ON ISOTHERMAL FLOW
OF VISCOUS LIQUIDS
THROUGH SCREW PUMPS

PROEFSCHRIFT

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1. Introduction to Screw Pump Theory

1.1 Introduction

Screw pumps have become increasingly more important with the growth of the plastics and synthetic fibers industries. Screw pumps are used almost exclusively to pump liquids with very large viscosities - in the liquid-filled discharge zone of a plasticating extruder, for example. Polymeric materials are melted in plasticating extruders and discharged at high pressure through a die to form the desired shape.

This Thesis is concerned with the isothermal flow of very viscous liquids through screw pumps. Present theories are based on a number of simplifying assumptions or restrictions. A more general theory should correctly account for the true geometry of the channels, the physical properties of the liquid, the temperature changes in the liquid, and the operating conditions. For the present, such a general theory cannot be given.

For instance, in present theories the helical channel is almost always replaced by a straight prismatic channel with a rectangular cross section. The assumed inlet and discharge channel geometry differs in these theories from the actual geometry. The simplified geometry can lead to discrepancies between calculated and observed results.

The non-Newtonian properties of the liquids tend to be blamed for some of these differences. Such reasoning is acceptable only when the flow analysis accounts for the correct geometry of the screw pump channels.
One purpose of this Thesis is to provide a more complete theory through the elimination of some of the simplifying assumptions about the screw pump channel geometry. The influence of the differences in channel geometry on screw pump performance can best be studied when other factors are constant. These influences will therefore be investigated for isothermal flow of a Newtonian liquid.

Flow through a helical channel bounded by cylindrical surfaces and by helical flights will be discussed in Chapter 2. The influence of the oblique inlet and discharge ends of the channel on screw pump performance is given in Chapter 4.

A comparison between these new theories and the current theory requires a fuller discussion of some elements of the screw pump theory based on isothermal flow through rectangular channels. Discussion of the existing theory is preceded here by a description of the geometry of screw pumps.

1.2 Screw Pump Geometry

Screw pumps generally consist of a single screw rotating in a cylindrical barrel. Twin parallel screws are sometimes used in plasticating extruders but never in screw pumps.

A schematic diagram of a screw pump is shown in Fig. 1.1.
**Fig. 1.1 - Schematic Diagram of Screw Pump**

**Fig. 1.2 - Typical Barrel and Screw with 2 Parallel Flights**
A positive inlet pressure is used to insure that the inlet zone of the pump is completely filled. The screw fits with a small clearance \( c \) in the cylindrical barrel. Rotation of the screw causes the liquid to move to the discharge port.

Figure 1.2 shows a cross section of a typical barrel and screw. The screw has an axial flight land width \( e \).

Screws can have more than one parallel flight, as this illustration shows. Two parallel channels are formed between the barrel, the screw root surface, and the flights. The channel depth \( h \) is the radial distance between the barrel and the screw root. Both channels are machined with a lead \( t \). The pitch then equals \( t/n \) for a screw with \( n \) parallel flights.

The clearance is generally very small compared to the barrel diameter \( D \). The helix angle \( \phi_o \) at the flight tip is for a negligible clearance given by

\[
\phi_o = \tan^{-1} \left( \frac{t}{\pi D} \right) \tag{1.1}
\]

The helix angle of the screw flights changes along the depth of the channel. The helix angle at a radius \( r \) is

\[
\phi_r = \tan^{-1} \left( \frac{t}{2\pi r} \right) = \tan^{-1} \left[ \frac{D}{2r} \tan \phi_o \right] \tag{1.2}
\]

The local helix angle \( \phi_r \) increases when \( r \) decreases. The channel width \( b_o \), developed at the flight tips is, for a negligible clearance, given by

\[
b_o = \left( \frac{t}{n} - e \right) \cos \phi_o \tag{1.3}
\]
A similar channel width, developed at radius \( r \), becomes

\[
\bar{b}_r = \left( \frac{t}{n - c} \right) \cos \varphi_r
\]

(1.4)

The effective, or flighted, axial length of the screw is defined as \( L \). The length of a flight tip is then

\[
L_{co} = \frac{L}{\sin \varphi_o}
\]

(1.5)

This length is also the length of the channel at the barrel diameter. The channel length developed at a radius \( r \) becomes

\[
L_r = \frac{L}{\sin \varphi_r}
\]

(1.6)

The channel length then decreases towards the screw root, since \( \sin \varphi_r \) increases with a decrease of \( r \).

A fillet radius is always used at the transition from screw root to screw flight. The fillet radius of the trailing flight is not always the same as that of the leading flight.

A number of other screw geometries are possible. Screws have been made with a lead that varies along the length of the screw. The depth \( h \) varies in some screws along its length. These special geometries, generally restricted to melting zones of extruders, will not be considered in this analysis.

The analysis will also be restricted to screws with only one flight. Corrections for additional parallel flights will be given wherever necessary.
1.3 Screw Pump Theory for Prismatic Rectangular Channels

Isothermal flow of very viscous liquids through screw pumps and metering zones of plasticating extruders has been studied by various investigators, a few of whom are listed in the references (1, 2, 3, 4, 7). A screw pump equation which relates the flow rate and the generated pressure to speed, screw pump geometry and physical properties of the liquid is derived in each theory.

In the existing theory the helical screw channel is replaced by a straight prismatic channel with a rectangular cross section. A very large channel aspect ratio or large width to depth ratio of the rectangular cross section, is assumed in the simplest theory. Flow rates for that case can be based on well known velocity distributions for flow between parallel plates.

The theory based on flow between parallel plates (1) will further be referred to as the simplified or flat plate theory. It leads to a relatively simple screw pump equation which is very useful as a reference equation. The results of more refined theories are usually compared to the screw pump equation of the simplified theory by the introduction of correction factors.

The rectangular channel theory, which for very large aspect ratios degenerates to the simplified theory, is first discussed. That discussion is not restricted to the published literature. The screw pump equations will be derived and terminology common to all theories will be introduced.
1.3.1 Assumptions for Rectangular Channel Theory

The following restrictions are imposed in the isothermal rectangular channel theory:

a. The viscosity of the liquid is uniform and not influenced by shear rates.

b. Inertia and body forces are neglected.

c. The liquid is incompressible.

d. Velocity distributions in channel cross sections with planes perpendicular to the centerline are congruent.

e. The helical channel can be replaced by a straight prismatic channel with a rectangular cross section. One side of that rectangle is equal to the developed width $b_0$ (1.3), the other is equal to the channel depth $h$.

f. The channel has no rounded corners.

g. End effects are neglected.

For now it is further assumed that the leakage flow rate through the flight clearances can be neglected. This leakage flow rate will be treated separately in Chapter 3.

1.3.2 Frame of Reference

The velocity distributions in the channel can be described relative to any desired frame of reference. The analysis is simplified when all velocities are described relative to the screw. Such a velocity distribution is the one seen by an observer who rotates with the screw. The screw appears to him to be at standstill, while the barrel and the gravity field seem to rotate in the opposite direction.
The velocities relative to the screw are identical, when either the barrel is at standstill and the screw rotates, or the screw is at standstill and the barrel rotates with an equal speed in the opposite direction, provided inertial and body forces can be ignored. This is precisely the second assumption made - that Reynolds' number must be small and viscous forces, compared with gravity forces, large.

Velocity distributions will always be described relative to the screw.

1.3.3 Development of Rectangular Channel

The liquid occupies an annular space between two coaxial cylinders, except for the space occupied by the flights. For large channel aspect ratios this annular space is very similar to the space between two parallel plates. This similarity suggested development of both barrel and screw root surface into flat plates to early investigators.

The development of the cylinder with diameter D, which contains the flight tips, is shown in Fig. 1.3.

In this theory it is assumed that the identical development applies to the flights at the root diameter, D-2h. An error is introduced here since a cylinder with a smaller diameter is actually developed. The error increases with the depth of the channel.
FIG. 1.3. DEVELOPMENT OF SCREW FLIGHTS FOR SINGLE FLIGHTED SCREW
1.10

The flights have a uniform helix angle in this development, which is equal to the true helix angle at the flight tips. The channel width perpendicular to the flights is now independent of the depth in the channel and equal to the width \( b_0 \) (1.3) at the periphery.

The screw channel can also be figuratively unwrapped from the screw. A perspective projection of the straight prismatic channel is shown in Fig. 1.4. The barrel, which now becomes a flat plate, moves with a uniform velocity \( \Pi DN \) over the flight tips. The direction of that velocity makes an angle \( \varphi_o \) with the flights. The barrel velocity can be resolved into components \( U_0 \) and \( V_0 \), respectively parallel with and perpendicular to the flights.

1.3.4 Equations of Motion

A Cartesian coordinate system \((x, y, z)\) is introduced in Fig. 1.4. Velocity components in the \(x, y, z\) directions are respectively \(u, v, w\). The velocities at \(z = h\) must be equal to the barrel velocities and must vanish at all other channel walls. The velocity distributions must satisfy a specified pressure difference between inlet and discharge.

In the absence of body and inertia forces, flow of an incompressible liquid is governed by the Navier-Stokes equations
\[
\frac{dp}{dx} = \mu \nabla^2 u \tag{1.7a}
\]
\[
\frac{dp}{dy} = \mu \nabla^2 v \tag{1.7b}
\]
\[
\frac{dp}{dz} = \mu \nabla^2 w \tag{1.7c}
\]

where
\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]

One of the assumptions states that velocity distributions in planes perpendicular to the centerline of the screw are congruent. The velocities are then the same at all points on the line TT\textsubscript{1} parallel to the x-axis and are independent of x. Thus
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = \frac{\partial^2 u}{\partial x^2} = \ldots = 0 \tag{1.8}
\]

From (1.7) and (1.8)
\[
\frac{\partial}{\partial x} \left( \frac{\partial p}{\partial x} \right) = \mu \nabla^2 \frac{\partial u}{\partial x} = 0
\]
\[
\frac{\partial}{\partial y} \left( \frac{\partial p}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial p}{\partial x} \right) = \mu \nabla^2 \frac{\partial v}{\partial x} = 0 \tag{1.9}
\]
\[
\frac{\partial}{\partial x} \left( \frac{\partial p}{\partial z} \right) = \frac{\partial}{\partial z} \left( \frac{\partial p}{\partial x} \right) = \mu \nabla^2 \frac{\partial w}{\partial x} = 0
\]

The derivative \( \frac{dp}{dx} \) is then independent of x, y, and z and must be a constant. Substitution of (1.8) in (1.7) reduces the Navier-Stokes equations to
\[
\frac{dp}{dx} = \mu \left[ \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] = \text{constant} \tag{1.10a}
\]
\[
\frac{dp}{dy} = \mu \left[ \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] \tag{1.10b}
\]
\[
\frac{dp}{dz} = \mu \left[ \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] \tag{1.10c}
\]
The distance between points \( T \) and \( T_1 \) respectively at inlet and discharge (Fig. 1.4) is for \( y \) and \( z = \text{constant} \) always equal to the developed channel length \( L_{CO} \) (1.5). The pressure difference between such points is the same for any value of \( y \) or \( z \) since \( \frac{\partial p}{\partial x} = \text{constant} \). In the rectangular channel theory that difference is equated to the pressure difference \( P_0 \) generated by the screw pump. We will show later (1.3.9) that this theory does not predict uniform pressures along the inlet and discharge ends. The effect of the ends has here been neglected.

The uniform pressure gradient in the channel direction becomes with (1.5)

\[
\frac{\partial p}{\partial x} = \frac{P_0}{L/\sin \psi_0} = \frac{P_0 \sin \psi_0}{L} = A_0/\mu
\]  

(1.11)

where \( A_0 \) is introduced to cancel \( \mu \) in (1.10a).

1.3.5 Flow Through Channel with Rectangular Cross Section

The flow rate through the screw pump is the flow rate relative to the screw in the \( x \)-direction. This flow rate \( Q \) is equal to

\[
Q = \int_0^h \int_0^{b_0} u \, dy \, dz
\]  

(1.12)
The partial differential equation for \( u \) (1.10a) is independent of \( x \), since \( \partial u / \partial x = \text{constant} \). The boundary conditions are also independent of \( x \). The \( u \) component is then a function, \( u(y,z) \), that is completely determined by this differential equation and its boundary conditions

\[
\begin{align*}
  u(y,0) &= u(0,z) = u(B_0,z) = 0 \\
  u(y,h) &= u_0 
\end{align*}
\]  

(1.13)  
(1.14)

The remaining differential equations need not be solved, if one is only interested in the flow rate. The problem is then reduced to the two-dimensional problem of determining the solution to

\[
\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0
\]  

(1.15)

for the boundary conditions (1.13, 1.14).

1.3.6 Separation Into Simpler Solutions

The problem can be simplified by resolving (1.15) into two simpler equations. Consider determination of the solution to

\[
\nabla^2 u_0 = A_0
\]  

(1.16)

for a set of boundary conditions, here symbolized by \( BC_0 \).

Assume now that the solutions to \( m \)-different Poisson equations of the form

\[
\nabla^2 u_i = A_i
\]  

(1.17)

with associated boundary conditions \( BC_i \) and associated constants \( A_i \) \((i = 1, m)\) can be determined. Let \( u_i \) \((i = 1, m)\) be the set of \( m \)-solutions. The desired solution \( u_0 \) is then

\[
u_0 = \sum_{i=1}^{m} u_i
\]  

(1.18)
provided that  
\[ A_o = \sum_{i=1}^{m} A_i \]  
(1.19) 

and also  
\[ B C_o = \sum_{i=1}^{m} B C_i \]  
(1.20) 

This can be proved by direct substitution.

For our purpose, the problem is separated then into finding the solution to these two partial differential equations and associated boundary conditions

\[ \nabla^2 u_1 = 0, \quad BC_1 = \begin{cases} 
  u_1(0, z) = 0 \\
  u_1(b_0, z) = 0 \\
  u_1(y, 0) = 0 \\
  u_1(y, h) = u_0 
\end{cases} \] 
\[ \nabla^2 u_2 = A_o, \quad BC_2 = \begin{cases} 
  u_2(0, z) = 0 \\
  u_2(b_0, z) = 0 \\
  u_2(y, 0) = 0 \\
  u_2(y, h) = 0 
\end{cases} \]  
(1.21) 
(1.22) 

The required solution is  
\[ u_0 = u_1 + u_2. \]  
(1.23) 

The separate problems have the following simple physical meaning. In the first problem, the barrel \( z = h \) is dragged over the channel with a velocity \( U_o \) in the absence of a pressure gradient. For this case, the integral (1.12) of the velocities is called the drag flow rate, \( Q_D \).

The second problem describes the familiar case of a velocity distribution in a conduit with a rectangular cross section and stationary walls caused by a uniform pressure gradient in the channel direction. Velocities are negative for a positive pressure gradient. The pressure back flow rate \( Q_p \) is now commonly defined as the flow rate caused by a negative pressure gradient, i.e., \( Q_p > 0 \) when in opposite direction to \( Q \) and \( Q_D \).
The total flow rate, neglecting the leakage flow rate, is with this definition

\[ Q = Q_D - Q_P \]  \hspace{1cm} (1.24)

Both differential equations (1.21, 1.22) can be solved by the method of separation of variables. The first problem has an analog in heat transfer (9). The second problem has an analog in Airy's stress function distribution in a twisted bar with a rectangular cross section (10). The resulting equations will be shown after a discussion of the simplified theory for flow through rectangular channels with large aspect ratios.

1.3.7 Simplified Flat Plate Theory

A very large aspect ratio \( b_0/h \), which approaches an infinite ratio, is assumed in the simplified theory. Later it will be shown that there is only a small difference between the solution for large aspect ratios \( (b_0/h > 10) \) and that for an infinite aspect ratio. The theory then applies to large aspect ratios.

The differential equations (1.10) can be simplified when aspect ratios are large. The channel then closely resembles the space between parallel plates. For that case, all velocities are parallel to the barrel, except in small regions near the flights. The width of these regions is of the order of \( h \). The influence of these regions becomes insignificant for very large aspect ratios. The velocity component \( w \), therefore, vanishes for a sufficiently large aspect ratio in all but an insignificant region.
Further, the $u$ component is zero at the root $z = 0$ and equal to $U_0$ at the barrel $z = h$. The derivative $\partial u/\partial y$ is insignificant compared with $\partial u/\partial z$. The pressure gradient $\partial p/\partial z$ vanishes for $w = 0$.

The partial differential equations (1.10) then reduce to

$$\frac{\partial p}{\partial x} = \mu \frac{d^2u}{dz^2} = \mu A_0 \quad (1.25a)$$

$$\frac{\partial p}{\partial y} = \mu \frac{d^2v}{dz^2} \quad (1.25b)$$

$$\frac{\partial p}{\partial z} = 0 \quad (1.25c)$$

Here the partial velocity derivatives were replaced by regular derivatives. The pressure is not a function of $z$, since $\partial p/\partial z$ vanishes, while $\partial p/\partial x$ is constant. The right-hand side of (1.25b) is a function of $z$ only. That equation can be satisfied only when $\partial p/\partial y$ is also a constant. Both equations can then be solved by simple integration.

The solution to equation (1.25a), in analogy to the two-dimensional case, can be formally regarded as the superposition of a drag flow and a pressure back flow distribution. The following equations must then be solved.

$$\frac{d^2u_1}{dz^2} = 0 \quad \begin{cases} u_1(0) = 0 \quad (1.26) \quad \frac{d^2u_2}{dz^2} = A_0 \quad \begin{cases} u_2(0) = 0 \quad u_2(h) = 0 \end{cases} \end{cases} \quad (1.27)$$

Integration yields

$$u_1 = \left( \frac{z}{h} \right) U_0 \quad (1.28a)$$

$$u_2 = \frac{1}{2} z (z-h) A_0 = \frac{1}{2} z (z-h) \frac{P_0 \sin \kappa}{\mu L} \quad (1.28b)$$

The linear distribution $u_1$ yields the drag flow

$$Q_{do} = \frac{1}{2} h E_0 U_0 \quad (1.29)$$
The parabolic distribution \( u_2 \) results in a pressure back flow

\[
Q_{p0} = \frac{h^3 b_0}{12} \frac{P_0 \sin \phi_0}{\mu L} \quad (1.30)
\]

Substitution of

\[
\begin{align*}
T_0 &= (t-e) \cos \phi_0 \\
U_0 &= \Pi D N \cos \phi_0 \\
t &= \Pi D \tan \phi_0
\end{align*}
\]

in the equations for \( Q_{DO} \) and \( Q_{P0} \) results in

\[
\begin{align*}
Q_{DO} &= \frac{\Pi^2}{2} \sin \phi_0 \cos \phi_0 (1-e/t) h D^2 N \\
Q_{P0} &= \frac{\Pi}{12} \sin^2 \phi_0 (1-e/t) h^3 D \frac{P_0}{\mu L}
\end{align*}
\]

The screw pump equation for the simplified theory without leakage flow is now

\[
Q = \frac{\Pi^2}{2} \sin \phi_0 \cos \phi_0 (1-e/t) h D^2 N - \frac{\Pi}{12} \sin^2 \phi_0 (1-e/t) h^3 D \frac{P_0}{\mu L} \quad (1.34)
\]

This is the reference equation with which results of other theories are compared.

1.3.8 Shape Factors of Rectangular Channel Theory

The partial differential equations (1.21, 1.22) for a rectangular channel yield a drag flow rate \( Q_D \) and a pressure back flow rate \( Q_P \) that can be expressed in the corresponding flow rates of the simplified theory through the introduction of correction factors \( F_D \) and \( F_P \). Then

\[
\begin{align*}
Q_D &= F_D Q_{DO} \quad (1.35a) \\
Q_P &= F_P Q_{P0} \quad (1.35b)
\end{align*}
\]

For that case, these correction factors are called the shape factors. The shape factors (4) are the following functions of the aspect ratio
These shape factors are shown in fig. 1.5 as function of the reciprocal aspect ratio $h/h_0$. They reduce for large aspect ratios to

$$F_D = 1.0 - 0.55 \left( \frac{h}{b_0} \right) \quad (1.38)$$

$$F_P = 1.0 - 0.63 \left( \frac{h}{b_0} \right) \quad (1.39)$$
These equations, and Fig. 1.5, show that the equations of the rectangular channel theory approach those of the simplified theory for larger aspect ratios.

1.3.9 Pressure Distribution of Simplified Theory

It was shown before that the gradients $\partial p/\partial x$ and $\partial p/\partial y$ are both constant in the simplified theory, while $\partial p/\partial z$ vanishes. The pressure distribution in the channel is then a linear function of $x$ and $y$. An expression will now be derived for $\partial p/\partial y$.

The transverse velocity distribution can be obtained by integration of (1.25b) for the boundary conditions

$$\nu(y, 0) = 0 \quad (1.40a)$$
$$\nu(y, h) = \nu_0 \quad (1.40b)$$

This distribution becomes

$$\nu = \left( \frac{2}{h} \right) \nu_0 + \frac{1}{2} z (z - h) \left( \frac{1}{\mu} \frac{\partial p}{\partial y} \right) \quad (1.41)$$

The transverse flow rate per unit flight length is then

$$q_y = \int_0^h \nu \, dz = \frac{1}{2} h \nu_0 - \frac{h^2}{12 \mu} \frac{\partial p}{\partial y} \quad (1.42)$$

This flow rate is independent of $y$ and applies equally well to the flight face $y = 0$. This flow rate must be zero when no leakage is considered, since there cannot be a flow rate crossing the flight. Then

$$\frac{\partial p}{\partial y} = \frac{6\mu \nu_0}{h^2} \quad (1.43)$$

The pressure distribution is given by

$$p = x \frac{\partial p}{\partial x} + y \frac{\partial p}{\partial y} + p_i \quad (1.44)$$
where $P_1$ is the pressure at $x = 0, y = 0$. The pressure along an isobar is constant. The equation for an isobar is then given by

$$\frac{\partial P}{\partial x} \, dx + \frac{\partial P}{\partial y} \, dy = 0$$

The slope of the isobars is from (1.11, 1.43)

$$\frac{dy}{dx} = -\frac{\partial P/\partial x}{\partial P/\partial y} = -\frac{P_0 \, h^2}{G \mu \, DNL} \quad (1.45)$$

The isobars of the simplified theory are parallel lines with a slope that is fully determined by the geometry of the pump and the operating condition.

We now introduce the back flow coefficient $\beta$, defined as the ratio of the back flow $Q_{p0}$ to the drag flow $Q_{d0}$. It is from (1.32, 1.33)

$$\beta = \frac{Q_{p0}}{Q_{d0}} = \frac{P_0 \, h^2 \tan \phi_0}{G \mu \, DNL} \quad (1.46a)$$

This coefficient characterizes the mode of operation of the screw pump. It is zero for pure drag flow and becomes 1.0 when the discharge is shut. The drag flow is equal to the pressure back flow when the pump operates against the shut-off pressure $P_S$.

The pressure back flow coefficient is then also

$$\beta = \frac{Q_{p0}}{Q_{p0} (P = P_S)} = \frac{P_0}{P_S} \quad (1.46b)$$

It can be regarded as a dimensionless pressure equal to the ratio of the operating pressure $P_0$ to the theoretical shut-off pressure $P_S$ that could be obtained at the same speed if the viscosity could be kept constant.

Comparison of (1.46a) to (1.45) shows that the slope of the isobars is also

$$\frac{dy}{dx} = -\frac{\beta}{\tan \phi_0} \quad (1.47)$$
The slope of the isobars is constructed in Fig. 1.6 for a series of \( \beta \) values for a screw pump with a helix angle of 17.65° \((t/D = 1)\).

![Diagram showing slopes of isobars for different values of \( \beta \)](image)

**Fig. 1.6 - SLOPES OF ISOBARS AS FUNCTION OF MODE OF OPERATION FOR \( \phi_0 = 17.65^\circ \)**

This figure shows how the slopes of the isobars change with operating conditions. The slope \( (1.47) \) corresponds to the slope of the ends, \( \tan \phi_0 \), only when

\[
\beta = -\tan^2 \phi_0 \quad (1.48)
\]
Such an operating condition demands $\beta = -0.1015$ for $\psi_0 = 17.65^\circ$, or a discharge pressure that is lower than the inlet pressure. The slope of the isobars can, for normal operating conditions ($\beta > 0.1$), never match the slope of the channel ends.

In actual operation, one must expect constant pressures at the channel inlet and discharge ends, when the connections to the screw pump housing do not restrict flow. The simplified theory, and for that matter the rectangular channel theory, do not satisfy these boundary conditions. An analysis is given in Chapter 4 for channels with large aspect ratios for flow with uniform pressures along inlet and discharge ends.

1.3.10 Dissipated Energy of Simplified Theory

In the previous analysis, it was assumed that the temperature of the liquid does not change along the length of the screw. Very viscous liquids are sheared and one must expect a temperature increase, unless the pump is cooled. The assumption that the pump operates isothermally is generally incorrect. It is important to make an order of magnitude estimate for the expected temperature increase.

In the following, we will discuss the average temperature rise for a uniform viscosity liquid that is pumped without heat exchange to the surroundings. Leakage flow is again neglected.
1.24

The energy dissipated in the liquid can be conveniently calculated from the dissipation function which, for the remaining shear rates \(\partial u/\partial z\) and \(\partial v/\partial z\), is given by

\[
\phi = \mu \left[ \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right]
\] (1.49)

The energy per unit time dissipated in the channel is

\[
E_1 = \int_0^h \int_0^L_c \int_0^\infty \phi \, dx \, dy \, dz
\] (1.50)

The dissipated energy is for the same flow rate and the same generated pressure influenced by the magnitude of the back flow coefficient \(\beta\). To show this we first use (1.46a) to eliminate \(P_0\) from (1.28). The velocity distribution becomes

\[
u = U_0 \left[ \left( \frac{z}{h} \right)(1-3\beta) + 3\beta \left( \frac{z}{h} \right)^2 \right]
\] (1.51a)

The distribution of the \(v\)-component, given by (1.41) and (1.43), reads

\[
v = U_0 \left( \frac{z}{h} \right) \left[ 3 \left( \frac{z}{h} \right) - 2 \right]
\] (1.51b)

The shear rates are then

\[
\frac{\partial u}{\partial z} = \frac{U_0}{h} \left[ 1 - 3\beta \left( 1 - \frac{2z}{h} \right) \right]
\] (1.52)

\[
\frac{\partial u}{\partial z} = 2 \frac{U_0}{h} \left[ 3 \left( \frac{z}{h} \right) - 1 \right]
\] (1.53)

Substitution in (1.50), integration and subsequent substitution of equations for \(b_0\), \(U_0\), \(V_0\), and \(t\) yields

\[
E_1 = \frac{3\pi}{h} \mu D^3 N^2 L \left( \frac{1}{c} \right) \left[ (1+3\beta^2) \cos^2 \varphi + 4 \sin^2 \varphi \right]
\] (1.54)

The energy \(E_p\), per unit time available at the discharge, is now

\[
E_p = Q P_o = (1-\beta) Q_{D_0} P_o
\] (1.55)
Substitution of (1.32) and elimination of $P_0$ with (1.46a) yields

$$E_p = \frac{3 \pi \mu D^2 N L (1 - \frac{\epsilon}{\pi})}{h} \beta (1 - \beta) \cos^2 \phi_0$$

(1.56)

Comparison of (1.54) and (1.56) shows that $E_1$ can be written as

$$E_1 = Q P_0 \left[ \frac{(1 + 3 \beta^2) \cos^2 \phi_0 + 4 \sin^2 \phi_0}{3 \beta (1 - \beta) \cos^2 \phi_0} \right]$$

(1.57)

Write

$$K_o = \frac{(1 + 3 \beta^2) \cos^2 \phi_0 + 4 \sin^2 \phi_0}{3 \beta (1 - \beta) \cos^2 \phi_0}$$

(1.58)

Then

$$E_1 = K_o E_p$$

The energy $E_p$ is also the energy dissipated in a pipe or other restriction, when a flow rate $Q$ requires a pressure drop $P_0$ between inlet and discharge. The average temperature increase for flow through a pipe can be calculated from the heat balance

$$Q P_0 = \varphi C_p Q \Delta T_p$$

(1.59)

or

$$\Delta T_p = \frac{P_0}{\varphi C_p}$$

(1.60)

This average temperature increase is of the order of 2-3°C. for polymers for $P_0 = 1,000$ psi ($7 \times 10^6$ N/m²). The average temperature increase in the screw pump from the energy dissipated in the channel is $K_o$ times that temperature increase.

Additional energy is generated in the flight clearances. The velocity distribution in the clearances is again the superposition of a drag flow and a pressure back flow distribution. This back flow is generally small compared with the drag flow,
1.26

as will be seen in Chapter 3. This energy can then be based on
the drag flow shear rate along, which is

\[ \dot{\gamma}_F = \pi D N / c \quad (1.61) \]

Shear rates in the clearances are at least one magnitude larger
than shear rates in the channel. A non-Newtonian liquid will
have different viscosities in clearances and channels.

Here we deviate from the original assumptions and introduce a
viscosity \( \mu_F \) for the liquid contained in the clearances. The
volume in the clearances is

\[ V_c = \pi D L c \left( \frac{e}{t} \right) \quad (1.62) \]

The energy per unit time dissipated in the clearances is

\[ E_F = \frac{V_c \mu_F \dot{\gamma}_F^2}{\pi^3 \mu_F D^3 N^2 L \left( \frac{e}{t} \right)} \quad (1.63) \]

The relative importance of this energy is shown by the ratio

\[ \xi_F = \frac{E_F}{E_1} = \frac{(\mu_F/\mu)(h/c)(e/t)/(1-e/t)}{(1 + 3 \beta^2) \cos^2 \phi_0 + 4 \sin^2 \phi_0} \quad (1.64) \]

For example, this ratio is \( \xi_F = 0.08 \) for \( \mu_F/\mu = c/h = e/t = 0.1, \beta = 0, \phi_0 = 17.65^\circ \). It can assume larger or smaller values.

The average temperature increase from \( E_1 \) and \( E_F \) becomes

\[ \Delta T = (1 + \xi_F) K_0 \Delta T_p = K \Delta T_p \quad (1.65) \]

The energy coefficient \( K_0 \) \((1.58)\) is a function of \( \beta \) and \( \phi_0 \). Fig. 1.7 shows that coefficient for different t/D ratios. The
energy is then a minimum for back flow coefficients of the order
of 0.3. The change of \( K_0 \) is small near that minimum. A nega-
tive \( \beta \) corresponds to a negative pressure \( P_0 \). The parameter
\( K_0 \) is negative for that case. The dissipated energy is then
positive, as it should be, while \( E_F \) is negative.
Fig. 1.7 - Energy coefficient $K_0$ as function of back flow coeff. $\beta$. 
FIG. 1.8 - HYDRAULIC EFFICIENCY OF SCREW PUMP, NEGLECTING ENERGY DISSIPATED IN FLIGHT CLEARANCES.
An order of magnitude estimate of the average temperature increase can be made from Fig. 1.7 together with (1.63), when \( P_0, \beta \) and the parameters in (1.63) are known. This increase is of the order of 8-10°C. per 1,000 psi (per \( 7.10^6 \) N/m\(^2 \)).

The efficiency of the screw pump becomes for \( \beta > 0 \)

\[
\eta = \frac{E_p}{E_p + E_1 + E_F} = \frac{1}{1 + K}
\]

(1.66)

This efficiency is shown in Fig. 1.8 for \( E_F = 0 \) and no leakage flow. The position of the maximum efficiency is only slightly dependent on the helix angle. This maximum occurs at values of \( \beta \) between 0.33 and 0.40.

Note that the pump becomes a prime mover, when both \( \beta < 0 \) and \( K > -1 \). The excess energy is in that case

\[
E_M = -(1 + K) Q P_0
\]

(1.67)

The efficiency as a prime mover becomes for \( E_F = 0 \)

\[
\eta_M = 1 + K
\]

(1.68)

The energy equations can readily be converted into equations for calculation of the drive horsepower.

1.4 Other Theories

A number of other theories have appeared in the literature. McKelvey (6) discussed adiabatic operation of screw pumps, defined as operation without heat exchange with the surroundings. Performance is then influenced by the temperature increase and associated viscosity reduction.

Colwell and Nicholls (14) analyzed flow through screw pumps with large aspect ratios when screw and barrel surfaces are controlled at arbitrary temperatures and the viscosity is a
function of temperature and shear stress. They considered shear stresses in the channel direction only. De Haven (13) derived equations for isothermal flow of a pseudoplastic liquid through an extruder.

A number of theories on flow of non-Newtonian liquids (13, 14, 15, 16) have been published. These theories have in common that shear rates in the channel direction only are considered. Such a one-dimensional model is incorrect, since transverse shear rates are significant. Consider, for instance, the shear rates for a Newtonian liquid, given in equations (1.52, 1.53). The transverse shear rate at the barrel is larger for $\beta = 0$ and $t/D = 1.0$ than the shear rate in the channel direction by a factor 1.30. A substantial error is introduced when transverse shear rates are ignored.

1.5 Scope of Investigation

The existing theory, discussed in this chapter, is based on a channel geometry that differs from the actual geometry. An improved theory that accounts for some of these geometric differences will be given in the following chapters. The influence of these differences is derived for isothermal flow of Newtonian liquids. Four distinct problems are treated in this Thesis.

The first problem deals with the influence of the curvature of the walls of screw pump channels. In existing theories, the helical channel is replaced by a straight prismatic channel
with a rectangular cross-section. Flow through channels bounded by cylindrical barrel and screw root surfaces and by two helical flight surfaces is discussed in Chapter 2.

The second problem is concerned with leakage flow through flight clearances. The existing theory is extended in Chapter 3. Leakage is responsible for the formation of a thin liquid layer, that remains near the barrel surface. The layer thickness, which must influence heat transfer to the liquid, is shown to be a function of the geometry of the screw, of a viscosity ratio, and of the mode of operation of the pump.

The third problem deals with the effect of the oblique channel ends on the performance of screw pumps. Existing theories do not yield a channel pressure distribution that satisfies the pressure boundary conditions along the oblique channel ends. Flow through channels with large aspect ratios is discussed in Chapter 4 for the proper boundary conditions. Comparison with flow rates of the simplified theory leads to the introduction of end effect correction factors. Pressure distributions and end effect factors for pressure back flow were experimentally investigated. Test results are discussed in Chapter 5.

The fourth problem deals with the numerical solution of linear partial differential equations. Calculation of end effect factors requires the solution of Laplace’s equation in a large number of regions. A computer technique was developed for the rapid calculation of linear partial differential equations in large families of regions. That method is discussed in Chapter 6.
2. Influence of Channel Curvature on Performance of Screw Pumps

2.1 Introduction

In existing theories, the helical channel is replaced by a straight prismatic channel with a rectangular cross section. This substitute channel resembles the actual channel reasonably well, when channels have large aspect ratios. The channel width \( b_0 \), channel length \( L_0 \) and helix angle \( \phi_0 \) of the prismatic channel are all uniform and equal to the width, length, and helix angle at the barrel diameter. These dimensions are quite different at the root diameter of the actual channel, particularly when channels are deep. The correction factors \( F_D \) and \( F_P \) of the rectangular channel theory account for the finite aspect ratio, but they do not account for the changes in width, length, and helix angle with channel depth.

In this chapter, flow through helical channels will be discussed. This analysis differs from previous theories by Squires (17) and McKelvey (18). Some of the results given here are the same as those previously given by the author (5).

A solution was obtained for flow through helical channels with large aspect ratios only. That solution differs from the solution predicted by the simplified flat plate theory.

The simplified flat plate theory predicts an equation for the flow rate which is incorrect when used for a finite aspect ratio. The shape factors \( F_D \) and \( F_P \) are in the rectangular channel theory introduced to correct for the reduced velocities near the short sides of the channel.
2.2

The equations, based on flow through helical channels with large aspect ratios, contain a similar error, when applied to channels with large aspect ratios. Another set of correction factors $F_{D1}$ and $F_{P1}$ would result if exact flow rate equations for helical channels with finite aspect ratios could be compared to equations for flow through helical channels with large aspect ratios. These exact factors cannot be given, since the exact solution was not obtained. The assumption is then made, that $F_{D1} = F_D$ and $F_{P1} = F_P$. Such an approximation is reasonable for small $h/b_0$ values. No upper bounds can be given for the $h/b_0$ ratios for which this approximation is valid.

2.2 Assumptions and frame of reference

In this chapter, flow of a very viscous liquid through helical channels is investigated for the following assumptions:

a. The viscosity of the liquid is uniform.

b. Inertia and body forces can be neglected.

c. The liquid is incompressible.

d. Velocity distributions are congruent in channel cross sections with planes perpendicular to the center line.

e. The screw channel is bounded by the cylindrical barrel and screw root surfaces and by two sides of a helical flight.

f. The channel has no rounded corners.

g. End effects are neglected.
FIG. 2.1 - SCHEMATIC DIAGRAM OF SCREW PUMP WITH INFINITELY THIN FLIGHTS
2.4

In this analysis, the leakage flow rate through the clearances between flight tips and barrel is neglected. The leakage is treated separately in Chapter 3.

Velocity distributions are again studied relative to the screw. A schematic diagram of a section of the screw is shown in Fig. 2.1. The analysis is initially made for screw pumps with one infinitely thin flight. Corrections for multiple flights and finite flight land widths are discussed at the end of the chapter.

Cylindrical coordinates $r$, $\Theta$, $z$ are introduced with corresponding velocity components $u$, $v$, and $w$. The coordinate system and some of the nomenclature is shown in Fig. 2.1.

2.3 Equations of Motion

The velocity distributions and screw pump equations must be based on Navier-Stokes' equations for flow of an incompressible viscous fluid. These equations are, in cylindrical coordinates and in the absence of body and acceleration forces

$$\frac{\partial p}{\partial r} = \mu \left[ \nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2} \frac{\partial v}{\partial \Theta} \right] \quad \text{(2.1a)}$$

$$\frac{1}{r} \frac{\partial p}{\partial \Theta} = \mu \left[ \nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial r} \right] \quad \text{(2.1b)}$$

$$\frac{\partial p}{\partial z} = \mu \nabla^2 w \quad \text{(2.1c)}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \Theta^2} + \frac{\partial^2}{\partial z^2} \quad \text{(2.1d)}$$

The continuity equation for an incompressible fluid is

$$\frac{1}{r} \frac{\partial (ur)}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \Theta} + \frac{\partial w}{\partial z} = 0 \quad \text{(2.1e)}$$
We assumed congruent velocity distributions in channel cross sections perpendicular to the screw axis. That assumption leads to additional conditions that must be satisfied by the velocity and pressure derivatives.

Velocities are then constant at all points on a helical curve with the same lead as the flights. Such a locus of corresponding points is formed by the intersection of a cylinder with radius \( r \) and a helical surface with lead \( t \). The differential equation for that helical curve is

\[
dz = (r \tan \varphi_r) \, d\theta = (R_0 \tan \varphi_0) \, d\theta \tag{2.2}
\]

The change of the component \( u \) can be expressed for \( r = \text{constant} \) as

\[
du = \frac{\partial u}{\partial \theta} \, d\theta + \frac{\partial u}{\partial z} \, dz
\]

That change must vanish when the increments \( dz \) and \( d\theta \) satisfy the equation for the helical curve. Substitution of (2.2) then gives the additional condition for the velocity derivative

\[
du = \left[ \frac{\partial u}{\partial \theta} + R_0 \tan \varphi_0 \frac{\partial u}{\partial z} \right] \, d\theta = 0 \tag{2.3}
\]

The term in brackets must vanish since \( d\theta \neq 0 \). Then

\[
\frac{\partial u}{\partial \theta} = - R_0 \tan \varphi_0 \frac{\partial u}{\partial z}
\]

Similar conditions can be derived for the derivatives of \( v \) and \( w \). The congruence of velocity distributions leads for all velocity components to the condition

\[
\frac{\partial}{\partial \theta} = - R_0 \tan \varphi_0 \frac{\partial}{\partial z} \tag{2.4}
\]
Differentiation of (2.4) yields the additional condition
\[
\frac{\partial^2}{\partial \theta^2} = \frac{R_0^2 \tan^2 \theta}{\rho} \frac{\partial^2}{\partial z^2} 
\]
(2.5)

A similar condition can be derived for the pressure gradients. All velocity derivatives at corresponding points are constant when velocity distributions are congruent. Substitution of these constants in (2.1) yields for points on such a helical curve

\[
\frac{\partial p}{\partial r} = \text{constant} \quad \frac{\partial p}{\partial \theta} = \text{constant} \quad \frac{\partial p}{\partial z} = \text{constant} 
\]
(2.6)

Two helical curves with the same r are shown in Fig. 2.2. Now

\[
P_D = p_A + \left( \frac{\partial p}{\partial \theta} \right)_A d\theta \\
P_C = p_B + \left( \frac{\partial p}{\partial \theta} \right)_B d\theta
\]

or

\[
P_D - P_C = p_A - p_B + \left[ \left( \frac{\partial p}{\partial \theta} \right)_A - \left( \frac{\partial p}{\partial \theta} \right)_B \right] d\theta
\]

From (2.6)

\[
\left( \frac{\partial p}{\partial \theta} \right)_A = \left( \frac{\partial p}{\partial \theta} \right)_B
\]

Then

\[
P_D - P_C = p_A - p_B 
\]
(2.7a)

The pressure difference between any pair of corresponding points on helical curves with the same radius r is the same when the difference \( \Delta z \) between the coordinates of the points is the same.

Two other curves on one helical surface with lead t are shown in Fig. 2.3. The curve through A has the radius r, that through F a radius \( r + \text{dr} \).
Fig. 2.2 - Development of cylinder with two helical curves.

Fig. 2.3 - Helical surface with lead equal to that of flights.
Now
\[ P_F = P_A + \left( \frac{\partial P}{\partial r} \right)_A \, dr \]
\[ P_E = P_B + \left( \frac{\partial P}{\partial r} \right)_B \, dr \]
From (2.6)
\[ \left( \frac{\partial P}{\partial r} \right)_A = \left( \frac{\partial P}{\partial r} \right)_B \]
or
\[ P_F - P_E = P_A - P_B \quad (2.7b) \]
The pressure difference between any pair of corresponding points with the same difference \( \Delta z \) between the coordinates of that pair is then the same on any helical curve with lead \( t \). Then
\[ \left( \frac{\partial P}{\partial z} \right)_{\text{along a helical curve}} = \text{constant} \quad (2.8) \]
The change in pressure for \( r = \text{constant} \) can be expressed as
\[ dp = \frac{\partial P}{\partial \theta} \, d\theta + \frac{\partial P}{\partial z} \, dz \]
Substitution of (2.2) yields the change in pressure along a helical curve
\[ dp = \left[ \frac{1}{R_0 \tan \phi_0} \frac{\partial P}{\partial \theta} + \frac{\partial P}{\partial z} \right] dz \quad (2.9) \]
The condition to be satisfied by the pressures is from (2.8) and (2.9)
\[ \left( \frac{\partial P}{\partial z} \right)_{\text{along a helical curve}} = \left[ \frac{1}{R_0 \tan \phi_0} \frac{\partial P}{\partial \theta} + \frac{\partial P}{\partial z} \right] = \text{const.} \quad (2.10) \]
Integration of (2.10) along a helical curve between corresponding points at inlet and discharge yields a constant pressure difference. This difference is equated to the difference \( P_0 \) generated by the screw pump, in analogy to the method followed in the rectangular channel theory (1.3.4).
Then
\[ P_0 = \left[ \frac{1}{R_0 \tan \phi_0} \frac{\partial P}{\partial \theta} + \frac{\partial P}{\partial z} \right] L \]

or
\[ \frac{\partial P}{\partial \theta} = R_0 \tan \phi_0 \left[ \frac{P_0}{L} - \frac{\partial P}{\partial z} \right] \quad (2.11) \]

Later we show that the resulting pressure distribution does not give uniform inlet and discharge pressures (section 2.9). End effects are in this analysis neglected, as they were in the simplified theory.

Elimination of the \( \theta \) coordinate with (2.4, 2.5) and substitution of the condition (2.11) for the pressure derivatives reduces the Navier-Stokes equations for flow with congruent velocity distributions to

\[ \frac{\partial P}{\partial r} = \mu \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + G(r) \frac{\partial v}{\partial z} - \frac{u}{r^2} + F(r) \frac{\partial v}{\partial z} \right] \quad (2.12a) \]

\[ \frac{R_0 \tan \phi_0}{r} \left( \frac{P_0}{L} - \frac{\partial P}{\partial z} \right) = \mu \left[ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + G(r) \frac{\partial w}{\partial z} - \frac{v}{r^2} - F(r) \frac{\partial w}{\partial z} \right] \quad (2.12b) \]

\[ \frac{\partial P}{\partial z} = \mu \left[ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + G(r) \frac{\partial w}{\partial z} \right] \quad (2.12c) \]

where
\[ G(r) = 1 + R_0^2 \tan^2 \phi_0 / r^2 \quad (2.12d) \]
\[ F(r) = 2 R_0 \tan^2 \phi_0 / r^2 \quad (2.12e) \]

The continuity equation becomes with (2.1e)
\[ \frac{1}{r} \frac{\partial (ur)}{\partial r} - \frac{R_0 \tan \phi_0}{r} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial z} = 0 \quad (2.13) \]

These equations describe the velocity and pressure distributions in a plane \( \theta = \text{constant} \) as function of \( r \) and \( z \).
A typical cross section through one of these planes is bounded by the lines

\[ r = R_0 \text{ (barrel surface)} \]
\[ r = R_1 \text{ (screw root surface)} \]
\[ z = 0 \text{ (flight surface)} \]
\[ z = t \text{ (flight surface)} \]

Boundary conditions are for that cross section

\[
\begin{align*}
    u(R_1, z) &= 0, \quad v(R_1, z) = 0, \quad w(R_1, z) = 0 \\
    u(r, 0) &= 0, \quad v(r, 0) = 0, \quad w(r, 0) = 0 \\
    u(r, t) &= 0, \quad v(r, t) = 0, \quad w(r, t) = 0 \\
    u(R_0, z) &= 0, \quad v(R_0, z) = \eta DN, \quad w(R_0, z) = 0
\end{align*}
\]

2.4 Equations of Motion for Channels with Large Aspect Ratios

The differential equations (2.12) can only be solved for flow through channels with very large aspect ratios. Velocities are tangential to the barrel and screw root surfaces for that case, except in small regions near the flights. The component \( u \) then vanishes for a sufficiently large aspect ratio. The \( v \)-component vanishes at the screw root and is equal to \( V_0 = \eta DN \) at the barrel surface. The derivative \( \partial v / \partial r \) is much larger than \( \partial v / \partial z \). Similarly, \( \partial w / \partial z \) is insignificant compared with \( \partial w / \partial r \).

The velocity derivatives with respect to \( \theta \) also become very small, as shown by (2.4). Both \( v \) and \( w \) can then be regarded as functions of \( r \) only. A number of partial derivatives can now be replaced by regular derivatives.
The differential equations reduce for \( u = 0 \), \( v = v(r) \) and \( w = w(r) \) to

\[
\frac{\partial p}{\partial r} = 0 \quad (2.14a)
\]

\[
\frac{R \tan \psi}{r} \left( \frac{P_0}{L} - \frac{\partial p}{\partial z} \right) = \mu \left[ \frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} - \frac{u}{r^2} \right] \quad (2.14b)
\]

\[
\frac{\partial p}{\partial z} = \mu \left[ \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right] \quad (2.14c)
\]

The right hand side of (2.14c) is a function of \( r \), since \( w = w(r) \). The pressure is not a function of \( r \), since \( \partial p/\partial z = 0 \). This equality is only possible when \( \partial p/\partial z = \text{constant} \). Now let

\[
\frac{1}{\mu} \frac{\partial p}{\partial z} = \frac{R \tan \psi}{\mu} \left( \frac{P_0}{L} - \frac{\partial p}{\partial z} \right) = A \quad (2.15)
\]

\[
\frac{1}{\mu} \frac{\partial p}{\partial z} = \frac{B}{R_0} \quad (2.16)
\]

where \( A \) and \( B \) are constants with the dimension of reciprocal time for a given condition.

The following ordinary differential equations must be solved to determine the velocity distributions

\[
\frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} - \frac{v}{r^2} = \frac{A}{r} \quad (2.17)
\]

\[
\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} = \frac{B}{R_0} \quad (2.18)
\]

These differential equations apply rigorously when the channel has an infinite aspect ratio. In the derivation of velocity distributions and screw pump equations, we initially assume that the equations for infinite aspect ratios can be
used for channels with finite aspect ratios. The results are later corrected, using known shape factors for a prismatic rectangular channel.

2.5 Velocity Distributions

For the velocity distributions, equations will now be derived. The differential equation for the v-component can be written as

\[
\frac{d}{d\gamma} \left[ \frac{1}{r} \frac{d}{d\gamma} (vr) \right] = \frac{A}{r} \quad (2.19)
\]

Repeated integration yields

\[
\frac{d}{d\gamma} (vr) = A \left[ r \ln r + r C_0 \right] \quad (2.20a)
\]

and

\[
v = \frac{A}{2} \left[ r \ln r - r C_1 + C_2 R_0^2 / r \right] \quad (2.20b)
\]

Boundary conditions are

\[
r = R_0, \quad v = 2\pi R_0 N
\]

\[
r = R_1, \quad v = 0
\]

The ratio of screw root radius to barrel radius is introduced to simplify the equations. Then

\[
\alpha = R_1 / R_0 \quad (2.22)
\]

The constants of integrations become

\[
C_1 = \frac{\left[ \ln R_0 - \alpha^2 \ln R_1 - 4\pi N/A \right]}{(1-\alpha^2)} \quad (2.23a)
\]

\[
C_2 = \frac{\ln \alpha + 4\pi N/A}{1-\alpha^2} \quad (2.23b)
\]

Substitution yields

\[
v = \frac{Ar}{2(1-\alpha^2)} \left[ \ln \left( \frac{r}{R_0} \right) - \alpha^2 \ln \left( \frac{r}{R_1} \right) - \left( \frac{R_1}{r} \right)^2 \ln \alpha \right] + \frac{2\pi N \left( \frac{R_1^2}{4} \right)}{r (1-\alpha^2)} \quad (2.24)
\]
The differential equation for the w-component can be written as

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) = \frac{B}{R_0} \]  

(2.25)

Repeated integration results in

\[ \frac{dw}{dr} = \frac{Br}{2R_0} + C_3 \frac{\rho_0}{r} \]  

(2.26a)

and

\[ w = \frac{Br^2}{4R_0} + C_3 R_0 \ln r + C_4 \rho_0 \]  

(2.26b)

Boundary conditions are

\[ r = R_0 \, , \, w = 0 \]

\[ r = R_1 \, , \, w = 0 \]  

(2.27)

Substitution yields the constants of integration

\[ C_3 = \frac{B(1-\alpha^2)}{4 \ln \alpha} \]  

(2.28)

\[ C_4 = -\frac{B}{4} \left[ 1 + (1-\alpha^2) \ln R_0 / \ln \alpha \right] \]  

(2.29)

This velocity component is then

\[ w = \frac{Br_0}{4} \left[ (r/R_0)^2 + (1-\alpha^2) \ln (r/R_0)/\ln \alpha - 1 \right] \]  

(2.30)

The velocity distribution is fully described by (2.24) and (2.30), once the value of either parameter A or B is known. An additional equation is required to determine these parameters.

2.6 Screw Pump Equation

The velocity distribution must satisfy the condition that there cannot be a flow rate across the flights. That condition will yield the necessary equation for the calculation of A or B.
An expression is first derived for the total flow rate. The flow rate can be found by integration of \( w \) over the channel cross section. Then with (2.30):

\[
Q = 2\pi \int_{R_1}^{R} \omega r \, dr = \frac{\pi}{8} BR_o^3 F(\alpha) \tag{2.31}
\]

where

\[
F(\alpha) = (\alpha^2 - 1) \left[ 1 + \alpha^2 + (1 - \alpha^2) / \ln \alpha \right] \tag{2.32}
\]

The constant \( B \) is then

\[
B = \frac{8Q}{\pi R_o^3 F(\alpha)} \tag{2.33}
\]

The velocity distributions (2.24, 2.30) can never satisfy the requirement that no liquid crosses the flight for all values of \( r \). Equation 2.30 shows, for example, that \( w \) vanishes at \( r = R_o \) and at \( r = R_1 \), but does not vanish for most other values of \( r \). This difficulty is solved in the following way. We specify that there cannot be any net flow across the helical flight, instead of rigorously satisfying the condition that no liquid crosses the flight at any value of \( r \). The net flow across any helical surface with the same pitch \( t \) vanishes since \( v \) and \( w \) are functions of \( r \) only.

Figure 2.4 shows a volume element in the channel formed between screw root, barrel, a helical surface with lead \( t \), and planes \( \Theta = \) constant and \( z = \) constant. The flow rate through the section AA'B'B must be equal to the flow rate through the rectangle BB'C'C, since there is no flow through the other surfaces. The helical boundary AA'C'C requires again that

\[
dz = (R_o \tan \psi_o) \, d\Theta \tag{2.34}
\]
FLUID ELEMENT BOUNDED BY HELICAL SURFACE, ROOT AND BARREL DIAMETER AND PLANES $\theta$ AND $Z$ CONSTANT
2.16

The flow rate through the plane \( z = \) constant is now
\[
dQ = d\theta \int_{R_1}^{R_0} w r \, dr \tag{2.35}
\]

The flow rate through the plane \( \theta = \) constant is
\[
dQ = d\theta \int_{R_1}^{R_0} v \, dr = (R_0 \tan \psi_o) d\theta \int_{R_1}^{R_0} v \, dr \tag{2.36}
\]

Both flow rates are the same. Then
\[
\int_{R_1}^{R_0} w r \, dr = (R_0 \tan \psi_o) \int_{R_1}^{R_0} v \, dr \tag{2.37}
\]

Substitution of (2.31) yields
\[
Q = \frac{\pi}{8} BR_0^3 F(\alpha) = 2\pi R_0 \tan \psi_o \int_{R_1}^{R_0} v \, dr \tag{2.38}
\]

After integration then
\[
Q = \frac{\pi}{4} R_0^3 \tan \psi_o \left[ 8 \pi N K(\alpha) - AG(\alpha) \right] \tag{2.39}
\]

where
\[
G(\alpha) = (1 - \alpha^2) \left[ 1 - \left( \frac{2\alpha \ln \alpha}{1 - \alpha^2} \right)^2 \right] \tag{2.40a}
\]
\[
K(\alpha) = 1 + \frac{2\alpha^2 \ln \alpha}{1 - \alpha^2} \tag{2.40b}
\]

The constant \( A \) becomes from (2.39)
\[
A = 8\pi N \frac{K(\alpha)}{G(\alpha)} - \frac{4Q}{\pi R_0^3 G(\alpha) \tan \psi_o} \tag{2.41}
\]

The discharge pressure is from (2.15, 2.16):
\[
P_0 = \frac{\mu L}{R_0} \left[ A / \tan \psi_o + B \right] \tag{2.42}
\]

Substitution of (2.33, 2.41) yields:
\[
P_0 = \mu L \left[ \frac{8\pi N K(\alpha)}{(R_0 \tan \psi_o) G(\alpha)} + \frac{4Q}{\pi R_0^2} \left( \frac{2}{F(\alpha)} - \frac{1}{G(\alpha) \tan^2 \psi_o} \right) \right] \tag{2.42a}
\]
Here this equation is rearranged in the form of a screw pump equation

\[ Q = Q_D - Q_P \]

\[ = \frac{R_0^3}{N} S(\phi_0, \alpha) - \frac{P_0 P_0^+}{\mu L} T(\phi_0, \alpha) \]  \hspace{1cm} (2.43)

where:

\[ S(\phi_0, \alpha) = \frac{2 \pi^2 F(\alpha) K(\alpha) \tan \phi_0}{F(\alpha) - 2 G(\alpha) \tan^2 \phi_0} \]  \hspace{1cm} (2.43a)

\[ T(\phi_0, \alpha) = \frac{1}{4} \frac{\pi F(\alpha) G(\alpha) \tan^2 \phi_0}{F(\alpha) - 2 G(\alpha) \tan^2 \phi_0} \]  \hspace{1cm} (2.43b)

The right-hand side of (2.43) contains a term proportional to the speed \(N\), and another term proportional to \(P_0/\mu L\). These terms replace the drag flow and pressure back flow rates of the existing theory.

### 2.7 Curvature Factors

Curvature factors \(F_{DC}\) and \(F_{PC}\) are introduced to compare this screw pump equation to that of the simplified flat plate theory. Equation (2.43) then becomes

\[ Q = F_{DC} \left[ \frac{\pi^2}{2} (\sin \phi_0 \cos \phi_0) D^2 L N \right] - F_{PC} \left[ \frac{\pi \ell}{12} (\sin^3 \phi_0) h^3 \left( \frac{D}{L} \right) \left( \frac{P_0}{\mu} \right) \right] \]  \hspace{1cm} (2.44)

The choice of the helix angle \(\phi_0\) and the diameter \(D\) in (2.43) is very specific. The diameter is the barrel diameter, the helix angle is the helix angle \(\phi_0\) at that diameter. These curvature factors express the influence of curvature on drag and pressure back flow rates in comparison to similar quantities of the simplified flat plate theory. The new factors are functions of \(h/D = (1-\ell)/2\) and \(\phi_0\). Then
FIG-2.5. CURVATURE FACTOR $F_{DC}$ FOR DRAGFLOW

- The graph shows the variation of curvature factor $F_{DC}$ with relative channel depth $h/D$ for different helix angles.
- The y-axis represents $F_{DC}$, ranging from 0.4 to 1.1.
- The x-axis represents relative channel depth $h/D$, ranging from 0.05 to 0.50.
- Different helix angles are indicated on the graph, including 5°, 10°, 15°, 17.65°, 20°, 25°, 30°, 40°, and 50°.
- The helix angle of 60° is marked with a special notation.

$h/D$: Relative Channel Depth
FIG. 2-6 - CURVATURE FACTORS FOR PRESSURE FLOW

- $F_{PC}$
- $h/D$: Relative Channel Depth

- Curves for different helix angles (5°, 10°, 15°, 17.65°, 20°, 30°, 40°, 50°, 60°)

- Graph shows the relationship between relative channel depth and curvature factor for various helix angles.
Figure 2.5 shows $F_{DC}$ as function of $h/D$ for several values of $\phi_o$. The influence of curvature is rather small for the commonly used square pitch ($t/D=1, \phi_o = 17.65^\circ$). The influence is larger for large helix angles. The drag flow coefficients determined by Squires (17) are identical to the coefficients $F_{DC}$ for $\phi_o = 0$.

Figure 2.6 shows $F_{PC}$ as function of $h/D$ for a series of helix angles. The influence of curvature on $F_{PC}$ is large for the same square pitch.

The large influence of curvature on pressure back flow is due to the shorter average channel lengths. The channel length in the conventional theories is based on the barrel diameter (1.5). It would be more correct to use the length and width (Fig. 2.7) developed at the average diameter ($D-h$). Substitution of these average dimensions in (1.33) yields

$$Q_p = F_{P3} \left[ \frac{\pi}{12} (\sin^2 \phi_o) \frac{h^3}{L} \left( \frac{D}{L} \right) \left( \frac{P}{\mu} \right) \right]$$

(2.47)

This correction factor is

$$F_{P3} = \frac{1 - \frac{h}{D}}{(1 - \frac{h}{D})^2 \cos^2 \phi_o + \sin^2 \phi_o}$$

(2.48)
FIG. 2.7 - WIDTH AND LENGTH PER TURN AT AVERAGE DIAMETER USED IN CALCULATION OF $F_{p3}$
FIG. 2.8 - CORRECTION FACTOR FOR PRESSURE BACKFLOW BASED ON CHANNEL LENGTH AND WIDTH AT AVERAGE DIAMETER
The correction factor $F_{p3}$ is shown in figure 2.8. Comparison with Fig. 2.6 shows that the difference between $F_{PC}$ and $F_{p3}$ is small.

### 2.8 Influence of Side Walls

The correction factors $F_{DC}$ and $F_{PC}$ apply to channels with large aspect ratio. These factors must be corrected for the influence of the side walls for channels with large $h/D$ ratios.

An analytical expression for flow through helical channels with finite aspect ratios was not determined because equations (2.12) could not be solved. The screw pump equation (2.44) then does not account for the reduced velocities near the flights. One can then reason that the flow rate reduction from this side effect is similar to the flow rate reduction in a rectangular channel. The side effect is there introduced by multiplying the flow rates of the simplified theory by shape factors $F_D$ and $F_p$ (1.36, 1.37).

An approximation of the flow rates through helical channels can be obtained when the drag flow and pressure back flow of the simplified helical theory are multiplied by the correction factors $F_D$ and $F_p$ of the rectangular theory.

Combined correction factors are then equal to the products of the curvature factors $F_{DC}$ ($F_{PC}$) (2.45, 2.46) and the shape factors $F_D$ ($F_p$).

It is reasonable to base the calculation of $F_D$ and $F_p$ for this case on the channel width $b_1$ at the average diameter $D_1 = D - h$. For an infinitely thin flight land

$$b_1 = \pi (D - h) \sin \psi_1 ,$$

(2.49)
FIG. 2.0 - COMBINED CURVATURE FACTOR FOR DRAGFLOW

- REF. h/D - Relative Channel Depth

- HELIX-ANGLE = 60°

- Graph shows the combined curvature factor for dragflow with different helix angles.
FIG. 2.10 - COMBINED CURVATURE FACTOR FOR PRESSURE BACKFLOW
where \( \phi_1 \) is the helix angle at the diameter \( D_1 \). Substitution of (1.2, 1.4) yields

\[
\frac{h}{b_1} = \frac{(h/D) \sqrt{(1 - h/D)^2 + \tan^2 \phi_1}}{\Pi (1 - h/D) \tan \phi_1}
\] (2.50)

The combined factors for helical channels with finite aspect ratios then become

\[
F_{D2} = F_D(h/b_1) \times F_{DC}(\phi_1, h/D) \tag{2.51}
\]
\[
F_{P2} = F_p(h/b_1) \times F_{PC}(\phi_1, h/D) \tag{2.52}
\]

The combined shape factors \( F_{D2} \) and \( F_{P2} \) are shown in Figs. 2.9 and 2.10.

### 2.9 Pressure Distribution in Helical Screw Channels with Large Aspect Ratios

The calculated pressure in the screw pump channel is, for large aspect ratios, a linear function of \( \theta \) and \( z \), since the partial pressure derivatives to \( \theta \) and \( z \) are both constant. These pressures are at inlet and discharge then linear functions of \( \theta \). A screw pump with inlet and discharge connections that do not restrict the flow must have uniform inlet and discharge pressures. The calculated pressures at inlet and discharge do not satisfy this requirement. This difference results from the assumption that velocity distributions are congruent. The case studied is really that of flow through a section of an infinitely long screw pump. Calculated pressure distributions approach actual distributions in regions that are reasonably far removed from inlet and discharge.
The pressure difference across the flights is of interest in leakage flow calculations. That pressure drop $\Delta p_F$ is the pressure difference at $z = \text{constant}$ between points at $\theta = 0$ and $\theta = 2\pi$ for single flighted screws. This pressure drop is also the maximum pressure change along the ends of the screw pump. It is here compared with the pressure difference $\Delta p_T$ generated in a length equal to one pitch of the screw ($L' = t$). The pressure difference $\Delta p_T$ depends on the mode of operation of the screw pump.

The total pressure generated in the screw pump is a maximum when the discharge is closed. This theoretical shut off pressure, $P_S$, is given by the extruder equation for $Q = 0$. It is a theoretical pressure that would be reached when the viscosity could be kept constant. From (2.42 a)

$$P_S = 16\pi^2 \frac{\mu}{N} \left( \frac{L}{t} \right) \frac{K(\alpha)}{G(\alpha)}$$

(2.53)

The back flow coefficient $\beta$ is again defined as

$$\beta = \frac{Q_P}{Q_D}$$

(2.54)

The drag flow $Q_D$ (2.43) is equal to $Q_P$ when $Q = 0$ and $P_0 = P_S$. The back flow coefficient is from (2.43) then also

$$\beta = \frac{Q_P}{Q_D} = \frac{P_0}{P_S}$$

The discharge pressure is from this equation proportional to $\beta$.

Then

$$P_0 = \beta P_S = 16\pi^2 \frac{\mu}{N} \left( \frac{L}{t} \right) \frac{K(\alpha)}{G(\alpha)}$$

(2.55)
The pressure difference $\Delta p_T$, generated in one turn, is then

$$\Delta p_T = 16 \pi^2 \beta N \frac{K(\alpha)}{G(\alpha)} \tag{2.56}$$

The pressure difference $\Delta p_F$ across the flights is from (2.15)

$$\Delta p_F = 2 \pi \frac{\beta \rho}{\rho_0} = 2 \pi \mu A \tag{2.57}$$

The constant $A$ is known from (2.41). That expression contains the flow rate $Q$, which with (2.43) can be written as

$$Q = (1 - \beta) R_0^3 N \sin (\theta, \alpha) \tag{2.58}$$

Substitution of (2.58) in (2.41) results in

$$A = 8 \pi N \frac{K(\alpha)}{G(\alpha)} \left\{ \frac{G(\alpha) \tan^2 \theta_0 / F(\alpha)}{1 - G(\alpha) \tan^2 \theta_0 / F(\alpha)} \right\} \tag{2.59}$$

The pressure difference $\Delta p_F$, expressed as a fraction $\alpha_{pl}$ of $\Delta p_T$, is from (2.46, 2.57, 2.59)

$$\alpha_{pl} = \frac{\Delta p_F}{\Delta p_T} = \frac{\beta - 2 G(\alpha) \tan^2 \theta_0 / F(\alpha)}{\beta \left[ 1 - 2 G(\alpha) \tan^2 \theta_0 / F(\alpha) \right]} \tag{2.60}$$

This ratio is shown in Fig. 2.11 as function of the back flow coefficient $\beta$ for a screw with a square pitch ($\theta_0 = 17.65^\circ$). The ratio is always larger than 1 in the region $0 < \beta < 1$. The ratio increases with a decrease of the back flow coefficient. Fig. 2.11 again shows that the calculated pressures along the ends of the screw are far from uniform.
\[ \alpha_{Pl} = \frac{\Delta P_F}{\Delta P_T} = \frac{\text{PRESSURE DIFF. ACROSS CHANNEL}}{\text{PRESSURE GENERATED IN ONE TURN}} = \frac{P_B - P_A}{P_B - P_C} \]

**FIG. 2.11 - PRESSURE RATIO \( \alpha_{Pl} \) AS FUNCTION OF BACKFLOW COEFFICIENT, RELATIVE CHANNEL DEPTH AND HELIX ANGLE**
The magnitude of \( \Delta p_F \) is strongly influenced by the mode of operation of the screw pump. This can be shown when the magnitude of \( \Delta p_F \) for closed discharge is compared with that for open discharge. The resulting ratio \( \alpha_{p2} \) is

\[
\alpha_{p2} = \frac{\Delta p_F (\beta = 1)}{\Delta p_F (\beta = 0)} = 1 - \frac{F(\alpha) / G(\alpha)}{2 \tan^2 \varphi_o} \tag{2.61}
\]

This ratio is shown in Figure 2.12 as function of the relative depth \( h/D \) for several helix angles. The change in \( \alpha_{p2} \) with a change in the mode of operation is particularly large for screw pumps with small relative channel depth.

### 2.10 Energy Dissipation for Large Aspect Ratios

The energy dissipated in the screw pump channel is also influenced by the curvature of the channel. The energy equations of the simplified theory (1.54, 1.56) were shown in chapter 1. Similar equations will be derived for helical channels with infinite aspect ratios. The results are then compared to those of the simplified theory.

The energy is determined by integration of the dissipation function over the liquid volume in the screw channel. The energy dissipated between flight tips and barrel will not be considered here. The contributions of that energy are identical to that of the simplified theory (1.63), since the clearance is small compared to the channel depth.

The equation for the dissipation function (19) reduces for \( u = v (r) \), \( w = w (r) \) to:

\[
\phi = \mu \left[ \left( \frac{dv}{dr} - \frac{v}{r} \right)^2 + \left( \frac{dw}{dr} \right)^2 \right] \tag{2.62}
\]
\[ \frac{\Delta P_F (\beta = 1)}{\Delta P_F (\beta = 0)} = 1 - \frac{F}{G} \frac{2 \tan^2 \theta_c}{2 \tan^2 \theta_c} \]

**FIG. 2.12 - RATIO OF PRESSURE DIFFERENCE IN CHANNEL BETWEEN CLOSED AND OPEN DISCHARGE**

- **h/D: Relative Channel Depth**
- **HELIX ANGLE = 30°**
The energy, \( E_{C1} \), dissipated in the channel per unit time, is then
\[
E_{C1} = \int_0^{2\pi} \int_0^L \int_{R_1}^R \phi \ r \ dr \ d\theta \ dz
\]

Substitution of the velocity distributions (2.24, 2.30) and integration yields
\[
E_{C1} = 2\pi \mu L R_0^2 \left[ \frac{1}{8} A^2 G(\alpha) - \frac{1}{16} B^2 F(\alpha) + 8\pi^2 N^2 \alpha^2 / (1 - \alpha^2) \right] \quad (2.63)
\]

Substitution of \( A \) and \( B \) from (2.33, 2.59) yields
\[
E_{C1} = 16\pi^3 \mu \mu R_0^2 L N^2 \left[ \frac{k^2(\alpha)}{G(\alpha)} \left\{ \frac{\beta^2 - 2G(\alpha) \tan^2 \psi_0 / F(\alpha)}{1 - 2G(\alpha) \tan^2 \psi_0 / F(\alpha)} + \frac{\alpha^2}{1 - \alpha^2} \right\} \right] \quad (2.64)
\]

This equation reduces for small values of \( h/D \) to the energy equation for the simplified theory (1.54).

The pumping energy \( E_{PC} \) is at the discharge
\[
E_{PC} = P_0 Q = \beta Q P_S \quad (2.65)
\]

Substitution of \( Q \) from (2.58) and \( P_S \) from (2.53) yields
\[
E_{PC} = 16\pi^3 \mu \mu R_0^2 L N^2 \left[ \frac{k^2(\alpha)}{G(\alpha)} \frac{\beta(1 - \beta)}{1 - 2G(\alpha) \tan^2 \psi_0 / F(\alpha)} \right] \quad (2.66)
\]

The total energy is then
\[
E_{TC} = E_{C1} + E_{PC} = 16\pi^3 \mu \mu R_0^2 L N^2 \left[ \frac{k^2(\alpha)}{G(\alpha)} \left\{ \frac{\beta - 2G(\alpha) \tan^2 \psi_0 / F(\alpha)}{1 - 2G(\alpha) \tan^2 \psi_0 / F(\alpha)} + \frac{\alpha^2}{1 - \alpha^2} \right\} \right] \quad (2.67)
\]
This equation converges for small values of \( h/D \) to the corresponding equations of the simplified theory, viz. \((1.54) + (1.56)\)

\[
E_T = 8 \pi^3 \mu N^2 R_0^3 \left( \frac{h}{h} \right)^3 \left[ (1 + 3 \beta) \cos^2 \phi_0 + 4 \sin^2 \phi_0 \right] (2.68)
\]

The dissipated and total energies predicted by this theory can be compared with similar quantities from the simplified theory by the introduction of energy coefficients \( H_V \) and \( H_{VT} \). Neglecting the energy dissipated in the flights these coefficients are

\[
H_V = \frac{E_{c1}}{E_1} \quad (eq. 2.64) \quad (2.69)
\]

\[
H_{VT} = \frac{E_{tc}}{E_T} \quad (eq. 2.67) \quad (2.70)
\]

Figs. 2.13 and 2.14 show respectively \( H_V \) and \( H_{VT} \) as function of the back flow coefficient for a screw with a square pitch \((t/D = 1)\). The cylindrical theory predicts lower energies for large relative channel heights. The influence of the flights on velocity distributions was not accounted for in the derivation of these equations. Figs. 2.13 and 2.14 then only show the trend but they are not correct for channels with large \( h/D \) ratios.

### 2.11 Corrections for Finite Land Width and Multiple Flights

The previous equations were based on infinitely thin flights and single flighted screws. Corrections can be made for
finite land width and multiple parallel flights. Figure 2.15 shows the development of the outside diameter of a screw with multiple flights and finite land widths. Flow rates (2.31, 2.38) were based on integrations over a total angle \( \Delta \theta = 2\pi \).

Figure 2.15 shows that this angle reduces to

\[
\Delta \theta = 2\pi \left( 1 - \frac{n e}{t} \right)
\]

(2.71)

for a screw with \( n \) multiple flights, each with an axial land width \( e \). Drag and pressure back flow rates (2.44) must, for that case, be multiplied by a factor \( 1 - \frac{n e}{t} \).

The energy was determined by integration through the volume between screw root and barrel. The actual volume is smaller by the factor \( 1 - \frac{n e}{t} \). For that case, energy equations must be multiplied by the same factor.

Calculation of shape factors \( F_{DL} \) and \( F_{PL} \) for multiple channels and finite land widths require a correction for the average aspect ratio of (2.50).
PERIPHERY OF SCREW DEVELOPED INTO PLANE

PARALLEL FLIGHTS

FIG. 2.15- CORRECTION \((1 - \eta e/t)\) FOR FINITE LAND WIDTH 'e' AND 'n' MULTIPLE FLIGHTS
3. Leakage Flow in Screw Pumps

3.1 Introduction

Screw pumps always have a small radial clearance between flight tips and barrel. This clearance is necessary for assembly and running of the screw pump. Further, it must be large enough to allow for thermal expansion and distortion of barrel and screw. The magnitude of the clearance influences the leakage flow rate, the heat transfer coefficient, and the energy generated in the flight clearances (1.63).

The flight tips drag liquid through the flight clearances in addition to the flow caused by the pressure difference between the sides of the flight tips. This internal leakage flow rate reduces the flow rate that would otherwise be obtained.

The existence of a pressure gradient (1.43) in a direction perpendicular to the flights was not realized in earlier theories (1, 2, 3). Drag flow across the flight tips was there neglected also. Mohr and Mallouk (7) introduced leakage caused by drag of the flight tips and at the same time accounted for the influence of the transverse pressure gradient.

The present analysis is an extension of the analysis by Mohr and Mallouk. It is restricted to isothermal flow of Newtonian liquids through screw pump channels with large aspect ratios. The developed channel is again a space between two parallel plates, similar to that used in the simplified theory. The
space in the clearances between flights and barrel also becomes a space between parallel plates. Again velocities are investigated relative to the screw. Only screws with a single flight and uniform channel depth are considered.

To assume equal viscosities in the channel and in the flight clearances is incorrect in this case. The shear rates in the clearances are of larger magnitude than those in the channel. Most polymers have different apparent viscosities at these two shear rate levels. It is then better to use a uniform viscosity $\mu_F$ in the clearances and a different uniform viscosity $\mu_C$ in the channel.

In rectangular channels with large aspect ratios, velocities are parallel to the barrel, except in small regions near the flight faces. The velocities in the clearances are similarly parallel to the barrel when the land width of the flight is large compared with the flight clearance. In this analysis we assume that both the channel aspect ratio and the flight width to clearance ratio are large. We can then assume velocities parallel to the barrel in the whole channel and in all flight clearance spaces.

3.2 Definition of Leakage Flow Rate.

The leakage flow rate is the loss in flow rate from flow through the clearances. In the following we define theoretical flow rates as flow rates through a screw pump without clearance.
These flow rates are given by the screw pump equation (1.34) of the simplified theory, here expressed as

\[ Q_T = Q_{DT} - Q_{PT} = \frac{1}{2} U_0 h b_0 - \frac{h^2 b_0}{12 \mu_c} \left( \frac{\partial P}{\partial x} \right)_c \]  

(3.1)

where the index T defines the theoretical case and the index C refers to the channel. The flow rate \( Q_T \) is a linear function of the pressure difference \( P_0 \) (Fig. 3.1) for a given speed \( N \). The clearance \( c \) between flight tips and barrel does not change the radial channel depth \( h \) between barrel and screw root, according to the definition of \( h \) (Fig. 1.2). A pump with a clearance \( c \) and the same channel depth \( h \) has a different performance characteristic. The resulting flow rate \( Q \) is shown in Fig. 3.1 as function of \( P_0 \) for the same speed \( N \).

Here the leakage flow rate \( Q_L \) is defined as the difference \( Q_T - Q \) for the same speed and the same pressure difference \( P_0 \).

### 3.3 Pressure Distribution with Leakage Flow

The pressure distribution in the channel was investigated in section 1.3.9 for zero leakage flow rate. The transverse flow rate (1.42) now does not vanish when liquid leaks across the flights. A different pressure distribution results when a leakage rate is assumed.

A detail of the developed screw pump channel is shown in Fig. 3.2. Here the line \( A_1A_2 \) is a full cross section through the screw pump channel with a plane perpendicular to the center line of the screw. Points \( A_1 \) and \( A_2 \) are physically the same point, since the analysis is based on a single flighted screw.
Fig. 3.1 - Definition of leakage flowrate $Q_L$

Fig. 3.2 - Control region
Flow through the channel is again governed by the reduced Navier-Stokes equations (1.25), now in the form

\[
\left( \frac{\partial p}{\partial x} \right)_c = \mu_c \frac{d^2 u}{d z^2} \quad (3.2a)
\]

\[
\left( \frac{\partial p}{\partial y} \right)_c = \mu_c \frac{d^2 v}{d z^2} \quad (3.2b)
\]

It was shown before that the pressure gradients are constants in channels with large aspect ratios. Similar equations apply to flow through the clearances. The pressure gradient \( (\partial p/\partial x)_F \) in the flight clearance is identical to that in the clearance \( (\partial p/\partial x)_C \), since the change of the pressure along the line \( B_1C_1 \) in the channel and in the clearance must be the same. The pressure difference between \( A_1 \) and \( B_2 \) is equal to that between \( A_2 \) and \( B_2 \), since \( A_1 \) and \( A_2 \) are one and the same point. Then

\[
B_o \left( \frac{\partial p}{\partial y} \right)_C + (\epsilon \cos \gamma_o) \left( \frac{\partial p}{\partial y} \right)_F + (\pi D \cos \gamma_o) \left( \frac{\partial p}{\partial x} \right)_C = 0
\]

or

\[
\left( \frac{\partial p}{\partial y} \right)_F = -\left( \frac{B_o}{\epsilon \cos \gamma_o} \right) \left( \frac{\partial p}{\partial y} \right)_C - \left( \frac{\pi D}{\epsilon} \right) \left( \frac{\partial p}{\partial x} \right)_C \quad (3.3)
\]

The pressure gradient \( (\partial p/\partial x)_C \) is identical to the pressure gradient (1.11), derived in the simplified theory

\[
\left( \frac{\partial p}{\partial x} \right)_C = \left( \frac{\partial p}{\partial x} \right)_F = \frac{P_o \sin \gamma_o}{L} \quad (3.4)
\]

Equation (3.3) contains two unknown transverse pressure gradients. An additional equation, required to determine these gradients, is obtained by equating the transverse flow rate in the channel to the transverse flow rate in the flight clearance.
The flow rate leaving the channel across \( B_1 C_1 \) is per unit flight length

\[
q_c = \frac{1}{2} U_0 h - \frac{h^3}{12} \frac{1}{\mu_c} \left( \frac{\partial p}{\partial \eta} \right)_c
\quad (3.5)
\]

A similar flow rate entering the flight clearance is

\[
q_F = \frac{1}{2} U_0 C - \frac{C^3}{12} \frac{1}{\mu_F} \left( \frac{\partial p}{\partial \eta} \right)_F
\quad (3.6)
\]

For \( q_F = q_C \) then

\[
\frac{1}{2} U_0 (h - c) = \frac{h^3}{12} \frac{1}{\mu_c} \left( \frac{\partial p}{\partial \eta} \right)_c - \frac{C^3}{12} \frac{1}{\mu_F} \left( \frac{\partial p}{\partial \eta} \right)_F
\quad (3.7)
\]

The gradient can now be calculated, but the results are rather cumbersome. To simplify the analysis let

\[
\alpha_F = \frac{\left( \frac{\partial p}{\partial \eta} \right)_F}{\left( \frac{\partial p}{\partial \eta} \right)_c}, \quad (3.8a)
\]

\[
\alpha_C = \frac{\left( \frac{\partial p}{\partial \eta} \right)_c}{\left( \frac{\partial p}{\partial \eta} \right)_c}, \quad (3.8b)
\]

Introduce further the back flow coefficient \( \beta_T \) for the screw pump without clearance, defined as

\[
\beta_T = \frac{Q_P}{Q_D_T} \quad (3.9)
\]

This coefficient is from (3.1)

\[
\beta_T = \frac{h^2}{6 U_0 \mu_c} \left( \frac{\partial p}{\partial x} \right)_c \quad (3.10)
\]

The barrel velocity component \( V_0 \) can, with (3.10), be expressed as

\[
V_0 = U_0 \tan \Phi_0 = \frac{h^2 + \tan \Phi_0}{6 \beta_T \mu_c} \left( \frac{\partial p}{\partial x} \right)_c \quad (3.11)
\]

Substitution of (3.8, 3.11) in (3.7) yields, after rearrangement

\[
\alpha_c = (\frac{c}{h})^3 \left( \frac{\mu_c}{\mu_F} \right) \alpha_F + (1 - \frac{c}{h}) \tan \Phi_0 / \beta_T \quad (3.12)
\]
Equation (3.3) can be expressed in a number of geometric ratios using the relations

\[ E_0 = (t - e) \cos \psi_0 \]  
\[ \Pi D = t / \tan \psi_0 \]  

The result is

\[ \alpha_c = - \frac{e/t}{1 - e/t} \alpha_F - \frac{1}{(1 - e/t) \tan \psi_0} \]  

Let now

\[ \varepsilon = \frac{e}{t} \]  
\[ \sigma = \left( \frac{e}{h} \right)^3 \left( \frac{m_c}{m_F} \right) \]  

Elimination of \( \alpha_c \) from (3.12, 3.14) yields the expression for \( \alpha_F \)

\[ \alpha_F = - \frac{1 + (1 - \varepsilon)(1 - c/h)\tan^2 \psi_0 / \beta_T}{[ (1 - \varepsilon) \sigma + \varepsilon ] \tan \psi_0} \]  

Similarly

\[ \alpha_c = \frac{\varepsilon (1 - c/h) \tan^2 \psi_0 / \beta_T - \sigma}{[ (1 - \varepsilon) \sigma + \varepsilon ] \tan \psi_0} \]  

A detailed discussion of the pressure distribution will not be given since it does not differ a great deal from that in a pump without leakage when the leakage flow rate is small. The above results will now be used to calculate the leakage flow rate \( Q_L \).

3.4 Leakage Flow Rate

The analysis of the leakage flow rate is simplified with the introduction of the control region \( A_1A_2B_2 \) of Fig. 3.2, where \( A_1A_2 \) is the full developed cross section through the screw pump. The flow rate leaving the control region through that cross section is equal to the total flow rate \( Q \) through the screw pump.
The sum of the flow rates entering the region across the boundaries $A_1B_2$ and $B_2A_2$ must be equal to $Q$. Now it is much simpler to determine $Q$ from the separate flow rates across $A_1B_2$ and $B_2A_2$.

Each flow rate can again be regarded as the superposition of a drag flow rate and a pressure back flow rate. The equations for these drag flow and pressure back flow rates are quite similar to those of the simplified theory.

The control region is further subdivided into the channel region $A_1C_1B_1$ and the flight region $C_1A_2B_2B_1$. Flow rates in the $x$-direction are investigated first.

The channel drag flow rate $Q_{DC}$ is identical to the theoretical drag flow rate

$$Q_{DC} = \frac{1}{2} U_0 T_0 \cdot h = Q_{DT} \quad (3.18)$$

The channel pressure back flow rate $Q_{PC}$ is similarly

$$Q_{PC} = Q_{PT} = \frac{k^3 T_0}{12} \frac{1}{\mu_c} \left( \frac{\partial P}{\partial x} \right)_c \quad (3.19)$$

The flight clearance drag flow rate $Q_{DF}$ and the pressure back flow rate $Q_{PF}$ are defined as flow rates across the line $B_1B_2$. They are

$$Q_{DF} = \frac{1}{2} U_0 c (e \cos \gamma_o) \quad (3.20)$$

$$Q_{PF} = \frac{c^3}{12} (e \cos \gamma_o) \frac{1}{\mu_F} \left( \frac{\partial P}{\partial x} \right)_c \quad (3.21)$$

Next we investigate flow in the $y$-direction. The flow rates that cross the line $B_2A_2$ are

$$Q_{DF_y} = \frac{1}{2} U_0 c (\Pi D \cos \gamma_o) \quad (3.22)$$

$$Q_{PF_y} = \frac{c^3}{12} (\Pi D \cos \gamma_o) \frac{1}{\mu_F} \left( \frac{\partial P}{\partial y} \right)_F \quad (3.23)$$
The total drag flow rate entering the region across both boundaries $A_1B_1$ and $B_2A_2$ is then

$$Q_D = Q_{DC} + Q_{DF} - Q_{DF_2} \quad (3.24)$$

Substitution of (3.18, 3.20, 3.22) and

$$V_0 = U_0 \tan \varphi_0$$
$$t = \Pi D \tan \varphi_0$$
$$b_0 = (t - e) \cos \varphi_0$$

yields the total drag flow rate

$$Q_D = \frac{1}{2} (h - c) T_0 U_0 \quad (3.25)$$

This equation shows that the drag flow through a pump with a clearance can be obtained from the equation of the simplified theory when the channel depth $h$ is replaced by an effective channel depth $h - c$.

Next it will be shown that the flow rates resulting from the pressure gradients can be expressed in $Q_{TD}$. The back flow coefficient is first used to eliminate the pressure gradient.

From (3.10)

$$\left( \frac{\partial p}{\partial x} \right)_c = \frac{G M_c U_0 \beta T}{h^2} \quad (3.26)$$

Substitution in (3.21) yields with (3.16)

$$Q_{PF} = \frac{1}{2} U_0 h (e \cos \varphi_0) \sigma \beta T \quad (3.27)$$

A similar substitution in (3.23) results in

$$Q_{PFy} = \frac{1}{2} U_0 h (\Pi D \cos \varphi_0) \sigma \beta T \alpha_F \quad (3.28)$$

The leakage flow rate $Q_L$ is then

$$Q_L = Q_{DT} - Q_{PF} - Q_D + Q_{PC} + Q_{TF} - Q_{PFy}$$
$$= \frac{1}{2} c T_0 U_0 + Q_{PF} - Q_{PFy} \quad (3.29)$$
Both $Q_{PF}$ (3.27) and $Q_{PFY}$ (3.28) now contain the theoretical drag flow rate $Q_{DT}$ as a factor. In leakage flow calculations it is advantageous to express the leakage flow as a fraction of that drag flow. This fraction is

$$ k_o = \frac{Q_L}{Q_{DT}} \quad (3.30) $$

Substitution of (3.27, 3.28, 3.29) in (3.30) leads to the leakage flow coefficient

$$ k_o = \left( \frac{c}{h} \right) + \frac{\sigma}{1-\varepsilon} \left[ \varepsilon \beta_T + \frac{\beta_T + (1-\varepsilon) \left( (1-c/h) \tan \phi \right)}{\left( (1-\varepsilon) \sigma + \varepsilon \right) \tan \phi} \right] \quad (3.31) $$

This coefficient is from (3.15, 3.16) a function of the ratios $c/h$, $e/t$, the helix angle $\phi$, the viscosity ratio and the back flow coefficient $\beta_T$.

It consists of a fraction $k_c$ that does not change with the mode of operation

$$ k_c = \frac{c}{h} + \frac{\sigma(1-c/h)}{\sigma(1-\varepsilon) + \varepsilon} \quad (3.32) $$

and a fraction that is proportional to $\beta_T$

$$ k_\beta = \beta_T \left( \frac{\sigma}{1-\varepsilon} \right) \left[ \varepsilon + \frac{1}{\sigma(1-\varepsilon) + \varepsilon \tan \phi} \right] = \beta_T k_{\beta_o} \quad (3.33) $$

The influence of different parameters on leakage flow can effectively be shown in a dimensionless performance diagram (Fig. 3.3), which relates the dimensionless flow rate

$$ f = \frac{Q}{Q_{DT}} \quad (3.34) $$

to the dimensionless pressure

$$ \beta_T = \frac{P_o}{P_{ST}} = \frac{Q_{PT}}{Q_{DT}} \quad (3.35) $$
$P_{ST}$ is the shut-off pressure for the pump without clearance. The dimensionless characteristic for that pump is the straight line

$$f_T = 1 - \beta_T \quad (3.36)$$

The characteristic for a pump with clearance is then

$$f = \frac{Q_T - Q_L}{Q_{DT}} = 1 - \beta_T - k_c \quad (3.37)$$

Substitution of (3.32, 3.33) results in

$$f = (1 - k_c) - \beta_T (1 + k_{\beta_0}) \quad (3.38)$$
The dimensionless characteristic of a pump with clearance is then also a straight line. It intersects the f-axis at a point \(1-k_c\), the \(\beta_T\) axis at a point \(\beta_T = (1-k_c)/(1+k_{\beta_0})\).

The coefficients \(k_c\) and \(k_{\beta_0}\) are functions of \(c/h\), \(e/t\), \(\lambda_C/\mu_F\) and \(\phi_0\). A number of dimensionless diagrams were constructed for \(\phi_0 = 17.65^\circ\). The other parameters were one at the time varied in each diagram.

Figs. 3.4 and 3.5 show the influence of clearance on performance. Both diagrams were constructed for constant viscosity ratios. A viscosity ratio was previously introduced to account for the non-Newtonian character of polymers. Now, a change in the clearance ratio \(c/h\) will then also change the viscosity ratio. The influence of screw wear on performance can be predicted from these diagrams.

Fig. 3.6 shows the influence of the viscosity, while Fig. 3.7 shows the influence of the land width ratio \(e/D\). The influence of the width appears to be small.

The diagrams show that leakage flow can become negative. This is only possible when the inlet pressure is much larger than the discharge pressure. Such an operating condition is uncommon in screw pumps. It can occur in extruders when the melting zone generates a large pressure which is partly throttled in the metering zone.
Fig. 3.4 - Influence of $C/h$ on leakage for $M_c/M_F = 1.0$

Fig. 3.5 - Influence of $C/h$ on leakage for $M_c/M_F = 3.0$
Fig. 3.6 - Influence of viscosity ratio on leakage

Fig. 3.7 - Influence of bandwidth on leakage
3.5 Leakage Flow Rate and Heat Transfer

So far it was assumed that all velocities everywhere in the channel are parallel to the barrel. That assumption is reasonable, except in small regions near the flight faces. The exact solution for the transverse velocity distributions in a rectangular channel results in a circulation, schematically shown in Fig. 3.8. The projections of the stream line are shown here as closed curves for a screw without a leakage flow rate. These stream lines are parallel to the barrel in most of the channel.

The stream lines must be different for a pump with a leakage flow rate, as shown in Fig. 3.9. The leakage flow rate through the flight clearance at $A_1$ must be the same as that through the flight clearance at $A_2$. That leakage flow rate occupies the space between the barrel and the connecting stream line $A_1A_2$.

This stream line is parallel to the barrel, except in the small regions near the flight faces. The liquid leaking through the flight clearances is then contained in a thin wall layer with a thickness $c^*$. The layer thickness is uniform in most of the channel width. Later it will be shown that this thickness $c^*$ differs from the clearance $c$.

The liquid contained in the wall layer flows in the proximity of the barrel and does not take part in the circulation. The stream line $A_1A_2$ separates the liquid in the channel in a circulating stream and a stream that always remains in the wall
FIG. 3.8 - SCHEMATIC DIAGRAM OF TRANSVERSE CIRCULATION IN CHANNEL WITHOUT CLEARANCE

FIG. 3.9 - FORMATION OF LAYER C* NEAR BARREL IN SCREW PUMP WITH LEAKAGE
layer. In other words, once each revolution the barrel is wiped clean except for the wall layer. The velocities in the wall layer are substantially equal to the barrel velocity.

This wall layer plays a dominant role in heat transfer to the liquid in the channel. In a heated screw pump, heat is continuously transferred to this wall layer. The innerface $A_1A_2$ of the wall layer is exposed to the circulating flow rate once each revolution.

Jepson (20) calculated heat transfer coefficients for the following model. The film that is stagnant relative to the barrel can be treated as a thermal resistance without appreciable heat capacity. In the time required for one revolution, this film is exposed to the circulating flow. The time interval for such a cycle is so short that the liquid in the channel can be treated as a slab of infinite thickness. The amount of heat transferred to the slab in that interval is then calculated by Jepson for a given wall temperature and a given uniform initial slab temperature. This amount of heat can be converted to a heat transfer coefficient.

Jepson calculated heat transfer coefficients as a function of the screw flight clearance and as a function of the number of wipes per minute.

It was shown in the previous section that the leakage flow rate $Q_L$, for a given screw pump geometry and a given viscosity ratio, changes with the mode of operation. One must then expect that the thickness of the wall layer also changes with the same parameters.
The intent is not to give a detailed analysis of heat transfer, but merely to point out that heat transfer coefficients cannot be based solely on the nominal flight clearance. They must be based on the thickness $c^*$ of the wall layer and, therefore, are influenced by the geometry of the screw, the viscosity ratio and the mode of operation of the screw pumps. The magnitude of the wall layer thickness is discussed in the next section.

3.6 Leakage Layer on Inside of Barrel

In heated or cooled screw pumps, the thickness of the wall layer is also influenced by the temperature distribution in the liquid. Here these temperature distributions are ignored. We again assume different viscosities $\mu_F$ in the flight clearance and $\mu_C$ in the channel to account for the influence of shear rate on viscosity. The same viscosity $\mu_C$ is assumed for the circulating liquid and the liquid contained in the wall layer, while that layer is in the channel. A different viscosity would otherwise invalidate all previous equations.

The wall layer thickness is derived from the transverse velocity distribution. That distribution is obtained by integration of (3.2b) for the boundary conditions

$$
\begin{align*}
\nu(y, 0) &= 0 \\
\nu(y, h) &= \nu_o
\end{align*}
$$

Then

$$
\nu = \nu_o \left( \frac{z}{h} \right) - \frac{1}{2} z (h-z) \frac{1}{\mu_c} \frac{\partial \nu}{\partial y} \left( \frac{\partial \nu}{\partial y} \right)_c
$$

(3.39)

The leakage flow rate $q_c$ per unit flight length is from (3.5, 3.8b) and (3.26)

$$
q_c = \frac{1}{2} \nu_o h - \frac{1}{2} \nu_o h \beta + \nu_c
$$

(3.40)
This flow rate can also be expressed as

\[ q_c = \frac{1}{2} U_o h \left[ 1 - \frac{\beta_T \alpha_c}{\tan \psi_0} \right] = \frac{1}{2} U_o h \left( 1 - \beta_y \right) \tag{3.41} \]

The coefficient \( \beta_y \) is a transverse back flow coefficient similar to the back flow coefficient for flow in the channel direction.

Equation (3.39) can be expressed in analogy to (1.51 a) as

\[ \nu^* = U_o \left[ \left( \frac{c}{h} \right) \left( 1 - 3 \beta_y \right) + 3 \beta_y \left( \frac{c}{h} \right)^2 \right] \tag{3.42} \]

The flow rate in the barrel layer is

\[ q_c = \int_{h-c^*}^{h} \nu \, dz \]

Substitution of (3.42) and integration yields

\[ q_c = \frac{1}{2} U_o h \left[ \frac{c^*}{h} (2 - \frac{c^*}{h}) - \beta_y \frac{c^*}{h} (3 - 2 \frac{c^*}{h}) \right] \tag{3.43} \]

The results of (3.41) and (3.43) must be the same. Then

\[ 1 - \beta_y = \frac{c^*}{h} \left( 2 - \frac{c^*}{h} \right) - \beta_y \left( \frac{c^*}{h} \right)^2 (3 - 2 \frac{c^*}{h}) \tag{3.44} \]

This equation has a double root at \( c^* = h \). The remaining root is

\[ \frac{c^*}{h} = \frac{1}{2} \left( \frac{1}{\beta_y} - 1 \right) = \frac{1}{2} \left[ \frac{\tan \psi_0}{\beta_T \alpha_c} - 1 \right] \tag{3.45} \]

Substitution of \( \alpha_c \) from (3.17 b) yields the relative barrel layer thickness

\[ \frac{c^*}{h} = \frac{1}{2} \frac{\sigma (1 - \varepsilon + \beta_T / \tan^2 \psi_0) + \varepsilon (c/h)}{\varepsilon (1 - c/h) - \beta_T \sigma / \tan^2 \psi_0} \tag{3.46} \]

The relative thickness is expressed again in the geometrical parameters \( c/h \), \( e/t \), and \( \psi_0 \), the viscosity ratio, and the back flow coefficient \( \beta_T \). A number of cases were calculated for a helix angle of 17.65° (\( t/D = 1 \)). One of the parameters was varied in each case - the others kept constant.
The operating conditions are varied in figs. 3.10 and 3.11 by varying $\beta_T$. A Newtonian liquid is used in fig. 3.10. The curves show that the thickness $c*$ is significantly influenced by the mode of operation of the screw pump for relative clearances larger than 0.04. Jepson's heat transfer model is based on the dotted line $c = c*$. For normal operating conditions, the actual wall layer thickness is smaller than the clearance used by Jepson.

Fig. 3.11 shows similar curves for a viscosity ratio $\mu_c/\mu_F = 5$. The influence of the mode of operation now becomes significant at a smaller relative clearance $c/h$.

The influence of the remaining parameters is shown in Figs. 3.12 and 3.13 for a back flow coefficient $\beta = 0.2$. The influence of the viscosity ratio is shown in Fig. 3.12. Thicker wall layers result for larger viscosity ratios. Viscosity ratios smaller than 1 are included to show the trend for cooled screw pumps.

Finally, Fig. 3.13 shows the influence of the land width for a viscosity ratio $\mu_c/\mu_F = 3$. The layer thickness does not change a great deal with a change in $e$, as would be predicted from Fig. 3.7.

Fig. 3.11 indicates that the calculated values for $c*/h$ can become negative. This occurs when the transverse flow rate $q_C$ of (3.40, 3.41) becomes negative, which is only possible when the pressure gradient in the transverse direction is so large, that $\beta y > 1$. 

**FIG. 3.10** - RELATIVE WALL LAYER THICKNESS $C^*/h$ FOR $M_c/M_F = 1.0$

INFLUENCE OF MODE OF OPERATION

**FIG. 3.11** - RELATIVE WALL LAYER THICKNESS $C^*/h$ FOR $M_c/M_F = 5.0$

INFLUENCE OF MODE OF OPERATION
Fig. 3.12 - Influence of viscosity ratio $\mu_e/\mu_f$ on relative wall layer thickness for $\beta_t = 0.2$, $\varepsilon/D = 0.1$.

Fig. 3.13 - Influence of relative flight width $\varepsilon/D$ on relative wall layer thickness for $\beta_t = 0.2$, $\mu_e/\mu_f = 3$. 
Negative wall layers cannot exist. Instead, a layer is formed on the screw root surface as shown schematically in Fig. 3.14. Some of the liquid is still dragged by the barrel through the flight clearance, but a larger quantity flows back into the channel. The equation for the relative wall layer thickness (3.46) does not apply to negative leakage flow rates. That layer thickness $c_{1}^{*}$ must be calculated from (3.41) and

$$q_{c_{1}} = \int_{0}^{c_{1}^{*}} v \, dz$$

(3.47)

Substitution of (3.42) and integration yields

$$q_{c_{1}} = \frac{1}{2} V_{o} h \left( \frac{c_{1}^{*}}{h} \right)^{2} \left[ 1 - \beta_{y} (3 - 2 \frac{c_{1}^{*}}{h}) \right]$$

(3.48)

This flow rate must again be equal to the one given by (3.41). The resulting equation has one root at $c_{1}^{*} = h$. The other roots are

$$\frac{c_{1}^{*}}{h} = \frac{(\beta_{y} - 1) \pm \sqrt{(\beta_{y} - 1)(9 \beta_{y} - 1)}}{4 \beta_{y}}$$

(3.49)

The transverse back flow coefficient $\beta_{y}$ must be larger than 1 for this case. Only the plus sign in (3.49) results in positive values for $c_{1}^{*}$. The occurrence of a screw root layer will be the exception rather than the rule, since it requires some combination of a large clearance, an inlet pressure higher than the discharge pressure, a very small land width or a large viscosity ratio. The relative screw root layer tends to be large when it occurs, since the velocities are small near the screw root. A number of examples are given in Table 3.1.
Table 3.1
Relative Screw Root Layer Thickness

for $\varphi_0 = 17.65^\circ$, $e/D = 0.1$

<table>
<thead>
<tr>
<th>$\beta_T$</th>
<th>$\mu_c/\mu_f$</th>
<th>$c/h$</th>
<th>$c_1^*/h$</th>
<th>$c_1^*/h$</th>
<th>$c_1^*/h$</th>
<th>$c_1^*/h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.8</td>
<td>5</td>
<td>0.150</td>
<td>0.6181</td>
<td>0.5386</td>
<td>0.4119</td>
<td>0.1217</td>
</tr>
<tr>
<td>-0.6</td>
<td>5</td>
<td>0.125</td>
<td>0.5052</td>
<td>0.4230</td>
<td>0.2982</td>
<td>0.0094</td>
</tr>
<tr>
<td>-0.4</td>
<td>5</td>
<td>0.100</td>
<td>0.3625</td>
<td>0.2849</td>
<td>0.1669</td>
<td></td>
</tr>
<tr>
<td>-0.2</td>
<td>5</td>
<td>0.075</td>
<td>0.1997</td>
<td>0.1289</td>
<td>0.0053</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.150</td>
<td>0.5046</td>
<td>0.4188</td>
<td>0.2856</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.125</td>
<td>0.3839</td>
<td>0.3000</td>
<td>0.1682</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.100</td>
<td>0.2453</td>
<td>0.1655</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4. End Effects in Channels with Large Aspect Ratios

4.1 Introduction

Screw pump channels are in conventional theories (1, 2, 3, 4, 6, 7) replaced by prismatic channels with rectangular cross sections. No mention is in these theories made of the boundary conditions along the oblique ends of the screw pump channel (Fig. 4.1). Congruent velocity distributions are assumed in all cross sections perpendicular to the center line of the screw. The pressure is in the simplified flat plate theory a linear function in x- and y (1.44). The pressures along the oblique ends, predicted by that theory, are then not constant, except for one special combination of drag flow and pressure back flow. The same applies to the rectangular channel analysis.

The screw pump must have uniform pressures along these oblique ends when the inlet and discharge connections of the pump do not restrict flow. The conventional analysis does not satisfy the actual pressures along the channel—inlet and discharge ends.

Flow through screw pumps is in this Chapter studied for uniform pressures along the oblique inlet and discharge ends. The resulting drag flow and pressure back flow rates differ from those determined in existing theories. These new flow rates are again expressed as fractions of the corresponding flow rates of the simplified theory. End effect correction factors are defined as the ratios of the new flow rates to those of the simplified flat plate theory.
This analysis is restricted to isothermal flow of very viscous incompressible liquids with uniform viscosities through channels with large aspect ratios. Large aspect ratios are used to keep the problem two dimensional. The majority of screw pumps and metering zones of extruders have aspect ratios in excess of 10. The results of this theory are then useful for most cases.

Acceleration and gravity forces can be neglected for slow flow of very viscous liquids. Velocity distributions are again studied relative to the screw. Leakage flow across the flights is neglected in this analysis.

4.2 Geometry of Screw Pump Channel

The helical channel with large aspect ratio is, in the simplified theory, replaced by a straight prismatic channel with a rectangular cross section. The same prismatic channel is used in this analysis, now however with the specific requirement that the ends of the channel make angles $\phi_0$ with the channel direction. The schematic development of this channel is shown in Fig. 4.1. The barrel becomes a flat plate that moves with a velocity $W_0 = \frac{1}{2} D N$ over channel and flight tips.

We must then determine the velocity distribution in the region $A_1A_1B_1B$ for uniform pressures along the inlet and discharge ends while the barrel moves with a velocity $W_0$ in the direction shown in Fig. 4.1.
FIG. 4.1 - SCREW PUMP CHANNEL DEVELOPED IN A PLANE, SHOWING OBLIQUE INLET AND DISCHARGE ENDS.
4.3 Average Velocities and Velocity Potential

All velocities in a channel with a large aspect ratio are parallel to the barrel except in small regions close to the flights. These regions become very small when the aspect ratio is very large. The velocity component \( w \) perpendicular to the barrel vanishes everywhere in the channel except in these small regions. In this analysis we assume such a large aspect ratio that these regions become insignificant and \( w \) vanishes in the whole channel.

Steady flow is, in the absence of acceleration and body forces, governed by the equations of motion

\[
\begin{align*}
\frac{\partial p}{\partial x} &= \mu \nabla^2 u \\
\frac{\partial p}{\partial y} &= \mu \nabla^2 v \\
\frac{\partial p}{\partial z} &= 0
\end{align*}
\]

(4.1) (4.2) (4.3)

The component \( u \) vanishes at the screw root, \( z = 0 \), and is equal to the barrel velocity component \( U_0 \) at \( z = h \). The derivative \( \partial u / \partial z \) is then much larger than the derivatives with respect to \( x \) and \( y \). In a similar way the derivative \( \partial v / \partial z \) is much larger than \( \partial v / \partial x \) and \( \partial v / \partial y \). Equations (4.1, 4.2) are then approximately equivalent to

\[
\begin{align*}
\frac{\partial p}{\partial x} &= \mu \frac{\partial^2 u}{\partial z^2} \\
\frac{\partial p}{\partial y} &= \mu \frac{\partial^2 v}{\partial z^2}
\end{align*}
\]

(4.4) (4.5)

The pressure is independent of \( z \) while the terms on the right hand side of both equations are approximately independent of \( x \) and \( y \). The equations can then be integrated.
For example

\[ u = \left( \frac{1}{2 \mu} \frac{\partial p}{\partial x} \right) z^2 + C_1 z + C_2 \]  

(4.6)

Boundary conditions are

\[ z = 0, \quad u = 0 \]
\[ z = h, \quad u = U_0 \]

The constants \( C_1 \) and \( C_2 \) of (4.6) can be evaluated yielding an equation for \( u \) in the form

\[ u = \left( \frac{1}{2 \mu} \frac{\partial p}{\partial x} \right) [ z ( z - h ) ] + U_0 \left( \frac{z}{h} \right) \]  

(4.7)

Similarly

\[ v = \left( \frac{1}{2 \mu} \frac{\partial p}{\partial y} \right) [ z ( z - h ) ] + U_0 \left( \frac{z}{h} \right) \]  

(4.8)

The average velocity components \( \bar{u} \) and \( \bar{v} \) are now determined by integration of (4.7, 4.8). For example

\[ \bar{u} = \frac{1}{h} \int_0^h u \, dz = - \frac{h^2}{12 \mu} \frac{\partial p}{\partial x} + \frac{U_0}{2} \]  

(4.9)

Similarly

\[ \bar{v} = \frac{1}{h} \int_0^h v \, dz = - \frac{h^2}{12 \mu} \frac{\partial p}{\partial y} + \frac{V_0}{2} \]  

(4.10)

These mean velocities can, following a principle used by Hele Shaw (8), be regarded as the velocities for a velocity potential defined by

\[ \psi = \frac{h^2 p}{12 \mu} - \frac{1}{2} U_0 x - \frac{1}{2} V_0 y \]  

(4.11)

The mean velocities are then

\[ \bar{u} = - \frac{\partial \psi}{\partial x}, \quad \bar{v} = - \frac{\partial \psi}{\partial y} \]  

(4.12)

The equation of continuity for the mean velocities can be written as

\[ \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \]  

(4.13)
Substitution of the velocities from (4.12) yields
\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{4.14}
\]
The velocity potential satisfies Laplace's equation. Calculation of the mean velocities requires the solution of Laplace's equation for the proper boundary conditions.

Boundary conditions are formulated for the frame of reference of Fig. 4.2. The mean velocity \( \bar{v} \) must vanish along the flights \( y = \pm \frac{1}{2} b_0 \) because no liquid can cross the flights in the absence of a leakage flow rate. Then
\[
y = \pm \frac{b_0}{2}, \quad \bar{u} = \left( \frac{\partial \psi}{\partial y} \right)_{\pm \frac{b_0}{2}} = 0 \tag{4.15}
\]
In Chapter 1 it was shown that velocity distributions in rectangular channels with large aspect ratios differ from those of the simplified theory only in small regions near the flights. The simplified theory predicts a uniform \( \bar{u} \), even along the flights \( y = \pm \frac{1}{2} b_0 \).

The average velocity vanishes along the flights in the rectangular channel theory, since a viscous liquid cannot slip over the flight faces. In that theory it becomes equal to the average velocity of the simplified theory at a small distance from the flight faces. This distance, which is of the order of \( h \), will in the analysis of end effects be ignored. At the flights we then accept \( \bar{u} \neq 0 \).

The oblique ends are described by the lines
\[
x = \pm a + y / \tan \psi_0 \tag{4.16}
\]
Fig. 4.2 - General Boundary Conditions

- $\psi = 0$
- $\frac{\partial \psi}{\partial y} = 0$
- $x = \alpha$
- $y = -\frac{E_0}{2}$
- $y = \frac{E_0}{2}$
- $P_1 = \text{constant}$
- $P_2 = \text{constant}$

FLIGHT
Let $p_1$ and $p_2$ be the uniform pressures at inlet and discharge. The potential $\psi_2$ at the discharge end is then

$$\psi_2 = \frac{h^2p_2}{12\mu} - \frac{1}{2} U_o a - \frac{1}{2} y \left( \frac{U_o}{\tan\psi_o} + U_o \right) \quad (4.17)$$

Along the inlet similarly

$$\psi_1 = \frac{h^2p_1}{12\mu} + \frac{1}{2} U_o a - \frac{1}{2} y \left( \frac{U_o}{\tan\psi_o} + U_o \right) \quad (4.18)$$

The velocity potential has singular points at the four corners $A$, $A_1$, $B_1$, and $B$.

The velocity potential could be determined for the general boundary conditions. It is more useful to determine the influence of the ends on drag flow and pressure back flow separately. We will show that the general case can be regarded as the superposition of several simpler cases.

### 4.4 Superposition of Simpler Cases

The general case can be regarded as the superposition of simpler cases. These cases will be so selected that the drag flow and pressure flow end correction factors can be determined immediately from the solution of two of these cases. Use is made of the superposition principle previously discussed in section 1.3.6.

The general case is here separated into three cases, each satisfying the boundary condition $\partial\psi/\partial y = 0$ along the flights. The boundary conditions for the three cases are shown in the following table.
TABLE 4.1

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Barrel Velocity Components</th>
<th>Uniform Pressures at Inlet</th>
<th>Discharge</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( U_0 )</td>
<td>( V_0 )</td>
<td>( \frac{1}{2} (p_1+p_2) )</td>
</tr>
<tr>
<td>1</td>
<td>( U_0 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>( V_0 )</td>
<td>( p_0=(p_1-p_2)/2 )</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>( p_1 )</td>
</tr>
<tr>
<td>SUM</td>
<td>( U_0 )</td>
<td>( V_0 )</td>
<td>( p_1 )</td>
</tr>
</tbody>
</table>

The first case was so selected that it is identical to the drag flow case of the simplified theory. Pressure gradients then vanish yielding a uniform pressure in the channel. The pressures along both oblique ends is then equal to that uniform pressure. This uniform pressure was here equated to the average pressure. The velocity potentials (4.17, 4.18) along the ends become

\[
\psi_1 = \psi_2 = -\frac{1}{2} U_0 x
\]  

(4.19)

The velocity potential in the region is further from (4.11)

\[
\psi = -\frac{1}{2} U_0 x
\]  

(4.20)

The average velocity component \( \bar{u} \) is \( \bar{u} = \frac{1}{2} U_0 \) (4.21)

while \( \bar{v} \) vanishes. The drag flow rate is

\[
Q_{DO} = \frac{1}{2} U_0 V_0 h
\]  

(4.22)

The velocity potentials along inlet and discharge for the second case are

\[
\psi_1 = \psi_2 = -\frac{1}{2} V_0 y
\]  

(4.23)

Then from (4.11)

\[
\frac{\partial \psi}{\partial y} = \left( \frac{h^2}{12 \mu} \right) \frac{\partial x}{\partial y} - \frac{1}{2} V_0
\]  

(4.24)
Fig. 4.3 shows schematically how the liquid flows through the channel when the barrel moves with a velocity \( V_0 \). The barrel drags liquid into the discharge end and removes liquid at the inlet end. The resulting flow rate \( Q_{D1} \) tends to reduce the total flow rate. This flow rate must be proportional to \( V_0 \) while it is independent of the pressures \( p_1 \) and \( p_2 \). The flow rate for the last case of Table 4.1 is clearly independent of \( U_0 \) and \( V_0 \). \( Q_{DO} \) and \( Q_{D1} \) are then the only flow rates that have the character of a drag flow. The total drag flow for channels with oblique ends is equal to

\[
Q_{BE} = Q_{DO} + Q_{D1} \tag{4.25}
\]

Calculation of this drag flow requires the solution of Laplace's equation for case 2 only, since \( Q_{DO} \) is known (4.22).
The last case of Table 4.1 corresponds to flow through a prismatic channel with oblique ends and stationary walls from uniform pressures along inlet and discharge ends. The velocity potentials along the ends are from (4.11)

\[ \psi_1 = - \psi_2 = \frac{1}{2} \left( p_1 - p_2 \right) \frac{h^2}{12\mu} = - \frac{h^2 p_0}{12\mu} \] (4.26)

The inlet and discharge pressures were so selected that the pressure vanishes at \( x = 0 \) and \( y = 0 \). The resulting pressure back flow rate \( Q_{PB} \) differs from the back flow \( Q_{PO} \) of the simplified theory. The latter was based on a constant pressure gradient \( \partial p/\partial x \), while \( \partial p/\partial y \) vanished. Case 3 and the corresponding simplified pressure back flow case must have identical inlet and discharge pressures at \( y = 0 \). Comparison of \( Q_{PB} \) and \( Q_{PO} \) yields the end correction factor for pressure back flow.

The analysis of the general case has now been reduced to the analysis of the simpler cases 2 and 3. It will later be shown that the results of case 2 can be predicted from the results of case 3.

4.5 Symmetry Considerations

Symmetry allows a further simplification of the problem. The drag flow velocity distribution induced by \( V_0 \) is treated first. Consider two points A and B (Fig. 4.4a), symmetrically located relative to the origin on a line through the origin. The mean velocities \( \bar{u}_A \) and \( \bar{u}_B \) in A and B change signs when the barrel velocity component \( V_0 \) is reversed (Fig. 4.4b).
Fig. 4.4 - Symmetry of Velocity Distribution
Subsequent rotation of the whole system around the z-axis through the origin through an angle of 180° interchanges the points A and B. The boundary conditions after this rotation (Fig. 4.4c) are identical to the original boundary conditions (Fig. 4.4a). The magnitudes and signs of \( \bar{u}_A \) and \( \bar{u}_B \) must then be the same. The mean velocities \( \bar{v}_A \) and \( \bar{v}_B \) must, for the same reason, be identical.

The gradients of the velocity potential must then from (4.12) be equal in points that are symmetrically located on a line through the origin.

The velocity potential difference between B and 0 is equal to the similar difference between 0 and A. This also applies to points \( \mathcal{O} \) and D located at inlet and discharge. Then

\[
\psi_0 - \psi_D = \psi_C - \psi_0 \tag{4.27}
\]

From (4.23), however,

\[
\psi_D = - \psi_C
\]

since

\[
\psi_D = - \psi_C
\]

Substitution in (4.27) shows that the velocity potential vanishes at the origin. Thus, any two points diametrically located on a line through the origin have equal but opposite velocity potentials.

Laplace's equation then requires solution in half the channel region only. The additional boundary condition

\[
x = 0 , \quad \psi(0, y) = - \psi(0, -y) \tag{4.28}
\]
can be used to determine that solution. A similar symmetry applies to the pressure back flow case since the velocity potentials at inlet and discharge (4.26) have equal and opposite values.

4.6 End Correction Factors

The drag flow rate \( Q_{DE} \) for channels with oblique ends can be expressed in the corresponding simplified drag flow rate \( Q_{DO} \) through the introduction of the drag flow end correction factor \( F_{DE} \). Then

\[
Q_{DE} = Q_{DO} F_{DE}
\]  

(4.29)

From (4.22, 4.25)

\[
F_{DE} = 1 + \frac{2Q_{DI}}{U_0 b_0 h}
\]  

(4.30)

where

\[
Q_{DI} = h \int_{-b_0/2}^{b_0/2} \bar{u} \, d\eta = -h \int_{-b_0/2}^{b_0/2} \left( \frac{\partial \psi}{\partial x} \right)_{x=0} \, d\eta
\]  

(4.31)

Introduce now the dimensionless velocity potential

\[
Z_D = \psi / U_0 b_0
\]  

(4.32)

and the dimensionless coordinates

\[
X = x / b_0, \quad Y = y / b_0
\]  

(4.33)

The reduced potentials along the inlet and discharge ends are from (4.23)

\[
Z_{D1} = Z_{D2} = -\frac{V_0 y}{2U_0 b_0} = -\frac{1}{2} Y
\]  

(4.34)

while \( \partial Z_D / \partial Y \) vanishes along the flights.
Further \[ \frac{U_0}{U_o} = \tan \phi_o \]

Substitution of (4.31, 4.32, 4.33) in (4.30) yields

\[ F_{DE} = 1 - 2 \tan \phi_o \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{\partial Z}{\partial X} \right) \, dY \]  

(4.35)

The regions, expressed in the dimensionless coordinates X and Y, are identical for geometrically similar screw pump channels, while they have the same dimensionless potential \( Z_D \) along the boundaries. The integral in (4.35) is then a constant for geometrically similar channels. The end correction factor is, therefore, a function only of the factors that describe the dimensionless region. It is then a function of the helix angle \( \phi_o \) and the ratio \( a/b_0 \).

The pressure back flow rate \( Q_{PE} \) can, in a similar way, be related to the simplified pressure back flow rate \( Q_{PO} \). The pressure back flow end correction factor \( F_{PE} \) is given by

\[ Q_{PE} = F_{PE} Q_{PO} \]  

(4.36)

The simplified pressure back flow rate must be based on pressures \( \pm p_0 \) at points \( y = 0, x = \pm a \), where \( p_0 = \frac{1}{2}(p_2 - p_1) \).

For that case

\[ \frac{\partial p}{\partial x} = \frac{p_0}{a} \]  

(4.37)

The simplified pressure back flow rate is then

\[ Q_{PO} = \frac{h^3 b_0}{12 \mu} \left( \frac{p_0}{a} \right) \]  

(4.38)
The flow rate through the channel with oblique ends is

$$ Q_{PE} = h \int_{-\frac{E_0}{2}}^{+\frac{E_0}{2}} \left( \frac{\partial \psi}{\partial x} \right)_{x=0} \, dy $$  \hspace{1cm} (4.39)$$

Let now

$$ Z_p = \psi / \left( \frac{h^2 \rho_0}{12 \mu} \right) , \quad X = x / E_0 , \quad Y = y / E_0 $$  \hspace{1cm} (4.40)$$

then

$$ F_{PE} = \frac{\alpha}{E_0} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \left( \frac{\partial Z_p}{\partial X} \right)_{x=0} \, dY $$  \hspace{1cm} (4.41)$$

At the oblique ends from (4.26, 4.40)

$$ Z_{p1} = - Z_{p2} = -1 $$  \hspace{1cm} (4.42)$$

while \( \partial Z_p / \partial Y \) vanishes along the flights. The integral (4.41) is again a constant for geometrically similar channels. Thus, the end correction factor \( F_{PE} \) is also a function of the helix angle \( \phi_0 \) and the ratio \( a/b_0 \).

4.7 Numerical Method

The analytical solution to the problem was not obtained. The method of finite differences was used instead to obtain approximate solutions.

The factors \( F_{DE} \) and \( F_{PE} \) must be determined for a range of helix angles and a large number of \( a/b_0 \) ratios. It is not essential to know the full solution of \( \psi \) through all regions when only \( F_{DE} \) and \( F_{PE} \) are required.
A numerical method was devised for the solution of large numbers of partial differential equations for problems with the following properties.

a - the region, for which the desired property of the differential equation must be calculated, is one of a family of regions. The family of regions consists of regions with one identical dimension, while the other dimension increases systematically.

b - the required property can be expressed as a geometric property of that differential equation in that region.

The end effect factors (4.35, 4.41) are typical examples of geometric properties of a differential equation in a region. The family of regions in that case consists of channels with the same helix angle $\phi_0$ and different ratios $a/b_0$.

The shape factors $F_D$ and $F_P$ (1.36, 1.37) for flow of viscous liquids through channels with rectangular cross sections are also geometric properties of Laplace's and Poisson's equations respectively, now in a family of rectangular regions with different aspect ratios.

The numerical method is explained and illustrated in Chapter 6. There it is shown that calculation of the required geometric property does not require calculation of the function through the whole region.
4.8 Functional Relation Between $F_{DE}$ and $F_{PE}$

A simple relation exists between the end correction factor for drag flow and that for pressure back flow. That relation can be derived in two different ways.

The total flow rate and pressure distribution of the simplified theory are compared in the first method with similar quantities of the theory of end effects for the same operating conditions $U_0$, $V_0$, and $P_0$. From the simplified theory

$$Q_1 = Q_{DO} - Q_{P0}$$

where $Q_{P0}$ is proportional to $P_0$. The theory of end effects predicts a total flow rate

$$Q_2 = Q_{DE} - Q_{PE} = F_{DE} Q_{DO} - F_{PE} Q_{P0}$$

Fig. 4.5 shows both $Q_1$ and $Q_2$ as functions of $P_0$ for the same screw pump running at the same speed. Now $F_{DE} < 1$, since $Q_{DL}$ reduces the drag flow (Fig. 4.3), while $F_{PE} > 1$, since the resistance of a channel with oblique ends is smaller than that of the channel with square ends, used in the simplified theory. The two characteristics must then intersect.

The equation for the velocity potential (4.11) for the total flow rate applies in both theories, but the sum of the conditions of Table 4.1 is in general not satisfied in the simplified theory. We know from the pressure distribution of the simplified theory (1.3.9) that the sum of these conditions is only satisfied when $\beta = -\tan^2\phi_0$ (1.48). The velocity distribution for that case satisfies the equation for the velocity potential, the differential equation (4.14) and the sum of the conditions of Table 4.1.
The intersect of \( Q_1(P_0) \) and \( Q_2(P_0) \) must then satisfy (1.48).

From (1.48, 4.43)

\[
Q_1 = Q_{DO} \left( 1 + \tan^2 \psi_0 \right)
\]  
(4.45)

while from (1.48, 4.44)

\[
Q_2 = Q_{DO} \left( F_{DE} + F_{PE} \tan^2 \psi_0 \right)
\]  
(4.46)

From \( Q_1 = Q_2 \) at that back flow coefficient then

\[
1 + \tan^2 \psi_0 = F_{DE} + F_{PE} \tan^2 \psi_0
\]  
(4.47)

This equation shows a simple relation between the end effect correction factors.

The same result can be obtained by the superposition of a number of cases with different boundary conditions, which all satisfy the condition \( \partial \psi / \partial y = 0 \) along the flights \( y = \pm b_0/2 \).

The flow rate in each of these cases is governed by the distribution of the velocity potential along the inlet and discharge ends.
First consider flow through the channel for uniform pressures \( \pm p_0 \) along the ends. The flow rate is \( Q_{PE} \); the velocity potential along the ends is given by (4.26). The uniform velocity potential at the discharge end is then

\[
\psi_{p1} = -\frac{h^2b_0}{12\mu} \quad (4.48)
\]

The pressure at \( x = a, y = 0 \), is equal to \( p_0 \); that at \( x = 0, y = 0 \), is zero.

Next consider the velocity potential for the pressure back flow rate of the simplified theory where again \( p = p_0 \) at \( x = a, y = 0 \), and \( p = 0 \) at \( x = y = 0 \). The pressure gradient in the channel direction is

\[
\frac{dp}{dx} = \frac{b_o}{\alpha} \quad (4.49)
\]

The average velocity \( \bar{u}_0 \) for this case becomes from (4.9)

\[
\bar{u}_0 = -\frac{h^2}{12\mu} \frac{b_o}{\alpha} \quad (4.50)
\]

while the velocity potential is

\[
\psi_{p2}'' = -\frac{h^2b_o}{12\mu} \left( \frac{x}{\alpha} \right) \quad (4.51)
\]

The flow rate for this case is equal to \( Q_{p0} \). The velocity potentials \( \psi_{p1} \) and \( \psi_{p2}'' \) have the same magnitudes at the point \( x = a, y = 0 \). The difference between the flow rates \( Q_{PE} \) and \( Q_{p0} \) is from (4.36)

\[
Q_{PE} - Q_{p0} = (F_{PE} - 1)Q_{p0} \quad (4.52)
\]
This difference, $Q_{PE} - Q_{PO}$, is due to the difference between the potentials $\psi'_p$ and $\psi''_p$ along the discharge end. Then

$$\Delta \psi_p = \psi'_p - \psi''_p = - \frac{k^2 p_0}{12 \mu} (1 - \frac{x}{\alpha}) \quad (4.53)$$

Substitution of (4.16) yields

$$\Delta \psi_p = \frac{k^2 p_0}{12 \mu \alpha} \left( \frac{y}{\tan \phi_0} \right) \quad (4.54)$$

The drag flow rate $Q_{D1}$ was from (4.23) due to a velocity potential at the discharge end

$$\psi_D = - \frac{1}{2} \psi_0 y \quad (4.55)$$

That flow rate is also

$$Q_{D1} = Q_{DE} - Q_{DO} = (F_{DE} - 1) Q_{DO} \quad (4.56)$$

The velocity potentials $\Delta \psi_p$ and $\psi_D$ are both proportional to $y$. The flow rates (4.52, 4.56) due to these potentials must then be related by

$$\frac{(F_{PE} - 1) Q_{PO}}{(F_{DE} - 1) Q_{DO}} = \frac{\Delta \psi_p}{\psi_D} \quad (4.57)$$

Substitution of (4.54, 4.55) and equations (1.29, 1.30) for $Q_{DO}$ and $Q_{PO}$, respectively, yields the relation

$$\frac{F_{PE} - 1}{F_{DE} - 1} = - \frac{1}{\tan^2 \phi_0} \quad (4.58)$$

or

$$F_{DE} + F_{PE} \tan^2 \phi_0 = 1 + \tan^2 \phi_0$$

This result is identical to (4.47). This relation is shown in Fig. 4.6 for a number of helix angles. Commonly used helix angles are of the order of 17° or $\tan \phi_0 \approx 1/3$. Fig. 4.6 then shows that the ends must have a larger effect on pressure
Fig. 4.6 - Relation between $F_{PE}$ and $F_{DE}$ for different helix angles
back flow, while the end correction factor for drag flow must be close to unity.

4.9 Calculation Results

The finite difference calculation of end effect correction factors is described in Chapter 6. The integrals (4.35, 4.41) are calculated with that method from Laplace's equation and the boundary conditions. The accuracy of the numerical method was ascertained for a similar problem with a known analytical solution. The accuracy of the calculated end effect factors was further tested by varying the mesh size. Both F_{DE} and F_{PE} were calculated to investigate the agreement with the relation (4.47) between these factors and \( \phi_0 \).

The numerical method breaks down when the number of meshes in the y-direction becomes too large. The width \( b_0 \) must then be divided into fewer meshes when the factors are calculated for large \( a/b_0 \) ratios.

End correction factors can either be calculated as functions of the L/D ratio of the screw, or as functions of the channel ratio \( L_{CO}/b_0 \). The channel ratio is here defined as the ratio of the developed channel length \( L_{CO}(=2a) \) to the channel width \( b_0 \). The following relation exists between L/D and \( L_{CO}/b_0 \) for a single flighted screw with a small land width

\[
\frac{L_{CO}}{b_0} = \frac{L}{\pi D \sin \phi_0} = \left( \frac{L}{D} \right) \frac{1}{\pi \sin \phi_0} (4.59)
\]
Table 4.2 shows calculation results for $F_{PE}$ for different helix angles as function of the channel ratio $L_{CO}/b_0$. The channel width was here divided into $n$ equal meshes. Results are shown for different $n$ values. Small helix angles require a larger number of meshes in the $y$-direction. The factors could, for small helix angles, only be calculated for a channel width divided into a small number of equal meshes, since the required number of meshes in the $y$-direction would otherwise become too large. The table shows an insignificant increase in accuracy for $n$ larger than 6.

Table 4.3 shows similar results for the drag flow end correction factor $F_{DE}$. The end effect correction factors are shown in Figs. 4.7 and 4.8 as function of $L_{CO}/b_0$ and in Fig. 4.9 also as function of $L/D$. The calculated values agree with the relation (4.47) between $F_{DE}$, $F_{PE}$ and the helix angle, as shown in Table 4.4.

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<th>$n$</th>
<th>$L_{CO}/b_0$</th>
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<th>$F_{PE}$</th>
<th>$1+\tan^2 \phi$</th>
<th>$F_{DE}+F_{PE} \tan^2 \phi$</th>
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### Table 4.2 - Pressure Backflow End Correction

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## Table 4.3 - Dragflow End Correction Factors

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Fig. 4.7 - Pressure Back Flow
End Effect Factors $F_{pe}$

(Channel width divided into $n$ equal finite differences)
FIG. 4.8 - DRAG FLOW
END EFFECT FACTORS $F_{de}$

(CHANNEL WIDTH DIVIDED INTO $n$ EQUAL MESHES)
FIG. 4.9 - DRAG FLOW END EFFECT FACTORS $F_{DE}$

(CHANNEL WIDTH DIVIDED INTO 'n' EQUAL MESHES)
4.10 End Effect for Large Channel Ratios

Oblique ends cause a pressure distribution near the channel ends that is quite different from the one predicted by the simplified theory. One must expect smaller differences between these two distributions at points in the channel that are farther removed from these ends.

Consider, for example, flow through a long channel with oblique ends due to a pressure difference between inlet and discharge (Fig. 4.10). Assume that the pressure gradient \( \frac{\partial p}{\partial y} \) becomes insignificant at a distance \( L*/2 \) from the inlet and from the discharge. The influence of the ends is then not observed in the zone with length \( L_1 \). The pressure changes in the zone \( L_1 \) in the x-direction only. Let \( (\partial p/\partial x)_1 \) be the uniform pressure gradient in zone \( L_1 \) for a flow rate \( Q \).

![Diagram of a channel with oblique ends and labeled dimensions](image-url)
Consider now the pressure distribution in the inlet and discharge zones with lengths $L^*/2$ for a flow rate $Q$. The distributions in these zones are for a constant flow rate $Q$ independent of the length of zone $L_1$. The pressure distributions in the inlet and discharge zones are then the same even when $L_1 = 0$. The total channel length for $L_1 = 0$ is equal to $L^*$. Let $F_{PE}^*$ be the end correction factor for a total channel length $L^*$.

A pressure drop

$$\Delta p^* = L^* \left( \frac{\partial p}{\partial x} \right)_1$$

would yield a flow rate $Q F_{PE}^*$ in a channel with oblique ends and a total channel length $L^*$, according to the definition for $F_{PE}$. A flow rate $Q$ would require a pressure drop

$$\Delta p_2 = L^* \left( \frac{\partial p}{\partial x} \right)_1 / F_{PE}^*$$

(4.60)

The pressure drop in zone $L_1$ is for the same flow rate $Q$

$$\Delta p_1 = L_1 \left( \frac{\partial p}{\partial x} \right)_1$$

(4.61)

The total pressure drop for a channel with a total length $L_{CO} = L^* + L_1$ is then

$$\Delta p_0 = \left( \frac{\partial p}{\partial x} \right)_1 \left[ L_1 + L^*/F_{PE}^* \right]$$

(4.62)

Let $F_{PE}$ be the correction factor based on the full length $L_{CO}$. The total pressure drop is then also

$$\Delta p_0 = L_{CO} \left( \frac{\partial p}{\partial x} \right)_1 / F_{PE}$$

(4.63)
From (4.62, 4.63) then

\[ \frac{L_{CO}}{F_{PE}} = L_1 + \frac{L^*}{F_{PE}^*} \]  \hspace{1cm} (4.64)

Substitution of \( L_1 = L_{CO} - L^* \) yields

\[ \left( \frac{L_{CO}}{b_0} \right) \left[ 1 - \frac{1}{F_{PE}} \right] = \frac{L^*}{b_0} \left[ 1 - \frac{1}{F_{PE}^*} \right] \]  \hspace{1cm} (4.65)

The right hand side of (4.65) is a constant if it can be assumed that the influence of the ends is insignificant at distances from the ends larger than \( L^*/2 \). This constant, \( C(\phi_0) \), assumes different values for different helix angles. Then for \( L_{CO} > L^* \)

\[ \frac{L_{CO}}{b_0} \left[ 1 - \frac{1}{F_{PE}} \right] = C(\phi_0) = C_\phi \]  \hspace{1cm} (4.66)

The parameter \( C_\phi \) was determined from the calculated \( F_{PE} \) values listed in Table 4.2. The results are shown in Table 4.4. The parameter \( C_\phi \) appears to be independent of the channel ratio \( L_{CO}/b_0 \), except for small channel ratios. It is, for instance, constant for \( \tan \phi_0 = 1/3 \) and channel ratios larger than 5.

Fig. 4.11 shows the change of \( C_\phi \) with helix angle for large \( L_{CO}/b_0 \) ratios. An almost straight line results when \( C_\phi \) is plotted as function of \( \cot \phi_0 \) and \( \phi_0 < 30^\circ \). The data yield the equation

\[ C_\phi = 0.96 \cot \phi_0 - 0.64 \]  \hspace{1cm} (4.67)

Screw pumps generally have channel ratios \( L_{CO}/b_0 \) in excess of the critical value \( L^*/b_0 \) and helix angles smaller than 30°.
FIG. 4.11 - CHANGE OF $C_{\phi}$ WITH $\cot \phi_0$

FOR LARGE CHANNEL RATIOS

($L_{co} > L^*$)
### Table 4.4 - Coefficient $C_p$

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Equations (4.66) and (4.67) can for those cases be combined to express the end correction factor for pressure back flow as a function of \( \frac{L_{CO}}{b_0} \) and \( \varphi_0 \). Then

\[
F_{PE} = \frac{L_{CO}/b_0}{0.64 + L_{CO}/b_0 - 0.96 \cot \varphi_0} \tag{4.68}
\]

As a function of \( L/D \) with (4.61)

\[
F_{PE} = \frac{L/D}{0.64 \pi \sin^2 \varphi_0 + L/D - 0.96 \pi \sin \varphi_0 \cos \varphi_0} \tag{4.69}
\]

Correction factors can with (4.68, 4.69) be calculated for any helix angle smaller than 30°. The end correction factor from (4.69) is shown in Fig. 4.12 as a function of the screw pitch to diameter ratio.

We now introduce the effective channel length \( L_e \), defined as the length of a channel with square ends that has the same resistance as the channel with length \( L_{CO} \) and with oblique ends. Then

\[
L_e = \frac{L_{CO}}{F_{PE}} \tag{4.70}
\]

From (4.68) for channels with \( L^* < L_{CO} \) and \( \varphi_0 < 30° \)

\[
\frac{L_e}{b_0} = 0.64 + \frac{L_{CO}}{b_0} - 0.96 \cot \varphi_0 \tag{4.71}
\]

The length \( L_2 \) defined in Fig. 4.10 is further

\[
L_2 = L_{CO} - b_0 \cot \varphi_0 \tag{4.72}
\]

The difference between the effective length and \( L_2 \) can with (4.71, 4.72) be expressed as

\[
\frac{\Delta L}{b_0} = \frac{L_e - L_2}{b_0} = 0.64 + 0.04 \cot \varphi_0 \tag{4.73}
\]
Fig. 4.12 - Pressure backflow end effect factors from Eq. (4.69)
The effective length ratio exceeds $L_2/b_0$ by an amount of the order of 0.75 for commonly used helix angles when $L^* < L_{CO}$. The drag flow end correction factor is close to 1 when $L^* < L_{CO}$ and helix angles are of the order of 17°. For instance, $F_{PE} = 1.16$ for $\varphi_0 = 17.6°$ and $L/D = 5$. The corresponding drag flow correction factor from (4.47) is 0.985. The same critical length $L^*$ applies to the pressure back flow and drag flow case since $F_{PE}$ and $F_{DE}$ are related by (4.47). The numerical value for $L^*/b_0$ is discussed at the end of the next section.

4.11 Pressure Distribution and Streamlines

Channel pressure distributions were calculated with the numerical method of Chapter 6. Fig. 4.13 shows a typical pressure distribution for pressure back flow for a channel ratio $L_{CO}/b_0 = 4$. Pressures at all mesh points were calculated for uniform pressures along the channel ends and for $p = 0$ at the mid point of the channel. Isobars were determined graphically from that distribution.

Also shown are the streamlines. The slope of the streamlines is given by

$$\left(\frac{dy}{dx}\right)_s = \frac{\nu}{u} \tag{4.74}$$

For pressure back flow at any value of $z$ from (4.7, 4.8, 4.9, 4.10)

$$\frac{\nu}{u} = \frac{\partial p/\partial y}{\partial p/\partial x} = \frac{\nu}{u} = \left(\frac{dy}{dx}\right)_s \tag{4.75}$$
The slope of the streamlines at any point \((x,y)\) is independent of \(z\) and is also equal to the slope of the streamlines of the average velocities. The isobars of Fig. 4.13 satisfy the equation

\[
dp = \frac{\partial p}{\partial x} \, dx + \frac{\partial p}{\partial y} \, dy = 0 \quad (4.76)
\]

The slope of the isobars is then

\[
\left( \frac{dy}{dx} \right)_p = - \frac{\partial p/\partial x}{\partial p/\partial y} \quad (4.77)
\]

The product of the slope of the streamlines and that of the isobars is then from (4.75, 4.77) at a point \((x, y)\)

\[
\left( \frac{dy}{dx} \right)_p \left( \frac{dy}{dx} \right)_s = -1
\]

Isobars are thus perpendicular to streamlines. For streamlines from (4.74)

\[
\overline{v} \, dx - \overline{u} \, dy = 0 \quad (4.78)
\]

The continuity equation for this case can be expressed as

\[
\frac{\partial \overline{v}}{\partial y} = - \frac{\partial \overline{u}}{\partial x} \quad (4.79)
\]

which shows that the expression \(\overline{v} \, dx - \overline{u} \, dy\) is the exact differential of a function \(\Omega\), called the stream function. Streamlines are the lines of constant stream function. For such a line

\[
d\Omega = \frac{\partial \Omega}{\partial x} \, dx + \frac{\partial \Omega}{\partial y} \, dy = 0 \quad (4.80)
\]

Then from (4.78, 4.80)

\[
\overline{u} = - \frac{\partial \Omega}{\partial y} , \quad \overline{v} = \frac{\partial \Omega}{\partial x} \quad (4.81)
\]
FIG-4·13 - ISOBARS AND STREAMLINES FOR PRESSURE BACK FLOW

FIG-4·14 - VELOCITY POTENTIAL DISTRIBUTION AND STREAMLINES DUE TO $V_0$
The stream function is constant along the flights, since the flights are streamlines. Let $\Omega = 0$ along the lower flight ($y = -b_0/2$) of Fig. 4.13. The stream function for a given value $x = x_0$ is

$$\Omega(x_0, y) = \int_{-\frac{b_0}{2}}^{y} \frac{\partial \Omega}{\partial y} \, dy = \int_{-\frac{b_0}{2}}^{y} \frac{\partial \Omega}{\partial y} \, dy$$

(4.82)

At $x = 0$

$$\Omega(0, \frac{b_0}{2}) = - \int_{-\frac{b_0}{2}}^{\frac{b_0}{2}} \frac{\partial \Omega}{\partial y} \, dy = - \frac{\Omega}{h}$$

(4.83)

The stream function of (4.82) can be expressed as a fraction of the stream function along the flight $y = b_0/2$. Then

$$S(x_0, y) = - \frac{\Omega(x_0, y)}{\Omega/h}$$

(4.84)

The stream function for pressure back flow also can be expressed with (4.9, 4.81, 4.82) as

$$\Omega(x_0, y) = \frac{k^2}{12\mu} \int_{-\frac{b_0}{2}}^{y} \frac{\partial P}{\partial x} \, dy$$

(4.85)

Then

$$S(x_0, y) = \left[ \int_{-\frac{b_0}{2}}^{y} \frac{\partial P}{\partial x} \, dy \right]_{x=x_0}$$

(4.86)

This reduced stream function was determined by numerical integration from the calculated pressure distribution. Streamlines are then lines of $S = \text{constant}$. A graphical construction was used to determine the streamlines from the $S$ distribution.

Fig. 4.14 shows the distribution of the velocity potential for drag flow due to $V_0$ alone.
For this case from (4.7, 4.8) for $U_0 = 0$

$$\frac{\bar{u}}{u} = \frac{(\frac{1}{2\mu} \frac{\partial \psi}{\partial y}) \bar{z}(z-h) + U_0(z)}{(\frac{1}{2\mu} \frac{\partial \psi}{\partial x}) \bar{z}(z-h)} = f(z) \quad (4.87)$$

Particles traveling at different $z$ describe different streamlines. For the average velocities from (4.12)

$$\frac{\bar{u}}{u} = \frac{\partial \psi/\partial y}{\partial \psi/\partial x} \quad (4.88)$$

Streamlines for the average velocities are again perpendicular to lines of constant velocity potential. A streamfunction, similar to the one used before, can be introduced. It is also constant along the flights. Fig. 4.14 shows how only part of the liquid entering the channel from the oblique end passes through the channel, while the other part leaves through the same channel end.

The slopes of the streamlines at $x = 0$ (Fig. 4.13) change with the channel ratio $L_{CO}/b_0$. Streamlines at $x = 0$ are almost parallel to the flights when $L_{CO}$ exceeds the critical length $L^*$. The pressure gradient $\partial p/\partial x$ at $x = 0$ is independent of $y$ in that case. The average slope of the streamlines at $x = 0$

$$\left(\frac{dy}{dx}\right)_{s(\text{AVG.})} = \frac{1}{b_0} \int_{-b_0/2}^{b_0/2} \left(\frac{dy}{dx}\right)_{s} dy \approx \frac{\int_{-b_0/2}^{b_0/2} \frac{\partial \psi}{\partial y} dy}{b_0 \left(\frac{\partial \psi}{\partial x}\right)_{x=0}} \quad (4.89)$$

was calculated by numerical integration for different ratios $L_{CO}/b_0$. Fig. 4.15 shows the results for different helix angles.
FIG. 4.15 - DECREASE OF \((\frac{\partial P}{\partial y})_{\text{avg.}}\) AT CENTER (X=0) WITH INCREASE OF \(\frac{L_{CO}}{B_0}\)
The average slope becomes insignificant when \( L_{CO} \) exceeds \( L^* \).

We now introduce the length \( L^{**} \) defined in Fig. 4.10. Then

\[
L^{**} = L^* - B_0 \cot \varphi_0 \quad (4.90)
\]

From Fig. 4.15 we selected values \( L^*/b_0 = 4, 5 \) for \( \cot \varphi_0 = 2, 3 \) respectively, as the channel ratios for which the slopes are insignificant. Substitution of these ratios yields a constant ratio \( L^{**}/b_0 = 2 \). The influence of the ends then ceases to be significant at a distance from point P (Fig. 4.10) equal to \( b_0 \) when \( L^{**}/b_0 > 2 \). The critical length is then

\[
L^*/b_0 \approx 2 + \cot \varphi_0 \quad (4.91)
\]
5. Experimental Verification of End Effect

5.1 Introduction
Experiments were carried out to measure the end effect in screw pump channels. It was outside our scope to measure end effects on rotating equipment. The pressure distribution, the end effect factor, and the streamlines were determined for a stationary channel with a rectangular cross section and an oblique end.

5.2 Model Channel
The general arrangement of the test model is shown in fig. 5.1a. A channel with a rectangular cross section is formed between two clear plastic parallel plates which are separated by two steel bars. The channel is at one end connected to a feed well, at the oblique end to a discharge well.

Corn syrup flows by gravity from a 2 inch diameter stand pipe into the feed well. The channel inlet cross section is at the feed well perpendicular to the sides of the channel. Fig. 5.1b shows a cross section through the feed well and through the entrance zone of the channel. A gate (fig. 5.1c) is installed in the discharge spout of the discharge well. The gate is set to provide a submerged discharge for the oblique end.
5.2

The gap between the parallel plates is 0.125 inches. The channel is 6 inches wide and has an average length of 15 inches. The aspect ratio of the rectangular cross section is 48/1. The slope of the oblique end (1/3) corresponds to a helix angle $\varphi_0 = 18.5^\circ$. Six small (0.032" diameter) injection ports were drilled in the top plate at intermediate distances of 1 inch. These ports were in the first test used for injection of small colored corn syrup streams.

Stand pipes were later cemented to the top plate of the channel. Each 0.5" ID pipe was connected to the channel through a 1/8" diameter hole. Fig. 5.1d shows the layout of 17 stand pipes in the oblique end and two stand pipes on top of the feed well.

5.3 Simulation of Screw Pump Channel

The model is an analog of the oblique end of a screw pump channel with a large aspect ratio. The comparable screw pump channel has a developed length $L_{CO} = 30"$ or a channel length ratio $L_{CO}/b_0 = 5$.

Fig. 5.2 shows the calculated pressure distribution in that screw pump channel. A similar calculated distribution for the model channel is shown in fig. 5.3. The pressures were in both cases adjusted for a zero discharge pressure. The differences between the isobars for the model and for the screw pump channel are very small. The calculated end effect factor for the screw pump is 1.8582, that for the model channel 1.8588. Both factors were calculated by finite
FIG. 5.1A - PLAN VIEW

FIG. 5.1B - SECTION 'AA'

FIG. 5.1C - MODEL CHANNEL

FIG. 5.1D - LAYOUT OF STAND PIPES
difference methods for a channel width divided into 6 meshes. The extrapolated factor for the model channel is 1.836.

The small difference between model and screw pump channel could have been predicted from the small relative transverse pressure gradient shown in Fig. 4.15. The model simulates a screw pump channel with a length equal to the critical length \( L^* \) discussed in section 4.10. It is, therefore, a reasonable analog for the end of screw pump channels that have lengths larger than the critical length.

5.4 Streamlines

Streamlines were made visible by the injection of small colored streams through small ports in the inlet zone of the channel. A main stream of clear corn syrup with a viscosity of 25 Poises was poured into the 2" stand pipe. Colored corn syrup with the same viscosity was pumped from a blow case. Fig. 5.4 shows the model during this test.

We showed in Chapter 4 that streamlines must be perpendicular to isobars. This not only applies to the average streamlines but also to the streamlines at any depth in the channel. The streamlines of Fig. 5.5 were photographed against a background of 1" square meshes. The streamlines are not influenced by the magnitude of the flow rate since inertia forces are insignificant at the flow rates used.
STREAMLINES PASS THROUGH
THE 6 INJECTION PORTS

FIG-5.2 - PRESSURE DISTRIBUTION AND STREAMLINES
IN SCREW PUMP CHANNEL

FIG 5.3 - CALCULATED ISOBARS, EXPERIMENTAL AND THEORETICAL
STREAMLINES FOR MODEL CHANNEL
The photograph shows a widening of the colored streams towards the discharge end. This is caused by a reduction of the velocities and not by diffusion.

Theoretical streamlines were calculated for the screw pump channel (Fig. 5.2) and for the model channel from the pressure distributions. The solid lines in Fig. 5.3 are the observed, the dotted lines the theoretical streamlines. Only streamlines passing through the location of the injection points are shown in both figures. The flow rate between adjacent streamlines is 1/6th of the total flow rate, except in the two outside lanes, where the flow rate between the streamline and the flight is 1/12th of the flow rate. The small differences between observed and theoretical streamlines near the discharge end could be due to inaccuracies in the photographic technique and due to the difficulty to measure streamlines precisely in the region where the colored streams fan out.

5.5 Pressure Distribution

Stand pipes and a separate 1/2" inlet connection to the main stand pipe were installed for the pressure distribution test. Colored corn syrup with a viscosity of 8.9 Poises was pumped from a blow case into the main stand pipe. The level in that pipe was manually controlled at 12" ± 1/8" above the top plate of the channel by regulating the air pressure in the blow case. The level in the discharge well was kept at 1-3/4" above the top plate. Fig. 5.6 shows the resulting pressure distribution against a background of 1" square meshes.
Fig. 5.4 Model Channel During Injection Test

Fig. 5.5 Experimental Streamlines
5.8

The levels were measured in all stand pipes after an equilibrium condition had been reached. These measured pressures are compared with the calculated pressures in Fig. 5.7. The calculated pressures were the result of a finite difference calculation for a channel width divided into 6 equal meshes. The measured pressures were reduced in Fig. 5.7 to give a pressure difference of 10" between inlet and discharge. They show good agreement between theory and experiment.

5.6 Experimental End Effect Factor

The experimental end effect factor, calculated from the flow rate, the pressure drop, the viscosity, and the dimensions of the model, was 1.88. This factor is about 2-1/2% larger than the theoretical value.

The measured flow rate was 11.5 cm$^3$/sec. It corresponds for a viscosity of 8.9 Poises and a density of 1.34 gram/cc to a Reynolds number of Re = 0.15. The length for the pressure drop per unit length to reach a constant value is for the inlet zones of circular tubes

$$z = 0.0288 \ D \ Re$$

where D is the diameter of the tube. Hartnett et al. (23) determined the constant in this equation for rectangular tubes with aspect ratios of 10:1, 5:1, and 1:1. The diameter D must then be replaced by the hydraulic diameter. Hydraulic diameters $D_H$ are for these aspect ratios respectively 1.81 h, 1.66 h, and h, where h is the short side of the rectangle.
Fig. 5.6 Pressure Distribution Test

Fig. 5.7 - Comparison of Theoretical and Measured Pressure Distributions
5.10
Hahnemann and Ehret (21) investigated a channel with an aspect ratio 50:1. For our purpose, it is more convenient to use the constant in the following equation

$$z = C h \text{Re} = C h \left( \frac{\rho U D h}{\mu} \right)$$

The resulting values of the constants are 0.0125, 0.060, 0.076, and 0.057 for the aspect ratios of 50:1, 10:1, 5:1, and 1:1, respectively. The hydrodynamic entrance length is negligible for the low Reynolds number used in our test. The gap between the parallel plates was made to a tolerance of ±0.001" which could at most introduce an error of 2-1/2%. The levels in the stand pipes were measured with an accuracy of ±1/8", corresponding to a possible error in the end effect factor of ±1%. The viscosity was measured in a Brookfield viscometer. Pressures in the model were kept low to avoid deflection of the plates. The experimental results appear to agree with the theory.
CHAPTER 6


6.1 Introduction

Calculation of the end effect factors of Chapter 4 requires the solution of Laplace's equation in a number of families of regions. For instance, all screw pump channels with the same helix angle belong to a family of regions with the same width and increasing length to width ratios. Different families exist for different helix angles.

A new numerical method was developed for the solution of linear partial differential equations in families of regions. Courant (24,25) discussed some of the elements in his lecture notes before electronic computers were used. He believed that the method could only be used for symmetrical regions. Other methods require evaluation of the function throughout the region by relaxation (11) or other iterative processes (12). The solution to a set of finite difference equations is in the described method obtained in closed form. Iteration is not required.

In Chapter 4, we showed how the calculation of end effect factors is reduced to the evaluation of two dimensionless integrals (4.34, 4.41) in a dimensionless region. The study of many linear partial differential equations is aimed at the evaluation of similar integrals or similar geometric properties. For instance, Timoshenko (10) "shows that the torque M transmitted by a bar with a rectangular cross section is related to the angle of twist \( \Theta \), the modulus of rigidity \( G \), and the dimensions 'a' and 'b' of the cross section by

\[
M = k_1 G \Theta a^2 b
\] (6.1)
The dimensionless coefficient $k_1$ is such a geometric property or shape factor for Poisson's equation in a rectangular region. This shape factor is a function of the aspect ratio $b/a$. The construction of equations of the type (6.1) can be predicted by dimensional analysis. The shape factor can then always be cast in the form of a dimensionless integral for a region or a boundary.

Twistings of bars with rectangular cross sections of different aspect ratio is a typical example of a problem involving a family of regions. The family consists of regions with one fixed side 'a' and a systematically increasing width 'b'.

The new method is always shorter than an iteration method when a given partial differential equation must be solved in families of regions. This is particularly true when the desired result is a geometric property or shape factor, rather than the value of the function in the whole region.

The principles of the method are first discussed. Next we show the advantages when the method is applied to families of regions. A short discussion on dimensional analysis and shape factors is then given.

The accuracy of results obtained with the new method is associated with the precision of the calculation. The accuracy is demonstrated for a problem with a known analytical solution. Slow viscous flow through a tube with a rectangular cross section was selected to demonstrate the method and the influence of precision on errors. This is followed by a discussion of the calculation of the end effect factors of Chapter 4.
6.2 Principles of Method

The method is a finite difference method which has many elements in common with existing numerical methods. The region to be investigated is first covered with a network of regularly spaced straight lines, each line being parallel to one of the coordinate axes. The partial differential equation is at each mesh point replaced by a finite difference equation. The resulting linear algebraic equations relate the value of the unknown function at that mesh point to values at neighboring points. The solution of these linear algebraic equations approaches the solution of the partial differential equation when the size of the meshes are decreased.

Figure 6.1 shows a segment of a region divided into meshes. The region has straight boundaries AA' and BB' which however is by no means necessary. Square meshes are used to simplify the description. Each mesh point is defined by indices $i$ and $j$.

Fig. 6.1 - Segment of Region
Assume for this discussion that Laplace's equation
\[ \nabla^2 F = 0 \] (6.2)
must be solved for known boundary conditions. The values
\[ F(i, j) \]
are unknown at all interior points \((i, j)\). Either the
function or its derivative is known along the boundary. A
finite difference equation of the form
\[ F(i+1, j) + F(i-1, j) + F(i, j+1) + F(i, j-1) - 4 F(i, j) = 0 \] (6.3)
can be written for the Laplacian at each mesh point.

Let the ranges for \(j\) and \(i\) be respectively \(j = 1, n, i = 1, m\).
The set \(F(i, j)\) can for every value of \(i\) be regarded as a
vector
\[ \{F\}^i = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} \] (6.4)

Vectors \(\{F\}^{i-1}\) and \(\{F\}^{i+1}\) at adjacent stations \(i-1\) and \(i+1\)
can similarly be expressed. Equation (6.3) can be rearranged to
\[ F(i+1, j) = 4 F(i, j) - F(i, j+1) - F(i, j-1) - F(i-1, j) \] (6.5)
One can then write a set of \(n\)-linear equations
\[ F(i+1, 1) = -F(i, 1) + 4F(i, 2) - F(i-1, 1) \]
\[ F(i+1, 2) = -F(i, 1) + 4F(i, 2) - F(i, 3) - F(i-1, 2) \] (6.6)
\[ F(i+1, 3) = -F(i, 2) + 4F(i, 3) - F(i, 4) - F(i-1, 3) \]
\[ F(i+1, n) = -F(i, n-1) + 4F(i, n) - F(i, n+1) - F(i-1, n) \]
The range \( j = 1, n \) does not allow use of the elements \( F(i,0) \) and \( F(i,n+1) \), since only \( n \) unknown values \( F(i,j) \) are considered. These quantities are either equal to known boundary values of the function, or they are related to the known derivatives at the boundary points \( j = 1 \) and \( j = n \). In the latter case

\[
\left( \frac{\partial F}{\partial y} \right)_{j=1} = \frac{F(i,2) - F(i,0)}{2h} \tag{6.7}
\]

\[
\left( \frac{\partial F}{\partial y} \right)_{j=n} = \frac{F(i,n+1) - F(i,n-1)}{2h}
\]

or

\[
F(i,0) = F(i,2) - 2h \left( \frac{\partial F}{\partial y} \right)_{1} \tag{6.8}
\]

\[
F(i,n+1) = F(i,n-1) + 2h \left( \frac{\partial F}{\partial y} \right)_{n}
\]

We will here assume that derivates rather than function values are given. Equations (6.6) then become

\[
F(i+1, 1) = 4F(i,1) - 2F(i,2) - F(i-1, 1) + 2h \left( \frac{\partial F}{\partial y} \right)_{1}
\]

\[
F(i+1, 2) = -F(i,1) + 4F(i,2) - F(i,3) - F(i-1, 2)
\]

\[
F(i+1, 3) = -F(i,2) + 4F(i,3) - F(i,4) - F(i-1, 3)
\]

\[
F(i+1, n) = -F(i,n-2) + 4F(i,n-1) - F(i,n) - F(i-1, n-1)
\]

\[
F(i+1, n) = -2F(i,n-1) + 4F(i,n) - F(i-1, n) - 2h \left( \frac{\partial F}{\partial y} \right)_{n}
\]
This set of equations can conveniently be written in matrix form

\[
\begin{bmatrix}
4 & -2 \\
-1 & 4 & -1 \\
-1 & 4 & -1 \\
-2 & 4
\end{bmatrix}
\begin{bmatrix}
\{F\}_{i+1} \\
\{F\}_{i}
\end{bmatrix}
= 
\begin{bmatrix}
2h\left(\frac{\partial F}{\partial y}\right)_{1} \\
0 \\
\vdots \\
0 \\
-2h\left(\frac{\partial F}{\partial y}\right)_{m}
\end{bmatrix}
\]

One can with this matrix equation always calculate \(\{F\}_{i+1} \) from \(\{F\}_{i}, \{F\}_{i-1}\) and the boundary conditions. Consider now for simplicity a case with a straight boundary at \(i = 1\) (Figure 6.2) and a uniform value \(F_{1}\) along that boundary.

**Fig. 6.2 - Introduction of unknowns \(x_{i}\) at \(i = 2\)**
Let further

\[
\{ B \} = \begin{bmatrix}
  4 & -2 & -1 & -1 \\
  -1 & 4 & -1 & -1 \\
  -1 & 4 & -1 & -2 \\
  -1 & 4 & -1 & 4
\end{bmatrix}
\]

Equation (6.10) becomes for \( \{ F \}_3 \)

\[
\{ F \}_3 = \{ B \} \{ F \}_2 - \{ F \}_1 + \begin{bmatrix}
  2h(\frac{\partial F}{\partial y})_1 \\
  \vdots \\
  -2h(\frac{\partial F}{\partial y})_n \\
\end{bmatrix}
\tag{6.11}
\]

The only unknowns in this equation are the elements \( F(2,j) \) in vector \( \{ F \}_2 \). Introduce now the unknowns

\[
F(2,j) = x_k
\tag{6.12}
\]

Equation (6.11) contains \( n \)-unknowns \( x_k \) and a set of constant coefficients. Formally we augment the vector \( \{ X \} \) with the element \( x_{n+1} = 1 \), to accommodate constants. Equation (6.11) then becomes

\[
\{ F \}_3 = \{ B \} \{ E \}_2 \{ X \} + \begin{bmatrix}
  \vdots & \vdots & \vdots & \cdots & \vdots & -F_i + 2h(\frac{\partial F}{\partial y})_1 \\
  \vdots & \vdots & \vdots & \cdots & -F_i \\
  \vdots & \vdots & \vdots & \cdots & -F_i \\
  \vdots & \vdots & \vdots & \cdots & -F_i - 2h(\frac{\partial F}{\partial y})_n \\
\end{bmatrix} \{ X \} \tag{6.13}
\]

\[
= \{ E \}_3 \{ X \}
\]
where

$$\{ X \} = \{ E \}_2 \{ X \} = \begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ \cdot & 1 & \cdots & \cdots \\ \cdot & \cdot & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \{ X \}$$

At $i = 4$ then

$$\{ F \}_4 = \{ B \} \{ E \}_3 \{ X \} - \{ E \}_2 \{ X \} + \begin{bmatrix} \cdot & \cdots & \cdots & \cdot & 2h (\frac{\partial F}{\partial y})_1 \\\\ \cdot & \cdots & \cdots & \cdot & \cdot \\\\ \cdot & \cdots & \cdots & \cdot & \cdot \\\\ \cdot & \cdots & \cdots & \cdot & \cdot \\\\ \cdot & \cdots & \cdots & \cdot & -2h (\frac{\partial F}{\partial y})_3 \end{bmatrix} \{ X \}$$

$$= \{ E \}_4 \{ X \} \quad (G.14)$$

The recursion equation (6.10) is used to produce a series of matrices $\{ E \}_i$. We will assume that the region ends at the boundary $i = m$ with a uniform derivative $\left( \frac{\partial F}{\partial x} \right) = F'_m$. At $i = m$, then

$$\frac{\partial F}{\partial x} = F'_m = \frac{F(m+1, \delta) - F(m-1, \delta)}{2h} \quad (G.15)$$

The matrix $\{ E \}_{m+1}$ must also be calculated.

Equation (6.15) becomes in matrix form

$$\left[ \{ E \}_{m+1} - \{ E \}_{m-1} \right] \{ X \} = \begin{bmatrix} \cdots & \cdots & 2h F'_m \\\\ \cdot & \cdots & 2h F'_m \\\\ \cdot & \cdots & 2h F'_m \\\\ \cdot & \cdots & 2h F'_m \\\\ \cdot & \cdots & 2h F'_m \end{bmatrix} \{ X \} \quad (G.16)$$
All coefficients in this equation are known. It can be written as

\[ \{ G \} \{ x \} = 0 \]  \hspace{1cm} (6.17)

where \( \{ G \} \) is an \( n \times (n+1) \) matrix.

This matrix equation represents a set of \( n \)-equations in \( n \)-unknowns \( x_k \) and a set of constants, since \( x_{n+1} = 1 \). The unknowns can be solved by any convenient method. Back substitution of the resulting values for \( x_k \) in

\[ \{ F \}_i = \{ E \}_i \{ x \} \]  \hspace{1cm} (6.18)

yields the values of the function at any mesh point.

A rectangular boundary was used to simplify the description. The method is not restricted to this simple geometry. A more general region is shown in Figure 6.3. Four unknowns, \( x_k \), are introduced at \( i = 1 \) and \( 2 \). The remaining unknown function values can at \( i = 2 \) be expressed in \( x_2, x_3 \), and the boundary values, using finite difference equations for irregular stars.

![Diagram of a rectangular boundary with labeled mesh points](image)
The unknown $x_1$ is eliminated from $\{E\}_{6}$, using the boundary condition at point $P$. The remaining unknowns are evaluated from the boundary conditions at points $Q_2$, $Q_3$ and $Q_4$. The method can in principle be applied to any shape of the contour and any set of mixed boundary conditions.

6.3 Extension to Families of Regions

The method is particularly suited to the solution of linear partial differential equations in families of regions. For instance, consider the family of regions of Figure 6.4, characterized by a uniform width $a$ and a variable length $L_x$. Assume that a given linear partial differential equation must be solved for this family of regions for the same boundary conditions. Let $m$ be the mesh number corresponding to the maximum length $L_m$ of that family.

Matrices of the type $\{E\}_i$ are for any length $L_x$ the same in the zone $0 \leq x \leq b$. A recursion equation of the type (6.10) applies in the zone $x \leq b$. Another series of matrices of the type $\{E\}_i$ can be formulated for the maximum range $b < x \leq L_m$. The number of matrices used for a given length $L_x$ then depends on the number of meshes in the $x$-direction contained in that length $L_x$. A coefficient matrix, calculated at a given value of $i$, does not change with the length $L_x$. An increased length requires additional matrices, but previously calculated matrices remain unaltered.
Fig. 6.4 - Typical Family of Regions
The function was at the boundary \( x = L_x \) assumed equal to a given value \( F_0 \) for any length \( L_x \). Let \( i = c \) be the mesh point at that boundary for an arbitrary length \( L_x \). The unknowns \( x_k \) then can be calculated from

\[
\begin{align*}
\{ E \}_c \{ x \}_c &= \{ F_0 \} \\
\end{align*}
\]

A similar set of unknowns can be calculated for a new region terminating at \( i = c + \alpha \), where the increment \( \alpha \) can have any desired value. Then

\[
\begin{align*}
\{ E \}_{c+\alpha} \{ x \}_{c+\alpha} &= \{ F_0 \} \\
\end{align*}
\]

This technique can be used to calculate \( x_k \) for a series of aspect ratios \( L_x/b \). Back substitution of any set \( x_k \) in the appropriate matrix equations \( (i \leq c) \) yields the solution of the differential equation for the required shape factor.

Solution of the differential equation in an additional region requires \( \alpha \) matrix operations of the type (6.14) and subsequent back substitution of \( x_k \) in the appropriate matrix equation. Additional techniques are described in Section 6.9.

6.4 Dimensional Analysis and Shape Factors

Differential equations are either solved to obtain the distribution of a function in a region, to derive an integrated form of that function in that region, or to derive an integrated form of the derivative on a boundary. The integrated form is often the more important result.
The dimensional construction of equations for integrated quantities can always be predicted by dimensional analysis of the differential equation. Exact equations for these quantities must always contain a factor expressing the influence of the relative shape of the region. These factors will further be referred to as shape factors. This will be illustrated with the following example.

The axial velocity \( w \) is for slow viscous flow through a long tube of constant cross section governed by Poisson's equation

\[
\nabla^2 w = \frac{1}{\mu} \frac{d p}{dz} = C_0
\]

where the pressure gradient is a constant.

Geometrically similar tubes are characterized by a similar dimension 'a' of the cross section. The cross sectional area is then proportional to \( a^2 \). Introduce now

\[
\tilde{w} = \frac{w}{a^2 \frac{d p}{dz}} , \quad \tilde{x} = \frac{x}{a} , \quad \tilde{y} = \frac{y}{a}
\]

Equation (6.19) then reduces to

\[
\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{w}}{\partial \tilde{y}^2} = 1
\]

This equation is identical for all geometrically similar tubes. The boundary condition \( \tilde{w} = 0 \) at the tube wall applies for all tubes. The solution \( \tilde{w}(\tilde{x}, \tilde{y}) \) of (6.21), satisfying the boundary conditions, is then the same for geometrically similar tubes.
The flow rate through any of these tubes can be written as

\[ Q = \int_A w \, dx \, dy = \frac{\alpha}{\mu} \frac{d\rho}{dz} \int_0^y w \, dx \, dy \]  

(6.22)

The integral of (6.22) has the same value \( k \) for geometrically similar tubes. The flow rate can then be written as

\[ Q = k \left( \frac{\alpha}{\mu} \frac{d\rho}{dz} \right) \]  

(6.23)

where \( k \) is a geometric or shape factor.

Next we investigate families of regions with the same characteristic dimension 'b' in one direction, but having a different characteristic dimension in the other direction. For instance, consider the family of regions of Fig. 6.4. All regions have the same height \( b \) but the length \( L_x \) is variable. The flow rates through tubes with these cross sections can always be expressed as (6.23). The geometric factor \( k \) is different for different ratios \( L_x/b \). It is then a function of that ratio.

Other important integral parameters appear in equations for flow or fluxes across boundaries. Integral constants of that type were used in the calculation of the end effect factors in screw pump channels (Chapter 4).

The matrix method can be used to calculate the shape factors from the coefficient matrices without having to determine the distribution of the function itself.
Calculation of the flow rate (6.22) from the matrices \[ E \] is here shown as an example. The flow rate through a square with sides 2h with point \((i, j)\) as center is by Simpson's rule

\[
\Delta Q_1 = \int \int w \, dx \, dy \approx \frac{h^2}{9} \left[ 16 w(i, j) + 4 \{ w(i, j-1) + w(i+1, j) + w(i-1, j) \} + w(i, j-1) + w(i, j+1) \right]
\]

\[
= \frac{h^2}{9} a_{kl} w_{kl}
\]

\[
\delta_{\ell} = i-1, i, i+1 \quad \delta_{k} = j-1, j, j+1
\]

(6.24)

where

\[
a_{kl} = \begin{bmatrix}
1 & 4 & 1 \\
4 & 16 & 4 \\
1 & 4 & 1
\end{bmatrix}
\]

(6.25)

The flow rate through the strip between \(i - 1\) and \(i + 1\) becomes for \(j = 1, n\)

\[
\Delta Q_0 = \frac{h^2}{9} a_{kl} w_{kl}
\]

where now

\[
a_{kl} = \begin{bmatrix}
1 & 4 & 1 \\
4 & 16 & 4 \\
4 & 16 & 4 \\
1 & 4 & 1
\end{bmatrix}
\]

(6.26)
The velocities at \(i\) contribute a quantity \(\Delta Q_i\) to the flow rate, given by

\[
\Delta Q_i = \left\{4, 16, 8, 16, \ldots, 8, 16, 4\right\} \{W_i\} = \{f\}_i \{W_i\} \quad (6.27)
\]

In analogy to equation (6.18), let

\[
\{W\}_i = \{E\}_i \{X\} \quad (6.28)
\]

then

\[
\Delta Q_i = \{f\}_i \{E\}_i \{X\} \quad (6.29)
\]

Similar expressions can be written for the contribution to the flow rate from \(\{W\}_{i-1}\) and \(\{W\}_{i+1}\). The summation of the flow rates through all strips in a region yields an equation for the total flow rate

\[
Q = \{F\}_i \{X\} \quad (6.30)
\]

where \(\{F\}_i\) is a row vector.

Equation (6.30) shows that flow rates and shape factors are linear equations in \(x_k\). They can be calculated once \(\{X\}\) is known. There is no need to calculate the velocity distribution by back substitution if the flow rate or shape factor is the only quantity of interest.

6.5 Accuracy of Numerical Method

The principles of the numerical method are straightforward. The method is clearly not attractive for manual calculation of a problem. The limitations to the method are demonstrated in the following sections. The key to success lies in the precision or number of digits retained in all operations.
The accuracy was tested for the problem of slow viscous flow through a long tube with a rectangular cross section, for which the analytical solution is known. This example is discussed in the next section. In addition, we calculated flow rates through triangular, circular and semicircular cross sections, for which shape factors are known.

6.6 Shape Factors for Tubes with Rectangular Cross Sections

Slow viscous flow of an incompressible liquid through a tube with a rectangular cross section has been studied by many investigators. The differential equation for viscous flow is analogous to the equation for torsion of bars with rectangular cross sections. The resulting flow equations are further used in some of the screw pump equations. For instance, Squires (4) uses the equation for the pressure back flow rate in the form

\[ Q_{P_{0}} = F_{p} \left[ \frac{a^{2}b}{12} \frac{1}{\mu} \frac{dP}{dz} \right] = F_{p} Q_{p*} \]  \hspace{1cm} (6.31)

for a channel with sides 'a' and 'b'. The quantity \( Q_{p*} \) is here the flow rate between parallel plates with a gap 'a', a plate width 'b', and the same pressure gradient. The so-called shape factor \( F_{p} \) is a function of the aspect ratio \( b/a \).

The difference between (6.23) and (6.31) is a result of a different definition for the geometric constant. Both \( k \) and \( F_{p} \) are functions of the aspect ratio, since

\[ F_{p} = 12 k \left( a/b \right) \]  \hspace{1cm} (6.32)
Flow through the tube is governed by Poisson's equation (6.19) for \( w = 0 \) at the tube wall. The analytical solution yields the shape factor of \((1.37)\).

Figure 6.5 shows the cross section of a typical rectangular tube covered with a network of lines parallel to \( x \)- and \( y \)-axes. It is only necessary to investigate one quadrant of the cross section since the solution in the remainder can be found from symmetry. The region has \( m \)-meshes in the \( x \)-direction, \( n \)-meshes in the \( y \)-direction. Interior mesh points are again defined by indices \( i \) and \( j \). Boundary conditions are

\[
\begin{align*}
\frac{\partial w}{\partial x}(x, 0) &= 0 \\
\frac{\partial w}{\partial y}(x, \frac{a}{2}) &= 0
\end{align*}
\]

Finite difference equations become for this case

\[
\begin{align*}
\frac{w(i+1, 1)}{4} &= w(i, 1) - w(i, 2) - w(i-1, 1) + h^2 C_0 \\
w(i+1, 2) &= -w(i, 1) + 4w(i, 2) - w(i, 3) - w(i-1, 2) + h^2 C_0 \\
w(i+1, n-1) &= -w(i, n-2) + 4w(i, n-1) - w(i, n) - w(i-1, n-1) + h^2 C_0 \\
w(i+1, n) &= -2w(i, n-1) + 4w(i, n) - w(i-1, n) + h^2 C_0
\end{align*}
\]

The vector \( \{ W \}_i \) for the velocities at \( i \) can be expressed as

\[
\{ W \}_i = \{ E \}_i \{ X \}
\]

where

\[
x_j = w(2, j)
\]
FIG. 6.9 - MESH POINTS IN TUBE WITH RECTANGULAR CROSSL-SECTION FOR CALCULATION OF ω FROM $\frac{\partial^2 \omega}{\partial t^2} = 0$
Matrices $\{E\}_i$ can be calculated with an equation similar to (6.10). Initial matrices at $i = 1$ and $i = 2$ are

$$\{E\}_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}, \{E\}_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \quad (6.37)$$

The center line of the cross section is at $i = m + 1$. The derivative $\partial w/\partial x$ vanishes along that line. The unknowns $x_k$ are then calculated from the matrix equation

$$\left[\{E\}_{m+2} - \{E\}_m\right]\{X\} = 0 \quad (6.38)$$

The flow rate equation (6.26) must be modified, since this region contains only half a strip. The velocity further vanishes at the boundary, while the other quadrants of the cross section must be accounted for. Different coefficients result for $n = \text{even}$ and $n = \text{odd}$. The flow coefficients are

For $m = \text{even}$

$$a_{kl} = \begin{bmatrix}
8 & 32 & 8 \\
32 & 128 & 32 \\
16 & 64 & 16 \\
32 & 128 & 32 \\
\end{bmatrix}$$

For $m = \text{odd}$

$$a_{kl} = \begin{bmatrix}
8 & 32 & 8 \\
32 & 128 & 32 \\
16 & 64 & 16 \\
32 & 128 & 32 \\
\end{bmatrix} \quad (6.39)$$
Let now

\[
\{k\} = \{8, 32, 16, \ldots, 32, 16, 32, 16, 8\} \quad \text{for } n = \text{even} \tag{6.40}
\]

\[
\{k\} = \{8, 32, 16, \ldots, 16, 32, 16, 16\} \quad \text{for } n = \text{odd}
\]

The incremental flow rate for \(i = \text{even}\) is then

\[
\Delta Q = \{k\}\{E_{i-1} + 2E_i\}\{x\} \tag{6.41}
\]

For \(i = \text{odd}\) it is

\[
\Delta Q = \{k\}\{2E_{i-1} + E_i\}\{x\} \tag{6.42}
\]

These alternating equations apply only when the derivative in the marching direction vanishes at the center. The flow rate \(Q^*_p\) of (6.31) is

\[
Q^*_p = \frac{C_0}{12} \alpha^3 b = \frac{4}{3} C_0 h^4 n^3 m \tag{6.43}
\]

There is no loss in generality when we assume \(C_0 = 1\), \(h = 1\) in (6.43) and in the finite difference solution of the problem. The shape factor is then calculated from (6.43) and the sum \(Q\) of the incremental flow rates (6.42). This sum can be expressed as

\[
Q = \{F\}\{x\} \tag{6.43a}
\]

The shape factor is then

\[
F_p = \frac{Q}{Q^*_p} = \frac{3Q}{4n^3 m} \tag{6.44}
\]

6.7 Errors and Precision

The accuracy of the numerical solution is influenced by the mesh size \(h\) and by the precision of the numerical calculations. Both the finite difference equation for Poisson's equation (6.3) and Simpson's rule have errors of the order of \(h^4\). Derivatives have errors of the order of \(h^2\). Shape factors for viscous flow through rectangular cross sections then have errors of the order of \(h^2\).
The mesh size $h$ is for a given width inversely proportional to the number of meshes. The exact shape factor $F_p$ can be expressed in the calculated factor $F_{PN}$, the number of meshes $n$, and an error coefficient $\varepsilon$. Then

$$F_p = F_{PN} + \frac{\varepsilon}{n^2} \quad (6.45)$$

For two different numbers of meshes $n_1$ and $n_2$

$$F_{PN1} - F_{PN2} = \varepsilon \left[ \frac{n_1^2 - n_2^2}{n_1^2 n_2^2} \right] \quad (6.46)$$

Substitution leads to the extrapolation equation

$$F_p = \frac{n_1^2}{n_1^2 - n_2^2} F_{PN1} - \frac{n_2^2}{n_1^2 - n_2^2} F_{PN2} \quad (6.47)$$

Extrapolated and calculated shape factors are shown in Table 6.1. Exact shape factors were calculated from the analytical solution (1.37) with errors smaller than $10^{-8}$.

The last two lines of Table 6.1 show the exact and extrapolated shape factors. The difference between exact and extrapolated factors is for $n_1 = 6$, $n_2 = 8$ of the order of $0.001\%$. It is of the order of $0.005\%$ when extrapolation must be based on $n_1 = 2$, $n_2 = 4$. Shape factors for other contours may require more complicated extrapolation equations. Calculation results can then be refined with a least square fit. Such a technique reduces the need for calculations with large numbers of meshes.

Inaccuracy can further be caused by accumulated round-off errors and by ill-conditioned matrices. Errors from both these sources
### Table 6.1 - Influence of Mesh Size on Shape Factors

All factors calculated with 16-precision.

<table>
<thead>
<tr>
<th>No. of Meshes</th>
<th>Aspect Ratio</th>
<th>0.100</th>
<th>0.666</th>
<th>0.500</th>
<th>0.400</th>
<th>0.333</th>
<th>0.250</th>
</tr>
</thead>
<tbody>
<tr>
<td>N = 2</td>
<td></td>
<td>0.39843750</td>
<td>0.5664212</td>
<td>0.66929257</td>
<td>0.73440324</td>
<td>0.77847639</td>
<td>0.83381757</td>
</tr>
<tr>
<td>N = 4</td>
<td></td>
<td>0.41590522</td>
<td>0.58214633</td>
<td>0.68189031</td>
<td>0.74470441</td>
<td>0.78711066</td>
<td>0.84030712</td>
</tr>
<tr>
<td>N = 6</td>
<td></td>
<td>0.41914600</td>
<td>0.585004626</td>
<td>0.68420329</td>
<td>0.74659297</td>
<td>0.78855752</td>
<td>0.90505584*</td>
</tr>
<tr>
<td>N = 8</td>
<td></td>
<td>0.42027823</td>
<td>0.58600208</td>
<td>0.68500912</td>
<td>0.74601236</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>Exact**</td>
<td></td>
<td>0.42173108</td>
<td>0.58728215</td>
<td>0.68604505</td>
<td>0.74809524</td>
<td>0.78995080</td>
<td>0.84243888</td>
</tr>
<tr>
<td>Extrapolated</td>
<td></td>
<td>0.42173395</td>
<td>0.58728499</td>
<td>0.68604758</td>
<td>0.74810182</td>
<td>0.78998875</td>
<td>0.84247030</td>
</tr>
</tbody>
</table>

* Insufficient precision
** Calculated from analytical solution

### Table 6.2 - Flow Through Tube with Rectangular Cross-Section

Influence of precision on solution of linear equations.

<table>
<thead>
<tr>
<th>Aspect Ratio</th>
<th>M1</th>
<th>X1 (8-precision)</th>
<th>X2 (16-precision)</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>2</td>
<td>79.598685</td>
<td>79.598707</td>
<td>-0.000,000,039</td>
</tr>
<tr>
<td>0.75</td>
<td>3</td>
<td>102.14011</td>
<td>102.140115</td>
<td>0.000,000,85</td>
</tr>
<tr>
<td>1.00</td>
<td>4</td>
<td>113.78687</td>
<td>113.786765</td>
<td>0.000,000,9</td>
</tr>
<tr>
<td>1.25</td>
<td>5</td>
<td>119.51310</td>
<td>119.512418</td>
<td>0.000,000,63</td>
</tr>
<tr>
<td>1.50</td>
<td>6</td>
<td>122.23602</td>
<td>122.23680</td>
<td>-0.000,013</td>
</tr>
<tr>
<td>1.75</td>
<td>7</td>
<td>123.50958</td>
<td>123.51115</td>
<td>0.000,44</td>
</tr>
<tr>
<td>2.00</td>
<td>8</td>
<td>124.15718</td>
<td>124.10229</td>
<td>-0.000,58</td>
</tr>
<tr>
<td>2.25</td>
<td>9</td>
<td>124.30282</td>
<td>124.37543</td>
<td>-0.001,2</td>
</tr>
<tr>
<td>2.50</td>
<td>10</td>
<td>124.34753</td>
<td>124.50140</td>
<td>0.011</td>
</tr>
<tr>
<td>2.75</td>
<td>11</td>
<td>125.88144</td>
<td>124.55945</td>
<td>-0.041</td>
</tr>
<tr>
<td>3.00</td>
<td>12</td>
<td>119.43261</td>
<td>124.58619</td>
<td>-0.11</td>
</tr>
<tr>
<td>3.25</td>
<td>13</td>
<td>110.81393</td>
<td>124.59851</td>
<td>-1.12</td>
</tr>
<tr>
<td>3.50</td>
<td>14</td>
<td>-15.60829</td>
<td>124.60418</td>
<td></td>
</tr>
</tbody>
</table>

* Based on quadrant with N=4 unknowns.
** M1 = Number of meshes in marching direction.
are reduced when a higher precision is used. Precision is here defined as it is used in digital computers. A number can be expressed as \( \pm m_p \times 10^q \) where the mantissa \( m_p \) is the largest decimal fraction smaller than 1, while \( q \) is an integer. The precision \( p \) is the number of digits retained in the mantissa.

The round-off error of the mantissa is of the order of \( 10^{-(p+1)} \). The relative round-off error of the number is of the same order.

The accumulated round-off error increases with the number of elementary operations. That number increases rapidly with \( n \) and \( m \). For instance, the number of operations required for the solution of \( n \) linear equations in \( n \) unknowns approaches \( n^3/3 \) when \( n \) is large. The number of operations for the calculation of \( \{ E \}_i \) is proportional to the product \( n \times m \). The growth of the error can become prohibitive when precision is insufficient.

The magnitude of errors was investigated for the problem of viscous flow through the rectangular tube by comparing calculation results for different precisions \( p = 4, 8, \) and 16. Round-off errors can occur or can be propagated in the calculation of \( \{ E \}_i \), in the solution of the set of linear equations, and the subsequent back substitution of \( x_k \) in flow rate equations.

The series \( \{ E \}_i \) starts with two matrices with binary coefficients that have no round-off errors. The same holds true for the finite difference equation or for the matrix \( \{ B \} \). The coefficients of subsequent matrices have no round-off errors until one of the elements exceeds \( 10^p \). The typical increase of the
**FIG. 6.6 - INCREASE OF THE ABSOLUTE VALUES OF THE COEFFICIENTS IN \{E\}_i^\alpha WITH i**
coefficients of \( \{ E \}_i \) is shown in Figure 6.6. Here we plotted the maximum and minimum absolute values of the coefficients for \( k = 1, n \) as function of the matrix number \( i \). The boundary matrices have no round-off errors when the corresponding \( \{ E \}_i \) matrices have no errors.

These boundary matrices were solved with 8 and 16 precision to determine to what degree they are ill conditioned. The matrices are precisely identical for \( m < 13 \) as shown in Figure 6.6. The solution \( (x_k)_8 \) for 8 precision differs from \( (x_k)_{16} \) for \( p = 16 \). Matrices are more ill conditioned when the relative difference between these solutions is larger. Table 6.2 shows the relative error for \( x_1 \) and \( n = 4 \) for increasing values of \( m \).

The coefficients \( \{ F \} \) of the flow rate equation (6.43a) show an increase similar to that of \( \{ E \}_i \). The errors in \( x_k \) are then magnified in the flow rate equation. The flow rate coefficients and the relative error in \( x_k \) increase simultaneously. The calculation of shape factors quite suddenly becomes unstable as shown in Figure 6.7. The point of breakdown is very much influenced by the precision.

The influence of the number of unknowns \( n \) was for the same problem investigated for a uniform precision \( p = 16 \). The results are shown in Figure 6.8, indicating that the problem becomes unstable at the same value \( m = 20 \) for any value of \( n \).

The ill condition of matrices is associated with the distribution of the coefficients. The character of the square part of the matrices can be described easier for flow through the full
FIG. 6.7 - INFLUENCE OF PRECISION OF STABILITY OF SHAPE FACTOR CALCULATION FOR n = 4

FIG. 6.8 - INFLUENCE OF NUMBER OF UNKNOWNS ON STABILITY OF SHAPE FACTOR CALCULATION FOR PRECISION p = 16
cross section than through one quadrant. The number of unknowns \( n \) is there larger and the square matrices for the unknowns are symmetrical. A typical set of matrices is shown in Table 6.3. The elements on a line parallel to the principal diagonal are the same, but the signs change from line to line. The series of matrices starts with a zero matrix and an identity matrix. The next matrices are the tri-diagonal matrix \( \{B\} \), the five-diagonal matrix \( \{B\}_{[2]} \), etc. Every subsequent matrix is a higher polynomial in \( \{B\} \). The distribution of the coefficients is fully described by the distribution of the coefficients on the secondary diagonal, as shown in Table 6.4.

There is no loss in generality when the coefficients of each row are divided by the pivotal elements. The resulting reduced matrices have coefficients equal to 1.0 on the principal diagonal. The distribution of the absolute values of the reduced coefficients on the secondary diagonal is shown in Figure 6.9. The index \( s \) is here the distance of an element to its principal diagonal.

Matrices of a series with \( n \) unknowns become more ill conditioned when \( m \) is large. All reduced coefficients increase with \( m \). Inversely, matrices for \( n \) unknowns are more ill conditioned when the reduced coefficients are larger.

The calculation of shape factors becomes unstable at \( m = 20 \) for all values of \( n \). The distribution of the reduced coefficients near the principal diagonal is from Figure 6.9 the same for any value of \( n \). Inversely, one might say that the matrices become
TABLE 6.3  
SERIES OF MATRICES FOR FLOW THROUGH  
FULL RECTANGULAR CROSS SECTION

\[
I = 1
\]
\[
\begin{bmatrix}
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
\end{bmatrix}
\]

\[
I = 2
\]
\[
\begin{bmatrix}
1 & * & * & * & * & * & * & * \\
* & 1 & * & * & * & * & * & * \\
* & * & 1 & * & * & * & * & * \\
* & * & * & 1 & * & * & * & * \\
* & * & * & * & 1 & * & * & * \\
* & * & * & * & * & 1 & * & * \\
\end{bmatrix}
\]

\[
I = 3
\]
\[
\begin{bmatrix}
4 & -1 & * & * & * & * & * & * \\
-1 & 4 & -1 & * & * & * & * & * \\
* & -1 & 4 & -1 & * & * & * & * \\
* & * & -1 & 4 & -1 & * & * & * \\
* & * & * & -1 & 4 & -1 & * & * \\
* & * & * & * & -1 & 4 & -1 & * \\
\end{bmatrix}
\]

\[
I = 4
\]
\[
\begin{bmatrix}
80 & -49 & 12 & -1 & * & * & * & * \\
-49 & 80 & -49 & 12 & -1 & * & * & * \\
12 & -49 & 80 & -49 & 12 & -1 & * & * \\
-1 & 12 & -49 & 80 & -49 & 12 & -1 & * \\
* & -1 & 12 & -49 & 80 & -49 & 12 & -1 \\
* & * & -1 & 12 & -49 & 80 & -49 & 12 \\
* & * & * & -1 & 12 & -49 & 80 & -49 \\
I = 5
\]

\[
\begin{bmatrix}
401 & -280 & 97 & -16 & 1 & * & * & * \\
-280 & 401 & -280 & 97 & -16 & 1 & * & * \\
97 & -280 & 401 & -280 & 97 & -16 & 1 & * \\
-16 & 97 & -280 & 401 & -280 & 97 & -16 & 1 \\
1 & -16 & 97 & -280 & 401 & -280 & 97 & -16 \\
* & 1 & -16 & 97 & -280 & 401 & -280 & 97 \\
* & * & 1 & -16 & 97 & -280 & 401 & -280 \\
I = 6
\]

\[
\begin{bmatrix}
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
\end{bmatrix}
\]

6.29
**Fig. 6.9 - Distribution of Reduced Elements on Secondary Diagonal**

**Table 6.4 Coefficients on Secondary Diagonal**

<table>
<thead>
<tr>
<th>Matrix No</th>
<th>Distance from Principal Diagonal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
</tr>
<tr>
<td>5</td>
<td>401</td>
</tr>
<tr>
<td>6</td>
<td>2084</td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>
ill conditioned when the reduced coefficients on the first line parallel to the principal diagonal exceed 0.92.

The instability is here discussed in terms of the instability of shape factors. The errors in $x_k$ are magnified in the flow calculation. This is not so for the calculation of end effect factors where instability for $p = 16$ occurs at $i = 30$ for any number of unknowns. The limitations of the method then depend on the end use. For instance, calculation of the function in a region can be very inaccurate while the calculation of the flux across a boundary is at the same time very precise. Calculation of the velocity at $i = 12$, $j = 4$ in the quadrant of the rectangular cross section requires by way of example substitution of the unknowns $x_k$ in the equation

$$\omega(12, 4) = -0.49 \times 10^{-7} x_1 + 0.11 \times 10^{-8} x_2 - 0.17 \times 10^{-8} x_3 + 0.99 \times 10^{-9} x_4 + 0.20 \times 10^{-6}$$

where

- $x_1 = 1.246$
- $x_2 = 1.975$
- $x_3 = 2.366$
- $x_4 = 2.491$

The individual products are in this equation very much larger than the velocity that must be calculated. Accurate results can only be obtained with the proper precision.

No rules can be given for the solution of a general problem. The values of the reduced determinants $D_n$ are sometimes used as an index for the degree of ill conditioning.

The shape factor calculation becomes unstable when

$$-\frac{1}{n} \ln |D_n| > 0.6p$$

(6.48)
where $D_n$ is the reduced determinant of the boundary matrix. That index is almost linear to $m$, as shown in Fig. 6.10. Then

$$-\frac{1}{m} \ln |D_n| \approx 0.6 (m^{-3})$$

The required precision for $m$ is for this problem approximately $p \approx m^{-3}$.

A number of unknowns are introduced at $i > 2$ when similar problems are solved in closed regions with convex boundaries. The reduced coefficients are for the same $n$ and $m$ smaller than those for the circumscribed rectangle, which would tend to make the problem more stable.

Round-off errors in $\{E\}_1$ are rather small. The relative error is for two elements of $\{E\}$ for the shape factor problem shown in Fig. 6.11. Here we show again the relative difference between the elements calculated with 8 and 16 precision. These errors were in this case calculated for boundary matrices that have no round-off errors until one element exceeds $10^P$.

The matrix $\{E\}_1$ of the end effect calculation has round-off errors. All matrices $\{E\}_1$ for that problem then have round-off errors. The influence of round-off errors in $\{E\}_1$ was investigated for the shape factor calculation.

That problem can also be solved using $0.1 \{I\}$ for $\{E\}_2$ instead of $\{I\}$. The matrix $\{E\}_2$ and all subsequent matrices then have round-off errors when the calculation is performed on a binary computer. This is so because $0.1$ is a rounded binary number.
Fig. 6.10 - Change of $-\frac{1}{m} \ln |D_n|$ with $m$.

Fig. 6.11 - Relative error in coefficients $e_{11}$ and $e_{24}$ of $\{E\}_i$ as function of matrix No. i.
The matrices \{E\}_i have from Figure 6.6 no round-off errors for \(i < 23\), when based on \(p = 16\) and \(\{I\}\). Table 6.5 shows the difference in \(F_p\) for the two cases, indicating a minor influence of the round-off errors of \{E\} on the final result.

One can then conclude that instability of the calculation is governed by the degree of ill conditioning of the matrices. This degree will be different for other partial differential equations. Matrices are for the same partial differential equation and the same boundary conditions, for \(n\) unknowns and \(m\) steps less ill conditioned the smaller the number of interior points in the region. Integrated results for fluxes (4.31, 4.35), which are based on derivatives, can be calculated more accurately than functions or integrals of those functions. It should be noted that the coefficients for the constant \(x_{n+1}\) influence the stability of the solution. For instance, the end effect factor (Section 6.8) for pressure back flow becomes unstable slightly sooner than that for drag flow. The square parts of the matrices are here the same. Different boundary conditions yield different coefficients for \(x_{n+1}\) for that case.

### 6.8 Calculation of End Effect Factors

The end effect factors of Chapter 4 have to be calculated for a number of families of regions. Each family consists of regions with the same helix angle \(\psi\), and different channel ratios \(L_0/b_0\). Helix angles are restricted to those with mesh points at the intersections of the oblique boundary with horizontal mesh lines. Half of the region of Figure 4.2 is in each
<table>
<thead>
<tr>
<th>M#</th>
<th>H/B</th>
<th>F_p 0.1</th>
<th>F_p for {I}</th>
<th>H/B</th>
<th>F_p 0.1</th>
<th>F_p for {I}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4.0000</td>
<td>0.05211360</td>
<td>0.05211360</td>
<td>5.0000</td>
<td>0.03468210</td>
<td>0.03468210</td>
</tr>
<tr>
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<td>0.10660515</td>
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<td>0.07257348</td>
<td>0.07257348</td>
</tr>
<tr>
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<td>0.17047258</td>
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<td>0.11915271</td>
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<tr>
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<td>0.23752143</td>
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<td>0.17084757</td>
</tr>
<tr>
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<td>0.36454339</td>
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</tr>
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<td>7</td>
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<td>0.30315946</td>
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<td>8</td>
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<td>0.55227729</td>
<td>0.9091</td>
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<td>0.58600206</td>
<td>0.8333</td>
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<tr>
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<td>0.61559419</td>
<td>0.7692</td>
<td>0.53029763</td>
<td>0.53029763</td>
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<tr>
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<td>0.5714</td>
<td>0.64162979</td>
<td>0.64162979</td>
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</tr>
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<td>15</td>
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<tr>
<td>16</td>
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<td>0.68500707</td>
<td>0.68500707</td>
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<td>0.61041903</td>
<td>0.61041903</td>
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<tr>
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<td>0.70316112</td>
<td>0.5882</td>
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<tr>
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<td>0.5556</td>
<td>0.65158767</td>
<td>0.65158767</td>
</tr>
<tr>
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<td>0.73430325</td>
<td>0.5263</td>
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<td>0.74676106</td>
<td>0.5000</td>
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<td>0.68427083</td>
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* M = NUMBER OF MESHES IN MARCHING DIRECTION
Fig. 6.12 - Typical Region of Screw Pump Channel

Fig. 6.13 - Skew Symmetry Along Boundary 'AE'
case considered. A typical region with boundary conditions for pressure back flow is shown in Figure 6.12. The marching direction is from E to C. The discussion is restricted to the pressure back flow case.

The matrices satisfy in zone ED recursion equation of type (6.9). Those in the oblique zone DC must be determined in a different way.

Unknowns $x_k (k = 1, n)$ are introduced at $i = 2$ (Figure 6.13). From symmetry at $i = 1$ then

$$x_{k,0} = -x_{(n-k+1,2)} \quad (6.50)$$

Let $z_j (j = 1, n)$ be the set of function values at $i = 1$. This set can be expressed in $x_{k,0}$ and $x_{k,2}$ from finite difference equations. For instance, for square meshes and $n = 7$

$$\begin{align*}
4z_1 - 2z_2 &= x_1 - x_7 \\
-z_1 + 4z_2 - 2z_3 &= x_2 - x_6 \\
-z_2 + 4z_3 &= x_3 - x_5
\end{align*} \quad (6.51)$$

The coefficients on the left side of these equations form a tri-diagonal matrix. Any $z_j$ can with the routine of Thomas (12) be expressed in $x_k$. The matrix $[E]_1$ is then known while $[E]_2$ is the identify matrix.

The matrices in zone ED, which are independent of the helix angle and independent of the channel ratio $L_{CO}/b_0$, are calculated in a separate computer program. These matrices are square, do not contain constants, and are antisymmetrical in this sense that the elements satisfy the relation

$$\epsilon (j, k) = \epsilon (n-j+1, k) \quad (6.52)$$
The matrices for zone DC are calculated in a second computer program. Two previously calculated matrices are there used as initial matrices. Let \( \{E\}_{s-1} \) and \( \{E\}_s \) be these two matrices. The index \( i = s \) must correspond to the index at point D of Figure 6.12.

The function is at the boundary BC equal to the constant \( \Psi_0 \). It is at D equal to the linear equation in \( x_k \) given by the coefficients of the last row of \( \{E\}_s \). The boundary condition is then

\[
e(n,1)x_1 + e(n,2)x_2 + \ldots + e(n,n)x_n = \Psi_0 \tag{6.53}
\]

where \( e(n,k) \) are elements of \( \{E\}_s \). This equation is the first of \( n \) boundary equations. One unknown could at this point be eliminated. That elimination is, however, postponed until all boundary equations have been determined.

The calculation of \( \{E\}_{s+1} \) is similar to that of previous matrices. The range for \( j \) is now reduced by 1. The matrix \( \{E\}_{s+2} \) has that same range.

Calculation of the \( (n-1) \)th row of \( \{E\}_{s+2} \) must be based on an equation for the irregular star of Figure 6.14. That star is shown with an arm \( \zeta = 0.5 \). Other helix angles may require arms of a different length. For this star

\[
\begin{align*}
\Psi_c &= \Psi_c + \xi k (2 \frac{\partial \Psi}{\partial y})_c + \frac{1}{2} \xi^2 k^2 (\frac{\partial^2 \Psi}{\partial y^2})_c \\
\Psi_B &= \Psi_c - k (2 \frac{\partial \Psi}{\partial y})_c + \frac{1}{2} k^2 (\frac{\partial^2 \Psi}{\partial y^2})_c
\end{align*} \tag{6.54a,b}
\]
Elimination of \( \left( \frac{\partial^2 \psi}{\partial y^2} \right)_c \) yields
\[
\left( \frac{\partial^2 \psi}{\partial y^2} \right)_c = \frac{2}{h^2 \xi (1 + \xi)} \left[ \psi_0 + \xi \psi_B - (1 + \xi) \psi_c \right] \quad (6.54c)
\]
while also
\[
\frac{\partial^2 \psi}{\partial x^2}_c = \frac{1}{h^2} \left( \psi_0 - 2 \psi_c + \psi_L \right) \quad (6.54d)
\]

The finite difference equation for the Laplacian becomes
\[
2 \left( 1 + \frac{1}{\xi} \right) \psi_c - \frac{2}{\xi + 1} \psi_B - \psi_L - \left[ \frac{2}{\xi (\xi + 1)} + 1 \right] \psi_0 = 0 \quad (6.55)
\]

The matrix \( \{B\} \) is now reduced to a \((n-1)\times(n-1)\) matrix. Elements of the \((n-1)\)th row are
\[
B(n-1, n-2) = -\frac{2}{\xi + 1} \quad ; \quad B(n-1, n-1) = \frac{2(1 + \xi)}{\xi} \quad (6.56)
\]

The \((n-1)\)th row of \( \{E\}_{b+2} \) yields the second boundary equation.

The irregular star for the last point C (Fig. 6.15) is again different since the derivative vanishes along LR. Here
\[
\psi_o = \psi_c + \frac{1}{2} \xi h^2 \left( \frac{\partial^2 \psi}{\partial y^2} \right)_c \quad (6.57a)
\]
while also
\[
h^2 \left( \frac{\partial^2 \psi}{\partial x^2} \right)_c = \psi_o - 2 \psi_c + \psi_L \quad (6.57b)
\]
FIG. 6.15 - IRREGULAR STAR NEAR END OF REGION FOR UNIFORM PRESSURE ALONG DISCHARGE END

\[ 2 \left(1 + \frac{1}{\xi} \right) \psi_c - \frac{2}{\xi+1} \psi_B - \psi_L - \psi_R - \frac{2}{\xi(\xi+1)} \psi_0 = 0 \]

FIG. 6.16 - ADDITIONAL IRREGULAR STARS FOR UNIFORM PRESSURE ALONG DISCHARGE END
The finite difference equation for the Laplacian becomes

\[ 2 \left( 1 + \frac{1}{\xi^2} \right) \psi_c - \psi_L - \psi(1 + \frac{2}{\xi^2}) = 0 \quad (6.58) \]

Similar equations for smaller helix angles are shown in Figure 6.16. The n-boundary equations are finally collected in one matrix equation

\[ \{G\} \{x\} = 0 \quad (6.59) \]

The solution of this set of equations yields \( x_k \).

The flow rate is by Simpson's rule expressed as

\[
q \approx \frac{\Delta x}{3} \left[ x_1 + n + 4 \left( x_2 + x_{n-1} \right) + 2 \left( x_3 + x_{n-2} \right) + \ldots \right.
\]
\[
\left. + 2 \left( x_{n-1} + x_1 \right) \right] + 4 \left( x_{n-1} + x_2 \right) + x_n + x_1 \]
\[
= \frac{2 \Delta x}{3} \left[ x_1 + 4x_2 + 2x_3 + \ldots + 2x_{n-1} + 4x_{n-1} + x_n \right] \quad (6.60)
\]

Equation (6.60) is similar to (6.30). The flow rate coefficients of the row vector \( \{F\} \) are for this case very small. Errors in \( x_k \) are then not propagated in the calculation of end-effect factors. This is not true for the calculation of the function in the region. The coefficients in \( \{E\}_i \) show an increase similar to that of Figure 6.6. Small errors can then cause large errors in the calculation of the function, particularly when \( i \) is large. This is true in Figure 4.15 where back substitution of \( x_k \) yields function values in the tip of the region that do not match the original constant pressures at the boundary. The function values are here correct at points where the coefficients of \( \{E\}_i \) are small.
The difference between the exact and calculated end effect factors is again a function of the mesh size \( h \). Equations for derivatives have errors of the order of \( h^2 \). The final error is a combination of errors of the order of \( h \) and \( h^2 \), since the region has a singular point at B (Figure 6.17).

![Figure 6.17 - Error proportional to \( h \) in zone Q'B.](image)

The derivative must vanish along the sides of the channel. The function must be constant along the boundary. These requirements cannot then be satisfied at B. The derivative is satisfied at P and Q but not at B. This means that the derivative between Q and B does not vanish. The finite difference method yields an additional flow rate across the channel wall between Q and B. That flow rate is of the order of \( h \).

The calculated end effect factors were for that reason analyzed by the least squares method using a second order polynomial in \( h \) as a model. The extrapolated results were used to determine the coefficients \( C \_\phi \) (4.67).
6.9 **Additional Examples**

A number of additional problems were solved by the same technique. For instance, a program was developed for the solution of Poisson's equation in the arbitrary region of Figure 6.3. This program was developed to calculate pressure gradients for viscous flow through tubes of any cross section.

Shape factors $F_p$ (6.31) were with this program determined for a number of cross sections with known analytical solutions for $p = 8$. Table 6.6 lists some of the results.

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1) Number of unknowns.

2) Number of steps in marching direction.

Improved results could have been obtained by extrapolation or calculation at higher precision. The results are without these refinements well within the accuracy normally required for engineering calculations.

Another program was developed to simultaneously solve Laplace's and Poisson's equations in the same region for different boundary conditions. This program was developed to calculate drag flow and pressure back flow coefficients for channels with one flat side and a symmetrical contour for the remainder of the boundary.
Laplace's equation governs the drag flow. Here the flat side has a uniform velocity in the channel direction, while the remainder of the boundary is at a standstill. Poisson's equation governs pressure back flow. The whole boundary is at a standstill for that case. The two differential equations differ only in the value of the constant. The constant in Laplace's equation is zero. Boundary conditions differ along the flat side of the contour.

The matrices were for this case augmented to accommodate coefficients for \( x_{n+1} \) and \( x_{n+2} \). The coefficients for \( x_k \) (\( k = 1, n \)) are in \( \{E\}_1 \) precisely the same for both differential equations. The coefficients for \( x_{n+1} \) account for the Poisson constant; those for \( x_{n+2} \) accommodate the coefficients for the boundary velocity for drag flow. The unknowns \( x_k(L) \) for Laplace and \( x_k(P) \) for Poisson can with Crout's method (22) be determined simultaneously.

Some problems require solutions in the same region for different boundary conditions. These different solutions can almost simultaneously be obtained when the boundary matrix is generated at that boundary. This method was used to determine the temperature distribution in a hollow cylinder (Figure 6.18.)
The inside of the cylinder is partly insulated. The remainder of the interior is kept at a uniform temperature of \(100^\circ C\). The external boundary is held at a uniform temperature of \(0^\circ\). The influence of the length of the internal insulation on the temperature distribution is determined.

A material is assumed with different thermal conductivities in the radial and in the axial direction. The temperature distribution must satisfy

\[
\frac{\partial^2 T}{\partial y^2} + \frac{1}{r} \frac{\partial T}{\partial r} + C_k \frac{\partial^2 T}{\partial z^2} = 0
\]

where \(C_k\) is the ratio of thermal conductivities.
Unknowns $x_k$ ($k = 1, n$) are introduced at $i = 2$ close to the outside radius. Matrices are calculated in the range $i = 3$ to $m + 1$ where $m$ corresponds to the internal radius. Boundary conditions are

$$0 < z < s : \frac{\partial T}{\partial z} = 0$$
$$s \leq z \leq \frac{L}{2} : T = T_o$$

These conditions yield $n$-equations in the unknowns $x_k$ and the temperature $T_o$ for any value of $s$. Here $z = s$ must coincide with a mesh point. The same matrices can be used to calculate different sets of unknowns $x_k$ for any acceptable value for $s$. Figure 6.19 shows a typical temperature distribution for $C_k = 2$.

The temperature gradients $\frac{\partial T}{\partial z}$ are small in part of the region. Calculations can in that case be significantly reduced using a larger mesh size in that part of the region where gradients are expected to be small. This program was designed to accommodate different mesh sizes in the $z$ direction. Figure 6.20 shows the calculation results for two different sets of mesh sizes. The problem was first solved for $n = 12$ with a square mesh, then for a combination of four single and four double meshes. The underscored temperatures refer to the last calculation. The minute temperature differences show the utility of that approach, which reduces the calculation time by a factor of approximately 3.
FIG. 6.20—INFLUENCE OF INCREASED MESH SIZE ON TEMPERATURE DISTRIBUTION

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SUMMARY

Screw pumps are used to pump very viscous liquids. The walls of the helical screw pump channel are formed by two cylindrical surfaces, the barrel and screw root surfaces, and by two helical flight surfaces. These channels have oblique inlet and discharge ends. This channel geometry is considerably simplified in the conventional theories for flow of viscous liquids through screw pumps.

A straight prismatic channel with a rectangular cross section is assumed in existing theories. The true geometry of the channel ends is ignored. Identical velocity distributions are assumed in parallel planes perpendicular to the channel direction, which result in uniform pressure gradients in the channel direction and in the direction perpendicular to the flights.

Flow through rectangular channels with large aspect ratios is, in the Simplified Theory, based on flow between parallel plates. The influence of the short sides of the rectangular cross section on the velocity distribution can be neglected in that theory. The resulting performance equation is useful as a reference equation. The drag flow and pressure back flow rates of other theories can be compared to that equation by the introduction of correction factors.

Shape factors are introduced in the rectangular channel theory to account for the reduced velocities along the short sides of the cross section. The true cross section is, however, not rectangular. It departs more from the rectangular cross section the deeper the channel.

The actual channel cross section is used in the analysis of Chapter 2. Here performance equations for isothermal flow of a Newtonian liquid are compared with those of the simplified theory for screw pumps with large \( L/D \) ratios, where the channel end geometry can be neglected. Congruent velocity distributions are again assumed. An analytical solution is given for the velocity distributions in helical channels with large aspect ratios. Correction factors \( F_{DC} \) for drag flow and \( F_{PC} \) for pressure back flow are derived as functions of the helix angle and the relative depth.
The differential equations cannot be solved for flow through deep channels. The correction factors then do not account for the reduced velocities near the flights. These reductions must, for the helical channel, be quite similar to those of the rectangular channel theory. Shape factors for the helical channel are approximately equal to those for rectangular channels with a width equal to the average width of the helical channel. The effect of curvature and flights is shown as combined factors $F_{D2}$ ($F_{P2}$), which are the products of these shape factors and $F_{DC}$ ($F_{PC}$).

The combined factor for drag flow, $F_{D2}$, is always smaller than 1.0. Drag flow is smaller than predicted by the simplified theory, but it is also smaller than that of the rectangular channel theory. This factor decreases for all helix angles with increased depth. That decrease is smallest at $\varphi_0 = 20^\circ$. Smaller or larger helix angles show a more rapid decrease with depth. The similar factor $F_{P2}$ is about 1.0 for $\varphi_0 = 20^\circ$ and is almost independent of the channel depth for that angle. It decreases for smaller or larger helix angles with increased depth. The decrease is more rapid the more the helix angle differs from $20^\circ$. The pressure back flow rate is larger than predicted by the rectangular channel theory. The energy dissipated in a helical channel is also less than the one derived in the simplified theory.

The conventional leakage flow analysis is for isothermal flow extended in Chapter 3. The leakage flow consists of one part proportional to the screw speed, another part proportional to the discharge pressure. The magnitude of the leakage flow rate is influenced by the screw geometry, the mode of operation of the screw pump, and the ratio of the viscosities in channel and flight clearances. Non-Newtonian viscosities can have a large viscosity ratio, which increases the leakage flow rate significantly.

The leakage flow rate causes formation of a thin wall layer on the inside of the barrel. The thickness of that layer is influenced by the same parameters that influence leakage flow. That layer plays an important role in heat transfer. Heat transfer coefficients must then be based on the thickness of this layer and not on the flight clearance, as is done in the conventional theory.
The inlet and discharge pressures of screw pumps are substantially uniform. Only for one operating condition can the uniform pressure gradients of the conventional theories satisfy a uniform pressure at the oblique inlet and discharge ends of the channel. Isothermal flow of a Newtonian liquid through channels with oblique ends and large aspect ratios is discussed in Chapter 4 for uniform inlet and discharge pressures.

The average channel velocity components are there expressed as the derivatives of a velocity potential. That potential can be regarded as the sum of a potential for pressure back flow and one for drag flow. The drag flow potential can again be separated into one caused by the barrel velocity component in the channel direction, and one caused by its transverse component. The boundary conditions for each case are simple. The potential for each case must satisfy Laplace's equation.

These equations are solved by numerical techniques. The end effect factors for drag flow, $F_{DE}$, and that for pressure back flow, $F_{PE}$, are determined for a series of helix angles and a series of $L/D$ ratios. A simple relation exists between $F_{DE}$ and $F_{PE}$, which shows that the factor for drag flow cannot be important for the usual screw designs.

The ends cause a pressure distribution that differs from that of the simplified theory only in a small region near the oblique ends. The transverse pressure gradients vanish for the pressure back flow case in parts of the channel removed from the oblique ends. Regions without significant transverse gradients exist when the channel length exceeds a critical length. Equations are given for $F_{PE}$ for channels with a greater than critical length. The end correction factor for pressure back flow is significant for the usual helix angles and $L/D$ ratios.

Experiments, carried out to measure the end effect in channels with oblique ends, are described in Chapter 5. It was not possible to measure end effects in rotating equipment. A stationary model was built to simulate a screw pump channel with a helix angle of $18.5^\circ$ and a channel ratio $L_{CO}/b_0 = 5$. The length of that channel is about equal to the critical length.

The difference between the theoretical and experimental end effect factor was of the order of 2%. Streamlines were made visible by injection of colored corn syrup streams in a clear main stream. Observed streamlines differed slightly from calculated streamlines. Seventeen stand pipes in the oblique end were used to measure the pressure distribution, which agreed well with the calculated pressure distribution.
A new numerical technique was developed for the solution of second order partial differential equations in families of regions for the same boundary conditions. Some of the elements of that method were described by Courant before electronic computers were used, but it appears that this method was never fully developed. The technique, which does not require iteration, is described in Chapter 6. It does not require evaluation of the function in the region when a flow rate or other integral function is the only desired result.

A limited number of unknown function values are introduced in this method at mesh points near one boundary. All other function values can be expressed as linear equations in these unknowns. The function values are equated to known boundary values at the opposite boundary. The number of boundary equations is equal to the number of unknowns. Similar equations can be formulated when the derivatives at the opposite boundary are known. The unknowns are then solved from a set of linear boundary equations.

Flow rates and other integral functions can, by Simpson's rule, also be expressed as linear equations in the unknowns. These integrals can be evaluated once the unknowns have been solved. Evaluation of the function in the region is, in that case, not required.

The set of boundary equations can be expressed as a matrix equation. Matrices tend to become ill-conditioned when the number of meshes between opposite boundaries becomes large. Thus solution is only possible when calculations are performed with the required precision.

The influence of precision on the stability of the calculation is demonstrated for viscous flow through a tube with a rectangular cross section. The number of meshes, for which the calculation becomes unstable, increases with the precision. That number is, for this example, independent of the number of unknowns.

Linear equations for integral functions and for function values have alternating positive and negative coefficients that increase rapidly when the number of meshes between boundaries increases. Errors in the unknowns are then propagated in the evaluation of these functions. This does not apply to the calculation of end effect factors. The coefficients in the flow rate equation are there small since that equation is based on the derivatives at the boundary where the unknowns are introduced. The magnification of errors in the unknowns does not occur here.
The calculation of shape factors for flow through tubes with rectangular cross sections and that of end effect factors is shown to illustrate the method. The method is much shorter than existing iteration techniques when integral functions must be evaluated in families of regions.
VI

SAMENVATTING

Schroefpompen worden gebruikt voor het verpompen van hoog viscose vloeistoffen. Het schroefpomp kanaal is de ruimte tussen de pompom, de schroefkernom en de twee radiale schroefvormige oppervlakken van de schroefas. Dit kanaal eindigt aan de inlaat en uitlaat zijde met een scherpe punt.

De exacte kanaalvorm wordt in bestaande theorieën niet gebruikt. Het kanaal wordt daar altijd vervangen door een recht prismatisch kanaal met een rechthoekige doorsnede. Men neemt dan aan, dat de snelheidsvordelingen in alle doorsneden loodrecht op de kanaalwanden congruent zijn. Deze veronderstelling geeft uniforme drukgradienten, zowel in de kanaalrichting als loodrecht op de schroefas. De schuine einden hebben in die beschouwing geen invloed omdat de druk aan de kanaaleinden nu vastgelegd is door de drukgradienten.

Kanaaldoorsneden met grote aspectverhoudingen worden in de Vereenvoudigde Theorie behandeld. Snelheidsvordelingen zijn daar, onder verwaarlozing van de invloed van de korte zijden van de rechthoekige doorsnede, gebaseerd op vergelijkingen voor viscose stroming tussen evenwijdige platen. De hieruit volgende vergelijking voor de pomp karakteristiek kan als een standaard vergelijking gebruikt worden. De schuine stroming en de drukstroming als gevolg van de drukgradient, die uit andere theorieën volgen, kunnen door invoering van correctie factoren uitgedrukt worden in de overeenkomstige grootheden van de Vereenvoudigde Theorie.

Vormfactoren worden in de rechthoekige prismatische kanalen met eindige aspectverhoudingen gebruikt om de invloed van de korte zijden op de opbrengst weer te geven. De doorsnede van het schroefvormige kanaal is echter niet rechthoekig. Het verschil met een rechthoekige doorsnede wordt groter naarmate het kanaal dieper is.

De invloed van de juiste kanaalvorm op de schroefpomp karakteristiek wordt in dit proefschrift geanalyseerd voor isotherme stroming van Newtonse vloeistoffen. De juiste kanaaldoorsnede wordt in hoofdstuk 2 gebruikt. Karakteristieken worden daar berekend op grond van de veronderstellingen, dat de pomp een grote L/D verhouding heeft en dat de snelheidsvordelingen in doorsneden loodrecht op de schroefas congruent zijn. De invloed van de schuine kanaaleinden is hier verwaarloosd. Een analytische oplossing kan slechts gegeven
worden voor kanalen met grote aspectverhoudingen. Correctiefactoren $F_{DC}$ voor de schuifstroom en $F_{PC}$ voor de drukstroom worden berekend. Deze factoren geven echter niet de invloed van de radiale kanaaldammen op de stroming.

De vormfactoren voor een kanaal met een rechthoekige doorsnede zijn ongeveer gelijk aan de vormfactoren voor het schroefvormig kanaal. De gecombineerde invloed van de radiale wanden en de ronding van het kanaal wordt dan benaderd door het gebruik van gecombineerde correctiefactoren $F_{D2}$ ($F_{P2}$), die gelijk zijn aan de producten van $F_{DC}$ ($F_{PC}$) en vormfactoren. De schuifstroom is kleiner dan die volgens de Vereenvoudigde Theorie, maar ook kleiner dan die voor prismatische rechthoekige kanalen met eindige aspectverhoudingen. De factor $F_{D2}$ wordt voor alle schroefhoeken kleiner voor diepere kanalen. De invloed van de kanaaldiepte is het kleinst voor $\varphi_0 = 20^\circ$. De factor $F_{P2}$ is voor $\varphi_0 = 20^\circ$ ongeveer 1.0 en wordt dan niet sterk beïnvloed door de kanaaldiepte. Deze factor wordt voor andere hoeken kleiner voor grotere kanaaldiepte. De invloed van de kanaaldiepte is groter naarmate de schroefhoek meer verschilt van 20°.

De bestaande analyse van de isotherme lekstroom is in hoofdstuk 3 uitgebreid. Deze lekstroom kan verdeeld worden in een gedeelte dat evenredig is met de schroefsnellheid, en een gedeelte dat evenredig is met de druk. Beide gedeelten zijn functies van de afmetingen van de schroef, de bedrijfswaarden en de verhouding van de viscositeit in het kanaal tot die in de schroefspeiling. De lekkage wordt aanzienlijk verhoogd wanneer de niet-Newtonse viscositeit in de speiling kleiner is dan die in het schroefkanaal.

De lekstroom vormt een dunne vloeistoflaag aan de binnenkant van de pomp. De dikte van deze laag wordt beïnvloed door dezelfde factoren. Deze laag speelt een voormade rol in warmte overdracht. Het is dan onjuist de coefficienten van warmte overdracht te baseren op de speiling, zoals dit gedaan wordt in bestaande theorieën. De coefficienten moeten gebaseerd worden op de dikte van deze laag.

De invloed van de schuine inlaat- en uitlaateinden van het kanaal op de pompkarakteristiek wordt in hoofdstuk 4 behandeld. De drukken aan die einden moeten vrijwel constant zijn. De bestaande theorieën, die gebaseerd zijn op uniforme drukgradienten, geven alleen in een uitzonderlijk geval uniforme einddrukken. De nieuwe theorie behandelt isotherme stroming door rechthoekige kanaalen met grote aspect-verhoudingen, waarbij het kanaal eindigt in schuine einden.
De gemiddelde snelheidscomponenten in het kanaal kunnen worden uitgedrukt als afgeleiden van een snelheidspotentiaal. De totale potentiaal kan worden beschouwd als de superpositie van een schuifstroming potentiaal voor de cylinder snelheidscomponent in de kanaalrichting, een voor de cylinder snelheidscomponent loodrecht op de schroefdraad, en één voor de drukstroom. Elke elementaire potentiaal moet voldoen aan de vergelijking van Laplace voor eenvoudige randvoorwaarden.

De correctiefactoren $F_{DE}$ voor de schuifstroming en $F_{PE}$ voor de drukstroom worden berekend met een numerieke methode. Deze factoren zijn functies van de schroefhoek en de relatieve lengte van het kanaal. Er bestaat een eenvoudige relatie tussen $F_{DE}$, $F_{PE}$ en de schroefhoek, waaruit men af kan leiden, dat de schuifstroming niet erg beïnvloed kan worden door de schuine kanaaleinden. De snelheidsverdeling in een kanaal met schuine einden is voor gelijke opbrengst alleen verschillend van die volgens de Vereenvoudigde Theorie in gebieden dichtbij de inlaat en uitlaat van het kanaal. Een formule kan afgeleid worden voor eindeffect-factoren voor kanaals met een lengte groter dan een critische lengte. De factor voor de drukstroom is van betekenis in schroefpomp berekeningen.

Experimenten voor de verificatie van het eindeffect zijn beschreven in hoofdstuk 5. Het was niet mogelijk eindeffecten te meten aan roterende machines. Een schroefpomp kanaal met een lengteverhouding $L_{CO}/b_0 = 5$ en een schroefhoek van 18.5° werd gesimuleerd in een modelkanaal. Deze kanaallengte is iets groter dan de critische lengte. Het verschil tussen de theoretische en gemeten eindeffect factor bedroeg 2%. Stroomlijnen konden bestudeerd worden door injectie van gekleurde stroop in een transparante hoofdstroom. De gemeten drukverdeling en gemeten stroomlijnen kwamen goed overeen met de berekende waarden.

Een nieuwe methode werd ontwikkeld voor het oplossen van tweede orde lineaire partiële differentiaalvergelijkingen in families van gebieden voor dezelfde randvoorwaarden. Deze methode is beschreven in hoofdstuk 6. Sommige elementen werden reeds voor de invoering van electronische rekenmachines door Courant beschreven in een college dictaat, maar de methode werd blijkbaar nooit ontwikkeld tot een bruikbare methode. De methode vereist geen iteratie. De functie hoeft niet overal berekend te worden wanneer alleen de integraal van de functie het verlangde resultaat is.

Een beperkt aantal onbekenden, gelijk aan de functiewaarden, wordt in deze methode ingevoerd in knooppunten nabij een van de randen van het gebied. De functie kan dan in elk ander knooppunt uitgedrukt worden als een lineaire vergelijking.
in deze onbekenden. Dergelijke vergelijkingen worden dan aan de tegenovergestelde rand van het gebied gelijk gesteld aan de functiewaarden aan die rand. De oplossing van een stel lineaire vergelijkingen geeft dan de waarden van de onbekenden. De integraal van de functie kan met de regel van Simpson ook uitgedrukt worden als een lineaire vergelijking in de onbekenden. De numerieke waarde van de integraal kan dan bepaald worden zonder dat de verdeling van de functie bekend is.

De lineaire randvoorwaarde-vergelijkingen vormen een matrixvergelijking. Deze matrices worden slechter geconditioneerd naarmate het aantal mazen tussen de twee randen groter wordt. Problemen kunnen dan alleen opgelost worden wanneer berekeningen uitgevoerd worden met de juiste precisie.

De invloed van de precisie op de stabiliteit wordt gedemonstreerd voor een visceuze stroming door een buis met een rechthoekige doorsnede. Het toelaatbare aantal mazen tussen de randen van het gebied wordt groter naarmate de gebruikte precisie groter is. Dit aantal is onafhankelijk van het aantal ingevoerde onbekenden.

De berekeningen van vormfactoren voor rechthoekige doorsneden en eindeffect-factoren voor schroefpomp kanalen worden uitvoerig besproken. De methode is veel korter dan de bestaande iteratieve methodes, vooral wanneer een integraal moet worden berekend in een familie van gebieden.
### Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Constant in eq. (2.15)</td>
</tr>
<tr>
<td>$A_0$</td>
<td>Constant in eq. (1.11)</td>
</tr>
<tr>
<td>$B$</td>
<td>Coefficient matrix (2.16)</td>
</tr>
<tr>
<td>$b_r$</td>
<td>Channel width at radius $r$</td>
</tr>
<tr>
<td>$b_0$</td>
<td>Channel width at barrel diameter</td>
</tr>
<tr>
<td>$b_1$</td>
<td>Channel width at average diameter</td>
</tr>
<tr>
<td>$C_1, C_2, \ldots$</td>
<td>Constants of integration</td>
</tr>
<tr>
<td>$c$</td>
<td>Radial clearance</td>
</tr>
<tr>
<td>$c^*$</td>
<td>Leakage wall layer thickness</td>
</tr>
<tr>
<td>$c_1^*$</td>
<td>Screw root layer thickness</td>
</tr>
<tr>
<td>$c_p$</td>
<td>Specific heat</td>
</tr>
<tr>
<td>$C_w$</td>
<td>Coefficient defined in (4.66)</td>
</tr>
<tr>
<td>$D$</td>
<td>Barrel diameter</td>
</tr>
<tr>
<td>$D_n$</td>
<td>Determinant of reduced matrix</td>
</tr>
<tr>
<td>$e$</td>
<td>Axial flight land width</td>
</tr>
<tr>
<td>$E_1, E_{Cl}$</td>
<td>Energy dissipated in channel per unit time</td>
</tr>
<tr>
<td>$E_F$</td>
<td>Energy dissipated in flight clearance per unit time</td>
</tr>
<tr>
<td>$E_P, E_{PC}$</td>
<td>Energy at pump discharge per unit time</td>
</tr>
<tr>
<td>$E_T, E_{TC}$</td>
<td>Total energy per unit time</td>
</tr>
<tr>
<td>$E$</td>
<td>Matrix defined in (6.13)</td>
</tr>
<tr>
<td>$F(\alpha)$</td>
<td>Function of $\alpha$ (2.32)</td>
</tr>
<tr>
<td>$F_D, F_P$</td>
<td>Shape factors for rectangular cross section</td>
</tr>
<tr>
<td>$F_{DC}, F_{PC}$</td>
<td>Curvature factors</td>
</tr>
<tr>
<td>$F_{DL}, F_{P1}$</td>
<td>Shape factors for helical channel</td>
</tr>
<tr>
<td>$F_{D2}, F_{P2}$</td>
<td>Combined correction factors</td>
</tr>
<tr>
<td>$F_{P3}$</td>
<td>Back flow correction factor based on average diameter</td>
</tr>
<tr>
<td>$F_{DE}, F_{PE}$</td>
<td>End effect correction factors</td>
</tr>
<tr>
<td>$F(r)$</td>
<td>Factor defined in (2.12e)</td>
</tr>
<tr>
<td>$f, f_T$</td>
<td>Flow factors, dimensionless</td>
</tr>
<tr>
<td>$G(r)$</td>
<td>Function defined in (2.12d)</td>
</tr>
<tr>
<td>$G(\alpha)$</td>
<td>Function of $\alpha$ (2.40a)</td>
</tr>
<tr>
<td>$h$</td>
<td>Channel depth</td>
</tr>
</tbody>
</table>
\textbf{H}_V, \textbf{H}_VT \quad \text{Energy coefficients (2.69, 2.70)}

h \quad \text{Mesh size in finite difference calculation}

\textbf{I} \quad \text{Identity matrix}

K, K_0 \quad \text{Energy coefficients (1.58, 1.65)}

K (\alpha) \quad \text{Function of } \alpha \text{ (2.40b)}

k_C, k_\beta, k_\rho_0 \quad \text{Leakage fractions (3.32, 3.33)}

k_\rho_0 \quad \text{Leakage coefficient}

L \quad \text{Axial length of screw pump}

L^{CO} \quad \text{Developed channel length at barrel diameter}

L^* \quad \text{Critical channel length (4.60)}

L^{**} \quad \text{Length defined in Fig. 4.10}

L_1, L_2 \quad \text{Defined in Fig. 4.10}

n \quad \text{Number of parallel flights}

N \quad \text{Screw speed}

n, m \quad \text{Number of meshes}

p, P \quad \text{Pressure}

P_0 \quad \text{Pressure difference generated by screw pump}

P_S, P_{ST} \quad \text{Shut-off pressure}

\Delta p_F \quad \text{Pressure difference across flights}

\Delta p_T \quad \text{Pressure generated in one turn}

Q, Q_T \quad \text{Total flow rate}

Q_{DT}, Q_{DO} \quad \text{Drag flow rate of Simplified Theory}

Q_{PT}, Q_{PO} \quad \text{Pressure back flow rate of Simplified Theory}

Q_D, Q_{DC} \quad \text{Drag flow rate}

Q_P, Q_{PC} \quad \text{Pressure back flow rate}

Q_{DE} \quad \text{Drag flow rate in channel with oblique ends}

Q_{PE} \quad \text{Pressure back flow rate in channel with oblique ends}

Q_{D1} \quad \text{Drag flow rate due to } V_0

q \quad \text{Flow rate per unit length}

r \quad \text{Cylindrical coordinate}

R_0 \quad \text{Radius at barrel}

R_1 \quad \text{Radius at screw root surface}

S (\phi_0, \alpha) \quad \text{Function defined in (2.43a)}

S \quad \text{Dimensionless stream function}
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>Function defined in (2.43b)</td>
</tr>
<tr>
<td>$t$</td>
<td>Lead of screw</td>
</tr>
<tr>
<td>$T$</td>
<td>Temperature</td>
</tr>
<tr>
<td>$U_0$</td>
<td>Barrel velocity component, x-direction</td>
</tr>
<tr>
<td>$u, v, w$</td>
<td>Velocity components</td>
</tr>
<tr>
<td>$\bar{u}, \bar{v}$</td>
<td>Average channel velocities</td>
</tr>
<tr>
<td>$V_0$</td>
<td>Barrel velocity component y- or $\theta$ direction</td>
</tr>
<tr>
<td>$V_C$</td>
<td>Volume in clearance</td>
</tr>
<tr>
<td>$X, Y$</td>
<td>Dimensionless coordinate</td>
</tr>
<tr>
<td>$x, y$</td>
<td>Cartesian coordinates</td>
</tr>
<tr>
<td>$z$</td>
<td>Cartesian or cylindrical coordinate</td>
</tr>
<tr>
<td>$Z_D, Z_P$</td>
<td>Dimensionless velocity potential</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Ratio $R_1/R_0$ (2.22)</td>
</tr>
<tr>
<td>$\alpha_e, \alpha_F$</td>
<td>Defined in (3.8)</td>
</tr>
<tr>
<td>$\alpha_{p1}$</td>
<td>$\Delta P_F/\Delta P_T$ (2.60)</td>
</tr>
<tr>
<td>$\alpha_{p2}$</td>
<td>Pressure ratio (2.61)</td>
</tr>
<tr>
<td>$\beta, \beta_T$</td>
<td>Back flow coefficient in channel direction</td>
</tr>
<tr>
<td>$\beta_y$</td>
<td>Transverse back flow coefficient</td>
</tr>
<tr>
<td>$\dot{\gamma}_F$</td>
<td>Shear rate in clearance</td>
</tr>
<tr>
<td>$E_F$</td>
<td>Flight energy fraction (1.64)</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>$e/t$ (3.15)</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Efficiency</td>
</tr>
<tr>
<td>$\eta_m$</td>
<td>Efficiency in prime mover</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>Cylindrical coordinate</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Viscosity</td>
</tr>
<tr>
<td>$\mu_c$</td>
<td>Viscosity of liquid in channel</td>
</tr>
<tr>
<td>$\mu_F$</td>
<td>Viscosity of liquid in flight clearances</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Defined in (3.16)</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Dissipation function</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>Velocity potential</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>Helix angle at barrel diameter</td>
</tr>
<tr>
<td>$\phi_r$</td>
<td>Helix angle at radius $r$</td>
</tr>
<tr>
<td>$r_f$</td>
<td>Reduced arm coefficient</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Stream function</td>
</tr>
</tbody>
</table>
References


STELLINGEN

I.

Een schroefpomp kan beschouwd worden als een speciaal soort viscometer. De effectieve schroefpomp viscositeit kan dan berekend worden uit proefresultaten met behulp van de schroefpomp vergelijking. De viscositeit kan ook in een capillair gemeten worden voor een schuifsnelheid gelijk aan de specifieke schroefpomp schuifsnelheid \( \pi DN/h \) (\( D = \) diameter, \( N = \) snelheid, \( h = \) schroefgang diepte). Aange­toond kan worden, dat de effectieve viscositeit voor niet Newtonse polymeren kleiner is dan de capillaire viscositeit. Dit verschil wordt groter naarmate het polymeer meer afwijkt van een Newtonse vloeistof.

II.

De snelheids verdeling van de schuifstroming in schroefpomp kanalen met grote aspect verhoudingen is voor niet Newtonse vloeistoffen niet lineair. Deze schuifstroming is kleiner dan die voor Newtonse vloeistoffen. De schuifstroming wordt kleiner naarmate het polymeer meer afwijkt van een Newtonse vloeistof.

III.

De door Jacobi afgeleide formules voor stroming van niet Newtonse vloeistoffen door schroefpomp kanalen zijn onbetrouwbaar omdat het door hem gebruikte superpositie principe niet geldt voor niet Newtonse vloeistoffen en omdat de schuifsnelheid in een richting loodrecht op de schroefdraad ten onrechte verwaarloosd is in de berekening van de viscositeit.

H. R. Jacobi, "Grundlagen der Extruder Technik",
Carl Hanser Verlag, Muenchen, 1960
IV.

Aangetoond kan worden, dat men de invloed van de afrondingsstraal in de hoeken van schroefpomp kanalen niet kan verwaarlozen wanneer de kanaal doorsnede een kleine aspectverhouding heeft.

V.

De door Squires afgeleide vergelijking voor de stroming van visceuze vloeistoffen in gedeeltelijk gevulde schroefpomp kanalen is niet juist voor kanalen met kleine aspectverhoudingen.


VI.

De afmetingen van lange overhangende assen, zoals gebruikt in roerwerken en pompen, worden meestal gebaseerd op een veilige marge tussen het critische toerental en het bedrijfstoerental. Het over te brengen draaimoment geeft in die gevallen schuifspanningen, die een fractie zijn van de toelaatbare spanningen. Het critisch toerental kan in zulke gevallen verhoogd worden door het aseïnd te verjongen of hol uit te voeren.

VII.

Het is gebruikelijk het mondstuk van een matrijs voor de extrusie van dunne platen te voeden vanuit een lang ingebouwd kanaal, dat evenwijdig loopt met de opening van het mondstuk. Men kan aantonen dat de gelijkmatigheid van de extrusie voor hetzelfde drukverlies beter wordt wanneer dat kanaal vervangen wordt door twee evenwijdige kanalen met een tussen geschakelde hydraulische weerstand.
VIII.

Sommige digitale electronische rekenmachines zijn zo gearrangeerd, dat de berekening toch voortgezet wordt wanneer de machine gevraagd wordt een vierkantswortel van een negatief getal te berekenen. Het is voor gebruikers van dergelijke machines onaanvaardbaar dat de machine niet onmiddellijk stopt en een boodschap geeft met de reden voor het beeindigen van de berekening.

IX.

Het is meer economisch problemen, die in aanmerking komen voor oplossing op digitale electronische rekenmachines, te programmeren en op te lossen op de snelste machine met het grootste geheugen. Vooral het programmeren is voor machines met een grote geheugen-capaciteit veel eenvouder.

X.

Het door Muijderman voor het bepalen van de invloed van de schuine einden van de groeven van axiale spiraallagers op de drukverdeling gebruikte analogon geeft een onjuiste oplossing. Toepassing van dit analogon tot de inlaatzijde van de groef geeft een positieve druk correctie, terwijl toepassing tot de uitlaatzijde voor dezelfde lineaire drukverdeling een negatieve druk correctie geeft. De drukverdeling is niet continue, wanneer terzelfder tijd aparte analogons worden gebruikt, respectievelijk voor het inlaat- en voor het uitlaateinde van een groef. Deze discontinue drukverdeling kan niet de juiste oplossing zijn.

E. A. Muijderman, "Spiral Groove Bearings", Proefschrift, Delft 1964
XI.

Commerciele gegevens over de viscositeit van polymeren zijn altijd gebaseerd op gemiddelde of schijnbare schuifsnelheden. De effectieve viscositeit hangt voor stroming door machines en apparaten af van het stromingsbeeld. Deze effectieve viscositeit kan dan alleen berekend worden wanneer de bepalings methode van de viscositeit volledig bekend is. Commerciele gegevens zijn in zulke gevallen alleen van waarde wanneer volledige gegevens over die methode en de berekening van die viscositeit bekend zijn.

XII.

In de USA is de vooropleiding van aankomende studenten, door de gedecentraliseerde organisatie van het middelbaar onderwijs, zo weinig uniform dat vrijwel alle universiteiten het eerste jaar besteden aan een herhaling van de stof, die in de High Schools had moeten worden behandeld. Het invoeren van een uniforme standaard en een uniforme onderwijs inspectie zou de achterstand met andere opleidingen aanzienlijk kunnen bekorten.

XIII.

De schroefpomp theorie wordt in de literatuur dikwijls verward met de extruder theorie. De schroefpomp theorie is van weinig waarde voor het ontwerp en de bedrijfs analyse van extruders, omdat de karakteristiek van een extruder vrijwel uitsluitend bepaald wordt door de druk die opgewekt wordt in de gedeelten van de extruder, waar het polymeer gesmolten wordt. De theorie voor het smelten van polymeren in extruders is nog niet ontwikkeld.

XIV.

Het is voor landen met een grote export van academisch personeel van belang te komen tot een internationale gelijkstelling van diplomas en titles.