CHAPTER 15

DEFORMATION OF SOLITARY WAVES ON A 45-DEG SLOPE

Presented by

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This is a presentation of recent computer results from a previously published theory of the deformation of solitary waves on a sloping beach. No attempt will be made here to review or derive the pertinent equations, and only those geometrical aspects of the theory will be introduced which are necessary for interpretation of the results.

The method used here in solving Laplace's equation for the velocity potential \( \varphi \) of the steady-state solitary wave and for its free surface determined jointly by a kinematic condition on the surface particles and by Bernoulli's equation with the pressure set equal to zero is illustrated in Fig. 1. The velocity potential is first defined inside a fixed region, the infinite strip, which will contain the solitary wave as a subdomain. Neumann conditions, \( \partial \varphi / \partial n = 0 \), are assigned to both the upper and lower boundaries of this strip of width \( b \). Singularities producing the velocity field can be located anywhere between the free surface of the fluid and the upper edge of the strip; but, for convenience, the singularities of the trial potential function were constrained to the upper boundary. It was found that the simplest singularity distribution consistent with the symmetry of the

solitary-wave flow field was a dipole which exactly reproduced McCowan's velocity potential.

With the requirement of a dipole singularity, the velocity field is functionally defined but contains three arbitrary parameters: the strip width, b; the dipole velocity (to the left in the figure), -U; and the dipole strength, \( \mu \). Explicit time dependence of the free-surface boundary conditions is removed by replacing time, \( t \), with the dipole velocity and the coordinates of the dipole \( (x_0, b) \) with respect to a coordinate system fixed in the strip. Assignment of a functional form to \( \Phi \) provides sufficient information to solve the first-order partial differential equation of the kinematic free-surface condition for its integral curves, \( f(x, x_0, \eta_1; U, \mu, b) = 0 \). Similarly, a second set of algebraic curves is generated from Bernoulli's equation with \( p = 0 \). These are \( g(x, x_0, \eta_2; U, \mu, b) = 0 \). Both sets of curves are required to pass through the undisturbed free surface of depth \( h \) for \( |x| \to \infty \) and both are required to pass through a pre-assigned maximum amplitude, \( \eta_{\text{max}} \), at \( x = x_0 \). Of the three undetermined parameters only one may be arbitrarily chosen since the other two are fixed by the equations of constraint, \( f = 0 \) and \( g = 0 \).

Solution of the steady-state solitary wave is finally obtained by varying the free parameter until the distance between the curves \( \eta_1 \) and \( \eta_2 \) is minimized at the point in \( x \) of their maximum separation. It is only in this regard that this treatment of the solitary wave differs from McCowan's. But this difference is essential to accurate treatment of the higher waves.

The dimensional parameters and distance measures are not suited to either calculation or geometrical interpretation. Dimensionless forms, first introduced by Munk, are preferred. These are shown on the right in Fig. 1.

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DIRECTION OF WAVE PROPAGATION

**H-UNDISTURBED FLUID DEPTH**

**KINEMATIC CONDITION**

**BERNOULLI'S EQUATION**

**DIMENSIONAL QUANTITIES TRANSFORMED TO**

- $\mu = \text{dipole strength}$
- $U = \text{dipole speed}$
- $h = \text{undisturbed water depth}$
- $x_0, b = \text{dipole source position}$
- $x, y = \text{field point}$
- $\eta_1, \eta_2 = \text{height of kinematic and Bernoulli curve above undisturbed free surface}$

**DIMENSIONLESS QUANTITIES**

- $N = \frac{\mu \pi}{bU}$
- $F = \frac{U^2}{gh}$
- $M = \frac{\pi h}{b}$
- $X_0, Y_0 = \frac{\pi x_0}{b}, \frac{\pi}{b}$
- $X, Y = \frac{\pi x}{b}, \frac{\pi y}{b}$
- $Y_1, Y_2 = \frac{\pi}{b(\eta_1 + h)}, \frac{\pi}{b(\eta_2 + h)}$

**Fig. 1. Schematic of the Steady-State Solitary Wave**
The infinite strip is held to the fixed width, \( \pi \), and all linear dimensions are expressed as fractions of the actual strip width, \( b \), times \( \pi \). Thus,

\[
X = \frac{\pi x}{b}, \quad Y = \frac{\pi y}{b}, \quad \text{etc.}
\]

Instead of the parameters \( b \), \( \mu \), and \( U \), the new dimensionless groups

\[
M = \frac{\mu h}{b}, \quad N = \frac{\mu U}{bU}, \quad F = \frac{u^2}{gh}
\]

are most convenient and occur naturally in the equations for the free-surface boundary conditions. \( M \) is purely geometric and represents the fraction of the infinite strip occupied by the undisturbed fluid. \( F \) is purely dynamical. It is the square of the dipole Froude number and occurs only in Bernoulli's equation as expressed at the free surface. \( N \) contains all three of the original parameters. Most of the results which follow will be expressed in terms of these dimensionless parameters and the dimensionless distance measures which accompany them.

The purpose in developing this structure for the solitary wave was to permit treatment of wave deformation in consequence of changes in the geometry. Moreover, continuous deformation can be followed from a state of steady propagation in water of uniform depth. The method is illustrated in Fig. 2, where the transformation of the lines \( X = \text{const.} \) and \( Y = \text{const.} \) from the infinite strip to the shoaling domain of the 45-deg-slope-terminated strip is shown. The velocity potential of the single dipole transforms from the infinite strip domain of the symmetric wave to satisfy the same boundary conditions along rigid surfaces in the new domain. As with the steady wave, it is possible in concept to solve the kinematic equation in the physical space of the \( W \)-plane and to proceed as outlined previously for the steady wave. In fact, the shoaling domain is so chosen that when the dipole is at
Fig. 2. Transformation of the Infinite Strip to the 45-deg Slope-Terminated Strip: \( W/2 = \tan^{-1}(\tanh Z/2)^{1/4} \) - \( \tan^{-1}(\tanh Z/2)^{1/4} \)
sufficient distance from the foot of the slope the wave shape and velocity field are not influenced by the slope. Hence, the wave is in steady-state propagation. Each new choice of the dipole position \((A_o, \eta)\) closer to the slope results in a more distorted instantaneous wave shape, and a sequence of solutions for these dipole positions permits tracing the continuous deformation of the wave.

The calculations are not done in this order only because the mapping function (shown in Fig. 2) cannot be inverted for \(Z\) as an explicit function of \(W\). Instead, the kinematic differential equation and Bernoulli’s equation as written for the \(W\)-plane are transformed to the infinite strip, where the velocity field is known explicitly. Solutions for the wave shape are found in the infinite strip, and the curves are transformed point by point back into the \(W\)-plane. Complete specification of a wave form requires an input consisting of the undisturbed fluid depth, \(M\); an amplitude point, \(Y_{\text{max}}\) through which pass both a curve of the kinematic equation and Bernoulli’s free-surface equation, \((Y_{\text{max}} > M)\); and an instantaneous dipole position, \((X_o, \eta)\). It can be seen from Fig. 2 that for a small value of \(X_o\), the dipole can considerably lead the wave crest to the slope.

Figure 3 shows a wave calculated in the infinite strip from the indicated input parameters compared to its physical image in the shoaling domain. The upper curve in both cases is the solution to the kinematic equation, and the lower curve is Bernoulli’s equation with \(P = 0\). The kinematic curve will be taken throughout as the shape of the free surface. The separation between the two curves on the slope side of the strip is typical of the computer results, where – for this initial survey – no account has been taken of derivatives of \(M, N, \) and \(F\) with respect to \(X_o\), and the trial potential is restricted to that of a single dipole.

For both waves in water of uniform depth and shoaling waves, the breaking point criterion is diagrammed in Fig. 4. The velocity field of the dipole alone is anti-symmetric in the infinite strip about the line \(X = X_o\). Thus, \(v\), the vertical component of velocity is 0 along \(X = X_o\). When the translational velocity of the dipole is superimposed, it can be seen that there is
Fig. 3. Comparison of Wave Shape As Computed in the Infinite Strip With Its Physical Image in the Slope-Terminated Strip
**Fig. 4.** Determination of the Breaking Point as the Critical Point of the Dipole Field
a point at which the horizontal component of the dipole field cancels the translational velocity to give $u = 0$. When the free surface of a wave contains this critical point ($u = 0, v = 0$) the surface is cusped, and this wave will be said to be breaking. This theory cannot follow the wave motion beyond the breaking state.

The position of the critical point in the infinite strip is a function only of $N$ as given by the equation in Fig. 5. The entire sequence of solitary waves in water of uniform depth is shown as the locus of points labelled according to their maximum amplitude divided by the undisturbed-depth ratio, $\eta_{\text{max}}/h$. These waves are called stable in the illustration, although the correctness of this usage may be in doubt for waves of amplitude greater than about 70 percent of the undisturbed fluid depth, as will be discussed. From this diagram the highest wave can be estimated at about 88 percent of the fluid depth as calculated from the single-dipole approximation.

At this point it is desirable to digress into some computed results for steady-state waves, not only for their intrinsic interest but also because fluid motions on the 45-deg slope can be identified as shoaling solitary waves only by their connection with stable waves through conservation of volume and total energy.

Figure 6 shows a typical solitary wave in water of uniform depth, the wave computed by varying $M$ until the separation between the two independent curves of the boundary conditions was minimized. Experimental data points from Daily and Stephan\(^5\) are shown for comparison. The vertical scale is relative wave height, but the horizontal scale is in the dimensionless units of $\pi x/b$. If the undisturbed water depth is taken as 1 ft, then the horizontal scale can be expressed in feet by $x(\text{ft}) = X/M$.

The error or degree of departure from an exact solution can best be expressed by a table of the pressure deviation from zero along the curve of the

Fig. 5. Critical Point Curve

\[ Y_C = \pi - \cos^{-1}(1 - N) \]
Fig. 6. Free Surface Profile of a Stable Solitary Wave: \( \eta_{\text{max}}/h = 0.6 \)
kinematic condition taken as the representation of the free surface. This is shown in the table of Fig. 7, where the maximum absolute pressure decrement in units of $P/\rho gh$ is 0.0090. This maximum deviation decreases with decreasing wave height and is somewhat larger for higher waves up to the point of breaking.

Figure 8 shows the square of the Froude member plotted against relative maximum wave height. Again, experimental data from Daily and Stephan are shown for comparison. Results of McCowan are also shown. Though his equations were identical to these in the single-dipole approximation, the poor convergence of the series expansions he used to evaluate the parameters accounts for the discrepancy between the curves.

Figure 9 depicts three of the highest waves. The dimensionless scale of $(X, Y)$ is maintained since in this scale the undisturbed water depth is given by the parameter $M$. This permits separation of the wave shapes in the figure in proportion to the difference in values of $M$ appropriate to each. In consonance with the critical-point curve discussed earlier, the wave (whose amplitude is 88.5 percent of the fluid depth) has a cusp formed at the crest; but unlike the waves of even slightly smaller amplitude, the separation between the kinetic and Bernoulli curves is not small. This wave appears to actually extend beyond the critical point because the cusp angle is less than the 90 deg known to occur at the critical point of a dipole field.

It will also be noticed that as the waves get higher, they become more slender without appreciable change of width at the base. As a result, the parameters, though single-valued in relative wave height, appear to be double-valued when plotted against the volume of fluid above the undisturbed free surface (see Fig. 10). A solitary wave higher than about 70 to 72 percent of the undisturbed depth shares the same fluid volume but possesses more energy than another wave of amplitude ratio less than 0.70. These results are consistent with experimental observation in that no stable solitary wave has been observed of amplitude ratio greater than 0.72, nor has the cusped wave ever
### SINGLE-DIPOLE APPROXIMATION

<table>
<thead>
<tr>
<th>( \eta/h )</th>
<th>Static Pressure at the Free Surface (( P/\rho g h ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.600</td>
<td>0.0</td>
</tr>
<tr>
<td>0.580</td>
<td>-0.0020</td>
</tr>
<tr>
<td>0.530</td>
<td>-0.0057</td>
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<tr>
<td>0.467</td>
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<tr>
<td>0.434</td>
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<tr>
<td>0.403</td>
<td>-0.0090</td>
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<tr>
<td>0.343</td>
<td>-0.0074</td>
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<tr>
<td>0.290</td>
<td>-0.0050</td>
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<td>-0.0019</td>
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<tr>
<td>0.203</td>
<td>+0.0008</td>
</tr>
<tr>
<td>0.169</td>
<td>+0.0027</td>
</tr>
</tbody>
</table>

Fig. 7. Pressure Decrement at the Free Surface of the Stable Solitary Wave:  \( \eta_{max}/h = 0.60 \)
Fig. 8. $F(\theta^2/g) \text{ vs Maximum Relative Wave Height for Stable Waves}$
\[ \eta_{\text{max}} = \frac{Y}{X} - 1 \]

**Fig. 9. The Highest Stable Waves in the Single-Dipole Approximation**
Fig. 10. Stable Wave Parameters
been observed. In spite of the consistently low pressure decrement at the calculated wave surfaces up to amplitude ratios of 0.80, there still remains the possibility that a still higher order of approximation to the wave might cause the curves to become single-valued in the volume up to the breaking point. But in any case, the curve of $\frac{\eta_{\text{max}}}{h}$ would be steep, and the question of stable existence, even with careful generation, of wave amplitude ratios greater than 0.72 would still be present.

It is now possible to return to solitary waves deforming on the 45-deg slope and to discuss their properties in terms of a connection to stable waves through conservation of total fluid volume. As mentioned earlier, the process of computing a deformed wave requires specifying a dipole position, $(X_0, \pi)$; a dimensionless undisturbed fluid depth, $M$; and a point $(X_0, Y_{\text{max}})$ to lie in the free surface. Computations are all done in the infinite strip. The symmetry of the stable waves caused the kinematic and Bernoulli curves to share three points, $(X_0, Y_{\text{max}})$ and $(\pm \infty, M)$. In deformed waves only two points are common to both curves $(X_0, Y_{\text{max}})$ and $(\pm \infty, M)$. Thus, variation in $M$ does not produce minimum separation between the free-surface curves of a particular wave, but instead generates new wave forms. To re-establish a minimizing procedure, new parameters would have to be introduced, either by adding another dipole to the specification of the trial potential function or by taking into account the derivative terms $\partial M/\partial X_0$, $\partial N/\partial X_0$, and $\partial F/\partial X_0$.

Figure 11 is a three-dimensional plot of all the parameters for the particular dipole position of $X_0 = 1.0$ — being almost directly above the origin in the $W$-plane, as Fig. 2 will substantiate. The straight constant $Y_{\text{max}}$ generate a surface which contains specification of humped wave motions at this dipole location. The breaking line in the crest of this surface, dividing it into two parts, and its projection in the $N, Y_{\text{max}}$ plane is the critical curve of Fig. 5. All fluid motions are contained on the left side of this breaking line. Identification of a deformed wave as being the image of a particular steady-state solitary wave requires matching both total energy and volume. To date, only very
Fig. 11. Diagram of the Parameters for the Case X₀ = 1.0
roughly, the iso-energy lines, then those points of intersection with the iso-volume lines which correspond to the same properties of a stable wave form a single curve on the parameter surface. This, then, would be the curve of parameters for deformed solitary waves. Any point of the surface not on this line represents a fluid motion whose history cannot be traced to a solitary wave. For illustration the next two figures constitute a set of nine wave forms calculated from the parameters along a line of roughly constant $N$. They are actually connected by constancy of the amplitude-to-depth ratio in the infinite strip, viz., $\frac{\tau_{\text{max}}}{h} = \frac{Y_{\text{max}}}{M - 1} = 0.25$. Along such a line in the parameter surface, the wave forms typically possess large volume and energy but low wave-crest amplification factors relative to stable waves of identical fluid volume when $M$ is small. As $M$ increases, energy and volume decrease while the amplification factor increases, as shown in Figs. 12 and 13. These waves were actually computed for a dipole position of $X_0 = 2.5$, but the qualitative behavior of the wave forms is the same for any dipole position.

Finally, in Fig. 14, two waves are shown superimposed at a dipole position of $X_0 = 1.0$. These are not necessarily solitary waves, but they demonstrate the characteristic wave shape near the breaking point. The kinematic and Bernoulli curves of the larger wave, shown breaking, are solid, while the curves for the smaller wave, just short of breaking, are dotted.
Fig. 12. Wave Forms at $X_0 = 2.5$
Fig. 13. Wave Forms at \( x_0 = 2.5 \)