PRECONDITIONING APPLIED TO EULERIAN–EULERIAN GAS-SOLID FLOW CALCULATIONS WITH VARIABLE DENSITY

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Abstract. Local preconditioning for Eulerian–Eulerian gas–solid flow calculations with variable gas density is investigated. The performance of simultaneous solution algorithms for low-Mach calculations is strongly related to preconditioning. The gas–solid drag source terms are at the origin of a solid volume fraction and frequency dependency of the mixture speed of sound. Whereas the solid volume fraction dependency of the mixture speed of sound is straightforward to account for in the gas–solid preconditioner, this is not the case with the frequency dependency, the frequency not being an extra calculation variable. The preconditioners used so far in the gas–solid flow literature do not account for or eliminate the frequency dependency of the mixture speed of sound and transfer the problem to the numerical speed of sound. Not accounting for the frequency dependency of the mixture speed of sound in the gas–solid preconditioner results, however, in drastic convergence slow down, even when a fully implicit treatment of the drag source terms is taken. Possible approaches to account for the frequency dependency of the mixture speed of sound in the gas–solid preconditioner are investigated. It is shown that the gas–solid preconditioner does not need to remove the frequency dependency of the mixture speed of sound, but should account for it and be scaled according to the mixture speed of sound at the highest frequency calculated, that is the filter frequency mixture speed of sound, which logically depends on the local mesh resolution. Accounting for the filter frequency mixture speed of sound in the gas–solid preconditioner as good as eliminates the reduction of the convergence speed by the gas–solid drag source terms. The addition of a drag history force to the gas–solid preconditioner to properly rescale frequencies lower than the filter frequency hardly alters the convergence behavior. The convergence speed is determined by propagation at the highest, i.e. filter frequency.
1 INTRODUCTION

Although since the late sixties a lot of research has been performed on the development of numerical techniques for the calculation of single phase flows, not much attention has been paid to the development of numerical techniques for multi-phase flows. The latter are not necessarily a straightforward extrapolation of the former, due to the interactions between the different phases, e.g. momentum transfer, complicating both the physical and the mathematical behavior of multi-phase flows.

A typical example of the influence of interphase momentum transfer by drag on the physical behavior of multi-phase flows is given by the behavior of the mixture speed of sound, i.e. the propagation speed of gas phase pressure waves in the mixture of the phases, one of the characteristic speeds in multi-phase flows. Focusing on gas–solid flows, experimental observations show that the mixture speed of sound may decrease drastically with increasing solids fraction. Comparing to the single gas phase speed of sound, the decrease of the mixture speed of sound can amount to more than a factor 30. Furthermore, the mixture speed of sound was observed to depend strongly on the sound frequency \( \omega \), that is the frequency of a gas phase pressure wave in the gas–solid mixture, the reduction of the mixture speed of sound being the most pronounced at lower frequencies \((\omega < 0.1 \text{Hz})\) (Ref. 2, 3). At high frequencies \((\omega > 10^6 \text{ Hz})\), gas phase pressure waves propagate quasi-undisturbed through the gas–solid mixture and the mixture speed of sound equals the single gas phase speed of sound, independent of the solid volume fraction. In most industrial gas–solid flow applications, gas phase pressure waves with frequencies from \(10^6\) Hz down to \(10^{-2}\) Hz occur.

Preconditioners (e.g. Ref. 4) are used to improve the convergence for low-Mach flow calculations with simultaneous solution algorithms by introducing a numerical speed of sound that scales according to the convective velocity, removing the stiffness in the flow direction. In the present paper, preconditioning for Eulerian–Eulerian gas–solid flow calculations is investigated. An eigenvalue analysis of the one-dimensional inviscid non-preconditioned and preconditioned gas–solid flow models is performed to investigate the mixture speed of sound and numerical mixture speed of sound behavior and the role of the gas–solid drag source terms. Possible approaches for the gas–solid preconditioner to account for the impact of the gas–solid drag source terms on the mixture speed of sound are derived and investigated.

2 CONTINUUM GAS–SOLID FLOW MODELS

In a Eulerian–Eulerian approach, both the phases are treated as entirely mixed continua. The basic Eulerian–Eulerian model equations describe the conservation of mass and momentum for each phase:\(^5\)

\[
\frac{\partial}{\partial t}(\varepsilon, \rho_v) + \frac{\partial}{\partial x}(\varepsilon, \rho_v v) = 0
\]

\[
\frac{\partial}{\partial t}(\varepsilon, \rho_u) + \frac{\partial}{\partial x}(\varepsilon, \rho_u u) = 0
\]

\[
\frac{\partial}{\partial t}(\varepsilon, \rho_v vv) + \frac{\partial}{\partial x}(\varepsilon, \rho_v vv) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left( \varepsilon, \rho_v v v \right) - \varepsilon, \rho_u - \varepsilon, \rho_p + \beta \left( u - v \right)
\]
\[
\frac{\partial}{\partial t} \left( \varepsilon_g \rho_g \vec{u} \right) + \frac{\partial}{\partial r} \left( \varepsilon_s \rho_s \vec{u} \right) = \frac{\partial P}{\partial r} \varepsilon_s + \frac{\partial}{\partial r} \left( \varepsilon_s \rho_s \vec{u} \right) + \varepsilon_s \frac{\partial P}{\partial r} - \beta s (\vec{u} - \vec{v})
\] (4)

These equations represent mass conservation of the solid phase (Eq. 1), mass conservation of the gas phase (Eq. 2), momentum conservation of the solid phase (Eq. 3), and finally momentum conservation of the gas phase (Eq. 4). \( \varepsilon_g \) and \( \varepsilon_s \) are the volume fractions of the solid and the gas phase, respectively.

Analogous to the gas phase pressure \( P \) and the gas phase shear stress \( s_s \), a solid phase pressure \( s_P \) and solid phase shear stress \( s_s \) are introduced in Eq. (3). The latter can be calculated via empirical correlations or via the Kinetic Theory of Granular Flow (KTGF), requiring the integration of an additional granular temperature transport equation.

The last two terms in Eqs. (3) and (4) are source terms describing the gas–solid momentum transfer, that is, the gas–solid interaction forces. The latter consist of an acoustic gas–solid interaction force, i.e. the solids volume fraction of the gas phase pressure gradient \(- \varepsilon_s \frac{\partial P}{\partial r}\), and a drag force \( \beta_s (\vec{u} - \vec{v}) \), corresponding to model A as defined by Ref. 7. Most commonly, the Ergun equation and the equation of Wen and Yu are used for the calculation of the interphase momentum transfer coefficient \( \beta \) in the drag force. For the constitutive equations, reference is made to Ref. 10.

3 SOLUTION ALGORITHM

3.1 Pseudo-time stepping

The set of model equations (1)–(4) is solved simultaneously, point- or linewise. The equation set is written in matrix formulation as:

\[
\frac{\partial Q}{\partial t} - \frac{\partial F}{\partial r} + K = 0,
\]

where \( Q \) is the vector of the four conservative variables (i.e. \( \varepsilon_g, \rho_g, \varepsilon_s, \rho_s, \varepsilon_s, \rho_s, \varepsilon_s, \rho_s, u \)), \( F \) includes both the convective and acoustic fluxes, and \( K \) contains the gravity and drag source terms.

For the integration, a pseudo-time stepping technique is used. A numerical stepper, the pseudo-time \( \tau \), is introduced for the solution of the non-steady problem:

\[
\frac{\partial Q}{\partial \tau} + \frac{\partial Q}{\partial t} = -\frac{\partial F}{\partial r} + K,
\]

or for the steady-state problem:

\[
\frac{\partial Q}{\partial \tau} = -\frac{\partial F}{\partial r} + K.
\]

An iteration loop is set up in pseudo-time. To further increase the numerical stability, a fourth-order Runge-Kutta scheme is used in the pseudo-time stepping. At convergence,
the pseudo-time step term vanishes. In what follows, the steady-state equations will be focused on. The results are, however, applicable to the unsteady state equations.

The equations are linearized and solved for the viscous variables $X$, i.e. $\varepsilon$, $P$, $v$, and $u$ instead of the conservative variables:

$$\frac{\partial Q}{\partial X} \frac{\partial X}{\partial \tau} = -\frac{\partial F}{\partial X} \frac{\partial X}{\partial \tau} + K$$  \hspace{0.5cm} (8)

### 3.2 Preconditioning

Using an explicit approach in the streamwise direction, the maximum allowable pseudo-time step scales inversely with the largest characteristic propagation speed in the streamwise direction. For low-Mach flows, the propagation of the gas phase pressure usually drastically restricts the pseudo-time step as:

$$\Delta \tau_{\text{max}} = CFL \frac{\Delta x}{u + c_m}$$  \hspace{0.5cm} (9)

CFL is about 1, but by coupling a fourth-order Runge-Kutta scheme to the pseudo-time stepping, CFL-values up to $2\sqrt{2}$ can be taken.

The integration scheme suffers from stiffness when the Mach-number is low, as the maximum allowable pseudo-time step is very small for the phenomena that are propagated with the convective speeds. A fully implicit approach is computationally expensive. A cheaper solution to reduce the stiffness in the streamwise direction is to apply local preconditioning. Preconditioning rescales the eigenvalues of the set of equations (1)–(4):

$$\gamma_{i,2}^{\text{num}} = u' \pm c_m^{\text{num}},$$

$$\gamma_{3,4}^{\text{num}} = v' \pm c_m^{\text{num}},$$

so that they become all of the same order of magnitude. It should be remarked that the numerical mixture speed of sound $c_m^{\text{num}}$ should be larger than the modified convective gas phase velocity $u'$, to guarantee backward moving pressure waves in the system, i.e. to avoid a numerically supersonic behavior induced by the preconditioner.

The pseudo-time derivative term is pre-multiplied with the preconditioner, (8) being transformed into:

$$\Gamma \frac{\partial X}{\partial \tau} = -\frac{\partial F}{\partial X} \frac{\partial X}{\partial \tau} + K$$  \hspace{0.5cm} (11)

Hence, the preconditioner affects the convergence behavior only and not the solution.

The preconditioner of Weiss and Smith, developed for single-phase gas flows, was applied by De Wilde et al. to gas-solid flows:

$$\Gamma = \begin{bmatrix} \rho_v & 0 & 0 & 0 \\ -\rho_v & \varepsilon v / u_{\text{ref}} & 0 & 0 \\ \rho_v & 0 & \varepsilon \rho_v & 0 \\ -\rho_v & \varepsilon u / u_{\text{ref}} & 0 & \varepsilon \rho_v \end{bmatrix}$$  \hspace{0.5cm} (12)
Hence, the single gas phase speed of sound is rescaled according to the gas phase convective speed. Neglecting the viscous contribution to $u_{\text{ref}}$:

$$
u_{\text{ref}} = \begin{cases} |v| & \text{if } |v| < \sqrt{\frac{\rho_g}{\rho_s}} \\ \sqrt{\frac{\rho_g}{\rho_s}} & \text{if } |v| > \sqrt{\frac{\rho_g}{\rho_s}} \end{cases}$$

(13)

Experimental observations\cite{13} show, however, at low frequencies, a remarkable gradual decrease of the mixture speed of sound with increasing solid volume fraction. To account for the reduction of the mixture speed of sound $c_m$ by the presence of solid particles, De Wilde et al.\cite{11,14} introduced the mixture speed of sound obtained from a mixture momentum equation:

$$c_{m}^{\text{mix}} = \sqrt{\frac{\rho_g}{\rho_g \rho_s + \rho_s \rho_s}}$$

in the preconditioner:

$$
u_{\text{ref}} = \begin{cases} |v| \frac{c_{g}^{\text{mix}}}{c_{m}^{\text{mix}}} & \text{if } |v| < \sqrt{\frac{\rho_g}{\rho_s}} \\ \sqrt{\frac{\rho_g}{\rho_s}} & \text{if } |v| > \sqrt{\frac{\rho_g}{\rho_s}} \end{cases}$$

(15)

Although improvement in the convergence behavior is observed in some cases, in particular with increasing mesh size, the improvement is not general, raising more questions about the functioning of preconditioning in gas–solid flows and the speed of sound to be accounted for.

Figure 1 shows the convergence behavior for a typical gas–solid flow calculation with the simultaneous solution algorithm\cite{10,11} respectively in the absence and in the presence of gas–solid drag source terms. Despite the fully implicit treatment of the gas–solid drag source terms\cite{10,11} and preconditioning\cite{10}, convergence in the absence of drag is much faster than in the presence of drag.

In the next paragraph, it is shown that the gas–solid flow preconditioner concepts of Refs. 10 and 11 fail by a frequency dependency of the numerical mixture speed of sound induced by the presence of the gas–solid drag source terms.
4 EIGENVALUE ANALYSIS

An eigenvalue analysis is performed on the non-preconditioned and preconditioned inviscid one-dimensional solid phase and gas phase mass and momentum conservation equations (1)–(4)) and assuming a uniform steady-state flow field.

4.1 Non-preconditioned model

Two of the four characteristic speeds obtained from the eigenvalue analysis are related to the propagation of the gas phase variables, whereas the two others are related to the propagation of the solid phase variables. Furthermore, the characteristic speeds can be decomposed as:

\[
\left[ \text{Re}(\omega_{s,\pm}) \right] = \hat{\omega} \pm c_m(\omega, \epsilon, \rho_s, \rho_g)
\]

\[
\left[ \text{Re}(\omega_{s,\pm}) \right] = \hat{\omega} \pm c_s(\rho_s)
\]

where \(c_m\) is the mixture speed of sound and \(c_s\) is the granular speed of sound, related to the propagation of the solid phase pressure \(P_s\). Hence, the mixture speed of sound \(c_m\) is calculated from:

\[
c_m = (s_1 - s_2)/2
\]

Figure 2: Mixture speed of sound as a function of the sound frequency and the solid volume fraction calculated with the non-filtered model A ((1)–(4)) and (16)–(17)). The calculated behavior is in agreement with the experimentally observed behavior. Higher frequency pressure waves, typically with a frequency higher than 10^6 Hz, propagate quasi-undisturbed through the gas–solid mixture, i.e. the high frequency mixture speed of sound equals the single gas phase speed of sound. As the frequency decreases, the mixture speed of sound is gradually decreased by the presence of solid particles. A low frequency limit behavior is observed for frequencies below 0.1 Hz. At such low frequencies, the mixture
speed of sound gradually decreases from the single gas phase speed of sound for solid volume fractions lower than $10^{-5}$ to a minimum mixture speed of sound for solid volume fractions higher than about 0.1. An analytical expression for the mixture speed of sound as a function of the sound frequency and the solid volume fraction was derived by Refs. 2, 3.

### 4.2 Preconditioned model

In case preconditioning is applied, the characteristic speeds are given by (10).

![Figure 3: Numerical mixture speed of sound resulting from preconditioning as a function of the frequency and the solid volume fraction.](image)

Non-filtered model A (1)–(4). Preconditioner based on Weiss and Smith, Eq. (12):

(a) based on the single gas phase speed of sound Eq. (13), Refs. 10, 16
(b) based on the low frequency mixture speed of sound Eqs. (14) and (15), Refs. 11, 14.

Figure 3 shows the ratio of the calculated numerical speed of sound and the desired numerical speed of sound, respectively for the preconditioner based on the single gas phase speed of sound (13), Refs. 10 and 16, and for the preconditioner based on the low frequency mixture speed of sound (14)–(15), Refs. 11 and 14. In both cases the desired numerical speed of sound equals the gas phase convective velocity $u$. Independent of the formulation for $u_{\text{ref}}$ that is used ((13) or (15)), the preconditioner (12) does not remove the frequency dependency in the numerical mixture speed of sound (Figure 3).

Using (13) in (12), the highest frequency numerical speed of sound is scaled properly by the preconditioner, but the lower frequency numerical speeds of sound are reduced below the desired numerical speed of sound, i.e. below the gas phase convective velocity. Hence, for low-frequency gas phase pressure waves, over-preconditioning induces supersonic behavior (Figure 4). One way to avoid this problem is to choose a reference velocity $u_{\text{ref}}$ in (13) that is larger than the convective velocity ($u_{\text{ref}} > u$). This, however, reduces the gain from applying preconditioning due to the poor scaling of the high frequencies. It is investigated further in this paper to what extent a subsonic behavior of the high frequency pressure waves combined with a supersonic behavior of the low frequency pressure waves induced by the preconditioner, is undesirable to the integration algorithm.
It should be noted that the frequency range covered in a simulation depends on the domain size and the mesh resolution and is as such not the entire frequency range. This is the basis for the design and analysis of an improved gas–solid preconditioner in the next paragraphs.

Using (14) and (15) in (12) (Figure 3b), the low frequency numerical speeds of sound are scaled properly and, at low frequencies, the dependence of the numerical speed of sound on the solid volume fraction and the frequency is removed. The high frequency numerical speeds of sound, on the other hand, are poorly scaled, in particular at higher solid volume fractions. A solid volume fraction and frequency dependency of the numerical speed of sound at high frequencies is induced by the preconditioner. Compared with the preconditioner based on (13), the preconditioner based on (14) and (15) does not induce numerical speeds of sound that are lower than the convective velocity. However, if higher frequencies are to be calculated, the numerical speed of sound to be used for the determination of the allowable pseudo-time step (9) is the largest, that is high frequency numerical speed of sound, which is a multiple of the convective velocity. In such case, using (14) and (15), the gain from applying preconditioning will be drastically reduced. The preconditioner (12) using (14) and (15) (Refs. 11 and 14) may have an improved performance compared to the preconditioner using (13) (Refs. 10 and 16) if coarse meshes, either spatially or temporally, and/or filtered models are used for the calculation, filtering out the higher frequencies.

In general, the frequency dependency of the mixture speed of sound remains an issue with respect to gas–solid preconditioning. In the next paragraphs, it is investigated which frequencies are important with respect to preconditioning. Furthermore, possible approaches for preconditioning accounting for the gas–solid drag source terms are derived and tested.

5 PRECONDITIONING ACCOUNTING FOR THE GAS–SOLID DRAG SOURCE TERMS

5.1 Preconditioning accounting for the highest (filter) frequency mixture speed of sound

Gas–solid flow calculations use spatial grid sizes from $10^{-4}$ m (Ref. 20) to $10^{-2}$ m (Ref. 10) and pseudo-time steps of $10^6$ s (Ref. 20) to $10^3$ s (Ref. 11). Typical domain sizes range from $10^{-2}$ m to $10^1$ m and $10^0$ s to $10^3$ s or even steady state. Hence, the variation of the mixture and numerical speed of sound with the frequency occurs in a frequency range that is typically important in the calculation of gas–solid flows. The preconditioners used so far for the
calculation of gas–solid flows,\textsuperscript{10,11,14,16} do not account for the frequency dependency of the mixture speed of sound. As shown in the previous paragraphs, the numerical speed of sound resulting from applying such preconditioners depends also strongly on the frequency. Hence, the speed of sound cannot be scaled properly, that is according to the convective velocity, for all possible frequencies, but only for a fixed frequency.

The highest frequency calculated, the filter frequency $\omega_f$, is directly related to the mesh resolution used ($\Delta x$ and $\Delta t$). As follows from a stability analysis of the numerical algorithm,\textsuperscript{11} the highest frequency waves are critical for the propagation of information through the calculation domain. In fact, each pseudo-time step or iteration, information is propagated from each cell to its neighbors, but not further on. Therefore, it is crucial to scale the preconditioner properly for the highest (filter) frequency waves, i.e. according to the filter frequency mixture speed of sound $c_m(\omega_f)$. The filter frequency mixture speed of sound $c_m(\omega_f)$, which is also the maximum mixture speed of sound in the calculation (Figure 2), is a factor $\alpha(\omega_f) (< 1)$ of the single gas phase speed of sound $c_g$:

$$c_m(\omega_f) = \alpha \cdot c_g \quad (18)$$

Hence, instead of using (13), the preconditioner (12) should be scaled according to:

$$u_{ref} = \frac{|u|}{\alpha} \quad \text{if} \quad |u| < c_m(\omega_f)$$

$$= \tilde{\alpha} \cdot |u| \quad (19)$$

The filter frequency $\omega_f$ is local and can vary over the calculation domain. Whereas (13) is the limit case of (18) and (19) for very high filter frequencies, i.e. for very fine calculation meshes, (14) and (15) are the limit case for very low filter frequencies, i.e. for very coarse calculation meshes.

One-dimensional test simulations are carried out on a 1 m domain with a spatial mesh size of 0.02 m, i.e. with a spatial filter frequency $\omega_f^*$ of 50 m$^{-1}$, with an average and inlet solids volume fraction of 0.10 and gas velocity of 5 m/s. The solids density $\rho_s$ is 1500 kg m$^{-3}$ and the solid particle diameter is 60 µm. It should be remarked that the corresponding temporal filter frequency $\omega_f^*$ and filter frequency mixture speed of sound $c_m(\omega_f)$ can be calculated from $c_m(\omega_f) = \omega_f^*/\omega_f^*$ and are respectively about 7400 Hz and 148 m/s (compared to $c_g \approx 340$ m/s).
Figure 5 compares the convergence speed using (13) and using (18)–(19) with intermediate and optimal $\alpha^*$ for CFL = 1 and for CFL maximum. The asymptotic convergence speed is seen from the slope of the residual versus iteration number curve. The optimal $u_{ref}$ found from the simulations ($u_{ref}^{opt} = |u|$ or $\alpha = 2$ in (19)) corresponds within 10% to the value expected from the mesh resolution $\Delta x$ or the filter frequency $\omega_f$ used. Figure 5 shows that the convergence speed drastically improves (by about a factor 5) by accounting for the filter frequency mixture speed of sound $c_m(\omega_f)$ (18)–(19) in the preconditioner (12), instead of using the single gas phase speed of sound $c_g$ (13). In fact, using a gas–solid preconditioner based on $c_m(\omega_f)$ (12), (18)-(19), the convergence speed approaches the reference convergence speed in the absence of gas–solid interaction source terms with the $c_g$ based preconditioner (12)–(13) (Figure 1). In general, the gain in convergence speed that can be obtained by rescaling $u_{ref}$ according to Eq. (19) depends on the mesh resolution or filter frequency, the solids volume fraction (Figure 2), the solids-to-gas density ratio, and the convective gas phase velocity, the gain being the most pronounced with coarse meshes (low filter frequency), high solid volume fractions, high solids-to-gas density ratios, and low convective gas phase velocities. Interestingly, the gain in convergence speed is more than proportional with $\alpha^*$ (19).

It should be remarked that with increasing $u_{ref}$ (i.e. increasing $\alpha^*$), the maximum CFL or pseudo-time step $\Delta \tau_{max}$ (9) for which numerical stability was guaranteed decreases. As seen from Figure 5, the overall convergence speed is, however, still drastically improved when accounting for the filter frequency mixture speed of sound $c_m(\omega_f)$ (18)–(19) in the gas–solid
preconditioner (12).

5.2 Preconditioning accounting for the mixture speed of sound at frequencies equal to or lower than the filter frequency

In practice, gas phase pressure waves over a wide frequency range are encountered in gas–solid flow fields. As mentioned before, the highest frequency calculated is determined by the mesh resolution. The lowest frequency calculated, on the other hand, is determined by the domain size. The pressure values stored in the grid cells of a calculation domain can be attributed to different of the frequencies calculated. It is investigated if the propagation of gas phase pressure waves at frequencies lower than the filter frequency plays an important role in the convergence behavior and if the mixture speed of sound at frequencies lower than the filter frequency should be properly rescaled by the gas–solid preconditioner.

The frequency dependency of the mixture speed of sound is due to the presence of the gas–solid drag source terms in the gas and solid phase momentum equations ((3) and (4)). Therefore, the possibility of incorporating the gas–solid drag source terms in the gas–solid preconditioner and eliminating the frequency dependency is investigated. Performing a Fourier transform,\(^{21}\) imposing a sinusoidal perturbation with amplitude \(\tilde{A}\), temporal frequency \(\omega\) and characteristic propagation speed \(s\):

\[
\delta X = \tilde{A} e^{i(\omega t - \omega t)} , \tag{20}
\]

Eq. (8) transforms into:

\[
\left( -i \omega \frac{\partial Q}{\partial X} + \omega \frac{\partial F}{\partial X} - \frac{\partial K}{\partial X} \right) \delta X = 0 \tag{21}
\]

The preconditioner acts on the pseudo-time-derivative term. Therefore, to eliminate the effects of the gas–solid drag source terms \(K\), \(\left[ \frac{\partial K}{\partial X} \right]_{\omega} \) is added to the left hand side of (21):

\[
\left( -i \omega \left(\frac{\partial Q}{\partial X} - \frac{1}{i \omega} \frac{\partial K}{\partial X} \right) + i \frac{\omega}{s} \frac{\partial F}{\partial X} - \frac{\partial K}{\partial X} \right) \delta X = 0 \tag{22}
\]

An inverse Fourier transform according to (20) results in:

\[
\frac{\partial Q}{\partial X} \frac{\partial (\delta X)}{\partial \tau} + \int_0^\tau \frac{\partial K}{\partial X} \frac{\partial (\delta X)}{\partial \tau'} d\tau' + \frac{\partial F}{\partial X} \frac{\partial (\delta X)}{\partial \tau} = \frac{\partial K}{\partial X} (\delta X) , \tag{23}
\]

where the initial conditions are chosen such that \(\delta K^{(0)} = \delta K(\tau = 0) = 0\). The extra term introduced \(\int_0^\tau \frac{\partial K}{\partial X} \frac{\partial (\delta X)}{\partial \tau'} d\tau'\), does not result in a modified preconditioner and modifies the solution, as it is written in terms of \(\frac{\partial (\delta X)}{\partial \tau'}\) instead of \(\frac{\partial (\delta X)}{\partial \tau}\). Indeed, writing \(\delta K^{(0)}\) as:
\[ \partial K = K + \left( \frac{\partial K}{\partial X} \frac{\partial (\Delta X)}{\partial \tau} \right) \Delta \tau + \left( \frac{\partial K}{\partial X} \frac{\partial (\Delta X)}{\partial \tau} \right) \Delta \tau^{(i)} + \ldots 
\]

\[ + \left( \frac{\partial K}{\partial X} \frac{\partial (\Delta X)}{\partial \tau} \right) \Delta \tau^{(i-1)} = K^{(i)} + \int_0^\tau \frac{\partial K}{\partial X} \frac{\partial (\Delta X)}{\partial \tau} d\tau'. \]

with:

\[ \Delta \tau^{(i)} = \tau^{(i)} - \tau^{(i-1)} \]

it can be seen that (23) amounts to:

\[ \frac{\partial Q}{\partial X} \frac{\partial (\Delta X)}{\partial \tau} + \frac{\partial F}{\partial X} \frac{\partial (\Delta X)}{\partial x} = \partial K^{(0)} = 0 \]

The extra term \( \int_0^\tau \frac{\partial K}{\partial X} \frac{\partial (\Delta X)}{\partial \tau} d\tau' \) introduced in (23), does, however, suggest a strategy for preconditioning. Provided that the updates are correlated in pseudo-time:

\[ \left( \frac{\partial (\Delta X)}{\partial \tau} \right)^{(i)} = R(\tau, \tau') \left( \frac{\partial (\Delta X)}{\partial \tau} \right)^{(i)}, \]

a possible preconditioner accounting for the effects of the gas–solid drag source terms could be:

\[ \left[ \frac{\partial Q}{\partial X} + \int_0^\tau \frac{\partial K}{\partial X} R(\tau, \tau') d\tau', \frac{\partial F}{\partial X} \frac{\partial (\Delta X)}{\partial x} \right] \frac{\partial (\Delta X)}{\partial \tau} + \frac{\partial F}{\partial X} \frac{\partial (\Delta X)}{\partial x} = \frac{\partial K}{\partial X} (\Delta X) \]

The new term introduced \( \int_0^\tau \frac{\partial K}{\partial X} R(\tau, \tau') d\tau' \) has a history force nature and requires the storage of \( \partial K/\partial X \) over the iterations in pseudo-time. The functional form of the correlation function \( R(\tau, \tau') \) is to be further determined. It should be remarked that, according to (23), (24) and (28), a fully implicit treatment of the gas–solid drag source terms, as proposed by De Wilde et al.\textsuperscript{11}, appears as a limit case in which the minimum number of time steps, that is the last time step, is accounted for. As shown by De Wilde et al.\textsuperscript{11}, the fully implicit treatment of the gas–solid drag source terms drastically improves the numerical stability.
At convergence, i.e. when a solution is reached, \( \tau \delta \frac{\partial \rho}{\partial \tau} \) vanishes. Hence, the gas–solid drag history force introduced does not alter the solution. During convergence, however, the new term introduced is expected to eliminate at least partially the impact of the gas–solid drag source terms on the mixture speed of sound at frequencies lower than the filter frequency and on the related convergence behavior.

The more the updates of the variables are correlated in pseudo-time, the more the impact of the gas–solid drag source terms on the convergence behavior can be eliminated. Therefore, in the next paragraph, the correlation of the updates is investigated.

### 5.2.1. Correlation of the updates

The applicability of the preconditioning approach presented in the previous paragraph (28) depends on the correlation of the updates of the variables in pseudo-time. Figure 6a shows a typical plot of the evolution in pseudo-time of the calculated updates of the solid volume fraction. To quantify the correlation and to determine the distance in number of pseudo-time steps or iterations between two uncorrelated updates, the correlation function is calculated:

\[
R(\Delta \tau) = \lim_{\tau \to \infty} \int \left( \frac{\partial X^{(i)}}{\partial \tau} \right) \left( \frac{\partial X^{(i+\Delta \tau)}}{\partial \tau} \right) d\tau
\]

Figure 6b shows the correlation function corresponding to the solid volume fraction updates shown in Figure 6a. Most importantly, Figure 6b shows that the updates are indeed correlated in pseudo-time. The correlation is significant up to a distance of about 40 iterations and vanishes from a distance of about 70 iterations. Hence, the preconditioner approach (28) can indeed be taken. However, as the number of pseudo-time steps of which the updates are correlated and can be coupled back is finite and limited and as the correlation is partial, the effects of the gas–solid drag source terms on the convergence behavior can only be eliminated...
partially:

\[
\frac{\partial Q}{\partial X} + \int_{\tau}^{\infty} \frac{\partial K}{\partial X} \cdot R(\tau, \tau') d\tau \cdot \frac{\partial (\partial X)}{\partial \tau} + \frac{\partial F}{\partial X} \cdot \frac{\partial (\partial X)}{\partial \tau} = \frac{\partial K}{\partial X} \cdot (\partial X),
\]

(30)

with \(\tau_c\) the cut-off pseudo-time.

More precisely, the frequency dependency of the numerical speed of sound induced by the gas–solid drag source terms is captured from the high frequencies on. The more pseudo-time steps accounted for in the gas–solid drag history force in the preconditioner (30), the lower the frequencies down to which the scaling of the numerical speed of sound behavior is improved, but the more storage of \(\frac{\partial K}{\partial X}\) is required.

Because the correlation of the updates is limited in pseudo-time, very low frequencies cannot be treated adequately by the new preconditioner, despite the gas–solid drag history force contribution to the preconditioner (30). In the next paragraph, simulations are carried out to determine the potential gain in convergence speed from a gas–solid drag history force contribution to the gas–solid preconditioner and to determine the actual need of accounting for frequencies lower than the filter frequency in the gas–solid preconditioner.

5.2.2. Convergence behavior with a gas–solid drag history force contribution to the preconditioner

Simulations are carried out with a constant interphase momentum transfer coefficient in the drag force. The latter allows to reduce the storage requirements of \(\frac{\partial K}{\partial X}\) and to analytically integrate the gas–solid drag history force integral in (30) for any cut-off pseudo-time \(\tau_c\). The one-dimensional test simulation conditions are as described in the previous paragraph. The solid volume fraction \(\varepsilon_s\) being 0.10, the filter frequency of being 7400 Hz, and the ratio of the highest (filter) and the lowest frequency calculated in the test simulations being 50, allows to investigate the impact on the convergence behavior of frequencies lower than the filter frequency in a frequency range in which the mixture speed of sound depends strongly on the frequency (Figure 2) and, furthermore, allows, in view of the expected correlation between the updates (Figure 6), to account maximally for the frequencies lower than the filter frequency in the simulation domain. Accounting for these aspects, in the test simulations, the gain in convergence speed from the gas–solid drag history force contribution to the preconditioner (30) is expected to be small but, if any, observable.
Figure 7: Convergence behavior with a gas–solid drag history force contribution to the preconditioner (30) based on the single gas phase speed of sound (13) for $CFL = 1$. Number of iterations $\Delta i$ (see Figure 6) accounted for in the gas–solid interaction history force contribution respectively 1 (which amounts to an implicit treatment of the gas–solid interaction source terms, see (24)), 10 and 100. $1-D$ test case, domain size: 1 m, $\Delta x = 0.02$ m, $\epsilon_s = 0.10$, $\rho_s = 1500$ kg m$^{-3}$, $d_p = 60$ µm, $u = 5$ m/s.

Figure 7 shows the convergence behavior obtained with increasing gas–solid drag history force contribution to the preconditioner (30). The number of iterations $\Delta i$ accounted for in the gas–solid drag history force contribution to the preconditioner (30), is respectively 1 (which amounts to an implicit treatment of the gas–solid interaction source terms, see (24)), 10 and 100. The corresponding cut-off pseudo-time $\tau_c$ in (30) is found from $\tau_c = \tau - (\Delta i \cdot \Delta \tau_{\text{max}})$, $\Delta \tau_{\text{max}}$ being given by (9). Figure 7 shows that, by accounting for frequencies lower than the filter frequency in the preconditioner, the initial convergence behavior is altered, but not the asymptotic convergence speed. Hence, whereas accounting for the filter frequency mixture speed of sound in the preconditioner is crucial and results in a major improvement of the asymptotic convergence speed (Figure 5), accounting for the mixture speed of sound at frequencies lower than the filter frequency is not useful. Hence, the gas–solid preconditioner should account for the frequency dependency of the mixture speed of sound, but does not need to remove the frequency dependency of the numerical mixture speed of sound. As assumed in a stability analysis of the integration algorithm, the real convergence behavior is found to be determined by the error amplification factor per iteration and, hence, by the propagation speed of the highest (filter) frequency waves only.

6 CONCLUSIONS

The gas–solid drag source terms are at the origin of a significant reduction of the convergence speed of simultaneous solution algorithms for Eulerian–Eulerian gas–solid flow
calculations, even when a fully implicit treatment of the drag source terms is taken. The performance of simultaneous solution algorithms for low-Mach calculations is strongly related to preconditioning. The development of a local gas–solid preconditioner is not straightforward. The gas–solid drag source terms induce a solid volume fraction and frequency dependency of the mixture speed of sound. The latter is not easily accounted for in the gas–solid preconditioner. The preconditioners used so far in the gas–solid flow literature do not account for or eliminate the frequency dependency of the mixture speed of sound and transfer the problem to the numerical speed of sound. Different approaches for the preconditioner to account for the frequency dependency of the mixture speed of sound are theoretically developed and applied in test simulations. The results show that the gas–solid preconditioner should account for the frequency dependency of the mixture speed of sound and should be based on the highest or filter frequency mixture speed of sound, which logically depends on the local mesh resolution. For a typical mesh size of 2 cm, a solids density of 1500 kg m\(^{-3}\), and a solid volume fraction of 10%, this results in a convergence speed-up factor 5. The convergence speed-up factor is more than proportional with the rescale factor \([\text{single gas phase speed of sound} / \text{filter frequency mixture speed of sound}]\) used in the gas–solid preconditioner. With coarser meshes, higher solids densities, or higher solid volume fractions, even higher convergence speed-up factors are obtained. By accounting for the filter frequency mixture speed of sound in the preconditioner, the impact of the gas–solid drag source terms on the convergence speed is as good as eliminated.

To properly rescale the mixture speed of sound at frequencies lower than the filter frequency, a gas–solid drag history force is added to the preconditioner. The drag history force approach taken is an extension of the fully implicit treatment of the drag source terms. The degree to which the gas–solid drag can be accounted for in the gas–solid preconditioner depends on the correlation of the updates in pseudo-time. Test simulations show that accounting for the mixture speed of sound at frequencies lower than the filter frequency in the gas–solid preconditioner does not result in any convergence speed-up. Whereas it is essential for the gas–solid preconditioner to account for the frequency dependency of the mixture speed of sound, i.e. by accounting for the filter frequency mixture speed of sound, the gas–solid preconditioner does not need to remove the frequency dependency in the numerical mixture speed of sound and does not need to account for the mixture speed of sound at frequencies lower than the filter frequency. The convergence behavior is determined by propagation at the highest, i.e. filter frequency.

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