Frequency analysis of fluids and fluid-structure systems
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Dr. C. Cuvelier heeft in belangrijke mate bijgedragen aan de totstandkoming van dit proefschrift. Het College van Dekanen heeft hem derhalve aangewezen als Toegevoegd Promotor.
Voorwoord

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aan Janet
Preface

Parts of the work presented in this thesis have been published, are to be published or have appeared as a research report, as listed below.

1. An efficient method for solving capillary free-boundary problems
   R.M.S.M. Schulkes & C. Cuvelier

2. On the computation of normal modes of a rotating, viscous, incompressible fluid with a capillary free boundary
   R.M.S.M. Schulkes & C. Cuvelier

3. Eigenfrequencies of a viscous fluid in a flexible container
   R.M.S.M. Schulkes

4. Some numerical methods for the computation of free-boundary problems governed by the Navier-Stokes equations
   C. Cuvelier & R.M.S.M. Schulkes

5. Interactions of a viscous fluid with an elastic solid: eigenmode analysis.
   R.M.S.M. Schulkes
Contents

1. General Introduction 1

2. Capillary free-boundary problems
   2.1 Introduction 5
   2.2 The TL Method 8
   2.3 Properties of the TL method 11
   2.4 A capillary free-boundary problem 13
   2.5 Numerical solution 17
   2.6 Numerical experiments 19
   2.7 Conclusions 24

3. Inviscid capillary oscillations with edge constraints
   3.1 Introduction 25
   3.2 Problem formulation 27
   3.3 Eigenfrequencies in limiting cases 32
   3.4 Free-surface deflection 34
   3.5 Liquid bridge oscillations 35
   3.6 Conclusions 38

4. Eigenmodes of a viscous, rotating fluid with a capillary free boundary
   4.1 Introduction 39
   4.2 Problem formulation 42
   4.3 Properties of the spectrum 48
   4.4 Numerical solution of the spectral problem 50
   4.5 Numerical experiments with $We = 0$ 57
   4.6 Rotational effects 63
   4.7 Axisymmetric liquid bridge oscillations 69
   4.8 Conclusions 72

5. Eigenmodes of a viscous fluid in a flexible container
   5.1 Introduction 75
   5.2 Problem formulation 77
   5.3 Static deflection of the membrane 79
5.4 Inviscid fluid oscillations 82
5.5 Viscous oscillations: properties of the spectrum 86
5.6 Numerical results 91
5.7 Conclusions 96

6. Interaction of a viscous fluid with an elastic solid 99
   6.1 Introduction 99
   6.2 Problem formulation 100
   6.3 Variational formulation and the spectral problem 104
   6.4 Numerical solution by the finite-element method 110
   6.5 Numerical results 116
   6.6 Conclusions 123

7. Concluding remarks 126
   Appendix A 129
   References 130
   Summary 137
   Samenvatting 138
   Curriculum Vitae 140
Chapter 1

General introduction

Oscillating fluids are all about us. Everyday experience make most people aware that the free surface of a fluid may undergo large deformations. These deformations may be induced purposely to be used to their advantage, or may be unwanted so that their occurance is to be avoided. In either case it is advantageous to understand the motion of the fluid, necessitating the development of mathematical models so that predictions can be made.

Of particular interest in this thesis are the eigenfrequencies (or resonance frequencies) of a system which consists of a (viscous) fluid, possibly in combination with a flexible mechanical structure. By an eigenfrequency of the system we understand the frequency at which, given some periodic external force, the system reacts 'excessively'. Questions related to the dynamic behaviour of a system are, among many others:

(i) what are the eigenfrequencies of the system and how does the system behave when an eigenmode is excited;

(ii) how quick do disturbances grow when the system is excited in one of its resonance modes;

(iii) how quick do disturbances damp out when the external force is removed.

Question (i) can generally be answered by neglecting sources of damping. In order to answer questions (ii) and (iii) the dissipation of energy has to be taken into account. Regarding the fluid this means that viscous effects have to be incorporated.

The motivation of the work in the first part of this thesis, in which we study the static configuration and dynamic behaviour of a free liquid surface,
is derived from the aerospace industry. On earth surface-tension forces may often be neglected due to the dominance of the gravitational force. In space, however, surface tension is generally the dominant force due to the absence of significant body forces. The static and dynamic behaviour of the free liquid surface is in that case quite different from that on earth. Fluids in space are common occurrence nowadays and understanding their behaviour in this 'micro-gravity' environment is desirable - not in the last place because it is hard to perform micro-gravity experiments on earth.

Fluids in propellant tanks of spacecraft provide a first example of liquids in a micro-gravity environment. Propellant tanks of spacecraft may contain large masses of liquid fuel (in comparison with the total mass of the craft) which, when excited, may have a profound effect on the motion of the spacecraft. Thus, for the control and stability of the spacecraft it is important to avoid nutation frequencies which are close to the characteristic (resonance) frequencies of the fluid. Failing to do so may lead to fluid-induced instabilities of which a graphic account is given in Abramson (1966).

A second example in which both the static and dynamic behaviour of a capillary free-surface is important, is derived from industrial processes in which liquid bridges are used. A liquid bridge is a volume of liquid anchored between two rigid parts and which is held together by surface tension. Liquid bridges are used in, for example, the area of crystal growth and purification of solids. The stability of liquid bridges is determined by the balance of surface-tension and gravitational (hydrostatic pressure) forces. Gravitational forces limit the size and therefore the use of liquid bridges on earth. The (near) absence of hydrostatic pressure forces in a micro-gravity environment increases the potential of processes in which liquid bridges are used. However, spacecraft in orbit almost always experience vibrations due to operating equipment or actions of the crew. These vibrations can, for example, disrupt the growth of crystals due to the vibration-induced fluid motion, see for example Delucas et al. (1990). Knowledge of the eigenfrequencies of a liquid bridge is clearly desirable.

In the second part of this thesis we investigate the motion of a viscous fluid in a container of which the walls are flexible. The flexible container walls may be deflected under the action of fluid forces exerted on the walls. Since the motion of the fluid is, in turn, effected by the moving container wall, this type of problem is generally known as a fluid-structure interaction problem. When eigenmodes of the coupled fluid-structure system are considered, a number of aspects are of interest. How does the fluid effect
the flexible structure when it oscillates in one of its eigenmodes and, indeed, how are the eigenmodes of the structure effected by the presence of the fluid. Interesting is also what happens when an eigenmode of the structure is near an eigenmode of the fluid.

Fluid oscillations in a flexible container occur in a wide variety of situations. In the aerospace industry the interaction between the liquid propellants and the non-rigid propellant tanks is a problem of great importance. Due to the weight constraints, propellant tanks are often thin-walled shells which can experience bending and breathing motions, coupled with oscillations of the liquid propellants. Infamous in this context are the so-called "pogo"-oscillations: accordion-like longitudinal vibrations of the spacecraft, possibly enhanced by liquid propellant sloshing.

The importance of the fluid-structure interaction problem is, however, by no means restricted to the aerospace industry. Other areas of interest are for example, oil and water oscillations in storage tanks, vibrations of heat exchangers, condensers and fuel-element assemblies in reactor cores and even some physiological applications like membrane oscillations in the inner ear.

The material presented in this thesis is ordered as follows. In chapter 2 we investigate a method for solving capillary free-boundary problems. We investigate, in particular, the speed of convergence of the method and present results for a number of representative cases. Chapter 3 deals with oscillations of an inviscid liquid with a capillary free-boundary on which edge constraints are imposed. It is shown how the problem, hitherto only solved by numerical means, may be solved analytically. Oscillations of a viscous fluid with a capillary free-boundary are investigated in chapter 4. It is shown how the problem can be treated numerically using the finite-element method in conjunction of an inverse-iteration procedure to solve the eigenvalue problem. Effects of rotation are investigated and compared with results obtained using other methods.

In chapter 5 we consider a simple fluid-structure interaction problem. Here the structure is modelled by a membrane. Properties of the spectrum are derived and some approximate analytical results, based on inviscid theory, are given. Chapter 6 deals with a more advanced fluid-structure interaction problem. The structure is in this case modelled by an elastic solid. A variational formulation of the problem is derived which allows general statements concerning the spectrum to be obtained. Finally, in chapter 7 some overall conclusions are presented and recommendations for further work are given.

Since the material presented in this thesis covers a number of related but,
more or less, separate topics, each chapter is preceded by an introduction in which an overview of the literature in each area is given. A vast amount of literature has appeared on each of the topics, making it next to impossible to provide an exclusive reference list. Where possible, references are given to review articles in which a detailed study of the literature has been made.
Chapter 2

Capillary free-boundary problems

2.1 Introduction

The rise of a liquid in a narrow tube and the curvature of a liquid surface in the neighbourhood of a solid boundary, has attracted attention of scientists for centuries. Some of the first measurements related to these phenomena go back to the early eighteenth century, and were carried out by Taylor (1712) and Hauksbee (1712). However, it was not until the beginning of the nineteenth century that one became aware of the fact that these and many other phenomena are all manifestations of something that happens whenever two different materials are adjacent to each other but do not mix. If one of the materials is a fluid which forms an interface with another fluid (or gas), then this interface is referred to as a capillary surface.

The first theory attempting to explain capillary phenomena was formulated by Young (1805). He proceeded from the analogy between an elastic membrane and a free surface of liquid having surface tension. Laplace (1806) placed the theory on a more secure foundation by considering the interactions between neighbouring particles which have a non-zero resultant on the free surface. Gauß (1830) perfected Laplace’s theory in certain respects and it became the object of intensive study by some of the leading researchers of that time, among who was Poisson (1831).

The advances that had been made on the subject of capillary phenomena were such that it prompted the following remark of the Russian scientist Davidov (1851):
"The outstanding contributions made by Poisson and Laplace to the mathematical theory of capillary phenomena have completely exhausted this subject and brought it to such a level of perfection that there is hardly anything to be gained by their further investigation".

And indeed, the theory of capillary phenomena has remained virtually unchanged up to the present day, at least where macroscopic effects are concerned. However, solving the non-linear (partial) differential equations that describe the equilibrium configuration of a capillary surface posed a serious problem. Only a few closed-form solutions are known and asymptotic solutions are remarkably hard to obtain even for the simplest geometric configurations, see for example Rayleigh (1916). During the first half of the twentieth century the capillary problem lost most of its appeal, undoubtedly partly due to the fact that the hurdle of solving the equations could not be taken. However, the practical importance of capillary phenomena in the industry in general and the aerospace industry in particular, lead, together with the development of digital computers and new mathematical techniques, to a renewed interest in the problem.

For a comparison of experimental and theoretical capillary surface shapes we refer to Leslie (1985). Additional references on capillary surface problems can be found in Finn (1986) and Myshkis et al. (1986) who give a comprehensive survey of analytical approaches together with existence and uniqueness results. The difficulty of finding analytical solutions to the capillary problem has resulted in the emphasis being placed on numerical solution techniques. Interesting is the fact that Bashfort & Adams (1883) and Runge (1895) performed some of their early work on the numerical solution of differential equations by calculating meniscus shapes. For a combined numerical-analytical treatment of a relatively simple one-dimensional capillary problem we refer to Concus (1968) who uses a shooting technique and Siekmann et al. (1981) who employ a Runge-Kutta procedure.

Solving the one-dimensional capillary problem in arbitrary configurations can be complicated by the fact that the domain on which the differential equation is to be solved, may not be known a priori. Problems in which this is the case are generally known as free-boundary problems: the domain of solution has to be found as part of the solution. This means that on the unknown free-boundary an extra boundary condition is prescribed in order to be able to find a unique solution to the problem. Concus & Pereyra (1983) have dealt with the 1-D capillary free-boundary problem by
parameterizing the governing equations to yield a system of six coupled, non-linear ordinary differential equations. The parameterization has been chosen such that the differential equations are solved on a fixed domain. The method suffers from the disadvantage that the parameterization depends on the shape of the vessel containing the fluid: each different vessel yields a different set of equations making the method cumbersome and hard to generalize. Hung et al. (1988) write the equations describing the free surface in integral form. The equation is then solved iteratively. The iterative procedure, however, appears to be rather ad hoc, no mention is made about the speed of convergence. Also, the method appears hard to extend to general container shapes.

Two dimensional capillary surfaces without a free boundary have been studied by a number of researchers. A finite-element approach to the problem can be found in Orr et al. (1975), Brown (1979), Brown et al. (1980) and Cuvelier (1987a). Babenko & Yur’ev (1980) have used a minimization technique to solve the 2-D capillary problem. The only paper known by us to have dealt with a 2-D capillary problem in which a free boundary is present is that by Orr et al. (1977). They use a finite-element technique to solve the differential equation, the position of the free boundary is approximated successively using a trial-method type of iterative procedure which is discussed below.

Cuvelier (1987a,b) discusses three techniques for solving free-boundary problems numerically, namely, the trial method, Newton’s method and the total-linearization method. In the case of the trial method the position of the free boundary is guessed and the differential equation is solved while disregarding one of the boundary conditions on the free boundary. The position of the free boundary is then updated by satisfying the relaxed condition as closely as possible. The trial method is applicable to a wide class of problems but has a linear rate of convergence. The speed of convergence can be accelerated by introducing a parameter (or set of parameters) by which the free-boundary update is multiplied. However, the convergence speed is quite sensitive to the parameter choice; choosing the parameter too large may result in a diverging iterative process. In Newton’s method the unknown position of the free boundary is introduced as a degree of freedom in the discretized problem. This yields a system of nonlinear algebraic equations that may be solved by the Newton-Raphson method with a quadratic rate of convergence. Newton’s method suffers from the drawback that a complete account of the variations with respect to the free-boundary degrees of freedom must be incorporated in the Jacobian of the system of equations.
These variations have a non-local character so that the method does not fit into standard finite-element codes.

In the Total Linearization (TL) method the problem is linearized with respect to the position of the free boundary. The influence of the geometric unknowns is thus reduced to boundary integrals in the weak formulation of the problem. The iterative procedure is then like in the trial method: guess the position of the free boundary, solve the equations using the TL-boundary conditions on the free boundary and update the estimate of the position of the free boundary using the relaxed boundary condition. The TL-method has a second order rate of convergence but it does not suffer from the disadvantages of Newton’s method.

In this chapter the TL-method applied to a capillary free-boundary problem will be discussed. In section 2.2 we discuss a generalization of the TL-method as introduced by Cuvelier (1987a). Some general characteristics of the method, like the speed of convergence, are established in section 2.3. The equations governing the capillary free-boundary problem are derived in 2.4 and the application of the TL procedure to this problem is outlined in section 2.5. In section 2.6 we perform some numerical experiments to establish numerical properties of the TL method applied to the capillary free-boundary problem. Calculations are performed for a number of representative cases.

2.2 The TL method

The free boundary problem we will consider in this section is as follows:

\[ -u''(x) = f(x), \quad (2.2.1) \]

with boundary conditions at \( x = a \)

\[ u(a) = A_1(a), \quad u'(a) = A_2(a), \]

and at \( x = b \)

\[ u(b) = B_1(b), \quad u'(b) = B_2(b), \]

where \( A_i, B_i \ i = 1,2 \) are functions of the unknowns \( a \) and \( b \) respectively. Note that for the unique solvability of the differential equation (2.2.1) on a fixed interval, it is sufficient to have one boundary condition at both \( a \) and \( b \), at least one of which must be a Dirichlet condition. The additional
boundary conditions are necessary in order to be able to determine the geometric unknowns $a$ and $b$. We now consider the weak formulation of problem (2.2.1) viz.,

find $u(x)$ such that for all $\phi$

\[
\int_a^b u'\phi'\,dx + A_2(a)\phi(a) - B_2(b)\phi(b) - \int_a^b f\phi\,dx = 0, \tag{2.2.2}
\]

\[u(a) = A_1(a),\]

\[u(b) = B_1(b).\]

Let $a = a_0 - \delta$ and $b = b_0 + \epsilon$ and linearize with respect to $a_0$ and $b_0$. This gives:

\[
\int_a^b u'\phi'\,dx = \int_{a_0}^{b_0} u'\phi'\,dx + \delta u'(a_0)\phi'(a_0) + \epsilon u'(b_0)\phi'(b_0) + O(\delta^2, \epsilon^2),
\]

\[
\int_a^b f\phi\,dx = \int_{a_0}^{b_0} f\phi\,dx + \delta f(a_0)\phi(a_0) + \epsilon f(b_0)\phi(b_0) + O(\delta^2, \epsilon^2),
\]

\[\phi(a) = \phi(a_0) - \delta \phi'(a_0) + O(\delta^2),\]

\[\phi(b) = \phi(b_0) + \epsilon \phi'(b_0) + O(\epsilon^2),\]

\[\vdots\]

etc.

Substitution of the linearized expressions into the weak formulation (2.2.2) yields the following equation to first order in $\epsilon$ and $\delta$,

\[
\int_{a_0}^{b_0} u'\phi'\,dx - \int_{a_0}^{b_0} f\phi\,dx + A_2(a_0)\phi(a_0) - B_2(b_0)\phi(b_0) - \delta \phi(a_0)[f(a_0) + A'_2(a_0)] - \epsilon \phi(b_0)[f(b_0) + B'_2(b_0)] = 0,
\]

and the linearized boundary conditions are

\[u(a_0) = A_1(a_0) + \delta[A_2(a_0) - A'_1(a_0)],\]
\[ u(b_0) = B_1(b_0) - \epsilon[B_2(b_0) - B'_1(b_0)]. \]

The parameters \( \epsilon \) and \( \delta \) can be found by using the linearized Dirichlet boundary conditions at \( a \) and \( b \) respectively, we get

\[ \delta = \frac{u(a_0) - A_1(a_0)}{A_2(a_0) - A'_1(a_0)}, \]

\[ \epsilon = -\frac{u(b_0) - B_1(b_0)}{B_2(b_0) - B'_1(b_0)}. \]

Substitution for \( \epsilon \) and \( \delta \) yields the linearized form of the weak formulation (2.2.2):

find \( u(x) \) such that for all \( \phi \).

\[
\int_{a_0}^{b_0} u' \phi' \, dx - \int_{a_0}^{b_0} f \phi \, dx + \phi(a_0) \left[ A_2(a_0) - \frac{u(a_0) - A_1(a_0)}{A_2(a_0) - A'_1(a_0)} (f(a_0) + A'_2(a_0)) \right] \\
- \phi(b_0) \left[ B_2(b_0) - \frac{u(b_0) - B_1(b_0)}{B_2(b_0) - B'_1(b_0)} (f(b_0) + B'_2(b_0)) \right] = 0. \tag{2.2.3}
\]

Problem (2.2.3) can be considered to be the weak formulation of the following differential equation:

\[ -u''(x) = f(x) \text{ on } (a_0, b_0), \tag{2.2.4} \]

with boundary conditions

\[ u'(a_0) = A_2(a_0) - \frac{u(a_0) - A_1(a_0)}{A_2(a_0) - A'_1(a_0)} (f(a_0) + A'_2(a_0)), \tag{2.2.5} \]

\[ u'(b_0) = B_2(b_0) - \frac{u(b_0) - B_1(b_0)}{B_2(b_0) - B'_1(b_0)} (f(b_0) + B'_2(b_0)). \tag{2.2.6} \]

Note that the interval on which differential equation (2.2.4) has to be solved is fixed. Since the TL-boundary conditions at \( a_0 \) and \( b_0 \) are of the Robbins-type it follows that equation (2.2.4) can be solved uniquely on \((a_0, b_0)\). The iterative procedure is now as follows: make an initial guess of the geometric unknowns \( a_0 \) and \( b_0 \), and solve differential equation (2.2.4) on the interval \((a_0, b_0)\) together with boundary conditions (2.2.5) and (2.2.6). The estimates \( a_0 \) and \( b_0 \) are then updated using \( a_1 = a_0 - \delta \) and \( b_1 = b_0 + \epsilon \) respectively, where \( \delta \) and \( \epsilon \) are as defined earlier. The differential equation is next solved on the interval \((a_1, b_1)\). This procedure is repeated until convergence is attained.
2.3 Properties of the TL method

To investigate the speed of convergence of the TL-iterative procedure we consider the updating process of the left-hand free-boundary point. In general this reads

$$a_{n+1} = a_n - \frac{u(a_n) - A_1(a_n)}{A_2(a_n) - A'_1(a_n)}.$$

Suppose that $\epsilon_n$ is the difference between the exact value $a$ and the position of the free boundary after $n$ iterations, i.e. $a_n = a + \epsilon_n$. Substitution for $a_n$ into the above equation and expanding $u$, $A_1$, $A'_1$ and $A_2$ about $a$ yields the following expression for the error after $n + 1$ iterations

$$\epsilon_{n+1} = \epsilon_n^2 \frac{f(a) + 2A'_2(a) - A''_1(a)}{2(A_2(a) - A'_1(a))}. \quad (2.3.1)$$

Since $\epsilon_{n+1}$ is of $O(\epsilon_n^2)$ we have a quadratically converging process. One important point to note however, is the fact that the radius of convergence may be quite small. From (2.3.1) one readily obtains

$$\epsilon_n = (\epsilon_0^2 G(a))^{2^{n-1}},$$

where $G(a)$ denotes the fraction in (2.3.1). The convergence criterion is thus

$$\epsilon_0 < G(a)^{-1/2}. \quad (2.3.2)$$

This condition can be very restrictive, for example when $A_2(a)$ is close to $A'_1(a)$. This point will be exemplified in subsequent sections. In case it is not possible to find an estimate of $a$ such that (2.3.2) is satisfied, one could use a continuation procedure by solving the differential equation with boundary conditions which successively approach the desired boundary condition. Alternatively one could use a Trial-method for the first few iterations and then switch to the TL procedure - the Trial method has a larger radius of convergence.

Next we will briefly consider an alternative approach to obtain the same results as those found by using the TL-method. Let us, for the moment, concentrate on the boundary conditions at $x = a$, being

$$u(a) = A_1(a), \quad (2.3.3)$$

and

$$u'(a) = A_2(a). \quad (2.3.4)$$
When the trial method is applied, the differential equation is solved using only one of the boundary conditions at \( x = a \), (2.3.3) say. An update of the estimate of \( a \) is obtained by satisfying the relaxed boundary condition (2.3.4) as closely as possible. The result of this update is however, that (2.3.3) is in general no longer satisfied. To overcome this problem we apply a procedure suggested by Garabedian (1956). To that end we define a new set of boundary conditions as follows:

\[
R_1 u \equiv u'(a) - A_2(a) + \tau(u(a) - A_1(a)) = 0, \tag{2.3.5}
\]

and

\[
R_2 u \equiv u(a) - A_1(a) = 0. \tag{2.3.6}
\]

Now solve the differential equation with boundary condition (2.3.6), and update the estimate of \( a \) using (2.3.5). By choosing the parameter \( \tau \) in (2.3.5) appropriately, we can ensure that both boundary conditions remain satisfied to second order accuracy after the free boundary update. All we need to do is to choose \( \tau \) such that \( R_1 u \) is stationary at \( x = a \), i.e.

\[
\left[ \frac{\partial R_1 u}{\partial x} \right]_{x=a} = 0.
\]

Using the fact that \(-u''(a) = f(a)\), this leads to

\[
\tau = \frac{f(a) + A_2'(a)}{u(a) - A_1'(a)},
\]

which, on substitution into (2.3.5), yields a boundary condition identical to the TL-boundary condition (2.2.6). We conclude that the TL-method is basically a trial free-boundary-type iterative procedure with a second order rate of convergence. Being a trial-method type procedure is in general an advantage due to the ease of implementation when compared with the equally fast converging Newton method.

In view of the derivation of the TL-boundary condition given in this section, the derivation in the previous section may appear to be unnecessarily complicated. For one-dimensional problems this is indeed true. However, the procedure of Garabedian can not be extended easily to higher dimensions wereas the procedure of section 2.2 is in principle applicable to 2 or 3-D problems. It is, in addition, important to realize that domains on which capillary problems are to be solved are often quite complicated (c.f. Orr et al. 1975, 1977). A finite-element approach is in that case generally
preferable to a finite-difference approach due to the relative ease with which complicated Neumann boundary conditions are handled in the finite-element case (Neumann boundary conditions may lead to complicated interpolations on the free boundary when a finite-difference method is applied). It will be clear that the approach of section 2.2 is natural when a finite-element technique is used.

2.4 A capillary free-boundary problem

In this section we introduce the equations which determine the shape of a static, capillary liquid-gas interface of an incompressible fluid, partly filling a container. The expressions are derived using the method proposed by Gauß (1830). Gauß based his reasoning on the principle of virtual work, according to which the energy of a system in equilibrium is unvaried under arbitrary virtual displacements consistent with constraints. In order to find the total energy of the system we consider four different terms separately:

i) Free-surface energy. The energy associated with the free surface is proportional to the surface area. We get

$$E_s = \alpha |S|,$$

where $|S|$ denotes the area of the free surface $S$ and $\alpha$ the coefficient of surface tension;

Figure 2.1: Schematic diagram of container with fluid
ii) Adhesion energy. The total adhesion energy is, like the free surface energy, proportional to the surface area. We consider the adhesion energy between the fluid and the solid, and the adhesion energy between the gas and the solid. We find

$$\mathcal{E}_a = \alpha_1 |\Gamma| + \alpha_2 |\Gamma_0|,$$

where $|\Gamma|, |\Gamma_0|$ denote the areas of solid in contact with fluid and gas respectively (cf. figure 2.1), and $\alpha_1, \alpha_2$ are the corresponding coefficients of surface tension;

iii) Potential energy. Assuming conservative forces we obtain the total potential energy of the liquid by integrating the potential energy per unit mass, $\Pi$, over the volume occupied by the liquid, viz.

$$\mathcal{E}_p = \int_{\Omega} \rho_f \Pi d\Omega,$$

where $\rho_f$ is the density of the fluid;

iv) Volume constraint. We consider problems in which the total volume of the fluid remains constant (by virtue of the incompressibility of the fluid). The volume multiplied by a Lagrange parameter $c$ is introduced as a new energy term. We write

$$\mathcal{E}_v = cV = c \int_{\Omega} d\Omega.$$

Thus, the total energy, $\mathcal{E}$, of the system is the sum of the free-surface, adhesion and potential energy together with the energy term resulting from the volume constraint. In particular

$$\mathcal{E} = \mathcal{E}_s + \mathcal{E}_a + \mathcal{E}_p + \mathcal{E}_v$$

$$= \alpha |S| + \alpha_1 |\Gamma| + \alpha_2 |\Gamma_0| + \int_{\Omega} \rho_f \Pi d\Omega + c \int_{\Omega} d\Omega.$$

Now, according to the principle of virtual work, the system will be in equilibrium only if $\delta \mathcal{E} = 0$, i.e. the variation of $\mathcal{E}$ with respect to the equilibrium state must be zero. It can be shown (cf. Finn, 1986) that $\delta \mathcal{E}$ is given by

$$\delta \mathcal{E} = \int_S \left[ -\alpha \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + \rho_f \Pi + c \right] n_S \cdot \delta x ds +$$

$$\int_{\gamma} \left[ \alpha n_R \cdot n_S + (\alpha_1 - \alpha_2) \right] \tau \cdot \delta x d\gamma.$$
In here \( R_1, R_2 \) denote the principal radii of curvature of \( S \), \( \delta x \) is some arbitrary variation of the position of \( S \), the vectors \( n_\Gamma, n_S \) and \( \tau \) are as indicated in figure 2.1 and \( \gamma \) is the curve at which the free surface intersects the container wall. Since \( \delta x \) is arbitrary it follows that \( \delta E = 0 \) only when

\[
- \alpha \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + \rho_f \Pi + c = 0 \quad \text{on} \quad S, \tag{2.4.1}
\]

and

\[
- n_\Gamma \cdot n_S = \cos \delta \quad \text{on} \quad \gamma, \tag{2.4.2}
\]

where we have assumed that \( (\alpha_1 - \alpha_2)/\alpha = \cos \delta \) is some given constant. The angle \( \delta \) is known as the contact angle (cf. figure 2.1) and is dependent on the material properties of the fluid and the container wall. Equations (2.4.1) and (2.4.2) are supplemented with the volume constraint.

Let us next consider the specific form of the equations in spherical polar coordinates. The container and the free-surface are assumed to be rotationally symmetric. The container may be rotating so that the potential \( \Pi \) is of the form

\[
\Pi(\theta, r) = -\frac{1}{2} \omega^2 r(\theta)^2 \sin^2 \theta + \rho_f g r(\theta) \cos \theta
\]

in which \( r(\theta) \) is the function describing the capillary free surface, \( \theta \) the azimuthal angle and \( g \) is the gravitational acceleration, see figure 2.2. The

![Figure 2.2: Schematic diagram of rotationally symmetric container.](image)
total curvature of $S$ in spherical polar coordinates becomes

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{2r^2 + 3r'^2 - rr''}{(r^2 + r'^2)^{3/2}} - \frac{r'}{r(r^2 + r'^2)^{1/2}} \cot \theta$$

where the primes denote differentiation with respect to $\theta$. Substituting for $\Pi$ and the total curvature into (2.4.1) and writing the equations in dimensionless form we obtain

$$- \frac{2r^2 + 3r'^2 - rr''}{(r^2 + r'^2)^{3/2}} + \frac{r'}{r(r^2 + r'^2)^{1/2}} \cot \theta + B o r \cos \theta - \frac{1}{2} W e r^2 \sin^2 \theta + c = 0, \quad (2.4.3)$$

The Bond number $B o = \rho_f g L^2 / \alpha$ and Weber number $W e = \rho_f \omega^2 L^3 / \alpha$ are dimensionless quantities in which $L$ denotes a characteristic length.

Henceforth we assume that the fluid surface intersects the container wall only at $\theta = \theta_1$ (the situation as depicted in figure 2.2). The case in which the free surface intersects $\Gamma$ at $\theta = \theta_2$ is treated analogously. The boundary conditions are thus

$$r(\theta_1) = r_{\Gamma}(\theta_1), \quad (2.4.4)$$

$$\frac{rr(\theta_1)r_{\Gamma}(\theta_1) + r'(\theta_1)r'_{\Gamma}(\theta_1)}{(rr(\theta_1)^2 + r'^2(\theta_1)^2)^{1/2}(r(\theta_1)^2 + r'(\theta_1)^2)^{1/2}} = \cos \delta, \quad (2.4.5)$$

and

$$r'(\theta_2) = 0, \quad \theta_2 = \pi. \quad (2.4.6)$$

The volume constraint is given by

$$V_f = \frac{2\pi}{3} \int_{\theta_1}^{\theta_2} (r_{\Gamma}^3(\theta) - r^3(\theta)) \sin \theta d\theta. \quad (2.4.7)$$

Equations (2.4.3)-(2.4.7) completely determine the capillary free-boundary. Note that in general we have two boundary conditions at $\theta_1$; one is required for solving the differential equation, the other to determine $\theta_1$ which is not known a priori.
2.5 Numerical solution

In order to be able to solve the non-linear free boundary problem numerically, a linearization of the problem is necessary. To this end we apply Newton's Linearization method to equation (2.4.3). This yields a sequence of functions \( r_0(\theta), \ r_1(\theta), \ldots, \ r_n(\theta), \ r_{n+1}(\theta) \rightarrow r(\theta) \) defined by the linear problem

\[
Ar''_{n+1} + Br'_{n+1} + Cr_{n+1} + c = D, \quad (2.5.1)
\]

where \( A, B, C \) and \( D \) are nonlinear functions of \( r_n(\theta) \) (for details we refer to Cuvelier (1987a)). The boundary conditions are as follows

\[
r_{n+1}(\theta_1) = r_\Gamma(\theta_1), \quad r'_{n+1}(\theta_1) = \Psi(\theta_1), \quad r'_{n+1}(\theta_2) = 0, \quad \theta_2 = \pi, \quad (2.5.2)
\]

with

\[
\Psi(\theta_1) = \frac{1}{2} r_\Gamma \left[ \frac{(r_\Gamma^2 + r_n^2) \sin 2\delta - 2r_\Gamma r_n^2}{r_\Gamma^2 \sin^2 \delta - r_n^2 \cos^2 \delta} \right].
\]

The linearized volume condition reads

\[
\int_{\theta_1}^{\theta_2} r_{n+1}^2 \sin \theta d\theta = \frac{1}{3} \int_{\theta_1}^{\theta_2} (2r_n^3 + r_\Gamma^3) \sin \theta d\theta - \frac{V_f}{2\pi}. \quad (2.5.3)
\]

The second-order ordinary differential equation given by (2.5.1) can be solved subject to boundary conditions (2.5.2) and volume constraint (2.5.3).

We now apply the TL procedure discussed in section 2.2, to the free-boundary problem defined by equations (2.5.1)-(2.5.3). Let \( \theta_{1,0} \) be some estimate of \( \theta_1 \), then the TL boundary condition at \( \theta_{1,0} \) is given by

\[
r'_{n+1}(\theta_{1,0}) = \Psi(\theta_{1,0}) - \left[ \frac{\Psi'(\theta_{1,0}) + \frac{1}{A}(B\Psi(\theta_{1,0}) + C r_{n+1}(\theta_{1,0}) + c - D)}{\Psi(\theta_{1,0}) - r'_\Gamma(\theta_{1,0})} \right](r_{n+1}(\theta_{1,0}) - r_\Gamma(\theta_{1,0})),
\]

where \( A, B, C \) and \( D \) are evaluated at \( \theta = \theta_{1,0} \). Observe that the above equation is nonlinear in the unknowns \( r_{n+1} \) and \( c \). A linearization leads to a boundary condition of the form

\[
r'_{n+1}(\theta_{1,0}) = \alpha(\theta_{1,0}) - \beta(\theta_{1,0})r_{n+1}(\theta_{1,0}) - \gamma(\theta_{1,0})c, \quad (2.5.4)
\]
where \( \alpha, \beta \) and \( \gamma \) are functions of \( \theta \). Linearizing the volume condition (2.5.3) about \( \theta_{1,0} \) yields the TL volume condition

\[
\int_{\theta_{1,0}}^{\theta_2} r_{n+1} r_n^2 \sin \theta d\theta + \rho(\theta_{1,0}) r_{n+1}(\theta_{1,0}) = \\
\frac{1}{3} \int_{\theta_{1,0}}^{\theta_2} (2r_n^3 + r_\Gamma^3) \sin \theta d\theta - \tau(\theta_{1,0}) - \frac{V_f}{2\pi},
\]

(2.5.5)

in which \( \rho \) and \( \tau \) are functions of \( \theta \) evaluated at \( \theta_{1,0} \). The iterative procedure is now as follows: obtain estimates \( \theta_{1,0} \) of \( \theta_1 \) and \( r_0(\theta) \) of \( r(\theta) \) and solve differential equation (2.5.1) subject to volume condition (2.5.5), the TL boundary condition (2.5.4) at \( \theta = \theta_{1,0} \) and the homogeneous Neumann boundary condition at \( \theta = \pi \). After the Newton iterative procedure for solving the differential equation (2.5.1) on the fixed domain \( (\theta_{1,0}, \pi) \) has converged, we update the estimate of \( \theta_1 \) using the Dirichlet condition via

\[
\theta_{1,n+1} = \theta_{1,n} + \delta = \theta_{1,n} - \frac{r_{n+1}(\theta_{1,n}) - \tau_\Gamma(\theta_{1,n})}{\Psi(\theta_{1,n}) - \tau_\Gamma(\theta_{1,n})}.
\]

Repeat this procedure until convergence is attained. When the fluid surface also intersects the container wall at \( \theta = \theta_2 \) an iterative procedure, analogous to the one described above for \( \theta_1 \), is required for \( \theta_2 \). In section 2.2 we showed that in certain cases the TL procedure will not converge, unless the initial estimate of \( \theta_1 \) is very close to the true value. For the problem we are considering at the moment this situation arises when the contact angle \( \delta \ll 1 \), since in that case \( \Psi(\theta_1) \approx r_\Gamma(\theta_1) \). One can readily show that the convergence criterion (2.3.2) becomes \( \varepsilon_0 < G_1 \sqrt{\delta}, \) where \( G_1 \) is some function of \( \theta_1 \). For very small values of \( \delta \) the convergence criterion can be very restrictive, in which case the TL method is unlikely to be a satisfactory procedure for solving the present free-boundary problem. For the Newton iterative procedure to converge we also require a good initial estimate of \( r \). This is in general a non-trivial problem, in particular when container shapes other than cylinders or spheres are used or when arbitrary fill fractions are to be considered. To obtain a good estimate of \( r \) assume that the fluid surface in the container is horizontal (this is true when surface tension effects are neglected and \( We = 0 \)). Finding the position of the free fluid-surface reduces in that case to finding \( \theta_1 \) (see figure 2.2), such that the
following equation is satisfied

\[ V_f = \frac{2\pi}{3} \int_{\theta_1}^{\pi} \left[ r_1^2(\theta) - \left( \cos \theta_1 r_1(\theta_1) / \cos \theta \right)^3 \right] \sin \theta d\theta, \]

where \( V_f \) is the quantity of liquid in the container. This problem can be solved readily using the bisection method. In all of the following numerical experiments the initial estimates of \( \theta_1 \) and \( r(\theta) \) were found this way.

2.6 Numerical Experiments

We start this section with a comparison of some numerical and approximate analytical results. To that end we consider a right-circular cylinder with radius \( a \) and height \( H \), partly filled with fluid. Assume that the fluid surface does not intersect the top or the bottom surface of the cylinder. The constant \( c \) can in that case be shown to be equal to

\[ c = Bo(h - \frac{V_f}{a^2 \pi}) + \frac{1}{4} We a^2 + \frac{2}{a} \cos \delta, \]

where \( h \) denotes the position of the origin relative to the bottom of the container. Substituting for \( c \) it follows that equation (2.4.3) can be written like

\[ r \cos \theta - \frac{1}{2} Fr r^2 \sin^2 \theta + c_0 + Bo^{-1} \left[ \frac{2}{a} \cos \delta - \frac{1}{R_1} - \frac{1}{R_2} \right] = 0, \tag{2.6.1} \]

where the dimensionless number \( Fr = Bo^{-1} We \), called the Froude number, is a measure of the ratio of gravitational and centrifugal forces, and the constant \( c_0 \) is given by

\[ c_0 = h - \frac{V_f}{a^2 \pi} + \frac{1}{4} Fr a^2. \]

Note that the expression multiplied by \( Bo^{-1} \) in (2.6.1) contains all the terms which are related to surface tension effects. In the limit \( Bo \to \infty \) the curve defined by (2.6.1) is a parabola which is indeed the shape of a free-surface of a heavy (i.e. without surface tension) rotating liquid. Comparing (2.4.3) and (2.6.1) we see that the term multiplying \( Bo^{-1} \) contains all the first and second order derivatives. This implies that for \( Bo \) large the solution of (2.6.1) has a boundary-layer character, that is, capillary effects are reduced.
to a small region close to the solid boundary. In figure 2.3 free-surface shapes are shown for the case $Bo = \infty$, $Bo = 100$ and $Bo = 10$ and $Fr = 0$ (figure 2.3a) and $Fr = 2$ (figure 2.3b). We have taken $a = 1$, $V_f = \pi$ and $\delta = 0.3\pi$. For the numerical calculation the interval $(\theta_1, \pi)$ is divided up into 80 elements of equal size. The stopping criterion of the Newton iterative procedure for solving the differential equation is $|r_{n+1} - r_n| < 10^{-4}$ and for the TL procedure $|\theta_{1,n+1} - \theta_{1,n}| < 10^{-4}$. Observe the boundary-layer character of the solution for $Bo = 100$ and the marked deviation of the free-surface shape from the heavy-liquid surface when $Bo = 10$.

In the second experiment we investigate the speed of convergence of the TL-method. To that end we consider a sphere with its centre at $(0,1)$ and radius 1, half filled with fluid. We fix the origin at $(0, 1.95)$ and take $\delta = \pi/4$, $We = 0$ and the stopping criterion for the Newton procedure is $|r_{n+1} - r_n| < 10^{-10}$. We start with $Bo = 1000$ and decrease $Bo$ stepwise until $Bo = 1$. In all of the following experiments a continuation procedure is applied: starting from a horizontal free surface we calculate the free-surface shape belonging to a certain parameter set, after which one or more
<table>
<thead>
<tr>
<th>$Bo$</th>
<th>TL-iterations</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1000</td>
<td>$1.2 \times 10^{-2}$</td>
</tr>
<tr>
<td>500</td>
<td>$5.0 \times 10^{-3}$</td>
</tr>
<tr>
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<td>$2.0 \times 10^{-2}$</td>
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<tr>
<td>50</td>
<td>$1.4 \times 10^{-2}$</td>
</tr>
<tr>
<td>10</td>
<td>$4.8 \times 10^{-2}$</td>
</tr>
<tr>
<td>1</td>
<td>$5.8 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Table 2.1: The magnitude of the TL-updates.

parameters are changed and the new free surface is calculated using the surface shape associated with the old parameter set as an initial guess. In table 2.1 the values of $Bo$ are shown together with magnitude of the TL update for the first four iterations. Observe that for the Bond numbers 1000 and 500 the convergence rate is linear, for $Bo=100$, 50 the rate is super-linear but not quite quadratic and for $Bo=10$, 1 the convergence rate is quadratic. It is evident that the convergence rate is effected by changes in the Bond number. We suspect this to be due to the parameter $c$ which is sensitive to changes in $Bo$ and appears in the TL boundary condition (2.5.4) (see also the following experiment).

In the next experiment we consider a cylinder of radius 1, height 2, half filled with fluid. We take $Bo = 10$ and the contact angle $\delta = \pi/4$ and slowly increase the Weber number. The origin of the coordinate system is chosen at the point (0,1.95) and remains fixed throughout the iterative process. The stopping criteria for the Newton and TL procedures are $|r_{n+1} - r_n| < 10^{-4}$ and $|\theta_{i,n+1} - \theta_{i,n}| < 10^{-4}$, $i = 1, 2$ respectively. In the first and second column of table 2.2 we present the values of $We$ and the number of TL-iterations respectively. In the third column we show the number of iterations required when the same free-boundary problem is solved using the trial method (refer to Cuvelier (1987a)). Observe that the TL method requires considerably less iterations than the trial method. This is a significant improvement, also because the TL method does not require some parameter to be estimated as is the case with the trial method. The large number of iterations required when $We = 27.7$ is due to the fact that the fluid surface reaches the bottom of the container which results in a new set of boundary conditions at $\theta = \theta_2$. The free surface is shown in figure 2.4 for a number of representative values
<table>
<thead>
<tr>
<th>We</th>
<th>TL-iterations</th>
<th>trial method</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>15</td>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>18</td>
</tr>
<tr>
<td>25</td>
<td>5</td>
<td>9</td>
</tr>
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<td>27.7</td>
<td>8</td>
<td>20</td>
</tr>
<tr>
<td>30</td>
<td>6</td>
<td>26</td>
</tr>
<tr>
<td>31</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>3</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 2.2: Comparing speed of convergence of TL method with trial method.

of We.

The TL boundary condition (2.5.4) was found to be very sensitive to large variations in $Bo$ or $We$. This is due to the fact that the right-hand side in (2.5.4) depends on the unknown parameter $c$ which is of order $\max(Bo, We)$. The sensitivity could be reduced significantly by using the Neumann boundary condition (equation (2.5.2)) instead of the TL-condition (2.5.4) at $\theta_1$ in the first Newton iteration. The Newton iteration provides a good initial guess to $c$. In all of the numerical experiments presented here this procedure was applied.

We next investigate the case in which the contact angle $\delta \ll 1$. Recall that in this situation a very good estimate of $\theta_1$ is required in order to satisfy the convergence criterion (2.3.2). The container we will consider is a sphere with radius 1, the fill fraction is 0.7, $Bo = 30$, $We = 0$ and the origin is taken at $(0,1.75)$. The results of the iterative procedure are shown in table 2.3. Observe that only very small variations in $\delta$ are allowed - for larger variations the TL procedure would not converge, even though the actual solution does not change significantly. As the contact angle decreases, the radius of convergence decreases which is reflected in the progressively smaller permissible variations of the parameters ($\delta$ in this case) in the continuation procedure. The free surface of the liquid is shown in figure 2.5 for a number of values of $\delta$.

We finally present some capillary free-boundary shapes of liquids in propellant tanks used in spacecraft. Figure 2.6 shows the free boundary of a liquid partially filling a TV-SAT tank for various Bond numbers, and in figure 2.7 the capillary free-boundary in a cylinder with hemispherical ends
Figure 2.4: The static free liquid surface in a rotating cylinder. $Bo = 10$, $\delta = \pi/4$, fill fraction = 0.5.

Figure 2.5: The capillary free-surface in a sphere for various contact angles: $Bo = 30$, $We = 0$, fill fraction = 0.7.

Figure 2.6: The static liquid surface in a TV-SAT tank: $We = 0$, $\delta = \pi/2$, fill fraction = 0.6.

Figure 2.7: The static liquid surface in a rotating tank with hemispherical ends: $Bo = 0$, $\delta = \pi/10$, fill fraction = 0.7.
<table>
<thead>
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<th>$\delta/\pi$</th>
<th>TL-iterations</th>
</tr>
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<td>0.6</td>
<td>3</td>
</tr>
<tr>
<td>0.25</td>
<td>4</td>
</tr>
<tr>
<td>0.15</td>
<td>4</td>
</tr>
<tr>
<td>0.1</td>
<td>4</td>
</tr>
<tr>
<td>0.075</td>
<td>3</td>
</tr>
<tr>
<td>0.05</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 2.3: The number of TL iterations as function of $\delta$.

is shown for various Weber numbers.

### 2.7 Conclusions

In the preceding sections we have presented a method for solving a capillary free-boundary problem. We have seen that this rather complex free-boundary problem can be solved efficiently using the TL method. In certain situations the convergence rate of the iterative procedure as defined by the TL method is not quadratic. However, when compared with the speed of convergence of the Trial method (with a linear rate of convergence), we find that the rate of convergence for the TL method is superior. The radius of convergence of the TL method is determined by the boundary conditions of the free-boundary problem - it can be prohibitively small for certain boundary conditions in which case a Trial method may be preferable. In general we can conclude that the TL method provides an efficient procedure for solving the capillary free boundary problem.
Chapter 3

Inviscid capillary oscillations with edge constraints

3.1 Introduction

Finding the eigenfrequencies of oscillations of an inviscid, incompressible fluid in an open container, is a classical problem in fluid mechanics. Mathematically, the problem is to find the eigenvalues $\omega$ which satisfy

$$\nabla^2 \phi = 0 \text{ in } \Omega,$$

$$\frac{\partial \phi}{\partial n} = \omega^2 \phi \text{ on } S, \quad (3.1.1)$$

$$\frac{\partial \phi}{\partial n} = 0 \text{ on } \Gamma,$$

where $\phi$ is a velocity potential in a tank $\Omega$ with free surface $S$ and rigid walls $\Gamma$. The linear eigenvalue problem has been the subject of intensive study by numerous well-known mathematicians since the beginning of the nineteenth century. Among those studying the sloshing problem were Poisson (1816,1828), Green (1838) and Rayleigh (1876,1899). The sixth edition of Lamb’s standard work Hydrodynamics (1932), contains all the references on the sloshing problem up to the time of its publication.

Interest in the problem of oscillating liquids intensified after the second world war. The impetus was largely due to the tremendous importance of the liquid slosh problem in the aerospace industry. Namely, the desire to manufacture air- and spacecraft with an increasing range or lifetime, lead to an increase of the mass of the fuel as a fraction of the total mass of the craft.
The oscillations of fuel became an ever important factor when the stability of the air- or spacecraft was to be considered. For an overview of work in this period we refer the survey paper of Abramson (1963), the NASA report edited by him (1966) and the survey of Moisseev & Petrov (1968).

Despite the well-known names associated with the sloshing problem, exact solutions to (3.1.1) or good approximations or bounds on the eigenvalues, are scarce. For a limited number of types of regions, the technique of separation of variables may be applied to yield an exact solution to (3.1.1), refer to for example Moisseev & Rumyantsev (1968). Fox & Kuttler (1983) use the technique of conformal mappings to relate an arbitrary region to one of the standard regions on which an exact solution is known. In this way, upper and lower bounds can be derived for the eigenvalues in a number of different container shapes. McIver (1989) has solved the eigenvalue problem using a combined numerical-analytical approach by writing it as an integral equation. In particular does he consider the case of a horizontal circular cylinder, filled to an arbitrary depth.

When capillary effects on the free surface $S$ are taken into account, the boundary condition on $S$ becomes more complicated. Not only does a term of third-order mixed, normal and tangential partial derivatives appear in the boundary condition on $S$, but also edge constraints may have to be imposed. The edge constraint, like for example the condition that the free surface is stuck to the container wall on $S \cap \Gamma$, complicates the eigenvalue problem significantly. Motivated by a consistent under-estimate by the conventional theory of phase speeds of travelling capillary waves in a long narrow canal, lead Brook Benjamin & Scott (1979) to investigate capillary waves with edge constraints. The frequencies of these waves are found to be given by the zeros of a function defined by an infinite series, all terms of which depend on the frequency. In the limit of infinite wavelength the frequencies become frequencies of standing waves. The eigenfrequencies are calculated using a Rayleigh-Ritz method. In a similar way Hocking (1987) has obtained numerical approximations to eigenfrequencies of capillary waves with more general edge constraints. However, to the author’s knowledge, no exact solution to the capillary wave problem with edge constraints is known.

The problem of capillary oscillations with edge constraints also arises when eigenfrequencies of liquid bridges (figure 3.1) are considered. Bauer (1982) obtained an exact analytical expression for the eigenfrequencies, however, the boundary conditions were over simplified and non-physical.
Approximate results using a combined analytical / numerical approach have been obtained by Meseguer (1983) for the case of a slender liquid bridge and by Sanz (1985) for the more general liquid bridge. Schilling & Siekmann (1989) use the numerical technique of boundary-elements to calculate the eigenfrequencies of inviscid liquid bridge oscillations. Hitherto, no analytical expression for the eigenfrequencies is known.

The equations describing linear oscillations of an inviscid, incompressible fluid with a capillary free boundary, are given in section 3.2. It is shown how the complicated boundary conditions on the free surface may be simplified using Green’s functions. In this way analytical solutions to the eigenvalue problem with edge constraints may be obtained. In section 3.3 we investigate a number of limiting cases of the problem under consideration. Section 3.4 deals with the free-surface deflection of the normal-mode oscillations, and in section 3.5 the problem of eigenmodes of axisymmetric liquid bridge oscillations is considered.

### 3.2 Problem formulation

Consider a rotationally symmetric container of height $h$ and radius $a$. The container is filled to the brim with an inviscid, incompressible fluid bounded by a horizontal capillary surface at $z = h$. The container wall in contact with the fluid is denoted by $\Gamma$ as shown in figure 3.2. In this chapter we will state the equations governing the motion of the fluid and the boundary conditions. The reader is referred to the next chapter for the derivation of the equations in the more general case of a viscous fluid. We will suffice with the remark that the equations given below may be obtained from equations (4.2.12)-(4.2.16) by neglecting all terms related to viscous effects and assuming a horizontal free surface.

Assume that the fluid oscillations are axisymmetric, i.e. no azimuthal dependence, so that the problem under consideration is 2-dimensional ($r,z$)-plane. The equation of motion of the fluid in $\Omega$ is governed by the Euler
equation
\[ \frac{\partial u}{\partial t} + \nabla p = 0, \]
and the incompressibility condition
\[ \nabla \cdot u = 0, \]
where \( u \) denotes the fluid velocity and \( p \) the pressure. On the container wall \( \Gamma \) and on the axis of symmetry \( r = 0 \), we have the condition
\[ u \cdot n = 0, \]
where \( n \) denotes the outward unit normal to the boundary under consideration. On the free surface the dynamic condition, viz.
\[ -p = -Bo \eta + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \eta}{\partial r} \right) \]
and the kinematic condition, namely
\[ u \cdot n = \frac{\partial \eta}{\partial t} \]
are given. In the above equations \( \eta(r, t) \) denotes the vertical displacement of the free surface from the steady state configuration \( z = h \) (cf. figure 3.2) and \( Bo \) is the dimensionless Bond number as introduced in section 2.4. On the free surface we finally have the symmetry condition at \( r = 0 \)
\[ \frac{\partial \eta}{\partial r} = 0, \]
and the so-called stuck-edge condition at the point of intersection of the free surface and the container wall

\[ \eta(a, t) = 0. \]

Let us now assume that the fluid flow is irrotational initially. It is well-known that for inviscid fluids the flow remains irrotational so that we may write \( \mathbf{u} = \nabla \phi \), where \( \phi \) is a velocity potential. Assume in addition that \( \mathbf{u}, \ p, \ \eta \sim e^{i\omega t} \). Substituting for \( \mathbf{u} \) and eliminating \( \eta \) from the boundary conditions (by virtue of the kinematic condition on \( S \)), yields the following set of equations

\[
\begin{align*}
\nabla^2 \phi &= 0 \text{ in } \Omega, \\
\frac{\partial \phi}{\partial n} &= 0 \text{ on } \Gamma, \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) - B_0 \frac{\partial \phi}{\partial z} &= -\omega^2 \phi \text{ on } S, \\
\frac{\partial \phi}{\partial r} \left( \frac{\partial \phi}{\partial z} \right) &= 0 \text{ at } r = 0, \\
\frac{\partial \phi}{\partial n} &= 0 \text{ at } r = 0, \\
\frac{\partial \phi}{\partial z} &= 0 \text{ at } \Gamma \cap S.
\end{align*}
\]

A solution to Laplace's equation (3.2.1) subject to the boundary conditions (3.2.2)-(3.2.4), that is disregarding the edge constraint, can be found readily using separation of variables. We obtain

\[
\phi_n(r, z) = A_n J_0(\alpha_{1,n} r) \cosh(\alpha_{1,n} z),
\]

\[
\hat{\omega}_n^2 = (B_0 \alpha_{1,n} + \alpha_{1,n}^3) \tanh(\alpha_{1,n} h).
\]

Here \( n \) refers to the \( n \)th eigenmode, \( A_n \) is some constant, \( J_0 \) is the zeroth order Bessel function and \( \alpha_{1,n} = j_{1,n}/a \) in which \( j_{1,n} \) are the zeros of the Bessel function \( J_1 \).

Taking into account the edge constraint (3.2.5) complicates the problem considerably due to the fact that the free-surface elevation cannot be found in terms of a single separated-variable mode for each frequency. With the edge constraint imposed the complete set of such modes must be used and the eigenfrequencies are given by the zeros of a function defined by an infinite series, all terms of which depend on the frequency. The zeros of
this function have to be evaluated numerically, cf. Brook-Benjamin & Scott (1979) and Hocking (1987). This problem can, however, be overcome when the complicated boundary conditions on \( S \) are written in more convenient form. Write (3.2.3) and (3.2.5) like

\[
L(\psi) = -r \omega^2 \phi(r, h), \quad \psi'(0) = 0, \quad \psi(a) = 0, \tag{3.2.8}
\]

where \( L \) is a linear, self-adjoint operator defined by

\[
L(\psi) = (r \psi')' - r Bo \psi
\]

and \( \psi = \partial \phi / \partial z \) (the primes denote differentiation with respect to \( r \)). The solution to the differential equation (3.2.8), subject to the boundary conditions at \( r = 0 \) and \( r = a \), can be written formally as

\[
\psi = \omega^2 \int_{S_0} K(r, \xi) \phi(\xi, h) \xi d\xi,
\]

where the Green's function \( K(r, \xi) \) is symmetric since the operator \( L \) is self-adjoint (cf. Courant & Hilbert (1953), vol.1). In appendix A it is shown that

\[
K(r, \xi) = \begin{cases} 
I_0(r \sqrt{Bo}) \left( K_0(\xi \sqrt{Bo}) - \frac{K_0(a \sqrt{Bo})}{I_0(a \sqrt{Bo})} I_0(\xi \sqrt{Bo}) \right) & r \leq \xi \\
I_0(\xi \sqrt{Bo}) \left( K_0(r \sqrt{Bo}) - \frac{K_0(a \sqrt{Bo})}{I_0(a \sqrt{Bo})} I_0(r \sqrt{Bo}) \right) & r > \xi 
\end{cases}
\]

where \( I_0 \) and \( K_0 \) are zeroth order modified Bessel functions of the first and third kind respectively. The problem under consideration is thus reduced to finding \( \phi \) and \( \omega \) such that the following equations are satisfied

\[
\nabla^2 \phi = 0 \quad \text{in} \quad \Omega,
\]

\[
\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad \Gamma
\]

\[
\frac{\partial \phi}{\partial z} = \omega^2 \int_{0}^{a} K(r, \xi) \phi(\xi, h) \xi d\xi \quad \text{on} \quad S \tag{3.2.9}
\]

\[
\frac{\partial \phi}{\partial r} = 0 \quad \text{at} \quad r = 0,
\]

30
together with the compatibility condition
\[
\int_0^a \int_0^a K(r, \xi) \phi(\xi, h) \xi d\xi \: rdr = 0.
\]
Since the Bessel functions \( J_0(\alpha_1, n r) \) are complete and orthogonal on the interval \([0, a]\), it follows that the general solution to problem (3.2.9) may be obtained by writing \( \phi \) as a linear combination of the eigenfunctions (3.2.6), viz.
\[
\phi(r, z) = C + \sum_{i=1}^{\infty} A_i \phi_i(r, z).
\]
With this choice for \( \phi \) the boundary conditions on \( \Gamma \) and \( r = 0 \) are satisfied automatically. Only the boundary condition on \( S \) and the compatibility condition remain to be satisfied. The constant \( C \) can be expressed in terms of the eigenfunctions \( \phi_i \) using the compatibility condition. On substituting for \( \phi \) in the condition on \( S \), eliminating \( C \) from the equations and using the orthogonality relations for \( J_0(\alpha_1, n r) \), we find that the following equation holds
\[
\alpha_1 \tanh(\alpha_1, n h) \frac{a^2}{2} \left[ J_0(\alpha_1, n) \right]^2 = \omega_n^2 \left[ \int_0^a \int_0^a K(r, \xi) J_0(\alpha_1, n \xi) J_0(\alpha_1, n r) \xi d\xi \: rdr - \left( \int_0^a \int_0^a K(r, \xi) J_0(\alpha_1, n r) \xi d\xi \: rdr \right)^2 \right] / \left( \int_0^a \int_0^a K(r, \xi) \xi d\xi \: rdr \right).
\]
Evaluating the integrals (which is tedious but relatively straightforward) and solving for \( \omega_n^2 \), we obtain the surprisingly elegant analytical expression for the eigenfrequencies of an inviscid fluid with a capillary free boundary subjected to edge constraints, viz.
\[
\omega_n^2 = (Bo \alpha_1 + \alpha_1^3) \tanh(\alpha_1, nh) \times
\left[ 1 - \frac{2Bo}{Bo + \alpha_1^2} \frac{I_1(a \sqrt{Bo})}{a \sqrt{Bo} I_0(a \sqrt{Bo}) - 2I_1(a \sqrt{Bo})} \right]^{-1}.
\]
Comparison of (3.2.10) with (3.2.7) shows that the term in the square brackets is due to imposing the edge constraint at \( r = a \). Employing the properties of \( I_0(x) \) and \( I_1(x) \) (cf. Abramowitz & Stegun, 1970) one can show that for
\( \alpha_{1,n} > 2/a\sqrt{3} \) the quantity in square brackets in (3.2.10) is strictly positive and bounded above by 1. In fact, the first zero of the Bessel function \( J_1 \) is equal to \( j_{1,1} \approx 3.83 \). It follows that the eigenfrequencies of an inviscid fluid with a capillary boundary on which edge constraints are imposed, are higher than those of the corresponding frequencies in the free-slip case (equation (3.2.7)).

### 3.3 Eigenfrequencies in limiting cases

Let us investigate a number of limiting cases. First consider the case in which \( a\sqrt{Bo} \gg 1 \). Using the asymptotic expressions \( I_0(x), I_1(x) \sim e^x/\sqrt{2\pi x} \) for \( x \gg 1 \), it follows that (3.2.10) reduces to

\[
\omega_n^2 = Bo \alpha_{1,n} \tanh(\alpha_{1,n} h) \left[ 1 + \frac{2}{a\sqrt{Bo}} + O(1/a^2 Bo) \right].
\]

(3.3.1)

Observe that equations (3.2.7) and (3.2.10) have the same limit as \( Bo \to \infty \). That is, the influence of the edge constraint on the eigenfrequencies is small when \( Bo \) is large. Next consider the limit \( a\sqrt{Bo} \to 0 \). Using the polynomial expansions of \( I_0 \) and \( I_1 \) for small arguments, viz.

\[
I_0(x) \approx 1 + \frac{1}{4} x^2 + O(x^4), \quad I_1(x) \approx \frac{1}{2} x + \frac{1}{16} x^3 + O(x^5),
\]

it follows that (3.2.10) becomes

\[
\omega_n^2 = \alpha_{1,n}^3 \tanh(\alpha_{1,n} h) \left[ 1 - \frac{8}{j_{1,n}^2} \right]^{-1} + O(a\sqrt{Bo}).
\]

(3.3.2)

Since \( j_{1,n} \) increases monotonically for increasing \( n \), it follows from (3.3.2) that the lowest eigenmodes are effected most by the edge constraints. In particular for \( n = 1 \) we have \( j_{1,1} = 3.83 \), so that imposing the edge constraints increases the eigenfrequency of the lowest eigenmode by 50%. Figure 3.3 shows a plot of the ratio of the eigenfrequencies given by (3.2.10) and those given by (3.2.7) for \( Bo \) in the range \([0, 1000]\). Observe that for \( Bo \leq 100 \), the edge constraints increase the eigenfrequencies significantly, for larger Bond numbers the effect of the edge constraints diminishes steadily. Note, however, that even for \( Bo = 1000 \) the edge constraint increases \( \omega_1 \) by 5% when compared with the eigenfrequency in the case of a freely moving contact line.
An interesting physical aspect is covered by allowing $Bo$ to be negative, corresponding to an inverted container. From equation (3.2.7) we find that if $Bo < -\alpha^2_{1,1}$ then $\omega_1$ becomes imaginary so that disturbances grow exponentially in time, that is the system becomes unstable. It is interesting to find out what the effect is on the stability of the system, when edge constraints on the free-surface boundary are imposed. Let $Bo = -B$ ($B > 0$) so that $\sqrt{Bo} = i\sqrt{B}$. Using the fact that $I_0(ia\sqrt{B}) = J_0(a\sqrt{B})$ and $I_1(ia\sqrt{B}) = iJ_1(a\sqrt{B})$, it follows that (3.2.10) is given by

$$\omega_n^2 = (-B\alpha_{1,n} + \alpha^3_{1,n}) \tanh(\alpha_{1,n}h) \left[ 1 + \frac{2B}{-B + \alpha^2_{1,n} \sqrt{a\sqrt{B}J_0(a\sqrt{B}) - J_1(a\sqrt{B})}} J_1(a\sqrt{B}) \right]^{-1}.$$ 

Let $B = \alpha^2_{1,n} + \epsilon$, $\epsilon \ll 1$. Substituting for $B$ we find, on using $J_1(\alpha_{1,n}a) = J_1(j_{1,n}) = 0$, that the above expression reduces to

$$\omega_n^2 = -2\epsilon \alpha^2_{1,n} \tanh(\alpha_{1,n}h) \left[ 1 + \frac{J_1'(j_{1,n}a)}{J_0(j_{1,n}a)} \right]^{-1}$$

to first order in $\epsilon$. It follows that $\omega_n$ is imaginary and hence the system unstable if $\epsilon > 0$. Since the stability boundary of the system is determined by the smallest value of $B$ (the smallest absolute value of $Bo$) at which the system becomes unstable it follows, that for stability we require

$$Bo > -\alpha^2_{1,1}.$$
If $Bo > -\alpha_{1,1}^2$, $\omega_n$ is real for all $n$ so that the system is stable. We conclude that the stability condition for free-surface oscillations is unchanged by imposing the edge constraints. Brook Benjamin & Scott (1979) have shown, using functional analytic methods, that the above stability requirement in fact holds for axisymmetric containers of arbitrary configurations.

### 3.4 Free-surface deflection

Let us consider the deflection corresponding to the normal mode oscillations. Assume that $\eta(r)$ is of the form

$$\eta(r) = \sum_{n=1}^{\infty} D_n \eta_n(r).$$

From the kinematic condition on the free surface we find that the free surface deflection of the $n$'th normal mode is given by

$$\eta_n(r) = \frac{1}{i\omega_n} \frac{\partial \phi_n}{\partial z},$$

which, on substituting for $\phi_n$ yields

$$\eta_n(r) = N_n \left[ J_0(\alpha_{1,n} r) - J_0(\alpha_{1,n} a) \frac{2I_1(a\sqrt{Bo}) - a\sqrt{Bo}I_0(\sqrt{Bo})}{2I_1(a\sqrt{Bo}) - a\sqrt{Bo}I_0(a\sqrt{Bo})} \right], \quad (3.4.1)$$

where $N_n$ is some constant. Note that the edge constraint, viz. $\eta_n(a) = 0$, is satisfied. The expression for the deflection when no edge constraint is imposed is proportional to (3.4.1) without the fraction containing the modified Bessel functions. Consider the limit $Bo \to 0$. Using the expressions for $I_0$ and $I_1$ as given earlier we find that (3.4.1) reduces to

$$\eta_n(r) = N_n \left[ J_0(\alpha_{1,n} r) - J_0(\alpha_{1,n} a) \frac{2r^2 - a^2}{a^2} \right].$$

More interesting is the case when $Bo \to \infty$. The expression for the deflection in this limit is given by

$$\eta_n(r) = N_n \left[ J_0(\alpha_{1,n} r) - J_0(\alpha_{1,n} a) \frac{2 - a\sqrt{Bo} e^{-\sqrt{Bo}(a - r)} \sqrt{a/r}}{2 - a\sqrt{Bo}} \right].$$
The above equation is a typical boundary layer equation. Namely, if \((a - r) \gg 1/\sqrt{Bo}\), i.e. if we are not in the vicinity of the container wall, then \(\eta_n(r) \approx N_n J_0(\alpha_{1,n} r)\) corresponding to the deflection in the case of a freely-moving contact line. However, when \(a - r \leq 1/\sqrt{Bo}\) the second term in the square brackets becomes important forcing \(\eta_n(r)\) to satisfy the edge constraint. The thickness of the capillary boundary layer is of order \(\sqrt{Bo}\). In figure 3.4 we have plotted the deflection as given by (3.4.1) for \(n = 1\) and \(Bo = 0, 1000\). For reference purposes the deflection in the case of a freely moving contact line has been plotted also. Observe the boundary layer character of the solution for \(Bo = 1000\).

### 3.5 Liquid bridge oscillations

Consider a liquid bridge which is situated between two rigid discs, both with radii \(r = a\) and separated a distance \(h\) as shown in figure 3.5. The volume of the liquid bridge is assumed to be \(V = \pi a^2 h\), so that when \(Bo = 0\) \(\Omega\) is the rectangular region \(\{x = (r, z) \in R^2 : 0 \leq r \leq a, 0 \leq z \leq h\}\). In analogy with section 3.2 we find that the axisymmetric inviscid oscillations of a liquid bridge are given by

\[
\nabla^2 \phi = 0 \quad \text{in} \quad \Omega,
\]

\[
\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad r = 0 \quad \text{and} \quad z = 0, \quad z = h,
\]

\[
\frac{\partial \phi}{\partial r} = \omega^2 \int_{S_0} K(z, \xi) \phi(\xi, a) d\xi, \quad \text{on} \quad r = a
\]

35
where the Green’s function $K(z, \xi)$ is in this case

$$K(z, \xi) = \frac{\alpha}{\tan(h/a)} \begin{cases} \sin(z/a)[\tan(h/a) \cos(\xi/a) - \sin(\xi/a)] & z \leq \xi \\ \sin(\xi/a)[\tan(h/a) \cos(z/a) - \sin(z/a)] & z > \xi \end{cases}$$

Like in section (3.2), $\phi(r, z)$ is written as an infinite sum of the eigenfunctions $\phi_n$ which are obtained when Laplace’s equation is solved subject to the boundary conditions on $r = 0$ and $z = 0, h$. In particular

$$\phi(r, z) = C + \sum_{n=1}^{\infty} B_n \phi_n(r, z),$$

where

$$\phi_n(r, z) = I_0(n\pi r/h) \cos(n\pi z/h).$$

The solution procedure is as outlined before, for the eigenvalues $\omega_n$ we find

$$\omega_n^2 = \frac{n\pi}{h^3} [(n\pi)^2 - \beta^2] \frac{I_0(n\pi/\beta)}{I_0(n\pi/\beta)} \times \left[1 - \frac{4\beta}{(n\pi)^2 - \beta^2} \frac{[1 - (-1)^n] \sin \beta + \beta [(-1)^n - \cos \beta]}{2(1 - \cos \beta) - \beta \sin \beta}\right]^{-1}, \quad (3.5.2)$$

in which $\beta = h/a$ and $I_0$ is the modified Bessel function of the first kind. It is clear that the ratio of disc separation and disc radius is an important parameter. Using elementary techniques, it can be shown that the right hand side of equation (3.5.2) is positive for all $n$ if $\beta \leq 2\pi$. For $n = 1$, the right
Figure 3.6: Dependence of the first and second eigenmode on the aspect ratio $\beta$.

Hand side of (3.5.2) is negative for $\beta$ larger than $2\pi$. In that case $\omega_1$ is imaginary and hence the system is unstable, i.e. disturbances grow exponentially. This observation corresponds with the well-known result that the static liquid bridge configuration is stable only when $\beta \leq 2\pi$. Interesting is the fact that this stability condition, which is due to Rayleigh (1899), was derived by considering axisymmetric disturbances on an infinitely long circular jet of liquid: if the wavelength of disturbances exceeds the circumference of the jet, then these disturbances are unstable so that the jet disintegrates. Bauer (1982) who considered liquid bridge oscillations without edge constraints, found that the eigenfrequencies are given by (3.5.2), however without the term in square brackets. The stability condition is in that case given by $h/a < \pi$. In figure 3.6 we have plotted the values of the first and the second eigenmode as a function of $\beta$. We find that the numerical results of Schilling and Siekmann (1989) are generally in good agreement with our analytical results and likewise for the results of Meseguer (1983) for the case of slender bridges ($\beta \geq 5$). Experimental work by Sanz (1985) on the eigenmodes of liquid bridges shows that the trends of curves in figure 3.6 compare well with the measured trends. Our theory can not be applied directly to validate Sanz's experiments since the experiments were carried out by means of the neutral-buoyancy technique: the influence of the outer fluid bath on the eigenmodes is significant.
3.6 Conclusions

In this chapter we have shown how the problem of oscillations of a capillary surface subjected to edge constraints may be solved analytically. The idea is to write the complicated free surface conditions in a more convenient form using Green’s functions. This procedure allows the problem to be solved using standard separation of variable techniques when the domain under consideration is rectangular. It has been shown that imposing edge constraints increases the eigenfrequencies of capillary oscillations by up to 50%. The influence of edge constraints on the eigenfrequencies is most significant for the lowest modes and small values of the Bond number.

A special case of the problem of axisymmetric oscillations of liquid bridges can also be dealt with, when the free surface conditions are rewritten using Green’s functions. An interesting extension of this work would be the study of the dynamics of a liquid bridge in a neutral buoyancy configuration so that a comparison between theory and experiments is possible.
Chapter 4

Eigenmodes of a viscous, rotating liquid with a capillary free boundary

4.1 Introduction

It will be evident that the linear eigenvalue problem (3.1.1), or problem (3.2.9) when capillary effects are included, is often a useful first order approximation when small oscillations of a low-viscous fluid are studied. However, the assumption that the fluid flow be irrotational severely restricts the types of flow that can be studied. In addition, physically realistic features such as the viscous dissipation of energy and the no-slip condition on solid boundaries, are neglected. When the viscosity of the fluid is low, it is possible to estimate the damping of eigenmodes due to viscous effects by using boundary-layer techniques. Myshkis et al. (1986) give a general expression for the damping coefficients due to viscous effects. However, the complexity of the expression, together with the fact that the non-trivial eigenvalue problem for inviscid oscillations still has to be solved, limits the use of this approach. The boundary-layer approach is effective for simple container shapes as is shown by Case & Parkinson (1957) who find a good agreement between analytical and experimental damping coefficients.

If one does not want to impose the condition of irrotational flows and if viscous effects have to be included, then the Navier-Stokes equations for the fluid motion have to be considered. In its general form the Navier-Stokes equations are non-linear. The presence of a free surface adds to the diffi-
culty of the problem since the domain occupied by the fluid is not fixed, but varies with time. For work on numerical solution techniques regarding the non-linear sloshing problem in a steady container, the reader is referred to Ramaswamy et al. (1986) and Huerta & Liu (1988) among others. A numerical study of the complete non-linear Navier-Stokes equations including rotational effects, can be found in Veldman & Vogels (1984,1986). They apply a version of the Marker-and-Cell technique to track the large deformation of the capillary free-boundary during spin-up. The solution of the non-linear time-dependent problem is, however, beyond the scope of this thesis. We shall therefore restrict our discussion to work on the linearized problem of small oscillations of a viscous fluid.

Theoretical results concerning the existence and uniqueness of the solution of the linearized Navier-Stokes equations, together with general statements on the position of the eigenvalues in the complex plane, have been obtained by Krein (1964) and Askerov et al. (1964). They proved that, in case normal oscillations of the form $e^{\lambda t}$ are considered, the spectrum is discrete, has two limiting points ($\lambda = 0$ and $\lambda = \infty$) and all eigenvalues, except possibly a finite number, are real. For sufficiently high values of the viscosity, all eigenvalues are real. Modelling the behaviour of liquids in a vessel which is placed in a low-gravity environment requires the effects of surface tension to be taken into account. For an in depth and comprehensive mathematical treatment of the linear motion (viscous or non-viscous) of an incompressible fluid with a capillary free-boundary, the reader is referred to the encyclopedic work of Myshkis et al. (1986). The book contains most of the references on theoretical aspects of low-gravity fluid dynamics up to the time of its appearance. A numerical study on small oscillations of a viscous fluid with a capillary free-boundary is presented by Cuvelier (1985). By considering the time-dependent behaviour of the free surface he is able to estimate oscillation frequencies and damping coefficients.

For the study of eigenfrequencies of a viscous fluid in a rotating container, the Navier-Stokes equations are essential since the fluid flow is not irrotational thus prohibiting a velocity potential formulation. Characteristic of rotating fluids is the occurrence of internal (or inertial) waves: vortex-like motion in the entire region occupied by the fluid. Internal wave motion of a fluid entirely filling a closed, rotating container is fairly well understood, see for example Greenspan (1980). Free surface waves, as well as internal waves, may occur when the rotating vessel is only partly filled by a liquid. Miles (1963) and Miles & Ball (1963) have investigated, by analytical means, the eigenfrequencies of these waves in an inviscid liquid in a slowly
rotating container. Skalak & Conly (1964) have studied a somewhat similar problem in which the slow rotation condition can be removed. The analysis is, however, restricted to special container shapes since separation of variables is used in parabolic-cylindrical coordinates. In Myshkis et al. (1986) results concerning the existence of internal- and surface waves in a rotating, capillary, inviscid liquid are obtained, together with qualitative properties of their respective eigenfrequencies.

El-Raheb & Wagner (1981) have introduced the so-called homogeneous-vortex method to calculate the eigenfrequency of the principal free-surface mode. The method is based on the assumption that the vorticity of the fluid is constant and takes on the value of its average over the fluid volume. The homogeneous-vortex approach, although of questionable validity, has been applied recently to the problem of fluid oscillations in tanks of rotating spacecraft. For work in this direction the reader is referred to Beig (1984), Ebert (1984) and updated versions of the homogeneous-vortex method are presented by Guibert et al. (1985) and Ebert (1989).

In this chapter we aim to study the motion, and in particular normal oscillations, of a viscous liquid with a capillary surface. We assume the fluid velocity to be small, so that the non-linear convective terms in the Navier-Stokes equations can be neglected. In section 4.2 we show how the linearized equations can be written in the form of a spectral problem. Properties of the spectrum, based on work by Krein (1964) and Askerov et al. (1964), among others, are mentioned in §4.3. In section 4.4 it is shown how the discrete analogue of the spectral problem can be written as a quadratic eigenvalue problem. Properties of the discrete spectrum are derived, and a method based on inverse iteration is presented for solving the eigenvalue problem. Results from numerical experiments, for the case in which the container does not rotate, are presented in §4.5. The effect on the normal oscillations of viscosity, capillary phenomena and tank configuration, are considered and compared with known analytical results. In §4.6 we investigate rotational effects. Eigenfrequencies obtained numerically are compared with theoretical results, and those found using the homogeneous vorticity model. Finally, in section 4.7 we investigate oscillations of liquid bridges.
4.2 Problem formulation

The problem to be studied in this chapter is as follows. Consider a rotationally symmetric, closed container $C$ defined by

$$C = \{ \mathbf{x} = (r, \theta, z) \in \mathbb{R}^3 | 0 \leq r \leq R, 0 \leq z \leq H, 0 \leq \theta \leq 2\pi \}.$$ 

The container is placed in a gravitational field $g$ directed along the negative $z$-axis, as shown in figure 4.1. We assume that the container is partly filled with an incompressible, Newtonian fluid of density $\rho_f$ and viscosity $\mu$. The region occupied by the fluid at time $t$ is denoted by $\Omega(t)$. Its boundary is denoted by $\partial \Omega(t)$ and can be divided into two parts: the fluid surface in contact with the container wall $\Gamma(t) = \partial \Omega(t) \cap \partial C$ and the free surface $S(t) = \partial \Omega - \Gamma(t)$.

The equations which determine the motion of the fluid in a coordinate frame rotating with angular velocity $\omega$, are the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\omega \times \mathbf{u} + \omega \times (\omega \times \mathbf{r}) + \frac{1}{\rho_f} \nabla p = \nu \nabla^2 \mathbf{u} + \mathbf{f}, \quad (4.2.1)$$

and the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0. \quad (4.2.2)$$

Here $\mathbf{u}$ denotes the fluid velocity, $p$ the pressure, $\mathbf{r}$ the radial vector, $\mathbf{f}$ the body force and $\nu = \mu/\rho_f$ the kinematic viscosity. The axis of rotation coincides with the symmetry axis of the container which is taken to be the $z$-axis. The Cauchy stress tensor is given by

$$\sigma = -p \mathbf{I} + \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T),$$

where $\mathbf{I}$ is the unit diadic. Since we consider rotationally symmetric containers only, it is natural to work in cylindrical polar coordinates; the gradient,
divergence and Laplace operators take on their respective forms in cylindrical coordinates. For details of the form of (4.2.1), (4.2.2) and the stress tensor in cylindrical polar coordinates we refer to Batchelor (1967).

The unit normal and tangent vectors are denoted by \( \mathbf{n} \) and \( \mathbf{\tau} \) respectively. In describing the boundary conditions we use the following notation: the normal and tangential stresses are denoted by

\[
\sigma_n = (\sigma \cdot \mathbf{n}) \cdot \mathbf{n}, \quad \sigma_\tau = (\sigma \cdot \mathbf{n}) \cdot \mathbf{\tau},
\]

and the normal and tangential components of the velocity are denoted by

\[
u_n = \mathbf{u} \cdot \mathbf{n}, \quad u_\tau = \mathbf{u} \cdot \mathbf{\tau}.
\]

On the wetted part of the container wall \( \Gamma(t) \) we have either the no-slip condition

\[
u = 0,
\]

or the free-slip conditions

\[
u_n = 0, \quad \sigma_\tau = 0.
\]

Next consider the conditions to be satisfied on the free surface \( S(t) \). The gas above the fluid is assumed to be unable to resist tangential stresses, hence

\[
\sigma_\tau = 0.
\]

The normal stress on the free surface is equal to the pressure jump across the free surface as a result of surface tension effects. This gives

\[
\sigma_n = \alpha \left( \frac{1}{R_1} + \frac{1}{R_2} \right) - p_g,
\]

where \( \alpha \) denotes the coefficient of surface tension, \( R_1 \) and \( R_2 \) the principal radii of curvature of \( S(t) \) (cf. section 2.4) and \( p_g \) the outside gas pressure which we can choose to be equal to zero. Let \( \xi(x, t) = 0 \) be the equation of the free surface \( S(t) \) then the kinematic condition on the free surface reads

\[
\frac{D\xi}{Dt} = 0,
\]

where \( D/Dt \) is the Lagrangian derivative. On the curve \( S(t) \cap \Gamma(t) \) a contact angle condition is given, viz.

\[
\mathbf{n}_\Gamma \cdot \mathbf{n}_S = - \cos \delta,
\]
where \( \delta, n_R \) and \( n_S \) are as shown in figure 4.1. The last condition to be specified is the volume constraint - the quantity of liquid, \( V_f \), in the container remains constant so that

\[
\int_{\Omega(t)} dx = V_f. \tag{4.2.9}
\]

The non-linear equations (4.2.1) and (4.2.2) can, in principle, be solved subject to the boundary conditions (4.2.3)-(4.2.9) and appropriate initial conditions. This is, however, beyond the scope of this thesis. Here we restrict ourselves to the linearized forms of equations (4.2.1)-(4.2.9): only small free surface displacements and fluid velocities will be considered. In addition we assume the fluid flow to be rotationally symmetric (i.e. no azimuthal dependence) so that the domain \( \Omega(t) \) is two-dimensional (the \( \tau, z \)-plane). Note that the condition of rotational symmetry does not imply that the azimuthal component of the velocity vector vanishes (this is only the case when \( \omega = 0 \)). Hence, the fluid velocity has in general three non-zero components.

Let us first consider the steady state \( u = 0 \). The momentum equation, together with condition (4.2.6), yields the following expression on the steady state free surface \( S_0 \)

\[
\alpha \left( \frac{1}{R_r} + \frac{1}{R_\theta} \right) + \rho_f f \cdot R_0 + \frac{1}{2} \rho_f (\omega \times R_0)^2 = \text{constant}, \tag{4.2.10}
\]

where \( R_0 \) denotes the position vector of \( S_0 \) and \( R_r, R_\theta \) are the principal radii of curvature of \( S_0 \), \( R_\theta \) being the curvature corresponding to the lines of curvature for which \( r = \text{constant} \) and \( R_r \) to the lines of curvature for which \( \theta = \text{constant} \). We recall that equation (4.2.10), subject to the contact angle condition (4.2.8) and the volume constraint (4.2.9), determines the equilibrium capillary free-boundary shape, cf. section 2.4.

Let us now perturb the steady state free surface such that the position vector of \( S(t) \) is given by

\[
R = R_0 + \eta n_S, \tag{4.2.11}
\]

where \( \eta(x, t)n_S \) denotes a small normal deviation from \( S_0 \), as show in figure 4.2. The velocity resulting from the perturbation of the free surface will be small, so that the non-linear convective term in (4.2.1) can be neglected. The free surface condition (4.2.6) becomes

\[
\sigma_n = \alpha \left[ \eta \left( \frac{1}{R_r^2} + \frac{1}{R_\theta^2} \right) + \frac{1}{r} \frac{\partial}{\partial s_0} (r \frac{\partial \eta}{\partial s_0}) \right],
\]

44
which follows from the variation in the curvature due to the perturbation (4.2.11) viz.

\[
\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R_r} + \frac{1}{R_\theta} + \eta \left( \frac{1}{R_r^2} + \frac{1}{R_\theta^2} \right) + \frac{1}{r} \frac{\partial}{\partial s_0} \left( r \frac{\partial \eta}{\partial s_0} \right) + o(\eta, \frac{\partial \eta}{\partial s_0}),
\]

in which \(s_0\) is some curvilinear coordinate parallel to the rotationally symmetric surface \(S_0\). The kinematic condition (4.2.7) becomes on neglecting all products of small quantities,

\[ u_n = \frac{\partial \eta}{\partial t}, \]

and finally, the contact angle condition is given by

\[ \frac{\partial \eta}{\partial s_0} = -\left( \frac{\cot \delta}{R_r} - \frac{\csc \delta}{R_C} \right) \eta, \]

in which \(R_C\) is the radius of curvature of the container wall \(\partial C\) corresponding to the lines of curvature for which \(\theta = \text{constant}\) (cf. Myshkis et al. 1986). Note that when the no-slip condition on \(\Gamma\) is specified (\(\eta|_{\Gamma} = 0\)), the contact angle condition reduces to \(\partial \eta/\partial s_0 = 0\). In that case the contact angle \(\delta\) does not appear explicitly in the equations - its effect is reduced to the shape of the capillary free boundary.

The equations will now be written in dimensionless form. Let \(L\) denote the characteristic length of the container, \(T = \sqrt{\rho_f L^3/\alpha}\) some time scale, \(U = L/T\) a velocity scale and let \(P = \alpha/L\) be some pressure scale. Assume in addition that \(u, p\) and \(\eta\) exhibit a temporal behaviour of the form \(e^{\lambda t}\), where \(\lambda\) is some complex number. The linearized Navier-Stokes equations in dimensionless form can then be written as follows:

\[
\begin{align*}
\lambda \mathbf{u} + 2\sqrt{We} \hat{\mathbf{k}} \times \mathbf{u} + \nabla p - Oh \nabla^2 \mathbf{u} &= 0 \\
\nabla \cdot \mathbf{u} &= 0
\end{align*}
\]

\text{in } \Omega_0, \quad (4.2.12)
The (linearized) boundary conditions are

$$
\begin{align*}
  u &= 0 \quad \text{or,} \\
  u_n &= 0 \quad \sigma_r = 0
\end{align*} \quad \text{on } \Gamma,
$$

(4.2.13)

$$
\begin{align*}
  \sigma_r &= 0 \\
  \sigma_n &= -\Phi \eta + \frac{1}{r} \frac{\partial}{\partial s_0} (r \frac{\partial \eta}{\partial s_0}) \\
  u_n &= \lambda \eta
\end{align*} \quad \text{on } S_0,
$$

(4.2.14)

$$
\frac{\partial \eta}{\partial s_0} = -\left(\frac{\cot \delta}{R_r} - \frac{\csc \delta}{R_C}\right) \eta \quad \text{on } \Gamma \cap S_0,
$$

(4.2.15)

and the symmetry conditions

$$
u_r = 0, \quad u_\theta = 0, \quad \sigma_r = 0, \quad \frac{\partial \eta}{\partial s_0} = 0 \quad \text{at } r = 0.
$$

(4.2.16)

In the above equations

$$
\Phi = Bo \hat{k} \cdot n_s - \frac{1}{R_r^2} - \frac{1}{R_\theta^2} - We(\hat{k} \times R_0) \cdot (\hat{k} \times n_s),
$$

and \( \hat{k} \) is a unit vector directed along the positive z-axis. Three dimensionless quantities have been introduced, namely the Bond number \( Bo = \rho_f g L^2/\alpha \), the Weber number \( We = \rho_f \omega^2 L^3/\alpha \) and the Ohnesorge number \( Oh = \mu/\sqrt{\rho_f \alpha L} \). The quantities \( u, p \) and \( \eta \) are in general complex functions only dependent on the space coordinates. The problem is solved on the fixed domain \( \Omega_0 \) so that the volume condition is satisfied when the integral of the displacement over the free surface is zero. This condition is fulfilled when the incompressibility constraint is satisfied.

The problem defined by equations (4.2.12)-(4.2.16) can be written in a variational formulation. To this end we note that the momentum equation can be written as

$$
\lambda u + 2\sqrt{We} \hat{k} \times u = \nabla \cdot \sigma,
$$

where \( \sigma \) is the Cauchy stress tensor with \( \mu \) replaced by \( Oh \). Taking the scalar product of the above equation with a test function \( \nu \), integrating over \( \Omega_0 \) and applying Green’s theorem, yields

$$
\int_{\Omega_0} (\lambda u \cdot \nu + 2\sqrt{We} \hat{k} \times u \cdot \nu - p \nabla \cdot \nu) d\Omega + Oh a(u, \nu) =
$$

46
\[
\int_{\partial \Omega_0} (\sigma_n v_n + \sigma_r v_r) r ds = 0,
\]
(4.2.17)

where
\[
a(u, v) = \int_{\Omega_0} \left[ 2 \left( \frac{\partial u_r}{\partial r} \frac{\partial v_r}{\partial r} + \frac{u_r v_r}{r^2} + \frac{\partial u_z}{\partial z} \frac{\partial v_z}{\partial z} \right) + \left( \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right) \left( \frac{\partial v_{\theta}}{\partial r} - \frac{v_{\theta}}{r} \right) + \right.
\]
\[
\left. \left( \frac{\partial u_r}{\partial z} + \frac{u_r}{r} \right) \left( \frac{\partial v_r}{\partial z} + \frac{v_r}{r} \right) \right] r dx.
\]

The test function \(v\) is chosen such that on \(\Gamma\) we have \(v = 0\) or \(v_n = 0\), depending on whether the no-slip or free-slip boundary condition is applied. The functional \(a(u, u)\) is proportional to the rate of dissipation of energy in the liquid (cf. Landau & Lifschitz, 1959a). Multiplication of the incompressibility condition by a test function \(q\) yields, after integration over \(\Omega_0\)
\[
\int_{\Omega_0} q(\nabla \cdot u) r dx = 0.
\]
(4.2.18)

The functions \(u, v\) and \(q\) must be so smooth that the integrals (4.2.17) and (4.2.18) exist. We suffice with stating that this generally means that the components of both \(u\) and \(v\) together with their first partial derivatives and \(p, q\) are in \(L_2(\Omega)\). Substitution of the boundary conditions (4.2.13)-(4.2.16) into the boundary integral in (4.2.17) and eliminating \(\eta\) using the kinematic condition on \(S_0\), yields the weak formulation of problem (4.2.12)-(4.2.16), viz.

find \(u\) and \(p\) such that for all suitably smooth functions \(v\) and \(q\) the following equations are satisfied
\[
\int_{\Omega_0} (\lambda u \cdot v + 2\sqrt{\text{We}} k \times u \cdot v - p \nabla \cdot v) r dx + a(u, v) + \frac{1}{\lambda} b(u, v) = 0,
\]
(4.2.19)

where the bilinear form \(b(u, v)\) is given by
\[
b(u, v) = \int_{S_0} \left[ \Phi u_n v_n + \frac{\partial u_n}{\partial s_0} \frac{\partial v_n}{\partial s_0} \right] r ds + r \left( \cot \delta \frac{\partial}{\partial r} - \csc \delta \frac{1}{r} \right) u_n v_n |_{S_0 \cap \Gamma_w}.
\]

Problem (4.2.19) defines a boundary-value problem for the determination of the spectral parameter \(\lambda\).

47
4.3 Properties of the spectrum

Returning to problem (4.2.19) we investigate the properties of the spectrum of linear viscous fluid oscillations. Let us define a potential-energy operator according to

\[ B = B_0 \hat{k} \cdot \mathbf{n}_S - \frac{1}{R_r} - \frac{1}{R_\theta} - W e (\hat{k} \times \mathbf{R}_0) \cdot (\hat{k} \times \mathbf{n}_S) - \frac{1}{r} \frac{\partial}{\partial s_0} (r \frac{\partial}{\partial s_0}). \]

The quantity \( \int_{S_0} u_n B u_n r ds = b(u, u) \) is then proportional to the potential energy of small fluid oscillations. If the minimum eigenvalue of \( B \) is strictly positive, i.e. \( \lambda_{\text{min}}(B) > 0 \), then there exists a unique solution of the problem defined by (4.2.19), as is shown by Cuvelier (1985) for the case in which \( We = 0 \), and Myshkis et al. (1986) for the more general case in which \( We \neq 0 \). Moreover, problem (4.2.19) is stable with respect to infinitesimal perturbations from its equilibrium state, if \( \lambda_{\text{min}}(B) > 0 \). The properties of the spectrum will now be considered for two separate cases, namely \( We = 0 \) and \( We \neq 0 \).

In the case where \( We = 0 \), the following results concerning the spectrum of (4.2.19) can be obtained:

(i) If the potential energy operator \( B \) is positive definite, which is equivalent to requiring that \( \lambda_{\text{min}}(B) > 0 \), then \( \text{Re}(\lambda) < 0 \), i.e. the normal oscillations are asymptotically damped;

(ii) If the dissipation of energy is sufficiently large, i.e. if

\[ Oh^2 (a(u, u))^2 \geq 4 \int_{\Omega_0} |u|^2 r dx \cdot b(u, u), \]

then the associated eigenvalue is real, which corresponds to an aperiodic damping process;

(iii) If the dissipation of energy is low, i.e. if

\[ Oh^2 (a(u, u))^2 < 4 \int_{\Omega_0} |u|^2 r dx \cdot b(u, u), \]

then the eigenvalues appear in complex conjugate pairs. The real parts of the eigenvalues are negative so that the motion corresponds to damped oscillations;
(iv) The spectrum is discrete and has a unique limit point \( \lambda = \infty \). For an arbitrary fixed value of \( Oh \), the problem cannot have more than a finite number of non-real eigenvalues. For large values of \( Oh \) (cf. (ii)) these eigenvalues will lie on the real axis and, conversely, for small \( Oh \) (cf. (iii)) the number of non-real eigenvalues becomes indefinitely large. As \( Oh \to 0 \) the non-real eigenvalues approach the purely imaginary eigenvalues of the problem of oscillations in an ideal liquid while the remaining eigenvalues stay real and tend to zero.

For details leading to properties (i)-(iv) we refer to Krein (1964), Kopachevskii & Myshkis (1966) and Askerov et al. (1968). We note that for a cylinder with radius \( a \) containing a fluid with contact angle \( \delta = \pi/2 \) the condition \( \lambda_{\min}(B) > 0 \) in (i), can be shown to reduce to

\[
Bo > -\alpha_{1,1}^2,
\]

corresponding to the stability condition as derived in section 3.3 (\( \alpha_{1,1} \) is as introduced in equation (3.2.7))

We next consider the case in which \( We \neq 0 \). The following general result concerning the spectrum of (4.2.19) is shown in Myshkis et al. (1986):

(v) The spectrum of (4.2.19) is discrete, it does not have finite limit points and the eigenvalues have a negative real part, i.e. normal oscillations are asymptotically damped.

More rigorous statements in the case \( We \neq 0 \) concerning the properties of the spectrum (like (i)-(iv) in the case \( We = 0 \)) have not yet been obtained. In order to acquire more insight into the properties of the spectrum of a rotating fluid, we turn to the inviscid case. Indeed, it is natural to assume that the eigenvalues of a viscous fluid tend to those in the inviscid case as \( Oh \to 0 \). The following can be said concerning the effect of Coriolis forces on the (purely imaginary) eigenfrequencies \( \omega \) of an ideal capillary liquid:

(vi) The set of normal oscillations consists of internal wave motions with frequencies in the range \( |\omega| < 2\sqrt{We} \) and surface waves with frequencies \( |\omega| > 2\sqrt{We} \). The internal waves (characteristic for a rotating fluid) occur within the entire fluid, whereas the surface waves attenuate rapidly into the fluid;

(vii) The set of internal wave frequencies is dense on the interval \( |\omega| < 2\sqrt{We} \), while the surface waves have a limiting point at infinity;
(viii) As $We \to 0$, the entire spectrum of internal waves disappears while the oscillation frequencies of the surface waves reduce to those of a non-rotating liquid.

The distinct properties of the liquid motion, depending on whether $|\omega| > 2\sqrt{We}$ or $|\omega| < 2\sqrt{We}$, stem from the fact that the governing equation is elliptic or hyperbolic. For details leading to properties (vi)-(viii) we refer to Miles (1959,1963), Greenspan (1968) and Myshkis et al. (1986).

4.4 Numerical solution of the spectral problem

The spectral problem, as defined by equation (4.2.19) will be discretized using a finite-element procedure. The continuity equation leads, as it is, to a system of equations with unfavourable properties from a numerical point of view (matrix with large band width or partial pivoting has to be applied). In order to overcome this difficulty a penalty-function method is used. The basic idea behind the penalty-function method is to uncouple the momentum equation and the continuity equation in (4.2.19). To that end the continuity equation is perturbed by adding a small term containing the pressure, i.e.

$$\epsilon_p p + \nabla \cdot u = 0,$$

or in the variational formulation

$$\epsilon_p \int_{\Omega_0} pqr dx + \int_{\Omega_0} q(\nabla \cdot u)r dx = 0,$$

where $\epsilon_p \ll 1$. In order to construct a discrete system of equations, the velocity $u$ and the pressure $p$ are approximated by a linear combination of basis functions. The element type used is in the family of Crouzeix-Raviart elements. In particular, a triangular element is taken on which the basis functions for the velocity are extended quadratic polynomials (based on the three vertices, the three midpoints of the sides and the barycentre of the triangle). The basis functions for the pressure are linear polynomial functions (one nodal point including two derivatives), which are discontinuous across inter-element boundaries. An advantage of the Crouzeix-Raviart element is that the velocity unknowns in the centroid and the two pressure derivatives may be eliminated element-wise. This means that a significant reduction in the number of unknowns can be achieved without effecting the accuracy of the element.
Substituting for the approximate expressions for \( u \) and \( p \) in the momentum and penalized continuity equation, yield respectively

\[
\lambda M \ddot{u} + 2\sqrt{We} C \ddot{u} - L^T \ddot{p} + OhA \ddot{u} + \frac{1}{\lambda} B \ddot{u} = 0,
\]

and

\[
\varepsilon_p D \ddot{p} = -L \ddot{u},
\]

All matrices are real while \( \lambda \) and the vectors \( \ddot{u} \) and \( \ddot{p} \) (which contain the velocity and pressure unknowns in the nodal points) are in general complex. The matrix \( M \), termed the mass matrix, is related to the inertia of the fluid, the matrix \( A \) is related to the viscous dissipation of energy and the matrix \( B \) is related to the potential energy of oscillations on the capillary free surface. The matrices \( M \) and \( A \) are symmetric and positive definite. The matrix \( B \) is symmetric and positive semi-definite (\( B \) is singular since only degrees of freedom on the free surface give non-zero entries). The matrix \( C \) which is related to the Coriolis force, is skew symmetric (\( C^T = -C \)) and singular (the \( z \)-component of the velocity does not appear in the Coriolis-force term). Since the matrix \( D \) is non-singular, it may be inverted to express the pressure unknowns in terms of the velocity unknowns. Substituting for \( \ddot{p} \) in the momentum equation yields the quadratic eigenvalue problem

\[
(\lambda^2 M + \lambda S + B)\ddot{u} = 0,
\]

where the following substitution has been made

\[
S = 2\sqrt{We} C + \frac{1}{\varepsilon_p} L^T D^{-1} L + OhA.
\]

Note that the matrix \( S \), termed the stiffness matrix, is symmetric positive definite only when \( We = 0 \).

A disadvantage of the penalty-function approach is apparent when one realises that the matrix \( L^T D^{-1} L \) is singular. Namely, the stiffness matrix \( S \) consists of a singular matrix multiplied by a large parameter \( (1/\varepsilon_p) \) added to a regular matrix. Finite precision of digital computers results in a singular matrix \( S \) when \( \varepsilon_p \) is too small. For \( \varepsilon_p \) too large, however, the incompressibility constraint is not modelled correctly. Typically we require \( 10^{-9} \leq \varepsilon_p \leq 10^{-6} \). Details concerning the penalty-function approach and the finite-element discretization have been omitted since the method is well established. The reader is referred to, for example, Cuvelier et al. (1986) in
which an in depth discussion of the finite-element technique in relation to the Navier-Stokes equations is presented.

Before solution methods for (4.4.1) are considered, it is instructive to obtain some information concerning the properties of the discrete spectrum and the corresponding eigenvectors. To that end we proof the following theorem:

**Theorem 4.1** Consider the eigenvalue problem

\[(\alpha^2 I + \alpha Q + R)x = 0, \quad (4.4.2)\]

where the matrices are of order \(m\), \(Q\) is symmetric positive definite and \(R\) is a diagonal positive semi-definite matrix with rank equal to \(l < m\). Eigenvalue problem (4.4.2) has \(2m\) eigenvalues of which \(m - l\) eigenvalues are zero, \(m - l\) eigenvalues are real and negative and the remaining eigenvalues, \(2l\) in number, may be non-real.

**Proof:** We consider the locus of the eigenvalue \(\alpha_k\) as a function of \(\tau\) in the auxiliary eigenvalue problem

\[(\tau Q + R)x_k(\tau) = -\alpha_k^2(\tau)x_k(\tau),\]

where we take \(\tau \in \mathbb{R}\). Note that for \(\alpha_k = \tau\) we retain eigenvalue problem (4.4.2). Assuming that the vectors \(x_k(\tau)\) have been normalized it follows that the Rayleigh quotient is given by

\[\alpha_k^2(\tau) = -x_k^T(\tau Q + R)x_k, \quad (4.4.3)\]

from which we can deduce that

\[\frac{d}{d\tau}(\alpha_k^2) = -x_k^T Q x_k. \quad (4.4.4)\]

The eigenvalues \(q_i\) \((i = 1, \ldots, m)\) of \(Q\) are real and positive with \(0 < q_1 \leq q_2 \leq \cdots \leq q_m\) so that bounds on (4.4.4) are as follows

\[-q_m \leq \frac{d}{d\tau}(\alpha_k^2) \leq -q_1.\]

Integration of the above expression from 0 to \(\tau\) yields

\[-q_m \tau + \alpha_k^2(0) \leq \alpha_k^2 \leq -q_1 \tau + \alpha_k^2(0), \quad (4.4.5)\]

52
where $\alpha_k^2(0) = -r_k$ as follows from (4.4.3), with $r_k$ ($k = 1, \ldots, m$) the diagonal elements of $R$ given explicitly by $r_1 = r_2 = \cdots = r_{m-l} = 0 < r_{m-l+1} \leq \cdots \leq r_m$. Let us now consider the locus of $\alpha_k^2(\tau)$. We are in particular interested in points of intersection of the locus with the curve $\alpha_k(\tau) = \tau$. The two different cases $1 \leq k \leq m - l$ and $m - l + 1 \leq k \leq m$ are investigated separately.

First consider the case $1 \leq k \leq m - l$ so that $r_k = 0$. From (4.4.5) it follows that

$$-q_m \tau \leq \alpha_k^2 \leq -q_1 \tau. \quad (4.4.6)$$

The two curves $\alpha_k^2 = -q_1 \tau$, $\alpha_k^2 = -q_m \tau$ and the curve $\alpha_k^2 = \tau^2$ are shown in figure 4.3a. We see that the locus defined by (4.4.6) always intersects the curve $\alpha_k^2 = \tau^2$ in two distinct points (at $\tau = 0$ and $\tau < 0$). Hence the auxiliary eigenvalue problem has a fixed point equal to 0 for $m - l$ distinct values of $k$ so that $m - l$ eigenvalues are zero. In addition there is a fixed point at $\tau < 0$ for $m - l$ distinct values of $k$ so that $m - l$ eigenvalues are strictly negative.

![Diagram a.](image)

![Diagram b.](image)

Figure 4.3: The curve $\alpha_k^2 = \tau^2$ together with the bounds of the locus defined by (4.4.6) and (4.4.7) are shown in figures (a) and (b) respectively.

Next consider the case $m - l + 1 \leq k \leq m$ so that $r_k > 0$. Equation (4.4.5) yields the locus

$$-q_m \tau - r_k \leq \alpha_k^2 \leq -q_1 \tau - r_k, \quad (4.4.7)$$

as shown in figure 4.3b together with the curve $\alpha_k^2 = \tau^2$. We note that in general there does not exist a real $\tau$ for which the locus (4.4.7) intersects
the curve $\alpha_k^2 = \tau^2$. This implies that for this case the fixed points of the auxiliary eigenvalue problem lie in the complex plane occurring in complex conjugate pairs, i.e. $2i$ eigenvalues may be non-real. $\square$

**Corollary to theorem 4.1:** Eigenvalue problem (4.4.2) does not have any imaginary eigenvalues when $q_1 \geq 4r_k$.

In order to apply the results of theorem 4.1 to the eigenvalue problem (4.4.1), we note that (4.4.1) is equivalent to (4.4.2) when the following transformations are used:

$$
\begin{align*}
M &= GG^T \quad \text{(Choleski decomposition)} \\
S &= GUQU^T G^T \\
B &= GU^{-1}RU^{-T} G^T \\
u &= G^{-T} U x,
\end{align*}
$$

in which $U$ is a unitary matrix ($UU^T = I$). Let us assume that $\Omega_0$ has been discretized in such a way that the total number of degrees of freedom is $n$, and $n_B$ degrees of freedom on the free surface $S$ give non-zero entries in the matrix $B$ so that $\text{rank}(B) = n_B$. Take $We = 0$ so that $S$ is symmetric positive definite and hence theorem 4.1 can be applied directly. It follows that $n - n_B$ eigenvalues of (4.4.1) are zero and $n - n_B$ eigenvalues are real and negative. The remaining $2n_B$ eigenvalues are not necessarily real. Since eigenvectors corresponding to the non-real eigenvalues do not lie in the null-space of $B$, we expect these eigenvalues to be related to the free surface oscillations. The properties (i)-(iv) as defined in section 4.3 refer to these eigenmodes. We note that the corollary to theorem 4.1 is effectively the discrete analogue of property (ii) as formulated in section 4.3 since the eigenvalues of the stiffness matrix are proportional to $Oh$.

Next consider the case $We \neq 0$ so that $S$ is no longer symmetric and theorem 4.1 cannot be applied. We have not been able to derive rigorous results other than the fact that $n - n_B$ eigenvalues are zero. However, assuming that the eigenvalues can be divided into three sets like in theorem 4.1 and using properties (vi)-(viii) of section 4.3 it seems reasonable to make the following conjecture about properties of the eigenvalues for $We \neq 0$: all the non-zero eigenvalues ($n + n_B$ in number) may be complex. Moreover, $2n_B$ eigenvalues for which the corresponding eigenvectors do not lie in the null-space of $B$ correspond to free-surface vibrations and the remaining $n - n_B$ eigenvalues correspond to internal-oscillation modes.

Let us now consider solution methods for (4.4.1). When written in the form of a generalized eigenvalue problem, (4.4.1) may be solved using the
Figure 4.4: Plot of cpu time versus the number of unknowns for the QZ-algorithm.

QZ-algorithm. However, application of the QZ-algorithm is disadvantageous for a number of reasons. One is apparent when figure 4.4 is considered where the cpu time (on a Convex 240) versus the number of unknowns has been plotted for a model problem. The exponential growth of the curve shows that $n \approx 500$ is the largest number of unknowns that can be handled ($n \approx 500$ requires approximately one hour of cpu time). This number of unknowns corresponds to a $6 \times 6$ mesh of quadratic elements - far too coarse a mesh for all but the simplest problems to be considered. An additional disadvantage of the QZ-algorithm is that all eigenvalues are calculated while generally only the few related to lowest free surface modes and principal internal modes are of interest.

The foregoing considerations lead us to the Inverse Iteration procedure in which selected eigenvalues and eigenvectors are calculated. A recent review by Kerner (1989) of methods for solving large-scale complex eigenvalue problems, shows that the Inverse Iteration procedure is in fact one of but a few methods capable of dealing with large unsymmetric eigenvalue problems. Direct application of the standard Inverse Iteration algorithm to (4.4.1) yields the following iterative procedure

$$(\mu^2M + \mu S + B)v^{(n+1)} = Bu^{(n)} - \mu Mv^{(n)},$$
\[
\hat{u}^{(n+1)} = \frac{1}{\mu} (\hat{v}^{(n+1)} - u^{(n)}),
\]
(4.4.8)

\[
u^{(n+1)} = \frac{\hat{u}^{(n+1)}}{M}, \quad \nu^{(n+1)} = \frac{\hat{v}^{(n+1)}}{M}, \quad M = \max(||\hat{u}^{(n+1)}||, ||\hat{v}^{(n+1)}||).
\]

where \(\mu\) is some guess of the eigenvalue \(\lambda\). The iterative process (4.4.8) requires the solution of a system of equations for each iteration. This can be carried out efficiently, because the LU-decomposition of the matrix multiplying \(\hat{v}^{(n+1)}\) needs to be calculated only once.

It is well-known that the iterative procedure converges to the eigenvector of which the corresponding eigenvalue is nearest (in modulus) to the initial guess \(\mu\). Successive approximations of the corresponding eigenvalue are obtained by considering components of the vectors \(\hat{u}^{(n)}\) and \(\hat{v}^{(n)}\). In particular, the \(n\)'th approximation of the eigenvalue \(\lambda\) is obtained via

\[
\lambda^{(n)} = \frac{\hat{v}^{(n)}_k}{\hat{u}^{(n)}_k},
\]

where \(\hat{u}^{(n)}_k\) and \(\hat{v}^{(n)}_k\) denote the \(k\)'th component of the vectors \(\hat{u}^{(n)}\) and \(\hat{v}^{(n)}\) respectively and \(k\) is chosen such that \(|\hat{v}^{(n)}_k| > |\hat{v}^{(n)}_i|\) for all \(i \neq k\). By considering the convergence behaviour of \(\lambda^{(n)}\) it is, in addition, possible to obtain an estimate of the eigenvalue second nearest to \(\mu\). It can be shown that this eigenvalue, denoted by \(\tilde{\lambda}^{(n)}\), is approximated successively by

\[
\tilde{\lambda}^{(n)} = \mu + (\lambda^{(n)} - \mu) \frac{\lambda^{(n-1)} - \lambda^{(n-2)}}{\lambda^{(n)} - \lambda^{(n-1)}}.
\]

The iterative procedure is terminated when the convergence criterion \(|\lambda^{(n+1)} - \lambda^{(n)}| < c\) is satisfied for some given \(c\). Generally we take \(c = 10^{-4}\). We also terminate the iteration process when it has not converged after 20 iterations. It was found that restarting the iteration with the computed estimates \(\lambda^{(n)}\) or \(\tilde{\lambda}^{(n)}\) is more profitable than continuing the iteration. When \(\mu\) is updated after each iteration by substituting \(\lambda^{(n)}\) for \(\mu\), a cubic rate of convergence can be attained as shown by Wilkinson (1965). However, this update of \(\mu\) requires the calculation of the LU-decomposition after each iteration. This factorization was found to be far more expensive than the subsequent triangular solvers. We therefore opted to keep \(\mu\) fixed during 20 iterations after which it was updated had the process not converged.

The main drawback of the Inverse Iteration procedure is the fact that estimates of the eigenvalues have to be obtained beforehand. One way in
which this can be achieved is to use the $QZ$-algorithm on a coarse mesh to
give an estimate of the principle eigenmodes of the system. An alternative
procedure has been used recently by Cullum et al. (1989). They developed
a generalized non-symmetric Lanczos procedure for obtaining estimates of
eigenvalues in truly large scale complex eigenvalue problems ($n \sim O(10^5)$).
Since on the whole, only a few modes are of interest, we take a different
approach in all but one section of this thesis. In most cases it is possible to
obtain a good estimate of $\mu$ by considering eigenfrequencies in the inviscid
case. Once an eigenvalue has been obtained, we use the knowledge that
eigenvalues lie on curves in the complex plane to find subsequent eigenvalues.
A continuation procedure was found to be quite effective to track eigenvalues
when their dependence on a parameter (like $Oh$, $Bo$ or $We$) was investigated.

4.5 Numerical experiments with $We = 0$

We start this section with a comparison of some numerical and analytical
results. To that end let us consider a right-circular cylinder with radius
$R = 1$, height $H = 2$, half filled with fluid. We take the contact angle
to be $\delta = \pi/2$, so that the corresponding inviscid-fluid oscillation problem
readily admits an analytic solution as outlined in chapter 3. The ideal-fluid
situation is approached when free-slip boundary conditions on the wetted
part of the container wall are assumed and $Oh \ll 1$: we expect $\text{Im}(\lambda_i) \to \omega_i$
(where $\omega_i$ are given by equation (3.2.7)) as $Oh \to 0$. The only source of
damping when free-slip boundary conditions are applied, is the dissipation
of energy due to viscous forces in the body of the fluid. This results in a
damping coefficient, $\delta_i$, given by (refer to e.g. Case & Parkinson, 1957)

$$\delta_i = -2Oh\alpha_{1,i}^2.$$  \hfill (4.5.1)

The eigenfrequencies corresponding to the first five eigenmodes are shown
in table 4.1. The calculations are carried out on a mesh with $\Delta r = 0.1$, $\Delta z = 0.1$ (200 triangular elements) and $Bo = 10$, $Oh = 10^{-3}$. Observe
that the agreement between numerical and analytic results is good for the
lower eigenmodes. For higher eigenmodes there is still a good agreement
between $\omega_i$ and $\text{Im}(\lambda_i)$, but not so between $\delta_i$ and $\text{Re}(\lambda_i)$. Since $\text{Re}(\lambda_i)$
is determined by the fluid motion throughout the body of the fluid, a fine
discretization of the whole of $\Omega_0$ is required to calculate the damping co-
efficient accurately. This is in contrast with the free surface oscillations,
characterized by $\text{Im}(\lambda_i)$, which require only a fine discretization of the free

57
<table>
<thead>
<tr>
<th>eigenmode</th>
<th>( \omega_i(3.2.7) )</th>
<th>Im(( \lambda_i ))</th>
<th>( \delta_i(4.5.1) )</th>
<th>Re(( \lambda_i ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.73</td>
<td>9.73</td>
<td>-0.029</td>
<td>-0.032</td>
</tr>
<tr>
<td>2</td>
<td>20.4</td>
<td>20.4</td>
<td>-0.10</td>
<td>-0.13</td>
</tr>
<tr>
<td>3</td>
<td>34.0</td>
<td>34.1</td>
<td>-0.21</td>
<td>-0.31</td>
</tr>
<tr>
<td>4</td>
<td>50.0</td>
<td>50.5</td>
<td>-0.35</td>
<td>-0.54</td>
</tr>
<tr>
<td>5</td>
<td>68.1</td>
<td>69.3</td>
<td>-0.54</td>
<td>-0.80</td>
</tr>
</tbody>
</table>

Table 4.1: The first five eigenmodes: numerical and analytical results.

surface. Taking 450 elements to discretize \( \Omega_0 \) gives the following damping coefficients for the first three eigenmodes: -0.030, -0.11 and -0.25; they are close to the theoretical values given by (4.5.1). The results in table 4.1 were found to be virtually independent of the penalty parameter \( \epsilon_p \) when it was in the range \( 10^{-9} \leq \epsilon_p \leq 10^{-3} \). The value \( \epsilon_p = 10^{-6} \) is therefore used in all of the calculations presented here. Plots of the velocity vectors for the lowest two eigenmodes are shown in figure 4.5. Owing to symmetry, only half of the container is plotted.

![Image](a.png)

Figure 4.5: Vector plots of the first (a) and second (b) eigenmode for the parameter values \( Bo = 10 \), \( Oh = 10^{-3} \) and \( \delta = \pi/2 \).

In table 4.2 the eigenvalues are shown for a number of representative values of \( Oh \) and \( Bo \) with \( \delta = \pi/2 \). In the first two columns is shown how \( \lambda_1 \)
varies as $Oh$ increases, i.e. as the fluid becomes more viscous. Observe that
$\text{Im}(\lambda_1)$ decreases and in fact vanishes when $Oh$ exceeds some critical value.
This is in accordance with property (ii) as formulated in section 4.3. Note also that the damping coefficient $\text{Re}(\lambda_1)$ increases as $Oh$ increases and is maximum when $\text{Im}(\lambda_1)$ just vanishes. Increasing the viscosity further leads to a decrease in the damping coefficient. In the third and fourth column of

<table>
<thead>
<tr>
<th>$Oh$</th>
<th>$\lambda_1$</th>
<th>$Bo$</th>
<th>$\lambda_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>-1.20 + 9.50i</td>
<td>-1.0</td>
<td>-0.031 + 7.84i</td>
</tr>
<tr>
<td>0.5</td>
<td>-4.93 + 5.37i</td>
<td>-10.0</td>
<td>-0.029 + 4.16i</td>
</tr>
<tr>
<td>0.75</td>
<td>-6.40 + 3.00i</td>
<td>-14.0</td>
<td>-0.027 + 1.36i</td>
</tr>
<tr>
<td>1.0</td>
<td>-6.83 + 0i</td>
<td>-14.65</td>
<td>-3 x 10^{-5} + 0i</td>
</tr>
<tr>
<td>1.5</td>
<td>-6.00 + 0i</td>
<td>-14.67</td>
<td>6 x 10^{-7} + 0i</td>
</tr>
</tbody>
</table>

Table 4.2: The dependence of $\lambda_1$ on $Oh$ and $Bo$.

table 4.2 we consider the case in which $Bo$ is negative corresponding to an inver-
ted container. Observe that $\text{Re}(\lambda_1)$ becomes positive when $Bo \approx 14.66$. 
This result agrees well with the stability condition derived in section 3.3 
(namely, $j_{1,1}^2 = 14.68$, $j_{1,1}$ being the first zero of the Bessel function $J_1(x)$). 
So far we have only considered problems in which free-slip boundary 
conditions were assumed. Introduction of the no-slip boundary condition, 
which is characteristic of viscous flows, leads to fundamental questions con-
cerning the model we use. Namely, experimental observations show that 
the line of intersection of the free surface and the container wall (contact line), 
is generally not fixed when the free surface oscillates. This is not what one 
would expect when the no-slip boundary condition would be satisfied on $\Gamma$. 
The physics behind this phenomenon is not well understood and various ad-
hoc models have been suggested to explain the experimental observations, 
see e.g. Brook Benjamin & Scott (1979), Lowndes (1980) and Ngan & Dus-
san V. (1982). The approach of Lowndes will be investigated in some more 
detail. The following form for the boundary conditions on the container wall 
was suggested

$$u_n = 0, \quad \sigma_\tau = -\frac{1}{\chi} u_\tau,$$

(4.5.2)

where $\chi$ is some parameter with $0 < \chi < \infty$. For $\chi \to 0$ the usual no-slip 
boundary condition in obtained, in the limit $\chi \to \infty$ the free-slip bound-
ary condition results. For intermediate values of $\chi$ the boundary condition
(4.5.2) implies that the tangential stress is proportional to the fluid velocity which introduces an extra source of damping in addition to the viscosity of the fluid. The finite-element approach as outlined earlier is not altered by this boundary condition, the only difference is that extra terms appear in the stiffness matrix of the fluid.

It is of interest to investigate the dependence of the eigenmodes on the parameter $\chi$. To that end we consider a cylinder with radius $R = 1$ filled with a viscous fluid to a depth $h = 1$. We take $Oh = 10^{-2}$ and $\delta = \pi/2$. In figures 4.6a,b we have plotted the real and (scaled) imaginary parts respectively of the first eigenvalue versus $\chi$ (note the logarithmic scale), for $Bo = 10, 100, 1000$. Consider figure 4.6a. Note the sharp decrease of $\text{Re}(\lambda)$ in the interval $10^{-3} \leq \chi \leq 10^{-1}$. We observe that for increasing values of $Bo$ the minimum attained by the curve $\text{Re}(\lambda)$ increases somewhat. However, for $Bo > 1000$ the dependence of $\text{Re}(\lambda)$ on $\chi$ did not differ much from the curve corresponding to $Bo = 1000$. In figure 4.6b we have plotted ratio $\text{Im}(\lambda) / \text{Im}(\lambda_0)$ versus $\chi$ for the three values of $Bo$ (we define $\lambda_0 = \lambda|_{\chi=10^3}$). Note the sudden decrease in the curves when $\chi \approx 10^{-2}$. The interval in which the ratio $\text{Im}(\lambda) / \text{Im}(\lambda_0)$ changes rapidly, corresponds to the interval of $\chi$ at which $\text{Re}(\lambda)$ decreases sharply. For $\chi \gg 10^{-1}$ or $\chi \ll$
we find that $\text{Im}(\lambda)$ is virtually independent of $\chi$: for $\chi \gg 10^{-1}$ $\text{Im}(\lambda)$ assumes the value in the corresponding free-slip case, for $\chi \ll 10^{-2}$ $\text{Im}(\lambda)$ is equal to the corresponding value when no-slip conditions are imposed. Note that for increasing values of $Bo$, the relative change in $\text{Im}(\lambda)$ as a function of $\chi$ decreases. This dependence on $Bo$ is in accordance with the results of chapter 3. There is was shown that for $Bo \leq 100$ the effect of edge constraints is significant, for larger values of $Bo$ the influence of the edge constraints reduces rapidly (see figure 3.3).

The position of the minimum in the curves in figure 4.6a was found to be slightly dependent on $Oh$: for $Oh = 10^{-1}$ the minimum occurred at $\chi = 2.0 \times 10^{-2}$, for $Oh = 10^{-3}$ the minimum occurred at $\chi = 3.5 \times 10^{-2}$. For higher modes the position of the minimum occurred at slightly larger values of $\chi$ as compared with the position of the minimum in the case of the lowest mode. The physical meaning of the "fudge parameter" $\chi$ is somewhat unclear and evenso the reason why the over-damped behaviour occurs for $\chi \sim 10^{-2}$. Hocking (1987) investigated, by analytical means, the effect on the eigenfrequencies of boundary conditions at the contact line. Taking the time-derivative of the free-surface displacement at the contact line proportional to the free-surface displacement, yields results quite similar to those presented in figures 4.6.

We conclude this section with an example of normal modes in a complicated geometrical tank. To that end the fluid motion in a propellant tank of a TV-SAT satellite is considered. In table 4.3 we show the eigenfrequencies for various fill fractions of the tank and parameter values $Oh = 10^{-3}$, $Bo = 100$, $\delta = \pi/2$. No-slip boundary conditions are prescribed on the solid boundaries. Note that the oscillatory part of the eigenfrequencies varies considerably with the fill fraction of the propellant tank. In figure 4.7 we show streamlines for the first two eigenvalues and various fill fractions.

<table>
<thead>
<tr>
<th>fill fraction</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>-0.24+44.7i</td>
<td>-0.72+82.9i</td>
</tr>
<tr>
<td>0.70</td>
<td>-0.26+22.5i</td>
<td>-0.35+45.1i</td>
</tr>
<tr>
<td>0.50</td>
<td>-0.29+41.0i</td>
<td>-0.61+77.3i</td>
</tr>
</tbody>
</table>

Table 4.3: First and second eigenmode of fluid in a TV-SAT tank for various fill fractions.
Figure 4.7: Streamlines of the fluid flow in a TV-SAT tank for the first (a,b) and second (c,d) eigenmodes with $Bo = 100$, $\delta = \pi/2$ and fill fractions 0.75 (a,c) and 0.5 (b,d).
4.6 Rotational effects

In this section we investigate the validity of the homogeneous-vortex method as presented by El-Raheb & Wagner (1981) who use it to study the motion of a rotating fluid. The method is in principle based on the assumption that the vorticity of the fluid is constant and takes on the value of its average over the fluid volume. For fluids entirely filling a closed, rotating container this assumption is true (Lamb 1932), but only approximately so when one of the boundaries of the fluids is free. In order to investigate the validity of the various assumptions made by El-Raheb & Wagner we shall compare results obtained using their model, with analytical results, and numerical results obtained from our model. Since the homogeneous-vorticity method in the rotational symmetric case differs somewhat from the approach taken by El-Raheb & Wagner, we give an outline of the main idea behind the method. For details, and arguments justifying various approximations, the reader is referred to the aforementioned paper. Surface tension effects are neglected for simplicity; obviously, this does not affect the homogeneous-vorticity arguments.

The dimensionless equations describing the motion $\mathbf{u}$ of an inviscid, heavy fluid in a rotating container are, using the notation of section 4.2:

$$\begin{cases}
\frac{\partial \mathbf{u}}{\partial t} + 2\sqrt{Fr} \hat{k} \times \mathbf{u} + \nabla p = 0, \\
\nabla \cdot \mathbf{u} = 0,
\end{cases} \quad \text{in } \Omega_0$$

Together with the boundary conditions

$$\mathbf{u} \cdot \mathbf{n}_\Gamma = 0, \text{ on } \Gamma$$

$$\mathbf{u} \cdot \mathbf{n}_S = \frac{\partial \eta}{\partial t},$$

$$p = [\hat{k} \cdot \mathbf{n}_S - Fr(\hat{k} \times \mathbf{R}_0) \cdot (\hat{k} \times \mathbf{n}_S)]\eta \quad \text{on } S_0.$$

The dimensionless Froude number $Fr = \omega^2 L/g \ (= Bo^{-1} \text{We})$ is as introduced in section 2.6. The steady state free surface $S_0$ is, on neglecting surface tension, given by the equation

$$z - \frac{1}{2} r^2 Fr = \text{constant},$$

where the constant depends on the volume of liquid in the container. The velocity vector $\mathbf{u}$ is now written as follows

$$\mathbf{u} = -\nabla \Phi + \Omega \times \mathbf{r},$$

(4.6.1)
where $\Phi$ is a velocity potential and $\Omega$ is a vorticity vector. The basic assumption of the homogeneous-vortex model is, that the vorticity vector $\Omega$ is only a function of time, i.e. $\Omega = \Omega(t)$, and assumes the values of its average over the fluid volume. The incompressibility condition is satisfied when $\Phi$ satisfies Laplace’s equation. The dynamic condition on $S_0$ together with the condition on $\Gamma$ are satisfied if we choose $\Phi$ such that

$$\frac{\partial \Phi}{\partial n} = \Omega \cdot (r \times n) \text{ on } \Gamma, \quad \frac{\partial \Phi}{\partial n} = \Omega \cdot (r \times n) - \frac{\partial \eta}{\partial t} \text{ on } S_0.$$  

Substitution of (4.6.1) into the momentum equation yields, on neglecting the Coriolis term and the term $\dot{\Omega} \times r$, a complete differential (neglecting terms which are not known to be small a priori is however questionable!). The use of Bernoulli’s theorem leads then to the following kinematic condition on $S_0$

$$\frac{\partial \Phi}{\partial t} = g_R \eta,$$

where $g_R = (1 + Fr^2r^2)^{1/2}$. We finally need an equation for the vorticity. Taking the curl of the momentum equation, using the fact that $\nabla \times \mathbf{u} = 2\Omega$ and averaging spatially dependent terms over the volume of the fluid yields the desired vorticity equation. To summarize: the homogeneous-vortex assumptions reduce the problem of calculating the motion of a fluid partly filling a rotating container to solving the velocity potential problem

$$\nabla^2 \Phi = 0 \text{ in } \Omega_0,$$  

subject to the boundary conditions

$$\frac{\partial \Phi}{\partial n} = \Omega \cdot (r \times n) \text{ on } \Gamma,$$

$$\frac{\partial \Phi}{\partial n} = \Omega \cdot (r \times n) - \frac{1}{g_R} \frac{\partial^2 \Phi}{\partial t^2} \text{ on } S_0,$$

coupled with the vorticity equation

$$\dot{\Omega} + \sqrt{Fr} k \times \Omega = -2\pi \frac{\sqrt{Fr}}{V_f} \int_{\Omega_0} \nabla (\frac{\partial \Phi}{\partial z}) r \, dx.$$  

(4.6.3)

Equations (4.6.2)-(4.6.3) will be solved using the finite-element technique. Hence, consider the weak formulation:
find $\Phi$ such that for all $v$

$$\int_{\Omega_0} \nabla v \cdot \nabla \Phi \, dx = \int_{\partial \Omega_0} v \Omega \cdot (r \times n) \, ds - \int_{S_0} \frac{1}{g_R} \frac{\partial^2 \Phi}{\partial t^2} \, ds.$$ 

We take the usual finite-element approximation for $\Phi$, namely $\Phi = \sum_i \alpha_i(t) \phi_i$, where $\phi_i$ are quadratic basis functions. The discrete system of equations thus obtained, reads

$$S\alpha = \Omega_0 b - B\ddot{\alpha}, \quad (4.6.4)$$

where $S$ is a stiffness matrix and $B$ a matrix related to the integral over the free boundary and the components of the vector $b$ are given by

$$b_i = \int_{\partial \Omega_0} \phi_i (zn_r - rn_z) \, ds,$$

$n_r$ and $n_z$ denoting the $r$- and $z$-components of the outward unit normal respectively. Note that due to the rotational symmetry, only the $\theta$-component of the vorticity appears in (4.6.4). Rotational symmetry uncouples the $z$-component of the vorticity equation from the $r$- and $\theta$-components. The $r$-component may be eliminated to yield the equation for the vorticity

$$\ddot{\Omega}_\theta + Fr \Omega_\theta = \gamma^T \alpha, \quad (4.6.5)$$

where the components of $\gamma$ are given by

$$\gamma_i = 2\pi \frac{Fr}{V_f} \int_{\partial \Omega_0} \frac{\partial \phi_i}{\partial z} n_r \, ds.$$

Assuming $\Omega$ and $\alpha$ to be of the form $e^{i\omega t}$, it follows that (4.6.4) and (4.6.5) yield the following generalized eigenvalue problem

$$\begin{pmatrix} Fr & -\gamma^T \\ -B & S \end{pmatrix} \begin{pmatrix} \Omega_\theta \\ \alpha \end{pmatrix} = \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \Omega_\theta \\ \alpha \end{pmatrix} . \quad (4.6.6)$$

We should bear in mind that the averaging procedure in the vorticity equation effectively means that only the principal eigenmode is selected.

In the case where surface tension effects are neglected and the angular velocity of the container is small, Miles (1963) has obtained analytic expressions for the surface and internal waves frequencies of a rotating inviscid
fluid. The eigenfrequency corresponding to the principal surface mode of a liquid half filling a cylinder of height $H = 2$ and radius $R = 1$, are approximated by

$$
\omega_{2,1}^2 = \omega_{0,1}^2 + 2Fr + O(Fr^2), \quad (4.6.7)
$$

where $\omega_{0,1}^2 = \alpha_{1,1} \tanh \alpha_{1,1}$ is the lowest eigenfrequency in the case of a non-rotating fluid. Before we proceed with some numerical experiments we mention the changes that occur in the spectral problem (4.2.19) when surface tension effects are neglected. The Weber number is replaced by the Froude number and the Ohnesorge number by the reciprocal of the Reynolds number. The bilinear form $b(u,v)$ reduces to

$$
b(u,v) = \int_{S_0} g_R u_n v_n r ds.
$$

In figure 4.8 the principal eigenvalues, as obtained by the homogeneous vorticity model and our model (with $Re = 10^4$ and free-slip boundary conditions), have been plotted against $Fr$. For reference purposes we have also plotted the theoretical values as given by equation (4.6.7). We note

![Figure 4.8: Plot of the eigenfrequencies of the principal free surface mode versus $Fr$.](image)

that the theoretical results of Miles and the results obtained by means of the quadratic eigenvalue problem, are asymptotically the same for $Fr \ll 1$. Although the homogeneous-vortex results agree reasonably well with those from the other two approaches, we observe a different asymptotic behaviour.
The anomaly suggests a fundamental difference between the homogeneous-vortex model and the other two approaches. In our opinion this is due to the omittance by El-Raheb & Wagner of the Coriolis term - a first-order term regarding rotational effects on the fluid motion. An important disadvantage of the homogeneous vortex method as presented by El-Raheb & Wagner, is the fact that ficticious instabilities occur when the volume of fluid is less than a certain amount, dependent on \( Fr \). We found, for example, that in the case of the container under consideration and \( Fr = 0.4 \), the first eigenmode becomes unstable when the fluid volume is less than 0.4π. This instability is clearly non-physical and is an artefact of the homogeneous-vortex method. Recently, Ebert et al. (1989) have reached the same conclusion. They found that the ficticious instability was due to neglecting the Coriolis term and the term \( \Omega \times r \) in the momentum equation. Taking into account these terms removed the instability.

To conclude this section, let us investigate the eigenmodes of a rotating, viscous fluid with a capillary free boundary, that is problem (4.2.19) will be solved. To that end we consider the case of a spherical container placed in a low-gravity environment with \( Bo = 10 \). The fill fraction is 0.4, \( Oh = 10^{-2} \), \( \delta = \pi/4 \) and no-slip boundary conditions are prescribed. We know that internal modes occur when the fluid rotates, so that in addition to complex eigenvalues referring to free-surface oscillations, complex eigenvalues associated with internal modes will be present. In figure 4.9 eigenvalues with modulus \( |\lambda| \leq 40 \) have been plotted in the complex plane. The eigenvalues in figure 4.9a and 4.9b correspond to \( We = 10 \) and \( We = 35 \) respectively. For reference purposes the line \( \text{Im}(\lambda) = 2\sqrt{We} \) has also been plotted (the dotted line). The eigenvalues shown have been obtained using the QZ-algorithm on a coarse mesh.

Consider figure 4.9a. It is, in this case, easy to differentiate between the free-surface and internal-oscillation modes. By considering plots of the eigenvectors we find that the eigenvalues close to the axis \( \text{Re}(\lambda) = 0 \) and above the line \( \text{Im}(\lambda) = 2\sqrt{We} \) correspond to free-surface modes, the remaining eigenvalues (with \( \text{Im}(\lambda) \leq 2\sqrt{We} \)) are related to the internal modes. Using inverse iteration it is hard to separate the internal oscillation modes since the moduli of eigenvalues corresponding to different modes may be quite close. It was, therefore, found profitable to use the QZ-algorithm on a coarse mesh to obtain estimates of the eigenvalues, which could then be used in the inverse iteration procedure on a fine mesh. Generally, the eigenvalues of the principal internal modes are estimated well on a coarse mesh and they are easily identified since their real parts are small.
Figure 4.9: Positions of eigenvalues in the complex plane for $We = 5$ (a) and $We = 35$ (b)

Next, consider figure 4.9b. The line $\text{Im}(\lambda) = 2\sqrt{We}$ ($=11.8$) does in this case not separate the eigenvalues corresponding to internal-oscillation modes ($\lambda_I$) and free-surface modes ($\lambda_S$). Namely, the eigenvalue $\lambda = -0.86 + 10.1i$ was found to correspond to the principal free-surface mode. It is clear that property (vi), as formulated in section 4.3 and applicable to inviscid oscillations, does not hold for viscous fluids. We did find that the imaginary parts of eigenvalues corresponding to internal-oscillation modes, were bounded above in all cases by a constant of the form $C\sqrt{We}$. In particular, for $Oh = 10^{-2}$ $C = 1.75$, for $Oh = 10^{-3}$ $C = 1.88$ and for $Oh = 10^{-4}$, $C = 1.96$. In analogy with property (vi) one might ask whether or not $\text{Im}(\lambda_S)$ is bounded below by $2\sqrt{We}$. In figure 4.9b we see that this is not quite the case on a coarse mesh: the imaginary part corresponding to at least one of the internal modes is somewhat larger than that of the lowest free-surface mode. Careful calculations on a fine mesh indicate, however, that the upper bound of $\text{Im}(\lambda_I)$ is indeed very close to the lower bound of $\text{Im}(\lambda_S)$ (to within 2% in the case under consideration). These results lead to the following conjecture concerning the spectrum of eigenfrequencies of a rotating, viscous fluid:

The set of normal oscillations of a rotating, viscous, incompressible fluid consists of free-surface modes with eigenfrequencies $\lambda_S$ for which $|\text{Im}(\lambda_S)| > C\sqrt{We}$ and internal-oscillation modes with
eigenfrequencies $\lambda_I$ for which $|\text{Im}(\lambda_I)| < C \sqrt{\text{We}}$. The constant $C$ is dependent on the viscosity of the fluid with $C < 2$ and $C \to 2$ in the limit of vanishing viscosity.

Since $\text{Im}(\lambda_S)$ may be close to $\text{Im}(\lambda_I)$ for sufficiently large values of $\text{We}$ we can expect some interaction between free-surface and internal modes. By this we mean modes with a fluid motion akin to that of internal modes in the body of the fluid but where the free surface is not a stream line, i.e. modes with a mixed internal and free-surface character. Fultz (1962) has reported the occurrence of such mixed modes in experiments but we are not aware of a detailed study of the properties of such modes.

In figure 4.10a a vector plot of the velocity field corresponding to the first free surface eigenmode is shown for $\text{We} = 0$. Figure 4.10b shows a vector plot of an internal oscillation mode for $\text{We} = 25$. Observe that the fluid motion corresponding to this mode occurs entirely within the body of the fluid: the free surface is a streamline, i.e. the corresponding eigenvector lies in the null-space of the matrix B. In figure 4.10c we see a vector plot of a mixed free-surface internal mode for $\text{We} = 35$. Note the vortex-like fluid motion near the symmetry-axis and the fact that the free-surface is not a streamline. Finally, figure 4.10d shows a plot of the principal free-surface mode for $\text{We} = 55$.

4.7 Axisymmetric liquid bridge oscillations

In the final section of this chapter we consider oscillations of liquid bridges which consist of a viscous fluid held between two rigid discs, not necessarily with equal radii (see figure 3.1). We start our numerical experiments with an investigation into the effect of viscosity on the eigenmodes and stability of liquid bridges. Consider to that end the configuration as studied analytically in section 3.4, i.e. the rigid discs have equal radii (equal to $a$), are separated a distance $L$ and the volume contained is $\pi a^2 L$ so that the liquid bridge is cylindrical when $Bo = 0$.

In figure 4.11 the values of $\text{Im}(\lambda)$ and $\text{Re}(\lambda)$ have been plotted versus $L$ ($a = 1$), where $\lambda$ is the lowest eigenmode of the liquid bridge. First consider figure 4.11a, in which we have plotted $\text{Im}(\lambda)$ versus $L$ for $Oh = 0.2$, 0.5 and 0.6. For reference purposes we have also plotted $\omega_1$ versus $L$ (dotted line) where $\omega_1$ is as obtained from inviscid theory (equation (3.5.2)). It was found that for $Oh \leq 0.1$ the values of $\text{Im}(\lambda)$ were virtually identical to those given by the inviscid theory. $\text{Im}(\lambda)$ decreases as $Oh$ increases when compared with
Figure 4.10: Vector plots corresponding to the principal surface mode for $We = 0$ (a), an internal mode for $We = 25$ (b), a mixed surface-internal mode for $We = 35$ (c) and the principal free-surface mode for $We = 55$. 
inviscid results. Of interest is the fact that the value of $L$ at which $\text{Im}(\lambda)$ becomes zero, decreases as $Oh$ increases. For example $\text{Im}(\lambda)$ vanishes at $L = 2\pi$ in the inviscid limit, while for $Oh = 0.6$ $\text{Im}(\lambda)$ vanishes at $L = 4.2$ and remains zero for $L \geq 4.2$. We note that for viscous fluids, unlike the case for inviscid fluids, the value of $L$ at which $\text{Im}(\lambda)$ vanishes does not mark the transition from stable to unstable oscillations. This is evident from figure 4.11a in which $\text{Re}(\lambda)$ versus $L$ has been plotted. We observe, for example, that even though $\text{Im}(\lambda) = 0$ for $L \geq 4.2$ when $Oh = 0.6$, $\text{Re}(\lambda)$ remains negative. This corresponds to an aperiodic damping process. Note that for all values of $Oh$, the real part of the eigenvalue starts to increase more rapidly when $\text{Im}(\lambda)$ has vanished (points indicated by a dot in figures 4.11a,b). Of interest is the fact that all the curves of $\text{Re}(\lambda)$ pass through the line $\text{Re}(\lambda) = 0$ at one particular value of $L$. This value of $L$ was in fact found to be $2\pi$ (to numerical accuracy). This indicates that the length at which the liquid bridge becomes unstable is unaffected by the viscosity of the liquid.

![Graphs](image)

Figure 4.11: Plots of $\text{Im}(\lambda)$ (a) and $\text{Re}(\lambda)$ (b) versus the liquid bridge length $L$ for $Bo = 0$ and different values of $Oh$.  

71
Let us finally consider oscillations of a rotating liquid bridge. A rotating liquid bridge occurs for example in the process of purifying solids where a bar of solid is heated locally so as to melt a small part of the bar which then forms a liquid bridge. The solid bar, including the molten zone, is rotated in order to maintain a uniform temperature in the melt. We remark that rotational effects have a destabilizing effect on the liquid bridge due to centrifugal forces. We investigate the case of a liquid bridge between two discs with unequal radii. The radius of the bottom disc is 0.75, the top disc has radius 0.5 and the two discs are separated a distance 1. The volume of fluid between the discs is $0.35\pi$ and we set the parameter values $Bo = 5$ and $Oh = 0.1$. Figure 4.12 shows the shape of the capillary free surface for various values of the Weber number. It was found that no stable steady-state configuration is possible for $We > 7.3$, i.e. surface tension forces could not balance the centrifugal forces for values of $We$ exceeding $We = 7.3$. In figure 4.13 vector plots are shown for the case $We = 7.0$, of the first (4.13a) and second (4.13b) free-surface modes, and two rotation induced internal modes (4.13c,d).

### 4.8 Conclusions

In this chapter we have studied the problem of normal oscillations of a viscous, incompressible fluid with a capillary free boundary. Starting from the steady state configuration of the capillary free surface, we linearize the governing Navier-Stokes equations by considering small perturbations with respect to the steady state solution. Writing the resulting equations in a variational formulation yields, using a finite-element discretization, a quadratic eigenvalue problem of the form (4.4.2). In section 4.4 we derive some properties of the quadratic eigenvalue problem. It is argued that the eigenvalue problem can be solved efficiently using the inverse iteration procedure since

![Figure 4.12: Shapes of the capillary free surface for $Bo = 5$ and various Weber numbers.](image)
Figure 4.13: Vector plots of the first (a) and second (b) free-surface modes and of two internal modes (c,d) for the parameter values $Bo = 0.5$, $Oh = 0.1$ and $We = 7.0$. 

73
generally only a few eigenvalues are of interest. We have seen that numerical results compare well with analytical results when sufficient elements are used to discretize the region $\Omega_0$. We are able to calculate eigenvalues for a large variety of parameter values and vastly different container shapes.

The dependence of the eigenvalues on the angular velocity of the container has been investigated in section 4.6. We found that the homogeneous-vortex model as presented by El-Raheb & Wagner (1981) may be used to obtain a first order approximation to the eigenvalue of the principal free-surface mode of a rotating inviscid fluid. However, the presence of non-physical instabilities limits the use of the method. The imaginary part of the eigenvalues corresponding to the internal modes of a rotating viscous fluid were found to be bounded above by a constant of the form $C\sqrt{\text{We}}$, which was also found to be the lower bound for the imaginary part of eigenvalues corresponding to free-surface modes. The constant $C$ satisfies $C < 2$ and $C \to 2$ as the viscosity tends to zero. A proof of the conjecture in section 4.6, which is based on numerical results, would be interesting. Mixed internal-free surface oscillation modes were found to be present when the angular velocity of the container was sufficiently large. Fultz (1962) reports the existence of such modes but the author is not aware of any detailed investigation into the properties of mixed modes. This aspect of oscillations in a rotating fluid certainly deserves attention.

In the final section of this chapter we have investigated oscillations of liquid bridges. We found the stability of liquid bridges to be unaffected by the viscosity of the fluid. Interesting is the fact that before disturbances become unstable, the oscillations change qualitatively in nature. Namely, as the stability boundary is approached the motion of the liquid bridge changes from damped periodic to damped aperiodic and finally, when the stability boundary is crossed, to undamped aperiodic (or exponential growth).
Chapter 5

Eigenmodes of a viscous fluid in a flexible container

5.1 Introduction

So far we have studied oscillations of a viscous fluid in a rigid container. However, often the container walls are not truly rigid and the interaction between the fluid and the container wall may have to be considered. Various aspects of this so-called fluid-structure interaction problem will be studied in this and the following chapter.

One of the difficulties one encounters when dealing with the fluid-structure interaction problem, is how to model the system of fluid and structure. Depending on the characteristics of the structure it may be modelled by for example a membrane, a shell, a thick plate or a solid obeying the equations of elasticity. Likewise, depending on the type of fluid interacting with the structure, viscous effects may either be neglected or may have to be taken into account and the compressibility constraint may or may not have to be imposed. It will be clear that any combination of the structure and fluid models will lead to problems with different characteristics and solutions with different properties.

A vast amount of literature exists dealing with various aspects of the fluid-structure interaction problem. The literature can be divided into two main areas, namely papers dealing with non-linear fluid-structure interaction problems and papers dealing with the linearized problem. We will not be concerned with non-linear interaction problems, for work in this area the reader is referred to Liu & Chang (1985) among many others. Regarding
literature dealing with linear fluid-structure interaction problems a further subdivision is possible. We identify the class of papers in which the governing equations are solved using numerical integration techniques to evaluate the time-dependent motion of the fluid and the solid and, on the other hand, there are papers in which the time-dependent problem is transformed to a problem in a frequency domain. We will restrict ourselves to the latter class of problems, for an overview of the former the reader is referred to Belytschko (1977).

When dealing with the linear fluid-structure interaction problem in a frequency domain one is interested in the influence of the fluid on the eigenmodes of the structure and vice versa. Also of interest is how the structure is deflected when the fluid oscillates in one of its eigenmodes and what happens when one of the principal eigenmodes of the fluid is near a principal mode of the structure. In Table 5.1 an overview is given of some fluid-structure interaction papers in which eigenmodes of the coupled system are considered. In the papers the general emphasis is on calculating modal deflections and eigenfrequencies of the structure. The fluid motion as a result of the structural modal vibration has received much less attention, we refer to Hamdi et al. (1978). We note that a variety of models for the structure-part of the problem have been used. On the other hand, the fluid-part of the problem has been dealt with mainly by neglecting viscous effects. When damping characteristics of the fluid-structure system are to be investigated, viscosity

<table>
<thead>
<tr>
<th>Author</th>
<th>fluid model</th>
<th>structure model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Miles (1958)</td>
<td>inviscid, incompressible</td>
<td>simple beam</td>
</tr>
<tr>
<td>Coale (1969)</td>
<td>inviscid, incompressible</td>
<td>shell</td>
</tr>
<tr>
<td>Boujot (1972)</td>
<td>inviscid, incompressible</td>
<td>elastic solid</td>
</tr>
<tr>
<td>Berger et al. (1975)</td>
<td>inviscid, incompressible</td>
<td>elastic solid</td>
</tr>
<tr>
<td>Hamdi et al. (1978)</td>
<td>inviscid, compressible</td>
<td>elastic solid</td>
</tr>
<tr>
<td>Morand &amp; Ohayon (1979)</td>
<td>inviscid, compressible</td>
<td>elastic solid</td>
</tr>
<tr>
<td>Su (1981)</td>
<td>viscous, compressible</td>
<td>shell</td>
</tr>
<tr>
<td>Denenuvy (1988)</td>
<td>inviscid, compressible</td>
<td>thick plate</td>
</tr>
<tr>
<td>Capodanno (1988)</td>
<td>inviscid, incompressible</td>
<td>membrane</td>
</tr>
</tbody>
</table>

Table 5.1:
is, however, important. Su (1981) argues, in addition, that the hydrodynamic forces on the structure due to the boundary layer which forms in the vicinity of the structure, may also affect vibration characteristics of the system.

In this chapter we will consider the motion of an incompressible, viscous fluid in an open flexible container. The flexible container wall is modelled by a membrane. In section 5.2 the problem to be considered is introduced and the governing equations are presented. Section 5.3 deals with the deflection of the membrane due to the hydrostatic pressure. In section 5.4 we consider the related problem of inviscid fluid oscillations and are able to show analytically, that under certain assumptions the eigenfrequencies decrease when a rigid wall is replaced by a membrane. In section 5.5 a number of qualitative properties of the spectral problem, which results from the linearized Navier-Stokes equations, are derived. We find that three types of normal modes exist namely, modes related to free surface oscillations, modes related to membrane vibrations and modes with a coupled free-surface and membrane character. The three types of modes have different damping characteristics and depend differently on the parameters. In section 5.6 numerical results are presented. Particular emphasis is placed on the fluid motion resulting from normal mode oscillations and the damping characteristics of various types of normal modes. Both of these aspects of fluid-structure interaction problems have not received a great deal of attention so far. We also investigate the case in which the eigenfrequencies of the free surface oscillations and membrane vibrations are close. The Reynolds number is found to be an important parameter in this case.

5.2 Problem formulation

Let us consider a two-dimensional container $C$ with boundary $\partial C$, partly filled with a viscous fluid. Unlike in the previous chapter, rotational and surface tension effects are neglected, but it is assumed instead that part of the container wall $\partial C$ may be flexible. We assume a Cartesian coordinate system and denote the rigid part of the container wall by $R$, the flexible part by $F$ and the free surface of the liquid by $S$, as shown in figure 5.1

The equations describing the fluid motion are, as before, given by (4.2.1) (with $\omega = 0$) and (4.2.2). On the rigid boundary $R$ the no-slip condition (4.2.3) is maintained. The free surface conditions (4.2.5) and (4.2.6) are simplified when surface tension forces are neglected. The conditions reduce
to

\[ \sigma_n = -p_g, \quad \sigma_r = 0. \] (5.2.1)

The dynamic condition (4.2.7) is retained while the contact angle condition (4.2.8) need not be taken into account. The flexible part \( F \) of the container wall is modelled by a membrane which yields the normal stress condition

\[ \sigma_n = -p_g + \frac{T}{R_c} - \rho_m \frac{\partial^2 \gamma}{\partial t^2}, \] (5.2.2)

in which \( R_c \) denotes the radius of curvature of the membrane considered positive when the corresponding centre of curvature lies outside the liquid. The equation of the membrane is given by \( \gamma(x, t) = \mathbf{n} \cdot (\mathbf{x}_F - \mathbf{x}) = 0 \) in which \( \mathbf{x}_F \) denotes the position vector of \( F \) relative to some origin. We assume the tension in the membrane to be constant. This implies that the membrane can not experience displacements in its plane, and hence the fluid motion tangential to the membrane is zero, that is

\[ \mathbf{u} \cdot \mathbf{\tau} = 0 \quad \text{on} \quad F \] (5.2.3)

On \( F \) we also have the kinematic condition

\[ \frac{D}{Dt} \gamma(x, t) = 0. \] (5.2.4)

The end points of the membrane are assumed fixed to rigid parts of the container wall so that at the points of intersection of \( F \) and \( R \) the condition \( \mathbf{u} = 0 \) is prescribed. The last condition to be prescribed is the volume constraint as given by equation (4.2.9).

Like in the previous chapter, we first determine the steady-state configuration of the membrane and the free surface. The problem is then linearized.
by considering small perturbations relative to this steady state. The procedure is analogous to that outlined in the section 4.2 so that we will suffice with stating the result. The linearized equations in dimensionless form describing the fluid motion, are given by

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \nabla p &= \frac{1}{Re} \nabla^2 \mathbf{u} \\
\nabla \cdot \mathbf{u} &= 0 \\
\end{align*}
\] in \( \Omega_0, \) (5.2.5)

together with the boundary conditions

\[
\begin{align*}
\sigma_n &= -\eta \\
\sigma_t &= 0 \\
\mathbf{u} \cdot \mathbf{n} &= \frac{\partial \eta}{\partial t} \\
\end{align*}
\] on \( S_0, \) (5.2.6)

\[
\mathbf{u} = 0 \text{ on } R, \quad \sigma_n = -\xi \hat{Y} \cdot \mathbf{n}_F + \frac{1}{\beta} \left( \frac{\xi}{R \zeta_0} + \frac{\partial^2 \xi}{\partial s^2} \right) - r_p \frac{\partial^2 \xi}{\partial t^2} \\
\mathbf{u} \cdot \mathbf{r} &= 0 \\
\mathbf{u} \cdot \mathbf{n} &= \frac{\partial \xi}{\partial t} \\
\end{align*}
\] on \( F_0. \) (5.2.8)

In the above equations the subscript 0 refers to steady-state parameters, \( \eta(x,t) \) and \( \xi(x,t) \) denote the small normal displacements of the free surface and membrane respectively and \( s \) is a curvilinear coordinate along \( F_0. \) We have introduced the dimensionless numbers \( \text{Re} = lU/\nu \) being the Reynolds number (\( l \) is a length scale and \( U = \sqrt{gl} \) a velocity scale), \( \beta = \rho_f gl^2 / t \) and \( r_p = \rho_m / l \rho_f. \) The quantity \( \beta \) is akin to the Bond number introduced earlier, and is a measure of the ratio of gravitational forces and tension forces in the membrane.

As mentioned earlier, we have assumed that the tension \( T \) in the membrane is constant. However, since the membrane is not massless, the gravitational force will create a tension difference in the membrane which is of order \( O(\beta), \) so that strictly speaking only small values of \( \beta \) may be considered. For simplicity this effect will be neglected.

### 5.3 Static deflection of the membrane

Before equations (5.2.5)-(5.2.8) are solved we need to determine the static deflection of the membrane due to the hydrostatic pressure of the fluid.
Note that since surface tension effects have been neglected, the free surface $S_0$ will be flat and perpendicular to the direction of the gravitational field (the $y$-axis). From the momentum equation with $u \equiv 0$ and the normal stress relation on $F$ it follows that the shape of the membrane is defined by

$$\frac{1}{R_{C_0}} = -\beta \hat{k} \cdot (x_{F_0} - x_{S_0}) \quad (5.3.1)$$

where $\hat{k}$ is a unit vector in the direction of the positive $y$-axis and $x_{S_0}$ denotes the position vector of the free surface. Together with the volume constraint, viz.

$$\int_{\Omega_0} dx = V_f, \quad (5.3.2)$$

equation (5.3.1) defines the shape of $F_0$ and the position of $S_0$. The problem to be solved is somewhat analogous to the capillary free-boundary problem dealt with in chapter 2. However, since the end points of the membrane are fixed, equations (5.3.1) and (5.3.2) are solved on a fixed domain thus reducing the difficulty of the problem.

Let us assume that the position of the free surface in the container is kept fixed (by adjusting the volume of fluid) when the tension in the membrane is varied. In the specific case as depicted in figure 5.2, equation (5.3.1) is equivalent to

$$\frac{f''(y)}{(1 + f(y))^3/2} = \beta(y - h), \quad f(0) = L, \quad f(a) = 0, \quad (5.3.3)$$

where the primes denote differentiation with respect to $y$. In the case where $\beta \ll 1$ we can approximate the solution to (5.3.3) analytically. A perturbation approach where $f(y)$ is expanded in powers of $\beta$ yields

$$f(y) = L(1 - y/a) + \frac{\beta}{6}[1 + (L/a)^2][y^3 - 3hy^2 + a(3h - a)y] + O(\beta^2). \quad (5.3.4)$$

We note that in case the volume constraint is to be satisfied, the free-surface height $h$ is obtained by substituting (5.3.4) in equation (5.3.2) which then yields an expression of the fluid volume in terms of $h$.

For the numerical solution of (5.3.3) we define the function

$$F(f) = f'' - \beta(y - h)(1 + f'^2)^{3/2},$$

where $F(f) = 0$ is the equation to be solved subject to appropriate boundary conditions. In the case where the membrane deflection is written as a
function of $x$ a similar expression holds. The non-linear function $F(f) = 0$ is solved using Newton’s method. To that end we define the sequence $f^{(1)}(y), f^{(2)}(y), \ldots, f^{(n)}(y) \rightarrow f(y)$ by the following linear problem

$$F'(f^{(n)})[f^{(n+1)}] = F'(f^{(n)})[f^{(n)}] - F(f^{(n)}),$$

(5.3.5)

where $F'(\phi)[\psi]$ denotes the Gateaux derivative of $F$ defined by

$$F'(\phi)[\psi] = \lim_{\delta \to 0} \frac{1}{\delta} [F(\phi + \delta \psi) - F(\phi)].$$

The linear problem defined by (5.3.5) can be solved using standard finite difference techniques. Each iteration requires the solution of a system of equations, while the sequence $f^{(1)}, f^{(2)}, \ldots$ converges quadratically. In figure 5.3 we show shapes of the membrane for various values of $\beta$. Figure 5.3a shows the membrane shapes for $L = 0.3$, $a = 0.9$ and $h = 1.0$. Analytical results correspond to the dotted lines. The numerical procedure converges for values of $\beta$ up to $\beta \approx 3.85$. Near that value of $\beta$ the derivative of $f(y)$ at $y = 0$ tends to infinity. For larger values of $\beta$ a Cartesian function representation of the membrane is no longer possible since the function would be multi-valued for $y \leq 0$. The approach outlined in chapter 2, where the equations are written in polar coordinates, is essential in that case. Figure 5.3b shows the shape of a vertically loaded membrane. For $\beta > 2.41$ the numerical procedure diverges indicating the stability limit of the problem. This stability limit is analogous to that of a pendant liquid drop. For a certain value of the surface tension and the gravitational field (essentially the Bond number) there is a maximum drop size - larger drops are not stable.
Figure 5.3: Plots of the deflected membrane for values of $\beta$ as indicated in the plots. The dotted lines in (a) correspond to the analytical result (5.3.4).

## 5.4 Inviscid fluid oscillations

Before we proceed to solve problem (5.2.5)-(5.2.8) numerically, we investigate the related problem of an inviscid fluid oscillating in a flexible container. Under certain conditions we are able to solve the equations analytically which gives us more insight into the problem. In this section we consider the inviscid equivalent of problem (5.2.5)-(5.2.8), i.e. neglect all the terms containing the Reynolds number and replace the no-slip condition on $R$ by the non-permeability condition. In analogy with chapter 3 we find that the problem for the velocity potential $\phi$ with a temporal behaviour of the form $e^{i\omega t}$ is as follows:

\begin{align*}
\nabla^2 \phi &= 0 \quad \text{in } \Omega_0, \\
\frac{\partial \phi}{\partial n} &= \omega^2 \phi \quad \text{on } S_0, \\
\frac{\partial \phi}{\partial n} &= 0 \quad \text{on } R, \\
\frac{\partial^2}{\partial s^2} \left(\frac{\partial \phi}{\partial n}\right) + \left(\frac{1}{R_{E_0}^2} + \beta(r_p\omega^2 - \dot{y} \cdot n_F)\right) \frac{\partial \phi}{\partial n} &= -\beta \omega^2 \phi \quad \text{on } F_0, \\
\frac{\partial \phi}{\partial n} &= 0 \quad \text{on } R \cap F_0.
\end{align*}

(5.4.1)  
(5.4.2)  
(5.4.3)  
(5.4.4)  
(5.4.5)
Note that $\eta$ and $\xi$ have been eliminated using the kinematic conditions on $S_0$ and $F_0$.

We will attempt to solve (5.4.1)-(5.4.5) within the context of perturbation theory with $\beta \ll 1$ as the perturbation parameter. However, the fact that the domain $\Omega_0$ depends on $\beta$ through the shape of the wall $F_0$ poses a problem: functions defined on the zeroth order domain will in general not be defined on the whole of the domain of the first order problem. This is clearly undesirable. Let us therefore assume in this section that the shape of $F_0$ does not depend on $\beta$, i.e. we neglect the bending of the membrane due to the hydrostatic pressure. Note that the boundary conditions on $F_0$ have a form similar to the capillary boundary conditions as defined in section 3.2. We use the same procedure as outlined in appendix A to rewrite conditions (5.4.4) and (5.4.5). Let the membrane be fixed at the points $s = 0$ and $s = a$ (the begin and end points of $F_0$), conditions (5.4.4) and (5.4.5) can then be shown to satisfy

$$
\frac{\partial \phi}{\partial n} = \beta \omega^2 \int_{F_0} K_\beta(s, \theta) \phi(\theta) d\theta \quad \text{on} \quad F_0,
$$

in which

$$
K_\beta(s, \theta) = \begin{cases} 
\frac{1}{\kappa \tan(\kappa a)} \sin(\kappa s)[\tan(\kappa a) \cos(\kappa \theta) - \sin(\kappa \theta)], & s \leq \theta \\
\frac{1}{\kappa \tan(\kappa a)} \sin(\kappa \theta)[\tan(\kappa a) \cos(\kappa s) - \sin(\kappa s)], & s > \theta,
\end{cases}
$$

with

$$
\kappa = \sqrt{\beta (r_\rho \omega^2 - \hat{y} \cdot \mathbf{n}_F)^{1/2}}.
$$

Multiplying equation (5.4.1) by $\phi$, integrating over $\Omega_0$, applying Green's theorem and substituting for the boundary conditions (5.4.2), (5.4.3) and (5.4.6) yields

$$
\omega^2 = \frac{\int_{\Omega_0} |\nabla \phi|^2 dx}{\int_{S_0} \phi^2 ds + \beta \int_{F_0} \phi(s) \int_{F_0} K_\beta(s, \theta) \phi(\theta) d\theta ds}.
$$

(5.4.7)

Since $K_\beta(s, \theta)$ is a non-linear function in $\omega^2$, it follows that, given $\phi$, equation (5.4.7) is a non-linear equation of which the roots correspond to the eigenfrequencies $\omega^2$. Let us consider the case in which $\beta \ll 1$, and assume that $\phi$ can be expanded in a power series of $\beta$ like,

$$
\phi = \phi^{(0)} + \beta \phi^{(1)} + \ldots
$$

83
Substituting for $\phi$ into (5.4.7) yields, on neglecting all terms of second order in $\beta$,

$$
\omega^2 = \omega_0^2 \left[ 1 - \beta \frac{\int_{F_0} F_0 \int K_0(s, \theta)\phi^{(0)}(\theta)d\theta ds}{\int_{S_0} \phi^{(0)}(\theta)^2 ds} + O(\beta^2) \right],
$$

(5.4.8)

where

$$
\omega_0^2 = \frac{\int_{\Omega_0} |\nabla \phi_0|^2 dx}{\int_{S_0} \phi_0^2 ds},
$$

is the familiar expression for the eigenfrequencies in terms of the eigenfunctions of an inviscid fluid in a rigid container. The kernel $K_0(s, \theta)$ is given by

$$
K_0(s, \theta) = \begin{cases} 
  s(a - \theta)/a, & s \leq \theta \\
  \theta(a - s)/a, & s > \theta.
\end{cases}
$$

Depending on the properties of the kernel $K_0(s, \theta)$ the eigenfrequencies $\omega$ in (5.4.8) will be greater or less than $\omega_0$. The integral operator

$$
K_0 f = \int_{s=0}^{s=a} K_0(s, \theta)f(\theta)d\theta,
$$

is in fact positive as can be shown readily. It follows that $\omega < \omega_0$ since the term containing $\beta$ in (5.4.8) is strictly positive. Hence, the eigenvalues of an inviscid fluid in a flexible container are smaller than the corresponding eigenvalues in a similarly shaped rigid container. Note that this result can be generalized easily to the case in which the container wall consists of a number of distinct membranes each with fixed end points. The integral in the numerator of (5.4.8) is in that case replaced by the sum of the integrals over the membranes. The fact that the modulus of the eigenvalues decreases when part of the rigid container wall is replaced by a membrane is what one might have expected on intuitive grounds since decreasing the rigidity of a system generally lowers the eigenfrequencies.

Next we investigate the specific case of the rectangular container of length $L$ and filled with fluid to a height $h$. The side wall at $x = 0$ and the bottom wall at $y = 0$ are assumed rigid, the side wall at $x = L$ is taken to be flexible in the interval $y \in (0, a)$ and rigid for $y \in [a, h]$. The zeroth
order problem ($\beta = 0$) can be solved readily using separation of variables to give
\[
\phi_k^{(0)} = \cos \frac{k\pi x}{L} \cosh \frac{k\pi y}{L}, \quad \omega_k^{(0)} = \frac{k\pi}{L} \tanh \frac{k\pi h}{L},
\]
where the subscript $k$ refers to the $k$'th eigenmode. Substituting $\phi_k^{(0)}$ and $\omega_k^{(0)}$ into (5.4.8) yields
\[
\omega_k = \omega_k^{(0)} \left[ 1 - \frac{\beta a^3}{2L q^2 \cosh^2(k\pi h/L)} \left( \frac{\sinh 2q}{2q} - 1 - \frac{2}{q^2} (\cosh q - 1)^2 \right) \right],
\]
in which $q = k\pi a/L$. In figures 5.4a,b we show plots of $\Delta \omega_k = (\omega_k^{(0)} - \omega_k)/\omega_k^{(0)}$ versus the container length for various values of $a$ and $k = 1, 2$. Observe first of all that the influence of the membrane on the eigenfrequencies decreases as $a$ decreases and that the interaction of the fluid and the structure is strongest for the first eigenmode. In order to explain these ob-

![Figure 5.4: Plots of $\Delta \omega_k$ versus $L$ for $h = 1$ and various values of $a$, for the first (a) and second (b) eigenmode.](image)

servations we use the well-known fact that the fluid motion resulting from free-surface oscillations penetrates the fluid to a depth of about one wavelength (cf. Lamb, 1932) and that the fluid velocity is maximal near the free surface. It is evident that as $a$ decreases the effect of the membrane on the eigenfrequencies will decrease simply because a smaller part of the container wall is flexible. In addition however, for small values of $a$ the membrane will be in contact with a region of fluid in which relatively low pressure gradients occur (as a result of the low fluid velocity) so that the fluid is capable of
deflecting the membrane only slightly. The change of the eigenfrequencies of the fluid will be correspondingly small. A similar argument holds in order to explain the observation that the interaction is strongest for the first mode and weaker for subsequent modes.

Interesting is the observation that $\Delta \omega$ has a well-defined maximum for some value of $L$. The position of the maximum and its height are dependent on both $a$ and $k$: for $a = 1.0, k = 1$ the maximum occurs at $L = 3.02$ while for $a = 1.0, k = 2$ the maximum is situated at $L = 6.03$. It is easy to show that $\Delta \omega_k$ tends to zero in the limits $q \to 0$ and $q \to \infty$. A remark is in place regarding the results presented in figures 5.4a,b. We have taken $h = 1$ and calculated $\Delta \omega_k$ for container lengths up to $L = 9$. The linear theory of small free-surface deflections is, however, valid only when the elevation of the free surface is small compared with the depth of the fluid. Hence, the results in figures 5.4a,b for $L \geq 3$ should be treated with caution. However, we do expect $\Delta \omega_k$ to decrease as $L$ increases because the potential energy of the fluid increases (proportional to $L$) while the potential energy of the membrane remains constant if we increase $L$. Thus the effect of the membrane on the fluid motion will decrease.

We conclude this section by considering the case in which the side walls of the rectangular container are rigid but the entire bottom wall consists of a membrane. The solution of the zeroth order problem is as in the previous case. Substituting for $\phi_k^{(0)}$ and $\omega_k^{(0)}$ in (5.4.8), where the integrals in the numerator are now over the bottom wall, we obtain

$$\omega_k = \omega_k^{(0)} \left[ 1 - \frac{\beta}{2} \left( \frac{L}{n\pi} \right)^2 \frac{1}{\cosh^2(n\pi h/L)} \left( 1 - \frac{2}{(n\pi)^2} [1 - (-1)^n]^2 \right) \right].$$

Note that $\Delta \omega_k \to 0$ as $L \to 0$, but unlike the previous case $\Delta \omega_k \propto L^2$ for large $L$, i.e. the interaction between fluid and membrane motion increases as $L$ increases. This may be explained by the fact that the total energy of both the fluid and the membrane increases as $L$ increases.

### 5.5 Viscous oscillations: properties of the spectrum

We assume, as before, that all time-dependent quantities exhibit a temporal behaviour of the form $e^{\lambda t}, \lambda \in C$. In order to obtain the variational formulation of the problem defined by (5.2.5)-(5.2.8), the Green formula (4.2.7)
is used. On substituting the boundary conditions and eliminating the free-
surface and membrane displacements by virtue of the kinematic conditions,
it follows that the variational formulation of problem (5.2.5)-(5.2.8) is given by

\[
\int_{\Omega_0} (\lambda u \cdot v - p \nabla \cdot v) dx + \frac{1}{Re} a(u, v) + \int_{S_0} \frac{1}{\lambda} u_n v_n ds + \\
\int_{\partial \Omega_0} \left( \lambda r \nabla u_n \cdot v_n + \frac{1}{\lambda} u_n v_n \hat{y} \cdot n_F + \frac{1}{\lambda \beta} \left( \frac{\partial u_n}{\partial s} \frac{\partial v_n}{\partial s} - \frac{u_n v_n}{Rc_0} \right) \right) ds = 0, \quad (5.5.1)
\]

\[
\int_{\Omega_0} q \nabla \cdot u dx = 0, \quad (5.5.2)
\]

where \( u \) and \( p \) are in general complex vector functions and the functional
\( a(u, v) \) is defined by

\[
a(u, v) = \frac{1}{2} \sum_{i,j=1}^{2} \int_{\Omega_0} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) dx, \quad x_1 = x, \quad x_2 = y.
\]

It is important to note that all degrees of freedom related to the mem-
brane have been eliminated from the problem by virtue of the kinematic
condition. We are left with a problem in which only quantities related to
the fluid (fluid velocity and pressure) are unknowns.

Let us now investigate the spectrum of the problem defined by equations
(5.5.1) and (5.5.2). To that end we consider the quadratic functional

\[
\Phi = \lambda^2 [(u, u)_{\Omega_0} + r(u_n, u_n)_{\partial \Omega_0}] + \lambda \frac{1}{Re} a(u, u) + (u_n, u_n)_{S_0} + \frac{1}{\beta} b(u_n, u_n).
\]

The functional \( \Phi \) takes on a stationary value (equal to zero) when \( u \) is a
solution of equations (5.5.1), (5.5.2). We have introduced the notation

\[
(u, u)_{\Omega_0} = \int_{\Omega_0} |u|^2 dx, \quad \cdots \quad \text{etc.},
\]

87
\[ b(u_n,u_n) = \int_{F_0} \left( u_n u_n (\beta \dot{y} \cdot n_F - \frac{1}{R_{C_0}^2}) + \frac{\partial u_n}{\partial s} \frac{\partial u_n}{\partial s} \right) ds. \]

Set \( \Phi \) equal to zero and treat the resulting equation as a quadratic in \( \lambda \). Solving for \( \lambda \) we can deduce the following qualitative properties concerning vibration modes of a viscous fluid in an elastic container:

(i) The problem as defined by (5.5.1), (5.5.2) is stable with respect to infinitesimal perturbations when \( (u_n,u_n)_{s_0} + \frac{1}{\beta} b(u_n,u_n) > 0; \)

(ii) If
\[
\frac{1}{Re^2} (a(u,u))^2 \geq 4[(u,u)_{\Omega_0} + r_p(u_n,u_n)_{F_0}][(u_n,u_n)_{s_0} + \frac{1}{\beta} b(u_n,u_n)]
\]
then the eigenvalues corresponding to the eigenfunctions \( u \) are real and negative. This corresponds to an aperiodic damping process;

(iii) If
\[
\frac{1}{Re^2} (a(u,u))^2 < 4[(u,u)_{\Omega_0} + r_p(u_n,u_n)_{F_0}][(u_n,u_n)_{s_0} + \frac{1}{\beta} b(u_n,u_n)]
\]
then the eigenvalues are complex occurring in complex conjugate pairs. The real part of the eigenvalues is negative so that oscillations are damped.

It is easy to show that the stability condition in (i) is satisfied for sufficiently small values of \( \beta \) (by virtue of the fact that \( 1/R_{C_0} \sim \beta \), cf. section 5.3). For large values of \( \beta \) condition (i) may not be satisfied.

Let us consider the expression for \( \lambda \) in some more detail when \( Re \) is large. Retaining only terms to first order in \( 1/Re \) we obtain

\[
\lambda \approx \frac{-a(u,u)}{2Re[(u,u)_{\Omega_0} + r_p(u_n,u_n)_{F_0}]} \pm i \left( \frac{(u_n,u_n)_{s_0} + \frac{1}{\beta} b(u_n,u_n)}{(u,u)_{\Omega_0} + r_p(u_n,u_n)_{F_0}} \right)^{1/2}. \text{(5.5.3)}
\]

Now assume that for a certain set of parameter values eigenfunctions \( u \) exist such that the terms associated with the inertia and bending energies of the membrane, \( (u_n,u_n)_{F_0} \) and \( b(u_n,u_n) \) respectively, are small in comparison with the fluid inertia term \( ((u,u)_{\Omega_0}) \) and the free-surface energy term.
\((u_n, u_n)_{S_0}\). Neglecting the terms related to the membrane motion, we find that (5.5.3) reduces to

\[
\lambda \approx \frac{-a(u, u)}{2Re(u, u)_{\Omega_0}} \pm i \left( \frac{(u_n, u_n)_{S_0}}{(u, u)_{\Omega_0}} \right)^{1/2}.
\] (5.5.4)

Equation (5.5.4) is the familiar expression for the eigenfrequencies of weakly damped free-surface oscillations of a viscous fluid in a rigid container expressed in terms of the corresponding eigenfunction, see for example Kopachevskii & Myshkis (1966). In sequel these modes will be termed free-surface oscillation modes.

Assume, on the other hand, that eigenfunctions exist such that the free-surface energy term is small in comparison with the term referring to the bending of the membrane. Neglecting the free-surface energy term in (5.5.3) yields

\[
\lambda \approx \frac{-a(u, u)}{2Re[(u, u)_{\Omega_0} + r_\rho(u_n, u_n)_{F_0}]} \pm i \left( \frac{1/\beta b(u_n, u_n)}{(u, u)_{\Omega_0} + r_\rho(u_n, u_n)_{F_0}} \right)^{1/2}.
\] (5.5.5)

Observe that \(\text{Im}(\lambda)\) in (5.5.5) is proportional to \(1/\sqrt{\beta}\). The functional dependence on \(\beta\) is in fact the same for a freely vibrating membrane (recall that eigenfrequencies of a membrane are proportional to the square root of the tension). We conclude that eigenmodes, for which the energy of free-surface oscillations is small compared with the bending energy of the membrane, are associated with normal mode oscillations of the membrane. These modes will be called membrane-vibration modes. We note that a vibrating membrane necessarily causes the fluid to vibrate as well. Membrane-vibration modes should therefore be regarded as normal modes in which the displacement of the membrane is akin to that of normal modes of a freely vibrating membrane. Observe that \(\text{Re}(\lambda)\) in (5.5.5) contains an extra term in the denominator as compared with \(\text{Re}(\lambda)\) in (5.5.4). This indicates that the membrane-vibration modes may be damped less than the free-surface oscillation modes. A third type of mode can be expected namely, a mode in which both the free surface and the membrane experience modal deflections simultaneously. Properties other than the mixed character of these modes, are not easily identifiable.

Of interest is the term representing the inertia of the fluid in the denominator of \(\text{Im}(\lambda)\) in (5.5.5). It is not hard to show that the eigenfrequencies of a freely vibrating membrane in terms of the eigenfunctions are given
by the expression $\text{Im}(\lambda)$ in (5.5.5), however without the fluid inertia term ($\text{Im}(\lambda)$ is in that case just the Rayleigh quotient where $u_n$ corresponds to the deflection of the membrane). It follows that the eigenfrequencies of a membrane in contact with a fluid will be less than those of a freely vibrating membrane. It appears, therefore, that the effect of the fluid is essentially to increase the mass of the membrane. This observation is related to the concept of added-mass. Namely, when solving fluid-structure interaction problems numerically one can, in certain cases, eliminate the unknowns related to the fluid motion by adding an "added-mass" matrix to the mass matrix of the dry structure, refer to for example Deruntz & Geers (1978) and Müller (1981).

The finite-element discretization of equations (5.5.1) and (5.5.2) proceeds along the same lines as outlined in section 4.4. The resulting eigenvalue problem is, once again, of the form

\[(\lambda^2 M_T + \lambda S + B_T)\ddot{u} = 0, \tag{5.5.6}\]

in which

\[M_T = M + M_{F_0}, \quad S = \frac{1}{\epsilon_p}L^TD^{-1}L + \frac{1}{Re}A, \quad B_T = B_{S_0} + B_{F_0}.\]

All matrices are real and symmetric, and $M_T, S$ are positive definite. $M$ and $M_{F_0}$ are mass matrices corresponding to the fluid and membrane motions respectively; $M_{F_0}$ is singular. $B_{S_0}$ and $B_{F_0}$ are matrices representing the potential energies of the perturbed free surface and membrane respectively and are both singular since only the degrees of freedom corresponding the free surface and the membrane give non-zero entries.

Assume a spatial discretization such that $n_S$ degrees of freedom on $S_0$ give non-zero entries in $B_{S_0}$ so that $\text{rank}(B_{S_0}) = n_S$. Likewise assume that $\text{rank}(B_{F_0}) = n_F$ and that the total number of degrees of freedom is equal to $n$. Since $\text{rank}(B_T) = \text{rank}(B_{S_0}) + \text{rank}(B_{F_0}) = n_S + n_F$, direct application of theorem 4.1 shows that $n - n_S - n_F$ eigenvalues of (5.5.6) are zero, $n - n_S - n_F$ are real and negative and the remaining $2n_S + 2n_F$ eigenvalues may be non-real. Those eigenvalues refer to free-surface oscillation modes ($2n_S$ in number) and membrane-vibration modes ($2n_F$ in number). The properties of these modes are as formulated earlier in this section.
5.6 Numerical results

We start our numerical experiment with a comparison of numerical results and the analytical results of section 5.4. To that end we consider a rectangular container of length $L$, height $h = 1$ and a partly flexible side wall with $a = 0.9$ (cf. the notation in section 5.4). We take $\beta = 0.1$ and assume that the container is entirely filled with a viscous fluid with $Re = 5 \times 10^2$. The region $\Omega_0$ is discretized by 504 triangular elements, refined at the bound-

![Graph of $\Delta \text{Im}(\lambda_1)$ and $\Delta \omega_1$ versus $L$.](image)

Figure 5.5: Plots of $\Delta \text{Im}(\lambda_1)$ and $\Delta \omega_1$ versus $L$. The solid line is the analytical curve for $\Delta \omega_1$ obtained from (5.4.9), the values of $\Delta \text{Im}(\lambda_1)$ are denoted by a $\Box$.

aries in order to capture the boundary layer. In the numerical calculation the bending of the membrane due to the hydrostatic pressure is not neglected so that the shape of $F_0$ is given by the solution of (5.3.1). In figure 5.5 we have indicated (by a square) the value of $\Delta \text{Im}(\lambda_1)$ for various values of the container length $L$, where $\Delta \text{Im}(\lambda_1)$ is given by

$$\Delta \text{Im}(\lambda_1) = \frac{\text{Im}(\lambda_r) - \text{Im}(\lambda_1)}{\text{Im}(\lambda_r)},$$

in which $\lambda_1$ is the first eigenmode of the fluid and $\lambda_r$ the corresponding mode in a similarly shaped rigid container. We have also plotted the curve $\Delta \omega_1$ versus $L$ where $\Delta \omega_1$ is obtained from (5.4.9). Observe that, eventhough viscous effects and bending of the membrane are neglected in the analytical calculation, there is a close agreement between the numerical and analytical results.

As was pointed out in section 5.5, we expect to observe at least two different types of normal oscillation modes namely, free-surface modes and
membrane-vibration modes. We now investigate some of the characteristics of these modes. Consider the same container as in the previous experiment with $L = 1$ and choose the parameter values $Re = 5 \times 10^2$, $\beta = 0.2$ and $\tau_p = 1$. In figure 5.6 vector plots of the fluid velocity are shown of the first two free-surface modes (5.6a,b) and the first and second membrane-vibration modes (5.6c,d). Observe that, for the present parameter values, the membrane is hardly deflected in the case of the free-surface modes. In the case of the principal membrane-vibration mode the membrane is deflected significantly causing the free surface to be displaced as well. In the case of the second membrane-vibration mode the membrane is deflected significantly while the free surface is hardly effected. The calculated eigenvalues corresponding to both types of oscillation modes are shown in table 5.2. Note that the damping coefficients of the membrane-vibration modes are signifi-

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>type of mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.097 + 1.76i</td>
<td>first free-surface mode</td>
</tr>
<tr>
<td>-0.21 + 2.49i</td>
<td>second free-surface mode</td>
</tr>
<tr>
<td>-0.038 + 6.20i (7.81)</td>
<td>first membrane-vibration mode</td>
</tr>
<tr>
<td>-0.052 + 14.1i (15.6)</td>
<td>second membrane-vibration mode</td>
</tr>
</tbody>
</table>

Table 5.2: Eigenfrequencies of free-surface and membrane-vibration modes. The numbers in brackets are eigenfrequencies of a freely vibrating membrane for the same parameter values.

cantly less than those of the free-surface modes corresponding to the same modal number. The number in brackets behind the eigenfrequencies of the first and second membrane vibration modes in table 5.2, are the first and second eigenfrequencies respectively of a freely vibrating membrane. We observe that the inertia of the fluid lowers the imaginary part of the eigenfrequencies. These results are in accordance with the qualitative remarks made in section 5.5.

We next investigate how eigenfrequencies corresponding to the free-surface and membrane-vibration modes vary as a function of the $\beta$. Since the functional dependence is, as we shall find, critically dependent on the Reynolds number, we perform the calculations for two representative values of the Reynolds number, namely $Re = 5 \times 10^2$ and $Re = 5 \times 10^3$. The container to be considered is as in the previous calculations with $L = 1$, $a = 0.9$ and $\tau_p = 1$. We restrict our investigation to the dependence on $\beta$ of the
Figure 5.6: Velocity vector plots of the first (a) and second (b) free-surface modes and the first (c) and second (d) membrane-vibration modes.
second free-surface mode (denoted by $\lambda_f$) and the first membrane-vibration mode (denoted by $\lambda_s$). $\beta$ is taken in the range $0.9 \leq \beta \leq 1.6$. In figure 5.7a we have plotted $\text{Im}(\lambda_f)$ (the dotted line) and $\text{Im}(\lambda_s)$ (the drawn line) versus $\beta$ for the case $Re = 5 \times 10^2$. Observe that $\text{Im}(\lambda_f)$ is virtually independent of $\beta$. In fact $\text{Im}(\lambda_f) \approx 2.5$ which is equal to the imaginary part of the second free-surface mode in a rigid rectangular container with the same dimensions. The drawn line in figure 5.7a is a curve proportional to $1/\sqrt{\beta}$ and intersects the curve of $\text{Im}(\lambda_f)$ near $\beta \approx 1.2$. The dependence of the real parts of $\lambda_s$ and $\lambda_f$ on $\beta$ is shown in figure 5.7b. Note that $\text{Re}(\lambda_f)$ and $\text{Re}(\lambda_s)$ do not change significantly with $\beta$ and that $\text{Re}(\lambda_f) \ll \text{Re}(\lambda_s)$.

Let us next consider the case in which $Re = 5 \times 10^3$. In figure 5.8a we have plotted $\text{Im}(\lambda)$ corresponding to the free-surface and membrane-vibration modes as a function of $\beta$. Figure 5.8b shows the dependence on $\beta$ of the corresponding values of $\text{Re}(\lambda)$. We observe that for $\beta < 1.2$ the dotted curve behaves quite similar to the dotted curve in figure 5.7a, i.e. it represents the free-surface mode which is independent of $\beta$. However, near $\beta \approx 1.2$ (the point at which the curves in figure 5.7a intersect) the dotted curve starts to decrease and its shape starts to resemble that of the drawn curve in figure 5.7a. This suggests that the free-surface mode has changed over into a membrane-vibration mode. Inspection of the corresponding eigenfunctions reveals that this is indeed the case. Regarding the drawn line in figure 5.8a we observe the opposite behaviour: it first represents membrane-vibration modes but changes over into a branch representing free-surface modes after $\beta \approx 1.2$. When we consider $\text{Re}(\lambda)$ of the corresponding modes (figure 5.8b), we observe a rapid decrease in $\text{Re}(\lambda)$ as the free-surface mode changes over into a membrane vibration mode (dotted line) and the opposite behaviour when the membrane mode changes over into a free-surface mode (drawn line). The qualitative statements of section 5.5 regarding free-surface and membrane-vibration modes are, apart from the behaviour in the transition region near $\beta = 1.2$, still valid. Namely, free-surface modes are independent of $\beta$ and damped more than the membrane-vibration modes, the membrane-vibration modes display a $1/\sqrt{\beta}$ behaviour. In the transition region near $\beta \approx 1.2$ the eigenfunctions have a coupled free-surface structural-vibration character, i.e. both the membrane and the free surface experience modal vibrations. We find that these coupled modes occur for both values of the Reynolds number, regardless of whether the branches in figures 5.7a or 5.8a intersect or not.

The bifurcation behaviour of the curves in figure 5.8a is typical for the case when branches, representing two types of eigenmodes of a system with-
Figure 5.7: Figure a shows the dependence on $\beta$ of $\text{Im}(\lambda_f)$ (indicated by the dotted line) and $\text{Im}(\lambda_s)$ (indicated by the drawn line). The curves $\text{Re}(\lambda_f)$ (dotted line) and $\text{Re}(\lambda_s)$ (drawn line) versus $\beta$ are shown in figure b. Calculations are for the case $Re = 5 \times 10^2$.

Figure 5.8: Figures a and b show the dependence on $\beta$ of the imaginary and real parts respectively of the eigenfrequencies corresponding to the free-surface and first membrane-vibration mode for $Re = 5 \times 10^3$. The imaginary part indicated by the dotted line in a corresponds to the real part indicated by the dotted line in b and likewise for the drawn lines.
out damping, meet for a certain parameter value. Su (1981) found that the bifurcation is not removed when viscous effects are included, agreeing with our results. We find that only for sufficiently high damping the bifurcation behaviour vanishes (figure 5.7a, low Reynolds number). Note that in the complex plane the branches corresponding to free-surface and membrane-vibration modes always remain separated since for no value of $\beta$ are the real and imaginary parts of the two modes identical.

Let us finally consider the case of a container with height $h = 0.5$ and length $L = 1.0$ of which the side walls are rigid and the bottom consists of a membrane. The tension is varied and assume that the volume of the fluid is adjusted such that the position of the free surface remains unaltered. The steady state configuration of the membrane is as shown in figure 5.3b for various values of $\beta$. Recall that the static configuration was stable for $\beta$ up to $\beta \approx 2.41$: for larger values of $\beta$ no static solution exists. Figure 5.9 shows a plot of the imaginary parts of eigenvalues corresponding to the first and second membrane-vibration modes as a function of $\beta$. We note the decrease in $\text{Im}(\lambda)$ as the stability boundary is approached. Figure 5.10 shows plots of the streamlines of the first (5.10a) and second (5.10b) membrane-vibration modes for the case $\beta = 2.4$.

5.7 Conclusions

In this chapter we have studied the normal modes of a system consisting of an open vessel with a flexible wall containing a viscous, incompressible fluid. We have shown that the eigenfrequencies related to free surface oscillation modes of an inviscid fluid in a container decrease when part of the rigid container wall is replaced by a membrane. From the expression for the eigenvalues of a weakly damped viscous fluid in a flexible container we are
able to deduce a number of qualitative results. Namely, if normal modes exist such that the membrane is not displaced significantly, then the eigenfrequencies of this mode are approximately the same as the eigenfrequencies of a viscous fluid in a similarly shaped rigid container. These modes are termed free-surface oscillation modes and are independent of the parameter $\beta$. On the other hand, if modes exist such that the free surface is displaced only slightly then the imaginary parts of the corresponding eigenfrequencies are proportional to $1/\sqrt{\beta}$. These modes are termed membrane vibration modes, their eigenfrequencies are less than those of a freely vibrating membrane. Structural vibrations may experience less damping than the free surface oscillations.

We find that the results obtained for an inviscid fluid are also true when viscous effects are included. We find, furthermore, that free-surface and membrane-vibration modes indeed exist. The properties of these modes agree well with the qualitative analytical results mentioned above. Of interest is the case in which an eigenfrequency of the fluid is close to an eigenfrequency of the membrane, and in particular its dependence on $\beta$. We find that the Reynolds number is an important parameter in this case. For small values of $Re$ the eigenfrequencies of the free-surface modes are found to be independent of the $\beta$, while the imaginary parts of the eigenfrequencies of the membrane-vibration modes are proportional to the reciprocal of the square root of $\beta$. For a certain "critical" value of $\beta$ the imaginary parts of both modes coincide. For large values of $Re$ the graphs of the imaginary parts of the two different modes are globally the same as in the case of small
Re. However, near the aforementioned critical value of $\beta$ to two curves display a bifurcation-type of behaviour. Namely, for a particular value of $\beta$ the two different branches correspond to either the free-surface of membrane-vibration modes, increasing $\beta$ will cause the branch representing free-surface modes to change over into a branch representing membrane-vibration modes and \textit{vice versa}. The change-over occurs in the neighbourhood of the critical value of $\beta$. In the vicinity of the critical value of $\beta$ the normal modes of the fluid and the membrane are coupled, i.e. simultaneous free-surface and membrane deflections are observed, both closely resembling the corresponding modal deflections in the uncoupled problem.
Chapter 6

Interactions of a viscous fluid with an elastic solid

6.1 Introduction

The membrane model for the flexible container wall, as applied in the previous chapter, is advantageous for a number of reasons. The first advantage is the simplicity of the model, permitting analytical solution techniques to be applied in certain cases. The second advantage is the fact that degrees of freedom related to the membrane can be eliminated so that only quantities related to the fluid motion appear as unknowns. However, the fact that a membrane has no inherent stiffness may be a severe drawback in certain cases, limiting the use of the membrane model. We therefore consider a more advanced model of the structure by assuming that the structure consists of a solid obeying the equations of linear elasticity. We shall find that some qualitative results obtained in the previous chapter still hold when the structure is modelled by a elastic solid. The extension of the structure model to that of an elastic solid does, however, allow new problems to be studied in which stiffness is important.

This chapter is organized as follows. In section 6.2 the problem to be considered is introduced and the governing equations are presented. In section 6.3 we derive a variational formulation of the fluid-structure interaction problem. A number of qualitative properties of the spectral problem, which results from the linearized Navier-Stokes equations coupled with the elasticity equations, are derived. Like in the previous chapter, we find that three types of normal modes exist, namely modes related to free surface oscillations...
tions, modes related to solid vibrations and coupled fluid-solid modes. The three types of modes have different damping characteristics and depend differently on the parameters. Section 6.4 deals with the discrete eigenvalue problem obtained after a finite element discretization has been applied. The eigenvalue problem is solved using an inverse iteration procedure. In section 6.5 numerical results are presented. The validity of an approximate expression due to Tong (1966) to obtain symmetric matrices is investigated numerically, and phenomena related to the viscosity of the fluid are studied.

### 6.2 Problem formulation

Consider a region $\Omega$ which is divided into two regions $\Omega_1$ and $\Omega_2$ such that $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = G$, as shown in figure 6.1. Let the region $\Omega_1$ be occupied by a viscous fluid and assume that $\Omega_2$ is occupied by an elastic solid. The boundary of $\Omega_1$, denoted by $\partial \Omega_1$, consists of a free surface $S$, a rigid boundary $W_f$ and the interfacial boundary $G$, i.e. $\partial \Omega_1 = S \cup W_f \cup G$. Likewise, the boundary of $\Omega_2$, $\partial \Omega_2$, consists of a rigid part $W_s$, an unconstrained surface $F$ and the interfacial boundary $G$, i.e. $\partial \Omega_2 = G \cup W_s \cup F$. Let $n$ denote the outward unit normal to $\Omega$ and let $\tau$ be the unit tangent to $\partial \Omega$ directed counter-clockwise from $n$.

The equations describing the fluid motion in $\Omega_1$ are the Navier-Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{1}{\rho_f} \nabla p = \nu \nabla^2 u + f,$$  \hspace{1cm} (6.2.1)
and the incompressibility condition

\[ \nabla \cdot \mathbf{u} = 0. \quad (6.2.2) \]

In the above equations \( \mathbf{u} \) denotes the fluid velocity, \( p \) the pressure, \( \mathbf{f} \) the external body force, \( \rho_f \) the density and \( \nu = \mu/\rho_f \) the kinematic viscosity of the fluid. On the free surface \( S \) we have the conditions that the normal stress in the fluid is equal to the outside gas pressure and the gas is not able to resist tangential stresses, thus

\[ \sigma_n = -p_g, \quad \sigma_\tau = 0. \quad (6.2.3) \]

Here \( p_g \) denotes the outside gas pressure and \( \sigma_n, \sigma_\tau \) denote the normal and tangential stresses respectively. We have used the notation

\[ \sigma_n = \sum_{i,j=1}^{2} \sigma_{ij} n_i n_j, \quad \sigma_\tau = \sum_{i,j=1}^{2} \sigma_{ij} n_i \tau_j, \]

in which \( n_i, \tau_i, i = 1, 2 \) are the components of the outward unit normal and unit tangent respectively, and

\[ \sigma_{ij} = -p \delta_{ij} + \mu (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \]

denotes the Cauchy stress tensor. On the free surface we also have the kinematic condition

\[ \frac{D}{Dt} \alpha(\mathbf{x}, t) = 0, \quad (6.2.4) \]

in which \( \alpha(\mathbf{x}, t) = \mathbf{n} \cdot (\mathbf{x}_S - \mathbf{x}) = 0 \) is the equation of the free surface \( S \) (\( \mathbf{x}_S \) is the position vector of \( S \)) and \( D/Dt \) denotes the Lagrangian derivative. On the rigid boundary \( W_f \) the no-slip condition characteristic of viscous flows is prescribed, viz.

\[ \mathbf{u} = 0. \quad (6.2.5) \]

The fluid is assumed to be incompressible so that the total quantity of liquid, \( V_f \), remains constant, hence

\[ \int_{\Omega_1} \, dx = V_f. \quad (6.2.6) \]
Next consider the equations governing the motion of the elastic solid in $\Omega_2$. Assuming linear elasticity relations we find, cf. Landau & Lifshitz (1959b)

$$
\rho_s \frac{\partial^2 \mathbf{d}}{\partial t^2} - \nabla \cdot \mathbf{T} = \mathbf{f},
$$

where $\mathbf{d}$ is the displacement vector of a particle in $\Omega_2$, $\rho_s$ is the density of the solid and

$$
T_{ij} = \frac{E}{1 + \sigma} e_{ij} + \frac{E \sigma}{(1 + \sigma)(1 - 2\sigma)} \epsilon_{kk} \delta_{ij}, \quad e_{ij} = \frac{1}{2} \left( \frac{\partial d_i}{\partial x_j} + \frac{\partial d_j}{\partial x_i} \right)
$$

denotes the stress tensor of the solid. $E$ is Young's modulus and $\sigma$ denotes Poisson's ratio. On the fixed boundary $W_s$ the elastic solid is not displaced so that

$$
\mathbf{d} = 0,
$$

while on the unconstrained surface $F$ the following conditions are given

$$
T_n = -p_g, \quad T_\tau = 0.
$$

The same notation as for the stress tensor of the fluid is used.

Finally consider the conditions on the interfacial boundary $G$. Let $\mathbf{n}_1$ denote a unit normal to $G$ directed outward from $\Omega_1$ and let $\mathbf{\tau}_1$ denote a unit tangent to $G$ directed counter-clockwise from $\mathbf{n}_1$. Likewise, $\mathbf{n}_2$ is a unit normal to $G$ directed outward from $\Omega_2$ and $\mathbf{\tau}_2$ is a unit tangent to $G$ in the counter-clockwise direction form $\mathbf{n}_2$. Demanding continuity of stresses across the interface $G$ yields the conditions

$$
\sigma_{n_1} = T_{n_1}, \quad \sigma_{\tau_1} = T_{\tau_1}.
$$

Note that we have used $\mathbf{n}_1, \mathbf{\tau}_1$ to define the normal and tangential stresses on $G$. However, since $\mathbf{n}_1 = -\mathbf{n}_2$ and $\mathbf{\tau}_1 = -\mathbf{\tau}_2$ it follows that the subscripts $n_1, \tau_1$ in (6.2.10) may be replaced by $n_2$ and $\tau_2$. In addition to the continuity of stresses we have the no-cavitation condition, i.e. the fluid remains in contact with the solid at all times, hence

$$
\frac{D}{Dt}(\mathbf{x}_G - \mathbf{x}) = 0,
$$

where $\mathbf{x}_G$ denotes the position vector of $G$.

As in the previous chapters, we will not attempt to solve the non-linear set of equations (6.2.1)-(6.2.11). Let us therefore first consider the equations
of the static problem after which small perturbations with respect to this static configuration are investigated. Putting \( \mathbf{u} \equiv 0 \) we obtain from (6.2.1)

\[
\frac{1}{\rho_f} \nabla p^0 = \mathbf{f},
\]  

(6.2.12)

the conditions on \( S^0 \) reduce to

\[
p^0 = p_g,
\]

(6.2.13)

and the conditions on the static interface \( G^0 \) read

\[
-p^0 = T_{n_1}^0, \quad T_{r_1}^0 = 0.
\]

(6.2.14)

The elasticity equation becomes

\[
- \nabla \cdot \mathbf{T}^0 = \mathbf{f},
\]

(6.2.15)

with the condition on \( F^0 \)

\[
T_n^0 = -p_g
\]

(6.2.16)

In the above and subsequent equations the superscript 0 refers to static parameters. Equations (6.2.12)-(6.2.16) together with the volume constraint (6.2.6) completely define the pressure at any point in the fluid, the stresses in the solid and the static configuration of the boundaries \( F^0, G^0 \) and \( S^0 \). It will be evident that solving (6.2.12)-(6.2.16) is a non-trivial problem even for simple configurations. We will not attempt to solve this problem but assume instead that a static configuration is given.

We take \( \mathbf{f} = -g \mathbf{k} \) where \( \mathbf{k} \) is the unit vector pointing along the positive \( y \)-axis and \( g \) the gravitational acceleration. Surface tension effects on \( S^0 \) are neglected so that the free surface is flat and perpendicular to the \( y \)-axis. Let us now perturb the static configuration such that the position vectors of \( S \) and \( F \) are given by

\[
\mathbf{x}_S = \mathbf{x}_{S^0} + \eta \mathbf{n}_S,
\]

and

\[
\mathbf{x}_G = \mathbf{x}_{G^0} + \mathbf{d}_G.
\]

Here \( \eta \) is the small normal displacement from \( S^0 \), \( \mathbf{n}_S \) is a unit normal to \( S^0 \) and \( \mathbf{d}_G \) is a small displacement from the static interface. Substituting the above expressions for \( \mathbf{x}_S \) and \( \mathbf{x}_G \) into (6.2.4) and (6.2.11), using the relations (6.2.12)-(6.2.16) and neglecting quantities which are of second order
smallness, we obtain the linearized equations governing the motion of the fluid and the elastic solid. In dimensionless form we have for the fluid

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \nabla p &= \frac{1}{Re} \nabla^2 \mathbf{u} \\
\nabla \cdot \mathbf{u} &= 0
\end{align*}
\] in \( \Omega_1^0 \), \hspace{1cm} (6.2.17)

\[
\begin{align*}
\sigma_n &= -\eta \\
\sigma_r &= 0 \\
\mathbf{u} \cdot \mathbf{n} &= \frac{\partial \eta}{\partial t}
\end{align*}
\] on \( S^0 \), \hspace{1cm} (6.2.18)

\[
\mathbf{u} = 0 \text{ on } W_f,
\] \hspace{1cm} (6.2.19)

for the solid

\[
\frac{\partial^2 \mathbf{d}}{\partial t^2} = \ddot{E} \nabla \cdot \mathbf{T} \text{ in } \Omega_2^0,
\] \hspace{1cm} (6.2.20)

\[
T_n = 0, \quad T_r = 0 \text{ on } F^0,
\] \hspace{1cm} (6.2.21)

\[
\mathbf{d} = 0, \text{ on } W_s,
\] \hspace{1cm} (6.2.22)

and for the interface

\[
\begin{align*}
\sigma_{n_1} &= \ddot{E} r_p T_n - k \cdot \mathbf{d}_G \\
\sigma_{r_1} &= \ddot{E} r_p T_r \\
\frac{\partial \mathbf{d}_G}{\partial t} &= \mathbf{u}
\end{align*}
\] on \( G^0 \), \hspace{1cm} (6.2.23)

In the above equations \( Re = UL/\nu \) denotes the Reynolds number (\( L \) is a length scale and \( U = \sqrt{gL} \) is a velocity scale), \( r_p = \rho_s/\rho_f \) is the ratio of densities of the solid and the fluid and \( \ddot{E} = E/\rho_s gL \) is a measure of the ratio of elastic bending forces and gravitational forces.

### 6.3 Variational formulation and the spectral problem

Equations (6.2.17) and (6.2.20) subject to boundary conditions (6.2.18), (6.2.19) and (6.2.21)-(6.2.23) will be written in a variational formulation. To that end the following Green formula are used: for the Stokes equations

\[
\int_{\Omega_1^0} \left( \frac{\partial \mathbf{u}}{\partial t} + \nabla p - \frac{1}{Re} \nabla^2 \mathbf{u} \right) \cdot \mathbf{v} \, dx = \int_{\Omega_1^0} \left( \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} - p \nabla \cdot \mathbf{v} \right) \, dx +
\]

104
\[
\frac{1}{Re} a(u, v) - \int_{\partial \Omega^0_1} (\sigma_{n_1} v_{n_1} + \sigma_{r_1} v_{r_1}) ds,
\] (6.3.1)

in which

\[
a(u, v) = \frac{1}{2} \sum_{i,j=1}^{2} \int_{\Omega_0} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) dx, \quad x_1 = x, \ x_2 = y.
\]

and for the linear elasticity relations

\[
\int_{\Omega^0_2} (\frac{\partial^2 d}{\partial t^2} - \bar{E} \nabla \cdot T) \cdot e dx = \int_{\Omega^0_2} \frac{\partial^2 d}{\partial t^2} \cdot e dx + \bar{E} b(d, e) - \int_{\partial \Omega^0_2} (\bar{E} T_{n_2} e_{n_2} + \bar{E} T_{r_2} e_{r_2}) ds,
\] (6.3.2)

in which

\[
b(d, e) = \frac{1}{2} \sum_{i,j=1}^{2} \int_{\Omega^0_2} T_{ij} \left( \frac{\partial e_i}{\partial x_j} + \frac{\partial e_j}{\partial x_i} \right) dx.
\]

Let us assume that the time-dependence of \( u, p, \eta \) and \( d \) is of the form \( e^{\lambda t} \) where \( \lambda \) is in general some complex quantity. Employing the Green formula (6.3.1) and (6.3.2) and substituting the boundary conditions on \( \partial \Omega^0_1 \) and \( \partial \Omega^0_2 \) we obtain the variational formulation of equations (6.2.17) - (6.2.23), namely:

find \( u \in V_1, \ p \in Q \) and \( d \in V_2 \) such that for all \( v \in V_1, \ q \in Q \) and \( e \in V_2 \) the following equations are satisfied:

\[
\int_{\Omega^0_1} (\lambda u \cdot v - p \nabla \cdot v) dx + \frac{1}{Re} a(u, v) + \frac{1}{\lambda} \int_{\partial \Omega^0} u_n v_n ds - \int_{G^0} \left[ (\bar{E} r_0 T_{n_1} - k \cdot d) v_{n_1} + \bar{E} r_0 T_{r_1} v_{r_1} \right] ds = 0,
\] (6.3.3)

\[
\int_{\Omega^0_1} q \nabla \cdot udx = 0,
\] (6.3.4)

\[
\lambda d = u \quad \text{on} \ G^0,
\] (6.3.5)
\[ \int_{\Omega_2} \lambda^2 \mathbf{d} \cdot \mathbf{e} \, dx + \tilde{E} b(\mathbf{d}, \mathbf{e}) - \int_{G^0} (\tilde{E} T_{n_2} \varepsilon_{n_2} + \tilde{E} T_{\tau_2} \varepsilon_{\tau_2}) \, ds = 0, \quad (6.3.6) \]

We have introduced the spaces of vector functions \( V_1 = \{ \mathbf{u} \in \Omega_1^0 : \mathbf{u} = 0 \text{ on } W_f \} \), \( V_2 = \{ \mathbf{d} \in \Omega_2^0 : \mathbf{d} = 0 \text{ on } W_s \} \) and the space of scalar functions \( Q = \{ q \in \Omega_1^0 \} \). The functions in \( V_1, V_2 \) and \( Q \) must be so smooth that the integrals (6.3.3)-(6.3.6) exist. Observe that equations (6.3.3) and (6.3.6) both contain integrals over the interfacial boundary \( G^0 \) in which normal and tangential stresses appear. The terms containing the stresses can be eliminated in the following way. Take equation (6.3.6), multiply it by the density ratio \( r_\rho \) and add the result to (6.3.3), giving

\[
\int_{\Omega_1^0} (\lambda \mathbf{u} \cdot \mathbf{v} - p \nabla \cdot \mathbf{v}) \, dx + \frac{1}{R \varepsilon} a(\mathbf{u}, \mathbf{v}) + \frac{1}{r_\rho} \int_{\Omega_2^0} \lambda^2 \mathbf{d} \cdot \mathbf{e} \, dx + \tilde{E} b(\mathbf{d}, \mathbf{e}) + \frac{1}{\lambda} \int_{G^0} \mathbf{v}_n \mathbf{v}_n \, ds -
\]

\[
\int_{G^0} \left[ \tilde{E} r_\rho (T_{n_1} \mathbf{v}_{n_1} + T_{n_2} \varepsilon_{n_2} + T_{\tau_1} \mathbf{v}_{\tau_1} + T_{\tau_2} \varepsilon_{\tau_2}) - k \cdot \mathbf{d} \mathbf{v}_n \right] \, ds = 0.
\]

By virtue of the fact that \( \mathbf{n}_1 = -\mathbf{n}_2 \) and \( \tau_1 = -\tau_2 \) on \( G^0 \), it follows that the terms containing normal and tangential stresses in the integral over \( G^0 \) cancel when we demand that the test functions are identical on \( G^0 \), i.e. \( \mathbf{v}|_{G^0} = \mathbf{e}|_{G^0} \).

Let us introduce a new variable \( \mathbf{w} \), where \( \mathbf{w} \) is defined as follows: \( \mathbf{w} = \mathbf{u} \) in \( \Omega_1^0 \) and \( \mathbf{w} = \frac{\partial \mathbf{d}}{\partial t} = \lambda \mathbf{d} \) in \( \Omega_2^0 \). The introduction of this new variable is motivated by dimensional considerations. Namely, with this choice of \( \mathbf{w} \) in \( \Omega_1^0 \) and \( \Omega_2^0 \) we obtain a variable which has the dimensions of velocity \( (\text{length} \cdot \text{time}^{-1}) \) in the whole of the domain under consideration. In this way the problem of having to deal with two fundamentally different unknowns can be eliminated and a unified variational principle for the governing equations of both the fluid and the solid may be obtained. In particular, equations (6.3.3)-(6.3.6) can be written like

find \( \mathbf{w} \in V \) and \( p \in Q \) such that for all functions \( \tilde{\mathbf{v}} \in \tilde{V} \) and \( q \in Q \) the following equations are satisfied:

\[
\int_{\Omega_1^0} (\lambda \mathbf{w} \cdot \tilde{\mathbf{v}} - p \nabla \cdot \tilde{\mathbf{v}}) \, dx + \frac{1}{R \varepsilon} a(\mathbf{w}, \tilde{\mathbf{v}}) + \frac{1}{r_\rho} \int_{\Omega_2^0} \lambda \mathbf{w} \cdot \tilde{\mathbf{v}} \, dx +
\]

106
\[ \frac{1}{\lambda} r_p \tilde{E}b(w, \hat{v}) + \frac{1}{\lambda} \int_{\Omega_1^0} w_n \hat{v}_n ds + \frac{1}{\lambda} \int_{G_0} k \cdot w \hat{v}_n ds = 0. \quad (6.3.7) \]

\[ \int_{\Omega_1^0} q \nabla \cdotwdx = 0, \quad (6.3.8) \]

where \( V = \{w \in V_1 \cup V_2\} \) and \( \hat{V} = \{v \in V_1, \ e \in V_2, \ v|_{G_0} = e|_{G_0}\} \). Note that the test functions \( \hat{v} \) are not necessarily of the same form in the two domains \( \Omega_1^0 \) and \( \Omega_2^0 \), all we demand is that on the interfacial boundary \( G_0 \) the test functions are identical. For more fundamental work on unified variational principles for fluid-structure interaction problems we refer to the recent paper by Van Groesen & Verstappen (1990).

General properties of the spectrum can be derived if we are able to write (6.3.7), (6.3.8) in the form of a functional which has to be minimized. The term containing the integral over \( G_0 \) in (6.3.7) is, however, non-symmetric which means that equation (6.3.7), as it stands, can not be written as a minimization problem. To overcome this difficulty we apply the so-called Tong-hypothesis (cf. Tong, 1966), i.e. we assume that \( k \cdot w \) may be replaced by \( (\hat{k} \cdot n_1)(w \cdot n_1) \). Note that when the unit normal \( n_1 \) is parallel to \( \hat{k} \) the Tong-hypothesis is satisfied identically, in all other cases it is an approximation of questionable validity. Assume for the moment that the Tong-hypothesis is a reasonable first order approximation. Solving equations (6.3.8) and (6.3.7) is then equivalent to finding the pair \((w, \lambda)\) such that the functional

\[ \Phi(w, \lambda) = \lambda^2[(w, w)_{\Omega_1^0} + r_p(w, w)_{\Omega_2^0}] + \lambda \frac{1}{Re} a(w, w) + r_p \tilde{E}b(w, w) + (w_n, w_n)_{S_0} + (w_n, w_n)_{G_0}, \]

assumes a stationary value equal to zero. We have introduced the notation

\[ (w, w)_{\Omega_1^0} = \int_{\Omega_1^0} |w|^2 dx, \quad (w_n, w_n)_{S_0} = \int_{S_0} |w_n|^2 ds, \]

\[ (w_n, w_n)_{G_0} = \int_{G_0} \hat{k} \cdot n_1 |w_n|^2 ds, \quad \ldots \text{etc.} \]

Take \( \Phi \equiv 0 \) and treat the resulting equation as a quadratic function in \( \lambda \). Solving for \( \lambda \) we can derive the following qualitative properties concerning the spectrum of the fluid-structure interaction problem:

107
(i) If 

\[ r_\rho \tilde{E}b(\mathbf{w}, \mathbf{w}) + (w_n, \mathbf{w})_{G^0} + (w_n, w_n)_{G^0} > 0, \]

then problem (6.3.8), (6.3.7) is stable with respect to infinitesimal perturbations;

(ii) If 

\[ \frac{1}{Re^2} a(\mathbf{w}, \mathbf{w})^2 \geq 4 \left[ (\mathbf{w}, \mathbf{w})_{\Omega^0_1} + r_\rho (\mathbf{w}, \mathbf{w})_{\Omega^0_2} \right] \times \\
[ r_\rho \tilde{E}b(\mathbf{w}, \mathbf{w}) + (w_n, \mathbf{w})_{G^0} + (w_n, w_n)_{G^0} ] , \]

then the eigenvalues corresponding to the eigenfunctions \( \mathbf{w} \) are real and negative, which corresponds to an aperiodic damping process;

(iii) If 

\[ \frac{1}{Re^2} a(\mathbf{w}, \mathbf{w})^2 < 4 \left[ (\mathbf{w}, \mathbf{w})_{\Omega^0_1} + r_\rho (\mathbf{w}, \mathbf{w})_{\Omega^0_2} \right] \times \\
[ r_\rho \tilde{E}b(\mathbf{w}, \mathbf{w}) + (w_n, \mathbf{w})_{G^0} + (w_n, w_n)_{G^0} ] , \]

then the eigenvalues are complex, occurring in complex-conjugate pairs. The real part of the eigenvalues is negative so that the oscillations are damped.

The term \( \tilde{E}b(\mathbf{w}, \mathbf{w}) \) corresponds to the bending energy of the solid while \((w_n, w_n)_{G^0}\) is a measure of the energy required to displace the interfacial boundary \( G^0 \) against gravitational forces. Condition (i) will be satisfied when \( \tilde{E}b(\mathbf{w}, \mathbf{w}) + (w_n, w_n)_{G^0} > 0 \) which will be the case when \( \tilde{E} \) is sufficiently large. Condition (i) is always satisfied if \( k \cdot \mathbf{n}_1 \) is positive at any point on \( G^0 \), i.e. if \( G^0 \) is concave upwards, since \((w_n, w_n)_{G^0}\) will then be strictly positive. Recall that a similar stability condition was given in section 5.5. Note that the damping of the normal modes is characterized by the term \( \frac{1}{Re} a(\mathbf{w}, \mathbf{w}) \) which is proportional to the energy dissipation due to viscous effects. This is the only source of damping since we have not included damping effects in the structure in our model.

In the previous chapter it was shown that three different types of eigen-modes can be identified when oscillations of a fluid in contact with a membrane are considered. The three types of modes were termed free-surface modes, membrane-vibration modes and coupled fluid-solid modes. The free-surface oscillation modes were characterized by a significant displacement of
the free-surface while the membrane was deflected only slightly. Membrane-vibration modes, on the other hand, are characterized by a significant displacement of the membrane while the free-surface is not necessarily affected significantly by the membrane motion. In order to investigate the properties of the eigenmodes in this chapter we assume that it is also possible here to make the distinction between free-surface and structural-vibration modes.

Let us first of all assume that normal modes exist such that \( w \) in \( \Omega^0_2 \) is small. Neglecting all terms which refer to quantities in \( \Omega^0_2 \) we obtain

\[
\lambda = \frac{-\frac{1}{Re} a(w, w) \pm \sqrt{\left(\frac{1}{Re^2} a(w, w)^2 - 4(w, w)_{\Omega^0_1}(w_n, w_n)_{G_0}\right)}}{2(w, w)_{\Omega^0_1}}. \tag{6.3.9}
\]

Equation (6.3.9) is the expression for the eigenfrequencies of a viscous fluid in a rigid container in terms of the corresponding eigenfunctions, cf. Kopachevskii & Myshkis (1966).

Assume, alternatively, that eigenmodes exist with corresponding eigenfunctions such that \( w \) in \( \Omega^0_1 \) is small. Neglecting terms in the discriminant which refer to quantities in \( \Omega^0_1 \) we obtain

\[
\lambda = \frac{a(w, w)}{2Re r_{\rho}(w, w)_{\Omega^0_2}} \pm i \sqrt{\frac{\tilde{E}b(w, w)}{(w, w)_{\Omega^0_2}}}. \tag{6.3.10}
\]

\( \text{Im}(\lambda) \) in (6.3.10) can readily be shown to be equivalent to the Rayleigh quotient for the case of a solid which experiences unconstrained vibrations. The eigenvalues given by (6.3.10) correspond therefore to eigenfunctions which represent structural vibrations which effect the fluid only slightly. Since \( w \) is assumed small in \( \Omega^0_1 \) it follows that \( \text{Re}(\lambda) \) is small and hence the corresponding modes are weakly damped.

Finally consider the case in which \( Re \) is large so that only terms to first order in \( 1/Re \) need be retained. Assume that normal modes exist such that on the interface \( G^0 \) the fluid and structure are displaced significantly while the free surface remains relatively undisturbed, i.e. we can neglect \( (w_n, w_n)_{G_0} \). Assume in addition that \( r_{\rho}\tilde{E}b(w, w) \gg (w_n, w_n)_{G_0} \) which can be achieved by choosing \( \tilde{E} \) sufficiently large. Under the above assumptions \( \lambda \) is approximated by

\[
\lambda = \frac{-a(w, w)}{2Re \left[ (w, w)_{\Omega^0_1} + r_{\rho}(w, w)_{\Omega^0_2} \right]} \pm i \sqrt{\frac{r_{\rho}\tilde{E}b(w, w)}{(w, w)_{\Omega^0_1} + r_{\rho}(w, w)_{\Omega^0_2}}}. \tag{6.3.11}
\]
Since $\text{Im}(\lambda) \sim \sqrt{E}$ it follows that the eigenmodes in (6.3.11) are related to structural vibration modes. The fluid inertia term $(w, w)_{\Omega^0_i}$ in the denominator of $\text{Im}(\lambda)$ in (6.3.11) is of interest. Namely, taking into account the fluid motion leads to an extra term in the denominator of $\text{Im}(\lambda)$ when comparing equations (6.3.10) and (6.3.11). It follows that the imaginary parts of normal modes of a freely-vibrating structure are larger than the corresponding modes of a structure in contact with a fluid due to the inertia of the fluid. We recall that a similar result was obtained in the previous chapter (cf. equation (5.5.5)).

### 6.4 Numerical solution by the finite-element method

Equations (6.3.7), (6.3.8) are solved using the finite-element technique. The incompressibility constraint will again be dealt with by using a penalty-function approach, i.e. the variational integral (6.3.8) is written like

$$
\epsilon_p \int_{\Omega^0_i} pq dx + \int_{\Omega^0_i} \nabla \cdot w dx = 0. \quad (6.4.1)
$$

Let us write equations (6.3.7) and (6.4.1) in the following abbreviated way

$$
F(w_i, \dot{v}_i) - P^i(p, \dot{v}_i) + E(w_i, \dot{v}_i) = 0, \quad i = 1, 2 \quad (6.4.2)
$$

$$
\epsilon D(p, q) + P^1(q, w_1) + P^2(q, w_2) = 0. \quad (6.4.3)
$$

We have used the notation

$$
F(w_i, \dot{v}_i) = \int_{\Omega^0_i} \lambda w_i \dot{v}_i dx + \frac{1}{Re} a(w_i, \dot{v}_i) + \frac{1}{\lambda} \int_{S^0} (w_i n_i) (\dot{v}_i n_i) ds,
$$

$$
E(w_i, \dot{v}_i) = r_\rho \lambda \int_{\Omega^0_i} w_i \dot{v}_i dx + \frac{1}{\lambda} r_\rho \bar{E} b(\dot{w}_i, \dot{v}_i) + \frac{1}{\lambda} \int_{G^0} (k_i w_i) (\dot{v}_i n_i) ds = 0,
$$

$$
P^i(p, \dot{v}_i) = \int_{\Omega^0_i} p \frac{\partial \dot{v}_i}{\partial x_j} dx, \quad D(p, q) = \int_{\Omega^0_i} pq dx
$$

We note that the functionals $F(w_i, \dot{v}_i)$, $P^i(p, \dot{v}_i)$ and $D(p, q)$ are defined only on the domain occupied by the fluid, i.e. on $\Omega^0_i$ including $G^0$. On the other hand, $E(w_i, \dot{v}_i)$ is a functional defined only on the domain occupied
by the elastic solid, i.e. on \( \Omega_2^0 \) including \( G^0 \). For the construction of the finite-element approximation of \( w \) and \( p \), we write \( w_i \) (\( i = 1, 2 \)) and \( p \) as the sum of usual finite-element basis functions with small support. The basis functions for \( w_i \), denoted by \( \phi_k(x) \), are divided into three sets. Namely, those which are non-zero at nodal points in \( \Omega_1^0 \setminus G^0 \) (\( K_f \) in number) and zero at the remaining nodal points, those non-zero only at the nodal points on \( G^0 \) (\( K_g \) in number) and those which are non-zero at nodal points in \( \Omega_2^0 \setminus G^0 \) (\( K_e \) in number) and zero elsewhere. Hence

\[
\begin{align*}
    w_i &= \sum_{k=1}^{K_f} \bar{w}_{ik} \phi_k(x) + \sum_{k=K_f+1}^{K_f+K_g} \bar{w}_{ik} \phi_k(x) + \sum_{k=K_f+K_g+1}^{K_f+K_g+K_e} \bar{w}_{ik} \phi_k(x) \\
    p &= \sum_{m=1}^{M} \tilde{p}_m \chi_m(x).
\end{align*}
\]

where the basis functions \( \chi_m(x) \) are non-zero only at the nodal points in \( \Omega_1^0 \) and on \( G^0 \) (since the pressure is defined only in the fluid domain). For the test functions \( \hat{v}_i \) we take \( \phi_n(x) \) and for \( q \) we take \( \chi_n(x) \). The discrete equivalents of equations (6.4.2) and (6.4.3) become on substituting for \( w_i \) and \( p \)

\[
\begin{align*}
    \sum_{k=1}^{K_f+K_g} \bar{w}_{ik} F(\phi_k, \phi_n) - \sum_{m=1}^{M} \tilde{p}_m P^i(\chi_m, \phi_n) + \sum_{k=K_f+1}^{K_f+K_g+K_e} \bar{w}_{ik} E(\phi_k, \phi_n) &= 0 \\
    i &= 1, 2 \quad n = 1, 2, \ldots, K_f + K_g + K_e \tag{6.4.4}
\end{align*}
\]

\[
\begin{align*}
    \varepsilon \sum_{m=1}^{M} D(\chi_m, \chi_n) + \sum_{k=1}^{K_f+K_g} \bar{w}_{1k} P^1(\chi_n, \phi_k) + \sum_{k=1}^{K_f+K_g+K_e} \bar{w}_{2k} P^2(\chi_n, \phi_k) &= 0 \tag{6.4.5} \\
    n &= 1, 2, \ldots, M.
\end{align*}
\]

Let \( \mathbf{w}_f^{(i)} \) (\( i = 1, 2 \)) denote the vectors containing the unknowns in the nodal points in \( \Omega_1^0 \setminus G^0 \). The vectors denoted by \( \mathbf{w}_g^{(i)} \) (\( i = 1, 2 \)) contain unknowns in the nodal points on \( G^0 \) and \( \mathbf{w}_e^{(i)} \) (\( i = 1, 2 \)) contain unknowns refering to nodal points in \( \Omega_2^0 \setminus G^0 \). Note \( \mathbf{w}_f^{(i)} \), \( \mathbf{w}_g^{(i)} \) and \( \mathbf{w}_e^{(i)} \) are vectors of length \( K_f \), \( K_g \) and \( K_e \) respectively. The explicit form of equation (6.4.4) is
then given by

\[
\begin{pmatrix}
F_{ff} & 0 & F_{fg} & 0 & 0 & 0 \\
0 & F_{ff} & 0 & F_{fg} & 0 & 0 \\
F_{gf} & 0 & F_{gg} & 0 & 0 & 0 \\
0 & F_{gf} & 0 & F_{gg} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
w^{(1)}_f \\
w^{(2)}_f \\
w^{(1)}_g \\
w^{(2)}_g \\
w^{(1)}_e \\
w^{(2)}_e
\end{pmatrix}
- \mathbf{P}^T \mathbf{p}
+ \begin{pmatrix}
w^{(1)}_f \\
w^{(2)}_f \\
w^{(1)}_g \\
w^{(2)}_g \\
w^{(1)}_e \\
w^{(2)}_e
\end{pmatrix} = 0 \quad (6.4.6)
\]

and the penalized incompressibility constraint can be written as

\[
cDp + \mathbf{P}
\begin{pmatrix}
w^{(1)}_f \\
w^{(2)}_f \\
w^{(1)}_g \\
w^{(2)}_g \\
w^{(1)}_e \\
w^{(2)}_e
\end{pmatrix}
= 0. \quad (6.4.7)
\]

The large matrices in equation (6.4.6) are square and of order \(2(K_f + K_g + K_e)\), the components of the sub-matrices are given by

\[
(F_{ff})_{ij} = F(\phi_i, \phi_j), \quad 1 \leq i, j \leq K_f
\]
\[
(F_{gg})_{ij} = F(\phi_i, \phi_j), \quad K_f + 1 \leq i, j \leq K_f + K_g
\]
\[
(E_{ee})_{ij} = E(\phi_i, \phi_j), \quad K_f + K_g + 1 \leq i, j \leq K_f + K_g + K_e
\]
\[\vdots\]

etc.
The matrix $\mathbf{P}$ in the momentum and the continuity equations is a non-square matrix of order $2(K_f + K_g + K_e) \times M$, only the first $2(K_f + K_g)$-columns containing non-zero elements. Thus

$$\mathbf{P} = (\mathbf{P}^1, \mathbf{P}^2, 0),$$

where $\mathbf{P}^n$ ($n = 1, 2$) are matrices of order $(K_f + K_g) \times M$ with components

$$(\mathbf{P}^n)_{ij} = P^n(\chi_i, \phi_j) \quad 1 \leq i \leq M, \quad 1 \leq j \leq K_f + K_g.$$  

The matrix $\mathbf{D}$ is square and non-singular so that it may be inverted. Hence, the continuity equation (6.4.7) may be solved for the pressure in terms of the velocity unknowns so that the pressure unknowns can be eliminated entirely from the momentum equation (6.4.6).

Substituting the expressions for the functionals $F(w_i, \dot{v}_i)$ and $E(w_i, \dot{v}_i)$ into the momentum equations, we obtain a discrete eigenvalue problem of the form

$$\left(\lambda^2 \mathbf{A} + \lambda \mathbf{B} + \mathbf{C}\right) \mathbf{x} = 0, \quad (6.4.8)$$

where

$$\mathbf{A} = \begin{pmatrix} M_{ff} & M_{fg} & 0 \\ M_{gf} & M_{gg} + M_{gg}^* & M_{ge}^* \\ 0 & M_{eg}^* & M_{ee}^* \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} A_{ff} & A_{fg} & 0 \\ A_{gf} & A_{gg} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} S_{ff} & S_{fg} & 0 \\ S_{gf} & S_{gg} + G_{gg} + B_{gg} & B_{ge} \\ 0 & B_{eg} & B_{ee} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{w}_f \\ \mathbf{w}_g \\ \mathbf{w}_e \end{pmatrix}.$$

We have used the notation (cf. equation (6.4.2))

$$F_{ff} = \lambda M_{ff} + A_{ff} + \frac{1}{\lambda} S_{ff},$$

$$\vdots$$

etc.

where $M$ is the mass matrix of the fluid, $A$ the stiffness matrix of the fluid including the penalty matrix $\frac{1}{\lambda} \mathbf{P}^T \mathbf{D}^{-1} \mathbf{P}$ and $S$ the matrix referring to the integral over the free surface of the fluid. Likewise, we have

$$E_{gg} = \lambda M_{gg}^* + \frac{1}{\lambda} B_{gg} + \frac{1}{\lambda} G_{gg},$$

113
\[ \vdots \]

etc.

where \( M^* \) is the mass matrix of the solid, \( B \) the stiffness matrix of the solid and \( G \) the matrix referring to the integral over the interface \( G^0 \). In the vector \( \mathbf{x} \) the components of \( \mathbf{w}_f \) refer to the fluid unknowns not including those on \( G^0 \), the components of \( \mathbf{w}_g \) refer to the unknowns on \( G^0 \) and the components of \( \mathbf{w}_e \) refer to the scaled unknowns in the elastic solid again excluding those on \( G^0 \). The matrices \( A \), \( B \) and \( C \) are of order \( 2(K_f + K_g + K_e) \) with \( A \) symmetric and non-singular, \( B \) symmetric and singular with rank \( 2(K_f + K_g) \). The matrix \( C \) is in general non-symmetric (due to the anti-symmetry of \( G_{gg} \) if the Tong-hypothesis is not applied) and is singular with rank equal to \( 2(K_g + K_e) + \text{rank}(S) \) (\( \text{rank}(S) < 2K_f \) since only degrees of freedom on the free-surface \( S^0 \) give non-zero entries).

So far we have not discussed the types of elements used to discretize the regions \( \Omega_1^0 \) and \( \Omega_2^0 \) nor have we dealt with the particular forms of the basis functions. Regarding the element types we shall restrict ourselves to triangular elements in \( \Omega_1^0 \) and \( \Omega_2^0 \). For the particular form of the basis functions on the triangular elements we first consider equations related to the fluid unknowns in \( \Omega_1^0 \). In the variational formulation of the Stokes equations \( (6.3.1) \), only first order partial derivatives occur, implying that linear basis functions for the approximation of \( w_i \) in \( \Omega_1^0 \) are in principle sufficient (to ensure continuous differentiability element-wise and piece-wise continuity in \( \Omega_2^0 \)). However, the obvious demand that the velocity approximation be not completely described by the continuity equation, forces higher order polynomials to be used for the basis functions for \( w_i \) in \( \Omega_1^0 \) (cf. Cuvelier \textit{et al.}, 1986). Hence, as in the previous chapters, for the finite-element approximation of the fluid velocity and the pressure the Crouzeix-Raviart element is used: the basis functions for the velocity are extended quadratic polynomials based on 7 nodal points (one on each corner of the triangle, one in each midpoint of the sides and one in the centroid), and the basis functions for the pressure are linear (one nodal point in the centroid and two derivatives), see figure 6.2a.

Next consider the finite-element approximation of \( w_i \) in \( \Omega_2^0 \). The variational formulation of the elasticity equations contains only first order partial derivatives so that linear basis functions based on the three vertices of the triangular element are sufficient to obtain a conforming finite-element approximation in \( \Omega_2^0 \), see figure 6.2b. There are no additional constraints which force higher order polynomials to be used. Recall that the derivation of the
Figure 6.2: Element for the fluid velocity and pressure (a) and for displacement unknowns (b).

Figure 6.3: Finite-element discretization.
variational form (6.3.7) imposed some conditions on the test functions at the interface $G^0$. In particular we demanded that the restriction to $G^0$ of test functions in $\Omega_1^0$ and $\Omega_2^0$ should be identical, i.e. $v = e$ on $G^0$. When an extended quadratic polynomial function is used for the velocity approximation in $\Omega_1^0$ and a linear basis function for the (scaled) displacements in $\Omega_2^0$, a discretization as depicted in figure 6.3 is obtained. We note that special element types are required for elements which lie in $\Omega_2^0$ and have one of their sides in common with the interfacial boundary $G^0$. This approach posed some problems since the special elements could not be incorporated easily in the finite-element package we used. We therefore opted to use quadratic polynomial functions on triangular elements for the basis functions in $\Omega_2^0$, thereby automatically satisfying the conditions imposed on the basis functions which lie on $G^0$.

6.5 Numerical results

In order to get some insight into the types of normal modes inherent to the system, we consider the following problem. A rectangular container of length $L = 1$, height $h = 1$ is half filled with a viscous fluid with $Re = 50$. The side walls of the container are assumed rigid while the bottom of thickness $\delta = 0.05$ is assumed flexible with $\bar{E} = 100$ and $\sigma = 0.25$. In figures 6.4a-f vector plots are shown of representative types of normal modes. The vector plots are made of the real parts of the components of the solution vector $(w_f, w_g, w_e)^T$. The first and second free-surface oscillation modes are shown in 6.4a,b respectively. Note that the free surface of the fluid is displaced significantly, but the flexible bottom is not affected noticeably by the fluid motion. Figures 6.4c,d show the first and second transverse oscillations modes of the structure. Observe that in the case of the first transverse mode the free surface of the fluid is also deflected significantly, the second transverse mode only slightly affects the free surface. In figures 6.4e,f vector plots are shown of the first and second longitudinal vibration modes. Observe that for the longitudinal modes a boundary layer forms in $\Omega_1$ near the interfacial surface $G^0$.

As mentioned, the vector plots in figures 6.4 were obtained from the solution vector $(w_f, w_g, w_e)^T$, that is, the plotted quantities in the solid refer to 'velocity unknowns'. We observe that the velocity vectors are continuous across the interfacial boundary $G^0$. Let us now re-scale the unknowns which refer to structural degrees of freedom such that they represent displacement
Figure 6.4: Vector plots of the solution vector corresponding to the first and second free-surface modes (a, b), the first and second transverse structural-vibration modes (c, d) and the first and second longitudinal structural-vibration modes (e, f).
rather than velocity vectors. Instead of just dividing the unknowns \( \mathbf{w}_g \) and \( \mathbf{w}_e \) by \( \lambda \) to obtain the vectors \( \mathbf{d}_g \) and \( \mathbf{d}_e \) which contain displacements, we re-scale as follows \( \mathbf{d}_g = \frac{|\lambda|}{\lambda} \mathbf{w}_g \) and likewise for \( \mathbf{d}_e \). This is done in order to obtain the same order of magnitude for the components of the displacement and velocity vectors (\( |\lambda| \) may be large). In all of the subsequent vector plots the structural unknowns are scaled this way. Figure 6.5a,b show vector plots of the real and imaginary parts respectively of components of the re-scaled solution vector corresponding to the second transverse structural vibration mode. In figure 6.5a we see that the elastic bottom is displaced downwards while the fluid velocity is upwards. Figure 6.5b shows a downward displacement of the elastic bottom, the fluid is directed downwards as well. The time-dependent motion of the system is a linear combination of the vector plots 6.5a and 6.5b.

![Figure 6.5: Vector plots of the real (a) and imaginary (b) parts of fluid velocity and the (scaled) structural displacements corresponding to the second transverse structural vibration mode.](image)

The eigenfrequencies corresponding to the various types of oscillation modes will be investigated in some detail. Regarding the eigenvalues of the free-surface oscillations we found that they were hardly effected by the flexibility of the bottom, i.e. they are virtually identical if a similarly shaped rigid container is used. This is not surprising since the vector plots show no interaction between the fluid and the solid in which case the eigenfrequencies are given by expression (6.3.9). The eigenfrequencies of a freely-vibrating plate in a gravitational field are well-known, cf. Landau & Lifshitz (1959b).
For a plate of thickness \( \delta \) and length \( L \) we have for the longitudinal waves

\[
\omega_n^{(l)} = \frac{n\pi}{L} \sqrt{\frac{\tilde{E}}{1 - \sigma^2}},
\]

and for the transverse waves approximately

\[
\omega_n^{(t)} = \sqrt{1 + \frac{\delta^2 \tilde{E}}{12(1 - \sigma^2)} \left(\frac{(2n + 1)\pi}{2L}\right)^4}.
\]

In figure 6.6a we have plotted the imaginary parts of the eigenvalues corresponding to the transverse structural vibrations. The curve with squares corresponds to the analytical values of a freely vibrating plate given by (6.5.2), the curve with the circles corresponds to the numerical results of an oscillating plate in contact with a viscous fluid. We observe that the oscillation frequencies of a freely vibrating plate are higher than the corresponding modes of a plate in contact with a fluid. This is in agreement with the remarks made in section 3 where we showed that the fluid inertia lowers the imaginary parts of the eigenvalues of structural vibration modes (equation (6.3.11)). Figure 6.6b shows a plot of the imaginary parts of the eigenvalues corresponding the the longitudinal oscillation modes. The curves with
squares denotes once again the eigenvalues of a freely vibrating solid (6.5.1), the curve with the circles those of a solid in contact with a fluid. We observe that the eigenvalues of a freely vibrating solid are again higher than those of the solid in contact with the fluid, however the difference between the curves is less than in figure 6.6a. The reason is that the fluid is only slightly effected by longitudinal vibration modes so that the fluid inertia term in (6.3.11) is small resulting in equation (6.3.10).

Let us briefly reconsider the plots in figures 6.4e,f. Namely, the 'wavy' shape of the solid in the case of the longitudinal vibration is of interest. Longitudinal vibrations of a freely vibrating plate do not display this transverse displacement super-imposed on the longitudinal motion so that the transverse displacement must be due to the presence of the fluid. Consider figure 6.4e in some more detail. We observe a slight transverse displacement with two maxima and two minima, i.e. a displacement akin to the fourth transverse vibration mode. One might expect that the presence of the fourth transverse vibration mode is due to the fact that the eigenfrequency of the first longitudinal mode is close to the eigenfrequency of the fourth transverse mode, resulting in a coupled modal vibration. This is indeed the case. However, with reference to figure 6.6 we note that the eigenfrequencies of a freely vibrating solid rather than those of the solid in contact with the fluid are the relevant frequencies to be considered. Namely, Im(\(\lambda\)) \(\approx\) 15 for the fourth transverse mode of the solid in contact with fluid while Im(\(\lambda\)) \(\approx\) 30 for the freely vibrating solid and Im(\(\lambda\)) \(\approx\) 30 for the first longitudinal mode. The mechanism behind the transverse motion is the pressure difference in the fluid which is the result of the fluid motion in the boundary layer near \(G^0\). We point out that the phenomenon described above is due to the viscosity of the fluid. Neglecting viscous effects does not lead to fluid motion as a result of longitudinal vibrations so that no pressure difference will be generated.

We next investigate the validity of the Tong-hypothesis. As was pointed out in section 6.3, the Tong-hypothesis is exactly satisfied in one special case only, in all other cases it is an approximation. A number of authors, like for example Boujot (1972) and Morand & Ohayon (1979), have applied the Tong-hypothesis in their calculations. We are, however, not aware

![Figure 6.7.](image-url)
of an investigation (numerical or analytical) into the error introduced when the the Tong-hypothesis is applied. It should be pointed out that in case the Tong-hypothesis is not applied the matrices \( A, B \) and \( C \) in equation (6.4.8) have to be stored non-symmetrically which may lead to storage difficulties when problems with a large number of degrees of freedom are considered. It is clearly advantageous when the Tong-hypothesis may be employed. In order to investigate the validity of the Tong approximation we consider an elastic hemispherical container fixed at the top. The radius of the outer boundary \( F \) is taken to be 1, the thickness of the container wall is \( \delta = 0.1 \) (see figure 6.7) and Poissons ratio is \( \sigma = 0.15 \). A fluid with \( Re = 200 \) fills the container entirely. In table 6.1 we present the eigenvalues of the first and second symmetric and anti-symmetric structural vibration modes for

<table>
<thead>
<tr>
<th></th>
<th>( \tilde{E} = 15 )</th>
<th></th>
<th>( \tilde{E} = 1500 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>without Tong</td>
<td>with Tong</td>
<td>without Tong</td>
</tr>
<tr>
<td>1st a-symm.</td>
<td>-0.007+0.69i</td>
<td>-0.013+0.81i</td>
<td>-0.052+3.05i</td>
</tr>
<tr>
<td>1st symm.</td>
<td>-0.040+1.34i</td>
<td>-0.040+1.34i</td>
<td>-0.047+5.97i</td>
</tr>
<tr>
<td>2nd a-symm.</td>
<td>-0.096+1.91i</td>
<td>-0.099+1.95i</td>
<td>-0.140+12.9i</td>
</tr>
<tr>
<td>2nd symm.</td>
<td>-0.150+2.68i</td>
<td>-0.150+2.70i</td>
<td>-0.180+21.8i</td>
</tr>
</tbody>
</table>

Table 6.1: Table of eigenvalues for \( \tilde{E} = 15 \) and \( \tilde{E} = 1500 \)

\( \tilde{E} = 15 \) and \( \tilde{E} = 1500 \). For both values of \( \tilde{E} \) we have calculated the eigenvalues using the exact expression on \( G^0 \) (columns 2 and 4) and using the Tong approximation (columns 3 and 5). We observe first of all that invoking the Tong-hypothesis increases the eigenvalues and in particular the imaginary parts thereof. Note also that the symmetric modes are effected less by the Tong-hypothesis than the anti-symmetric modes and that the effect of the Tong approximation diminishes for the higher modes. For \( \tilde{E} = 1500 \) we observe only a slight difference between the exact and approximate results. In fact, we found that for \( \tilde{E} \geq 1500 \) the boundary integral over \( G^0 \) could be neglected entirely without changing the eigenvalues by more than 3%. This might have been anticipated since the energy required to displace the interface, characterized by the boundary integral over \( G^0 \), is small compared with the bending energy of the solid when \( \tilde{E} \) is large. Thus, the question of whether or not the Tong-hypothesis may be applied is only important when \( \tilde{E} \) is small.
In the last numerical experiment we consider a problem which is derived from the aerospace industry. In propellant tanks of spacecraft and satellites one has, in a number of cases, placed baffles on the sides of the tank. The aim of the baffles is to prevent the occurrence of excessive fluid sloshing during lift of and flight and to keep the eigenfrequencies of the liquid (in particular those of the lowest anti-symmetric modes) well separated from the eigenfrequencies of the spacecraft, see for example Abramson (1966). Numerous baffle types and shapes exist ranging from rigid perforated rings to flexible slabs. Stephens (1966) found that the damping coefficients of free-surface modes in tanks with flexible baffles mounted on the sides, were larger than those in tanks without baffles or with rigid baffles. Experimental tests showed that the damping coefficients were dependent on the rigidity of the baffle and, in fact, attained a maximum for a certain value of the rigidity.

To verify these results numerically, we consider a rectangular elastic baffle of length $L = 0.5$ and thickness $\delta = 0.05$ fixed to the side of a rigid container. The container consists of a rectangular top part and a hemispherical bottom of radius 1 (see figure 6.9a). For the fluid we take $Re = 50$ and Poisson’s ratio is $\sigma = 0.25$. We consider the eigenvalues of the lowest eigenmodes of the system, i.e. the first free-surface mode and the lowest vibration mode of the baffle. In figure 6.8 the values of $\text{Im}(\lambda)$ and $\text{Re}(\lambda)$ have been plotted against $\tilde{E}$, for $\tilde{E}$ in the range $100 \leq \tilde{E} \leq 1500$. The values of $\text{Im}(\lambda)$ on the curve with squares in figure 6.8a correspond to the values $\text{Re}(\lambda)$ on the curve with squares in figure 6.8b and likewise for the curves with circles. The dependence on $\tilde{E}$ of the curves $\text{Im}(\lambda)$ is typical of a system with two different oscillatory components each of which depends on a common parameter. A branch corresponding to one type of modal vibrations changes over into a branch representing the other type of modal vibration as the common parameter, $\tilde{E}$ in this case, is varied (cf. chapter 5). Here, for $\tilde{E} \leq 500$ the branch with the squares corresponds to the free surface modes while for $\tilde{E} \geq 1300$ the branch with circles corresponds to free surface modes. For $\tilde{E} \approx 1000$ we have coupled free-surface and baffle vibrations. If one is interested in maximizing the damping coefficient of the first anti-symmetric fluid oscillation mode, it follows from figure 6.8b that the material properties of the baffle must be chosen such that $\tilde{E} \approx 1200$. Namely, for that value of $\tilde{E}$ the damping coefficient for the free-surface mode is $-0.4$, for all other values of $\tilde{E}$ the coefficient is larger and approaches $-0.2$ for values of $\tilde{E}$ far removed from the optimum value. Hence by an appropriate choice of the material properties of the baffle, the damping coefficient of the first
Figure 6.8: Im($\lambda$) and Re($\lambda$) plotted against $\tilde{E}$ in (a) and (b) respectively.

anti-symmetric eigenmode can be increased by a factor 2. In figure 6.9 vector plots are shown a baffle vibration mode for $\tilde{E} = 100$ (b), and of the free-surface mode for $\tilde{E} = 1000$ (c) and $\tilde{E} = 10^5$ (d).

6.6 Conclusion.

In this paper we have studied the interaction of a viscous fluid with an elastic solid. In particular we have investigated the eigenmodes of the coupled fluid-structure system. Under the assumption that the so-called Tong-hypothesis holds, we have shown that oscillation modes exist with a number of different properties. First of all there exist fluid-oscillation modes. These modes are characterized by a small deflection of the structure and a large displacement of the fluid particles. Secondly there exist normal modes in which the structure is deflected significantly not effecting the fluid to a large extend. Finally there are modes in which the fluid as well as the structure is displaced significantly. The existence of all three kinds of modes is verified numerically.

It was shown analytically that the extend to which the fluid effects the eigenvalues of the solid depends on the inertia of the fluid displaced by the solid. In all cases does the fluid lower the imaginary parts of the eigenvalues of the structural vibrations. Numerically we find that the presence of the fluid can lower the imaginary parts of the eigenvalues corresponding to the
Figure 6.9: Schematic diagram of a container with baffle (a). Vector plots of a baffle vibration mode for $\tilde{E} = 100$ (b), the first anti-symmetric surface mode for $\tilde{E} = 1000$ (c) and $\tilde{E} = 10^5$ (d).
transverse vibration modes by up to 50%. Eigenvalues of the longitudinal modes are effected less by the presence of the fluid due to the small fluid inertia term in that case.

The validity of the Tong-hypothesis is investigated numerically. It is found that application of the Tong-hypothesis increases the magnitude of the eigenvalues of the normal modes. The Tong-hypothesis mostly effects the lowest anti-symmetric eigenvalues. Symmetric modes and higher oscillation modes are effected less. Taking Young's modulus large enough also reduces the effect of the Tong-hypothesis.

We have shown numerically that there are cases in which the viscosity of the fluid plays an important role. This applies in particular to longitudinal vibrations in the solid which create a boundary layer in the fluid adjacent to the fluid-solid interface. A pressure difference along the interfacial boundary, which is the result of the fluid motion, causes the solid to undergo a transverse displacement. In problems where damping coefficients of the system are to be found it is obvious that viscous effects have to be included. We have seen that for a relatively complicated system of a viscous fluid in a rigid tank in which a flexible baffle is placed, the eigenmodes can be calculated. We found that the damping of the system is dependent on the rigidity of the baffle. In fact, there exists a value of the rigidity for which the damping is maximum. This is in agreement with experimental observations.
Chapter 7

Concluding remarks

The work presented in this thesis is centered around two topics. The first topic, comprising chapters 2, 3 and 4, deals with the static and dynamic behaviour of a capillary free boundary. The second topic, dealt with in chapters 5 and 6, regards an investigation into some aspects of fluid-structure interactions.

In all cases where the dynamic behaviour of the (capillary) free liquid surface or the fluid-structure system is considered, the governing equations are linearized. This means that at first the static equilibrium configuration of the system has to be determined. In chapter 2 it is shown how the equilibrium configuration of a rotationally-symmetric capillary free-boundary problem can be calculated in rather complex geometries. The calculations have been restricted to the one-dimensional case (the geometry is two-dimensional), the two-dimensional problem remains largely unsolved. The solution technique we have used can, however, be extended directly to two-dimensional problems. This is a distinct advantage over other methods that have been used for solving the capillary free-boundary problem.

Finding the eigenfrequencies of a liquid with a capillary free boundary by analytical means, is difficult due to the boundary conditions and complicated geometries involved. When viscous effects are neglected it is found that the boundary conditions on the capillary free boundary may be simplified by means of Green’s functions. This allows analytical solutions to be found in simple geometries (chapter 3). However, in complicated geometries or when viscous effects are to be incorporated, the employment of numerical solution techniques is necessary. For the discretization of the governing equations it is found that the finite-element technique is very effective. First
of all because the (complicated) boundary conditions are dealt with in a natural way and, secondly, because complicated geometries do not pose a problem. In conjunction with an inverse-iteration procedure to solve the quadratic eigenvalue problem, the finite-element technique is well-suited to calculate eigenfrequencies of a viscous, rotating fluid with a capillary free boundary (chapter 4). We feel that the approach presented in chapter 4 for the computation of eigenmodes of rotating fluids, offers a distinct advantage over other methods like the homogeneous-vortex approach. The main thrust of our argument is that no questionable simplifications need be made in our approach so that the physics of the system is modelled correctly.

The main extension of the work presented in chapter 4 lies in a generalization to real-life three-dimensional problems. Namely, the first antisymmetric eigenmode of free-surface oscillations is generally of principle interest and, furthermore, containers are often not rotationally symmetric or do not rotate about the symmetry axis of the container.

Two models have been considered for the structure in the fluid-structure interaction problem: a membrane model (chapter 5) and a linear elasticity model (chapter 6). It is found that with the simple membrane model some important characteristics of fluid-structure interaction problems can be established. General results, like the effect of the fluid on the eigenfrequencies of the structure, are qualitatively the same for the membrane and the linear-elasticity model. When the stiffness of the solid is important, the membrane model for the structure is not adequate and more advanced models have to be considered. A specific example of a system in which the stiffness of the solid is essential is given in the final section of chapter 6. We considered a viscous fluid in a rigid tank on the side of which a flexible baffle was mounted. The damping characteristics of the system was found to depend on the rigidity of the baffle, confirming experimental results. The finite-element discretization in conjunction with the inverse-iteration procedure is again found effective for the computation of eigenmodes of the fluid-structure system. Extensions of the work in chapters 5 and 6 lie in the generalization of the models to three dimensions and to application of the models to some real-life problems.

Modelling the dynamic behaviour of a viscous fluid in a low-gravity environment is a goal of the research presented in this thesis. It is known that various industrial processes involving liquids, benefit greatly from the absence of gravity so that fluids in a micro-gravity environment are likely to become increasingly common in the near future (see Walter, 1987). The potential of processes involving liquid bridges, to mention but one example,
increases significantly in space. However, liquid bridges occur often in the form of molten material anchored between two rigid parts. A more realistic model for the dynamics of a liquid bridge would therefore require the inclusion of phenomena like temperature dependence of the surface-tension coefficient, oscillations superimposed on convection-induced flows inside the liquid bridge, etc.. It will be evident that these phenomena are often non-linear thus complicating the problem significantly. Our work can only be regarded as a first step in the direction of solving these complicated problems.

Modelling the dynamic behaviour of a satellite containing liquid propellants and flexible parts, may be regarded as a second goal of the work presented here. Reaching this goal requires the aforementioned extensions of the models to three dimensions, incorporation of the conservation of momentum equations and coupling of the work in chapters 4 and 6. Clearly, extensions in each of these directions will be non-trivial and will require a great deal of effort.
Appendix A

For the construction of the Green's function for the operator $L(\psi)$ as defined in section 3.2, we have to find two linearly independent solutions of the problem $L(\psi) = 0$. One of the solutions must satisfy the boundary condition at $r = 0$ while the other satisfies the boundary condition at $r = a$. The general solution of the differential equation $L(\psi) = 0$ is given by

$$\psi(r) = AI_0(r\sqrt{Bo}) + BK_0(r\sqrt{Bo}),$$

where $A$ and $B$ are constants. The function $\psi(r)$ satisfying the boundary condition $\psi'(0) = 0$ is given by

$$\psi_1(r) = A_1 I_0(r\sqrt{Bo}).$$

Likewise, the solution of the homogeneous differential equation satisfying the boundary condition $\psi(a) = 0$ is given by

$$\psi_2(r) = A_2 \left[ I_0(r\sqrt{Bo}) - \frac{I_0(a\sqrt{Bo})}{K_0(a\sqrt{Bo})} K_0(r\sqrt{Bo}) \right].$$

The Green's function is then given by

$$K(r,\xi) = -\frac{1}{c} \begin{cases} \psi_1(r)\psi_2(\xi) & r \leq \xi \\ \psi_1(\xi)\psi_2(r) & r > \xi \end{cases}$$

where

$$c = \xi[\psi_1(\xi)\psi_2'(\xi) - \psi_1'(\xi)\psi_2(\xi)] = \frac{I_0(a\sqrt{Bo})}{K_0(a\sqrt{Bo})} = \text{constant}.$$ 

Substituting for $\psi_1$ and $\psi_2$ yields the desired expression.
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Summary

Fluid oscillations in rigid and flexible containers occur in a wide variety of practical situations. When a free fluid surface exists these oscillations may lead to large deformations of the free surface. These large deformations are often undesirable due to, for example, stability considerations (in air- and spacecraft) or conditions of quiescence required for industrial processes (crystal growth).

In this thesis we study the motion and in particular the eigenfrequencies of fluids in rigid and flexible containers. In all cases we consider the linearized equations - that is, starting from the static configuration of the system we consider only small perturbations with respect to this static state so that all non-linear products of small quantities may be neglected. This procedure implies that the static configuration of the system must be known. For a fluid with a capillary free surface this means that the shape of the static capillary free surface has to be determined. A solution procedure for this problem comprises the first part of this thesis.

Oscillations of an inviscid fluid subjected to edge constraints are studied analytically. We show how the problem can be treated when the conditions on the capillary surface are rewritten by means of Green's functions. The severe geometrical constraints of analytical methods and the restrictive conditions on flow types that can be studied analytically, lead to numerical solution techniques to be considered. We use the finite-element technique to discretize the equations governing the motion of an incompressible, viscous fluid with a capillary free surface. This procedure leads to a generalized eigenvalue problem which can be solved by means of an inverse-iteration procedure. It is shown that this procedure is effective in calculating eigenfrequencies of fluid oscillations in complex geometries.

Oscillations of a viscous fluid in a flexible container are studied for two different models of the container wall. First we consider a membrane model for the container wall. This model is interesting due to its simplicity while all basic features of the fluid-structure problem can be demonstrated. A more advanced structure model is considered subsequently. Namely, the structure is assumed to consist of an elastic solid obeying the equations of linear elasticity. The principal difficulty of this fluid-structure model is the derivation of a unified variational formulation. The different nature of the two media (fluid with velocity unknowns and solid with displacement unknowns), requires the introduction of a new variable such that in both media the unknowns have the dimension of velocity.
Samenvatting

Oscillaties van vloeistoffen in een starre of flexibele container komen veelvuldig voor. In geval er een vrij vloeistofoppervlak is, kunnen oscillaties leiden tot grote verplaatsingen van de vloeistof aan het vrije oppervlak. Deze vloeistofbewegingen zijn vaak ongewenst in verband met stabiliteits-eisen (in bijvoorbeeld vliegtuigen en ruimtevaartuigen) of in verband met voorwaarden die gesteld worden aan industriële processen (zoals in het geval van kristalgroei).

In dit proefschrift beschouwen we het dynamische gedrag en in het bijzonder de eigenfrequenties van vloeistoffen in een starre of flexibele container. In alle gevallen gaan we uit van de lineaire vergelijkingen. Dat wil zeggen, uitgaande van de configuratie van het systeem in de statische toestand beschouwen we slechts kleine verstoringen met betrekking tot deze statische configuratie zodat alle (niet lineaire) produkten van kleine groot- heden verwaarloosbaar zijn. Deze handelswijze impliceert dat de statische configuratie van het systeem bekend moet zijn. In geval van een vloeistof met een capillaire vrije rand betekent dit dat de vorm van de statische capillaire vrije rand allereerst bepaald moet worden. De oplossing van dit probleem beslaat het eerste deel van dit proefschrift.

Oscillaties van een niet-visceuze vloeistof waarbij op het vrije vloeistofoppervlak zogenaamde "edge-constraints" zijn voorgeschreven, zijn bestudeerd met behulp van analytische technieken. We laten zien hoe het probleem kan worden opgelost als de randvoorwaarden op de vrije rand worden herschreven met behulp van Greense functies. De toepasbaarheid van analytische oplosmethoden op praktische problemen is gering vanwege de geometrische beperkingen en als het gevolg van beperkingen die aan de vloeistofstroming opgelegd moeten worden. Dit leidt er toe dat numerieke oplosmethoden toegepast worden. In dit proefschrift wordt de eindige-elementen techniek gebruikt om de vergelijkingen die de beweging van een visceuze, niet compressibele vloeistof beschrijven te discretiseren. Dit resulteert in een gegeneraliseerd eigenwaardeprobleem dat opgelost wordt met behulp van inverse-iteratie. We laten zien dat eigenfrequenties van vloeistofoscillaties in gecompliceerde geometrieën efficiënt kunnen worden berekend op deze manier.

Oscillaties van een visceuze vloeistof in een flexibele container worden bestudeerd voor twee verschillende modellen voor de containerwand. Allereerst beschouwen we een membraanmodel voor de containerwand. Dit model is interessant vanwege de eenvoud terwijl de belangrijkste eigenschappen
van het vloeistof-vaste stof interaktie probleem aanschouwbaar zijn. Een geavanceerder model voor de vaste stof wordt vervolgens behandeld. De vaste stof wordt namelijk beschouwd als een elastische stof die voldoet aan de vergelijkingen voor lineaire elasticiteit. De voornaamste moeilijkheid met dit vloeistof-vaste stof model is het bepalen van een variationele formulering. De verschillende eigenschappen van de twee media (vloeistof met snelheidsonbekenden en de vaste stof met als onbekenden de verplaatsingen) noodza-ken tot de invoering van een nieuwe onbekende, zodanig dat in de twee media de onbekenden de dimensie van snelheid hebben. De gediscretiseerde vergelijkingen resulteren wederom in een gegeneraliseerd eigenwaardeprobleem.
Curriculum Vitae


Sinds oktober 1986 is de auteur verbonden aan de vakgroep Toegepaste Analyse van de Faculteit der Technische Wiskunde & Informatica van de Technische Universiteit Delft. Aldaar heeft hij onder begeleiding van Prof. Dr. A.E.P. Veldman en Dr. C. Cuvelier het onderzoek verricht dat heeft geleid tot de totstandkoming van dit proefschrift. In het kader van het promotieonderzoek werd werk gepresenteerd op congressen in Engeland, Finland, Nederland en de Verenigde Staten.

De auteur heeft zich naast zijn promotieonderzoek in Delft aktief beziggehouden met zaken aangaande het aio-stelsel. Hij is mede-oprichter en voorzitter van het Delfts AIO Overleg.