Vector-valued model structure validation test in direct closed-loop identification
On the basis of the cross-correlation function

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Vector-valued model structure validation test in direct closed-loop identification
On the basis of the cross-correlation function

MASTER OF SCIENCE THESIS

For the degree of Master of Science in Systems and Control at Delft University of Technology

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June 23, 2011
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Abstract

Many systems in life are required to be operated under the presence of feedback loop. The reasons for this can vary from safety requirements to the need of meeting the specified performance of a system. The presence of the feedback loop in a system requires the identification experiment to be conducted under the closed-loop configuration, which in most cases involve a (stabilizing) controller.

This thesis focusses on the model structure validation test of model estimate of a closed-loop system, where the model estimate is obtained by the direct closed-loop identification method. Two types of model structure validation test on the basis of cross-correlation function will be discussed in this thesis, namely the closed-loop point-wise test and the closed-loop vector-valued test. With the cross-correlation function being evaluated in these tests is the cross-correlation of the residual signal $\varepsilon$ and excitation signal $r$.

The point-wise test evaluates the cross-correlation function for each and every single time lags only, whereas the vector-valued test evaluates the cross-correlation function for the overall time lags. In this thesis we will see that the proposed vector-valued test is an accurate model validation test in detecting the presence of bias in a model estimate of a system operated in closed-loop configuration.

Keywords: Closed-loop model structure validation, closed-loop system identification, vector-valued model structure validation, direct closed-loop identification
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I would like to thank *dr.ir. X.J. A. Bombois*, *Prof.dr.P.M.J Van den Hof* and *dr.ir. A J. den Dekker* as my supervisors and advisors, for the time and effort spent during our discussions and on the process of correcting this thesis report. In the whole process of this thesis project I have accepted plenty of constructive and crucial corrections as well as many insightful comments and ideas.

Delft, University of Technology

June 23, 2011
I would like to thank the Lord Jesus Christ, for making this possible. Without God’s help and intervention, I would never have persevered.

-This thesis is dedicated to my family, especially my parents, for their never-ending love and support.-
Chapter 1

Introduction

In the first part of this chapter, the basic concept of system identification will be briefly explained. Then this chapter is continued with the objective and problem statement of this thesis project. The last part of this chapter will be the chapter outline of this thesis report.

1-1 Introduction

In this section the basic concept of a system, model, as well as system identification will be presented briefly.

1-1-1 Definition of a system

A system can be defined as an object that produces observable signals resulting from interactions between variables related to the system. The observable signal that is being the subject of interest for the observers are usually called output signal, while the external signals entering the system that are determined and can be manipulated by the observers are called input signal. The other external signals that can not be manipulated are called disturbances.

A dynamical system is a system where its current output value is not only affected by the current external stimuli, but also by the past values of the input. The output of a dynamical system where the measurement of the external stimuli is not available is called time-series, and a mathematical model that represents the behaviour of this type of output is called a time-series model [1].

A system can be operated in an open-loop or closed-loop closed-loop configuration. An open-loop system is a system which is operated under the condition where the signal entering the system(input signal) is directly influenced by the operator, where in the closed-loop case the input signal to the system is already affected by the presence of a feedback loop.
1-1-2 Types of models

The basic feature of a model is that they attempt to link observations together to form some logical pattern. This logical pattern explains how the variables in the system relate one to another. Models may be presented in various shapes and in varying degrees of mathematical formalism. A model which does not need any mathematical formalism is called mental model. An example of this type of model is the knowledge of a human being of driving a car. To drive a car it requires that a human mind should be able to perceive the consequence of turning the steering wheel with relation to the whole position of the car. Different models which involve tables or plots are called graphical models.

A mathematical model is required to be able to further explain a system to be used in more advanced applications. A mathematical model involves difference or differential equations, fuzzy models and neural networks. These mathematical models are used for simulation and prediction purposes. The other type of model which is built up by many interconnected subroutines and lookup tables in computerized descriptions is called a software model. This type of model has a very important role in decision making for complicated systems [2].

1-1-3 System identification

A mathematical model can be obtained either from a process of modelling which does not involve any experimentation on the actual system, or through a system identification process. System identification mainly focuses on an attempt to obtain a mathematical model of a real-life actual system, strictly by using only measured data gathered from experimentation, without the need of having knowledge of physical properties or physical structure of the system. The main objective of system identification is thus to find a mathematical model based on the available measured data, where the estimated model should be able to represent the true system the best, relevant to its intended purpose. This type of modelling is also often referred to black-box modelling.

System identification becomes extremely useful in the case where it is impossible to model a particular system using first-principle model, i.e. systems that do not have first principal relations, such as modelling of brain pattern, or econometric modelling. Another case where black-box modelling can become extremely handy is in the case where first-principle models are too complex, e.g. modelling a huge industrial plant that involves high number of equations, and high level of model complexity.

1-2 Main aspects in a system identification process

This section mainly discusses about the main aspects that are involved in the identification process of an LTI (Linear Time Invariant) system. Consider the schematic diagram in Figure 1-1. In this schematic diagram, \( u \) and \( y \) are the input and the output signal respectively, where \( v \) is a nonmeasurable disturbance on the system’s output, \( S \) is the true system to be modelled. A parameterized set of models is denoted by \( M(\theta) \), where the parameter \( \theta \) range over a parameter set \( \Theta \subset \mathbb{R}^d \).
1-2 Main aspects in a system identification process

Figure 1-1: Schematic representation of system identification

Figure 1-2: Identification procedure
4 Introduction

When confronting a model $M$ in the model set $M(\theta)$ with the input-output measurement data, $(y, u)$, an error or also called residual signal will be present. This error or the residual signal is denoted by $\varepsilon(\theta)$. The existence of the residual signal is caused by the fact that the measurement data is always influenced by noise disturbance and the effect caused by unmodelled dynamics of the system. As a result, the model estimate will never be able to fully represent the true system.

1-2-1 Identification experiment

In order to conduct identification of a system, measurement of the input signal entering the system and the output signal needs to be gathered, and this input-output data is represented by a data set $Z^N := \{y(t), u(t)\}_{t=1, \ldots, N}$. Choosing and designing the right input signal is a crucial matter.

An input signal for identification experiment should be able to excite the system properly, such that the experiment yields a data set that can be used for the identification process, which results a model estimate that consistently represents the relevant dynamics of the system. This aspect of choosing and designing a suitable input signal is called experiment design.

1-2-2 Model set and identification criterion

In Figure 1-1, $M(\theta)$ represents a set of parameterized models, where the parameter estimate $\theta$ can range over a parameter set $\Theta \subset \mathbb{R}^d$. For example we have a model that is represented by the following difference equation:

$$y(t) + a_1 y(t-1) + a_2 y(t-2) = b_1 u(t-1) + b_2 u(t-2) + e(t) \quad (1-1)$$

Where the parameter vector $\theta \in \mathbb{R}^4$ is composed by four parameters, $\theta = [a_1, a_2, b_1, b_2]^T$. We can see from this simple example that by varying the coefficients of the difference equation over the parameter set $\Theta \subset \mathbb{R}^d$, a model set $M(\theta)$ is obtained.

When the model set $M(\theta)$ is confronted with the input and output signal as in Figure 1-1, an error or residual signal will be present. Each model from the model set $M(\theta)$ will have its own residual signal, which means this residual signal is not only a function of time, it is also a function of the parameter vector $\theta$.

Following from the previous example, the corresponding residual signal is as follows:

$$\varepsilon(t, \theta) = y(t) + a_1 y(t-1) + a_2 y(t-2) - b_1 u(t-1) - b_2 u(t-2) \quad (1-2)$$

This residual signal can then be used as a criterion in construction the model estimate. The most common choice is using the power of the residual signal as a cost function:

$$V_N(\theta) = \frac{1}{N} \sum_{t=1}^{N} \varepsilon^2(t, \theta) \quad (1-3)$$

The parameter estimate is obtained by finding a value $\theta$ that minimizes the cost function $V_N(\theta)$.

$$\hat{\theta}_N = \arg \min \theta V_N(\theta, Z^N) \quad (1-4)$$
The relation above shows that the parameter estimate \( \hat{\theta}_N \) is the value of \( \theta \) that minimizes the cost function \( V_N(\theta) \).

More about this identification criterion will be explained in Chapter 2

1-2-3 Model validation

After the three previous aspects are passed, now we should already have one model candidate in the model set that best describes the data set according to the chosen criterion (minimization of the cost function). To verify whether this model candidate is valid for its intended purposes, model validation test needs to be conducted. Model validation test involves various procedures to test how the model relates to the measured data, to the prior knowledge of the system, and to its intended use. A good model estimate will then be accepted at a certain level of confidence, and an inadequate or bad model estimate will be rejected.

Even a good model estimate can never fully describe the true system. Thus the best a model estimate can only be, is only as far as it is being able to represent well or satisfactorily certain aspects of our particular interest in the system. When a model is invalidated, redesigning identification experiment and the change in the model set and in the identification criterion can improve the quality of a model relevant to a certain aspect being the particular interest.

Prior knowledge of the system plays an important role in affecting all these identification aspects. We can then summarize all these aspects in system identification by a schematic diagram in Figure 1-2.

1-3 Objective of this research

In system identification, in order to verify whether a model candidate is valid for its intended purposes, model validation test needs to be conducted. Based on the result of the validation test, good model estimate will then be accepted at a certain level of confidence, and an inadequate or bad model estimate will be rejected.

The validation process in this research specifically aims to a model structure validation test of the model estimate. This means the main interest of the validation process is in testing whether or not a model estimate is a consistent estimate of the true system. It can also alternatively be explained as testing whether the true system is contained in the model set \( (S \in M) \) or it is not \( (S \notin M) \), which also means whether or not we have bias in our model estimate. It may be confusing for some readers who are unfamiliar with the basic system identification concept regarding the use of term, 'consistent estimate', 'model estimate', 'true system\( (S) \)', 'model set\( (M) \)' in this introduction chapter. However, explanation of these basic system identification terms will be treated in Chapter 2 of this thesis report.

The available model structure validation methods mainly only deal with the model validation for an open-loop system. For example, the most widely used standard validation test [1], which is the one used in Matlab’s system identification toolbox [3] only deals with validation of an open-loop model. At this point we might also aware that many varieties of systems in life need to be operated under the presence of a feedback loop such as for stability reasons,
fulfilling the safety requirements and the need of meeting the specified performance of a system. The identification process of these systems therefore needs to be conducted under the presence of feedback loop. Based on the given facts, we can see that there is a clear need to develop a model structure validation technique for the closed-loop identification problem.

In this research, two closed-loop model structure validation techniques will be considered. The first validation method will be directly adapted from the the standard open-loop model validation [1, 2]. This first validation method considered in this thesis will be called as closed-loop point-wise model validation test. Since this closed-loop point-wise test is directly adapted from the standard validation test, the shortcomings in the standard validation test also applies to this closed-loop point wise test. For the details about the shortcomings in the standard open-loop validation test, the reader is suggested to consult [4].

In [4], an improved validation method is proposed in order to overcome the shortcomings in the standard open-loop validation test. This improved validation test uses vector-valued evaluation of a sample cross-correlation function. By applying the vector valued validation test, we can detect the presence of bias in our model estimate. Of course the reader at this stage is not expected to understand in detail what this vector-valued test really is. The purpose of mentioning the term vector-valued in this chapter is solely to introduce the reader to this term as early as possible, in order to familiarize the reader to the term that is the main subject of this thesis. Another application of the vector-valued test that is also motivated from [4] used for a change detection problem of a plant system operated in closed-loop, can be found in [5].

The improved method in the open-loop case as presented in [4] is the central subject that motivates this thesis. By considering and motivated by all the facts given above, the main objective of this thesis project is to propose some extension of the improved model validation of the open-loop case to the closed-loop case. That is, to try to evaluate the implementation and adaptation of the improved model validation test from the open-loop case to the closed-loop case. This is done by the evaluation from the theoretical point of view as well as from the “practical” point of view, conducted on the Matlab simulation environment.

By considering all the facts presented above, the following research question can be formulated:

**Research question:** How can we apply the concept of an accurate model validation test that is capable of detecting the presence of bias(if it exists) in our closed-loop model estimate?

### 1-4 Chapter outline

In Chapter 2 of this thesis report, the basic concept of prediction-error identification which covers the concept of one-step-ahead prediction, the identification criterion, black-box model structure, and basic model validation technique is explained. Then in Chapter 3, the basic concept of direct closed-loop identification will be presented. The closed-loop validation concept, the closed-loop point-wise model validation, and the closed-loop vector-valued validation test will be explained in Chapter 4. The concept of closed-loop model and theory presented in Chapter 4, will be simulated in Chapter 5. Then in the last chapter of this thesis report, conclusions will be given as well as possibilities of future research.
Chapter 2

Prediction-error identification

2-1 Introduction

Identification of parametric models is an extremely useful tool to represent dynamics in a system in a situation where non-parametric representation of a system such as frequency response, step or impulse response of a system are insufficient. This parametric model can serve several purposes such as for process simulation, predicting future behaviour of a system, and controller design purposes.

The main feature that the parametric model can offer is that the essential dynamics of a system can be represented by a limited number of coefficients. This makes tasks like designing controller or system simulation become much simpler and more efficient compared to when the non-parametric model are utilized to solve similar tasks.

In this chapter the concept of prediction error identification will be presented. The identification of Linear Time Invariant (LTI) system is the main focus of this chapter. The open-loop scheme of a system will be used in this entire chapter, to help understand the basic concept of prediction error identification.

The closed-loop scheme will not be discussed until the next chapter. In this research the identification phase for the closed-loop case utilizes direct closed-loop identification method, thus practically we can just use the similar open-loop identification concept in the closed loop case. Since in this direct closed-loop identification we can treat the data set as if it were collected from an open-loop identification. The concept of this direct closed-loop identification will be further discussed in the next chapter.

2-2 Systems and disturbances

A dynamical system or the true system that we want to estimate is represented by the following relation:
8 Prediction-error identification

This true system \( \mathcal{S} = \{G_0(z), H_0(z)\} \) is represented in block diagram Figure 2-1. Where \( G_0(z) \) is the true plant system, which is a strictly proper stable rational discrete-time transfer function, \( y(t) \) is the output signal, \( u(t) \) is the input excitation signal, \( v(t) \) is the disturbance signal. This disturbance signal can represent the measurement noise, the effect of stochastic disturbance, the effect of non-measurable input signal, etc. For the sake of simplicity the signal \( v(t) \) in this paper will always be referred as a measurement noise.

The measurement noise is comprised of the true noise filter \( H_0(z) \) and the white noise sequence \( e(t) \) with variance \( \sigma^2_e \). Where the true noise filter \( H_0(z) \) is stable, inversely stable and monic discrete-time transfer function. For example:

\[
H_0(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots}
\]  

is monic if \( a_0 = b_0 = 1 \)

The true system \( \mathcal{S} = \{G_0(z), H_0(z)\} \) is of course unknown and therefore needs to be estimated using the available data set gathered from this true system. The true system \( \mathcal{S} \) can be represented by the imaginary concept of having true parameter vector \( \theta_0 \). This true parameter vector \( \theta_0 \) is of course only exist in the asymptotical case or theoretical case of system identification, where we could consider having unlimited number of data points. As in the real practical application of system identification it will of course never be able to be found.

Based on the equation representing the data generating system Eq. (2-1), we can simply observe that the output signal is made up of two different contributions, the contribution from the input signal, and the contribution of the disturbance signal (measurement noise) which is independent from the input signal.

2-3 One-step-ahead prediction error

The main objective of prediction error identification is to find both plant model estimate \( G(z, \theta) \) and noise model \( H(z, \theta) \) that can represent the true system \( \mathcal{S} = \{G_0(z), H_0(z)\} \) the
best. One of the attributes of a model estimate is its capability to predict future values of the output signal and therefore lead to the construction of the residual signal which is used as the identification criterion and in model validation.

The input-output data $Z^N = \{u(t), y(t)|t = 1...N\}$ is generated by the following data generating system:

$$y(t) = G_0(z)u(t) + H_0(z)e(t)$$

(2-3)

Suppose that based on this data set we have computed a model $G(z, \theta)$ and $H(z, \theta)$, that exist in the model set $M$. The simulated plant and noise model is given by the following relation:

$$y(t, \theta) = G(z, \theta)u(t) + H(z, \theta)e(t)$$

(2-4)

when $\theta = \theta_0$, $y(t) - y(t, \theta) = 0$

Now we want to compute the signal $y(t, \theta)$, but since the white-noise sequence $e(t)$ is unknown, as a consequence the signal $y(t, \theta)$ needs to be predicted. The predictor $\hat{y}(t, \theta)$ should be chosen so that the power of $(y(t) - \hat{y}(t, \theta))$ is minimized. The derivation of the expression that can be used to calculate $\hat{y}(t, \theta)$ is explained in the following subsection.

### 2-3-1 One-step-ahead prediction of $y(t)$

The prediction of $y(t, \theta)$ can be calculated by first deriving a suitable expression.

The true system in Eq. (2-3) can be re-written as:

$$y(t) = G_0(z)u(t) + [H_0(z) - 1]e(t) + e(t)$$

(2-5)

By substituting $e(t) = H_0^{-1}(z)[y(t) - G(z)u(t)]$ in the second term of Eq. (2-5), the following relation is obtained:

$$y(t) = G_0(z)u(t) + [H_0(z) - 1]H_0^{-1}(z)[y(t) - G(z)u(t)] + e(t)$$

(2-6)

and by simplifying the previous equation we have:

$$y(t) = H_0^{-1}(z)G_0(z)u(t) + [1 - H_0^{-1}(z)]y(t) + e(t)$$

(2-7)

Having the assumption that the true noise filter $H_0(z)$ to be proper, monic, stable and inversly stable, the consequence is $H_0^{-1}$ is also proper, monic, and stable. Thus we will have:

$$H_0(z) = \frac{1}{1 + d_1z^{-1} + d_2z^{-2} + \cdots}$$

(2-8)

$$\frac{1}{H_0(z)} = 1 + d_1z^{-1} + d_2z^{-2} + \cdots$$

(2-9)
where \( \{d_i\} \) is a sequence tends to zero.

Assuming that \( G_0(z) \) is strictly proper (no direct feedthrough term), and having \( H_0(z) \) to be monic, proper, and stable, a direct consequence is that the expression

\[
H_0^{-1}(z)G_0(z)u(t) + [1 - H_0^{-1}(z)]y(t)
\]  

(2-10)

as in Eq. (2-7), is fully determined by the input-output data up until time \( t - 1 \) (past observations, \( y^{t-1} u^{t-1} \)). The one-step-ahead predictor of the output signal \( y(t) \) is then denoted as:

\[
\hat{y}(t|t - 1) = H_0^{-1}(z)G_0(z)u(t) + [1 - H_0^{-1}(z)]y(t)
\]  

(2-11)

as a result referring to Eq. (2-7):

\[
y(t) = \hat{y}(t|t - 1) + e(t)
\]  

(2-12)

What this equation represents is that the signal \( y(t) \) consists of the one-step-ahead predictor \( \hat{y}(t|t - 1) \) that is available at time \( t - 1 \) and the white noise sequence \( e(t) \) that is unknown.

### 2-3-2 One-step-ahead prediction error

From Eq. (2-12), the one-step-ahead prediction error (or residual signal) can be denoted as:

\[
\varepsilon(t) := y(t) - \hat{y}(t|t - 1)
\]  

(2-13)

By substitution of Eq. (2-11) to the Eq. (2-13), the prediction error is now:

\[
\varepsilon(t) := H_0^{-1}(z)[y(t) - G_0(z)u(t)]
\]  

(2-14)

Considering the plant model \( G(z, \theta) \) and the noise model \( H(z, \theta) \), the output predictor \( \hat{y}(t|t - 1) \) as in Eq. (2-11), now become:

\[
\hat{y}(t, \theta) = H^{-1}(z, \theta)G(z, \theta)u(t) + [1 - H^{-1}(z, \theta)]y(t)
\]  

(2-15)

and the prediction error (residual signal) \( \varepsilon(t, \theta) \) is defined as follows:

\[
\varepsilon(t, \theta) = y(t) - \hat{y}(t, \theta)
\]  

(2-16)

From Eq. (2-14) the residual signal now become:

\[
\varepsilon(t, \theta) := H^{-1}(z, \theta)[y(t) - G(z, \theta)u(t)]
\]  

(2-17)

By substituting \( y(t) \) in Eq. (2-17) with \( G_0(z)u(t) + H_0(z)e(t) \) the prediction error (or the residual signal) can be expressed as:
Consider an ideal case when $\theta = \theta_0$, the residual signal in Eq. (2-18) will then be exactly equal to the white noise sequence, $\varepsilon(t, \theta_0) = \epsilon(t)$.

2-4 Black box model structure

Using set of measured data $Z^N$ that is generated by the true system, we want to find the best parametric models $G(z, \theta)$ and $H(z, \theta)$ that estimates the unknown transfer function $G_0$ and $H_0$.

The prediction error identification method is defined by a set of transfer function models $G(z, \theta)$ and $H(z, \theta)$. These models are considered as candidates of the optimal model that best represents the true system. The model set is denoted as $\mathcal{M}$, and defined as collections of models:

$$\mathcal{M} := \{(G(z, \theta), H(z, \theta)) \mid \theta \in \Theta \subset \mathbb{R}^d\} \quad (2-19)$$

Where a real valued parameter vector $\theta$ ranges over real value numbers of dimension $d$, with the set of parameter vector $\Theta$. The parameterization that determines the specific relation between a parameter $\theta \in \Theta$ and model $(G(z, \theta), H(z, \theta))$ within $\mathcal{M}$ is defined by the following mapping:

$$\mu : \Theta \rightarrow \mathcal{M} \quad (2-20)$$

The set of models $\mathcal{M}$ have the same properties as explained before, which are the properness and stability of the transfer function $G_0(z)$, as well as the monicity, stability, and stability of the inverse of $H_0(z)$. The most common way to represent the parameterization in $G(z, \theta)$ and $H(z, \theta)$ is in terms of polynomials in $z^{-1}$ variable.

These polynomial representation will be further explained by depicting a case of ARX(Auto Regressive eXogenous) model set. An ARX model set is determined by two polynomials:
Prediction-error identification

\[ A(z, \theta) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{n_a} z^{-n_a} \]
\[ B(z, \theta) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{n_b-1} z^{-n_b+1} \] (2-21)

with parameter vector:
\[ \theta := [a_1 \ a_2 \ \cdots \ a_{n_a} \ b_0 \ b_1 \ \cdots \ b_{n_b-1}]^T \] (2-22)

Based on the parameter vector \( \theta \), the ARX model set \( \mathcal{M} \) is defined as:
\[ \mathcal{M} = \{(G(z, \theta), H(z, \theta)) | \theta \in \Theta \subset \mathbb{R}^{n_a+n_b} \} \] (2-23)

where \( G(z, \theta) = \frac{B(z, \theta)}{A(z, \theta)} \), \( H(z, \theta) = \frac{1}{A(z, \theta)} \)

The expression in Eq. (2-23) shows that for the ARX model structure, the parameter vector \( \theta \) ranges over real value numbers of dimension \( n_a + n_b \).

Thus the input-output relation of ARX model structure is given as:
\[ y(t) = \frac{B(z, \theta)}{A(z, \theta)} u(t) + \frac{1}{A(z, \theta)} e(t) \] (2-24)

The parameterization \( \mu \) in Eq. (2-20) determines whether or not there exist relation between the two rational functions \( G(z, \theta) \) and \( H(z, \theta) \). When ARX model is utilized, all models in the model set have common denominator in \( G(z, \theta) \) and \( H(z, \theta) \). This characterization of a model set is related to the structural property of a model set, therefore the name model structure is used.

\[ B(z, \theta) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{n_b-1} z^{-n_b+1} \]
\[ A(z, \theta) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{n_a} z^{-n_a} \]
\[ C(z, \theta) = 1 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_{n_c} z^{-n_c} \]
\[ D(z, \theta) = 1 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_{n_d} z^{-n_d} \]
\[ F(z, \theta) = 1 + f_1 z^{-1} + f_2 z^{-2} + \cdots + f_{n_f} z^{-n_f} \] (2-25)

Other than ARX model structure there exist several other model structures as in Table 2-1. Different polynomials in Eq. (2-25) are then utilized to characterize these different model structures. We can see from the set of polynomials in Eq. (2-25), the polynomials \( A, C, D, F \) are all monic.

The parameter vector for the model structures in Table 2-1, in general expression is defined as:
\[ \theta := [a_1 \ a_2 \ \cdots \ a_{n_a} \ b_0 \ b_1 \ \cdots \ f_{n_f}]^T \] (2-26)
### Table 2-1: Black-box model structures

<table>
<thead>
<tr>
<th>Model structure</th>
<th>$G(z, \theta)$</th>
<th>$H(z, \theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARX</td>
<td>$\frac{B(z, \theta)}{A(z, \theta)}$</td>
<td>$\frac{1}{A(z, \theta)}$</td>
</tr>
<tr>
<td>ARMAX</td>
<td>$\frac{B(z, \theta)}{A(z, \theta)}$</td>
<td>$\frac{C(z, \theta)}{A(z, \theta)}$</td>
</tr>
<tr>
<td>OE-Output Error</td>
<td>$\frac{B(z, \theta)}{F(z, \theta)}$</td>
<td>1</td>
</tr>
<tr>
<td>BJ - Box Jenkins</td>
<td>$\frac{B(z, \theta)}{F(z, \theta)}$</td>
<td>$\frac{C(z, \theta)}{D(z, \theta)}$</td>
</tr>
<tr>
<td>FIR - Finite Impulse Response</td>
<td>$B(z, \theta)$</td>
<td>1</td>
</tr>
</tbody>
</table>

Of course this does not mean that we can have all the polynomials in one particular parameter vector $\theta$. What this notation means is that for example when we have Box-Jenkins model structure, the parameter vector in this model structure does not contain the polynomial $A$, but it contains the polynomials $B$, $C$, $F$, and $D$. As also for the OE model structure for example, only contains the polynomial $B$ and $F$, thus the polynomials $A$, $C$, and $D$ are equal to zero, and so thus similar concept of formation combination of polynomial coefficients applies for other types of model structure.

The model set $\mathcal{M}$ in general is then defined as follows:

$$\mathcal{M} = \{(G(z, \theta), H(z, \theta)) | \theta \in \Theta \subset \mathbb{R}^{n_\theta}\}$$

(2-27)

with $n_\theta$ is the degree of the polynomials used in the parameter estimate $\theta$, e.g. for ARMAX model structure, $n_\theta = n_a + n_b + n_c$, where $n_a$ is the degree of polynomial $A$, $n_b$ for polynomial $B$, and $n_c$ for polynomial $C$.

#### 2-4-1 Parameter dependency of the model $G(z, \theta)$ and $H(z, \theta)$

One important property of black-box model structure that worth to mention is parameter dependency of $G(z, \theta)$ and $H(z, \theta)$ in the model structure.

The ARX and ARMAX model structure are dependently parameterized, since they have common denominator of $G(z, \theta)$ and $H(z, \theta)$, see Table 2-1. Unlike these two model structures, OE, BJ, and FIR model structures have the property that the plant model $G(z, \theta)$ and the noise model $H(z, \theta)$ are parameterized independently in the corresponding model sets, since there is no common polynomial factor in $G(z, \theta)$ and $H(z, \theta)$.

The parameter vector for these independently parameterized model structure can be decomposed into: $\theta = \begin{bmatrix} \rho \\ \eta \end{bmatrix}$ such that:

$$\mathcal{M} = \{(G(z, \rho), H(z, \eta)) | \rho \in \Gamma \subset \mathbb{R}^{d_1}, \eta \in \Delta \subset \mathbb{R}^{d_2}\}$$

(2-28)
The previous expression shows that $\rho$ is the parameter vector that represents the coefficients of the polynomials in the plant model $G(z, \theta)$, where $\eta$ is the coefficients of the polynomials in the noise model $H(z, \theta)$. These two parameter vectors $\rho$ and $\eta$ can have different dimensions. This property of parameter dependency is an important aspect to consider in the validation of the model estimate.

2-4-2 Choice of model structure

The choice of model structure is an important issue in the identification process. Choosing a wrong model structure will lead to a “bad” model estimate. The decision of using a specific model structure can be based on the prior information or knowledge about the system being modelled. In doing system identification process we might end up in one of the three following conditions:

- **Condition 1:** The true system is contained in the model set $S \in \mathcal{M}$

In this first condition, we consider a case where we have selected a correct type of model structure and the order of our model with respect to the true system. For example, when we have an ARX true system we will be in the condition where the true system is contained in the model set, $S \in \mathcal{M}$, when we have chosen an ARX, ARMAX or Box-Jenkins model structure with the same or higher order with respect to the true system. As a consequence, we will have $G(z, \theta^*) = G_0(z)$ and $H(z, \theta^*) = H_0(z)$, where $G(z, \theta^*)$ and $H(z, \theta^*)$ are the asymptotic plant and noise model, which can be represented by the asymptotic parameter estimate $\theta^*$. This asymptotic parameter estimate will be explained in detail in the next section.

- **Condition 2:** The true system is not contained in the model set $\{S \notin \mathcal{M} \text{ with } G_0 \notin \mathcal{G}\}$

In this condition suppose that we have chosen our model estimate which has lower order compared to the true system. This means that the true system is not contained in the model set, since our model estimate contains bias with respect to the true system. The expression $\mathcal{G}$ represents the model set for the plant model estimate $G(z, \theta)$. For example we are trying to identify an $n^{th}$ order ARX true system using any model structure as in Table 2-1 with order smaller than order $n$ of the true system $S = \{G_0(z), H_0(z)\}$. Since the order of our model estimate $\{G(z, \theta), H(z, \theta)\}$ will always be smaller than the order of the true system, this means that we will always have $G(z, \theta^*) \neq G_0(z)$ and $H(z, \theta^*) \neq H_0(z)$.

- **Condition 3:** The true system is not contained in the model set $\{S \notin \mathcal{M} \text{ with } G_0 \in \mathcal{G}\}$

In this case suppose that we are in a condition where we have chosen a certain type of model structure where the order or the plant model $G(z, \theta)$ is the same as or higher than the true plant system $G_0$, with $H(z, \theta^*) \neq H_0(z)$. Two different examples of this condition can be presented as follows:

1. For example we have an ARX true system with order $n$, and then in the identification process we select an OE model estimate with the same order $n$, thus in this case since OE is an independently parameterized model structure we can still have $G(z, \theta^*) = G_0(z)$ with $H(z, \theta^*) \neq H_0(z)$.
2. Another example, suppose that we have a Box-Jenkins true system with order $n$ for both the plant system $G_0(z)$ and noise filter $H_0(z)$, and we try using ARX or ARMAX model structure with the same order $n$, to find our model estimate. Since ARX and ARMAX are dependently parameterized model structures, thus, although the order of plant model $G(z, \theta)$ is the same as the order of $G_0(z)$, we will never have a consistent estimate of $G(z, \theta)$. As a consequence, in this type of situation we will always have $G(z, \theta^*) \neq G_0(z)$ and $H(z, \theta^*) \neq H_0(z)$. The detail explanation of the asymptotic parameter vector $\theta^*$ is given in the next section.

It is important to remember that at this point we are still discussing the identification of a system operated in an open-loop condition. The properties of having the case where $\{S \notin \mathcal{M} \text{ with } G_0 \in \mathcal{G}\}$ in the closed-loop identification that we will see in the next chapter, will be different from the open-loop case we are discussing at this moment.

In practice the Box-Jenkins model structure is the most widely used, since it is very rare for a real system in life to possess properties like in ARX or ARMAX model structure, where the denominator of the plant and the noise model are identical.

### 2-5 Identification criterion

Having the true system $S = \{G_0(z), H_0(z)\}$, using prediction error identification we want to find model estimate $G(z, \theta)$ and $H(z, \theta)$ that minimizes the cost function (power of the residual signal) denoted by $V(\theta)$. The residual signal $\varepsilon(t, \theta)$ is computed as in Eq. (2-17). Here we will see two cases of finding the minimizing parameter vector, for an ideal criterion (asymptotic case) as well as for a tractable criterion (practical case).

- **Ideal identification criterion**
  An asymptotic parameter vector that minimizes the cost function is denoted by $\theta^*$. This unique parameter vector $\theta^*$ is the solution of the following minimization problem:

  $$\theta^* = \arg \min_{\theta} \tilde{V}(\theta) \quad (2-29)$$

  with:

  $$\tilde{V}(\theta) = \bar{E}\varepsilon^2(t, \theta) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} E\varepsilon^2(t, \theta) \quad (2-30)$$

  Under the condition that the true system is contained in the model set, $S \in \mathcal{M}$, and by having a persistently exciting input signal for sufficient order, we will have $\theta^* = \theta_0$. Where $\theta_0$ is the true parameter vector that represents the true system $S = \{G_0(z), H_0(z)\}$.

- **Tractable identification criterion**
  In reality, since we always have limited $N$ number of measured data points, the ideal criterion of course will not be able to be found. Therefore the power of the residual
signal, which is used as a cost function, will need to be estimated from the available limited data points. The cost function for the tractable case estimated using available data set $Z^N$ is given as follows:

$$V(\theta, Z^N) = \frac{1}{N} \sum_{t=1}^{N} \varepsilon^2(t, \theta)$$

$$= \frac{1}{N} \sum_{t=1}^{N} (H^{-1}(\theta)(y(t) - G(\theta)u(t)))^2$$

The parameter estimate $\hat{\theta}_N$ is obtained by solving the following minimization problem:

$$\hat{\theta}_N = \arg\min_{\theta} V_N(\theta, Z^N)$$

This parameter estimate $\hat{\theta}_N$ is the estimate of the ideal parameter vector $\theta^*$, with property that $\hat{\theta}_N$ will asymptotically converge ($N \to \infty$) to normally distributed random variable with mean $\theta^*$ and variance, $P_\theta$:

$$\hat{\theta}_N \sim \mathcal{N}(\theta^*, P_\theta)$$

To put it in a simpler explanation, the parameter estimate $\hat{\theta}_N$ will converge asymptotically to $\theta^*$ with probability 1 when $N \to \infty$, where in this case the variance $P_\theta$ will become closer and closer to zero when $N \to \infty$, for more detail consult [1].

2-6 Model validation

The main objective of model validation is to arrive at a conclusion whether or not the user is satisfied with the performance of the identified model for its intended application. Of course to be able to know exactly whether or not an identified model performs well, the model needs to be directly applied for its intended purpose. However, before actually applying the model for its intended purpose, a model validation test should reveal whether the user can have confidence in the identified model. This is done by confronting the model with the prior knowledge of the real system that the user has in hand. Obviously when there is a persistent inconsistency of the model’s behaviour with respect to the real system, we can pretty much conclude that the identified model can not be used as a representation of the real system for the intended purpose. The inconsistency of a model can be checked by using applicable data-dependent tools. These data-dependent tools will be explained briefly as follows:

- **Input-output behaviour**

  The identified model can be confronted with characterization of the system such as using nonparametric models. Comparing the frequency response of the identified model with the nonparametric model(frequency plot) the user can confirm whether all the important dynamics of the system has been modelled. This is of course a representation
of an ideal case, since the nonparametric model depends strongly on the chosen windowing operation, thus this matter needs to be handled carefully. For example when the windowing operation is chosen to be too big, the dynamic of the system might be flatten out, and will not appear anymore(model bias). When it is too small, the nonparametric model will have high variance, and it will be difficult to determine the important dynamics of the system. Comparisson of time delay, modelled and measured step responses and other transient signals can also be used as validation tools.

- **Residual tests**
  
The residual signal provides a very useful information when it comes to a model structure validation in system identification. The main concern of model structure validation is to confirm whether or not the chosen model set can contain the true system ($S \in \mathcal{M}$ or $S / \in \mathcal{M}$). When we have consistent model estimate, $S \in \mathcal{M}$, the asymptotic plant and noise model will be equal to the true system, $G(\theta^*) = G_0$ and $H(\theta^*) = H_0$. In the case where the model set can not contain the true system, $S \notin \mathcal{M}$ we can have these two following cases:

- For the dependently parametrized model structure (ARX and ARMAX), in the case $S \notin \mathcal{M}$ the residual test can not detect whether we are in the case \{ $G(\theta^*) = G_0$ and $H(\theta^*) \neq H_0$ \} or \{ $G(\theta^*) \neq G_0$ and $H(\theta^*) \neq H_0$ \}.
- For the independently parameterized model structure (BJ,OE, and FIR model estimate), in the case $S \notin \mathcal{M}$ the residual test can detect whether we are in the case \{ $G(\theta^*) = G_0$ and $H(\theta^*) \neq H_0$ \} or \{ $G(\theta^*) \neq G_0$ and $H(\theta^*) \neq H_0$ \}.

The residual test used in the model structure validation test is done by evaluating the autocorrelation function of the residual signal $\varepsilon(t, \theta)$, $R_\varepsilon(\tau)$, and the cross-correlation of the residual signal with the input signal $u(t)$, $R_{\varepsilon u}(\tau)$. Suppose that we consider an asymptotic case, the identified parameter vector is then a unique optimum parameter vector $\theta^*$ that minimizes the power of the residual signal. Considering the expression of the residual signal in Eq. (2-18), the residual signal for the parameter vector $\theta^*$ is then:

$$\varepsilon(t, \theta^*) = G_0(z) - G(z, \theta^*) H(z, \theta^*) u(t) + H_0(z) e(t)$$  \hspace{1cm} (2-34)

From the expression in Eq. (2-34) we can see that when the asymptotic plant model is equal to the true plant system, $G(\theta^*) = G_0$, then the first term in Eq. (2-34) is equal to zero. This also means that the residual signal $\varepsilon(t, \theta^*)$ is independent from the input signal $u(t)$, which also mean that the cross-correlation between the residual signal and the input signal is equal to zero for all time lags $\tau$, $R_{\varepsilon u}(\tau) = 0$, $\forall \tau$. This cross-correlation function is defined as:

$$R_{\varepsilon u}(\tau) \triangleq \mathbb{E}(\varepsilon(t)u(t - \tau))$$  \hspace{1cm} (2-35)

Referring back to Eq. (2-34), when we have $R_{\varepsilon u}(\tau) = 0$, and the asymptotic noise model is equal to the true noise filter, $H(z, \theta^*) = H_0$, then the residual signal will be equal to the white noise sequence, $\varepsilon(t, \theta^*) = e(t)$. This also means that the autocorrelation of the residual signal become:
\[ R_e(\tau) = \sigma_e^2 \delta(\tau) = \begin{cases} \sigma_e^2 & \text{for } \tau = 0 \\
0 & \text{for } \tau \neq 0 \end{cases} \quad (2-36) \]

with \( \delta(\tau) \) a dirac delta function, \( \sigma_e^2 \) is the white noise variance and

\[ R_e(\tau) \triangleq \bar{E}(\varepsilon(t)\varepsilon(t-\tau)) \quad (2-37) \]

Based on these facts we can see that the validation of the identified model \( G(z, \theta) \) and \( H(z, \theta) \) can be done by evaluating both the cross-correlation \( R_{eu}(\tau) \) as well as the autocorrelation of the residual signal \( R_e(\tau) \).

For the practical (non-asymptotic) case where we only have limited data points, the residual test is done on the basis of the sample autocorrelation of the residual signal, \( \hat{R}_e(\tau) \) and the sample cross-correlation of the residual and the input signal, \( \hat{R}_{eu}(\tau) \). The sample auto and cross-correlation are given as follows:

\[ \hat{R}_e(\tau) := \frac{1}{N} \sum_{t=1}^{N} \varepsilon(t, \hat{\theta}_N)\varepsilon(t-\tau) \quad (2-38) \]

\[ \hat{R}_{eu}(\tau) := \frac{1}{N} \sum_{t=1}^{N} \varepsilon(t, \hat{\theta}_N)u(t-\tau) \quad (2-39) \]

with \( \tau = 0, \cdots n_\tau - 1 \), where \( n_\tau \) is a user chosen maximum number of lags.

Since we only have limited data points, even when the true system is contained in the model set, \( S \in \mathcal{M} \), the sample cross-correlation (\( \hat{R}_{eu}(\tau) \)) and the sample autocorrelation (\( \hat{R}_e(\tau) \) for \( \tau \neq 0 \)) will never be exactly equal to zero. Therefore to check whether they are actually close enough to zero, these values of the sample auto and cross-correlation function for different time lags must be compared with a certain bound. When these values are smaller than the defined bound then we can conclude with a certain probability that the residual signal is indeed uncorrelated with the input signal, and the residual signal is indeed a realization of a white noise signal. Which also means that we can confirm the validation our model estimate. The method of how to define the bound for the validation of open-loop case can be referred to [1, 2]

### 2-7 Important remarks

The residual model validation method explained in this chapter can be referred to as a standard model validation test or point-wise test [1],[2]. This validation test is the one that is used in the Matlab system identification toolbox [3] . This test is called point-wise because this test evaluates the value of the sample auto and cross-correlation with respect to the defined value of bound, for each values of time lags \( \tau \), instead of evaluating the auto or correlation function for overall \( \tau \) altogether.

To be a little bit more specific, the residual validation test explained in this chapter is used for system operating in open-loop configuration. Since in this thesis project our main concern
is on the validation of a model identified from closed-loop system using direct-identification (explained in the subsequent chapters), thus the model validation of the open-loop system will not be explained further in this thesis. However, the improvement made for the closed-loop model validation in this thesis is adapted from the similar concept that is used in the open-loop case. For details about the standard open-loop model validation method (used in Matlab system identification toolbox, [3]) please refer to [1],[2], and [6]. For the improved model validation test in open-loop case (using vector-valued test) consult [4].

2-8 Short summary

In this chapter the basic concept of prediction error identification for an open-loop system is presented. This covers the general concept of black-box model structure, choice of model structure, one-step-ahead prediction error, and identification criterion. Several different probable conditions regarding the choice of model structure with respect to the true system was also explained in section 2-4-2. In the latter part of this chapter we have also seen the basic concept of residual model validation test for the open-loop system identification.
Chapter 3

Closed-loop identification configuration

In this chapter the closed-loop system configuration used in this research is presented. The general approach of direct closed-loop identification and the situations where the direct closed-loop identification can or can not produce a consistent model estimate are also covered in this chapter. This chapter explains as well the decomposition of the residual signal $\varepsilon(t, \theta)$, which will be useful for comprehensive analysis purposes in the model validation step.

3-1 Closed-loop identification

Many systems in life such as industrial production processes, mechanical servo systems, or even biological processes, they typically exhibit unstable dynamical behaviour when operated under open-loop condition. Thus results in the need for these types of systems to be operated under feedback loop. Other reasons such as to fulfil the safety requirements of a process and the need of meeting the specified performance of a system can also be considered as strong factors for not allowing varieties of systems to be operated in open-loop configuration. The presence of the feedback loop in a system requires the identification experiment to be conducted under the system’s closed-loop configuration, which in most cases involve a (stabilizing) controller.

This thesis will focus mainly on the validation method of a model estimate of a true system which is operated under the presence of feedback loop, where this model estimate is obtained using direct closed-loop identification(explained in the subsequent section). Although the plant system $G_0(z)$ does not necessarily have to be a stable system, a stable plant system is the one being considered in this research. The configuration of the true system that is operated under a feedback loop can be shown by the closed-loop diagram in Figure 3-1.

The data generating system in Figure 3-1 can be expressed by the following expressions:

$$
y(t) = G_0(z)u(t) + H_0(z)e(t)$$
$$u(t) = C(z)[r(t) - y(t)]$$

(3-1)
By simple substitution and arranging the Eq. (3-1) the closed-loop data generating system is characterized by:

\[
y(t) = S_0(z)[G_0(z)C(z)r(t) + H_0(z)e(t)] \\
u(t) = S_0(z)C(z)[r(t) - H_0(z)e(t)]
\]

with:

\[
S_0(z) = (1 + C(z)G_0(z))^{-1} \\
S(z, \hat{\theta}_N) = (1 + C(z)G(z, \hat{\theta}_N))^{-1}
\]

**Important remarks**

Some important remarks regarding the closed-loop identification configuration given as follows:

- From Eq. (3-2) we can clearly see that unlike in the open-loop system, in the closed-loop configuration, the signal \(u(t)\) is not independent from the noise \(v(t) = H_0(z)e(t)\), where the correlation is represented by the transfer function \([-S_0(z)C(z)]\). However, we can also see that from the characterization (Eq. (3-2)) of the closed-loop system, the excitation signal \(r(t)\) is independent from noise \(v(t)\).

- For the case where neither the true plant system \(G_0(z)\) nor the controller \(C(z)\) contains a delay, the data generating system \(S\) and the model set \(M\) satisfy the condition that \(G_0(z)\) and \(G(z, \theta)\) are always strictly proper, which means \(G_0(\infty) = G(\infty, \theta) = 0\) for all \(\theta\). This weak assumption is introduced to avoid problems with algebraic loop in the closed-loop systems, which happens when neither the true plant system \(G_0(z)\) nor the controller \(C(z)\) contains a delay. The problem arises since, when neither \(G_0(z)\) nor \(C(z)\) contains a delay, the plant’s output \(y(t)\) will be dependent on \(u(t)\) which is also dependent on \(y(t)\), where we can see directly that this creates confusion, and therefore this makes finding consistent estimate to be impossible. For multivariable situations this restriction can be consulted in [7]

### 3-2 Direct closed-loop identification

Direct closed-loop identification is a closed-loop system identification method that employs a common open-loop identification approach. The difference with the standard open-loop
Identification is that the data set \( Z^N := \{y(t), u(t)\}_N \) is collected under the presence of feedback loop that can be represented by the data generating system in Figure 3-1.

The signal \( u(t) \) in this case is no longer the excitation signal as in the common open-loop identification, instead this signal is the controller’s output, and this system under closed-loop configuration is excited by the external excitation (reference) signal \( r(t) \). In fact, by considering Eq. (3-2), the signal \( u(t) \) in this case contains the effect of noise \( v(t) \), where the relation is shown by the following expression:

\[
 u(t) = \frac{C(z)}{1 + C(z)G_0(z)}[r(t) - v(t)] \tag{3-3}
\]

This direct identification method is a very interesting approach, since the presence of feedback can simply be ignored, and we can treat the data set as if it were collected from an open-loop system. In this case we are focussing only on a stable plant system under feedback loop.

The true system and the estimated plant and noise model can be represented by the block diagram in Figure 3-2, where the residual signal is computed by the following expression:

\[
 \varepsilon(t, \theta) = H(z, \theta)^{-1}[y(t) - G(z, \theta)u(t)] \tag{3-4}
\]

Elimination of the variable \( y(t) \) and \( u(t) \) in Eq. (3-4) is simply done by substituting \( y(t) \) and \( u(t) \) using Eq. (3-2). The expression of the residual signal now becomes:

\[
 \varepsilon(t, \theta) = \hat{G}(z)r(t) + \hat{H}(z)e(t) \tag{3-5}
\]

with:

\[
 \hat{G}(z, \theta) = \frac{S_0(z)C(z)}{H(z, \theta)} [G_0(z) - G(z, \theta)] \tag{3-6}
\]

\[
 \hat{H}(z, \theta) = \frac{S_0(z)}{S(z, \theta)H(z, \theta)} \tag{3-7}
\]

where:

\[
 S_0(z) = (1 + C(z)G_0(z))^{-1}
\]

\[
 S(z, \theta) = (1 + C(z)G(z, \theta))^{-1}
\]

\( S_0(z) \) and \( S(z, \theta) \) is the sensitivity function of the true plant system and the sensitivity of the plant model respectively.
The parameter estimate based on available data $Z^N := \{y(t), u(t)\}_N$ is obtained by solving the minimization problem with $N$-data points, given as follows:

$$\hat{\theta}_N = \arg \min_{\theta} V_N(\theta, Z^N) \quad V_N(\theta, Z^N) = \left( \frac{1}{N} \sum_{t=1}^{N} \varepsilon^2(t, \theta) \right)$$

(3-8)

In this case we consider the situation where the true system is operated under the presence of feedback loop (Figure 3-1) and there exist persistently exciting external excitation signal $r(t)$ of sufficient order applied to the system. In the next section we will observe different situations that might occur when conducting the direct closed-loop identification.

### 3-3 Different conditions in direct closed-loop identification

In prediction error identification, our main goal is to obtain consistent model estimate $G(z, \theta)$ and $H(z, \theta)$, with respect to the true system. An optimum model estimate $\{G(z, \theta), H(z, \theta)\}$ is obtained when $\varepsilon(t, \theta)$ reduces to white noise $e(t)$. This means by considering Eq. (3-5), the transfer function that correlates the residual signal and the excitation signal, $\tilde{G}(z)$, will be equal to zero asymptotically in $N$(number of data points), and the transfer function that filters the white-noise signal, $\tilde{H}(z)$, will be equal to 1, also asymptotically in $N$.

In conducting the identification process we might end up in one of the following possibilities:

- **The true system is contained in the model set $S \in \mathcal{M}$**

  Suppose that we have model set $\mathcal{M}$ with parameter set $\Theta \subset \mathbb{R}^d$ such that $S \in \mathcal{M}$. When we consider an asymptotic case where we have unlimited data points, the parameter estimate $\hat{\theta}_N$ converges to its asymptotic value $\theta^*$ determined by:

  $$\theta^* = \arg \min_{\theta \in \Theta} \tilde{V}(\theta) \quad \tilde{V}(\theta) = \tilde{E}\varepsilon^2(t, \theta)$$

  (3-9)

  When a persistently exciting external signal $r(t)$ for sufficiently high order is present, we will have $G(z, \theta^*) = G_0(z)$ and $H(z, \theta^*) = H_0(z)$. Thus the model estimate $\{G(z, \hat{\theta}_N), H(z, \hat{\theta}_N)\}$ is a consistent estimate of the true system $S = \{G_0(z), H_0(z)\}$.

- **Undermodelling case $S \notin \mathcal{M}$ with $G_0 \notin \mathcal{G}$**

  Now suppose that we are in the case where the true system $\{G_0, H_0\}$ is not contained in the chosen model set $\mathcal{M}$ with parameter set $\Theta \subset \mathbb{R}^d$. This means the parameter vector $\theta_0$ that represents the true systems is outside the parameter range $\Theta$.

  As a consequence, the asymptotic parameter estimate $\theta^*$ is not equal to the true parameter vector $\theta_0$, and this results $G(\theta^*) \neq G_0$ and $H(\theta^*) \neq H_0$, and by considering Eq. (3-5) to Eq. (3-7), $\tilde{G}(z, \theta^*) \neq 0$ and $H(z, \theta^*) \neq 1$. Therefore, the residual signal $\varepsilon(t, \theta)$ will never be equal to white-noise $e(t)$.

  Based on this reasoning, the identification process will result model estimate $\{G(z, \hat{\theta}_N), H(z, \hat{\theta}_N)\}$, which is inconsistent estimate(the model estimate contains bias) of the true system $\{G_0(z), H_0(z)\}$, since as we know, the parameter estimate $\hat{\theta}_N$ converges asymptotically to the parameter $\theta^*$, which in this case is not equal to the true parameter vector $\theta_0$. 

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• Undermodelling case \( S \notin \mathcal{M} \) with \( G_0 \in \mathcal{G} \)

Suppose that now we are in the case where \( S \notin \mathcal{M} \), with \( G_0 \in \mathcal{G} \), with \( H(\theta^*) \neq H_0 \). Consider the expression below:

\[
\varepsilon(t, \theta) = \frac{S_0(z)C(z)}{H(z, \theta)} \left[ G_0(z) - G(z, \theta) \right] r(t) + \frac{S_0(z)}{S(z, \theta) H(z, \theta)} \left[ H_0(z) - G(z, \theta) \right] \varepsilon(t) \tag{3-10}
\]

it can simply be rewritten as:

\[
\varepsilon(t, \theta) = e(t) + \frac{S_0(z)C(z)}{H(z, \theta)} \left[ G_0(z) - G(z, \theta) \right] r(t) + \frac{S_0(z)H_0(z) - S(z, \theta)H(z, \theta)}{S(z, \theta) H(z, \theta)} \varepsilon(t) \tag{3-11}
\]

with \( S_0(z) = (1 + C(z)G_0(z))^{-1} \) and \( S(z, \theta) = (1 + C(z)G(z, \theta))^{-1} \)

In prediction error identification essentially we want to find parameter vector \( \theta \) such that the residual signal \( \varepsilon(t, \theta) \) reduces to \( e(t) \). Thus considering Eq. (3-11), ideally we want to minimize both the term \( L_1(z, \theta) \) and \( L_2(z, \theta) \) to be equal to zero. We can see that \( L_1(z, \theta) \) and \( L_2(z, \theta) \) are in the same parameter set, since both of these terms contains both plant model \( G(z, \theta) \) and noise model \( H(z, \theta) \) which are a function of parameter vector \( \theta \).

Even when the plant and the noise model are independently parameterized such as when using Box-Jenkins model structure, where the model set can be defined as,

\[
\mathcal{M} = \{ G(z, \eta), H(z, \zeta) \mid \theta = [ \eta \ \zeta]^T \}
\]

both \( L_1 \) and \( L_2 \) are still function of the parameter \( \theta \), since both of these terms contains both \( G(z, \eta) \) and \( H(z, \zeta) \).

Thus when we are in the case \( S \notin \mathcal{M} \) with \( G_0 \in \mathcal{G} \), in this case \( H(\theta^*) \neq H_0 \), the optimum parameter vector \( \theta^* \) will never be equal to the true parameter \( \theta_0 \), which also means that the optimum parameter \( \theta^* \) does not minimize \( L_2(z, \theta) \) and \( L_1(z, \theta) \). Therefore \( G(\theta^*) \neq G_0 \), even when \( G_0 \in \mathcal{G} \). Consistent parameter estimate \( \theta^* = \theta_0 \) can therefore only be obtained when both the plant model \( G(z, \theta) \) and noise model \( H(z, \theta) \) are in full order, or in other words having \( S \in \mathcal{M} \).

### 3-4 \ residual signal decomposition

In this part, the residual signal \( \varepsilon(t, \hat{\theta}_N) \), computed by Eq. (3-4) will be separated in three terms based on different sources of contribution to the residual signal. This is necessary since we are going to utilize these terms for further analysis purposes, used in the subsequent part of this paper.

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This residual signal can be decomposed into three terms as shown by the following equation:

\[ \varepsilon(t, \hat{\theta}_N) = \beta_b(t, \hat{\theta}_N, G_0, \theta^*) + \beta_v(t, \hat{\theta}_N, \theta^*) + \varepsilon_v(t, \hat{\theta}_N) \]  

(3-12)

By simply equating and separating Eq. (3-12) with Eq. (3-5) to Eq. (3-7) we have three following terms:

- The first term corresponds to the error caused by asymptotic bias as a result from an undermodelling error (bias in the model estimate), which also means that this term exist only when we are in the case that the true system is not contained in the model set, \( S \not\in M \). This bias error is denoted by \( \beta_b \) and expressed by the following asymptotic expression:

\[ \beta_b(t, \hat{\theta}_N, G_0, \theta^*) = \frac{S_0(z)C(z)}{H(z, \hat{\theta}_N)} [G_0(z) - G(z, \theta^*)] r(t) \]  

(3-13)

This bias term is equal to zero when we have consistent model estimate, \{\( G_0, H_0 \)\} \( \in M \), since having consistent model estimate means having \( \theta^* = \theta_0 \), and thus \( G(\theta^*) = G_0 \).

- The second term of the error corresponds to the variance error in the parameter estimate as a result of limited number of data points being used in the identification process. This variance error is expressed by the following expression:

\[ \beta_v(t, \hat{\theta}_N, \theta^*) = \frac{S_0(z)C(z)}{H(z, \hat{\theta}_N)} [G(z, \theta^*) - G(z, \hat{\theta}_N)] r(t) \]  

(3-14)

- The third term corresponds to the error that is induced by the measurement noise, as well as the modelling error of both the plant model and the noise model, with respect to the true system. The expression is given as follows:

\[ \varepsilon_v(t, \hat{\theta}_N) = \frac{S_0(z)}{S(z, \hat{\theta}_N)} \frac{H_0(z)}{H(z, \hat{\theta}_N)} e(t) \]  

(3-15)

with:

\[ S_0(z) = (1 + C(z)G(z))^{-1} \]

\[ S(z, \hat{\theta}_N) = (1 + C(z)G(z, \hat{\theta}_N))^{-1} \]

We can see that when we add all these three term together we will have the original term Eq. (3-12).

**Remarks in relation with the open-loop case**

The term \( \varepsilon_v(t, \hat{\theta}_N) \) that is used in the open-loop identification contains the effect of measurement noise and only the modelling error of the noise model with respect to the true noise filter, \( \left( \begin{array}{c} H_0(z) \\ H(z, \hat{\theta}_N) \end{array} \right) \), for more details, please consult [4]. Unlike the expression of \( \varepsilon_v(t, \hat{\theta}_N) \) in the open-loop identification ([4]), since the term \( \varepsilon_v(t, \hat{\theta}_N) \) in closed-loop identification contains not only the effect of the modelling error of the noise model but also the modelling error of plant model with respect to the true plant system \( G_0 \), as shown in Eq. (3-15), where this contribution is represented by the term \( \frac{S_0(z)}{S(z, \hat{\theta}_N)} \).
3-5 Chapter summary

The configuration of a closed-loop system used in this research as well as the basic concept of direct closed-loop identification are presented in this chapter. We have seen in section 3-3, three different possible conditions that might occur on the process of identification of a system operated in closed-loop configuration. Based on the reasoning given in section 3-3 we have seen that a consistent parameter estimate, $\theta^* = \theta_0$, can only be obtained when both the plant model $G(z, \theta)$ and noise model $H(z, \theta)$ are in full order with respect to the true system, or in other words, having $S \in \mathcal{M}$. In the latter part of this chapter, we have also seen how the residual signal is decomposed in three different terms, where these terms are to be used in the following chapter, for detail analysis on the model validation test of a closed-loop system.
Chapter 4

Closed-loop model structure validation test

4-1 Introduction

In system identification, in order to verify whether a model candidate is valid for its intended purposes, we need to do a model validation test. Based on the validation tests, good model estimate will then be accepted at a certain level of confidence, and an inadequate or bad model estimate will be rejected. In this chapter, the validation of closed-loop model identified by direct closed-loop identification method will be presented extensively. Two different model validation test will be presented. The first validation test is the closed-loop point-wise validation test, and the second is the closed-loop vector-valued validation test.

The closed-loop point-wise test is adapted directly from the standard open-loop model validation method [1, 2], which is used in matlab system identification toolbox [3]. This standard open-loop validation test, evaluates the autocorrelation function of the residual signal $\varepsilon(t, \theta)$ as well as the cross-correlation function of the residual and past input $u(t - \tau)$. However, since in the closed-loop case the residual signal will always be correlated to the past input signal, as the result of the feedback loop, therefore in this closed-loop model validation, the cross-correlation test is done for the residual signal and the excitation signal $r(t)$. Meanwhile, the closed-loop point-wise validation test on the basis of autocorrelation will not be covered in this thesis, since the main focus of this thesis is the residual validation test on the basis of cross-correlation function.

The point-wise validation test in this case is basically done by evaluating the sample cross-correlation function between the residual signal $\varepsilon(t, \theta)$ and the excitation signal $r(t)$. The evaluation of the cross-correlation function is done one-by-one for each and every time lag $\tau$, and not for the overall time lags altogether. This may cause the bias existed in the model estimate to be undetected by the point-wise validation test. Since the closed-loop point-wise test used in this thesis is adapted directly from the standard open-loop model validation, therefore as stated in [4] the estimation of the noise variance in the point-wise test in this
case is not done consistently and properly, and also this test neglects the randomness in the parameter estimate.

Motivated by the improvement made by the use of the vector-valued test in the open-loop case [4], in this research the vector-valued model validation test for closed-loop identification is then proposed to overcome the shortcomings found in the closed-loop point-wise identification. As the name itself suggests, the vector-valued test evaluates the cross-correlation function in a vector form, where this cross-correlation vector is used in an expression of the test statistic evaluated for the model validation purpose. This also means that the evaluation of the sample cross-correlation function between the residual and the excitation signal, is done for the overall $\tau$ altogether, thus makes this test an accurate validation test in detecting the presence of undermodelling error that might be hidden in the point-wise evaluation. The vector-valued test also overcomes the weak points in the point-wise test by incorporating the effect of randomness in the parameter estimate by the use of a projection operator, as well as consistently estimating the noise variance.

In this thesis, we will only consider vector-valued validation test on the basis of the cross-correlation function of $\varepsilon(t, \theta)$ and $r(t-\tau)$, where the vector-valued test using the autocorrelation function of the residual signal for the time being is to be left for the possibility of future research. In the following part, point-wise model validation on the basis of cross-correlation function test is presented first then continued with the vector-valued model validation test. After that these two test are compared. Before going into the details of closed-loop model validation, the properties of the model validation in closed-loop system is first explained.

### 4.2 Model validation of closed-loop system

A closed-loop system is by definition a system that is operated under a feedback loop. As a consequence of a system being operated under a feedback loop, the past-input of the system $u(t-\tau)$ will always be correlated to the residual signal $\varepsilon(t, \theta)$. Therefore it is impossible to do a model validation by evaluating the sample cross-correlation between the residual and the input signal, which is done in an open-loop model validation (for details of open-loop model validation see [1, 2, 4]). The validation of a closed-loop model (Figure 4-1) in this case is therefore can be done by evaluating the cross-correlation function of the residual signal and the excitation signal ($r(t)$), as well as the evaluation of the autocorrelation function evaluating the whiteness of the residual signal.

The relation between the residual signal and the excitation signal as well as the term that filters the white noise, derived in the previous chapter is given below:

$$
\varepsilon(t, \hat{\theta}_N) = S_0(z) C(z)[G_0(z) - G(z, \hat{\theta}_N)] r(t) + \frac{S_0(z)}{S(z, \hat{\theta}_N)} \frac{H_0(z)}{H(z, \hat{\theta}_N)} e(t) 
\tag{4-1}
$$

with $S_0(z) = (1 + C(z)G_0(z))^{-1}$ and $S(z, \theta) = (1 + C(z)G(z, \theta))^{-1}$

By observing Eq. (4-1), we can grasp the concept that when we are in case of having full order model structure, $\mathcal{S} \in \mathcal{M}$,
1. The transfer function that correlates the residual signal and the excitation signal, \( \hat{G}(z) \), will be equal to zero, and the transfer function \( \hat{H}(z) = 1 \), when \( N \to \infty \).

2. Thus, the residual signal \( \varepsilon(t, \hat{\theta}_N) \) will reduce to white noise \( e(t) \), asymptotically in \( N \).

Based on this fact we can use the residual test on the basis of cross-correlation function between the residual and the excitation signal \( R_{\varepsilon r}(\tau) \), as well as on the basis of autocorrelation of the residual signal, \( R_{\varepsilon}(\tau) \). When we are in a condition where \( S \in M \), based on point 1. above when signal \( r(t) \) is persistently exciting, we can see that when the cross correlation \( \hat{R}_{\varepsilon r}(\tau) = 0 \), is the same as having \( G(\theta^*) = G_0 \), and based on point 2. having the autocorrelation \( \hat{R}_{\varepsilon}(\tau) = 0 \) for \( \tau \neq 0 \) is the same as having \( G(\theta^*) = G_0 \) and \( H(\theta^*) = H_0 \).

**Undermodelling case** \( S \notin M \)

By considering Eq. (4-1), it can simply be rewritten as:

\[
\varepsilon(t, \theta) = e(t) + \frac{S_0(z)C(z)}{H(z, \theta)} [G_0(z) - G(z, \theta)] r(t) + \frac{S_0(z)H_0(z) - S(z, \theta)H(z, \theta)}{S(z, \theta)H(z, \theta)} e(t) \tag{4-2}
\]

Based on the reasoning given in Chapter 3 on pg.25, for the condition where the true system is not contained in the model set, \( S \notin M \) with \( G_0 \in M \), we know that the optimum parameter \( \theta^* \) does not minimize \( L_2(z, \theta) \) and \( L_1(z, \theta) \) in Eq. (4-2), therefore \( G(\theta^*) \neq G_0 \), even when \( G_0 \in \mathcal{G} \). Consistent parameter estimate \( \theta^* = \theta_0 \) can therefore only be obtained when both the plant model \( G(z, \theta) \) and noise model \( H(z, \theta) \) are in full order, with respect to the order of the true system, or in other words having \( S \in M \). Or in other words, in the closed-loop case \( G(\theta^*) \) is never be equal to \( G_0 \) when \( H(\theta^*) \neq H_0 \).

Given this fact, when we are in the case where \( S \notin M \), the model validation test will not be able to distinguish whether we are in the situation where both the plant and noise model are not in full order, \( \{S \notin M \text{ with } G_0 \notin \mathcal{G}\} \), or we are in the case where \( S \notin M \), with plant model
in its full order, \( \{S \notin \mathcal{M} \text{ with } G_0 \in \mathcal{G} \} \), even when we are using independently parameterized model structure such as Box-Jenkins model structure. Therefore, we will not be able to do a validation procedure on the plant and the noise model separately as in the open-loop case. As a consequence to this, the validation of the model estimate in this case either validates or invalidates both the plant and the noise model altogether.

**Important remark**

Since in this scope of research the main focus is only on the residual test on the basis of the cross-correlation function \( \hat{R}_{\varepsilon r}(\tau) \), therefore in this chapter and after, we will not discuss the residual validation test on the basis of autocorrelation \( \hat{R}_\varepsilon(\tau) \).

### 4-3 Point-wise model validation

Viewing from an asymptotic case we can see that based on the Eq. (4-1), when a consistent model estimate that represents the true system is obtained \( G(z, \theta^*) = G_0(z) \), we will have \( \hat{G}(z) = 0 \), and therefore the cross-correlation \( R_{\varepsilon r}(\tau) \) is also equal to zero. Based on this concept, the cross-correlation function can then be used as a validation test of both the plant model \( G(z, \hat{\theta}_N) \) and the noise model \( H(z, \hat{\theta}_N) \), considering the fact that the validation of the plant and the noise model in the closed-loop system is inseparable, pointed out in the previous section. Since the model validation in closed-loop system can only detect the case where \( S \in \mathcal{M} \) or \( S \notin \mathcal{M} \), therefore it implies that by having the cross-correlation \( R_{\varepsilon r}(\tau) = 0 \), in this case is the same as having \( G(\theta^*) = G_0 \) and \( H(\theta^*) = H_0 \), thus \( \varepsilon(t, \hat{\theta}_N) \) converges asymptotically to \( \varepsilon(t) \). Also when \( R_{\varepsilon r}(\tau) \neq 0 \), implies that \( G(\theta^*) \neq G_0 \) and \( H(\theta^*) \neq H_0 \).

When we see the point-wise test in terms of hypothesis testing, the point-wise model validation test according to [4] evaluates this following hypothesis:

\[
H_0 : \beta_b(t, \hat{\theta}_N, G_0, \theta^*) + \beta_v(t, \hat{\theta}_N, \theta^*) = 0 \quad (4-3)
\]

When we consider Eq. (3-12) and by testing this hypothesis, once a model passes the point-wise validation test, we are actually assuming that \( \varepsilon(t, \hat{\theta}_N) = \varepsilon_v(t, \hat{\theta}_N) \). Thus we can see here that when a model has been validated by this test, the effect of undermodelling that might exist and is undetected by this test, will be indistinguishable from the effect of measurement noise. Having \( \varepsilon(t, \hat{\theta}_N) = \varepsilon_v(t, \hat{\theta}_N) \), also imply that this point-wise test neglects the randomness effect in the parameter estimate \( \hat{\theta}_N \), which is represented by the expression \( \beta_v(t, \hat{\theta}_N, \theta^*) \).

This hypothesis can be translated into a more intuitive expression as follows:

\[
H_0 : G(z, \hat{\theta}_N) = G_0(z) \iff H(z, \hat{\theta}_N) = H_0(z) \quad (4-4)
\]

The symbol \( \iff \), in the hypothesis represents the fact that the validation of both the plant and the noise model are inseparable.

We can also say that by evaluating this hypothesis, means that we do not check the consistency of the model estimate, since we are evaluating \( \hat{\theta}_N = \theta_0 \), instead of \( \theta^* = \theta_0 \). Therefore in this point-wise test, the best scenario is that once a model has passed this validation test, since the
effect of undermodelling is indistinguishable from the effect of measurement noise, it assumes that the undermodelling/bias is might as well not present. However, even though the effect of bias is indistinguishable, it does not mean that it is not there.

This is the main reason why the vector-valued test is proposed in the subsequent section. In the case where the reader is not familiar with the basic concept of hypothesis testing and test statistic, the reader is suggested to read [8].

The point-wise model structure validation of a (direct) closed-loop identification can be done by evaluating the sample cross-correlation function between the residual signal and the reference (excitation) signal. The sample cross-correlation between the residual and the reference signal is given by the following expression:

$$\hat{R}_{\varepsilon r}(\tau) := \frac{1}{N} \sum_{t=1}^{N} \varepsilon(t, \hat{\theta}_N)r(t - \tau)$$

(4-5)

with $\tau = 0, \cdots, n_\tau - 1$, where $n_\tau$ is a user chosen maximum number of lags.

If the residual signal $\varepsilon(t, \hat{\theta}_N)$ and the reference signal $r(t)$ are independent, by following similar reasoning as in [2], the application of the central limit theorem provides:

$$\sqrt{N}\hat{R}_{\varepsilon r}(\tau) \in \mathcal{N}(0, P)$$

(4-6)

where the variance $P$ is estimated by the following expression:

$$P = \sum_{\kappa=-n_\tau+1}^{n_\tau-1} \hat{R}_r(\kappa)\hat{R}_\varepsilon(\kappa)$$

(4-7)

The expression of the variance is motivated from the point-wise model validation in open-loop case presented in [1] and [4].

To check whether or not there is a correlation between the residual signal and the reference signal, the following condition is used:

$$\left| \hat{R}_{\varepsilon r}(\tau) \right| < N_\alpha \sqrt{\frac{P}{N}}$$

(4-8)

where $N_\alpha$ is the $\alpha$-level of the $\mathcal{N}(0, 1)$ distribution, i.e. $N_{0.99} = 2.57$

Only when the sample cross-correlation is smaller than the confidence bound for all $\tau$, we can then conclude that the residual signal and the reference signal are uncorrelated, and therefore the hypothesis in Eq. (4-4), is not invalidated.

**Important remarks regarding this closed-loop point-wise test**

It is important to mention that the point-wise test in this case is directly adapted from the standard open-loop model validation test [1, 2, 4], which is the one that is utilized in the Matlab system identification toolbox. Therefore in this chapter and after, whenever closed-loop point-wise test is discussed, it refers to the adaptation of the standard open-loop model validation to be used in the closed-loop model validation. To be more specific, in this point-wise test the estimation of the noise variance $\sigma^2_e$ for the validation purpose is not estimated consistently and properly, for detail analysis the reader is strongly suggested to read [4].
4-4 Vector-valued model validation

As explained in the previous section, the point-wise test does not check the consistency of the model estimate. We also know that by testing the hypothesis in Eq. (4-3), once a model passes the point-wise test, we are actually assuming that

\[ \varepsilon(t, \hat{\theta}_N) = \varepsilon_v(t, \hat{\theta}_N) \]

This means, when a model passes the point-wise validation test, the effect of undermodelling in the residual \( \varepsilon(t, \hat{\theta}_N) \) that might exist will be indistinguishable from the effect of measurement noise \( \varepsilon_v(t, \hat{\theta}_N) \), as a consequence this test presumes that \( S \in M \). By assuming \( \varepsilon(t, \hat{\theta}_N) = \varepsilon_v(t, \theta_0) \), also implies that this point-wise test neglects any random effect in the parameter estimate \( \hat{\theta}_N \).

The vector-valued test however, overcomes this weak point by incorporating the hypothesis test that can verify whether or not we have bias in our model (checking the consistency of the model estimate), as well as incorporating the randomness in the parameter estimate by the use of a projection operator.

The vector-valued model validation test evaluates this following hypothesis:

\[ H_0 : \beta_v(t, \hat{\theta}_N, G_0, \theta^*) = 0 \]

This hypothesis can be translated into a more intuitive expression as follows:

\[ H_0 : G(z, \theta^*) = G_0(z) \leftrightarrow H(z, \theta^*) = H_0(z) \]

Under the persistency of excitation conditions on the excitation signal, this hypothesis can be interpreted as having \( \hat{\theta}_N \) as a consistent estimate of \( \theta^* = \theta_0 \), rather than having \( \hat{\theta}_N = \theta_0 \) as in the point-wise test. A consistent estimate means that no bias is present in the parameter estimate \( \hat{\theta}_N \) with respect to the true system. In other words, this hypothesis conveys a condition where the true system is contained in the model set \( S \in M \). Meaning that once a model passes this vector-valued test, we can be sure at a certain probability that our model contains no bias and thus it is a consistent estimate of the true system.

In the subsequent part, the test statistic of the hypothesis \( H_0 \) is conducted on the basis of the sample cross-correlation function between the residual signal and the reference signal as in Eq. (4-5).

For the ease of readability, the sample cross-correlation vectors will be denoted as:

\[
\hat{R}_{er} := [\hat{R}_{er}(0) \cdots \hat{R}_{er}(n_r - 1)]^T \\
\hat{R}_{\xi r} := [\hat{R}_{\xi r}(0) \cdots \hat{R}_{\xi r}(n_r - 1)]^T
\]

When the hypothesis \( H_0 \) holds, the residual signal \( \varepsilon(t, \hat{\theta}_N) \) as in Eq. (3-12) becomes:

\[ \varepsilon(t, \hat{\theta}_N) = \varepsilon_v(t, \hat{\theta}_N) := \beta_v(t, \hat{\theta}_N, \theta^*) + \varepsilon_v(t, \hat{\theta}_N) \]

The vector form of the residual signals are denoted as:

\[
\varepsilon(\hat{\theta}_N) = [\varepsilon(1, \hat{\theta}_N) \cdots \varepsilon(N, \hat{\theta}_N)]^T \\
\varepsilon_v(\hat{\theta}_N) = [\varepsilon_v(1, \hat{\theta}_N) \cdots \varepsilon_v(N, \hat{\theta}_N)]^T
\]

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As a model structure has been chosen and the parameter estimate $\varepsilon(t, \hat{\theta}_N)$ was obtained from expression Eq. (3-8), one would like to know whether or not the true system is contained in the model class.

The validation of the current model estimate is done via hypothesis testing, where the main objective of this hypothesis test is to test whether the sample cross-correlation $\hat{R}_{\varepsilon r}$ is likely to be a realization of the random vector $\hat{R}_{\xi v}$. This validation test is based on the concept that under the hypothesis $H_0$, $\beta_b = 0$, thus $\varepsilon(t, \hat{\theta}_N)$ reduces to $\xi_v(t, \hat{\theta}_N)$, and therefore $\hat{R}_{\varepsilon r} = \hat{R}_{\xi v}$.

This hypothesis test is done by first having a general expression for $\varepsilon(\hat{\theta}_N)$ in terms of $\varepsilon(\theta^*)$ then the expression for $\xi_v(\hat{\theta}_N)$ in terms of $\varepsilon_v(\theta^*)$. The purpose of doing this is to incorporate the randomness in the parameter estimate, to be incorporated in the validation test. The statistical properties of $\hat{R}_{\xi v}$ are then derived to be used as a test that verifies whether the sample cross correlation $\hat{R}_{\varepsilon r}$ is likely to be a realization of $\hat{R}_{\xi v}$.

**4-4-1 Derivation of a general relation between $\varepsilon(\hat{\theta}_N)$ and $\varepsilon(\theta^*)$**

The parameter estimate $\hat{\theta}_N$ is defined as follows:

$$\hat{\theta}_N = \arg\min_{\theta} V_N(\theta, Z^N)$$

$$V_N(\theta, Z^N) = \left( \frac{1}{N} \sum_{t=1}^{N} \varepsilon^2(t, \theta) \right)$$

(4-12)

By definition, the prediction error gradient $\psi(t, \hat{\theta}_N)$ is expressed as the negative value of the partial derivative of the residual $\varepsilon$ with respect to the parameter $\theta$ and it is evaluated at the parameter values that belongs to parameter estimate $\hat{\theta}_N$. The expression is then denoted as follows:

$$\psi(t, \hat{\theta}_N) = -\frac{\partial \varepsilon(t, \theta)}{\partial \theta} \bigg|_{\theta = \hat{\theta}_N}$$

(4-13)

For example we have a parameter vector $\theta = [a \ b]^T$, with parameter estimate $\hat{\theta}_N = [a_1 \ b_1]^T$. In vector notation, the prediction error gradient $\psi(t, \hat{\theta}_N)$ is then written as:

$$\psi(t, \hat{\theta}_N) = \frac{\partial \varepsilon(t, \theta)}{\partial \theta} = \left[ \begin{array}{c} -\frac{\partial \varepsilon(t, \theta)}{\partial a} \bigg|_{a = a_1} \\ -\frac{\partial \varepsilon(t, \theta)}{\partial b} \bigg|_{b = b_1} \end{array} \right]$$

(4-14)

However since we want to have clear separation between the partial derivative of the residual signal with respect to each parameter, we will have each column of our vector, representing the partial derivative for each different parameters, instead of just stacking them up which will create confusion, since $\psi(t, \hat{\theta}_N)$ is also a function of time $t$. Thus the previous expression is better represented as:

$$\psi^T(t, \hat{\theta}_N) = \left[ -\frac{\partial \varepsilon(t, \theta)}{\partial a} \bigg|_{a = a_1}, -\frac{\partial \varepsilon(t, \theta)}{\partial b} \bigg|_{b = b_1} \right]$$

(4-15)
By using first order Taylor expansion on the residual $\varepsilon$ with respect to the parameter $\theta$, with parameter $\theta$ is evaluated at parameter estimate $\hat{\theta}_N$ we have the following expression:

$$
\varepsilon(t, \theta) \approx \varepsilon(t, \hat{\theta}_N) + \left. \frac{\partial \varepsilon(t, \theta)}{\partial \theta} \right|_{\theta = \hat{\theta}_N} (\theta - \hat{\theta}_N) \quad (4-16)
$$

Having the $\psi(t, \hat{\theta}_N)$ defined in Eq. (4-13), the previous expression is simply translated into:

$$
\varepsilon(t, \theta) \approx \varepsilon(t, \hat{\theta}_N) - \psi^T(t, \hat{\theta}_N)(\theta - \hat{\theta}_N) \quad (4-17)
$$

by re-arranging this equation, and considering the case where $\theta = \theta^*$ we have:

$$
\varepsilon(t, \hat{\theta}_N) \approx \varepsilon(t, \theta^*) - \psi^T(t, \hat{\theta}_N)(\hat{\theta}_N - \theta^*) \quad (4-18)
$$

Since we know that $\hat{\theta}_N$ has its optimum value at $\theta^*$, thus by considering Eq. (4-12), we have

$$
\frac{\partial V_N(\hat{\theta}_N)}{\partial \theta} = 0
$$

We can then write this expression as:

$$
\frac{1}{N} \sum_{t=1}^{N} \frac{\partial^2 \varepsilon(t, \hat{\theta}_N)}{\partial \theta} = 0 \quad (4-19)
$$

By applying chain-rule we obtain:

$$
\frac{1}{N} \sum_{t=1}^{N} \psi(t, \hat{\theta}_N) \cdot \varepsilon(t, \hat{\theta}_N) = 0 \quad (4-20)
$$

By substituting $\varepsilon(t, \hat{\theta}_N)$ with Eq. (4-18), we now have:

$$
\frac{1}{N} \sum_{t=1}^{N} \psi(t, \hat{\theta}_N) \left[ \varepsilon(t, \theta^*) - \psi^T(t, \hat{\theta}_N)(\hat{\theta}_N - \theta^*) \right] = 0 \quad (4-21)
$$

Considering vector notation analysis using:

$$
\Psi_{\hat{\theta}} = \begin{bmatrix} 
\psi^T(1, \hat{\theta}_N) \\
\vdots \\
\psi^T(N, \hat{\theta}_N) 
\end{bmatrix} \quad (4-22)
$$

we can discard the averaging term $\frac{1}{N} \sum_{t=1}^{N}$ from the previous expression, and now we end up with the following vector notation:

$$
\Psi_{\hat{\theta}}^T \left[ \varepsilon(\theta^*) - \Psi_{\hat{\theta}}(\hat{\theta}_N - \theta^*) \right] = 0 \quad (4-23)
$$

and by simply rearranging this expression the following expression is obtained:

$$
\hat{\theta}_N - \theta^* = (\Psi_{\hat{\theta}}^T \Psi_{\hat{\theta}})^{-1} \Psi_{\hat{\theta}}^T \varepsilon(\theta^*) \quad (4-24)
$$

We can as well write Eq. (4-18) in vector notation as:

$$
\varepsilon(\hat{\theta}_N) = \varepsilon(\theta^*) - \Psi_{\hat{\theta}}(\hat{\theta}_N - \theta^*) \quad (4-25)
$$
4-4 Vector-valued model validation

By substituting \((\hat{\theta}_N - \theta^*)\) as in Eq. (4-24) to this expression, and then rearranging it, the following expression that relates the residual signal \(\varepsilon(\hat{\theta}_N)\) with \(\varepsilon(\theta^*)\) is obtained:

\[
\varepsilon(\hat{\theta}_N) = \left[ I - \Psi_{\hat{\theta}}(\Psi_{\hat{\theta}}^T \Psi_{\hat{\theta}})^{-1} \Psi_{\hat{\theta}}^T \right] \varepsilon(\theta^*)
\]  
(4-26)

This matrix \(\Delta\) is called the projection matrix or similarly projection operator, which has dimension \([N \times N]\), with \(N\) is number of data points.

This derivation of the general expression relating the \(\varepsilon(\hat{\theta}_N)\) and \(\varepsilon(\theta^*)\) is similar to the derivation in [4]. The expression used in the computation of this prediction error gradient \(\psi\) will be explained later.

The analysis of the projection operator \(\Delta\) presented in [4] is further expanded in [9] and builds on alternative results on uncertainty quantification presented in [10, 11].

4-4-2 Expression of \(\xi_v\) and statistical properties of \(\hat{R}_{\xi_v r}\)

Under the hypothesis \(\mathcal{H}_0\) in Eq. (4-9), we will have \(\beta_b = 0\), thus \(\varepsilon(\hat{\theta}_N)\) reduces to \(\xi_v(\hat{\theta}_N)\), and for the unique minimum parameter estimate \(\theta^*\) where \(\beta_b = 0\) and \(\beta_v = 0\), and hence \(\varepsilon(\theta^*) = \varepsilon_v(\theta^*)\).

Thus based on the expression Eq. (4-26), \(\xi_v(\hat{\theta}_N)\) can then be expressed as:

\[
\xi_v(\hat{\theta}_N) = \left[ I - \Psi_{\hat{\theta}}(\Psi_{\hat{\theta}}^T \Psi_{\hat{\theta}})^{-1} \Psi_{\hat{\theta}}^T \right] \varepsilon_v(\theta^*)
\]  
(4-27)

We now go back to the expression of the sample cross-correlation \(\hat{R}_{\xi_v r}\). It can be re-written in matrix form as :

\[
\hat{R}_{\xi_v r} = \frac{1}{N} \begin{bmatrix}
    r(1) & r(2) & \cdots & r(N) \\
    \ddots & \ddots & \cdots & \ddots \\
    0 & r(1) & \cdots & r(N - n_r + 1)
\end{bmatrix}
\begin{bmatrix}
    \xi_v(1, \hat{\theta}_N) \\
    \vdots \\
    \xi_v(N, \hat{\theta}_N)
\end{bmatrix}
\]  
(4-28)

Having the fact that the signal \(r(t)\) is independent from the white-noise sequence \(e(t)\) (see Eq. (3-2)) as well as having the asymptotic property of the parameter estimate as follows:

\[
\hat{\theta}_N \rightarrow \theta^*, \quad \text{with probability 1 as } N \rightarrow \infty
\]  
(4-29)

and we also know that under the hypothesis \(\mathcal{H}_0\) in Eq. (4-9), \(\xi_v(\theta^*) = e(t)\), thus the sample cross-correlation vector \(\hat{R}_{\xi_v r}\) will asymptotically converge to a zero-mean Gaussian distribution denoted as follows:

\[
\hat{R}_{\xi_v r} \sim \text{AsN}(0, P_{\mathcal{H}_0})
\]  
(4-30)

with the expression of asymptotic covariance matrix as:
\[ P_{H_0} = E[\hat{R}_{e_{\theta}}\hat{R}_{e_{\theta}}^T] = \frac{1}{N^2} \text{Pr} \Lambda_{\hat{e}_v(\hat{\theta}_N)} \text{Pr}^T \]  

with \( \Lambda_{\hat{e}_v(\hat{\theta}_N)} = E[\hat{e}_v(\hat{\theta}_N)\hat{e}_v^T(\hat{\theta}_N)] \)

### 4.4.3 Cross-correlation test statistic

The hypothesis test in Eq. (4-9) is done by analysing the expression \( \hat{R}_{e_{\theta}}^T \text{Pr}^{-1} \hat{R}_{e_{\theta}} \). The hypothesis test verifies whether or not the sample cross-correlation \( \hat{R}_{e_{\theta}} \) is likely to be a realization of the random variable \( \hat{R}_{e_{\theta}} \). The basic definition of chi-square (central) distribution states that when, \( x_i \) is an independent normally distributed variable with mean 0 and variance 1, \( \sum_{i=1}^{n} x_i^2 \) is a central \( \chi^2 \)-distributed random variable with \( n \) degrees of freedom.

Thus when the hypothesis \( H_0 \) holds, \( \hat{R}_{e_{\theta}}^T \text{Pr}^{-1} \hat{R}_{e_{\theta}} \) is an element of the central chi-square distribution \( \chi^2(n_\tau) \) with \( n_\tau \) degrees of freedom, denoted as:

\[ \hat{R}_{e_{\theta}}^T \text{Pr}^{-1} \hat{R}_{e_{\theta}} \in \chi^2(n_\tau) \]  

Thus the hypothesis \( H_0 \) is unfalsified; or to put it in simpler words, the model estimate is accepted (not invalidated) when:

\[ \hat{R}_{e_{\theta}}^T \text{Pr}^{-1} \hat{R}_{e_{\theta}} \leq \mathcal{C}_\chi(\alpha, n_\tau) \]  

with \( \mathcal{C}_\chi(\alpha, n_\tau) \) the upper critical value of chi-square distribution with \( n_\tau \) degrees of freedom with the significance level \( \alpha \). That is for \( x \in \chi^2(n_\tau) \Rightarrow \text{Pr}(x > \mathcal{C}_\chi(\alpha, n_\tau) = \alpha). \) This test statistic is also known as an upper one-sided chi-square test.

The approximation of the asymptotic covariance matrix \( P_{H_0} \) can be done by substitution of the projection matrix \( \Delta \) as in expression Eq. (4-27) to the matrix \( \Lambda_{\hat{e}_v(\hat{\theta}_N)} \) in the asymptotic covariance matrix, which means that we also consider the effect of randomness in the parameter estimate \( \hat{\theta}_N \). \( P_{H_0} \) in Eq. (4-31) is then written as:

\[ P_{H_0} = \frac{1}{N^2} \text{Pr} \left[ I - \Psi_\hat{\theta}[\Psi_\hat{\theta}^T \Psi_\hat{\theta}]^{-1} \Psi_\hat{\theta}^T \right] \Lambda_{\hat{e}_v(\hat{\theta}^*)} \Delta^T \text{Pr}^T \]  

with \( \Lambda_{\hat{e}_v(\hat{\theta}^*)} = E[\hat{e}_v(\hat{\theta}^*)\hat{e}_v^T(\hat{\theta}^*)] \)

Since \( \Lambda_{\hat{e}_v(\hat{\theta}^*)} \) is an asymptotic expression, in practical case where we only have limited data points, this expression needs to be estimated from available data. The method for estimating \( \Lambda_{\hat{e}_v(\hat{\theta}^*)} \) for the closed-loop model validation case is explained in the next section. Using the estimated value of \( \hat{\Lambda}_{\hat{e}_v(\hat{\theta}^*)} \), the approximation of \( P_{H_0} \) is then expressed as follows:

\[ \hat{P}_{H_0} = \frac{1}{N^2} \text{Pr} \left[ I - \Psi_\hat{\theta}[\Psi_\hat{\theta}^T \Psi_\hat{\theta}]^{-1} \Psi_\hat{\theta}^T \right] \hat{\Lambda}_{\hat{e}_v(\hat{\theta}^*)} \Delta^T \text{Pr}^T \]  

The reliability of the \( \chi^2 \) test statistic with \( \hat{P}_{H_0} \) is presented in [12, 11].
4-4-4 Estimation of $\Lambda_{\varepsilon_v(\theta^*)}$

The hypothesis in Eq. (4-10) is equivalent to having $(G_0, H_0) \in \mathcal{M}$, this means $G(z, \theta^*) = G_0(z)$ and $H(z, \theta^*) = H_0(z)$, thus under the null hypothesis in Eq. (4-10), by considering Eq. (3-15), $\varepsilon_v(\theta^*)$ reduces to white-noise $e(t)$,

$$\varepsilon_v(\theta^*) = e(t)$$  (4-36)

Thus we can already see that the autocorrelation of $\varepsilon_v(\theta^*)$ becomes:

$$R_{\varepsilon_v}(\tau) = R_e(\tau) = \begin{cases} \sigma_e^2 & \text{for } \tau = 0 \\ 0 & \text{for } \tau \neq 0 \end{cases}$$  (4-37)

with $\delta(\tau)$ a dirac delta function, $\sigma_e^2$ is the white noise variance.

The value of $\Lambda_{\varepsilon_v(\theta^*)} = E[\varepsilon_v(\theta^*) \varepsilon_v^T(\theta^*)]$ can be expressed in matrix form, consisting of the autocorrelation function of $\varepsilon_v(\theta^*)$:

$$E[\varepsilon_v(\theta^*) \varepsilon_v^T(\theta^*)] = \begin{bmatrix} R_{\varepsilon_v}(0) & R_{\varepsilon_v}(-1) & \cdots & R_{\varepsilon_v}(-N + 1) \\ R_{\varepsilon_v}(1) & R_{\varepsilon_v}(0) & \cdots & R_{\varepsilon_v}(-N + 2) \\ \vdots & R_{\varepsilon_v}(1) & \ddots & \vdots \\ \vdots & \vdots & \ddots & R_{\varepsilon_v}(0) \\ R_{\varepsilon_v}(N - 1) & R_{\varepsilon_v}(N - 2) & \cdots & R_{\varepsilon_v}(0) \end{bmatrix}$$  (4-38)

By considering Eq. (4-37), this matrix becomes a diagonal matrix:

$$E[\varepsilon_v(\theta^*) \varepsilon_v^T(\theta^*)] = \begin{bmatrix} R_{\varepsilon_v}(0) & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \cdots & R_{\varepsilon_v}(0) \end{bmatrix}$$  (4-39)

having the estimate of the autocorrelation at $\tau = 0$:

$$\hat{R}_{\varepsilon_v}(0) = \hat{\sigma}_e^2$$  (4-40)

Based on this relation it is clear that the only important information to be estimated from data is the noise variance $\sigma_e^2$. Thus the term $\Lambda_{\varepsilon_v(\theta^*)} = E[ee^T]$ in the covariance matrix $\hat{P}_{H_0}$ is then estimated by:

$$\hat{\Lambda}_{\varepsilon_v(\theta^*)} = \begin{bmatrix} \hat{\sigma}_e^2 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \cdots & \hat{\sigma}_e^2 \end{bmatrix}$$  (4-41)

The estimation of the matrix $\Lambda_{\varepsilon_v(\theta^*)}$ in the vector-valued model validation in an open-loop identification will not be the same as Eq. (4-41), for more detail consult [4] and [5].

Accurate estimation of noise variance $\sigma_e^2$ is obtained by the use of an auxiliary model that ensures undermodelling error does not exist in the plant model. This is done by estimating...
an OE (Output-Error) model structure of the auxiliary plant model $G_a(z, \hat{\theta}_a)$ with a model order high enough to ensure that the effect of undermodelling can be neglected. Subsequently, the residual signal is described as:

$$\varepsilon(t, \theta_a) = H^{-1}(z, \theta_a)(y(t) - G(z, \theta_a)u(t))$$  \hspace{1cm} (4-42)

Since it is an OE model structure, $H^{-1}(z, \theta_a) = 1$, therefore the latter equation becomes:

$$\varepsilon(t, \theta_a) = y(t) - G(z, \theta_a)u(t)$$  \hspace{1cm} (4-43)

In this case we have, $\hat{\nu}(t) = \hat{H}(z)e(t)$, with:

$$\hat{H}(z) = \frac{S_0(z)H_0(z)}{S(z, \theta)}$$  \hspace{1cm} (4-44)

Since the auxiliary model has OE model structure, $\hat{H}(z)$ becomes:

$$\hat{H}(z, \hat{\theta}_a, G_0, H_0) = \frac{S_0(z)H_0(z)}{S(z, \theta_a)}$$  \hspace{1cm} (4-45)

and thus it follows that (considering Eq. (3-15)):

$$\hat{\nu}(t) = \varepsilon(t, \hat{\theta}_a) = \frac{S_0(z)H_0(z)}{S(z, \theta_a)}e(t)$$  \hspace{1cm} (4-46)

By neglecting the undermodelling effect the residual signal of this auxiliary model is described as:

$$\varepsilon(t, \theta_a) \approx \beta_v(t, \hat{\theta}_a, \theta^*_a) + \hat{\nu}(t)$$  \hspace{1cm} (4-47)

In vector form, the effect of variance $\beta_v(t, \hat{\theta}_a, \theta^*_a)$ is represented by the projection operator $\Delta$, and thus the residual signal is shown by the following equation:

$$\varepsilon(\hat{\theta}_a) = \begin{bmatrix} I - \Psi_\theta^{T}[\Psi_\theta^{T}\Psi_\theta]^{-1}\Psi_\theta^{T} \end{bmatrix}_{\Delta} \hat{\nu}$$  \hspace{1cm} (4-48)

where $\hat{\nu} = \begin{bmatrix} \hat{\nu}(1) \\ \vdots \\ \hat{\nu}(N) \end{bmatrix}$

Since the projection operator $[I - \Psi_\theta^{T}[\Psi_\theta^{T}\Psi_\theta]^{-1}\Psi_\theta^{T}]$ turns out to be singular [4], the latter equation can not be used to estimate $\hat{\nu}(t)$. Therefore the following approach is used to estimate the dynamic of $\hat{\nu}(t)$:

1. The noise filter $\hat{H}(z, \hat{\theta}_a, G_0, H_0)$ can be estimated by the identification of time-series model $H_a(z, \theta_a)$ that describes the dynamics of the residual signal $\varepsilon(\hat{\theta}_a)$. This time-series model is given by the following relation:
\[ \varepsilon(\hat{\theta}_a) = H_a(z, \hat{\theta}_v) \varepsilon(t) \] (4-49)

The estimate of \( \sigma^2_e \) can then be obtained by using the following expression:

\[ \hat{\sigma}^2_{e,\text{int}} = \frac{1}{N} \sum_{t=1}^{N} \left[ \tilde{H}_a(z, \hat{\theta}_v) \right]^{-1} \varepsilon(t, \hat{\theta}_a)^2 \] (4-50)

2. In order to take into account the projection operator the following scaling factor is introduced as a correction factor applied to the previously obtained estimate of the noise variance:

\[ \tilde{\sigma}^2_e = \frac{N}{N - n_a} \hat{\sigma}^2_{e,\text{int}} \] (4-51)

with \( N \) being the number of data, and \( n_a \) being the dimension of parameter estimate of the auxiliary model \( \hat{\theta}_a \). This scaling factor was motivated from the fact that \( \text{trace}[I - \Phi_\theta [\Phi_\theta^T \Phi_\theta]^{-1}] = N - n_a \) (consult [4]).

This implies that the effect of the projection operator is still being taken into account as a correction part even though the time-series model \( \tilde{H}_a(z, \hat{\theta}_v) \) and the estimate variance of the white noise \( e(t) \) are obtained from the residual signal of the auxiliary model \( \varepsilon(t, \hat{\theta}_a) \).

Other possible approaches to the estimation of noise properties in case of undermodelling are suggested in [13, 14, 15].

### 4-5 Computation of the prediction error gradient

In this section a method of calculating the prediction error gradient used in this research will be presented. The prediction error gradient \( \psi(t, \hat{\theta}_N) \) is defined as the negative of the partial derivative of the residual signal with respect to the parameter vector \( \theta \), with the result of the partial derivation evaluated at the value of the parameter estimate \( \hat{\theta}_N \):

\[ \psi(t, \hat{\theta}_N) = -\frac{\partial \varepsilon(t, \theta)}{\partial \theta} \bigg|_{\theta = \hat{\theta}_N} \] (4-52)

By substituting the residual signal \( \varepsilon(t, \theta) \) with the expression in Eq. (3-4), we get:

\[ \psi(t, \hat{\theta}_N) = -\frac{\partial}{\partial \theta} (H^{-1}(z, \theta)[y(t) - G(z, \theta)u(t)]) \] (4-53)

Then by applying product rule to the previous expression we get:

\[ \psi(t, \hat{\theta}_N) = \left[ \frac{1}{H(z, \theta)^2} \frac{\partial H(z, \theta)}{\partial \theta} \right] (y(t) - G(z, \theta)u(t)) + \left[ \frac{1}{H(z, \theta)} \frac{\partial G(z, \theta)}{\partial \theta} \right] u(t) \] (4-54)

By considering \( \varepsilon(t, \theta) = H(z, \theta)^{-1}[y(t) - G(z, \theta)u(t)] \) and simply re-organizing the previous equation leads us to:
\[
\psi(t, \hat{\theta}_N) = \left[ \frac{1}{H(z, \theta)} \frac{\partial H(z, \theta)}{\partial \theta} \right] \varepsilon(t, \theta) + \left[ \frac{1}{H(z, \theta)} \frac{\partial G(z, \theta)}{\partial \theta} \right] u(t)
\] (4-55)

To make it clearer, we consider the following example.

Suppose that we have a parameter vector \( \theta = [a \ b]^T \) with parameter estimate \( \hat{\theta}_N = [a_1 \ b_1]^T \), using Eq. (4-55) the prediction error gradient is then computed as follows:

\[
\psi(t, \hat{\theta}_N) = \frac{1}{H(z, \theta)} \left[ \frac{\partial H(z, \theta)}{\partial a} \bigg|_{a=a_1} \frac{\partial H(z, \theta)}{\partial b} \bigg|_{b=b_1} \right] \varepsilon(t, \theta) + \frac{1}{H(z, \theta)} \left[ \frac{\partial G(z, \theta)}{\partial a} \bigg|_{a=a_1} \frac{\partial G(z, \theta)}{\partial b} \bigg|_{b=b_1} \right] u(t)
\] (4-56)

Based on this example we have four transfer functions, which are the result of the partial derivation of the plant model \( G(z, \theta) \) and noise model \( H(z, \theta) \), with respect to the parameter \( a \) and \( b \). These transfer functions will then be evaluated at the value of the parameter estimate \( \hat{\theta}_N \) that minimizes the power of the residual signal, which are at value \( a_1 \) and \( b_1 \). Two of these set of transfer functions, filters the residual signal \( \varepsilon(t, \theta) \), and the other two, filters the input signal \( u(t) \).

Based on this, we can immediately see that the dimension of the prediction error gradient will be \([N \times 2]\), since we know that the residual signal \( \varepsilon(t, \theta) \) and input signal \( u(t) \) is a vector with size \( N \). To generalize, the prediction error gradient has dimension of \([N \times n_\theta]\), where \( N \) is the number of data, and \( n_\theta \) is the number of parameters in the parameter vector \( \theta \).

**Short summary on the vector-valued test**

Suppose that we already have a model estimate that we want to validate using the vector-valued test. The steps used in vector-valued validation test can be summarized as follows:

1. Computation of the sample cross-correlation \( \hat{R}_{\varepsilon r}(\tau) \) as in Eq. (4-5), building matrix \( P_r \) as in Eq. (4-28), and the prediction error gradient \( \psi(t, \theta) \) using expressions as explain in Section 4-5.

2. Computation of the estimate of the noise variance, \( \sigma^2_e \) by identifying an auxiliary OE plant model \( G(z, \hat{\theta}_a) \) and the residual signal \( \varepsilon(t, \theta_a) \) as in Eq. (4-43) and the time-series model \( \hat{H}_a(z, \hat{\theta}_v) \) as in Eq. (4-49), then we use Eq. (4-50) and Eq. (4-51) to compute \( \sigma^2_e \).

3. Using the estimate of noise variance \( \sigma^2_e \) obtained in the previous step, using Eq. (4-41) to get the estimate of the matrix \( \Lambda_{\varepsilon u}(\theta^*) \), that we will use in computing the covariance matrix \( \hat{P}_{H_0} \) as in Eq. (4-43).

4. Calculation of the covariance matrix \( \hat{P}_{H_0} \) using Eq. (4-43).
5. Evaluate the test statistic \( \hat{R}_{er}^T P_{H_0}^{-1} \hat{R}_{er} \) with the upper critical value of chi-square distribution \( C_\chi(\alpha, n_\tau) \), where the estimated covariance matrix \( \hat{P}_{H_0} \) obtained from the previous step is used in the computation of test statistic \( \hat{R}_{er}^T P_{H_0}^{-1} \hat{R}_{er} \). When the value of this test statistic is smaller than \( C_\chi(\alpha, n_\tau) \), the model is not invalidated by the vector-values test (Eq. (4-33)).

4-6 Comparing the vector-valued test with the point-wise test

Both the point-wise and the vector-valued closed-loop model validation test have been presented. Some important aspects related to the difference between the closed-loop model point-wise validation test on the basis of the cross-correlation, and the vector-valued test are presented as follows:

- The point-wise cross-correlation test only uses point-wise evaluation of the cross-correlation function \( \hat{R}_{er}(\tau) \). Meaning that the evaluation of \( \hat{R}_{er}(\tau) \) with respect to the confidence bound is done one-by-one at each time lag \( \tau \), whereas for the vector-valued test, the test statistic \( \hat{R}_{er}^T P_{H_0}^{-1} \hat{R}_{er} \) compares the cross-correlation function \( \hat{R}_{er}(\tau) \) in a vector form where it is used as a pre and post-multiplication of the covariance matrix \( P_{H_0}^{-1} \) and then compared with the upper critical value of chi-square distribution \( C_\chi(\alpha, n_\tau) \). This means the vector-valued test, evaluates the sample cross-correlation function for overall lags \( \tau \).

Based on the fact that the evaluation of the sample cross-correlation function in the point-wise test is not done for the overall lags \( \tau \), the false alarm rate (1-\( \alpha \)) corresponds to the confidence bound \( N_\alpha \) (see Eq. (4-8)) in the point-wise test is only true when the interpretation is applied for each and every time lag \( \tau \), and not for the combination of overall time lags, as in the vector-valued test. This means we might say that we have a 99% confidence bound, however this is meaningless, since it is only true that the 99% probability is only applied for each and every \( \tau \), however when the overall time lags is considered, it will no longer be 99%. Whereas for the vector-valued test, for example we are using 0.01 significance level for our chi square test statistic as in Eq. (4-33), we can be sure that we will have 1% probability that the value of \( \hat{R}_{er}^T P_{H_0}^{-1} \hat{R}_{er} \) will be exceeding the upper critical value of the chi-square distribution \( C_\chi(0.01, n_\tau) \).

- Under the persistency of excitation of the excitation signal \( r(t) \) used in the identification process, the hypothesis used in the point-wise test:

\[
\mathcal{H}_0 : \beta_0(t, \hat{\theta}_N, G_0, \theta^*) + \beta_v(t, \hat{\theta}_N, \theta^*) = 0
\]  

(4-57)

is the same as having \( \hat{\theta}_N = \theta_0 \) or \( \hat{\theta}_N = \theta^* \), instead of having \( \hat{\theta}_N \) as a consistent estimate of \( \theta_0 = \theta^* \) like in the vector-valued test. As a consequence, the test that is based on this hypothesis neglects any random effects (variance) in the parameter estimate, since \( \varepsilon(t, \hat{\theta}_N) \) given in equation Eq. (3-12), reduces to \( \varepsilon_v(t, \hat{\theta}_N) \). Also by having \( \varepsilon(t, \hat{\theta}_N) = \varepsilon_v(t, \hat{\theta}_N) \), means that once a model passes the point-wise validation test, the effect of undermodelling in the residual signal \( \varepsilon(t, \hat{\theta}_N) \) is indistinguishable from the effect of noise.

Since having this hypothesis also mean that \( \hat{\theta}_N = \theta^* \), thus under the null hypothesis of the point-wise test in Eq. (4-3) we have \( \varepsilon(t, \hat{\theta}_N) = \varepsilon_v(t, \hat{\theta}_N) = \varepsilon_v(t, \theta^*) \). Implicitly,
by considering the vector-valued point of view, this hypothesis also mean $\xi_v(t, \hat{\theta}_N) = \varepsilon_v(t, \theta^*)$, since $\beta_v = 0$ and $\varepsilon_v(t, \hat{\theta}_N) = \varepsilon_v(t, \theta^*)$.

Thus by considering the point-wise test in terms of the vector-valued test, by using the same expression of the covariance matrix as in Eq. (4-31), the covariance matrix that corresponds to the hypothesis in the point-wise test is shown as:

$$ P_{H_0} = \frac{1}{N^2} P_r \Lambda_{\varepsilon_v(\hat{\theta}_N)} P_r^T = \frac{1}{N^2} P_r \Lambda_{\varepsilon_v(\theta^*)} P_r^T = \frac{1}{N^2} P_r \Lambda_{\varepsilon(\hat{\theta}_N)} P_r^T $$

(4-58)

with $\Lambda_{\varepsilon(\hat{\theta}_N)} = E[\varepsilon(\hat{\theta}_N)\varepsilon^T(\hat{\theta}_N)]$

With similar properties as used in the open-loop point-wise validation([1],[4]), the variance $P$ in Eq. (4-7) is actually an estimate of the $(1,1)$ element of the covariance matrix,

$$ P_{H_0} = \frac{1}{N^2} P_r \Lambda_{\varepsilon_v(\hat{\theta}_N)} P_r^T $$

(4-59)

according to [4] with proof in [9].

By comparing Eq. (4-34) and Eq. (4-58), we can directly see that the covariance matrix corresponds to $H_0$ in point-wise test, leaves out the projection operator $\Delta$. This means that the point-wise test in this case neglects the randomness in the parameter estimate $\hat{\theta}_N$.

- We can also see from Eq. (4-58), that $\Lambda_{\varepsilon(\hat{\theta}_N)}$ is used as an estimate of $\Lambda_{\varepsilon_v(\theta^*)}$, which implies that the diagonal element of $\Lambda_{\varepsilon_v(\theta^*)}$ (see Eq. (4-38)):

$$ \hat{R}_{\varepsilon_v(\theta^*)}(0) = \hat{R}_{\varepsilon(\hat{\theta}_N)}(0) = \frac{1}{N} \sum_{t=1}^{N} \varepsilon^2(t, \hat{\theta}_N) $$

(4-60)

Considering Eq. (3-12), we can perceive that there is problem with this equation, moreover from Eq. (3-12) and

$$ \xi_v(t, \hat{\theta}_N) := \beta_v(t, \hat{\theta}_N, \theta^*) + \varepsilon_v(t, \hat{\theta}_N) $$

(4-61)

the autocorrelation of $\varepsilon_v(\theta^*)$ now becomes:

$$ \hat{R}_{\varepsilon_v(\theta^*)}(0) = \hat{R}_{\varepsilon(\hat{\theta}_N)}(0) \approx \frac{1}{N} \sum_{t=1}^{N} [\beta_v^2(t, \hat{\theta}_N, G_0, \theta^*) + \varepsilon^2_v(t, \hat{\theta}_N)] $$

(4-62)

The approximation sign is introduced as the cross terms can be neglected since their values tend to be equal to zero when $N$ approaches infinity, under the condition that $r(t)$ and $v(t)$ are uncorrelated.

It is obvious that the residual signal $\varepsilon(t, \hat{\theta}_N)$ will have a large undermodelling error $\beta_0$ for low order model estimate ($G(\hat{\theta}_N)$) (lower or much lower than the true system $G_0$). Large contribution of undermodelling error $\beta_0$ will then cause a big offset to the $\varepsilon(\hat{\theta}_N)$ estimating $\varepsilon_v(\theta^*)$ as shown in Eq. (4-62). Based on this reasoning we can immediately see the problem of estimating $\varepsilon_v(\theta^*)$ with the value of $\varepsilon(\hat{\theta}_N)$ that is already distorted with high contribution from $\beta_0$. 
While for high order model estimate (higher or much higher than the order of the true system), the effect of undermodelling error will be extremely small. However, some of the measurement noise $v(t)$ will be modelled in the plant model, thus they will no longer present in the residual error $\xi_v(t, \hat{\theta}_N)$. Where by considering Eq. (4-27), the residual error $\xi_v(t, \hat{\theta}_N)$ can be quantified as:

$$\frac{1}{N} \sum_{t=1}^{N} \xi_v^2(t, \hat{\theta}_N) = \frac{1}{N} \left\| \left[ I_N - \Psi_{\hat{\theta}} \Psi_{\hat{\theta}}^T \Psi_{\hat{\theta}} \right] \varepsilon_v(\theta^*) \right\|_2^2$$  \hspace{1cm} (4-63)

Moreover, when we consider an extreme case where the order of the model estimate is the same as the number of data $N$, the noise will be completely modelled in the plant model and considered as the dynamic of the system. As the consequence the residual error $\varepsilon(t, \hat{\theta}_N) = 0$, since the noise is completely modelled. Then at this point it will be an obvious mistake trying to estimate $\varepsilon_v(\theta^*)$ with the zero value $\varepsilon(t, \hat{\theta}_N)$, where as we know will never be the case in real life where noise is always present.

Based on the two cases of undermodelling and overfitting case, in order to avoid the problems presented above, the estimation of $\varepsilon_v(\theta^*)$ in the vector-valued test is therefore done by consistently estimating the noise variance $\sigma_e^2$ as explained in Section 4-4-4.

## 4-7 Chapter summary

In this chapter, the basic concept of model validation in a closed-loop identification is explained. Based on the reasoning found in section 4-2, we know that the model validation in the closed-loop case either validates or invalidates both the plant and noise model simultaneously. Which is different than the open-loop model validation, where the validation of the plant and noise model can be done separately by the evaluation of the cross-correlation and the autocorrelation function, respectively.

The theory for the closed-loop point-wise and vector-valued model structure validation test as well as the comparison between these two tests are explained in this chapter. In the last section of this chapter we have seen some detail explanation of how these two tests are compared.
In this chapter, simulation of the point-wise as well as vector-valued closed-loop model validation test on the basis of cross-correlation function $R_{et}(\tau)$ (as explained in Chapter 4) will be presented. We will first start with the details about the closed-loop system being simulated, which has configuration as depicted in Figure 5-1.

5-1 True system

The closed-loop true system that is simulated is a 4th order ARX data generating system. The (closed-loop) input-output (Figure 5-1) data set $Z^N := \{y(t), u(t)\}_{t=1\ldots N}$ of the true system is gathered during the simulation for the identification purpose.

4th order ARX true system

![Figure 5-1: Closed-loop data generating system](image)
The $4^{th}$ order ARX true system is given as:

$$S: \quad y(t) = \frac{G_0(z)}{1 - 1.976z^{-1} + 2.196z^{-2} - 1.849z^{-3} + 0.8881z^{-4}} u(t) + \frac{1}{1 - 1.976z^{-1} + 2.196z^{-2} - 1.849z^{-3} + 0.8881z^{-4}} e(t)$$

with controller:

$$C(z) = \frac{\text{numC}}{\text{denC}} = \frac{0.0362 - 0.0624z^{-1} + \cdots + 0.0286z^{-7}}{1 - 3.4546z^{-1} + \cdots - 0.2804z^{-7}}$$

where:

\begin{align*}
\text{numC} &= [0.0362 - 0.0624z^{-1} + 0.0678z^{-2} - 0.0375z^{-3} - 0.0213z^{-4} + 0.0558z^{-5} - 0.0431z^{-6} + 0.0286z^{-7}] \\
\text{denC} &= [1.0000 - 3.4546z^{-1} + 6.2550z^{-2} - 7.6388z^{-3} + 6.6578z^{-4} - 4.0734z^{-5} + 1.5346z^{-6} - 0.2804z^{-7}] 
\end{align*}

The bode plot of the plant plant system $G_0(z)$ and the closed-loop transfer function

$$y(t) = \frac{C(z)G_0(z)}{1 + C(z)G_0(z)} r(t)$$

of the true system can be seen in Figure 5-3.

In the next two sections we will consider two simulation examples with different data specification. In these simulation examples we will see how the point-wise and the vector-valued test performs in all three different possibilities of conditions, which are the case when $S \in \mathcal{M}$, $S \notin \mathcal{M}$ with $G_0 \notin \mathcal{G}$, and $S \notin \mathcal{M}$ with $G_0 \in \mathcal{G}$.

In these simulation examples, PRBS(Pseudo-Random Binary Signal) (consult [2],[6],[16]) signal is used as the excitation signal $r(t)$, thus the excitation signal is always considered to be persistently exciting for sufficiently high order.
5-2 Simulation example 1

The data specification in this simulation is explained as follows:

- \( N = 1000 \) data points
- PRBS excitation signal with amplitude 3 (Power \( r(t) = 9 \))
- Power of the noise \( v(t) = 3.6123 \)
- Power of the input signal \( u(t) = 0.6762 \)
- Variance white noise \( e(t) \), \( \sigma^2 = 0.1057 \)
- \( \text{SNR}_u = 0.1872 \)
- \( \text{SNR}_r = 2.4915 \)

In this closed-loop identification simulation, we can evaluate two different signal to noise ratio (SNR), which are the signal to noise ratio of the excitation signal \( r(t) \) with the measurement noise \( v(t) \), denoted by \( \text{SNR}_r \) and the SNR for the input signal \( u(t) \) with measurement noise \( v(t) \), denoted by \( \text{SNR}_u \), given by the following relation:

\[
\text{SNR}_r = \frac{\text{power of signal } r(t)}{\text{power of signal } v(t)}; \quad \text{SNR}_u = \frac{\text{power of signal } u(t)}{\text{power of signal } v(t)}
\]

Since in the closed-loop identification process, the data set \( Z^N := \{y(t), u(t)\} \) is collected under the presence of feedback loop, thus the excitation signal \( r(t) \) will get filtered by the
controller as well as the input signal $u(t)$ will get affected by the measurement noise $v(t)$. By considering $r(t)$ being a white-noise sequence, and it is filtered by the controller $C(z)$, as a consequence the power of the excitation signal $r(t)$ in this closed-loop case would have been reduced significantly by the time it enters the true plant system ($G_0$) as an input signal $u(t)$. Thus it is obvious that it is the reason why the value of SNR_u in this case is smaller than SNR_r.

The other consequence of having the presence feedback loop is that the signal $u(t)$ will also contain the dynamics of the true noise filter $H_0$, by the time it enters the plant system. The plot of the input signal, output signal and the excitation signal of this simulation example is shown in Figure 5-4.

Based on the input-output data $\{y(t), u(t)\}$, we use this data set in the model identification process. In this simulation example we consider three different possibilities that might occur in the process of identifying the model estimate. All three possible conditions are as follows:

- **Condition 1**: $S \in \mathcal{M}$
- **Condition 2**: $S \notin \mathcal{M}$ with $G_0 \notin \mathcal{G}$
- **Condition 3**: $S \notin \mathcal{M}$ with $G_0 \in \mathcal{G}$.

### 5-2-1 Condition 1: $S \in \mathcal{M}$

In this case we consider the case where we have chosen full order plant and noise model $G(z, \theta)$ and $H(z, \theta)$. This means we are using 4th order ARX model in estimating the 4th order ARX true system. The model estimate obtained by using the corresponding input-output data set is given below:

![Figure 5-4: Data set gathered from the 4th order ARX true system, for Simulation example 1](image-url)
The resulting frequency response of the plant model \( G(z, \hat{\theta}_N) \) and the noise model \( H(z, \hat{\theta}_N) \), compared with the frequency response of the true plant system \( G_0 \) and true noise filter \( H_0 \) is shown in Figure 5-5.

After identifying these models, now we want to do a model structure validation test using both point-wise and the vector-valued test on the basis of cross-correlation function \( \hat{R}_{\epsilon r}(\tau) \) as already explained in Chapter 4. The result of point-wise model validation test on the basis of cross-correlation function \( \hat{R}_{\epsilon r}(\tau) \) is inside the 99% confidence bound for all \( \tau \). Thus the point-wise model validation, as expected, is not invalidating the model estimate.

Now we do a model validation of the model estimate by using vector-valued validation test. As explained in Chapter 4, a model is not invalidated by this test when the test value, \( t_{test} = \hat{R}_{\epsilon r}^T P_{H_0}^{-1} \hat{R}_{\epsilon r} \) is smaller than the upper critical value of chi-square distribution \( C_\chi(\alpha, n_\tau) \). The vector-valued test in this case is conducted with the number of time lags is chosen to be \( n_\tau = 50 \), and with 0.01 significance level, and thus the value of \( C_\chi(0.01, 50) = 76.154 \). The resulting value of \( \hat{R}_{\epsilon r}^T P_{H_0}^{-1} \hat{R}_{\epsilon r} \) is 45.4617, thus it is smaller than the value \( C_\chi(0.01, 50) \), and based on this result our model estimate as expected is not invalidated by the vector-valued
5.2 Matlab closed-loop system simulation

Figure 5-6: Point-wise model validation of 4\textsuperscript{th} order ARX model, Simulation example 1, condition 1. The red-dashed line represents the 99\% confidence bound.

The result of these two validation tests shows us that both of these tests are able to not invalidate the correct model estimate, which is the 4\textsuperscript{th} order ARX model.

5.2.2 Condition 2: $S \notin \mathcal{M}$ with $G_0 \notin G$

Suppose in this condition, when we are doing an identification process we have chosen a correct type of model structure with wrong model order, that is an ARX model with order 3. The resulting model estimate is shown as follows:

\[
G(z, \hat{\theta}_N) = \frac{0.3506z^{-1} - 0.7812z^{-2} + 0.5847z^{-3}}{1 - 1.576z^{-1} + 1.16z^{-2} - 0.4417z^{-3}} \\
H(z, \hat{\theta}_N) = \frac{1}{1 - 1.576z^{-1} + 1.16z^{-2} - 0.4417z^{-3}} \tag{5-5}
\]

The bode plot of the plant model $G(z, \hat{\theta}_N)$ and the noise model $H(z, \hat{\theta}_N)$, compared with the frequency response of the true plant system $G_0$ and true noise filter $H_0$ is shown in Figure 5-7.

The result of point-wise model validation test on the basis of cross-correlation function for this model estimate can be seen in Figure 5-8. Based on this validation result we can see that the sample cross-correlation function $\hat{R}_{\varepsilon r}(\tau)$ is inside the 99\% confidence bound for all $\tau$. Thus this means, the point-wise model validation fails to invalidate the supposedly invalidated model estimate.

Now we do a model validation of the model estimate by using vector-valued validation test. The value $\hat{R}_{\varepsilon r}^T P_{H_0}^{-1} \hat{R}_{\varepsilon r}$ for this model estimate is 133.7509, which is larger than the upper critical value of chi-square distribution $C_\chi(0.01, 50) = 76.154$. Based on this result, our model estimate as expected is invalidated by the vector-valued test.
Figure 5-7: Bode plot for the 3rd order ARX model Simulation example 1, condition 2, $S \notin M$ with $G_0 \notin G$, and the bode plot of $G_0$ and $H_0$.

Figure 5-8: Point-wise model validation of 3rd order ARX model, Simulation example 1, condition 2. The red-dashed line represents the 99% confidence bound.
Evaluating the result of these two validation tests, we can see that the point-wise test has failed to invalidate the 3rd order ARX model estimate, whereas the vector-valued test can still give an accurate validation result by invalidating the 3rd order ARX model estimate.

**5-2-3 Condition 3: \( S \notin \mathcal{M} \text{ with } G_0 \in \mathcal{G} \)**

In this condition suppose that we use a 4th order OE model estimate in estimating the 4th order ARX true system. The resulting OE model estimate is shown as follows:

\[
G(z, \hat{\theta}_N) = \frac{0.3241z^{-1} - 0.3369z^{-2} - 0.1368z^{-3} + 0.5057z^{-4}}{1 - 1.596z^{-1} + 1.82z^{-2} - 1.488z^{-3} + 0.6179z^{-4}}
\]

\[H(z, \hat{\theta}_N) = 1\]  \hspace{1cm} (5-6)

The bode plot of the plant model \( G(z, \hat{\theta}_N) \) and the noise model \( H(z, \hat{\theta}_N) \), compared with the frequency response of the true plant system and true noise filter is shown in Figure 5-9. From this bode plot we can see that the second resonance peak of the OE plant model \( G(z, \hat{\theta}_N) \) seems to try to fit the dynamic of the noise filter \( H_0 \) at the same frequency as the second resonance peak of the noise filter \( H_0 \). This may be caused by the fact that since in this case we are using OE model where \( H(z, \hat{\theta}_N) = 1 \), therefore the noise dynamics of \( v(t) \) are modelled in the plant model \( G(z, \hat{\theta}_N) \).

The result of point-wise model validation test on the basis of cross-correlation function for this model estimate can be seen in Figure 5-10. Based on this validation result we can see
5-2 Simulation example 1

Figure 5-10: Point-wise model validation of 4th order OE model, Simulation example 1, condition 3. The red-dashed line represents the 99% confidence bound

that the sample cross-correlation function $\hat{R}_{er}(\tau)$ is inside the 99% confidence bound for all $\tau$. This means that the point-wise model validation fails to invalidate the 4th order OE model estimate.

Now model validation of the model estimate is done by using vector-valued validation test. The value $t_{test}$ for this 4th order OE model estimate is $792.7473$, which is much larger than the upper critical value of chi-square distribution $C_{\chi}(0.01, 50) = 76.154$. Based on this result our model estimate as expected is invalidated by the vector-valued test.

By evaluating the result of these two validation tests, we can see that the point-wise test has again failed to invalidate the 4th order OE model estimate, whereas the vector-valued validation test still has successfully invalidated the OE model estimate.

**Short simulation conclusion**

Based on the results for this simulation example we can see that the vector-valued validation test can always give a consistent validation results, whereas in this example the point-wise test has failed in invalidating the model estimates both in the case where $S \notin M$ with $G_0 \in G$, and $S \notin M$ with $G_0 \notin G$.

The vector-valued validation test for the ARX model order 1 to 10 is shown in Figure 5-11 and the vector-valued validation test for the OE model estimate for model order 1 to 10 is given in Figure 5-12. Based on these results we can see that the vector-valued test is able to successfully invalidate the ARX model estimate lower than 4, and validates all the model estimates of order 4 to 10. As for the OE model estimates, we can see in Figure 5-12 that the value of $\hat{R}_{er}^TP_{R_0}^{-1}\hat{R}_{er}$ for OE model estimates with order 1 to 10 are much larger than $C_{\chi}(0.01, 50)$. This means the vector-valued test is able to successfully invalidate all OE model estimates of order 1 to 10.
Figure 5-11: Vector-valued model validation of the ARX model estimate with order $n=[1,10]$, for the Simulation example 1. The solid blue line is the $\hat{R}_{xx}^T P_{xx}^{-1} \hat{R}_{xx}$ and the dashed red line is $C_{\chi}(0.01, 50)$. The right plot is the zoomed in version of the left plot.

Figure 5-12: Vector-valued model validation of the OE model estimate with order $n=[1,10]$, for the Simulation example 1. The solid blue line is the $\hat{R}_{xy}^T P_{xy}^{-1} \hat{R}_{xy}$ and the dashed red line is $C_{\chi}(0.01, 50)$.
5-3 Simulation example 2

Compared to the previous simulation example, in this simulation example the number of data \( N \) is increased as well as the power of the excitation signal \( r(t) \). The specification of the data used in this simulation is given as follows:

- \( N = 2000 \) data points
- PRBS excitation signal with amplitude 5 (Power \( r(t) = 25 \))
- Power of the noise \( v(t) = 2.8897 \)
- Power of the input signal \( u(t) = 1.5981 \)
- Variance white noise \( e(t), \sigma^2_e = 0.1021 \)
- \( \text{SNR}_u = 0.5530 \)
- \( \text{SNR}_r = 8.6514 \)

Just like in the previous simulation example, two different signal to noise ratio (SNR) are evaluated in this closed-loop identification simulation. These two SNR’s are evaluated by the following expressions:

\[
\text{SNR}_r = \frac{\text{power of signal } r(t)}{\text{power of signal } v(t)}; \quad \text{SNR}_u = \frac{\text{power of signal } u(t)}{\text{power of signal } v(t)}
\]

The plot of the input signal, output signal and the excitation signal of this simulation example is shown in Figure 5-13.

Similar to the previous simulation example, in this simulation example we consider three different possibilities that might occur in the process of identifying the model estimate as follows:

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Figure 5-14: Bode plot for the 4\textsuperscript{th} order ARX model Simulation example 2, condition 1, \( S \in \mathcal{M} \), and the bode plot of \( G_0 \) and \( H_0 \)

- Condition 1: \( S \in \mathcal{M} \)
- Condition 2: \( S \notin \mathcal{M} \) with \( G_0 \notin \mathcal{G} \)
- Condition 3: \( S \notin \mathcal{M} \) with \( G_0 \in \mathcal{G} \).

5-3-1 Condition 1: \( S \in \mathcal{M} \)

In this case we consider the case where we have chosen full order plant and noise model \( G(z, \theta) \) and \( H(z, \theta) \). This means we are using 4\textsuperscript{th} order arx model in estimating the 4\textsuperscript{th} order ARX true system. The model estimate obtained by using the corresponding input-output data set is given below:

\[
G(z, \hat{\theta}_N) = \frac{0.0001628z^{-1} + 0.002763z^{-2} + 0.05363z^{-3} + 0.1914z^{-4}}{1 - 1.965z^{-1} + 2.182z^{-2} - 1.844z^{-3} + 0.8802z^{-4}} \quad (5-7)
\]

\[
H(z, \hat{\theta}_N) = \frac{1}{1 - 1.965z^{-1} + 2.182z^{-2} - 1.844z^{-3} + 0.8802z^{-4}}
\]

The bode plot of the plant model \( G(z, \hat{\theta}_N) \) and the noise model \( H(z, \hat{\theta}_N) \), compared with the frequency response of the true plant system and true noise filter is shown in Figure 5-14.

The result of point-wise model validation test on the basis of cross-correlation function can be seen in Figure 5-15. Based on this validation result we can see that the sample cross-correlation function \( R_{\tau}\) is inside the 99\% confidence bound for all \( \tau \). Thus the point-wise model validation, as expected, is able to not invalidate the model estimate.
Similar to the previous simulation example, the vector-valued test in this case is conducted with the number of time lags is chosen to be $n_{\tau} = 50$, and with 0.01 significance level, and thus the value $C_{\chi}(0.01, 50) = 76.154$. The resulting value of $\hat{R}_{\epsilon r}^T P_{H_0}^{-1} \hat{R}_{\epsilon r}$ is $51.7447$, thus it is smaller than the chi-square value $C_{\chi}(0.01, 50)$, and based on this result our model estimate as expected is not invalidated by the vector-valued test. By seeing the result of these two validation tests, we can see that both of these test are able in not invalidating the correct model estimate, which is a 4th order ARX model.

5-3-2 Condition 2: $S \notin \mathcal{M}$ with $G_0 \notin \mathcal{G}$

Suppose in this case, when we are doing an identification process we have chosen a correct type of model structure with wrong model order, that is an ARX model with order 3. The resulting model estimate is shown as follows:

$$G(z, \hat{\theta}_N) = \frac{0.2622z^{-1} - 0.5962z^{-2} + 0.4767z^{-3}}{1 - 1.554z^{-1} + 1.166z^{-2} - 0.4748z^{-3}}$$

$$H(z, \hat{\theta}_N) = \frac{1}{1 - 1.554z^{-1} + 1.166z^{-2} - 0.4748z^{-3}}$$

(5-8)

The bode plot of the plant model $G(z, \hat{\theta}_N)$ and the noise model $H(z, \hat{\theta}_N)$, compared with the frequency response of the true plant system and true noise filter is shown in Figure 5-16.

The result of point-wise model validation test on the basis of cross-correlation function for this model estimate can be seen in Figure 5-17. Based on this validation result we can see that the sample cross-correlation function $\hat{R}_{\epsilon r}(\tau)$ at some of the positive lags are outside the 99% confidence bound. Thus this means, unlike in the previous simulation example, the point-wise model validation in this simulation where we are in the case $S \notin \mathcal{M}$ with $G_0 \notin \mathcal{G}$, successfully invalidates the supposedly invalidated 3rd order ARX model estimate.
Figure 5-16: Bode plot for the 3rd order ARX model Simulation example 2, condition 2, $S \notin \mathcal{M}$ with $G_0 \notin \mathcal{G}$, and the bode plot of $G_0$ and $H_0$.

Figure 5-17: Point-wise model validation of 3rd order ARX model, Simulation example 2, condition 2. The red-dashed line represents the 99% confidence bound.
Now we do a model validation of the model estimate by using vector-valued validation test. The value of test statistic for this model estimate is \( t_{test} = 669.8412 \), which is much larger than the upper critical value of chi-square distribution \( C_{\chi^2}(0.01, 50) = 76.154 \). Based on this result, our model estimate is as expected invalidated by the vector-valued test.

From the result of these two validation tests, we can see that both the point-wise test and the vector-valued validation test are able to give correct validation results by invalidating the 3rd order ARX model estimate.

5-3-3 Condition 3: \( S \notin \mathcal{M} \) with \( G_0 \in \mathcal{G} \)

In this case suppose that we use a 4th order OE model estimate in estimating the 4th order ARX true system. The resulting OE model estimate is shown as follows:

\[
G(z, \hat{\theta}_N) = \frac{0.1966z^{-1} - 0.4279z^{-2} + 0.3471z^{-3} + 0.1214z^{-4}}{1 - 1.956z^{-1} + 2.245z^{-2} - 1.888z^{-3} + 0.8408z^{-4}}
\]

\[
H(z, \hat{\theta}_N) = 1
\]

The bode plot of the plant model \( G(z, \hat{\theta}_N) \) and the noise model \( H(z, \hat{\theta}_N) \), compared with the frequency response of the true plant system and true noise filter is shown in Figure 5-18. From this bode plot, similar to the OE case in the previous simulation example, we can see that the second resonance peak of the OE plant model \( G(z, \hat{\theta}_N) \) seems to try to fit the dynamic of the noise filter \( H_0 \) at the same frequency as the second resonance peak of the noise filter \( H_0 \).

The result of point-wise model validation test on the basis of cross-correlation function for this model estimate can be seen in Figure 5-19. Based on this validation result we can see...
that the sample cross-correlation function $\hat{R}_{\epsilon \epsilon}(\tau)$ is inside the 99% confidence bound for all $\tau$. This means just like the previous simulation example, the point-wise model validation fails to invalidate the 4th order OE model estimate.

Now we see the case where the model validation of the model estimate is done by using vector-valued validation test. The value of the test statistic, $\hat{R}_{\epsilon \epsilon}^T P_{H_0}^{-1} \hat{R}_{\epsilon \epsilon}$, for this 4th order OE model estimate is $1422.399$, which is much larger than $C_{\chi^2}(0.01, 50) = 76.154$. Based on this result our model estimate as expected is invalidated by the vector-valued test.

**Short simulation conclusion**

Based on the results for this simulation example we can see that the vector-valued validation test can always give a consistent validation results, whereas in this simulation example we can see that by increasing the number of data points and the power of the excitation signal used in the identification experiment, the point-wise test was able to invalidate the model estimate in the case $S \notin M$ with $G_0 \notin G$. However, in the case where $S \notin M$ with $G_0 \in G$, the point-wise test still unable to produce a correct validation result, which is shown by the fact that it still fails in invalidating the 4th order OE model estimate.

The vector-valued validation test for the ARX model order 1 to 10 is shown in figure Figure 5-20 and for the OE model estimate for model order 1 to 10 is given in Figure 5-21. These results show us that just like in the previous simulation example, the vector-valued test is able to successfully invalidate the ARX model estimate lower than 4, and validates all the model estimates of order 4 to 10. As for the OE model estimates, we can see in Figure 5-21 that the value of $\hat{R}_{\epsilon \epsilon}^T P_{H_0}^{-1} \hat{R}_{\epsilon \epsilon}$ for all OE model estimates are much larger than $C_{\chi^2}(0.01, 50)$. This means the vector-valued test is able to successfully invalidate all OE model estimates of order 1 to 10.
Figure 5-20: Vector-valued model validation of the ARX model estimate with order n=[1,10], for the Simulation example 2. The solid blue line is the $\hat{R}_T P^{-1}_0 \hat{R}_T$ and the dashed red line is $C_\chi(0.01, 50)$. The right plot is the zoomed in version of the left plot.

Figure 5-21: Vector-valued model validation of the OE model estimate with order n=[1,10], for the Simulation example 2. The solid blue line is the $\hat{R}_T P^{-1}_0 \hat{R}_T$ and the dashed red line is $C_\chi(0.01, 50)$. 
5-4 More simulation analysis on the vector-valued test

In this section we will see the effect of varying the power of the excitation signal \( r(t) \) and varying the number of data points \( N \) to the value of test statistic \( \hat{R}^T_{er} P^{-1}_{H_0} \hat{R}_{er} \) for model orders that contain no bias. The model order being analyzed in this case is therefore model order 4 and higher, with ARX model structure.

5-4-1 Varying the power of the excitation signal

In this part we will consider two cases of simulation identification experiment. The first case is the condition where for every different experiment simulations, we use the exact same white noise signal \( e(t) \). The second case is the condition where for every different experiment simulations the white noise signal \( e(t) \) are generated new.

Case 1: Using identical white noise \( e(t) \)

Suppose that we have generated a PRBS signal with length 1000. Then we use the sequence of this PRBS signal and vary the amplitude, such that we have several PRBS signals with different amplitudes.

By considering this scenario, in this simulation example we will vary the amplitude of our PRBS excitation signal \( r(t) \) from amplitude 3 to 9. Then for each of these PRBS signals we will use it to do several simulations of identification experiments that result different input-output data sets. When conducting these several experiments, the white-noise sequence \( e(t) \) used in every simulation experiments in this case are identical. The purpose of doing this is to eliminate some of the uncertainty factors that are introduced by generating new random variable \( e(t) \) for every simulation experiment, that might hinder us from perceiving a relation between changing the power of \( r(t) \) and the change in test value \( \hat{R}^T_{er} P^{-1}_{H_0} \hat{R}_{er} \).

After conducting several simulation experiments for different amplitudes of PRBS signal, the resulting data sets are then used in the identification process that results different model estimates with the same order. For each of these model estimates and also for each model resulted from different data sets, the test statistics \( \hat{R}^T_{er} P^{-1}_{H_0} \hat{R}_{er} \) are then calculated and plotted. The plot of the value of \( \hat{R}^T_{er} P^{-1}_{H_0} \hat{R}_{er} \) for ARX model order 4 can be seen in Figure 5-22, for ARX model order 6 in Figure 5-23, and for model order 8 in Figure 5-24.

From the relation shown in Figure 5-22, Figure 5-23, and Figure 5-24 we can observe that for higher power of excitation signal, the value of the test statistic in this case, although the difference is not that significant, tends to become smaller. This could be caused by the fact that since when we have higher signal to noise ratio, in general case of identifying linear time invariant system, we will get more accurate model estimate. This means the residual \( \varepsilon(t, \hat{\theta}_N) \) will probably become closer to the realization of a white noise signal. Thus the relation between \( \varepsilon(t, \hat{\theta}_N) \) and \( r(t) \) will also become smaller and smaller, which leads to smaller \( \hat{R}_{er}(\tau) \), and smaller test value \( \hat{R}^T_{er} P^{-1}_{H_0} \hat{R}_{er} \). It is also worth to mention that all these values of \( \hat{R}^T_{er} P^{-1}_{H_0} \hat{R}_{er} \) are smaller than the chi-square value \( C_{\chi^2}(0.01, 50) = 76.154 \), since these ARX models are certainly not undermodelled.
5-4 More simulation analysis on the vector-valued test

**Figure 5-22:** Different values of test statistic $\hat{R}_r^T P^{-1} H_0^T \hat{R}_r$ of $4^{th}$ order ARX model estimate, for the varying amplitude of PRBS excitation signal $r(t)$ [amp 3 to 9], in Case 1

**Figure 5-23:** Different values of test statistic $\hat{R}_r^T P^{-1} H_0^T \hat{R}_r$ of $6^{th}$ order ARX model estimate, for the varying amplitude of PRBS excitation signal $r(t)$ [amp 3 to 9], in Case 1

**Figure 5-24:** Different values of test statistic $\hat{R}_r^T P^{-1} H_0^T \hat{R}_r$ of $8^{th}$ order ARX model estimate, for the varying amplitude of PRBS excitation signal $r(t)$ [amp 3 to 9], in Case 1
Figure 5-25: Different values of test statistic $\hat{R}_T^T P_{H_0}^{-1} \hat{R}_{e\nu}$ of 4th order ARX model estimate, for the varying amplitude of PRBS excitation signal $r(t)$ [amp 3 to 9], in Case 2

Figure 5-26: Different values of test statistic $\hat{R}_T^T P_{H_0}^{-1} \hat{R}_{e\nu}$ of 6th order ARX model estimate, for the varying amplitude of PRBS excitation signal $r(t)$ [amp 3 to 9], in Case 2

Figure 5-27: Different values of test statistic $\hat{R}_T^T P_{H_0}^{-1} \hat{R}_{e\nu}$ of 8th order ARX model estimate, for the varying amplitude of PRBS excitation signal $r(t)$ [amp 3 to 9], in Case 2
5-4 More simulation analysis on the vector-valued test

**Case 2: White noise signal \( e(t) \) are generated new**

In this part we will see how the value of test statistic changes for different amplitude of PRBS signal varies from 3 to 9. Similar to the previous case, we use PRBS signal of length 1000. However, in this case we will generate new white noise signal \( e(t) \) for every simulation experiment which is for varying PRBS amplitude of 3 to 9.

After conducting several simulation experiments for different amplitude of PRBS signal, the resulting data sets are then used in the identification process that results different model estimates with the same order. For each of these model estimates the test statistics \( \hat{R}_{\epsilon r}^TP_{H_0}^{-1}\hat{R}_{\epsilon r} \) are then calculated and plotted. The plot of the value of \( \hat{R}_{\epsilon r}^TP_{H_0}^{-1}\hat{R}_{\epsilon r} \) for ARX model order 4 can be seen in Figure 5-25, for ARX model order 6 in Figure 5-26, and for model order 8 in Figure 5-27.

By evaluating Figure 5-25, Figure 5-26, Figure 5-27, we can see that unlike the simulation in Case 1, in this case where the random signal \( e(t) \) are generated new for different simulation experiment, we do not see any relation between varying the excitation power and the change in value of the test statistic \( \hat{R}_{\epsilon r}^TP_{H_0}^{-1}\hat{R}_{\epsilon r} \). Although we can also see from these plots that all these values of \( \hat{R}_{\epsilon r}^TP_{H_0}^{-1}\hat{R}_{\epsilon r} \) are of course smaller than the chi-square value \( C_1(0.01, 50) = 76.154 \), since we know that these ARX models contain no bias with respect to the true system.

5-4-2 **Varying the number of data points**

Unlike in the case where we vary the power of the excitation signal, since in this case what varies is the number of data points, thus for every identification experiment it is impossible to have the same sequence of signal \( r(t) \) and the same white-noise sequence \( e(t) \). This means in this case we will only consider the case where we have to generate new random signal \( r(t) \) and \( e(t) \) for every identification experiment simulation.

In this case we will consider the case where we use PRBS amplitude 3. After conducting several simulation experiments for different number of data \( N = 1000, 1500, \cdots, 3000 \), the resulting input-output data sets are then used in the identification process that results different model estimates with the same order. For each of these model estimates the test statistic \( \hat{R}_{\epsilon r}^TP_{H_0}^{-1}\hat{R}_{\epsilon r} \) is then calculated and plotted. The plot of the value of \( \hat{R}_{\epsilon r}^TP_{H_0}^{-1}\hat{R}_{\epsilon r} \) for ARX model order 4 can be seen in Figure 5-28, for ARX model order 6 in Figure 5-29, and for model order 8 in Figure 5-30. From these plots we can see that unlike the Case 1 when we vary the power input signal, here we hardly see any relation between the changing number of data points and the value of test statistic.

Suppose that we now insanely increase the amplitude of the PRBS excitation signal to 50, and we do the simulation experiment for different number of data like we did previously. Then using these different data sets we conduct identification experiment using the same model order. The plot of the value of \( \hat{R}_{\epsilon r}^TP_{H_0}^{-1}\hat{R}_{\epsilon r} \) for ARX model order 4 can be seen in Figure 5-31, for ARX model order 6 in Figure 5-32, and for model order 8 in Figure 5-33.

From these plots in Figure 5-31, Figure 5-32, Figure 5-33, we can see that by increasing significantly the signal to noise ratio does not affect the result of the test statistic value in relation with the number of data. In other words, just like in our previous simulation, in this simulation we still do not see any relation between increasing the number of data and the
Figure 5-28: Different values of test statistic $\hat{R}_x^T P_{\hat{R}_x}^{-1} \hat{R}_x$ of 4th order ARX model estimate, for the varying number of data $N = 1000, 1500 \ldots 3000$, with PRBS amplitude 3

Figure 5-29: Different values of test statistic $\hat{R}_x^T P_{\hat{R}_x}^{-1} \hat{R}_x$ of 6th order ARX model estimate, for the varying number of data $N = 1000, 1500 \ldots 3000$, with PRBS amplitude 3
value of test statistic. This phenomena probably happens as a result of the uncertainty or stochastic effect induced by the random variable \( r(t) \) and \( e(t) \) those are generated new, for every single simulation experiment.

**5-5 Chapter summary**

In this chapter we have seen two different simulation examples with different data specification. The excitation signal \( r(t) \) used in the second simulation example is more powerful than the first simulation example. As also more data points are used in the second simulation example. For each of the simulation example, we consider three different conditions, \( S \in \mathcal{M} \), \( S \notin \mathcal{M} \) with \( G_0 \notin \mathcal{G} \), and \( S \notin \mathcal{M} \) with \( G_0 \in \mathcal{G} \). In the Simulation example 1, we have seen that the point-wise validation test on the basis of cross-correlation function \( \hat{R}_{r} \) has failed in invalidating both the case \( S \notin \mathcal{M} \) with \( G_0 \notin \mathcal{G} \) which is the 3\textsuperscript{rd} order ARX model estimate, and for the case \( S \notin \mathcal{M} \) with \( G_0 \in \mathcal{G} \), which is represented by the 4\textsuperscript{th} order OE model estimate. Whereas the vector-valued test in this case is able to deliver correct model validation, by successfully invalidating all the ARX model estimate lower than 4, and validates all the model estimates of order 4 to 10. As for the OE model estimates, the vector-valued test is able to successfully invalidate all OE model estimates of order 1 to 10.

In the Simulation example 2 however, the point-wise test performs a little bit better than the point-wise test done in Simulation example 1. Where in this case, it is able to invalidate a 3\textsuperscript{rd} order ARX model estimate, that represents the condition \( S \notin \mathcal{M} \) with \( G_0 \notin \mathcal{G} \), but it still fails to invalidate the 4\textsuperscript{th} order OE model estimate, which represents the condition where...
Figure 5-31: Different values of test statistic \( \hat{R}_r^T P_{H_0}^{-1} \hat{R}_r \) of 4\(^{th}\) order ARX model estimate, for the varying number of data \( N = 1000, 1500 \cdots 3000 \), with PRBS amplitude 50

Figure 5-32: Different values of test statistic \( \hat{R}_r^T P_{H_0}^{-1} \hat{R}_r \) of 6\(^{th}\) order ARX model estimate, for the varying number of data \( N = 1000, 1500 \cdots 3000 \), with PRBS amplitude 50
Figure 5-33: Different values of test statistic $\hat{R}^T_{er} P^{-1}_{H_0} \hat{R}_{er}$ of $8^{th}$ order ARX model estimate, for the varying number of data $N = 1000, 1500 \ldots 3000$, with PRBS amplitude 50.

In the last section of this chapter, we have also seen the effect of varying the amplitude of the excitation signal $r(t)$, as well as varying the number of data points $N$ to the change in value of the test statistic $\hat{R}^T_{er} P^{-1}_{H_0} \hat{R}_{er}$. When observing the effect of varying the amplitude of the excitation signal, where for each of the simulation experiment the white noise signal $e(t)$ being used are identical, we saw that the value of test statistic, although insignificant, tends to become smaller as the PRBS amplitude gets higher. For the same case of varying the amplitude of the PRBS excitation signal, where the white noise signal $e(t)$ are generated new for every simulation experiment, we have seen that in this case no relation can be perceived between varying the PRBS amplitude and the change in value of the test statistic $\hat{R}^T_{er} P^{-1}_{H_0} \hat{R}_{er}$.

The evaluation of the effect of varying the number of data points $N$ used in the identification process to the test value, in this case also showed no relation.
Chapter 6

Conclusions

6-1 Conclusions

Several important main points that can be concluded from this research are stated as follows:

- From the simulation presented in Chapter 5, we have seen that the closed-loop point-wise validation test on the basis of cross-correlation function has failed in performing invalidation for the case where \( S \notin \mathcal{M} \) with \( G_0 \in \mathcal{G} \), which is represented by the 4\(^{th}\) order OE model estimate, for both simulation examples. From these two examples we can see that even when we have significantly increased the number of data, as well as the power of the excitation signal \( r(t) \), the point-wise validation test still fail to invalidate the 4\(^{th}\) order OE model, estimating the 4\(^{th}\) order ARX system. Whereas the vector-valued test are able to always consistently perform correct model validation for all of the three possible conditions, \( S \in \mathcal{M} \), \( S \notin \mathcal{M} \) with \( G_0 \notin \mathcal{G} \), and \( S \notin \mathcal{M} \) with \( G_0 \in \mathcal{G} \), in both simulation examples. From this fact, we can then say that the closed-loop vector-valued model structure validation test in this case is an accurate validation test in detecting the presence of bias in the model estimate.

- Based on the simulation results we can also say that in this case, the vector-valued validation test is irreplaceable by the point-wise model validation test on the basis of cross-correlation function. Since the 4\(^{th}\) order OE model estimates which represent a condition \( \{ S \notin \mathcal{M} \text{ with } G_0 \in \mathcal{G} \} \) in the two different simulation examples, both are only invalidated by the vector-valued test.

- Although the closed-loop vector-valued validation test in this case is shown to be an accurate model structure validation in detecting the presence of bias, it requires much more computational effort compared to the point-wise test. This is caused by the fact that the vector-valued test involves large dimension inverse of matrices and multiplications, as well as the computation of the prediction error gradient \( \psi(t, \theta) \), which requires the computation of the partial derivative of the plant and noise model with respect to the parameter vector \( \theta \). This is an important aspect to consider especially when we
are dealing with a model validation involving large number data points and high order model.

6-2 Future research possibilities

Below are several suggestions for possibilities of future research:

1. In this scope of research, the vector-valued test is only conducted on the basis of the sample cross-correlation function, \( \hat{R}_{\epsilon r}(\tau) \) therefore in the future research it could be interesting to propose and investigate the vector-valued test on the basis of the sample autocorrelation of the residual signal \( \hat{R}_e(\tau) \).

2. Since in this case the simulation of the vector-valued test is only conducted only for ARX and OE models for relatively low order system, for future research the theory can be further tested and analyzed by using ARMAX or Box-Jenkins model structure with higher order model.

3. The other possibilities of future research would be to do the model validation for a model estimating an unstable plant \( G_0 \) using either direct closed-loop identification as explained in Chapter 3, or even using another closed-loop identification method such as \textit{Indirect closed-loop identification} method [2].

4. The possibility of further testing and improving the closed-loop point-wise test on the basis of cross-correlation function, by consistently estimating the noise variance \( \sigma^2_e \), to be incorporated in the calculation of variance \( P \) as in Chapter 4, can also be a good subject of research.
Bibliography


Glossary

List of Symbols

\( \Delta \)  
Projection matrix

\( \psi(t, \hat{\theta}_N) \)  
Prediction error gradient

\( \sigma^2_e \)  
Noise variance

\( \tau \)  
Time lags

\( \theta^* \)  
Asymptotic parameter vector

\( \theta_0 \)  
True parameter vector

\( \hat{\theta}_N \)  
Parameter estimate obtained using \( N \) data points

\( \varepsilon(t, \theta) \)  
Residual signal

\( G \)  
Plant model set

\( \mathcal{M} \)  
Model sets

\( S \)  
True system

\( e(t) \)  
White noise signal

\( G(z, \theta) \)  
Plant model

\( G_0(z) \)  
True plant system

\( H(z, \theta) \)  
Noise model

\( H_0(z) \)  
True noise filter

\( N \)  
Number of data points

\( P_\theta \)  
Parameter vector variance

\( r(t) \)  
Excitation(reference) signal

\( R_{e,r}(\tau) \)  
Cross-correlation function of \( \varepsilon \) and \( r \)

\( R_\varepsilon(\tau) \)  
Autocorrelation function of residual \( \varepsilon \)

\( S(z, \theta) \)  
Sensitivity function of \( G(z, \theta) \)

\( S_0(z) \)  
Sensitivity function of \( G_0(z) \)

\( u(t) \)  
Input signal

\( v(t) \)  
Measurement noise

\( y(t) \)  
Output signal