I. INTRODUCTION

Since the work of Weaver and Lobkis,\textsuperscript{1,2} many researchers have shown theoretically and experimentally that the cross correlation of the recordings of a diffuse wave field at two receiver positions yields the Green’s function between these positions. In most cases it is assumed that the diffuse wave field consists of normal modes (with uncorrelated amplitudes) in a closed system. Less attention has been paid to the theory of Green’s function retrieval in arbitrary inhomogeneous open systems. Nevertheless, the first result stems from 1968, albeit for one-dimensional media, when Claerbout\textsuperscript{3} showed that the seismic reflection response of a homogeneous open system can be obtained by cross correlating recordings of the wave field at two positions. One approach is based on physical arguments, exploiting the principle of time-reversal invariance of the acoustic wave equation. The other approach is based on Rayleigh’s reciprocity theorem. Using a unified notation, we show that the result of the time-reversal approach can be obtained as an approximation of the result of the reciprocity approach. © 2005 Acoustical Society of America. [DOI: 10.1121/1.2046847]

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II. TIME-REVERSAL APPROACH

In this section we summarize the time-reversal approach of Derode et al.\textsuperscript{8,9} for deriving expressions for Green’s function retrieval. Consider a lossless arbitrary inhomogeneous acoustic medium in a homogeneous embedding. In this configuration we define two points with coordinate vectors \( \mathbf{x}_A \) and \( \mathbf{x}_B \). Our aim is to show that the acoustic response at \( \mathbf{x}_B \) due to an impulsive source at \( \mathbf{x}_A \) [i.e., the Green’s function \( G(\mathbf{x}_B, \mathbf{x}_A, t) \)] can be obtained by cross correlating passive measurements of the wave fields at \( \mathbf{x}_A \) and \( \mathbf{x}_B \) due to sources on a surface \( S \) in the homogeneous embedding. The derivation starts by considering another physical experiment, namely an impulsive source at \( \mathbf{x}_A \) and receivers at \( \mathbf{x} \) on \( S \). The response at one particular point \( \mathbf{x} \) on \( S \) is denoted by \( G(\mathbf{x}, \mathbf{x}_A, t) \). Imagine that we record this response for all \( \mathbf{x} \) on \( S \), revert the time axis, and feed these time-reverted functions \( G(\mathbf{x}, \mathbf{x}_A, -t) \) to sources at all \( \mathbf{x} \) on \( S \). The superposition principle states that the wave field at any point \( \mathbf{x}' \) in \( S \) due to these sources on \( S \) is then given by

\[
    u(\mathbf{x}', t) \approx \int_S G(\mathbf{x}', \mathbf{x}_A, t) \ast \frac{G(\mathbf{x}, \mathbf{x}_A, -t) d^2 \mathbf{x}}{\text{source}}.
\]

where \( \ast \) denotes convolution and \( \approx \) “proportional to.” According to this equation, \( G(\mathbf{x}', \mathbf{x}, t) \) propagates the source function \( G(\mathbf{x}, \mathbf{x}_A, -t) \) from \( \mathbf{x} \) to \( \mathbf{x}' \) and the result is integrated over all sources on \( S \). Due to the invariance of the acoustic wave equation for time-reversal, the wave field \( u(\mathbf{x}', t) \) focuses for \( \mathbf{x}' = \mathbf{x}_A \) at \( t = 0 \). McMechan\textsuperscript{10} exploited this property in a seismic imaging method which has become known as reverse time migration. Derode et al.\textsuperscript{8,9} give a new interpretation to Eq. (1). Since \( u(\mathbf{x}', t) \) focuses for \( \mathbf{x}' = \mathbf{x}_A \) at \( t = 0 \), the
wave field \( u(x',t) \) for arbitrary \( x' \) and \( t \) can be seen as the response of a virtual source at \( x_\lambda \) and \( t=0 \). This virtual source response, however, consists of a causal and an anticausal part, according to

\[
\begin{align*}
u(x',t) = G(x',x_\lambda,t) + G(x',x_\lambda,-t).
\end{align*}
\]

(2)

This is explained as follows: the wave field generated by the anticausal sources on \( S \) first propagates to all \( x' \) where it gives an anticausal contribution, next it focuses in \( x_\lambda \) at \( t = 0 \), and finally it propagates again to all \( x' \) giving the causal contribution. The propagation paths from \( x' \) to \( x_\lambda \) are the same as those from \( x_\lambda \) to \( x' \), but are traveled in opposite direction. Combining Eqs. (1) and (2), applying source-receiver reciprocity to \( G(x,x_\lambda,-t) \) in Eq. (1), and setting \( x'=x_B \) yields

\[
\begin{align*}
G(x_B,x_\lambda,t) + G(x_B,x_\lambda,-t) \propto \int \left[ G(x_B,x,t) - G(x_B,x,-t) \right] d^2x.
\end{align*}
\]

(3)

The right-hand side of Eq. (3) can be interpreted as the integral of cross correlations of observations of wave fields at \( x_B \) and \( x_\lambda \), respectively, due to impulsive sources at \( x \) on \( S \); the integration takes place along the source coordinate \( x \). The left-hand side is interpreted as the superposition of the response at \( x_B \) due to an impulsive source at \( x_\lambda \) and its time-reversed version. Since the Green’s function \( G(x_B,x_\lambda,t) \) is causal, it can be obtained from the left-hand side of Eq. (3) by taking the causal part. The reconstructed Green’s function contains the ballistic wave as well as the coda due to multiple scattering in the inhomogeneous medium.

III. RECIPROCITY APPROACH

In this section we summarize our derivation based on Rayleigh’s reciprocity theorem.\(^{4-6} \) A reciprocity theorem relates two independent acoustic states in one and the same domain.\(^{11,12} \) Consider an acoustic wave field, characterized by the acoustic pressure \( p(x,t) \) and the particle velocity \( v_i(x,t) \). We define the temporal Fourier transform of a space- and time-dependent quantity \( p(x,t) \) as \( \hat{p}(x,\omega) = \int \exp(-j\omega t)p(x,t)dt \), where \( j \) is the imaginary unit and \( \omega \) the angular frequency. In the space-frequency domain the acoustic pressure and particle velocity in a lossless arbitrary inhomogeneous acoustic medium obey the equation of motion \( j\omega \rho \hat{p} + \nabla^2 \hat{v}_i = \hat{q} \), where \( \hat{v}_i \) is the partial derivative in the \( x_i \) direction (Einstein’s summation convention applies for repeated lower-case subscripts), \( \rho(x) \) the mass density of the medium, \( \kappa(x) \) its compressibility, and \( \hat{q}(x,\omega) \) a source distribution in terms of volume injection rate density. We introduce two independent acoustic states, which will be distinguished by subscripts \( A \) and \( B \), and consider the following combination of wave fields in both states: \( \hat{p}_A \hat{v}_{iA} - \hat{v}_{iA} \hat{p}_B \). Note that these products in the frequency domain correspond to convolutions in the time domain. Rayleigh’s reciprocity theorem is obtained by applying the differential operator \( \partial_i \), according to \( \partial_i [\hat{p}_A \hat{v}_{iA} - \hat{v}_{iA} \hat{p}_B] \), substituting the equation of motion and the stress-strain relation for states \( A \) and \( B \), integrating the result over a spatial domain \( V \) enclosed by \( S \) with outward pointing normal vector \( n=(n_1,n_2,n_3) \) and applying the theorem of Gauss. This gives

\[
\begin{align*}
\int_V \left[ \hat{p}_A \hat{v}_{iA} - \hat{v}_{iA} \hat{p}_B \right] d^3\!x = \int_S \left[ \hat{p}_A \partial_i \hat{v}_{iA} - \hat{v}_{iA} \partial_i \hat{p}_B \right] n_i d^2\!x.
\end{align*}
\]

(4)

Since the medium is lossless, we can apply the principle of time-reversal invariance.\(^{13} \) In the frequency domain time-reversal is replaced by complex conjugation. Hence, when \( \hat{p} \) and \( \hat{v}_i \) are a solution of the equation of motion and the stress-strain relation with source distribution \( \hat{q} \), then \( \hat{p}^* \) and \( -\hat{v}^*_i \) obey the same equations with source distribution \( -\hat{q}^* \) (the asterisk denotes complex conjugation). Making these substitutions for state \( A \) we obtain

\[
\begin{align*}
\int_V \left[ \hat{p}_A \hat{v}_{iA} + \hat{v}_{iA} \hat{p}_B \right] d^3\!x = \int_S \left[ \hat{p}_A \partial_i \hat{v}_{iA} + \hat{v}_{iA} \partial_i \hat{p}_B \right] n_i d^2\!x.
\end{align*}
\]

(5)

Next we choose impulsive point sources in both states, according to \( \hat{q}_A(x,\omega) = \delta(x-x_{\lambda A}) \) and \( \hat{q}_B(x,\omega) = \delta(x-x_{\lambda B}) \), with \( x_{\lambda A} \) and \( x_{\lambda B} \) both in \( V \). The wave field in state \( A \) can thus be expressed in terms of a Green’s function, according to

\[
\begin{align*}
\hat{p}_A(x,\omega) = \hat{G}(x,x_{\lambda A},\omega),
\end{align*}
\]

(6)

\[
\begin{align*}
\hat{v}_{iA}(x,\omega) &= - (j\omega \rho(x))^{-1} \partial_i \hat{G}(x,x_{\lambda A},\omega),
\end{align*}
\]

(7)

where \( \hat{G}(x,x_{\lambda A},\omega) \) obeys the wave equation

\[
\begin{align*}
\partial_i (\rho^{-1} \partial_i \hat{G}) + (\omega^2/\kappa(x)) \hat{G} = -j\omega \delta(x-x_{\lambda A}),
\end{align*}
\]

(8)

with propagation velocity \( c(x) = (\kappa(x)\rho(x))^{1/2} \); similar expressions hold for the wave field in state \( B \). Substituting these expressions into Eq. (5) and using source-receiver reciprocity of the Green’s functions gives

\[
\begin{align*}
2\Re\{\hat{G}(x_B,x_{\lambda A},\omega)\} = \int \left[ \frac{1}{j\omega \rho(x)} \right] (-\partial_i \hat{G}(x_B,x,\omega)) \hat{G}^*(x_{\lambda A},x,\omega) n_i d^2\!x,
\end{align*}
\]

(9)

where \( \Re \) denotes the real part. Note that the left-hand side is the Fourier transform of \( \hat{G}(x_B,x_{\lambda A},t) + \hat{G}(x_B,x_{\lambda A},-t) \); the products \( \partial_i \hat{G}^* \), etc., on the right-hand side correspond to cross correlations in the time domain. Expressions like the right-hand side of this equation have been used by numerous researchers (including the authors) for seismic migration in the frequency domain. Esmerroy and Oristaglio\(^{14} \) explained the link with the reverse time migration method, mentioned in Sec. II. What is new (compared with migration) is that Eq. (9) is formulated in such a way that it gives an exact representation of the Green’s function \( \hat{G}(x_B,x_{\lambda A},\omega) \) in terms of cross correlations of observed wave fields at \( x_B \) and \( x_{\lambda A} \). Note that, unlike in Sec. II, we have not assumed that the medium outside surface \( S \) is homogeneous. The terms \( \hat{G} \) and \( \hat{G}^* \) under the integral represent responses of monopole and dipole sources at \( x \) on \( S \); the combination of the two correlation products under the integral ensures that waves propagating outward from the sources on \( S \) do not interact with those propagating inward and vice versa. When a part of \( S \) is a free


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The accuracy of this approximation is demonstrated with a simple example. We consider a two-dimensional configuration with a single point diffractor at \((x_1, x_3) = (0, 600)\) m in a homogeneous medium with propagation velocity \(c = 2000\) m/s, see Fig. 1, in which \(C\) denotes the diffractor. Further, we define \(x_4 = (-500, 100)\) m and \(x_5 = (500, 100)\) m, denoted by A and B in Fig. 1. The surface \(S\) is a circle with its center at the origin and a radius of 800 m. The solid arrows in Fig. 1 denote the propagation paths of the Green’s function \(G(x_B, x_A, t)\). For the Green’s functions in Eq. (10) we use analytical expressions, based on the Born approximation (hence, the contrast at the point diffractor is assumed to be small). To be consistent with the Born approximation, in the cross correlations we also consider only the zeroth- and first-order terms. Figure 2(a) shows the time-domain representation of the integral of Eq. (10), convolved with a wavelet with a central frequency of 50 Hz. Each trace corresponds to a fixed source position \(x\) on \(S\); the source position in polar coordinates is \((\phi, r = 800)\). The sum of all these traces (multiplied by \(rd\phi\)) is shown in Fig. 2(b). This result accurately matches the time-domain version of the left-hand side of Eq. (10), i.e., \(G(x_B, x_A, t) + G(x_B, x_A, -t)\), convolved with a wavelet, see homogeneous medium as well as the source positions are accurately known), we approximate \(n_i d_c G\) by \(-j(\omega/c)\hat{G}\), hence

\[
2\Re\{\hat{G}(x_B, x_A, \omega)\} = \frac{2}{\rho c} \int_S \hat{G}(x_B, x, \omega) \hat{G}^{\ast}(x_A, x, \omega) d^2 x. \tag{11}
\]

This approximation is quite accurate when \(S\) is a sphere with very large radius so that all rays are normal to \(S\) (i.e., \(\alpha = 0\)). In general, however, this approximation involves an amplitude error that can be significant, see the numerical example in Sec. V. However, since this approximation does not affect the phase it is considered acceptable for many practical situations. Transforming both sides of Eq. (11) back to the time domain yields Eq. (3) (i.e., the result of Derode et al.8,9), with proportionality factor \(2/\rho c\).

**V. NUMERICAL EXAMPLE**

We illustrate Eq. (10) with a simple example. We consider a two-dimensional configuration with a single point diffractor at \((x_1, x_3) = (0, 600)\) m in a homogeneous medium with propagation velocity \(c = 2000\) m/s, see Fig. 1, in which \(C\) denotes the diffractor. Further, we define \(x_4 = (-500, 100)\) m and \(x_5 = (500, 100)\) m, denoted by A and B in Fig. 1. The surface \(S\) is a circle with its center at the origin and a radius of 800 m. The solid arrows in Fig. 1 denote the propagation paths of the Green’s function \(G(x_B, x_A, t)\). For the Green’s functions in Eq. (10) we use analytical expressions, based on the Born approximation (hence, the contrast at the point diffractor is assumed to be small). To be consistent with the Born approximation, in the cross correlations we also consider only the zeroth- and first-order terms. Figure 2(a) shows the time-domain representation of the integral of Eq. (10), convolved with a wavelet with a central frequency of 50 Hz. Each trace corresponds to a fixed source position \(x\) on \(S\); the source position in polar coordinates is \((\phi, r = 800)\). The sum of all these traces (multiplied by \(rd\phi\)) is shown in Fig. 2(b). This result accurately matches the time-domain version of the left-hand side of Eq. (10), i.e., \(G(x_B, x_A, t) + G(x_B, x_A, -t)\), convolved with a wavelet, see homogeneous medium as well as the source positions are accurately known), we approximate \(n_i d_c G\) by \(-j(\omega/c)\hat{G}\), hence

\[
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\]

This approximation is quite accurate when \(S\) is a sphere with very large radius so that all rays are normal to \(S\) (i.e., \(\alpha = 0\)). In general, however, this approximation involves an amplitude error that can be significant, see the numerical example in Sec. V. However, since this approximation does not affect the phase it is considered acceptable for many practical situations. Transforming both sides of Eq. (11) back to the time domain yields Eq. (3) (i.e., the result of Derode et al.8,9), with proportionality factor \(2/\rho c\).
VI. CONCLUSIONS

In the literature several derivations have been proposed for Green’s function retrieval from cross correlations of wave fields in inhomogeneous open systems. In this letter we compared a derivation based on the time-reversal approach with one based on Rayleigh’s reciprocity theorem. One of the conclusions is that the expression obtained by the time-reversal approach is an approximation of that based on Rayleigh’s reciprocity theorem.

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