t \} \]

\[ L_{11g_{s+3}}(\mathbf{D}) = \frac{R^t}{t} \chi_{cn}^t \]

\[ L_{11g_{s+4}}(\mathbf{D}) = -2L_{2c}(\mathbf{d}_{cn})^t + 2g_{24}^t L_{cn}(\mathbf{w}_{sy}, \psi)^t + 2\frac{tg_{26}^t}{R^t} \{ L_{cn}(\mathbf{w}_{sx}, \psi)^t + L_{cn}(\mathbf{w}_{sy}, \chi)^t \} \]

\[ L_{11g_{s+5}}(\mathbf{D}) = \frac{R^c}{t^c} u_{cn}^t + \frac{R^c}{t^c} u_{sn}^t - \frac{R^c}{2t^c} \chi_{cn}^t - \frac{R^c}{2t^c} \chi_{sn}^t \]

\[ L_{11g_{s+6}}(\mathbf{D}) = -\frac{1}{c} L_{1c}(\mathbf{d}_{cn})^b - L_{cn}(\mathbf{w}_{sx}, \chi)^b + \frac{g_{34}^b}{c} L_{cn}(\mathbf{w}_{sy}, \psi)^b +\frac{tg_{16}^b}{cR^b} \{ L_{cn}(\mathbf{w}_{sx}, \psi)^b + L_{cn}(\mathbf{w}_{sy}, \chi)^b \} \]

\[ L_{11g_{s+7}}(\mathbf{D}) = \frac{1}{c} L_{3c}(\mathbf{d}_{cn})^b - nu_{cn}^b + \frac{g_{34}^b}{c} L_{cn}(\mathbf{w}_{sy}, \psi)^b - \left(1 - \frac{tg_{36}^b}{cR^b}\right) \{ L_{sn}(\mathbf{w}_{sx}, \psi)^b + L_{sn}(\mathbf{w}_{sy}, \chi)^b \} \]

\[ L_{11g_{s+8}}(\mathbf{D}) = \frac{R^b}{t} \chi_{sn}^b \]

\[ L_{11g_{s+9}}(\mathbf{D}) = -2L_{2c}(\mathbf{d}_{cn})^b + 2g_{24}^b L_{cn}(\mathbf{w}_{sy}, \psi)^b + 2\frac{tg_{26}^b}{R^b} \{ L_{cn}(\mathbf{w}_{sx}, \psi)^b + L_{cn}(\mathbf{w}_{sy}, \chi)^b \} \]

\[ L_{11g_{s+N,1}}(\mathbf{D}) = \frac{1}{c} L_{1c}(\mathbf{d}_{sn})^t - L_{sn}(\mathbf{w}_{sx}, \chi)^t + \frac{g_{34}^t}{c} L_{sn}(\mathbf{w}_{sy}, \psi)^t +\frac{tg_{16}^t}{cR^t} \{ L_{sn}(\mathbf{w}_{sx}, \psi)^t + L_{sn}(\mathbf{w}_{sy}, \chi)^t \} \]

\[ L_{11g_{s+N,2}}(\mathbf{D}) = \frac{1}{c} L_{3c}(\mathbf{d}_{cn})^b + nu_{sn}^t + \frac{g_{34}^b}{c} L_{cn}(\mathbf{w}_{sy}, \psi)^b - \left(1 - \frac{tg_{36}^b}{cR^b}\right) \{ L_{sn}(\mathbf{w}_{sx}, \psi)^b + L_{sn}(\mathbf{w}_{sy}, \chi)^b \} \]

\[ L_{11g_{s+N,3}}(\mathbf{D}) = \frac{R^t}{t} \chi_{sn}^t \]

\[ L_{11g_{s+N,4}}(\mathbf{D}) = -2L_{2c}(\mathbf{d}_{sn})^b + 2g_{24}^b L_{sn}(\mathbf{w}_{sy}, \psi)^b + 2\frac{tg_{26}^b}{R^b} \{ L_{sn}(\mathbf{w}_{sx}, \psi)^b + L_{sn}(\mathbf{w}_{sy}, \chi)^b \} \]

\[ L_{11g_{s+N,5}}(\mathbf{D}) = -\frac{R^c}{t^c} u_{sn}^t + \frac{R^c}{t^c} u_{sn}^b - \frac{R^c}{2t^c} \chi_{sn}^t - \frac{R^c}{2t^c} \chi_{sn}^b \]

\[ L_{11g_{s+N,6}}(\mathbf{D}) = \frac{1}{c} L_{1c}(\mathbf{d}_{sn})^b - L_{sn}(\mathbf{w}_{sx}, \chi)^b + \frac{g_{34}^b}{c} L_{sn}(\mathbf{w}_{sy}, \psi)^b +\frac{tg_{16}^b}{cR^b} \{ L_{sn}(\mathbf{w}_{sx}, \psi)^b + L_{sn}(\mathbf{w}_{sy}, \chi)^b \} \]

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\[ L_{11\text{sn}(N-1,1)}(D) = \frac{1}{c}L_{3}(d_{n})^{b} + n u_{n}^{b} + \frac{g_{34}^{b}}{c}L_{Q}(\bar{w}_{1,y},\psi)^{b} \]
\[ - (1 - \frac{t}{cR})^{b} \{ L_{Q}(\bar{w}_{1,y'},\psi)^{b} + L_{Q}(\bar{w}_{y},\psi)^{b} \} \]
\[ L_{11\text{sn}(N-1,1,8)}(D) = \frac{R^{b}}{t} x_{sn}^{b} \]
\[ L_{11\text{sn}(N-1,1,9)}(D) = -2L_{2}(d_{sn})^{b} + 2g_{24}^{b}L_{Q}(\bar{w}_{y},\psi)^{b} + 2\frac{t g_{26}^{b}}{R} \{ L_{Q}(\bar{w}_{y},\psi)^{b} + L_{Q}(\bar{w}_{y},\chi)^{b} \} \]
\[ L_{121}(F) = \frac{1}{c}t_{11}(f_{co})^{t} \]
\[ L_{122}(F) = -\frac{1}{c}t_{12}(f_{co})^{t} \]
\[ L_{12e}(F) = -\frac{1}{c}t_{12}(f_{co})^{t} \]
\[ L_{12g}(F) = 0 \]
\[ L_{12h}(F) = \frac{1}{c}t_{12}(f_{co})^{t} \]
\[ L_{12m}(F) = -\frac{1}{c}t_{12}(f_{co})^{t} \]
\[ L_{12n}(F) = -\frac{1}{c}t_{12}(f_{co})^{t} \]
\[ L_{12o}(F) = 0 \]
\[ L_{12p}(F) = \frac{1}{c}t_{12}(f_{co})^{t} \]
\[ L_{12q}(F) = -\frac{1}{c}t_{12}(f_{co})^{t} \]
\[ L_{12r}(F) = 0 \]
\[ L_{12s}(F) = \frac{1}{c}t_{12}(f_{co})^{t} \]
\[ L_{12t}(F) = -\frac{1}{c}t_{12}(f_{co})^{t} \]
\[ L_{12u}(F) = 0 \]
\[ L_{12v}(F) = \frac{1}{c}t_{12}(f_{co})^{t} \]
\[ L_{12w}(F) = -\frac{1}{c}t_{12}(f_{co})^{t} \]
\[ L_{12x}(F) = 0 \]
\[ L_{12y}(F) = \frac{1}{c}t_{12}(f_{co})^{t} \]
\[ L_{12z}(F) = -\frac{1}{c}t_{12}(f_{co})^{t} \]

\[ L_{12N}(F) = \frac{R^{b}}{c t} x_{12}(f_{co})^{t} \]
\[ L_{12N}(F) = \frac{R^{b}}{c t} x_{12}(f_{co})^{t} \]
\[ L_{12N}(F) = \frac{R^{b}}{c t} x_{12}(f_{co})^{t} \]
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\[ L_{12N}(F) = \frac{R^{b}}{c t} x_{12}(f_{co})^{t} \]
\[ L_{12N}(F) = \frac{R^{b}}{c t} x_{12}(f_{co})^{t} \]
\[ L_{12N}(F) = \frac{R^{b}}{c t} x_{12}(f_{co})^{t} \]

\[ L_{11}^{(1)}(D, E) = \frac{R^{1}}{2 c t} L_{Q}(\psi, \psi)^{t} + \frac{g_{16}^{1}}{c} L_{Q}(\psi, \chi)^{t} - R^{1} \frac{L_{Q}(\chi, \chi)^{t}}{R^{1}} \]
\[ L_{12}^{(1)}(D, E) = \frac{R^{1}}{2 c t} L_{Q}(\psi, \psi)^{t} + \frac{g_{26}^{1}}{c} L_{Q}(\psi, \chi)^{t} - R^{1} \frac{L_{Q}(\chi, \chi)^{t}}{R^{1}} \]
\[ L_{14}^{(1)}(D, E) = \frac{R^{1}}{2 c t} L_{Q}(\psi, \psi)^{t} + \frac{g_{36}^{1}}{c} L_{Q}(\psi, \chi)^{t} - R^{1} \frac{L_{Q}(\chi, \chi)^{t}}{R^{1}} \]
\[ L_{15}^{(1)}(D, E) = \frac{R^{1}}{2 c t} L_{Q}(\psi, \psi)^{t} + \frac{g_{46}^{1}}{c} L_{Q}(\psi, \chi)^{t} - R^{1} \frac{L_{Q}(\chi, \chi)^{t}}{R^{1}} \]
\[ L_{16}^{(1)}(D, E) = \frac{R^{1}}{2 c t} L_{Q}(\psi, \psi)^{t} + \frac{g_{56}^{1}}{c} L_{Q}(\psi, \chi)^{t} - R^{1} \frac{L_{Q}(\chi, \chi)^{t}}{R^{1}} \]
\[ L_{17}^{(1)}(D, E) = \frac{R^{1}}{2 c t} L_{Q}(\psi, \psi)^{t} + \frac{g_{66}^{1}}{c} L_{Q}(\psi, \chi)^{t} - R^{1} \frac{L_{Q}(\chi, \chi)^{t}}{R^{1}} \]

\[ L_{11}^{(2)}(D, E) = \frac{R^{2}}{2 c t} L_{Q}(\psi, \psi)^{t} + \frac{g_{16}^{2}}{c} L_{Q}(\psi, \chi)^{t} - R^{2} \frac{L_{Q}(\chi, \chi)^{t}}{R^{2}} \]
\[ L_{12}^{(2)}(D, E) = \frac{R^{2}}{2 c t} L_{Q}(\psi, \psi)^{t} + \frac{g_{26}^{2}}{c} L_{Q}(\psi, \chi)^{t} - R^{2} \frac{L_{Q}(\chi, \chi)^{t}}{R^{2}} \]
\[ L_{14}^{(2)}(D, E) = \frac{R^{2}}{2 c t} L_{Q}(\psi, \psi)^{t} + \frac{g_{36}^{2}}{c} L_{Q}(\psi, \chi)^{t} - R^{2} \frac{L_{Q}(\chi, \chi)^{t}}{R^{2}} \]
\[ L_{15}^{(2)}(D, E) = \frac{R^{2}}{2 c t} L_{Q}(\psi, \psi)^{t} + \frac{g_{46}^{2}}{c} L_{Q}(\psi, \chi)^{t} - R^{2} \frac{L_{Q}(\chi, \chi)^{t}}{R^{2}} \]
\[ L_{16}^{(2)}(D, E) = \frac{R^{2}}{2 c t} L_{Q}(\psi, \psi)^{t} + \frac{g_{56}^{2}}{c} L_{Q}(\psi, \chi)^{t} - R^{2} \frac{L_{Q}(\chi, \chi)^{t}}{R^{2}} \]
\[ L_{17}^{(2)}(D, E) = \frac{R^{2}}{2 c t} L_{Q}(\psi, \psi)^{t} + \frac{g_{66}^{2}}{c} L_{Q}(\psi, \chi)^{t} - R^{2} \frac{L_{Q}(\chi, \chi)^{t}}{R^{2}} \]
\[ L_{1\mu_n+1}^{nl}(D, F) = \frac{R^t}{2c t} L_{cn}^Q(\psi, \chi)^t + \frac{g_{14}^t}{c} L_{cn}^Q(\psi, \chi)^t - \frac{R^t}{2t} L_{cn}^Q(\chi, \chi)^t \]

\[ L_{1\mu_n+2}^{nl}(D, F) = \frac{R^t}{2c t} L_{sn}^Q(\psi, \psi)^t - \frac{g_{32}^t}{c} L_{sn}^Q(\psi, \psi)^t + \frac{g_{16}^t}{c} L_{sn}^Q(\psi, \chi)^t \]

\[ L_{1\mu_n+3}^{nl}(D, F) = 0 \]

\[ L_{1\mu_n+4}^{nl}(D, F) = \frac{R^b}{2c t} L_{cn}^Q(\psi, \psi)^b + \frac{g_{32}^b}{c} L_{cn}^Q(\psi, \chi)^b - \frac{R^b}{2t} L_{cn}^Q(\chi, \chi)^b \]

\[ L_{1\mu_n+5}^{nl}(D, F) = 0 \]

\[ L_{1\mu_n+6}^{nl}(D, F) = \frac{R^b}{2c t} L_{sn}^Q(\psi, \psi)^b + \frac{g_{32}^b}{c} L_{sn}^Q(\psi, \chi)^b \]

\[ L_{1\mu_n+7}^{nl}(D, F) = 0 \]

\[ L_{1\mu_n+8}^{nl}(D, F) = 0 \]

\[ L_{1\mu_n+9}^{nl}(D, F) = \frac{R^b}{2c t} L_{cn}^Q(\psi, \psi)^b + \frac{g_{32}^b}{c} L_{cn}^Q(\psi, \chi)^b \]

\[ L_{1\mu_{n+1}}^{nl}(D, F) = \frac{R^t}{2c t} L_{sn}^Q(\psi, \psi)^t + \frac{g_{32}^t}{c} L_{sn}^Q(\psi, \chi)^t - \frac{R^t}{2t} L_{sn}^Q(\chi, \chi)^t \]

\[ L_{1\mu_{n+2}}^{nl}(D, F) = \frac{R^t}{2c t} L_{cn}^Q(\psi, \psi)^t + \frac{g_{32}^t}{c} L_{cn}^Q(\psi, \chi)^t \]

\[ L_{1\mu_{n+3}}^{nl}(D, F) = 0 \]

\[ L_{1\mu_{n+4}}^{nl}(D, F) = \frac{R^t}{2c t} L_{sn}^Q(\psi, \psi)^t + \frac{g_{32}^t}{c} L_{sn}^Q(\psi, \chi)^t \]

\[ L_{1\mu_{n+5}}^{nl}(D, F) = 0 \]

\[ L_{1\mu_{n+6}}^{nl}(D, F) = \frac{R^b}{2c t} L_{sn}^Q(\psi, \psi)^b + \frac{g_{32}^b}{c} L_{sn}^Q(\psi, \chi)^b - \frac{R^b}{2t} L_{sn}^Q(\chi, \chi)^b \]

\[ L_{1\mu_{n+7}}^{nl}(D, F) = 0 \]

\[ L_{1\mu_{n+8}}^{nl}(D, F) = 0 \]

\[ L_{1\mu_{n+9}}^{nl}(D, F) = \frac{R^b}{2c t} L_{sn}^Q(\psi, \psi)^b + \frac{g_{32}^b}{c} L_{sn}^Q(\psi, \chi)^b \]

\[ L_{211}^{nl}(D) = 0 \]

\[ L_{212}^{nl}(D) = -\frac{t}{R^t} L_{co}^Q(\bar{w}_y, L_4(d)) + \frac{t^2}{2c(R^t)^2} L_{co}^Q(\bar{w}_x, L_6(d)) + \frac{g_c c(R^t)^2}{R^t(t_r)^2} (r_1 v_{co}^t + r_2 v_{co}^b) \]

\[ + \frac{t g_{42}^t}{R^t} L_{co}^c(\psi, \bar{w}_y, \bar{w}_x) + \frac{t^2 g_{66}^t}{2c(R^t)^2} \{ L_{co}^c(\chi, \bar{w}_y, \bar{w}_x)^t + L_{co}^c(\psi, \bar{w}_x, \bar{w}_x)^t \} \]

\[ + \frac{t^2 g_{46}^t}{R^t(R^t)^2} L_{co}^c(\chi, \bar{w}_y, \bar{w}_x)^t + L_{co}^c(\psi, \bar{w}_x, \bar{w}_x)^t \]

\[ L_{213}^{nl}(D) = L_4(d_{co}) - g_{44} L_{co}^c(\psi)^t - \frac{t g_{46}^t}{R^t} \{ L_{co}^c(\bar{w}_x, \psi)^t + L_{co}^c(\bar{w}_y, \chi)^t \} \]

\[ + \frac{e_c c(R^t)^2}{R^t} \{ -\frac{t^2 C_{12}^t}{R^t} w_{co}^c + D_6 w_{co}^t + D_7 w_{co}^b \} \]

\[ - \frac{t^2 C_{12}^t}{R^t} w_{co}^c + D_6 w_{co}^t + D_7 w_{co}^b \]

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\[ L_{21a}(D) = \frac{t}{2cR^t} L^Q_{co}(w_{xy}, L_6(d))^t \]
\[ - \frac{t^2}{2cR^t} L^Q_{co}(\bar{w}, \bar{w}_{xy})^t - \frac{t^2}{2c(R^t)^2} \left\{ L^c_{co}(\chi, \bar{w}_{xy}, \bar{w}_{xy})^t + L^c_{co}(\psi, \bar{w}_{xy}, \bar{w}_{xy})^t \right\} \]

\[ L_{21a}(D) = -\frac{e_c R^c C_3}{t \epsilon_c} \left\{ \frac{t^c}{R^c} w_{co}^c + C_1 w_{co}^t + C_2 w_{co}^b \right\} \]

\[ L_{21a}(D) = 0 \]

\[ L_{21b}(D) = \frac{t}{2cR^b} L^Q_{co}(\bar{w}_{xy}, L_4(d))^b \]
\[ - \frac{t^2}{2cR^b} L^Q_{co}(\bar{w}, \bar{w}_{xy})^b - \frac{g_c(R^b)^2}{R^c t(r_1)^2 C_8} \left\{ r_1 v_{co}^t - r_2 v_{co}^b \right\} \]
\[ + \frac{t^2}{2c(R^b)^2} \left\{ L^c_{co}(\chi, \bar{w}_{xy}, \bar{w}_{xy})^b + L^c_{co}(\psi, \bar{w}_{xy}, \bar{w}_{xy})^b \right\} \]
\[ + \frac{t^2}{2c(R^b)^2} \left\{ L^c_{co}(\chi, \bar{w}_{xy}, \bar{w}_{xy})^b + L^c_{co}(\psi, \bar{w}_{xy}, \bar{w}_{xy})^b \right\} \]

\[ L_{21b}(D) = L_4(d_{co})^b - E_{44} L^Q_{co}(\bar{w}_{xy}, \psi)^b \]
\[ - \frac{t^2}{2cR^b} \left\{ L^c_{co}(\chi, \bar{w}_{xy}, \bar{w}_{xy})^b + L^c_{co}(\psi, \bar{w}_{xy}, \bar{w}_{xy})^b \right\} \]
\[ + \frac{e_c R^c}{R^t} \left\{ \frac{t^c}{R^c} C_1 w_{co}^c - D_{10} w_{co}^b - D_9 w_{co}^t \right\} \]

\[ L_{21c}(D) = \frac{t}{2cR^c} L^Q_{co}(\bar{w}_{xy}, L_6(d))^c \]
\[ - \frac{t^2}{2cR^c} L^Q_{co}(\bar{w}, \bar{w}_{xy})^c - \frac{t^2}{2c(R^c)^2} \left\{ L^c_{co}(\chi, \bar{w}_{xy}, \bar{w}_{xy})^c + L^c_{co}(\psi, \bar{w}_{xy}, \bar{w}_{xy})^c \right\} \]

\[ L_{21a_{+1}}(D) = \frac{nt}{2cR^t} L^Q_{co}(d_{sn})^t \]
\[ - \frac{ntg_{44}^t}{2cR^t} L^Q_{co}(\bar{w}_{xy}, \psi)^t - \frac{nt^2g_{66}^t}{2c(R^t)^2} \left\{ L^c_{co}(\chi, \bar{w}_{xy}, \bar{w}_{xy})^t + L^c_{co}(\psi, \bar{w}_{xy}, \bar{w}_{xy})^t \right\} \]

\[ L_{21a_{+2}}(D) = -n L_4(d_{cn})^t + \frac{nt}{2cR^t} \left\{ L^Q_{co}(\bar{w}, \bar{w}_{xy})^t \right\} \]
\[ + \frac{nt^2g_{46}^t}{2cR^t} \left\{ L^c_{co}(\chi, \bar{w}_{xy}, \bar{w}_{xy})^t + L^c_{co}(\psi, \bar{w}_{xy}, \bar{w}_{xy})^t \right\} \]
\[ + \frac{t^2g_{44}^t}{R^t} L^c_{sn}(\psi, \bar{w}_{xy}, \bar{w}_{xy})^t \]
\[ + \frac{t^2g_{66}^t}{2c(R^t)^2} \left\{ L^c_{sn}(\chi, \bar{w}_{xy}, \bar{w}_{xy})^t + L^c_{sn}(\psi, \bar{w}_{xy}, \bar{w}_{xy})^t \right\} \]
\[ + \frac{t^2g_{46}^t}{2c(R^t)^2} \left\{ L^c_{sn}(\chi, \bar{w}_{xy}, \bar{w}_{xy})^t + L^c_{sn}(\psi, \bar{w}_{xy}, \bar{w}_{xy})^t \right\} \]

\[ + \frac{g_c(R^t)^2}{R^c t(r_2)^2 (C_8 + n^2 g_c D_1)} \left\{ -r_1 v_{sn}^t + r_2 v_{sn}^b \right\} \]
Definition of the Operators in the Governing Equations

\[ L_{21\gamma_1}\left(\mathcal{D}\right) = L_4\left(d_{\gamma_1}\right) + \frac{n_g c R^4}{R t (r_2)^2 (C_8 + n_g^2 D_1)} \left( - \frac{t^4 D_4^c w_{\gamma_1}}{R^4} - \frac{t r_1}{2 R} D_2 w_{\gamma_1} - \frac{t r_2}{2 R} D_3 w_{\gamma_1} \right) \]

\[ L_{21\gamma_2\lambda_3}\left(\mathcal{D}\right) = L_4\left(d_{\gamma_1}\right)^t - \frac{n_i^2 g_{54}^c}{2 c R^4} L_5\left(d_{\gamma_1}\right)^t + \frac{n t L_Q^c}{R t} L_{\gamma_1}(w_{\gamma_1}, L_4(d))^t + \frac{n t^2}{2 c R^4} L_{\gamma_1}(w_{\gamma_1}, L_6(d))^t \]

\[ - \frac{n t g_{54}^c}{R} L_{\gamma_1}(\psi, \omega_{\gamma_1}, \eta)^t - \frac{n t g_{56}^c}{2 c R^4} L_{\gamma_1}(\chi, \omega_{\gamma_1}, \tau)^t \]

\[ - \frac{n t g_{54}^c}{R} L_{\gamma_1}(\psi, \omega_{\gamma_1}, \eta)^t - \frac{n t g_{56}^c}{2 c R^4} L_{\gamma_1}(\chi, \omega_{\gamma_1}, \tau)^t \]

\[ + \frac{e c R^4}{2 c R^4} \left( \frac{t^4 C_1^2}{2 c R^4} w_{\gamma_1}^c + D_6 w_{\gamma_1}^b + D_7 w_{\gamma_1}^b \right) \]

\[ + \frac{n g_{\gamma_1}^c}{2 c R^4} \left( \frac{1}{r_2} - (2 R D_5) / t \right) \left( - r_1 v_{\gamma_1}^b + r_2 v_{\gamma_1}^b \right) \]

\[ + \frac{n g_{\gamma_1}^c}{2 c R^4} \left( \frac{1}{r_2} - (2 R D_5) / t \right) \left( - \frac{t^4 D_4^c}{R^4} w_{\gamma_1}^c - \frac{t r_1}{2 R} D_2 w_{\gamma_1}^c - \frac{t r_2}{2 R} D_3 w_{\gamma_1}^c \right) \]

\[ L_{21\gamma_1\lambda_4}\left(\mathcal{D}\right) = L_4\left(d_{\gamma_1}\right)^t + \frac{n_t L_Q^c}{R t} L_{\gamma_1}(w_{\gamma_1}, L_6(d))^t \]

\[ - \frac{n g_{64}^c}{c R^4} L_{\gamma_1}(\psi, \omega_{\gamma_1}, \eta)^t - \frac{n g_{66}^c}{c R^4} L_{\gamma_1}(\chi, \omega_{\gamma_1}, \tau)^t \]

\[ - \frac{n g_{64}^c}{c R^4} L_{\gamma_1}(\psi, \omega_{\gamma_1}, \eta)^t - \frac{n g_{66}^c}{c R^4} L_{\gamma_1}(\chi, \omega_{\gamma_1}, \tau)^t \]

\[ L_{21\gamma_2\lambda_5}\left(\mathcal{D}\right) = - \frac{n c R^4}{2 c R^4} \left( \frac{t^4 C_1^2}{c R^4} w_{\gamma_1}^c + C_1 w_{\gamma_1}^b + C_2 w_{\gamma_1}^b \right) + \frac{n g_{\gamma_1}^c R C_3 C_4}{c R^4} \left( - r_1 v_{\gamma_1}^b + r_2 v_{\gamma_1}^b \right) \]

\[ + \frac{n g_{\gamma_1}^c}{c R^4} \left( \frac{1}{r_2} - (2 R D_5) / t \right) \left( - \frac{t^4 D_4^c}{R^4} w_{\gamma_1}^c - \frac{t r_1}{2 R} D_2 w_{\gamma_1}^c - \frac{t r_2}{2 R} D_3 w_{\gamma_1}^c \right) \]

\[ L_{21\gamma_2\lambda_6}\left(\mathcal{D}\right) = \frac{n t}{2 c R^4} L_6\left(d_{\gamma_1}\right)^b - \frac{n t g_{64}^c}{c R^4} L_{\gamma_1}(w_{\gamma_1}, \psi)^b - \frac{n t g_{66}^c}{c R^4} L_{\gamma_1}(w_{\gamma_1}, \chi)^b \]

\[ L_{21\gamma_2\lambda_7}\left(\mathcal{D}\right) = -n L_4\left(d_{\gamma_1}\right)^b + \frac{n t}{2 c R^4} L_6\left(d_{\gamma_1}\right)^b - \frac{t}{R^4} L_{\gamma_1}(w_{\gamma_1}, L_4(d))^b - \frac{t^2}{2 c R^4} L_{\gamma_1}(w_{\gamma_1}, L_6(d))^b \]

\[ + \frac{n g_{54}^c}{2 c R^4} L_{\gamma_1}(w_{\gamma_1}, \psi)^b + \frac{n t}{R^4} \left( g_{54}^c - g_{56}^c \right) \left( L_{\gamma_1}(w_{\gamma_1}, \psi)^b + L_{\gamma_1}(w_{\gamma_1}, \chi)^b \right) \]
\[ \begin{align*}
+ \ & \frac{t g_{44}^b}{R^b} l_{sn}^c(\psi, \bar{w}_y, \bar{w}_y) + \frac{t g_{66}^b}{2c(R^b)^3} \{ l_{sn}^c(\chi, \bar{w}_y, \bar{w}_x) + l_{sn}^c(\psi, \bar{w}_x, \bar{w}_x) \} +
+ \ & \frac{t^2 g_{46}^b}{R^b} \{ l_{sn}^c(\chi, \bar{w}_y, \bar{w}_x) + l_{sn}^c(\psi, \bar{w}_x, \bar{w}_y) \} + \frac{t^2 g_{64}^b}{2c(R^b)^2} l_{sn}^c(\psi, \bar{w}_y, \bar{w}_x) + \frac{t^2 g_{64}^b}{2c(R^b)^2} \} l_{sn}^c(\psi, \bar{w}_y, \bar{w}_x) + \frac{t^2 g_{64}^b}{2c(R^b)^2} \}

+ \ & \frac{g_y c(R^b)^2}{R^4 t(r_1)^2(C_8 + n^2 g_D)} \{ r_1 v_{sn}^t - r_2 v_{sn}^b \} + \frac{ng_y c(R^b)^2}{R^4 t(r_1)^2(C_8 + n^2 g_D)} \{ \frac{cD_4}{R} w_{cn} + \frac{cD_4}{R} w_{cn} + \frac{cD_4}{R} w_{cn} + \frac{cD_4}{R} w_{cn} \}
\end{align*} \]

\[ L_{21n_{s+n}}(D) = L_{4}(d_{cn})^b - \frac{n^2 t}{2cR^b} L_5(d_{cn})^b + \frac{nt}{R^b} l_{sn}^Q(\bar{w}_y, L_4(d))^b + \frac{nt}{2c(R^b)^2} l_{sn}^Q(\bar{w}_y, L_4(d))^b \]

\[ - \frac{g_{44}^b}{2cR^b} l_{cn}^Q(\bar{w}_y, \psi) - \frac{t}{R^b} \{ g_{66}^b - \frac{nt g_{56}^b}{2cR^b} \{ l_{sn}^Q(\bar{w}_x, \psi) + l_{sn}^Q(\bar{w}_x, \chi) \} \}
\]

\[ - \frac{nt g_{56}^b}{R^b} l_{sn}^c(\psi, \bar{w}_y, \bar{w}_x) + \frac{nt}{2c(R^b)^3} \{ l_{sn}^c(\chi, \bar{w}_y, \bar{w}_x) + l_{sn}^c(\psi, \bar{w}_x, \bar{w}_x) \} - \frac{nt g_{56}^b}{2c(R^b)^2} l_{sn}^c(\psi, \bar{w}_y, \bar{w}_x) - \frac{nt g_{56}^b}{2c(R^b)^2} l_{sn}^c(\psi, \bar{w}_y, \bar{w}_x) \}
\]

\[ + \frac{c t c(R^b)^2}{R^t t C_5} \{ l_{C_5}^c w_{cn} - D_{10} w_{cn} - D_9 w_{cn} \}
\]

\[ + \frac{ng_c c R^b}{R^4 t(r_1)^2 + (2R^b D_8)/t} \{ r_1 v_{sn}^t + r_2 v_{sn}^b \} + \frac{ng_c c R^b}{2C_8 + n^2 g_D} \{ \}
\]

\[ L_{21n_{s+n}}(D) = \frac{n L_6(d_{sn})^b}{c} + \frac{nt}{2cR^b} l_{cn}(\bar{w}_y, L_6(d))^b \]

\[ - \frac{n g_{64}^b}{c} l_{sn}(\bar{w}_y, \psi) + \frac{nt}{c} g_{66}^b \{ l_{sn}^Q(\bar{w}_x, \psi) + l_{sn}^Q(\bar{w}_x, \chi) \} \]

\[ - \frac{t g_{64}^b}{2cR^b} l_{cn}(\psi, \bar{w}_y, \bar{w}_y) + \frac{t^2 g_{64}^b}{2c(R^b)} \} l_{sn}(\chi, \bar{w}_y, \bar{w}_x) + l_{cn}(\psi, \bar{w}_x, \bar{w}_x) \}
\]

\[ L_{21n_{n+n+1}}(D) = \frac{nt}{2cR^t} l_{6}(d_{sn})^t + \frac{nt g_{64}^t}{2cR^t} l_{cn}(\bar{w}_y, \psi)^t + \frac{nt^2 g_{64}^t}{2c(R^t)} \}
\]

\[ L_{21n_{n+n+2}}(D) = n L_4(d_{sn})^t - \frac{nt}{2cR^t} l_{6}(d_{sn})^t - \frac{t L_{cn}(\bar{w}_y, L_4(d))^t}{2c(R^t)^2} l_{cn}(\bar{w}_y, L_4(d))^t \]

\[ - \frac{t^2 L_{cn}(\bar{w}_y, L_4(d))^t}{2c(R^t)^2} l_{cn}(\bar{w}_y, L_4(d))^t \]

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\[ \begin{align*}
- \left( \frac{g_{44}}{c} \right)^t [L^Q_{sn}(\vec{w}_{xy}, \psi)^t \left\{ t \frac{g_{56}}{2cR} \left( L^Q_{sn}(\vec{w}_{xy}, \psi)^t \right) + L^Q_{sn}(\vec{w}_{xy}, \chi)^t \right\} ] \\
+ \frac{t e_{44}}{R^t} L^c_{cn}(\psi, \vec{w}_{xy}, \vec{w}_{xy})^t \\
+ \frac{t^2 e_{44}}{2c(R^t)} \left( L^c_{cn}(\chi, \vec{w}_{xy}, \vec{w}_{xy})^t + L^c_{cn}(\psi, \vec{w}_{xy}, \vec{w}_{xy})^t \right) \\
+ \frac{e_{c} c(R^t)^2}{R^t (r_2)^2 (C_8 + n^2 g_{y} D_1)} \{ -r_1 v^t_{cn} + r_2 v^b_{cn} \} \\
+ \frac{n g_{y} c(R^t)^2}{R^t (r_2)^2 (C_8 + n^2 g_{y} D_1)} \left\{ t^4 D^c_{4w^c_{sn}} + \left( \frac{t r_1}{2R^t} - D_2 \right) w^t_{sn} + \left( \frac{t r_2}{2R^t} - D_3 \right) w^b_{sn} \right\} \\
L_{21_{4N,n_{an+4}^{(D)}}} = L_{4}(d^t_{sn}) - \frac{n t^2}{2R^t} L^0_{sn}(d^t_{sn}) - \frac{n t}{2R^t} \left( L^Q_{cn}(\vec{w}_{xy}, L_{4}(d))^t \right) - \frac{n t^2}{2c(R^t)} \left( L^Q_{sn}(\vec{w}_{xy}, \psi)^t \right) + L^Q_{sn}(\vec{w}_{xy}, \chi)^t \\
+ \frac{n t g_{44}}{R^t} L^c_{cn}(\psi, \vec{w}_{xy}, \vec{w}_{xy})^t + \frac{n t^2 g_{56}}{2c(R^t)} \left( L^c_{cn}(\chi, \vec{w}_{xy}, \vec{w}_{xy})^t + L^c_{cn}(\psi, \vec{w}_{xy}, \vec{w}_{xy})^t \right) \\
+ \frac{t^2 g_{46}}{2c(R^t)} \left( L^c_{cn}(\chi, \vec{w}_{xy}, \vec{w}_{xy})^t + L^c_{cn}(\psi, \vec{w}_{xy}, \vec{w}_{xy})^t \right) \\
+ \frac{e_{c} c(R^t)^2}{R^t C_5} \left\{ \frac{t^4 C^c_{12} w^c_{sn} + D_6 w^t_{sn} + D_7 w^b_{sn}}{2R^t (C_8 + n^2 g_{y} D_1)} \right\} \\
+ \frac{n g_{y} c(R^t)^2}{2R^t (C_8 + n^2 g_{y} D_1)} \left\{ \frac{1}{(r_2)^2} - \left( \frac{2R^t D_5}{t} \right) \right\} \left\{ r_1 v^t_{cn} - r_2 v^b_{cn} \right\} \\
L_{21_{4N,n_{an+4}^{(D)}}} = -\frac{n}{c} L_{6}(d^t_{cn}) + \frac{t}{2cR^t} \left( L^Q_{sn}(\vec{w}_{xy}, L_{6}(d))^t \right) \\
+ \frac{n g_{64}}{c} L^Q_{cn}(\vec{w}_{xy}, \psi)^t + \frac{n t g_{56}}{c(R^t)} \left\{ L^Q_{cn}(\vec{w}_{xy}, \psi)^t + L^Q_{cn}(\vec{w}_{xy}, \chi)^t \right\} \\
- \frac{t g_{64}}{2c R^t} L^c_{cn}(\psi, \vec{w}_{xy}, \vec{w}_{xy})^t - \frac{t^2 g_{66}}{2c(R^t)^2} \left( L^c_{sn}(\chi, \vec{w}_{xy}, \vec{w}_{xy})^t + L^c_{sn}(\psi, \vec{w}_{xy}, \vec{w}_{xy})^t \right) \\
L_{21_{6N,n_{an+3}^{(D)}}} = -\frac{n}{c} L_{6}(d^t_{cn}) - \frac{t e_{c} C^c_{5} t^c_{w^c_{sn}} + C^c_{w^t_{w^c_{sn}}} + C^c_{w^b_{w^c_{sn}}}}{t C_5 (C_8 + n^2 g_{y} D_1)} \left\{ r_1 v^t_{cn} - r_2 v^b_{cn} \right\} \\
+ \frac{n^2 g_{y} c R^t C^c_{5} C_4}{t C_5 (C_8 + n^2 g_{y} D_1)} \left\{ \frac{t^4 D^c_{4w^c_{sn}} - \left( \frac{t r_1}{2R^t} - D_2 \right) w^t_{sn} - \left( \frac{t r_2}{2R^t} - D_3 \right) w^b_{sn}}{2R^t} \right\} \\
+ \frac{n^2 g_{y} c R^t C^c_{5} C_4}{t C_5 (C_8 + n^2 g_{y} D_1)} \left\{ \frac{1}{(r_2)^2} - \left( \frac{2R^t D_5}{t} \right) \right\} \left\{ r_1 v^t_{cn} - r_2 v^b_{cn} \right\} \\
\end{align*} \]
\[ L_{21g(N_{s}=0)}(D) = - \frac{nt}{2cR^b} L_{6}(d_{cn})^b + \frac{ntg_{54}^b}{2cR^b} \frac{L_{cn}(\bar{w}, \bar{y}, \bar{\psi})^b}{2c(R^b)} + \frac{ntg_{56}^b}{2cR^b} \{L_{cn}(\bar{w}, \bar{y}, \bar{x})^b + L_{cn}(\bar{w}, \bar{y}, \bar{\chi})^b\} \]

\[ L_{21g(N_{s}=1)}(D) = nL_{4}(d_{sn})^b - \frac{nt}{2cR^b} L_{5}(d_{sn})^b - \frac{t}{R^b} L_{cn}(\bar{w}, \bar{y}, L_{4}(d))^b - \frac{t^2}{2c(R^b)} L_{cn}(\bar{w}, \bar{x}, L_{6}(d))^b \]

\[ - n(g_{44}^b - \frac{tg_{54}^b}{2cR^b}) L_{sn}(\bar{w}, \bar{y}, \bar{\psi})^b - \frac{nt}{R^b} (g_{46}^b - \frac{tg_{56}^b}{2cR^b}) \{L_{sn}(\bar{w}, \bar{x}, \bar{\psi})^b + L_{sn}(\bar{w}, \bar{y}, \bar{\chi})^b\} \]

\[ + \frac{tg_{54}^b}{R^b} \frac{L_{cn}(\bar{w}, \bar{y}, \bar{w}, \bar{\psi})^b}{2c(R^b)} + \frac{t^2 g_{56}^b}{2cR^b} \{L_{cn}(\bar{w}, \bar{x}, \bar{w}, \bar{x})^b + L_{cn}(\bar{w}, \bar{y}, \bar{w}, \bar{x})^b\} \]

\[ + \frac{t^2 g_{56}^b}{2cR^b} \{L_{cn}(\bar{w}, \bar{x}, \bar{w}, \bar{y})^b + L_{cn}(\bar{w}, \bar{x}, \bar{w}, \bar{x})^b\} + \frac{t^2 g_{54}^b}{2cR^b} \frac{L_{cn}(\bar{w}, \bar{y}, \bar{w}, \bar{x})^b}{2c(R^b)} \]

\[ + \frac{g_{y,c}^2(R^b)^2}{R^t(r_1)^2(C_8 + n^2 g_{y,c}^2 D_1)} \{r_1 v_{cn}^t - r_2 v_{cn}^t\} \]

\[ + \frac{ng_{y,c}(R^b)^2}{R^t(r_1)^2(C_8 + n^2 g_{y,c}^2 D_1)} \{\frac{t^2 D^c_{4w,s}^c - \frac{tr_1}{2R^t} D^c_{2w,s}^c - \frac{tr_2}{2R^t} D^c_{3w,s}^c}{2R^t} w_{sn}^b\} \]

\[ L_{21g(N_{s}=1)}(D) = L_{4}(d_{sn})^b - \frac{n^2 t}{2cR^b} L_{5}(d_{sn})^b - \frac{nt}{R^b} L_{cn}(\bar{w}, \bar{y}, L_{4}(d))^b - \frac{nt^2}{2c(R^b)^2} L_{cn}(\bar{w}, \bar{x}, L_{6}(d))^b \]

\[ - (g_{44}^b - \frac{n^2 tg_{54}^b}{2cR^b}) L_{sn}(\bar{w}, \bar{y}, \bar{\psi})^b - \frac{nt}{R^b} (g_{46}^b - \frac{n^2 tg_{56}^b}{2cR^b}) \{L_{sn}(\bar{w}, \bar{x}, \bar{\psi})^b + L_{sn}(\bar{w}, \bar{y}, \bar{\chi})^b\} \]

\[ + \frac{ntg_{54}^b}{R^b} \frac{L_{cn}(\bar{w}, \bar{y}, \bar{w}, \bar{x})^b}{2c(R^b)} + \frac{nt^2 g_{56}^b}{2cR^b} \{L_{cn}(\bar{w}, \bar{x}, \bar{w}, \bar{x})^b + L_{cn}(\bar{w}, \bar{y}, \bar{w}, \bar{x})^b\} \]

\[ + \frac{nt^2 g_{56}^b}{(R^b)^2} \{L_{cn}(\bar{w}, \bar{x}, \bar{w}, \bar{x})^b + L_{cn}(\bar{w}, \bar{x}, \bar{w}, \bar{y})^b\} + \frac{nt^2 g_{54}^b}{2c(R^b)^2} \frac{L_{cn}(\bar{w}, \bar{y}, \bar{w}, \bar{x})^b}{2c(R^b)^2} \]

\[ + \frac{e_{c}^2(cR^b)^2}{R^t t C_5} \{t^2 C_{15}^c - D_{10} w_{sn}^b - D_{9} w_{sn}^t\} \]

\[ + \frac{ng_{y,c} R^b}{R^t t (C_8 + n^2 g_{y,c}^2 D_1)} \{r_1 v_{cn}^t - r_2 v_{cn}^t\} \]

\[ + \frac{n^2 g_{y,c} R^b}{R^t t (C_8 + n^2 g_{y,c}^2 D_1)} \{\frac{t^2 D_{4w,s}^c - \frac{tr_1}{2R^t} D_{2w,s}^c - \frac{tr_2}{2R^t} D_{3w,s}^c}{2R^t} w_{sn}^b\} \]

\[ L_{21g(N_{s}=1)}(D) = - \frac{n}{c} L_{6}(d_{cn})^b + \frac{t}{2cR^b} L_{sn}(\bar{w}, \bar{y}, L_{6}(d))^b \]

\[ + \frac{ng_{54}^b}{c} L_{cn}(\bar{w}, \bar{y}, \bar{\psi})^b + \frac{ntg_{56}^b}{c R^b} \{L_{cn}(\bar{w}, \bar{x}, \bar{\psi})^b + L_{cn}(\bar{w}, \bar{y}, \bar{\chi})^b\} \]
Definition of the Operators in the Governing Equations

\[-\frac{t_{64}}{2cR^b} L_{6c}^b (\psi, \bar{w}_{\text{xy}}, \bar{w}_{\text{xy}}) + \frac{t_{66}^b}{2cR^b} L_{6c}^b (\chi, \bar{w}_{\text{xy}}, \bar{w}_{\text{xy}}) + L_{6c}^b (\psi, \bar{w}_{\text{xy}}, \bar{w}_{\text{xy}})\}

L_{22_1}(f) = \frac{(R^t)^2}{(R^c)^2 C_3 r_2} \hat{q}_x^c

L_{22_2}(f) = \frac{t}{R^t} L_{c0}^Q (\bar{w}_{\text{xy}}, L_4(f)) + \frac{t^2}{2c(R^t)^2} L_{c0}^Q (\bar{w}_{\text{xy}}, L_6(f)) + \frac{t}{R^t} L_{c0}^Q (\bar{w}_{\text{xy}}, n_{xy})^t

L_{22_3}(f) = L_4(f_{co})^t + 2\lambda_p^t

L_{22_4}(f) = \frac{R^t}{t} \hat{q}_x^c + \frac{t}{2cR^t} L_{c0}^Q (\bar{w}_{\text{xy}}, L_6(f))^t + L_{c0}^Q (\bar{w}_{\text{xy}}, n_{xy})^t + L_{c0}^Q (\bar{w}_{\text{xy}}, n_{xy})^t + \frac{(R^t)^2}{2(R^c)^2 C_3 r_2} \hat{q}_x^c

L_{22_5}(f) = 0

L_{22_6}(f) = -\frac{(R^t)^2}{(R^c)^2 C_3 r_1} \hat{q}_x^c

L_{22_7}(f) = \frac{t}{R^t} L_{c0}^Q (\bar{w}_{\text{xy}}, L_4(f) + \frac{t^2}{2c(R^t)^2} L_{c0}^Q (\bar{w}_{\text{xy}}, L_6(f)) + \frac{t}{R^t} L_{c0}^Q (\bar{w}_{\text{xy}}, n_{xy})^b

L_{22_8}(f) = L_4(f_{co})^b - 2\lambda_p^b

L_{22_9}(f) = \frac{R^b}{t} \hat{q}_x^b + \frac{t}{2cR^b} L_{c0}^Q (\bar{w}_{\text{xy}}, L_6(f)) + L_{c0}^Q (\bar{w}_{\text{xy}}, n_{xy})^b + L_{c0}^Q (\bar{w}_{\text{xy}}, n_{xy})^b + \frac{(R^b)^2}{2R^c C_3 r_1} \hat{q}_x^c

L_{22_{n+1}}(f) = nn^t_{xysin} + \frac{nt}{2cR^t} L_6(f_{sin})^t + \frac{(R^t)^2}{(R^c)^2 C_3 r_2} \hat{q}_x^c

L_{22_{n+3}}(f) = -nL_4(f_{cn})^t + \frac{nt}{2cR^t} L_5(f_{cn})^t - \frac{t}{R^t} L_{sn}^Q (\bar{w}_{\text{xy}}, L_4(f))^t - \frac{t^2}{2c(R^t)^2} L_{sn}^Q (\bar{w}_{\text{xy}}, L_6(f))^t - \frac{t}{R^t} L_{sn}^Q (\bar{w}_{\text{xy}}, n_{xy})^t

L_{22_{n+3}}(f) = L_4(f_{cn})^t + \frac{nt}{2cR^t} L_5(f_{cn})^t + \frac{nt}{R^t} L_{sn}^Q (\bar{w}_{\text{xy}}, n_{xy})^t + \frac{nt^2}{2c(R^t)} L_{sn}^Q (\bar{w}_{\text{xy}}, L_6(f))^t + L_{sn}^Q (\bar{w}_{\text{xy}}, n_{xy})^t + \frac{(R^t)^2}{(R^c)^2 C_3 r_2} \hat{q}_x^c

L_{22_{n+5}}(f) = 0

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\[ L_{22_{\text{a,b},n,0}}(F) = n n_{\text{b,ysn}}^b + \frac{nt}{2cR^b} L_6(f_{\text{sn}})^b - \frac{(R^b)^2}{(R^c)^2 C_3 r_1} q_{\text{xen}}^c \]

\[ L_{22_{\text{a,b},n,1}}(F) = -n L_4(f_{\text{cn}})^b + \frac{nt}{2cR} L_5(f_{\text{cn}})^b - \frac{t}{R^b} L_{\text{sn}}(\overline{w}_{s}, L_4(f))^b \]

\[ + \frac{t^2}{2c(R^c)^2} L_{\text{sn}}(\overline{w}_{s}, L_6(f))^b + \frac{t}{R^b} L_{\text{sn}}(\overline{w}_{s}, n_{sxy})^b \]

\[ L_{22_{\text{a,b},n,2}}(F) = L_4(f_{\text{cn}})^b - \frac{n^2 t}{2cR^b} L_5(f_{\text{cn}})^b + \frac{nt}{R^b} L_{\text{sn}}(\overline{w}_{s}, n_{sxy})^b \]

\[ + \frac{nt}{R^b} L_{\text{sn}}(\overline{w}_{s}, L_4(f))^b + \frac{nt^2}{2c(R^c)^2} L_{\text{sn}}(\overline{w}_{s}, L_6(f))^b \]

\[ L_{22_{\text{a,b},n,3}}(F) = \frac{R^b}{t} q_{\text{xen}}^b + \frac{n}{2cR^b} L_6(f_{\text{sn}})^b + \frac{t}{2cR^b} L_{\text{cn}}(\overline{w}_{s}, L_6(f))^b \]

\[ + L_{\text{cn}}(\overline{w}_{s}, n_{sxy})^b + L_{\text{cn}}(\overline{w}_{s}, n_{sxy})^b + \frac{(R^b)^2}{2(R^c)^2 C_3 r_1} q_{\text{xen}}^c \]

\[ L_{22_{\text{a,b},n+1,1}}(F) = -n n_{\text{ycn}}^b - \frac{nt}{2cR^t} L_6(f_{\text{cn}})^t + \frac{(R^t)^2}{(R^c)^2 C_3 r_2} q_{\text{xsn}}^c \]

\[ L_{22_{\text{a,b},n+1,2}}(F) = n L_4(f_{\text{sn}})^t - \frac{nt}{2cR^t} L_5(f_{\text{sn}})^t - \frac{t}{R^t} L_{\text{cn}}(\overline{w}_{s}, L_4(f))^t \]

\[ - \frac{t^2}{2c(R^c)^2} L_{\text{cn}}(\overline{w}_{s}, L_6(f))^t - \frac{t}{R^t} L_{\text{cn}}(\overline{w}_{s}, n_{sxy})^t \]

\[ L_{22_{\text{a,b},n+1,3}}(F) = L_4(f_{\text{sn}})^t - \frac{n^2 t}{2cR^t} L_5(f_{\text{sn}})^t - \frac{nt}{R^t} L_{\text{cn}}(\overline{w}_{s}, n_{sxy})^t \]

\[ - \frac{nt}{R^t} L_{\text{cn}}(\overline{w}_{s}, L_4(f))^t - \frac{nt^2}{2c(R^c)^2} L_{\text{cn}}(\overline{w}_{s}, L_6(f))^t \]

\[ L_{22_{\text{a,b},n+1,4}}(F) = -\frac{R^t}{t} q_{\text{xsn}}^t + \frac{n}{c} L_6(f_{\text{cn}})^t + \frac{t}{2cR^t} L_{\text{sn}}(\overline{w}_{s}, L_6(f))^t \]

\[ + L_{\text{sn}}(\overline{w}_{s}, n_{sxy})^t + L_{\text{sn}}(\overline{w}_{s}, n_{sxy})^t + \frac{(R^t)^2}{2(R^c)^2 C_3 r_2} q_{\text{xsn}}^c \]

\[ L_{22_{\text{a,b},n+1,5}}(F) = 0 \]

\[ L_{22_{\text{a,b},n+1,6}}(F) = -n n_{\text{ycn}}^b - \frac{nt}{2cR^b} L_6(f_{\text{cn}})^b - \frac{(R^b)^2}{(R^c)^2 C_3 r_1} q_{\text{xsn}}^c \]

\[ L_{22_{\text{a,b},n+1,7}}(F) = n L_4(f_{\text{cn}})^b - \frac{nt}{2cR^b} L_5(f_{\text{cn}})^b - \frac{t}{R^b} L_{\text{cn}}(\overline{w}_{s}, L_4(f))^b \]

\[ - \frac{t^2}{2c(R^c)^2} L_{\text{cn}}(\overline{w}_{s}, L_6(f))^b - \frac{t}{R^b} L_{\text{cn}}(\overline{w}_{s}, n_{sxy})^b \]
The text in the image contains mathematical equations related to the definition of operators in the governing equations. The equations are quite complex and involve multiple terms and variables. Here is a transcription of the equations as presented in the image:

\[ L_{22}^{(1,1,1,1,1)}(E) = L_4(f_{sn})^b - \frac{n^2 t}{2cR^b} L_5(f_{sn})^b - \frac{nt}{R^b} L_{cn}(\bar{w}_{xy}, n_{xy})^b \]
\[ - \frac{nt}{R^b} L_{cn}(\bar{w}_{xy}, L_4(f))^b - \frac{nt}{2c(R^b)^2} L_{cn}(\bar{w}_{xy}, L_6(f))^b \]
\[ L_{22}^{(1,2,1,1,1)}(F) = \frac{-R^b}{t} L_{sn}(\bar{w}_{xy}, n_{xy})^b + \frac{t}{2cR^b} L_{sn}(\bar{w}_{xy}, L_4(f))^b \]
\[ + L_{sn}(\bar{w}_{xy}, n_{xy})^b + L_{sn}(\bar{w}_{xy}, n_{xy})^b + \frac{(R^b)^2}{2 (R^c)^2 C_3 r_1} \]

The equations are numbered from 232 to 242.
\[ L_{2n}^{\text{nl}}(D, F) = -\frac{R_{\text{b}}^{\text{nl}}}{2t} L_{co}^Q(\psi, \chi)^t - g_{46}^b L_{co}^Q(\psi, \chi)^t \]

\[ L_{2n}^{\text{n2}}(D, F) = \frac{1}{2c} \left\{ t_{\text{b}}^Q L_{co}(\psi, \chi)^t + L_{co}^Q(\psi, \chi)^t \right\} + \frac{R_{\text{b}}^{\text{n2}}}{t} \left\{ t_{\text{b}}^Q L_{co}(\psi, \chi)^t + L_{co}^Q(\psi, \chi)^t \right\} \]

\[ + \frac{R_{\text{b}}^{\text{n2}}}{4ct} \left\{ L_{co}^C(\psi, \psi, \psi)^t - 3 \frac{t}{R_{\text{b}}} L_{co}^C(\omega_{\gamma}, \psi, \psi)^t \right\} \]

\[ + \frac{g_{66}^b}{2c} L_{co}^C(\chi, \psi, \psi)^t - \frac{t}{R_{\text{b}}} L_{co}^C(\omega_{\gamma}, \psi, \psi)^t - \frac{2}{R_{\text{b}}} L_{co}^C(\omega_{\gamma}, \chi, \psi)^t \right\} \]

\[ L_{2n+1}^{\text{n1}}(D, F) = -\frac{n g_{64}^t L_{sn}^Q(\psi, \psi)^t}{4c} - \frac{n t g_{66}^t L_{sn}^Q(\psi, \chi)^t}{2cR_{\text{b}}} \]

\[ L_{2n+2}^{\text{n1}}(D, F) = -L_{sn}^Q(\psi, L_{4}(d))^t - L_{sn}^Q(\psi, L_{4}(f))^t - \frac{t}{2cR_{\text{b}}} \left\{ L_{sn}(\chi, L_{6}(d))^t + L_{sn}(\chi, L_{6}(f))^t \right\} \]

\[ -\frac{t}{2cR_{\text{b}}} L_{sn}(\psi, \psi, \psi)^t - 3 \frac{t}{R_{\text{b}}} L_{sn}(\omega_{\gamma}, \psi, \psi)^t \]

\[ -(g_{46} + g_{64}^t)(L_{sn}^C(\chi, \psi, \psi)^t - \frac{t}{R_{\text{b}}} L_{sn}(\omega_{\gamma}, \psi, \psi)^t - 2 \frac{t}{R_{\text{b}}} L_{sn}(\omega_{\gamma}, \chi, \psi)^t \]

\[ \frac{t g_{66}^t L_{sn}(\psi, \psi, \psi)^t}{2cR_{\text{b}}} \]

\[ L_{2n+3}^{\text{n1}}(D, F) = n L_{sn}^Q(\chi, n_{xy})^t + \frac{nt}{2cR_{\text{b}}} \left\{ L_{sn}^Q(\chi, L_{6}(d))^t + L_{sn}^Q(\chi, L_{6}(f))^t \right\} \]

\[ -\frac{t}{2c} L_{sn}(\psi, \psi, \psi)^t + \frac{n t g_{66}^t}{4c} \left\{ L_{sn}^Q(\psi, \psi, \psi)^t + L_{sn}(\psi, L_{4}(d))^t + L_{sn}(\psi, L_{4}(f))^t \right\} \]

\[ -(g_{46} + g_{64}^t)(L_{sn}^C(\psi, \chi, \psi)^t - \frac{t}{R_{\text{b}}} L_{sn}(\omega_{\gamma}, \psi, \psi)^t - 2 \frac{t}{R_{\text{b}}} L_{sn}(\omega_{\gamma}, \chi, \psi)^t \]

\[ \frac{t g_{66}^t}{2cR_{\text{b}}} \]

\[ L_{2n+4}^{\text{n1}}(D, F) = \frac{1}{2c} \left\{ L_{cn}^Q(\psi, L_{6}(d))^t + L_{cn}^Q(\psi, L_{6}(f))^t \right\} + \frac{R_{\text{b}}^{\text{n1}}}{t} \left\{ L_{cn}^Q(\psi, \chi, \psi)^t + L_{cn}^Q(\psi, \chi)^t \right\} \]

\[ -\frac{n R_{\text{b}}^{\text{n1}}}{2ct} L_{cn}^Q(\psi, \psi)^t - \frac{n g_{66}^t}{4c} L_{sn}^Q(\psi, \chi)^t + \frac{R_{\text{b}}^{\text{n1}}}{4c} L_{cn}^C(\psi, \psi, \psi)^t - \frac{3}{R_{\text{b}}} L_{cn}^C(\omega_{\gamma}, \psi, \psi)^t \]

\[ + \frac{g_{66}^t}{2c} L_{cn}^C(\psi, \psi, \psi)^t - \frac{t}{R_{\text{b}}} L_{cn}(\omega_{\gamma}, \psi, \psi)^t - 2t \frac{t}{R_{\text{b}}} L_{cn}(\omega_{\gamma}, \chi, \psi)^t \]

\]

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\[ L_{2n+9}^{\text{n}}(D, F) = 0 \]
\[ L_{2n+9}^{\text{n}}(D, F) = \frac{\mathbf{g}_{\mathbf{64}}^b}{4c} L_{c(x, \psi)}^b - \frac{\mathbf{g}_{\mathbf{66}}^b}{2cR^b} \mathbf{L}_{c(x, \psi)}^b \]
\[ L_{2n+7}^{\text{n}}(D, F) = -L_{c(x, \psi)}^b L_{c(x, \psi)}^b \frac{t}{2cR^b} \left( \mathbf{L}_{c(x, \psi)}^b + \mathbf{L}_{c(x, \psi)}^b \right) \]
\[ \frac{R^b}{2t} \left( \mathbf{L}_{c(x, \psi)}^b - \frac{t}{R^b} \mathbf{L}_{c(x, \psi)}^b \right) \]
\[ \frac{g_{\mathbf{66}}^b}{4c} \mathbf{L}_{c(x, \psi)}^b - \frac{t}{R^b} \frac{\mathbf{L}_{c(x, \psi)}^b}{2cR^b} \frac{t}{R^b} \mathbf{L}_{c(x, \psi)}^b - \frac{t}{R^b} \mathbf{L}_{c(x, \psi)}^b \]
\[ L_{2n+5}^{\text{n}}(D, F) = nL_{c(x, \psi)}^b + \frac{\mathbf{g}_{\mathbf{66}}^b}{2cR^b} \left( \mathbf{L}_{c(x, \psi)}^b + \mathbf{L}_{c(x, \psi)}^b \right) \]
\[ \frac{n}{2cR^b} \left( \mathbf{L}_{c(x, \psi)}^b \right) \frac{t}{t} \mathbf{L}_{c(x, \psi)}^b \]
\[ \frac{g_{\mathbf{66}}^b}{4c} \mathbf{L}_{c(x, \psi)}^b - \frac{t}{R^b} \frac{\mathbf{L}_{c(x, \psi)}^b}{2cR^b} \frac{t}{R^b} \mathbf{L}_{c(x, \psi)}^b - \frac{t}{R^b} \mathbf{L}_{c(x, \psi)}^b \]
\[ \frac{g_{\mathbf{66}}^b}{2cR^b} \left( \mathbf{L}_{c(x, \psi)}^b \right) \frac{t}{t} \mathbf{L}_{c(x, \psi)}^b \]
\[ \frac{g_{\mathbf{66}}^t}{2cR^t} \left( \mathbf{L}_{c(x, \psi)}^t \right) \frac{t}{t} \mathbf{L}_{c(x, \psi)}^t \]

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\[
\frac{R^t g_{44}^t}{2t} \{L_{cn}^C(\psi, \psi, \psi)^t - 3 \frac{t}{R^t} \{L_{cn}^C(\bar{\omega}_{wy}, \psi, \psi)^t \}
\]

\[
-(g_{46}^t + \frac{g_{64}^t}{4c}) \{L_{cn}^C(\chi, \psi, \psi)^t - \frac{t}{R^t} \{L_{cn}^C(\bar{w}_{rx}, \psi, \psi)^t - 2 \frac{t}{R^t} \{L_{cn}^C(\bar{w}_{wy}, \chi, \psi)^t \}
\]

\[
- \frac{tg_{66}^t}{2cR^t} \{L_{cn}^C(\chi, \chi, \psi)^t - \frac{t}{R^t} \{L_{cn}^C(\bar{w}_{rx}, \chi, \psi)^t - \frac{t}{R^t} \{L_{cn}^C(\bar{w}_{wy}, \chi, \chi)^t \}
\]

\[
L_{2n, n+3}^{l1} (D, F) = -nL_{cn}^Q(\psi, n_{xy})^t - \frac{nt}{2cR^t} \{L_{cn}^Q(\chi, L_6(d))^t + L_{cn}^Q(\chi, L_6(f))^t \}
\]

\[
- n \{L_{cn}^Q(\psi, L_4(d))^t + L_{cn}^Q(\psi, L_4(f))^t \} - (\frac{R^t g_{44}^t}{2t} - \frac{n^2 g_{54}^t}{4c}) \{L_{sn}^Q(\psi, \psi)^t \}
\]

\[
- (g_{46}^t - \frac{n^2 g_{56}^t}{2cR^t}) \{L_{sn}^Q(\psi, \chi)^t - \frac{nR^t g_{44}^t}{2t} \{L_{cn}^C(\psi, \psi, \psi)^t - \frac{t}{R^t} \{L_{cn}^C(\bar{w}_{wy}, \psi, \psi)^t \}
\]

\[
- n(g_{40}^t + \frac{g_{64}^t}{4c}) \{L_{cn}^C(\chi, \psi, \psi)^t - \frac{t}{R^t} \{L_{cn}^C(\bar{w}_{rx}, \psi, \psi)^t - 2 \frac{t}{R^t} \{L_{cn}^C(\bar{w}_{wy}, \chi, \chi)^t \}
\]

\[
\frac{ntg_{66}^t}{2cR^t} \{L_{cn}^C(\chi, \chi, \psi)^t - 2 \frac{t}{R^t} \{L_{cn}^C(\bar{w}_{rx}, \chi, \psi)^t - \frac{t}{R^t} \{L_{cn}^C(\bar{w}_{wy}, \chi, \chi)^t \}
\]

\[
L_{2n, n+4}^{l1} (D, F) = \frac{1}{2c} \{L_{sn}^Q(\psi, L_6(d))^t + L_{sn}^Q(\psi, L_6(f))^t \} + \frac{R^t}{t} L_{sn}^Q(\chi, n_{x})^t + L_{sn}^Q(\psi, n_{xy})^t
\]

\[
+ \frac{nR^t g_{44}^t}{2ct} L_{cn}^Q(\psi, \psi)^t + \frac{ng_{66}^t}{c} L_{cn}^Q(\chi, \psi)^t + \frac{R^t g_{64}^t}{4ct} \{L_{sn}^C(\psi, \psi, \psi)^t - \frac{t}{R^t} \{L_{sn}^C(\bar{w}_{wy}, \psi, \psi)^t \}
\]

\[
+ \frac{g_{66}^t}{2c} \{L_{sn}^C(\chi, \chi, \psi)^t - \frac{t}{R^t} \{L_{sn}^C(\bar{w}_{rx}, \chi, \psi)^t - 2 \frac{t}{R^t} \{L_{sn}^C(\bar{w}_{wy}, \chi, \psi)^t \}
\]

\[
L_{2n, n+5}^{l1} (D, F) = 0
\]

\[
L_{2n, n+6}^{l1} (D, F) = \frac{ng_{64}^b}{4c} L_{cn}^Q(\psi, \psi)^b + \frac{ntg_{66}^b}{2cR^b} L_{cn}^Q(\psi, \chi)^b
\]

\[
L_{2n, n+7}^{l1} (D, F) = -L_{cn}^Q(\psi, L_4(d))^b - L_{cn}^Q(\psi, L_4(f))^b - \frac{t}{2cR^b} \{L_{cn}^Q(\chi, L_6(d))^b + L_{cn}^Q(\chi, L_6(f))^b \}
\]

\[
- L_{cn}^Q(\chi, n_{x})^b - \frac{R^b g_{44}^b}{2t} (g_{46}^b + \frac{tg_{54}^b}{2cR^b}) L_{sn}^Q(\psi, \psi)^b - n(g_{46}^b + \frac{tg_{56}^b}{2cR^b}) L_{sn}^Q(\psi, \psi)^b
\]

\[
- \frac{R^b g_{44}^b}{2t} \{L_{cn}^C(\psi, \psi, \psi)^b - 3 \frac{t}{R^b} L_{cn}^C(\bar{w}_{wy}, \psi, \psi)^b \}
\]

\[
- (g_{46}^b + \frac{g_{64}^b}{4c}) \{L_{cn}^C(\chi, \psi, \psi)^b - \frac{t}{R^b} L_{cn}^C(\bar{w}_{rx}, \psi, \psi)^b - 2 \frac{t}{R^b} L_{cn}^C(\bar{w}_{wy}, \chi, \psi)^b \}
\]

\[
- \frac{tg_{66}^b}{2cR^b} \{L_{cn}^C(\chi, \chi, \psi)^b - 2 \frac{t}{R^b} L_{cn}^C(\bar{w}_{rx}, \chi, \psi)^b - \frac{t}{R^b} L_{cn}^C(\bar{w}_{wy}, \chi, \chi)^b \}
\]

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\[
L_{2 \left( \frac{m}{2} \right), n \left( \frac{m}{2} \right), \psi}^{n!} (D, F) = -n L_{cb}^{Q}(\chi, n_{xy})^{b} - \frac{nt}{2cR} \{ L_{cb}^{Q}(\chi, L_{6}(d))^{b} + L_{cb}^{Q}(\chi, L_{6}(\hat{f}))^{b} \\
- n \{ L_{cn}^{Q}(\psi, L_{4}(d))^{b} + L_{cn}^{Q}(\psi, L_{4}(\hat{f}))^{b} \} - \frac{R_{bb}^{b} g_{b}^{b}}{2t} - \frac{n^{2} g_{b}^{b} g_{b}^{b}}{4c} L_{s}^{Q}(\psi, \psi)^{b} \\
- (g_{b}^{b} + \frac{n^{2} b_{b}^{b}}{2cR} L_{s}^{Q}(\psi, \psi)^{b} - \frac{R_{bb}^{b} g_{b}^{b}}{2t} L_{cn}^{C}(\psi, \psi)^{b} - 3 \frac{t}{R} L_{cn}^{C}(\psi, \psi)^{b} - \frac{t}{R} L_{cn}^{C}(\psi, \psi)^{b} - 2 \frac{t}{R} L_{cn}^{C}(\psi, \psi)^{b} \\
- n (g_{b}^{b} + \frac{g_{b}^{b}}{4c} (L_{cn}^{C}(\chi, \psi)^{b} - \frac{R_{bb}^{b} L_{cn}^{C}(\psi, \psi)^{b}}{2t} L_{cn}^{C}(\psi, \psi)^{b} - \frac{R_{bb}^{b} L_{cn}^{C}(\psi, \psi)^{b}}{2t} L_{cn}^{C}(\psi, \psi)^{b} - 2 \frac{t}{R} L_{cn}^{C}(\psi, \psi)^{b} \\
L_{2 \left( \frac{m}{2} \right), n \left( \frac{m}{2} \right), \psi}^{n!} (D, F) = \frac{L_{s}^{Q}(\psi, L_{6}(d))^{b} + L_{s}^{Q}(\psi, L_{6}(\hat{f}))^{b} \} + \frac{R_{bb}^{b} L_{s}^{Q}(\chi, n_{x})^{b} + L_{s}^{Q}(\psi, n_{xy})^{b} \\
+ \frac{R_{bb}^{b} g_{b}^{b} L_{s}^{Q}(\psi, \psi)^{b}}{2cR} + \frac{R_{bb}^{b} g_{b}^{b} L_{s}^{Q}(\psi, \psi)^{b}}{4cR} L_{s}^{C}(\psi, \psi)^{b} - \frac{R_{bb}^{b} L_{s}^{C}(\psi, \psi)^{b}}{2t} L_{s}^{C}(\psi, \psi)^{b} - 2 \frac{t}{R} L_{s}^{C}(\psi, \psi)^{b} \\
+ \frac{g_{b}^{b}}{2cR} (L_{s}^{C}(\chi, \psi)^{b} - \frac{R_{bb}^{b} L_{s}^{C}(\psi, \psi)^{b}}{2t} L_{s}^{C}(\psi, \psi)^{b} - 2 \frac{t}{R} L_{s}^{C}(\psi, \psi)^{b})
\]

where

\[
\begin{align*}
f_{co}^{j} &= \left[ n_{x,co}^{j}, m_{x,co}^{j}, n_{xy,co}^{j}, 0, 0, 0 \right] \\
f_{cn}^{j} &= \left[ n_{x,cn}^{j}, m_{x,cn}^{j}, n_{xy,cn}^{j}, 0, 0, 0 \right] \\
f_{sn}^{j} &= \left[ n_{x,sn}^{j}, m_{x,sn}^{j}, n_{xy,sn}^{j}, 0, 0, 0 \right] \\
d_{co}^{j} &= \left[ 0, 0, 0, -w_{co}^{j}, 0, 0 \right] \\
d_{cn}^{j} &= \left[ 0, 0, (n_{v,sn}^{j} - w_{cn}^{j}), \frac{t}{R} (n_{w,sn}^{j} - n v_{sn}^{j}), (n \frac{t}{R} u_{sn}^{j} + 2 n x_{sn}^{j}) \right] \\
d_{sn}^{j} &= \left[ 0, 0, (n_{v,sn}^{j} - w_{sn}^{j}), \frac{t}{R} (n_{w,sn}^{j} + n v_{sn}^{j}), (n \frac{t}{R} u_{sn}^{j} - 2 n x_{sn}^{j}) \right]
\end{align*}
\]
### Appendix B.6

#### B.6. Prebuckling and Buckling Boundary Conditions

**Table B-1. Global simply supported and clamped boundary conditions for the prebuckling state**

<table>
<thead>
<tr>
<th>Simply supported (SS3G)</th>
<th>Clamped (C4G)</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td><strong>Rigid stiffener</strong></td>
<td><strong>Rigid stiffener</strong></td>
</tr>
<tr>
<td>$n_{c0}^t + n_{c0}^b = \Lambda$; $u_{c0}^b - u_{c0}^t + \frac{(t_c^e + t)}{t} \chi_{c0}^t = 0$</td>
<td>$u_{c0}^t + u_{c0}^b = 2\Lambda$; $u_{c0}^b - u_{c0}^t + \frac{(t_c^e + t)}{t} \chi_{c0}^t = 0$</td>
</tr>
<tr>
<td>$v_{c0}^b (R_c^e - R_c^b)/R_c^t + v_{c0}^b (R_t^e - R_t^b)/R_c^b = 0$;</td>
<td>$v_{c0}^t (R_c^e - R_c^b)/R_c^t + v_{c0}^b (R_t^e - R_t^b)/R_c^b = 0$;</td>
</tr>
<tr>
<td>$v_{c0}^b R_c^t - v_{c0}^t R_c^b = 0$;</td>
<td>$v_{c0}^t R_c^t - v_{c0}^t R_c^b = 0$;</td>
</tr>
<tr>
<td>$w_{c0}^t = 0$; $w_{c0}^b - w_{c0}^t = 0$</td>
<td>$w_{c0}^t = 0$; $w_{c0}^b - w_{c0}^t = 0$</td>
</tr>
<tr>
<td>$m_{c0}^t + m_{c0}^b + \frac{(t_c^e + t)}{2t} (n_{c0}^t - n_{c0}^b) = 0$;</td>
<td>$u_{c0}^b - u_{c0}^t = 0$; $\chi_{c0}^t - \chi_{c0}^e = 0$</td>
</tr>
<tr>
<td>$\chi_{c0}^t - \chi_{c0}^e = 0$; $w_{c0}^t = 0$</td>
<td>$w_{c0}^e = 0$</td>
</tr>
</tbody>
</table>

**Table B-2. Global simply supported and clamped boundary conditions for the buckling state**

<table>
<thead>
<tr>
<th>Simply supported (SS3G)</th>
<th>Clamped (C4G)</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td><strong>Rigid stiffener</strong></td>
<td><strong>Rigid stiffener</strong></td>
</tr>
<tr>
<td>$n_{c0}^{(1)} + n_{c0}^{(1)} = 0$;</td>
<td>$u_{c0}^{(1)} + u_{c0}^{(1)} = 0$;</td>
</tr>
<tr>
<td>$u_{c0}^{(1)} - u_{c0}^{(1)} + (t^e + t)\chi_{c0}^{(1)}/t = 0$</td>
<td>$u_{c0}^{(1)} - u_{c0}^{(1)} + (t^e + t)\chi_{c0}^{(1)}/t = 0$</td>
</tr>
<tr>
<td>$v_{c0}^{(1)} (R_c^e - R_c^b)/R_c^t + v_{c0}^{(1)} (R_t^e - R_t^b)/R_c^b = 0$;</td>
<td>$v_{c0}^{(1)} (R_c^e - R_c^b)/R_c^t + v_{c0}^{(1)} (R_t^e - R_t^b)/R_c^b = 0$;</td>
</tr>
<tr>
<td>$v_{c0}^{(1)} R_c^t - v_{c0}^{(1)} R_c^b = 0$</td>
<td>$v_{c0}^{(1)} R_c^t - v_{c0}^{(1)} R_c^b = 0$</td>
</tr>
<tr>
<td>$w_{c0}^{(1)} = 0$; $w_{c0}^{(1)} - w_{c0}^{(1)} = 0$</td>
<td>$w_{c0}^{(1)} = 0$; $w_{c0}^{(1)} - w_{c0}^{(1)} = 0$</td>
</tr>
<tr>
<td>$m_{c0}^{(1)} + m_{c0}^{(1)} + (t^e + t)(n_{c0}^{(1)} - n_{c0}^{(1)})/2t = 0$;</td>
<td>$u_{c0}^{(1)} - u_{c0}^{(1)} = 0$;</td>
</tr>
<tr>
<td>$\chi_{c0}^{(1)} - \chi_{c0}^{(1)} = 0$; $w_{c0}^{(1)} = 0$</td>
<td>$\chi_{c0}^{(1)} - \chi_{c0}^{(1)} = 0$; $w_{c0}^{(1)} = 0$</td>
</tr>
</tbody>
</table>

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### Table B-3. Definition of various local boundary conditions for the prebuckling state

<table>
<thead>
<tr>
<th>SS3-free</th>
<th>C4-free</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="SS3-free Diagram" /></td>
<td><img src="image" alt="C4-free Diagram" /></td>
</tr>
</tbody>
</table>

**Top face (SS3):**
\[ n_{xc0}^i = 0; \quad v_{c0}^i = 0; \quad w_{c0}^i = 0; \quad m_{xc0}^i = 0 \]

**Bottom face (free):**
\[ n_{xc0}^b = 0; \quad n_{xyxc0}^b = 0; \quad q_{xc0}^b = 0; \quad m_{xc0}^b = 0 \]

**Core (free):**
\[ q_{xc0}^c = 0 \]

**free-SS3**

**Top face (free):**
\[ n_{xc0}^i = 0; \quad n_{xyxc0}^i = 0; \quad q_{xc0}^i = 0; \quad m_{xc0}^i = 0 \]

**Bottom face (SS3):**
\[ n_{xc0}^b = 0; \quad v_{c0}^b = 0; \quad w_{c0}^b = 0; \quad m_{xc0}^b = 0 \]

**Core (free):**
\[ q_{xc0}^c = 0 \]

**free-C4**

**Top face (free):**
\[ u_{c0}^i = 0; \quad v_{c0}^i = 0; \quad w_{c0}^i = 0; \quad \chi_{c0}^i = 0 \]

**Bottom face (free):**
\[ n_{xc0}^b = 0; \quad n_{xyxc0}^b = 0; \quad q_{xc0}^b = 0; \quad m_{xc0}^b = 0 \]

**Core (free):**
\[ q_{xc0}^c = 0 \]

For axial compression two conditions, i.e. the first condition of each face, are replaced by
\[ n_{xc0}^i + n_{xc0}^b = \lambda \Rightarrow N_x^G = \lambda \quad \text{or} \quad U^G = \Delta U \]
\[ m_{xc0}^i + m_{xc0}^b + (t^i + t)(n_{xc0}^i - n_{xc0}^b)/2t = 0 \Rightarrow M_x^G = 0 \]
Table B-4. Definition of various local boundary conditions for the buckling state

<table>
<thead>
<tr>
<th>SS3-free</th>
<th>C4-free</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Top face (SS3):</strong></td>
<td><strong>Top face (C4):</strong></td>
</tr>
<tr>
<td>$n_{xen}^{(1)} = 0; v_{sen}^{(1)} = 0;$</td>
<td>$u_{cen}^{(1)} = 0; v_{cen}^{(1)} = 0;$</td>
</tr>
<tr>
<td>$w_{cen}^{(1)} = 0; m_{xcen}^{(1)} = 0$</td>
<td>$w_{cen}^{(1)} = 0; \chi_{cen}^{(1)} = 0$</td>
</tr>
<tr>
<td><strong>Bottom face (free):</strong></td>
<td><strong>Bottom face (free):</strong></td>
</tr>
<tr>
<td>$n_{xcn}^{(1)} = 0; n_{xysn}^{(1)} = 0;$</td>
<td>$n_{xcn}^{(1)} = 0; n_{xysn}^{(1)} = 0;$</td>
</tr>
<tr>
<td>$b_{(1)}^{(1)} = b_{(1)}^{(1)}$</td>
<td>$b_{(1)}^{(1)} = b_{(1)}^{(1)}$</td>
</tr>
<tr>
<td>$q_{xcn}^{(1)} = 0; m_{xcn}^{(1)} = 0$</td>
<td>$q_{xcn}^{(1)} = 0; m_{xcn}^{(1)} = 0$</td>
</tr>
<tr>
<td><strong>Core (free):</strong></td>
<td><strong>Core (free):</strong></td>
</tr>
<tr>
<td>$q_{xcn}^{(1)} = 0$</td>
<td>$q_{xcn}^{(1)} = 0$</td>
</tr>
<tr>
<td><strong>free-SS3</strong></td>
<td><strong>free-C4</strong></td>
</tr>
<tr>
<td><strong>Top face (free):</strong></td>
<td><strong>Top face (free):</strong></td>
</tr>
<tr>
<td>$n_{xen}^{(1)} = 0; n_{xycen}^{(1)} = 0;$</td>
<td>$n_{xen}^{(1)} = 0; n_{xycen}^{(1)} = 0;$</td>
</tr>
<tr>
<td>$q_{xen}^{(1)} = 0; m_{xen}^{(1)} = 0$</td>
<td>$q_{xen}^{(1)} = 0; m_{xen}^{(1)} = 0$</td>
</tr>
<tr>
<td><strong>Bottom face (SS3):</strong></td>
<td><strong>Bottom face (C4):</strong></td>
</tr>
<tr>
<td>$n_{xcn}^{(1)} = 0; v_{cen}^{(1)} = 0;$</td>
<td>$u_{cen}^{(1)} = 0; v_{cen}^{(1)} = 0;$</td>
</tr>
<tr>
<td>$w_{cen}^{(1)} = 0; m_{xcen}^{(1)} = 0$</td>
<td>$w_{cen}^{(1)} = 0; \chi_{cen}^{(1)} = 0$</td>
</tr>
<tr>
<td><strong>Core (free):</strong></td>
<td><strong>Core (free):</strong></td>
</tr>
<tr>
<td>$q_{xcn}^{(1)} = 0$</td>
<td>$q_{xcn}^{(1)} = 0$</td>
</tr>
</tbody>
</table>
Prebuckling and Buckling Boundary Conditions
APPENDIX C

Numerical Solver

C.1. Parallel Solver Algorithm for a Block-tridiagonal Submatrices System

In this appendix, parallelization of Potter’s method for solving the matrix equations in equation (3-20) in chapter 3 and equations (6-17) and (6-19) in chapter 6 is presented. The so-called a bordered block-tridiagonal form in the last two equations differs from the standard block-tridiagonal form because of the presence of the right border column and the bottom border. However, it is possible to rewrite the bordered block-tridiagonal matrix to the standard one such that the Potter’s method [42] which exploits the special form of block-tridiagonal can be used.

\[
\begin{bmatrix}
D_0 & U_0 & \cdots & -g_0 & x_0 & y_0 \\
L_1 & D_1 & U_1 & -g_1 & x_1 & y_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
L_{N-1} & D_{N-1} & U_{N-1} & -g_{N-1} & x_{N-1} & y_{N-1} \\
L_N & D_N & -g_N & x_N & y_N \\
n_0^t & n_1^t & \cdots & n_{N-1}^t & n_N & n
\end{bmatrix}
\]

(C.1)

First, we can rewrite the system of equation (C.1) as follows

\[
Kx = y + qg
\]

(C.2)

\[
n^t x + qn = y
\]

(C.3)

where the matrix $K$ is the submatrix of coefficient matrix in equation (C.1) which is a standard block-tridiagonal matrix.

Next, solving for $x$ from equation (C.2) yields

\[
x = x^y + qx^g
\]

(C.4)

where $x^y$ and $x^g$ are the solution of

\[
K\{x^y, x^g\} = \{y, g\}
\]

(C.5)
The scalar $q$ can be solved from equation (C.3) by substituting $x$ from equation (C.4) which yields

$$q = \frac{y - n^t x^y}{n - n^t x^g} \quad (C.6)$$

Finally, the complete solution for the vector $x$ is obtained by back substitution of $q$ into equation (C.4).

Thus, the problem is reformulated such that one has to solve the sets of linear equation in equation (C.5) with a block-tridiagonal matrix as coefficient matrix, instead of a bordered block-tridiagonal matrix. To solve the standard block-tridiagonal matrix of system in equation (C.11), the parallelized Potters method will be presented here. The basic idea behind this technique is the so-called divide and conquer method of Wang [17, 61] which has been used to solve a tridiagonal system.

We consider again a block-tridiagonal system of equations. For convenience, it is assumed that $(N+1)$ can be factorized as $(N+1) = p^q k$, in which $p$ denotes the number of partition groups. For the sake of clarity we assume $p = 3$, then the block-tridiagonal matrix is partitioned as follows

$$
\begin{bmatrix}
D_0 & U_0 \\
L_1 & D_1 & U_1 \\
& \ldots & \ldots & \ldots \\
& & L_k & D_k & U_k \\
& & & L_{k+1} & D_{k+1} & U_{k+1} \\
& & & & \ldots & \ldots & \ldots \\
& & & & L_{2k} & D_{2k} & U_{2k} \\
& & & & & L_{2k+1} & D_{2k+1} & U_{2k+1} \\
& & & & & & \ldots & \ldots & \ldots \\
& & & & & & L_{3k} & D_{3k} \\
\end{bmatrix}
\begin{bmatrix}
y_0 \\
y_1 \\
\vdots \\
y_k \\
y_{k+1} \\
\vdots \\
y_{2k} \\
y_{2k+1} \\
\vdots \\
y_{3k} \\
\end{bmatrix} = \begin{bmatrix}
x \\
\vdots \\
\end{bmatrix} \quad (C.7)
$$

The parallelized solver for a tri-diagonal block submatrices system can be described by the following sequence of steps:

1. Eliminate recursively in the first group the lower diagonal submatrices $[L_2]$ to $[L_k]$, in the second group the lower diagonal submatrices $[L_{k+2}]$ to $[L_{2k}]$ and in the last group the lower diagonal submatrices $[L_{2k+2}]$ to $[L_{3k}]$. During this elimination nonzero submatrices $[F_i]$ are created, respectively below submatrix
\{D_0\}, \{D_k\} and \{D_{2k}\}. The elimination of the lower diagonal submatrices in a group is independent of the elimination process in the other groups, so, this step can be executed in parallel.

The system after this step becomes

\[
\begin{bmatrix}
D_0 & U_0 \\
F_1 & I - A_1 \\
& & \ddots \\
-F_k & E_k & U_k \\
& -F_{k+1} & I - A_{k+1} \\
& & \ddots \\
F_{2k} & E_{2k} & U_{2k} \\
& -F_{2k+1} & I - A_{2k+1} \\
& & \ddots \\
& & & \ddots \\
-F_{3k} & & & I - A_{3k} \\
& & & & E_{3k}
\end{bmatrix}
\begin{bmatrix}
x_0 \\
b_1 \\
& \ddots \\
b_k \\
& & \ddots \\
b_{k+1} \\
& & & \ddots \\
b_{2k} \\
& & & & \ddots \\
b_{2k+1} \\
& & & & & \ddots \\
b_{3k-1} \\
& & & & & & \ddots \\
b_{3k}
\end{bmatrix}
= \begin{bmatrix}
y_0 \\
b_1 \\
& \ddots \\
b_k \\
& & \ddots \\
b_{k+1} \\
& & & \ddots \\
b_{2k} \\
& & & & \ddots \\
b_{2k+1} \\
& & & & & \ddots \\
b_{3k}
\end{bmatrix}
\tag{C.8}
\]

where

\[
[F_{jk+i}] = -[D_{jk+i}]^{-1}[L_{jk+i}]
\]

\[
[A_{jk+i}] = -[D_{jk+i}]^{-1}[U_{jk+i}]
\]

\[
\{b_{jk+i}\} = [D_{jk+i}]^{-1}\{y_{jk+i}\}\quad \{i = 1 \text{ for } j = 0, 1, \ldots, 2\}
\]

\[
[F_{jk+i}] = -[[D_{jk+i} + [L_{jk+i}][A_{(jk+i)-1}]]^{-1}[L_{jk+i}][F_{(jk+i)-1}]
\]

\[
[A_{jk+i}] = -[[D_{jk+i} + [L_{jk+i}][A_{(jk+i)-1}]]^{-1}[U_{jk+i}]
\]

\[
\{b_{jk+i}\} = [[[D_{jk+i} + [L_{jk+i}][A_{(jk+i)-1}]]^{-1}\{y_{jk+i} - [L_{jk+i}][b_{(jk+i)-1}]\}
\}
\quad \{i = 2, \ldots, k-1 \text{ for } j = 0, 1, \ldots, 2\}
\]

\[
[E_{jk+k}] = [D_{jk+k} + [L_{jk+k}][A_{(jk+k)-1}]
\]

\[
[F_{jk+k}] = -[L_{jk+k}][F_{(jk+k)-1}]
\]

\[
\{b_{jk+k}\} = \{y_{jk+k}\} - [L_{jk+k}][b_{(jk+k)-1}]\quad \{j = 0, \ldots, 2\}
\]
Parallel Solver Algorithm for a Block-tridiagonal Submatrices System

2. Eliminate recursively in the first group the upper diagonal submatrices $[A_{k-2}]$ to $[u_0]$, in the second group the upper diagonal submatrices $[A_{2k-2}]$ to $[A_{k+1}]$ and in the last group the upper diagonal submatrices $[A_{3k-2}]$ to $[A_{2k+1}]$. During this elimination nonzero submatrices $[C_1]$ are created. This step can also be executed in parallel.

The system after this step becomes

\[
\begin{bmatrix}
E_0 & -C_0 \\
-B_1 & I & -C_1 \\
\vdots & \vdots & \vdots \\
-B_{k-1} & I & -C_{k-1} \\
-F_k & E_k & U_k \\
-B_{k+1} & I & -C_{k+1} \\
-B_{2k-1} & I & -C_{2k-1} \\
-F_{2k} & E_{2k} & U_{2k} \\
-B_{2k+1} & I & -C_{2k+1} \\
-B_{3k-1} & I & -C_{3k-1} \\
-F_{3k} & E_{3k} & U_{3k} \\
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
\vdots \\
x_{k-1} \\
x_k \\
\vdots \\
x_{2k-1} \\
x_{2k} \\
\vdots \\
x_{3k-1} \\
x_{3k} \\
\end{bmatrix} =
\begin{bmatrix}
b_0 \\
a_1 \\
\vdots \\
a_{k-1} \\
a_k \\
\vdots \\
a_{2k-1} \\
a_{2k} \\
\vdots \\
a_{3k-1} \\
a_{3k} \\
\end{bmatrix}
\]  

(C.9)

where

\[
[B_{jk+1}] = [F_{jk+1}] \\
[C_{jk+1}] = [A_{jk+1}] \\
\{a_{jk+1}\} = \{b_{jk+1}\} \\
\{i = (k-1) \text{ for } j = 0,\ldots, 2\}
\]

\[
[B_{jk+1}] = [F_{jk+1}] + [A_{jk+1}][B_{(jk+i)+1}] \\
[C_{jk+1}] = [A_{jk+1}][C_{(jk+i)+1}] \\
\{a_{jk+1}\} = \{b_{jk+1}\} + [A_{jk+1}][a_{(jk+i)+1}] \\
\{i = (k-2),\ldots, 1 \text{ for } j = 0,\ldots, 2\}
\]

\[
[E_0] = [D_0] + [U_0][B_1] \\
[C_0] = [U_0][C_1] \\
\{b_0\} = \{y_0\} - [U_0]\{a_1\}
\]

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3. Eliminate in the first group the upper diagonal submatrix \([U_k]\) by using the (k+1)th equations from the second group, which will create a nonzero submatrix \([U_k]\) in the first group. In the same way, the upper diagonal submatrix \([U_{2k}]\) of the second group is eliminated by using the (2k+1)th equations from the third group. Note that if the algorithm is implemented on distributed memory parallel computers, this step requires data transfers from one processor to another processor. This step can also be executed in parallel. The system after this step becomes

\[
\begin{bmatrix}
E_0 & -C_0 \\
-B_1 & I & -C_1 \\
. & . & . \\
-B_{k-1} & I & -C_{k-1} & -C_k \\
-F_k & G_k & -C_k \\
. & . & . \\
-B_{2k-1} & I & -C_{2k-1} & -C_{2k} \\
-F_{2k} & G_{2k} & -C_{2k} \\
. & . & . \\
-B_{3k} & I & -C_{3k-1} & -C_{3k} \\
-F_{3k} & E_{3k} & -C_{3k} \\
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
. \\
x_{k-1} \\
x_k \\
. \\
x_{2k-1} \\
x_{2k} \\
. \\
x_{3k-1} \\
x_{3k} \\
\end{bmatrix} = \begin{bmatrix}
b_0 \\
a_1 \\
. \\
a_{k-1} \\
a_k \\
. \\
a_{2k-1} \\
a_{2k} \\
. \\
a_{3k-1} \\
b_{3k} \\
\end{bmatrix}
\]

\[(C.10)\]

where

\[
[G_{jk}] = [E_{jk}] + [U_{jk}][F_{jk+1}]
\]

\[
[C_{jk}] = [U_{jk}][C_{jk+1}]
\]

\[
\{a_{jk}\} = \{b_{jk}\} - [U_{jk}][\{b_{jk+1}\}] \quad \{j = 1, \ldots, 2\}
\]

4. At this point, all interior unknowns are fully described in terms of boundary and/or coupling unknowns (i.e. \(x_1, x_k, x_{2k}, x_{3k}\)). These unknowns are governed by coupling and boundary equations which are uncoupled from the interior unknowns. Considering only these equations yields the following block tridiagonal system

\[
\begin{bmatrix}
E_0 & -C_0 \\
-F_k & G_k & -C_k \\
-F_{2k} & G_{2k} & -C_{2k} \\
-F_{3k} & E_{3k} & -C_{3k} \\
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_k \\
x_{2k} \\
x_{3k} \\
\end{bmatrix} = \begin{bmatrix}
b_0 \\
a_k \\
a_{2k} \\
b_{3k} \\
\end{bmatrix}
\]

\[(C.11)\]
This system of equations is solved by means of the standard Potters' method. As we already know, the Potters' method cannot be executed in parallel, but the system of equation (C.11) is much smaller than the original system of equation in equation (C.7).

5. Finally, the rest of unknowns can now easily be computed by substitution of the results from equation (C.11) into the interior equations in equation (C.10) which yields

\[ x_{jk+i} = a_{jk+i} + [B_{jk+i}]x_{jk} + [C_{jk+i}]x_{jk+k} \quad (i = 1,\ldots,(k-1) \text{ for } j = 0,\ldots,2) \quad (C.12) \]
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Curriculum Vitae

The author was born in Jakarta, Indonesia, on May 21, 1964. In 1983 he graduated from Senior High School at "Regina Pacis Bogor", Indonesia. From 1984 to 1991 he studied Aerospace Engineering at the Delft University of Technology, The Netherlands. The work described in his Master's thesis was carried out under the supervision of Prof. Dr. J. Arbocz.

In the end of 1991 he joined the Aerospace Structures and Computational Mechanics group as an Assistant in Opleiding. In 1992 when he met Prof. M Baruch, he started to work towards his doctor's degree on sandwich cylindrical shells. His works at the structures research group were supervised by Prof Dr. J. Arbocz and Prof. M. Baruch.