SEISMIC INVERSION
BY A RMS BORN APPROXIMATION
IN THE SPACE–TIME DOMAIN

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ABSTRACT

A new approximate method to calculate the space–time acoustic wave motion generated by an impulsive point source in a horizontally layered configuration is presented. The configuration consists of a stack of fluid layers between two acoustic half-spaces where the source and the receiver are located in the upper half-space. A distorted-wave Born approximation is introduced; the important feature of the method is the assumption of a background medium with vertical varying root-mean-square acoustic wave speed. A closed-form expression for the scattered field in space and time as a function of the contrast parameters is deduced. The result agrees closely with rigorously calculated synthetic seismograms. In the inverse scheme the wave speed and mass density can be reconstructed within a single trace. Results of the inversion scheme applied to synthetic data are shown.

1. INTRODUCTION
The ordinary Born approximation is widely used to simplify both forward and inverse problems of wave propagation models for seismic exploration (Cohen and Bleistein 1977, 1979; Phinney and Frazer, paper presented at 48th SEG meeting, San Francisco, 1978; Raz 1981a, b). Within this Born approximation, the solution of the acoustic wave equation is expressed as a perturbation about a known solution to the simple equations of a homogeneous background medium. The Born approximation is a low-contrast approximation in the integral equation governing the total scattering mechanism of the acoustic waves in the configuration. We derive a closed-

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form expression in the space–time domain. However, the inherent use of a constant background medium leads to incorrect arrival times.

In the distorted-wave approximation (Beylkin and Oristaglio 1985) we assume a more realistic inhomogeneous background medium. The field solution of a point source is then often difficult to obtain and complicated inversion schemes result (Clayton and Stolt 1981; Foster and Carrion 1984; Bleistein and Gray 1985; Weglein, Violette and Keho 1986).

Here we deal with a horizontally layered configuration and we take an inhomogeneous background medium such that a simple approximate solution to the problem of a point-source in this background medium is obtained. The precise assumption of the structure of the background medium is circumvented by deriving a closed-form low-contrast approximation of the analytical expression in the Laplace-Fourier transform domain of the Cagniard–de Hoop technique (de Hoop 1960; Aki and Richards 1980) with primaries only (Drijkonigen and Fokkema 1987). It appears that we have arrived at a distorted-wave Born approximation with some vertically varying root-mean-square acoustic wave speed. Some numerical results of the present forward problem are presented and compared with the results of the exact Cagniard–de Hoop technique.

Further, we show that our rms Born approximation leads to a simple inversion scheme, where the two constitutive parameters, the mass density and the wave speed, of the fluid layers can simultaneously be reconstructed within a single trace. We finally present some results of the inversion scheme applied to synthetic data of the Cagniard–de Hoop technique.

2. Description of Configuration

The Cartesian coordinates \( \{x_1, x_2, x_3\} \) locate a point in space, while \( t \) represents the time of observation. The summation convention applies to repeated subscripts. Partial differentiation with respect to \( x_i \) is denoted by \( \partial_i \); the symbol \( \partial_t \) denotes the partial derivative with respect to time.

We consider a horizontally stratified linear acoustic medium in the vertical \( x_3 \)-direction. The acoustic properties of the configuration are characterized by its volume density of mass \( \rho \) and its compressibility \( \kappa \). Both \( \rho \) and \( \kappa \) are functions of \( x_3 \). They are independent of the horizontal coordinates \( x_1, x_2 \), and the time coordinate \( t \). The functions \( \rho = \rho(x_3) \) and \( \kappa = \kappa(x_3) \) are taken piecewise constant (Fig. 1).

The related acoustic wave speed is given by

\[
c = (\rho c)^{-1/2} > 0.
\]

(1)

An impulsively point source at \( \{0, 0, x_3^{(0)}\} \) generates the acoustic waves. This source starts to act at \( t = 0 \). A receiver is located at \( \{x_1^{(r)}, x_2^{(r)}, x_3^{(r)}\} \). We consider the case that both source and receiver are located in the upper homogeneous half-space with \( \rho = \rho_0, \kappa = \kappa_0 \) and \( c = c_0 \) (Fig. 1). The latter model is important in land and marine seismics.
3. Basic Acoustic Wave Equations

The equations that govern the linearized acoustic wave motion in a fluid are the equation of motion

$$\partial_t p + \rho \partial_t v_k = f_k^i,$$  \hspace{1cm} (2)

and the deformation equation

$$\partial_t v_k + \kappa \partial_t p = q^i.$$  \hspace{1cm} (3)

In these equations $p$ is the acoustic pressure, $v_k$ is the particle velocity, $f_k^i$ is the volume source density of force, and $q^i$ is the volume density of external volume injection rate. We consider a point source located at \{0, 0, x_3^{(S)}\} with vertical force strength $F(0)$ and injection rate strength $Q(i)$. Hence,

$$f_1^i = 0, f_2^i = 0, f_3^i = F(i)\delta(x_1, x_2, x_3 - x_3^{(S)}),$$  \hspace{1cm} (4)

$$q^i = Q(i)\delta(x_1, x_2, x_3 - x_3^{(S)}).$$  \hspace{1cm} (5)
where the source-strength \( \{F(t), Q(t)\} \) is zero when \( t < 0 \). In the analysis of the solution of (2)-(5), we take advantage of the invariance of the configuration with respect to time and with respect to the horizontal coordinates.

4. Transformed Wave Equations

Using the one-sided Laplace transform with respect to time

\[
\tilde{f}(x_1, x_2, x_3, s) = \int_0^\infty f(x_1, x_2, x_3, t) \exp(-st) \, dt, \quad s > 0, \tag{6}
\]

and subsequently a Fourier transform with respect to the horizontal coordinates

\[
\tilde{f}(\alpha_1, \alpha_2, x_3, s) = \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} f(x_1, x_2, x_3, s) \exp\left[i\sigma_1 x_1 + \alpha_2 x_3\right] \, dx_1 \tag{7}
\]

with the inverse Fourier transform

\[
\tilde{f}(x_1, x_2, x_3, s) = \left(\frac{s}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\alpha_2 \int_{-\infty}^{\infty} \tilde{f}(\alpha_1, \alpha_2, x_3, s) \exp\left[-i\sigma_1 x_1 + \alpha_2 x_3\right] \, d\alpha_1, \tag{8}
\]

lead to the transformed acoustic wave equations for the fundamental acoustic quantities \( \tilde{p} \) and \( \tilde{v}_3 \)

\[
\partial_3 \tilde{p} + s \rho \tilde{v}_3 = \tilde{f}_3 \tag{9}
\]

and

\[
\partial_3 \tilde{v}_3 + s \gamma^2 \rho^{-1} \tilde{p} = \tilde{q}_3, \tag{10}
\]

in which \( \gamma = \gamma(x_3) \) is the vertical slowness given as

\[
\gamma = \left(c^{-2} + \alpha_1^2 + \alpha_2^2\right)^{1/2} > 0, \tag{11}
\]

and \( c = c(x_3) \) is the acoustic wave speed. In our transformed equations, the source distributions are the 'force'

\[
\tilde{f}_3 = \tilde{F}(s)\delta(x_3 - x_3^0), \tag{12}
\]

and the 'volume injection'

\[
\tilde{q}_3 = \tilde{Q}(s)\delta(x_3 - x_3^0), \tag{13}
\]

Next we consider some approximate methods to solve our transformed acoustic wave equations in the layered configuration at hand.

5. Contrast Sources in a Background Medium

We first observe that the total field \( \{\tilde{p}, \tilde{v}_3\} \) can be written as the sum of two field constituents:

\[
\tilde{p} = \tilde{p}' + \tilde{p}^*, \quad \tilde{v}_3 = \tilde{v}_3^* + \tilde{v}_3^*, \tag{14}
\]
where \{\bar{p}', \bar{v}'_3\} are the transformed quantities of the incident field in the (in)homogeneous background medium with \(\rho_B(x_3), \kappa_B(x_3)\), and where \{\bar{p}^s, \bar{v}^s_3\} denote the transformed quantities of the scattered field. The incident field satisfies
\[
\partial_3 \bar{p}' + s \rho_B \bar{v}'_3 = \bar{f}'_3
\]  
and
\[
\partial_3 \bar{v}'_3 + s \gamma_B \rho_B^{-1} \bar{p}' = \bar{q}',
\]
where \(\gamma_B = \gamma_B(x_3)\) is the vertical slowness in the background medium, given as
\[
\gamma_B = \left( c_B^{-2} + c_0^2 + \alpha_0^2 \right)^{1/2} > 0,
\]
and \(c_B = c_B(x_3)\) is the wave speed in the background medium. When the total field satisfies (9)-(10), the scattered field satisfies
\[
\partial_3 \bar{p}^s + s \rho_B \bar{v}^s_3 = \bar{f}^s_3
\]
and
\[
\partial_3 \bar{v}^s_3 + s \gamma_B \rho_B^{-1} \bar{p}^s = \bar{q}^s,
\]
in which the scattered field is understood to be excited in the background medium by the contrast source distributions
\[
\bar{f}^s_3 = -s(\rho - \rho_B) \bar{v}^s_3
\]
and
\[
\bar{q}^s = -s(\gamma_B \rho^{-1} - \gamma_B \rho_B^{-1}) \bar{p}.
\]
Note that in the expressions for the contrast source distributions of (20) and (21) the total field quantities \(\bar{p}\) and \(\bar{v}_3\) occur. From (18) and (19), the scattered acoustic pressure is obtained as
\[
\bar{p}(x_3) = \int_{-\infty}^{\infty} \bar{f}^s_3(x_3, x'_3) \text{d}x'_3 + \int_{-\infty}^{\infty} \bar{q}^s(x_3, x'_3) \text{d}x'_3,
\]
and
\[
\bar{v}^s_3(x_3) = \int_{-\infty}^{\infty} \bar{f}^s_3(x_3, x'_3) \text{d}x'_3 + \int_{-\infty}^{\infty} \bar{q}^s(x_3, x'_3) \text{d}x'_3,
\]
where \{\bar{p}^s, \bar{v}^s_3\} is the acoustic wavefield excited by a unit force in the \(x_3\)-direction at \(x_3 = x'_3\) and \{\bar{p}', \bar{v}'_3\} is the acoustic wavefield excited by a unit volume injection source at \(x_3 = x'_3\). These unit source fields satisfy
\[
\partial_3 \bar{p}' + s \rho_B \bar{v}'_3 = \delta(x_3 - x'_3),
\]
\[
\partial_3 \bar{v}'_3 + s \gamma_B \rho_B^{-1} \bar{p}' = 0,
\]
and
\[
\partial_3 \bar{p}^s + s \rho_B \bar{v}^s_3 = 0,
\]
\[
\partial_3 \bar{v}^s_3 + s \gamma_B \rho_B^{-1} \bar{p}^s = \delta(x_3 - x'_3).
\]
So far we have dealt with a rigorous analysis. As soon as the unit source solutions \( \{ \hat{p}_0, \hat{b}_0 \} \) of (15)--(16), \( \{ \hat{p}_s, \hat{b}_s \} \) of (24)--(25), and \( \{ \hat{p}_b, \hat{b}_b \} \) of (26)--(27) are calculated, we can take the total field in (20)--(21) to be the incident field and calculate the scattered field from (22)--(23). This procedure is known as the Born approximation.

6. First Born Approximation

In the First Born approximation we take the background medium to be the homogeneous medium with

\[
\rho_b = \rho_0, \quad c_b = c_0.
\]

The incident field \( \{ \hat{p}_i, \hat{b}_i \} \) in this homogeneous background medium is obtained as

\[
\hat{p}_i(x_3) = (2\gamma_0)^{-1}[\gamma_0 F + \rho_0 \hat{Q}] \exp \left[ -\gamma_0 x_3 - x_3^B \right],
\]

\[
\hat{b}_i(x_3) = (\gamma_0/\rho_0) \hat{p}_i(x_3),
\]

when \( x_3 > x_3^B \), and the unit-source solutions \( \{ \hat{p}_s, \hat{b}_s \} \) with source location at \( x_3 = x_3^B \) and observer location at \( x_3 = x_3^B \) as

\[
\hat{p}_s(x_3^B, x_3') = -(1/2) \exp \left[ \gamma_0 (x_3^B - x_3') \right],
\]

\[
\hat{b}_s(x_3^B, x_3') = -(\gamma_0/\rho_0) \hat{p}_s(x_3^B, x_3'),
\]

\[
\hat{p}_b(x_3^B, x_3') = (\rho_0/2\gamma_0) \exp \left[ \gamma_0 (x_3^B - x_3') \right],
\]

\[
\hat{b}_b(x_3^B, x_3') = -(\gamma_0/\rho_0) \hat{p}_b(x_3^B, x_3'),
\]

when \( x_3^B < x_3 \). Subsequently, we take in the contrast source distributions of (20)--(21) the total field to be equal to the incident field. This approximation is valid for small values of \( \rho - \rho_0 \) and \( c^{-2} - c_0^{-2} \) only. We then have

\[
\hat{p}_r(x_3^B) = \frac{s}{4} \int_{x_3^B}^{\infty} (\gamma_0 \hat{F} + \rho_0 \hat{Q}) \frac{\gamma_0\rho_0^{-1} - \gamma_0^2 \rho^{-1} \rho_0}{\gamma_0^2} \exp \left[ -\gamma_0 z \right] \, dx_3,
\]

\[
\hat{b}_r(x_3^B) = \frac{s}{4} \int_{x_3^B}^{\infty} (\gamma_0 \rho_0^{-1} \hat{F} + \hat{Q}) \frac{\gamma_0^2 \rho_0^{-1} - \gamma_0^2 \rho^{-1} \rho_0}{\gamma_0} \exp \left[ -\gamma_0 z \right] \, dx_3,
\]

in which \( \gamma_0 = (\rho_0^{-2} + \rho^{-2} + 2\rho_0^{-1} \rho)^{1/2} \) and the vertical travel distances \( z = z(x_3') \) between receiver and source via the reflection point \( x_3' \) is given by

\[
z = 2x_3' - x_3^B - x_3^B.
\]

Note that the integrals start at \( x_3^B \), since the actual medium is identical to the background medium for values of \( x_3 \) less than \( x_3^B \).

7. Space-Time Domain Results for a Volume Injection Source

In the following, we only consider the point source of the volume injection type, i.e. \( F = 0, \hat{Q} \neq 0 \). This type of source is of importance in marine seismics. We then
measure the pressure \( p \) only. We therefore continue our analysis with (25). For our configuration (Fig. 1), the integral of (35) can be calculated analytically. We then obtain

\[
\left( \frac{s}{2\pi} \right)^2 \tilde{p}(x_1, x_2, x_3, s) = \rho_0 s^2 \tilde{Q}(s) \tilde{G}(x_1, x_2, x_3, s),
\]

(38)
in which the spectral representation of Green's function \( \tilde{G}(x_1, x_2, x_3, s) \) is given by contributions of the interfaces \( x_3 = x_3^{(n)}, n = 0, 1, 2, \ldots, N, \) as

\[
\tilde{G} = \frac{1}{8\pi^2} \sum_{n=0}^N \left[ A_n + \frac{B_n}{4\gamma_0^2} \right] \exp \left[ -s\gamma_0 z_n \right]
\]

(39)
with

\[
z_n = z(x_3^{(n)}) = 2x_3^{(n)} - x_3^{(n-1)} - x_3^{(n+1)},
\]

(40)
while \( A_n \) and \( B_n \) are given by

\[
A_n = \frac{1}{4} (\rho_{n+1} \rho_{n-1} - \rho_n \rho_{n+1}^{-1}) - \frac{1}{4} (\rho_n \rho_{n-1} - \rho_0 \rho_{n+1}^{-1}) \approx \frac{\rho_{n+1} - \rho_n}{\rho_{n+1} + \rho_n}
\]

(41)
and

\[
B_n = (c_n^{-2} - c_0^{-2})\rho_0 \rho_n^{-1} - (c_{n+1}^{-2} - c_0^{-2})\rho_0 \rho_{n+1}^{-1} \approx c_n^{-2} - c_{n+1}^{-2}.
\]

(42)
The approximations made in (41) and (42) are consistent with the low-contrast approximations of the Born approximation. Note that the term \( A_n + B_n(2\gamma_0)^{-2} \) in (39) is the low-contrast approximation of the reflection factor of an acoustic wave in our transform domain. This reflection factor with respect to an interface at \( x_3^{(n)} \) is given by

\[
\Gamma_n = \frac{\gamma_n \rho_n^{-1} - \gamma_{n+1} \rho_{n+1}^{-1}}{\gamma_n \rho_n^{-1} + \gamma_{n+1} \rho_{n+1}^{-1}},
\]

(43)
in which

\[
\gamma_n = (c_n^{-2} + a_1^2 + a_2^2)^{1/2} = \gamma_0 \left( 1 + \frac{c_n^{-2} - c_0^{-2}}{\gamma_0^2} \right)^{1/2} \approx \gamma_0 \left( 1 + \frac{c_n^{-2} - c_0^{-2}}{2\gamma_0^2} \right).
\]

(44)
In the last expression of (44), we have used a low-contrast approximation. Using this approximation in the expression for the reflection factor, we obtain the low-contrast approximation for the reflection factor as

\[
\Gamma_n \approx \frac{\rho_{n+1} - \rho_n}{\rho_{n+1} + \rho_n} + \frac{c_n^{-2} - c_{n+1}^{-2}}{4\gamma_0^2} = A_n + \frac{B_n}{4\gamma_0^2}
\]

(45)
From (39) and (45), we observe the well-known feature that the first Born approximation takes into account the primary reflections only. Further, we remark that the approximation of the reflection factor given by (45) is only valid when \( \gamma_{n+1} \) is a regular function of \( \alpha_1 \) and \( \alpha_2 \). This excludes the possibility of head-wave occurrence. The latter waves only occur for large offsets, restricting the admissible values of \( r \). This is again consistent with the Born approximation.
Table I. The location of the interfaces, mass density and wave speed of the layers for the nine-layer configuration.

<table>
<thead>
<tr>
<th>n</th>
<th>$x_2^{(n)}$ (m)</th>
<th>$\rho_n$ (kg/m$^3$)</th>
<th>$c_n$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>70</td>
<td>1000</td>
<td>1500</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>1010</td>
<td>1600</td>
</tr>
<tr>
<td>2</td>
<td>135</td>
<td>1200</td>
<td>1700</td>
</tr>
<tr>
<td>3</td>
<td>175</td>
<td>1200</td>
<td>1800</td>
</tr>
<tr>
<td>4</td>
<td>210</td>
<td>1250</td>
<td>1700</td>
</tr>
<tr>
<td>5</td>
<td>260</td>
<td>1150</td>
<td>1600</td>
</tr>
<tr>
<td>6</td>
<td>340</td>
<td>1200</td>
<td>1900</td>
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<td>7</td>
<td>415</td>
<td>1300</td>
<td>2000</td>
</tr>
<tr>
<td>8</td>
<td>—</td>
<td>1500</td>
<td>2200</td>
</tr>
</tbody>
</table>

In order to arrive at the space–time domain representations, we use the results of Appendix A, namely

$$p^{(m)} = \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} \exp \left[ -\frac{s y_2^0 x_2}{2n y_0^2} \right] \exp \left[ -is(x_1 x_1 + x_2 x_2) \right] dx_1$$

$$= \int_{R_n}^{\infty} \frac{\exp \left[ -\frac{s t}{R_n} \right] \theta_{R_n} dt_n}{R_n}$$

(46)

where

$$\theta_n^{(1)} = 1,$$

$$\theta_n^{(2)} = R_n z_n t(t^2 - r^2/c_0^2)^{-3/2},$$

(47)

in which the horizontal offset $r$ between receiver and source is defined as

$$r = [((y_R^0)^2 + (y_0^0)^2)^{1/2} > 0$$

(48)

and the travel distance $R_n$ between receiver and source via the reflection point $x_3^{(n)}$ as

$$R_n = [r^2 + z_n^2]^{1/2} > 0.$$  

(49)

The time-domain representation of the scattered field $p^s$ is then recognized as

$$p^s(r, x_3^{(n)}, t) = \rho_0 \alpha_0^2 \theta_0^3 [Q(t)^*G(r, x_3^{(n)}, t)],$$

(50)

in which $*$ denotes the convolution and the space–time Green's function $G = G(r, x_3^{(n)}, t)$ denotes the impulse response of the system and is given by

$$G = \frac{1}{4\pi} \sum_{n=0}^{\infty} \left[ \frac{A_n}{R_n} + \frac{B_n x_1^0}{4(t^2 - r^2/c_0^2)^{3/2}} \right] H(t - R_n/c_0),$$

(51)

where $H$ is the unit step function defined by

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$$

(52)
Fig. 2. The first Born approximation of the space–time Green’s function for the nine-layer configuration of Table 1; $x_3^{(0)} = x_3^{(R)} = 7.5$ m and $r = 100$ m and $50$ m, respectively.

Now we shall present some numerical results for the nine-layer configuration of Table 1. For this configuration, Drijkoningen and Fokkema (1987) have computed the Green’s function of (51) by using the exact Cagniard–de Hoop technique with only primary reflections. We shall compare our approximate results with the ones obtained from this Cagniard–de Hoop technique. In Fig. 2, we present the results of the first Born approximation for two different values of the offset $r$. The dashed lines represent the exact results of the Cagniard–de Hoop technique with primaries only. We observe that the results of the Born approximation do not fit the arrival times very well.

8. RMS Born Approximation

In order to improve the Born approximation, we should use a more realistic background medium. The disadvantage of using a more realistic background medium is
the complication of the field solution. Therefore, we avoid the precise assumption of a realistic background medium. As a starting point, we have the solution of the Cagniard–de Hoop technique with primary reflection only. The Green’s function is obtained as (Drijkoningen and Fokkema 1987)

$$G = \frac{1}{8\pi^2} \sum_{n=0}^{N} \sum_{m=0}^{n} \exp \left[ -s \sum_{m=0}^{n} \gamma_m h_m \right],$$  

(53)

in which the reflection factor $\Gamma_n$ is given by (43) and where $h_m$ denotes the vertical wave path that the acoustic wave has transversed in the layer with wave speed $c_m$. Note that in (53), we have taken the transmission factors of all interfaces equal to 1. This approximation is consistent with the low-contrast approximations. The total vertical geometrical path is given by

$$z_n = \sum_{m=0}^{n} h_m,$$  

(54)

where

$$h_0 = 2(x_3^{(0)} - x_3^{(B)}),$$
$$h_m = 2(x_3^{(m)} - x_3^{(m-1)}), \ m = 1, 2, 3, \ldots.$$  

(55)

Further, we have an upper bound and a lower bound for the vertical wave propagation term $\sum_n \gamma_n h_n$. For real $\alpha_1$ and $\alpha_2$, we are able to write (cf. Appendix B)

$$(c_s^{-2} + \alpha_1^2 + \alpha_2^2)^{1/2} z_n \leq \sum_{m=0}^{n} \gamma_m h_m \leq (c_s^{-2} + \alpha_1^2 + \alpha_2^2)^{1/2} z_n,$$  

(56)

where $c_s'$ is the root-mean-square wave speed defined as

$$c_s' = \left( \sum_{m=0}^{n} c_m h_m \right)^{1/2} \left( \sum_{m=0}^{n} c_m^{-1} h_m \right)^{-1/2}$$  

(57)

and $c_s''$ is the root-mean-square wave speed defined as

$$c_s'' = \left( \sum_{m=0}^{n} h_m \right)^{1/2} \left( \sum_{m=0}^{n} c_m^{-2} h_m \right)^{-1/2}.$$  

(58)

Note that only the first definition of (57) for the root-mean-square wave speed is commonly used in seismics (cf. Helbig 1981, p. 159) to replace a stack of layers by a single layer with some replacement velocity. From (56) it is obvious that a good approximation for the vertical wave propagation is given by

$$\sum_{m=0}^{n} \gamma_m h_m \approx \gamma_n^{\text{rms}} z_n,$$  

(59)

with

$$\gamma_n^{\text{rms}} = ((c_s^{-2} + \alpha_1^2 + \alpha_2^2)^{1/2},$$  

(60)
where
\[ c_{\text{rms}} = (c_0^r c_0^m)^{1/2} \] (61)

is the mean value of \( c_0^r \) and \( c_0^m \). Using the result of (59) in our expression of the Green's function of (53), we obtain
\[
\tilde{G} = \frac{1}{8\pi^2} \sum_{n=0}^{N} \Gamma_n \exp \left[ -\frac{\gamma_{\text{rms}}^n x_n}{\gamma_0} \right].
\] (62)

At the interface at \( x_0^{(0)} \), in the same way as in (44), we write
\[
\gamma_n = (c_0^r - 2 + \alpha_1^2 + \alpha_2^2)^{1/2} = \gamma_{\text{rms}}^n \left( 1 + \frac{c_0^r - (c_{\text{rms}}^m)^{-2}}{(\gamma_{\text{rms}}^n)^2} \right)^{1/2}
\approx \gamma_{\text{rms}}^n \left( 1 + \frac{c_0^r - (c_{\text{rms}}^m)^{-2}}{2(\gamma_{\text{rms}}^n)^2} \right).
\] (63)

and in the same way we write \( \gamma_{n+1} \) as
\[
\gamma_{n+1} \approx \gamma_{\text{rms}}^n \left( 1 + \frac{c_0^r - (c_{\text{rms}}^m)^{-2}}{2(\gamma_{\text{rms}}^n)^2} \right).
\] (64)

Hence, the low-contrast approximation of the reflection factor is obtained as
\[
\Gamma_n \approx \frac{\rho_{n+1} - \rho_n}{\rho_{n+1} + \rho_n} + \frac{(c_0^r - 2) \gamma_{\text{rms}}^n}{4(\gamma_{\text{rms}}^n)^2} = A_n + \frac{B_n}{(2\gamma_{\text{rms}}^n)^2}.
\] (65)

For the validity of (63) and the following analysis, we refer to the remarks made after (45). With \( \gamma_0 \approx \gamma_{\text{rms}}^n \) at the interface \( x_0^{(0)} \) and the low-contrast result of (65), the low-contrast approximation of the Green's function of (62) is arrived at
\[
\tilde{G} = \frac{1}{8\pi^2} \sum_{n=0}^{N} \left[ A_n + \frac{B_n}{(2\gamma_{\text{rms}}^n)^2} \right] \exp \left[ -\frac{\gamma_{\text{rms}}^n x_n}{\gamma_0} \right].
\] (66)

Using the results of (46) and (47), with \( \gamma_0 \) replaced by \( \gamma_{\text{rms}}^n \), the space–time Green's function is obtained as
\[
G = \frac{1}{4\pi} \sum_{n=0}^{N} \left[ \frac{A_n}{R_n} + \frac{B_n z_n t}{4(t^2 - (r/c_{\text{rms}}^n)^2)^{3/2}} \right] \text{H}(t - R_n/c_{\text{rms}}^n).
\] (67)

Note that the traveltine
\[ T_n = R_n/c_{\text{rms}}^n \] (68)

is our approximation of the exact traveltine. Further, in the denominator of the second term in (67), we may replace \( c_{\text{rms}}^n \) by \( c_0 \). This is consistent with our previous approximations. We finally end up with
\[
G(t) = \frac{1}{4\pi} \sum_{n=0}^{N} \left[ \frac{A_n}{R_n} + \frac{1}{4} B_n z_n w(t) \right] \text{H}(t - T_n),
\] (69)

where
\[ A_n = (\rho_{n+1} - \rho_n)/(\rho_{n+1} + \rho_n) \] (70)
and

\[ B_n = c_n^{-2} - c_{n+1}^{-2}, \quad (71) \]

\[ w(t) = t/(r^2 - r^2/c_{0}^2)^{3/2}. \quad (72) \]

If, in the expression of the traveltimes of (68), we enforce \( c_n^{\text{rms}} \) to be equal to \( c_0 \), we obtain the space–time Green’s function of the first Born approximation with the homogeneous background of Section 7. The result of (69) can be interpreted as a distorted wave Born approximation using a background medium with our chosen vertically varying root-mean-square acoustic wave speed.

We subsequently present some numerical results for our nine-layer configuration of Table 1. We compare our approximate results with the ones obtained from the Cagniard–de Hoop technique. In Fig. 3, the dashed lines represent the exact results of the Cagniard–de Hoop technique with primaries only. We observe that the results of our simple expression of the rms Born approximation are in good agree-

![Fig. 3](image)

**Fig. 3.** The rms Born approximation of the space–time Green’s function for the nine-layer configuration of Table 1; \( x_3^{(0)} = x_3^{(0)} = 7.5 \text{ m} \) and \( r = 100 \text{ m} \) and 50 m, respectively.
ment with the much more complicated Cagniard–de Hoop technique for primaries only.

9. Velocity Inversion

We first consider the simplified case that there is no contrast in mass density, i.e. \( \rho_n = \rho_0 \), for all \( n \). In order to obtain the Green's function \( G(t) \), the inversion procedure consists of a deconvolution of the data with the known source signature. For \( t > r/c_0 \), we divide the values obtained by \( w(t) \). Then, we determine the location of the jumps in the curve \( G(t)/w(t) \). These time instants are the arrival times \( T_n, n = 0, 1, 2, \ldots, N \). The last time instant considered in the data is defined as \( T_{N+1} \).

With the results of (69) and \( A_n = 0 \), for all \( n \), the values of the wave speed are obtained from

\[
c_{n+1}^{-2} = c_n^{-2} - B_n, \quad n = 0, 1, \ldots, N - 1,
\]

where \( B_n \) is obtained from the recurrence relations

\[
B_n z_n = \frac{16\pi}{T_n} - \int_{T_n}^{T_{n+1}} g(t)/w(t) \, dt - \sum_{m=0}^{n-1} B_m z_m,
\]

\( z_n = (R_n^2 - r^2)^{1/2} \),

\( R_n = T_n c_{rms} \).

The time integral of (74) is computed numerically by a simple rectangular integration rule over the discrete time samples. The rms-velocity is determined recursively as

\[
c_{n}^{\text{rms}} = (c_n^i c_n^o)^{1/2},
\]

with

\[
c_n^i = \left( \sum_{m=0}^{n} c_m^i (T_m - T_{m-1}) \right)^{1/2} \left/ \left( \sum_{m=0}^{n} c_m^{-2} (T_m - T_{m-1}) \right)^{1/2} \right.
\]

and

\[
c_n^o = \left( \sum_{m=0}^{n} c_m^o (T_m - T_{m-1}) \right)^{1/2} \left/ \left( \sum_{m=0}^{n} c_m^{-2} (T_m - T_{m-1}) \right)^{1/2} \right.
\]

in which

\( T_{-1} = r/c_0 \).

Note that these definitions of \( c_n^i \) and \( c_n^o \) differ from (57) and (58), but it has been verified numerically that the final results for \( c_{n}^{\text{rms}} \) do not differ significantly. The present definitions are more advantageous in the inverse scheme, since we can only determine the arrival times from the data. As soon we have determined \( c_{n}^{\text{rms}} \), the precise locations of \( z_n \) (and hence \( h_n \)) can be obtained. In this way we arrive at a simple inversion scheme.

In Fig. 4, we present the synthetic data of the Cagniard–de Hoop method (with primaries only) of the Green's function for the configuration of Table 1, provided
there is no mass-density contrast. In the same figure, we also present the results of the function $G(t)/w(t)$, when $t > r/c_0$. Using these data, the reconstructed wave speeds are presented in Fig. 5. A very close agreement with the exact results (dashed line) is observed. In view of the simplicity of the inversion procedure, we have arrived at a very elegant single-pass seismic inversion scheme.

10. Complete Inversion

We now consider the complete inversion scheme. We observe that the expression of $G(t)$ of (69) consists of two constituents, where the first one contains mass-density information and the second one contains velocity information. When there is mass-density contrast we have to modify our inversion scheme as follows. We first determine the jumps in the curve of $G(t)$. These time instants are the arrival times $T_n$, $n = 0, 1, 2, \ldots, N$. The last time instant considered in the data is defined as $T_{N+1}$. In each time interval $T_n < t < T_{n+1}$, $n = 0, 1, 2, \ldots, N$, we then perform a simple least-squares error fit of the right-hand side of (69) to the data. Then, the values of $\sum_{m=0}^{n} A_m/R_m$ and $\sum_{m=0}^{N+1} B_m x_m$ are determined from the minimization procedure of
the quantities
\[
\left[ \int_{T_n}^{T_{n+1}} \right] 4\pi G(t) - \sum_{m=0}^{n} A_m/R_m - \left( \sum_{m=0}^{n} \frac{1}{2} B_m z_m \right) w(t) \right]^2 \, dt.
\]

The values of the mass density and the wave speed are then obtained as
\[
\rho_{n+1} = \rho_a (1 + A_n)/(1 - A_a),
\]
\[
\frac{c_{n+1}^2}{\rho_{n+1}} = c_n^2 - B_n,
\]
where \( A_n \) and \( B_n \) are found from the recurrence relations
\[
A_n/R_n = 4\pi \frac{a_{11} b_1 - a_{12} b_2}{a_{11} a_{22} - a_{12}^2} - \sum_{m=0}^{n-1} A_m/R_m,
\]
\[
B_n z_n = 16\pi \frac{a_{11} b_1 - a_{12} b_2}{a_{11} a_{22} - a_{12}^2} - \sum_{m=0}^{n-1} B_m z_m,
\]
in which
\[
a_{11} = \int_{T_n}^{T_{n+1}} dt,
\]
\[
a_{12} = \int_{T_n}^{T_{n+1}} w(t) \, dt,
\]
\[
a_{22} = \int_{T_n}^{T_{n+1}} w^2(t) \, dt,
\]
\[
b_1 = \int_{T_n}^{T_{n+1}} G(t) \, dt,
\]
\[
b_2 = \int_{T_n}^{T_{n+1}} G(t)w(t) \, dt.
\]
The integrals of (88) and (89) have to be calculated numerically, using e.g., a rectangular integration rule over the time samples. The integrals of (85)-(87) can be calculated analytically. However, it is more consistent to evaluate these integrals numerically in the same manner as the others.

Without any changes, the values of $z_n$, $R_n$ and $c_{n}^{\text{max}}$ follow from (75)-(80) of the previous section. This concludes our complete inversion scheme. In contrast to Raz (1981a), we are able to reconstruct within our approximations both the density and the velocity profile. The only a priori information we have used is the knowledge that the horizontally stratified medium consists of a stack of homogeneous layers. This assumption complies with the presence of jumps in the data corresponding to the arrival times $T_n$ of the reflected waves from the interfaces.

In Fig. 6, the reconstructed mass-density profile and the wave speed profile are shown. As a starting point, we have taken the synthetic data of Fig. 3, when the offset between source and receiver is equal to 50 m. In Fig. 6, the exact profiles are presented as the dashed lines. The reconstructed mass-density profile is in close agreement with the exact one. The reconstructed wave speed profile differs more from the exact one. The reason is that the second term of (69) with the velocity

![Graph showing reconstructed mass density and wave speed profiles](image-url)
contrast is substantially less than the first term with the density contrast. Hence, small errors in the reconstructed mass density have a great influence on the reconstructed wave speeds. To illustrate this effect, we apply our complete inversion scheme to the synthetic data of Fig. 4. These are the data of the configuration without any contrast in mass density. The reconstructed profiles are shown in Fig. 7. The inverted mass density indeed approximates the constant mass density \( \rho_0 = 1000 \) very well. But the small discrepancies in this curve are sufficient to yield an inverted wave speed profile deviating more from the exact result than the one of Fig. 5. In Fig. 5, we have used the a priori information that there is no density contrast.

11. Conclusions

We have derived a simple closed-form expression for the space-time domain scattered field in a layered structure. The results of the forward modelling are in good agreement with the synthetic data of the Cagniard–de Hoop technique for primaries only. Our closed-form expression consists of two constituents; in the first only the mass–density contrast occurs, while in the second only the velocity contrast is
present. Using a single trace, both the mass densities and the wave speeds of the layered structure can be reconstructed from synthetic data. At this stage, we have not investigated how our inversion scheme will perform on real data. For incomplete data and noisy data several traces are expected to be needed.

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Appendix A

Some Inverse Fourier Transforms

We consider the inverse Fourier transforms of the following type

$$f^{(m)} = \int_{-\infty}^{\infty} \, dz_2 \int_{-\infty}^{\infty} \frac{\exp \left( -syz \right)}{2\pi} \exp \left[ -is(\sigma_1 x_1 + \sigma_2 x_2) \right] \, dz_1,$$

(A1)
in which

$$\gamma = (e^{-2} + \alpha_1^2 + \alpha_2^2)^{1/2}.$$

(A2)

Using the Cagniard–de Hoop method (de Hoop 1960; Aki and Richards 1980, p. 224), we obtain

$$f^{(m)} = R^{-1} \int_{R(r)}^{\infty} \exp \left( -r \gamma^{(m)}(r) \right) \, dr,$$

(A3)

with

$$\gamma^{(m)}(r) = \frac{2}{\pi} \int_{0}^{\pi/2} \text{Re} \left\{ \gamma^{1-n} \right\} \, d\psi,$$

(A4)

$$R = (r^2 + z^2)^{1/2}, \quad r = (x_1^2 + x_2^2)^{1/2},$$

(A5)

and

$$\gamma = \frac{z}{R^2} \tau - \frac{r}{R^2} \left( z^2 - \frac{R^2}{c^2} \right)^{1/2} \cos (\psi).$$

(A6)

When $m = 1$, we directly observe that

$$\gamma^{(1)}(r) = 1.$$  

(A7)

In order to derive the expression for $m = 3$, we use the following result

$$\frac{2}{\pi} \int_{0}^{\pi/2} \text{Re} \left\{ (a - ib \cos (\psi))^{-1} \right\} \, d\psi = (a^2 + b^2)^{-1/2}.$$  

(A8)
Differentiating both sides of this equation with respect to $a$, we obtain

$$
\frac{2}{\pi} \int_0^\infty \text{Re} \left\{ \left[ a - ib \cos (\psi) \right]^{-2} \right\} d\psi = a(a^2 + b^2)^{-3/2}.
$$

(A9)

With

$$
a = \frac{2}{R^2} \tau, \quad b = \frac{R}{R^2} \left( \frac{\tau^2 - R^2}{c^2} \right)^{1/2}, \quad a^2 + b^2 = \frac{1}{R^2} \left( \frac{\tau^2 - c^2}{c^2} \right),
$$

(A10)

we find that

$$
g^{(\delta)(\tau)} = R \xi \tau \left( \tau^2 - \frac{\tau^2}{c^2} \right)^{-3/2}.
$$

(A11)

**APPENDIX B**

**Bounds of Vertical Wave Propagation**

We will prove that for real $\alpha$ and $\beta$, the vertical wave propagation term $\sum_m \gamma_m h_m = \sum_{m=0}^\infty \gamma_m h_m$ in the exponential function of (53) can be approximated within an upper bound and a lower bound as

$$
(c'^{-2} + \alpha_1^2 + \alpha_2^2)^{1/2} \sum_m h_m \leq \sum_m \gamma_m h_m \leq (c'^{-2} + \alpha_1' + \alpha_2')^{1/2} \sum_m h_m.
$$

(B1)

where $c' = c'_x$ and $c'' = c''_x$ are defined in (57) and (58).

From Cauchy's inequality and using the definition of $\gamma_m$ of (44), and

$$
\sum_m \gamma_m h_m \leq \left( \sum_m \gamma_m^2 \bar{h}_m \right)^{1/2} \left( \sum_m \bar{h}_m \right)^{1/2} = \left[ \left( \sum_m c_m^{-2} h_m \right) / \left( \sum_m h_m \right) + \alpha_1^2 + \alpha_2^2 \right]^{1/2} \sum_m h_m = \left[ c'^{-2} + \alpha_1^2 + \alpha_2^2 \right]^{1/2} \sum_m h_m,
$$

(B2)

which proves the right-hand inequality of (B1). From Cauchy's inequality, it also follows that

$$
\sum_m h_m \leq \left( \sum_m c_m^{-1} h_m \right)^{1/2} \left( \sum_m c_m h_m \right)^{1/2}.
$$

(B3)

Hence,

$$
\left( \sum_m c_m^{-1} h_m \right)^{1/2} \geq \left( \sum_m h_m \right) / \left( \sum_m c_m h_m \right)^{1/2},
$$

(B4)

or

$$
\sum c_m^{-1} h_m \geq \left( \sum_m \bar{h}_m \right) \left( \sum_m c_m^{-1} h_m \right)^{3/2} / \left( \sum_m c_m h_m \right)^{1/2} = c'^{-1} \sum_m h_m.
$$

(B5)

Further, the functions $(c_m^{-2} + \alpha_1^2 + \alpha_2^2)^{1/2}$ and $(c'^{-2} + \alpha_1' + \alpha_2')^{1/2}$ are monotonically increasing with increasing real $\alpha_1$ and $\alpha_2$, and these functions have the same asymptotic value when $\alpha_1^2 + \alpha_2^2$ tends to infinity. From (B5) we may then conclude that the
inequality
\[
\sum_m (c_m^2 + \alpha_1^2 + \alpha_2^2)^{1/2} h_m \geq (c^{-2} + \alpha_1^2 + \alpha_2^2)^{1/2} \sum_m h_m
\] (B6)

holds. Equation (B6) is the left-hand inequality of (B1).

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