DELFT UNIVERSITY OF TECHNOLOGY

REPORT 01-16

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WITH A LOW AND TIME-VARYING VELOCITY.

PART II: THE BEAM-LIKE CASE.

G. SUWEKEN AND W.T. VAN HORSSSEN

ISSN 1389-6520

Reports of the Department of Applied Mathematical Analysis

Delft 2001

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On The Transversal Vibrations of A Conveyor Belt with A Low and Time-Varying Velocity. Part II: The Beam-like Case

G. Suweken and W.T. van Horssen

Abstract

In this paper initial-boundary value problems for a beam equation (with string effect) are considered. These problems can be used as simple models to describe the vertical vibrations of a conveyor belt, for which the velocity is small with respect to the wave speed. In this paper the belt is assumed to move with a time-varying velocity $V(t) = \epsilon(V_0 + \alpha \sin(\Omega t))$. Formal asymptotic approximations of the solutions are constructed to show the complicated dynamical behavior of the belt. Complicated dynamical behaviour of the belt system occurs when the frequency Ω is the sum or difference of any two natural frequencies of the system with zero belt velocity. For special values of the belt parameters these sum type and difference type of internal resonances coincide giving rise to even more complicated dynamical behaviour. Some examples (including detuning cases) will be studied in detail.

1 Introduction

Axially moving systems are present in a wide class of engineering problems which arise in industrial, civil, aerospace, mechanical, electronic and automotive applications. Aerial cables, tram-ways, oil pipelines, magnetic tapes, power transmission belts, paper sheet and web processes, fiber winding and band saw blades are examples of cases where an axial transport of mass can be associated with transverse vibrations.

Investigating transverse vibrations of a belt system is a challenging subject which has been studied for many years (see [1-4] for a recent overview) and is still of interest today. The vibrations can be classified into two types, i.e. whether it is of a string-like type or of beam-like type, depending on the bending stiffness of the belt. If the bending stiffness can be neglected then the system is classified as string (wave)-like, otherwise it is classified as beam-like. The transverse vibration of a belt system (with constant belt velocity V) can be modeled as:

- string-like by

$$v_{tt} + 2Vv_{xt} + (\kappa V^2 - c^2)v_{xx} = 0, \quad \text{and} \quad (1)$$

- beam-like (with a string effect) by

$$v_{tt} + 2Vv_{xt} + (\kappa V^2 - c^2)v_{xx} + \frac{E_b I_y}{\rho A} v_{xxxx} = 0, \quad (2)$$

where $V, \kappa, c, E_b, I_y, \rho$, and A are constants which will be described in section 2 of this paper.

The main purpose of studying the dynamic behavior of a belt system is to know the *natural frequencies* of the vibrations. By knowing these natural frequencies, the so-called resonance-free belt system can be designed (see [3]). Resonances that can cause severe vibrations can be initiated by some parts of the belt system, such as the varying belt speed, the roll eccentricities, and other belt imperfections. The occurrence of resonances should be prevented since they can cause operational and maintenance problems including excessive wear of the belt and the support components, and the increase of energy consumption of the system.

In this paper we will study the vibrations of a moving belt which is modeled by a beam equation with a string effect. The belt speed is considered to be time-varying and to be small compared to the wave speed. In [5] a string-like model for a similar belt system has been studied. It will turn out that the beam-like model and the string-like model give rise to different behaviour of the solutions. It is assumed that the low and time-varying belt speed $V(t)$ is given by $\epsilon(V_0 + \alpha \sin(\Omega t))$, where ϵ, V_0, α , and Ω are constants with $0 < \epsilon \ll 1$ and $V_0 > |\alpha|$. It should be observed that the velocity changes periodically such that the belt moves in one direction. In fact the small parameter ϵ indicates that the belt speed $V(t)$ is small compared to the wave speed c . Recently the authors of [8] also studied the vibrations of an axially moving beam with a time-dependent velocity. As has been pointed out in [5] their application of the truncation method does not give approximations which are valid on long time-scales of order ϵ^{-1} . Moreover, their expansion or extension of some terms in the partial differential equation in terms of the eigenfunctions has been done incorrectly. The variation in $V(t)$ can be considered as some kind of excitation. In relation to excitations, some results in this area have been obtained by Sack [9] and Archibald and Emslie [10]. Sack considered the problem of a string moving with a constant velocity at which one of its end (i.e. $x = L$) is subjected to an harmonic excitation. In [9] the vibrations of the string at $x = L$ is forced to be $v(x, t) = v_0 \cos(\Omega t)$. Archibald and Emslie also studied the case where one end of the moving string is subjected to a harmonic excitation to represent the case of a belt traveling from an eccentric pulley to a smooth pulley. Whereas the case where both ends of the string are excited is studied by Mahalingham in [11]. He used a moving string model to study the transverse vibrations of power transmission chains. In all of these works, the belt movement is assumed to be constant.

This paper is organized as follows. In section 2 the equation of motion describing the dynamic behavior of a belt moving with a non-constant velocity is derived. The belt is assumed to be simply supported in vertical direction. Then in section 3, the two time-scales perturbation method is used to find approximations of the solution of the problem. It will turn out in section 3 that complicated dynamical behaviour of the belt system occurs when

the frequency Ω is the sum or the difference of any two natural frequencies ω_k and ω_n of the zero belt-velocity system. For special values of belt parameters these sum type and difference type of internal resonances can coincide giving rise to even more complicated dynamical behaviour. In section 4 of this paper we will study the (difference type) case $\Omega = \omega_2 - \omega_1$ and the detuned case $\Omega = \omega_2 - \omega_1 + \epsilon\phi$ with ϕ of order one. In section 5 we will study the (sum type) cases $\Omega = \omega_2 + \omega_1$, $\Omega = 2\omega_1$, and $\Omega = \omega_3 + \omega_2$. For some special values of the beam parameters we will also study the case (including detuning) for which a sum type and a difference type of internal resonance coincide: $\Omega = \omega_3 + \omega_2 = \omega_5 - \omega_2$. Finally in section 6 of this paper we will make some remarks and draw some conclusions.

2 A beam model

If the belt speed V is not constant but a function of t , then (2) becomes:

$$v_{tt} + (\kappa V^2 - c^2)v_{xx} + 2Vv_{xt} + V_tv_x + \frac{E_b I_y}{\rho A}v_{xxxx} = 0, \quad (3)$$

for $0 < x < L, t > 0$ and where:

- $v(x, t)$: the displacement of the belt in the y (vertical) direction,
- $V(t)$: the time-varying belt speed,
- c : the wave speed,
- E_b : Young's modulus,
- I_y : the moment of inertia with respect to the x (horizontal) axis,
- ρ : the mass density of the belt,
- A : the area of the cross section of the belt,
- κ : a constant representing the relative stiffness of the belt. Its value is in $[0, 1]$,
- x : coordinate in horizontal direction,
- t : time, and
- L : the distance between the pulleys.

Since the beam is assumed to be simply supported we have as boundary conditions

$$v(0, t) = v(L, t) = v_{xx}(0, t) = v_{xx}(L, t) = 0. \quad (4)$$

The initial values are given by:

$$v(x, 0) = f(x), \quad v_t(x, 0) = g(x), \quad (5)$$

where f is the initial displacement of the beam, and where g is the initial velocity of the beam. Considering the case where $V(t) = \epsilon(V_0 + \alpha \sin(\Omega t))$, in which V_0 and α are constants and $V_0 > |\alpha|$, equation (3) becomes:

$$\begin{aligned} v_{tt} - c^2 v_{xx} + \frac{E_b I_y}{\rho A} v_{xxxx} = & -\epsilon \Omega \alpha \cos(\Omega t) v_x - 2\epsilon (V_0 + \alpha \sin(\Omega t)) v_{xt} + \\ & -\epsilon^2 \kappa (V_0 + \alpha \sin(\Omega t))^2 v_{xx}. \end{aligned} \quad (6)$$

Note that (6) is essentially the same equation as the one studied by Öz and Pakdemirli in [8].

Solutions of the form $v(x, t) = \sum_{n=1}^{\infty} v_n(t) \sin(\frac{n\pi x}{L})$ certainly satisfy the boundary conditions. It should be observed that these solutions are odd and $2L$ -periodic in x . However, the first two terms in the right hand side of (6) are even, whereas other terms are odd. To make these two terms also odd, we have to multiply these two terms with $\mathcal{H}(x)$ (see also [5], [18], [20]) where $\mathcal{H}(x)$ is given by:

$$\begin{aligned}\mathcal{H}(x) &= \begin{cases} 1 & \text{if } 0 < x < L \\ -1 & \text{if } -L < x < 0 \end{cases} \\ &= \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin\left(\frac{(2j+1)\pi x}{L}\right).\end{aligned}\quad (7)$$

Then we obtain

$$\begin{aligned}v_{tt} - c^2 v_{xx} + \delta v_{xxxx} &= -\epsilon \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin\left(\frac{(2j+1)\pi x}{L}\right) \left[\alpha \Omega \cos(\Omega t) v_x + \right. \\ &\quad \left. 2(V_0 + \alpha \sin(\Omega t)) v_{xt} \right] - \epsilon^2 \kappa (V_0 + \alpha \sin(\Omega t))^2 v_{xx},\end{aligned}\quad (8)$$

where $\delta = \frac{E_b I_y}{\rho A}$. Substituting $v(x, t) = \sum_{n=1}^{\infty} v_n(t) \sin(\frac{n\pi x}{L})$ into (8) gives:

$$\begin{aligned}\sum_{n=1}^{\infty} \left[\ddot{v}_n + \left\{ \left(\frac{cn\pi}{L} \right)^2 + \delta \left(\frac{n\pi}{L} \right)^4 \right\} v_n \right] \sin\left(\frac{n\pi x}{L}\right) &= -\epsilon \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin\left(\frac{(2j+1)\pi x}{L}\right) \\ &\quad \left[\alpha \Omega \cos(\Omega t) \sum_{n=1}^{\infty} \frac{n\pi}{L} v_n \cos\left(\frac{n\pi x}{L}\right) + 2(V_0 + \alpha \sin(\Omega t)) \sum_{n=1}^{\infty} \frac{n\pi}{L} \dot{v}_n \cos\left(\frac{n\pi x}{L}\right) \right] + \\ &\quad \epsilon^2 \kappa (V_0 + \alpha \sin(\Omega t))^2 \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} \right)^2 v_n \sin\left(\frac{n\pi x}{L}\right).\end{aligned}\quad (9)$$

Multiplying both sides of (9) with $\sin(\frac{k\pi x}{L})$, and then integrating with respect to x from $x = 0$ to $x = L$ gives:

$$\begin{aligned}\ddot{v}_k + \left\{ \left(\frac{ck\pi}{L} \right)^2 + \delta \left(\frac{k\pi}{L} \right)^4 \right\} v_k &= \epsilon \left[\sum_{k=n-2j-1} - \sum_{k=2j+1+n} - \sum_{k=2j+1-n} \right] \\ &\quad \left[\frac{2\alpha \Omega n v_n \cos(\Omega t)}{(2j+1)L} + \frac{4n(V_0 + \alpha \sin(\Omega t)) \dot{v}_n}{(2j+1)L} \right] + \mathcal{O}(\epsilon^2),\end{aligned}\quad (10)$$

with $v_k(0) = \frac{2}{L} \int_0^L f(x) \sin(\frac{k\pi x}{L}) dx$, and $\dot{v}_k(0) = \frac{2}{L} \int_0^L g(x) \sin(\frac{k\pi x}{L}) dx$.

It should be observed that in order to obtain (10) the terms v_x and v_{xt} in (6) have been expanded in eigenfunctions (i.e. in $\sin(\frac{n\pi x}{L})$) of the boundary-value problem (4), (6) with $\epsilon = 0$. In [8] these terms were not expanded accordingly. In the next sections approximations of the solutions of (10) will be constructed for different Ω -values.

3 Application of the two time-scales perturbation method

Due to occurrence of so-called secular terms a straightforward perturbation method can not be used to solve (10) approximately. For that reason a two-time-scales perturbation method (with time scales $t_0 = t$ and $t_1 = \epsilon t$) is used. The introduction of these two time scales defines the following transformations:

$$\begin{aligned} v_k(t; \epsilon) &= w_k(t_0, t_1; \epsilon), \\ \frac{dv_k}{dt} &= \frac{\partial w_k}{\partial t_0} + \epsilon \frac{\partial w_k}{\partial t_1}, \\ \frac{d^2 v_k}{dt^2} &= \frac{\partial^2 w_k}{\partial t_0^2} + 2\epsilon \frac{\partial^2 w_k}{\partial t_0 \partial t_1} + \epsilon^2 \frac{\partial^2 w_k}{\partial t_1^2}. \end{aligned} \quad (11)$$

Substitution of (11) into (10) yields:

$$\begin{aligned} \frac{\partial^2 w_k}{\partial t_0^2} + 2\epsilon \frac{\partial^2 w_k}{\partial t_0 \partial t_1} + \epsilon^2 \frac{\partial^2 w_k}{\partial t_1^2} + \left\{ \left(\frac{ck\pi}{L} \right)^2 + \delta \left(\frac{k\pi}{L} \right)^4 \right\} w_k = \\ \epsilon \left[\sum_1 - \sum_2 - \sum_3 \right] \left[\frac{2\alpha\Omega n w_n \cos(\Omega t_0)}{(2j+1)L} + \frac{4n(V_0 + \alpha \sin(\Omega t_0))}{(2j+1)L} \frac{\partial w_n}{\partial t_0} \right] + \\ \mathcal{O}(\epsilon^2), \end{aligned} \quad (12)$$

where $\sum_1 = \sum_{k=n-2j-1}$, $\sum_2 = \sum_{k=2j+1+n}$, and $\sum_3 = \sum_{k=2j+1-n}$. Assuming that $w_k(t_0, t_1; \epsilon) = w_{k0}(t_0, t_1) + \epsilon w_{k1}(t_0, t_1) + \epsilon^2 w_{k2}(t_0, t_1) + \dots$ then (12) becomes:

$$\begin{aligned} \left[\frac{\partial^2 w_{k0}}{\partial t_0^2} + \epsilon \frac{\partial^2 w_{k1}}{\partial t_0^2} + \mathcal{O}(\epsilon^2) \right] + 2\epsilon \left[\frac{\partial^2 w_{k0}}{\partial t_0 \partial t_1} + \epsilon \frac{\partial^2 w_{k1}}{\partial t_0 \partial t_1} + \mathcal{O}(\epsilon^2) \right] + \mathcal{O}(\epsilon^2) + \\ \left\{ \left(\frac{ck\pi}{L} \right)^2 + \delta \left(\frac{k\pi}{L} \right)^4 \right\} (w_{k0} + \epsilon w_{k1} + \mathcal{O}(\epsilon^2)) = \epsilon \left[\sum_1 - \sum_2 - \sum_3 \right] \\ \left(\frac{2\alpha\Omega n w_{n0} \cos(\Omega t_0)}{(2j+1)L} + \frac{4n(V_0 + \alpha \sin(\Omega t_0))}{(2j+1)L} \frac{\partial w_{n0}}{\partial t_0} \right). \end{aligned} \quad (13)$$

By taking together terms of equal powers in ϵ we obtain from (13) the following $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ equations:

$$\begin{aligned} \mathcal{O}(1) : \quad \frac{\partial^2 w_{k0}}{\partial t_0^2} + \left[\left(\frac{ck\pi}{L} \right)^2 + \delta \left(\frac{k\pi}{L} \right)^4 \right] w_{k0} &= 0, \\ \mathcal{O}(\epsilon) : \quad \frac{\partial^2 w_{k1}}{\partial t_0^2} + 2 \frac{\partial^2 w_{k0}}{\partial t_0 \partial t_1} + \left[\left(\frac{ck\pi}{L} \right)^2 + \delta \left(\frac{k\pi}{L} \right)^4 \right] w_{k1} &= \left[\sum_1 - \sum_2 - \sum_3 \right] \\ &\quad \left(\frac{2\alpha\Omega n w_{n0} \cos(\Omega t_0)}{(2j+1)L} + \frac{4n(V_0 + \alpha \sin(\Omega t_0))}{(2j+1)L} \frac{\partial w_{n0}}{\partial t_0} \right). \end{aligned} \quad (14)$$

The $\mathcal{O}(1)$ equation can be easily solved, yielding:

$$w_{k0} = A_{k0}(t_1) \sin(\omega_k t_0) + B_{k0}(t_1) \cos(\omega_k t_0), \quad (15)$$

where:

$$\omega_k^2 = \left(\frac{ck\pi}{L}\right)^2 + \delta \left(\frac{k\pi}{L}\right)^4, \quad B_{k0}(0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx, \quad \text{and} \\ A_{k0}(0) = \frac{2}{\omega_k L} \int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right) dx. \quad (16)$$

The $A_{k0}(t_1)$ and $B_{k0}(t_1)$ in (15) are still arbitrary and can be used to avoid secular terms in the solution of the $\mathcal{O}(\epsilon)$ equation (14).

The $\mathcal{O}(\epsilon)$ equation now becomes

$$\frac{\partial^2 w_{k1}}{\partial t_0^2} + \omega_k^2 w_{k1} = -2\omega_k \left[\dot{A}_{k0}(t_1) \cos(\omega_k t_0) - \dot{B}_{k0}(t_1) \sin(\omega_k t_0) \right] + \\ \left[\sum_1 - \sum_2 - \sum_3 \right] \left(\frac{2\alpha\Omega n \cos(\Omega t_0)}{(2j+1)L} [A_{n0} \sin(\omega_n t_0) + B_{n0} \cos(\omega_n t_0)] + \frac{4n(V_0 + \alpha \sin(\Omega t_0))\omega_n}{(2j+1)L} [A_{n0} \cos(\omega_n t_0) - B_{n0} \sin(\omega_n t_0)] \right). \quad (17)$$

From (17) it can readily be seen that there are infinitely many values of Ω that can cause internal resonances. In fact these values are (in an $\mathcal{O}(\epsilon)$ neighbourhood of) $\omega_n + \omega_k, \omega_n - \omega_k, \omega_k - \omega_n$, and $-(\omega_n + \omega_k)$, where $k = n - 2j - 1$, or $k = 2j + 1 + n$ or $k = 2j + 1 - n$ (see also the summations in (17)). To show how the secular terms can be eliminated and how the belt system can behave, we will consider in section 4 the (difference type) case $\Omega = \omega_2 - \omega_1$ and the detuned case $\Omega = \omega_2 - \omega_1 + \epsilon\phi$ with ϕ of order one. In section 5 we will consider the (sum type) cases $\Omega = \omega_2 + \omega_1, \Omega = 2\omega_1$, and $\Omega = \omega_3 + \omega_2$. For some special values of the beam parameters we will also study in section 5 the case (including detuning) for which a sum type and a difference type of internal resonance coincide (in hat case $\Omega = \omega_3 + \omega_2 = \omega_5 - \omega_2$).

4 The case $\Omega = \omega_2 - \omega_1 + \epsilon\phi$

In this section we will study the case $\Omega = \omega_2 - \omega_1$, and the case $\Omega = \omega_2 - \omega_1 + \epsilon\phi$ with ϕ of order one.

4.1 The case $\Omega = \omega_2 - \omega_1$.

In appendix 1 it has been shown that for $\Omega = \omega_2 - \omega_1$ the equation $\Omega \pm \omega_n = \pm\omega_k$ only has the rather trivial solutions $n = 2$ and $k = 1$ if $\Omega - \omega_n = -\omega_k$, and $n = 1$ and $k = 2$ if

$\Omega + \omega_n = \omega_k$. Then, by taking apart those terms in the right-hand side of the $\mathcal{O}(\epsilon)$ -equation (17) that cause secular terms in $w_{k1}(t_0, t_1)$, we find that A_{k0} and B_{k0} have to satisfy

$$\begin{aligned}\dot{A}_{10} &= \frac{-2\alpha(\omega_1 + \omega_2)}{3\omega_1 L} B_{20}, \\ \dot{B}_{10} &= \frac{2\alpha(\omega_1 + \omega_2)}{3\omega_1 L} A_{20}, \\ \dot{A}_{20} &= \frac{-2\alpha(\omega_1 + \omega_2)}{3\omega_2 L} B_{10}, \\ \dot{B}_{20} &= \frac{2\alpha(\omega_1 + \omega_2)}{3\omega_2 L} A_{10},\end{aligned}\tag{18}$$

and for $k \geq 3$: $\dot{A}_{k0} = \dot{B}_{k0} = 0$.

System (18) can readily be solved, yielding

$$\begin{aligned}A_{10}(t_1) &= -\sqrt{\frac{\omega_2}{\omega_1}} B_{20}(0) \sin(\gamma t_1) + A_{10}(0) \cos(\gamma t_1), \\ B_{10}(t_1) &= \sqrt{\frac{\omega_2}{\omega_1}} A_{20}(0) \sin(\gamma t_1) + B_{10}(0) \cos(\gamma t_1), \\ A_{20}(t_1) &= A_{20}(0) \cos(\gamma t_1) - \sqrt{\frac{\omega_1}{\omega_2}} B_{10}(0) \sin(\gamma t_1), \quad \text{and} \\ B_{20}(t_1) &= B_{20}(0) \cos(\gamma t_1) + \sqrt{\frac{\omega_1}{\omega_2}} A_{10}(0) \sin(\gamma t_1),\end{aligned}\tag{19}$$

where $\gamma = \frac{2\alpha(\omega_1 + \omega_2)}{3L\sqrt{\omega_1\omega_2}}$. For $k \geq 3$, we have $A_{k0}(t_1) = A_{k0}(0)$ and $B_{k0}(t_1) = B_{k0}(0)$. For $n \geq 1$, $A_{n0}(0)$ and $B_{n0}(0)$ can be determined from (16). From (17) a solution $w_{k1}(t_0, t_1)$ can now be obtained without unbounded terms (that is without secular terms). So far we have constructed a formal approximation $w_{k0}(t_0, t_1) + \epsilon w_{k1}(t_0, t_1)$ of $v_k(t; \epsilon)$. And so, we have constructed an approximation $\sum_{k=1}^{\infty} w_{k0}(t_0, t_1) \sin(\frac{k\pi x}{L}) + \mathcal{O}(\epsilon)$ of the solution $v(x, t)$ of the initial-boundary value problem (3) - (5) with $\Omega = \omega_2 - \omega_1$.

4.2 The detuning case $\Omega = \omega_2 - \omega_1 + \epsilon\phi$.

When we detune the frequency of the belt velocity fluctuation by taking $\Omega = \omega_2 - \omega_1 + \epsilon\phi$ with $\phi = \mathcal{O}(1)$, then the $\mathcal{O}(\epsilon)$ -equation (17) becomes:

$$\begin{aligned}\frac{\partial^2 w_{k1}}{\partial t_0^2} + \omega_k^2 w_{k1} &= -2\omega_k \left[\dot{A}_{k0} \cos(\omega_k t_0) - \dot{B}_{k0} \sin(\omega_k t_0) \right] + \\ &\left[\sum_1 - \sum_2 - \sum_3 \right] \left(\frac{2\alpha\Omega_0 n \cos(\Omega t_0)}{(2j+1)L} \{A_{n0} \sin(\omega_n t_0) + B_{n0} \cos(\omega_n t_0)\} + \right. \\ &\quad \left. \frac{4n\omega_n(V_0 + \alpha \sin(\Omega t_0))}{(2j+1)L} \{A_{n0} \cos(\omega_n t_0) - B_{n0} \sin(\omega_n t_0)\} \right),\end{aligned}\tag{20}$$

where $\Omega_0 = \omega_2 - \omega_1$. Now, it should be observed that in (20)

$$\begin{aligned}\cos(\Omega t) &= \cos((\omega_2 - \omega_1)t_0 + \phi t_1) = \cos(\Omega_0 t_0) \cos(\phi t_1) - \sin(\Omega_0 t_0) \sin(\phi t_1), \text{ and} \\ \sin(\Omega t) &= \sin((\omega_2 - \omega_1)t_0 + \phi t_1) = \sin(\Omega_0 t_0) \cos(\phi t_1) + \cos(\Omega_0 t_0) \sin(\phi t_1).\end{aligned}$$

Then, by taking apart those terms in the right-hand side of (20) that give rise to secular terms in $w_{k1}(t_0, t_1)$, we find that in order to avoid secular terms, $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy:

$$\begin{aligned}\dot{A}_{10} &= -p \sin(\phi t_1) A_{20} + p \cos(\phi t_1) B_{20}, \\ \dot{B}_{10} &= -p \cos(\phi t_1) A_{20} - p \sin(\phi t_1) B_{20}, \\ \dot{A}_{20} &= -q \sin(\phi t_1) A_{10} - q \cos(\phi t_1) B_{10}, \\ \dot{B}_{20} &= q \cos(\phi t_1) A_{10} - q \sin(\phi t_1) B_{10},\end{aligned}\tag{21}$$

and for $k \geq 3$

$$\dot{A}_{k0} = 0 \quad \text{and} \quad \dot{B}_{k0} = 0,\tag{22}$$

where

$$p = \frac{-2\alpha(\omega_1 + \omega_2)}{3\omega_1 L} \quad \text{and} \quad q = \frac{2\alpha(\omega_1 + \omega_2)}{3\omega_2 L}.\tag{23}$$

Notice that for $\phi = 0$, (21) is reduced to (18). In appendix 2, the solution of system (21) has been derived. It turns out that the solution of system (21) is:

$$\begin{aligned}A_{10}(t_1) &= K_1 \sin(\beta_1 t_1) + K_2 \cos(\beta_1 t_1) + K_3 \sin(\beta_2 t_1) + K_4 \cos(\beta_2 t_1), \\ B_{10}(t_1) &= \frac{1}{pq\phi} [A_{10}^{(3)} + (\phi^2 - pq)\dot{A}_{10}], \\ A_{20}(t_1) &= \frac{-1}{p} [\dot{A}_{10} \sin(\phi t_1) + \dot{B}_{10} \cos(\phi t_1)], \\ B_{20}(t_1) &= \frac{1}{p} [\dot{A}_{10} \cos(\phi t_1) - \dot{B}_{10} \sin(\phi t_1)],\end{aligned}\tag{24}$$

and for $k \geq 3$, we find that $A_{k0} = A_{k0}(0)$, and $B_{k0} = B_{k0}(0)$. Note also that in (24) $\phi \neq 0$. In (24), K_1, K_2, K_3 , and K_4 are constants of integration, p, q are given by (23), $\beta_1 = \sqrt{\frac{1}{2}[\phi^2 - 2pq - \sqrt{\phi^4 - 4pq\phi^2}]}$, and $\beta_2 = \sqrt{\frac{1}{2}[\phi^2 - 2pq + \sqrt{\phi^4 - 4pq\phi^2}]}$. As in subsection 4.1 we have now constructed an approximation $\sum_{k=1}^{\infty} w_{k0}(t_0, t_1) \cdot \sin(\frac{k\pi x}{L}) + \mathcal{O}(\epsilon)$ of the solution $v(x, t)$ of the initial-boundary value problem (3) - (5) with $\Omega = \omega_2 - \omega_1 + \epsilon\phi$ and $\phi = \mathcal{O}(1)$.

For Ω in a neighbourhood of $\omega_2 - \omega_1$ we can now conclude (see also (24)) that no instabilities for the belt system will occur. A similar analysis as given in this section can be applied to other cases where Ω is of difference type (that is $\Omega = \omega_m - \omega_n$ for some m and n). However, in some of these cases instabilities can occur as will be explained in the next section.

5 Ω is a sum of two natural frequencies

In section 3, it has been shown that in order to remove secular terms we have to solve the equation $\Omega \pm \omega_n = \pm \omega_k$, where $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$. In the case $\Omega = \omega_2 - \omega_1$, it has been shown in section 4 that the only solutions of the problem $(\omega_2 - \omega_1) \pm \omega_n = \pm \omega_k$ (for an arbitrary value of $\mu^2 = \frac{\delta \pi^2}{c^2 L^2}$) are the trivial solutions $k = 1, n = 2$ and symmetrically $k = 2, n = 1$. For other values of Ω and for certain specific values of μ^2 we may have solutions other than the trivial ones. In this section we will consider the cases $\Omega = \omega_2 + \omega_1$, $\Omega = 2\omega_1$, and $\Omega = \omega_3 + \omega_2$. Other cases can be investigated similarly.

5.1 The case $\Omega = \omega_2 + \omega_1$.

First we have to solve the equation $\Omega \pm \omega_n = \pm \omega_k$. In fact we have to consider the following three cases:

- i) $-\omega_n - \omega_k = \Omega$, which obviously has no solution since the right-hand side is positive while the left hand side is negative,
- ii) $\omega_n + \omega_k = \Omega$, which obviously has only the trivial solution $k = 2, n = 1$ or $k = 1, n = 2$,
- iii) $\omega_k - \omega_n = \Omega$ (or equivalently $\omega_n - \omega_k = \Omega$) which may or may not have solutions depending on the value of μ^2 . From $\omega_k - \omega_n = \Omega$, it follows that $k\sqrt{1 + k^2\mu^2} = n\sqrt{1 + n^2\mu^2} + 2\sqrt{1 + 4\mu^2} + \sqrt{1 + \mu^2}$. Since $f(k) = k\sqrt{1 + \mu^2 k^2}$ is an increasing function it then follows from $k\sqrt{1 + \mu^2 k^2} > n\sqrt{1 + \mu^2 n^2}$ that $k > n \geq 1$. Then from $1 \leq n < k$ it follows that

$$\begin{aligned} k\sqrt{1 + k^2\mu^2} &= n\sqrt{1 + n^2\mu^2} + 2\sqrt{1 + 4\mu^2} + \sqrt{1 + \mu^2} \\ &< n\sqrt{1 + k^2\mu^2} + 2\sqrt{1 + k^2\mu^2} + \sqrt{1 + k^2\mu^2} \\ \Rightarrow n &< k < n + 3, \quad \Rightarrow k = n + 1 \quad \text{or} \quad k = n + 2. \end{aligned}$$

Since $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$ it follows that k can only be equal to $k = n + 1$. So, $\omega_k - \omega_n = \Omega = \omega_2 + \omega_1$ can only have solutions for $k = n + 1$. The possibility to have solutions turns out to be depending on the values of μ^2 . In Table 1 some of these solutions are given.

Assuming that $\Omega \pm \omega_n = \pm \omega_k$ only has the trivial solutions ($k = 2$ and $n = 1$, and $k = 1$ and $n = 2$) it turns out that no secular terms occur in the solution of (17) if $A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy

$$\begin{aligned} \dot{A}_{10} &= \frac{2\alpha(\omega_1 - \omega_2)}{3L\omega_1} B_{20}, & \dot{B}_{10} &= \frac{2\alpha(\omega_1 - \omega_2)}{3L\omega_1} A_{20}, \\ \dot{A}_{20} &= \frac{2\alpha(\omega_1 - \omega_2)}{3L\omega_2} B_{10}, & \dot{B}_{20} &= \frac{2\alpha(\omega_1 - \omega_2)}{3L\omega_2} A_{10}, \end{aligned} \tag{25}$$

Table 1: Values of k, n , and μ^2 for which $\omega_k = \omega_n + \Omega$ has solutions

$\Omega = \omega_2 + \omega_1$			$\Omega = 2\omega_1$			$\Omega = \omega_3 + \omega_2$					
k	n	μ^2	k	n	μ^2	k	n	μ^2	k	n	μ^2
3	2	-	2	1	0.7143	6	5	-	4	1	0.3851
4	3	0.3851	3	2	0.1664	7	6	-	5	2	0.0732
5	4	0.1607	4	3	0.0773	8	7	0.4588	6	3	0.0349
6	5	0.0926	5	4	0.0451	9	8	0.2124	7	4	0.0211
7	6	0.0613	6	5	0.0297	10	9	0.1321	8	5	0.0143
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

and for $k \geq 3$, $\dot{A}_{k0} = \dot{B}_{k0} = 0$. The solution of (25) can readily be determined, yielding

$$\begin{aligned}
A_{10}(t_1) &= A_{10}(0) \cosh(r_1 t_1) - \sqrt{\frac{\omega_2}{\omega_1}} B_{20}(0) \sinh(r_1 t_1), \\
A_{20}(t_1) &= A_{20}(0) \cosh(r_1 t_1) - \sqrt{\frac{\omega_1}{\omega_2}} B_{10}(0) \sinh(r_1 t_1), \\
B_{10}(t_1) &= -\sqrt{\frac{\omega_2}{\omega_1}} A_{20}(0) \sinh(r_1 t_1) + B_{10}(0) \cosh(r_1 t_1), \\
B_{20}(t_1) &= -\sqrt{\frac{\omega_1}{\omega_2}} A_{10}(0) \sinh(r_1 t_1) + B_{20}(0) \cosh(r_1 t_1),
\end{aligned} \tag{26}$$

where $r_1 = \frac{2\alpha(\omega_2 - \omega_1)}{3L\sqrt{\omega_1\omega_2}}$, and for $k \geq 3$: $A_{k0}(t_1) = A_{k0}(0)$ and $B_{k0}(t_1) = B_{k0}(0)$. It is obvious from (26) that instabilities for the belt system will occur. When for instance $\mu^2 \approx 0.3851$ it turns out that $\Omega \pm \omega_n = \pm\omega_k$ also has other solutions than the trivial ones (see Table 1). To avoid secular terms in the solution of (17) it then turns out that A_{10}, B_{10}, A_{20} , and B_{20} still have to satisfy (25), and that

$$\begin{aligned}
\dot{A}_{30} &= \frac{-12\alpha}{7L\omega_3}(\omega_3 + \omega_4)B_{40}, & \dot{B}_{30} &= \frac{12\alpha}{7L\omega_3}(\omega_3 + \omega_4)A_{40}, \\
\dot{A}_{40} &= \frac{-12\alpha}{7L\omega_4}(\omega_3 + \omega_4)B_{30}, & \dot{B}_{40} &= \frac{12\alpha}{7L\omega_4}(\omega_3 + \omega_4)A_{30},
\end{aligned} \tag{27}$$

and that for $k \geq 5$, $\dot{A}_{k0} = \dot{B}_{k0} = 0$. The solution of (27) can readily be determined, yielding

$$\begin{aligned}
A_{30}(t_1) &= -\sqrt{\frac{\omega_4}{\omega_3}} B_{40}(0) \sin(\beta t_1) + A_{30}(0) \cos(\beta t_1), \\
A_{40}(t_1) &= -\sqrt{\frac{\omega_3}{\omega_4}} B_{30}(0) \sin(\beta t_1) + A_{40}(0) \cos(\beta t_1), \\
B_{30}(t_1) &= \sqrt{\frac{\omega_4}{\omega_3}} A_{40}(0) \sin(\beta t_1) + B_{30}(0) \cos(\beta t_1),
\end{aligned}$$

$$B_{40}(t_1) = \sqrt{\frac{\omega_3}{\omega_4}} A_{30}(0) \sin(\beta t_1) + B_{40}(0) \cos(\beta t_1), \quad (28)$$

where $\beta = \frac{12\alpha(\omega_3+\omega_4)}{7L\sqrt{\omega_3\omega_4}}$ and for $k \geq 5$: $A_{k0}(t_1) = A_{k0}(0)$ and $B_{k0}(t_1) = B_{k0}(0)$. Also for $\mu^2 \approx 0.3851$ it is obvious that instabilities for the belt system will occur. It should be observed that for $\mu^2 \approx 0.3851$ we have $\Omega = \omega_2 + \omega_1 = \omega_4 - \omega_3$. So, for special values of the beam parameters, also frequency Ω of difference type can lead to instabilities.

5.2 The case $\Omega = 2\omega_1$.

As in subsection 5.1 we first have to solve the equation $\Omega \pm \omega_n = \pm\omega_k$. Again we have to consider three cases:

- i) $-\omega_n - \omega_k = \Omega = 2\omega_1$, which obviously has no solutions,
- ii) $\omega_n + \omega_k = \Omega = 2\omega_1$, which obviously only can be satisfied for $k = n = 1$. But since $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$, it follows that there is in this case no solution.
- iii) $\omega_k - \omega_n = \Omega = 2\omega_1$ (or equivalently $\omega_n - \omega_k = \Omega$), which may or may not have solutions depending on the value of μ^2 . From $\omega_k - \omega_n = 2\omega_1$ it follows that $k\sqrt{1 + \mu^2 k^2} = n\sqrt{1 + \mu^2 n^2} + 2\sqrt{1 + \mu^2}$. Since $f(k) = k\sqrt{1 + \mu^2 k^2}$ is an increasing function it then follows from $k\sqrt{1 + \mu^2 k^2} > n\sqrt{1 + \mu^2 n^2}$ that $k > n \geq 1$. Then, from $1 \leq n < k$ it follows that

$$\begin{aligned} k\sqrt{1 + \mu^2 k^2} &= n\sqrt{1 + \mu^2 n^2} + 2\sqrt{1 + \mu^2} \\ &< n\sqrt{1 + \mu^2 k^2} + 2\sqrt{1 + \mu^2 k^2} \\ &\Rightarrow n < k < n + 2 \Rightarrow k = n + 1. \end{aligned}$$

So, $\omega_k - \omega_n = 2\omega_1$ can only have solutions for $k = n + 1$. The possibility to have solutions turns out to be depending on the values of μ^2 . In Table 1 some of these solutions are given.

When μ^2 is not (in neighbourhood of) a value as listed in Table 1 then it easily follows from (17) that no secular terms occur in the solution if $\dot{A}_{k0} = \dot{B}_{k0} = 0$ for all $k \geq 1$. When for instance $\mu^2 \approx 0.7143$ it turns out that no secular terms occur in the solution of (17) if $A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy

$$\begin{aligned} \dot{A}_{10} &= \frac{-4\alpha}{3L\omega_1}(\omega_2 - \omega_1)B_{20}, & \dot{B}_{10} &= \frac{4\alpha}{3L\omega_1}(\omega_2 - \omega_1)A_{20}, \\ \dot{A}_{20} &= \frac{-8\alpha\omega_1}{3L\omega_2}B_{10}, & \dot{B}_{20} &= \frac{8\alpha\omega_1}{3L\omega_2}A_{10}, \end{aligned} \quad (29)$$

and for $k \geq 3$ $\dot{A}_{k0} = \dot{B}_{k0} = 0$. The solution of system (29) can readily be determined, yielding:

$$\begin{aligned} A_{10}(t_1) &= -CB_{20}(0) \sin(\theta t_1) + A_{10}(0) \cos(\theta t_1), \\ A_{20}(t_1) &= -\frac{1}{C}B_{10}(0) \sin(\theta t_1) + A_{20}(0) \cos(\theta t_1), \\ B_{10}(t_1) &= CA_{20}(0) \sin(\theta t_1) + B_{10}(0) \cos(\theta t_1), \\ B_{20}(t_1) &= \frac{1}{C}A_{10}(0) \sin(\theta t_1) + B_{20}(0) \cos(\theta t_1), \end{aligned} \quad (30)$$

where $C^2 = \frac{\omega_2(\omega_2 - \omega_1)}{2\omega_1^2}$, $\theta = \frac{4\sqrt{2}\alpha}{3L} \sqrt{\frac{\omega_2 - \omega_1}{\omega_2}}$ and for $k \geq 3$: $A_{k0}(t_1) = A_{k0}(0)$, $B_{k0}(t_1) = B_{k0}(0)$. Obviously no instabilities for the belt system will occur when $\mu^2 \approx 0.7143$ or when μ^2 is not (in neighbourhood of) a value as listed in Table 1. It should be remarked that a similar analysis can be performed if $\Omega = 2\omega_N$ for some fixed $N > 1$.

5.3 The case $\Omega = \omega_3 + \omega_2$.

As in the previous two subsections 5.1 and 5.2 we first have to solve the equation $\Omega \pm \omega_n = \pm \omega_k$. Again we have to consider three cases:

- i) $-\omega_n - \omega_k = \Omega = \omega_2 + \omega_3$, which obviously has no solution,
- ii) $\omega_n + \omega_k = \Omega = \omega_2 + \omega_3$, which obviously has the trivial solutions $k = 2$ and $n = 3$, or $k = 3$ and $n = 2$. In this case additional solutions can only occur if $\omega_k + \omega_1 = \Omega = \omega_2 + \omega_3$ has a solution. In appendix 1 (see the case $\omega_k = \omega_n + \omega_2 - \omega_1$) it has been shown that this is not possible.
- iii) $\omega_k - \omega_n = \Omega = \omega_2 + \omega_3$ (or equivalently $\omega_n - \omega_k = \Omega$), which may or may not have solutions depending on the value of μ^2 . From $\omega_k - \omega_n = \Omega$ it follows that $k\sqrt{1 + \mu^2 k^2} = n\sqrt{1 + \mu^2 n^2} + 3\sqrt{1 + 9\mu^2} + 2\sqrt{1 + 4\mu^2}$. Since $f(k) = k\sqrt{1 + \mu^2 k^2}$ is an increasing function it then follows that $k > n$ and $k > 3$. Then, from $k > n$ and $k > 3$ it follows that

$$\begin{aligned} k\sqrt{1 + \mu^2 k^2} &= n\sqrt{1 + \mu^2 n^2} + 3\sqrt{1 + 9\mu^2} + 2\sqrt{1 + 4\mu^2} \\ &< n\sqrt{1 + \mu^2 k^2} + 3\sqrt{1 + \mu^2 k^2} + 2\sqrt{1 + \mu^2 k^2} \\ &\Rightarrow n < k < n + 3 + 2 \Rightarrow k = n + 1 \text{ or } k = n + 2 \text{ or } k = n + 3 \text{ or } \\ &k = n + 4. \end{aligned}$$

Since $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$ it follows that k can only be equal to $n + 1$ or $n + 3$. The possibility to have solutions turns out to be depending on the values of μ^2 . In Table 1 some of these solutions are given.

Assuming that $\Omega \pm \omega_n = \pm \omega_k$ only has the trivial solutions ($k = 2$ and $n = 3$, and $k = 3$ and $n = 2$) it turns out that no secular terms occur in the solution of (17) if $A_{k0}(t_1)$ and $B_{k0}(t_1)$ satisfy

$$\begin{aligned}\dot{A}_{20} &= -\frac{6\alpha}{5L\omega_2}(\omega_3 - \omega_2)B_{30}, & \dot{B}_{20} &= -\frac{6\alpha}{5L\omega_2}(\omega_3 - \omega_2)A_{30}, \\ \dot{A}_{30} &= -\frac{6\alpha}{5L\omega_3}(\omega_3 - \omega_2)B_{20}, & \dot{B}_{30} &= -\frac{6\alpha}{5L\omega_3}(\omega_3 - \omega_2)A_{20},\end{aligned}\quad (31)$$

and for $k = 1, 4, 5, 6, \dots$ $\dot{A}_{k0} = \dot{B}_{k0} = 0$. The solutions of system (31) can readily be determined, yielding:

$$\begin{aligned}A_{20}(t_1) &= -\sqrt{\frac{\omega_3}{\omega_2}}B_{30}(0)\sinh(s_1 t_1) + A_{20}(0)\cosh(s_1 t_1), \\ A_{30}(t_1) &= -\sqrt{\frac{\omega_2}{\omega_3}}B_{20}(0)\sinh(s_1 t_1) + A_{30}(0)\cosh(s_1 t_1), \\ B_{20}(t_1) &= -\sqrt{\frac{\omega_3}{\omega_2}}A_{30}(0)\sinh(s_1 t_1) + B_{20}(0)\cosh(s_1 t_1), \\ B_{30}(t_1) &= -\sqrt{\frac{\omega_2}{\omega_3}}A_{20}(0)\sinh(s_1 t_1) + B_{30}(0)\cosh(s_1 t_1),\end{aligned}\quad (32)$$

where $s_1 = \frac{6\alpha(\omega_3 - \omega_2)}{5L\sqrt{\omega_3\omega_2}}$, and for $k = 1, 4, 5, 6, \dots$ $A_{k0} = A_{k0}(0)$ and $B_{k0}(t_1) = B_{k0}(0)$. From (32) it is obvious that instabilities for the belt system will occur. When for instance $\mu^2 = 0.0732$ it turns out that $\Omega \pm \omega_n = \pm \omega_k$ also has other solutions than the trivial ones (see Table 1). To avoid secular terms in the solution of (17) it now turns out that $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy

$$\begin{aligned}\dot{A}_{20} &= -\frac{6\alpha}{5L\omega_2}(\omega_3 - \omega_2)B_{30} - \frac{10\alpha}{21L\omega_2}(\omega_5 + \omega_2)B_{50}, \\ \dot{B}_{20} &= -\frac{6\alpha}{5L\omega_2}(\omega_3 - \omega_2)A_{30} + \frac{10\alpha}{21L\omega_2}(\omega_5 + \omega_2)A_{50}, \\ \dot{A}_{30} &= -\frac{6\alpha}{5L\omega_3}(\omega_3 - \omega_2)B_{20}, & \dot{B}_{30} &= -\frac{6\alpha}{5L\omega_3}(\omega_3 - \omega_2)A_{20}, \\ \dot{A}_{50} &= -\frac{10\alpha}{21L\omega_5}(\omega_5 + \omega_2)B_{20}, & \dot{B}_{50} &= \frac{10\alpha}{21L\omega_5}(\omega_5 + \omega_2)A_{20},\end{aligned}\quad (33)$$

and $\dot{A}_{k0} = \dot{B}_{k0} = 0$ for $k = 1, 4, 6, 7, 8, \dots$. The solution of (33) can readily be determined, yielding

$$\begin{aligned}A_{20}(t_1) &= K_1 \sin(s_2 t_1) + A_{20}(0) \cos(s_2 t_1), \\ B_{20}(t_1) &= K_2 \sin(s_2 t_1) + B_{20}(0) \cos(s_2 t_1),\end{aligned}$$

$$\begin{aligned}
A_{30}(t_1) &= \frac{d_1 K_2}{s_2 \omega_3} \cos(s_2 t_1) - \frac{d_1 B_{20}(0)}{s_2 \omega_3} \sin(s_2 t_1) + \left(A_{30}(0) - \frac{d_1 K_2}{s_2 \omega_3} \right), \\
B_{30}(t_1) &= \frac{d_1 K_1}{s_2 \omega_3} \cos(s_2 t_1) - \frac{d_1 A_{20}(0)}{s_2 \omega_3} \sin(s_2 t_1) + \left(B_{30}(0) - \frac{d_1 K_1}{s_2 \omega_3} \right), \\
A_{50}(t_1) &= \frac{d_2 K_2}{s_2 \omega_5} \cos(s_2 t_1) - \frac{d_2 B_{20}(0)}{s_2 \omega_5} \sin(s_2 t_1) + \left(A_{50}(0) - \frac{d_2 K_2}{s_2 \omega_5} \right), \quad \text{and} \\
B_{50}(t_1) &= \frac{-d_2 K_1}{s_2 \omega_5} \cos(s_2 t_1) + \frac{d_2 A_{20}(0)}{s_2 \omega_5} \sin(s_2 t_1) + \left(B_{50}(0) + \frac{d_2 K_1}{s_2 \omega_5} \right), \quad (34)
\end{aligned}$$

where $s_2 = \left[\left(\frac{10}{21} \right)^2 \frac{(\omega_2 + \omega_5)^2}{\omega_2 \omega_5} - \left(\frac{6}{5} \right)^2 \frac{(\omega_3 - \omega_2)^2}{\omega_3 \omega_2} \right]$, $K_1 = \frac{-1}{s_2} \left[\frac{d_1}{\omega_2} B_{30}(0) + \frac{d_2}{\omega_2} B_{50}(0) \right]$, $K_2 = \frac{1}{s_2} \left[\frac{-d_1}{\omega_2} B_{30}(0) + \frac{d_2}{\omega_2} B_{50}(0) \right]$, $d_1 = \frac{6\alpha(\omega_3 - \omega_2)}{5L}$, and $d_2 = \frac{10\alpha(\omega_5 + \omega_2)}{21L}$. And for $k = 1, 4, 6, 7, 8, \dots$, $A_{k0}(t_1) = A_{k0}(0)$ and $B_{k0}(t_1) = B_{k0}(0)$. From (34) it is obvious that now no instabilities for the belt system will occur. It should be observed that for $\mu^2 \approx 0.0732$ we have $\Omega = \omega_2 + \omega_3 = \omega_5 - \omega_2$. So, for special values of the beam parameters also frequencies Ω of sum type can lead to stable behaviour. To obtain more insight in the complicated dynamical behaviour of the belt system we will detune in the next section the beam parameter μ^2 (keeping Ω fixed).

5.3.1 The detuned Case $\mu^2 \approx 0.0732$

In the previous subsection 5.3 it has been shown that if $\Omega = \omega_2 + \omega_3$ (and μ^2 is not in the neighbourhood of a value as listed in Table 1) then the belt system is unstable. For $\mu^2 \approx 0.0732$, however, the belt system is stable. To obtain more insight in this different behaviour we will detune μ^2 in a neighbourhood of 0.0732. Observe that $\mu^2 = \frac{\delta \pi^2}{c^2 L^2}$ with $\delta = \frac{E_b I_y}{\rho A}$. So, to obtain a detuning of μ^2 we can detune δ in the original PDE (8). For that reason we put $\mu^2 = \mu_{cr}^2 + \epsilon \psi$ with $\mu_{cr}^2 = 0.0732$ and ψ an arbitrary constant of $\mathcal{O}(1)$, and $\delta = \delta_{cr} + \epsilon \nu$ with $\psi = \frac{\pi^2}{c^2 L^2} \nu$ and $\mu_{cr}^2 = \frac{\pi^2}{c^2 L^2} \delta_{cr}$. The frequency Ω is kept fixed, that is, $\Omega = \sqrt{\left(\frac{2c\pi}{L} \right)^2 + \delta_{cr} \left(\frac{2\pi}{L} \right)^4} + \sqrt{\left(\frac{3c\pi}{L} \right)^2 + \delta_{cr} \left(\frac{3\pi}{L} \right)^4}$. It should be observed that this type of detuning is different from the one studied in section 4. By replacing δ in (8) by $\delta_{cr} + \epsilon \nu$ the same analysis as presented in section 2 and section 3 can be repeated. To avoid secular terms in the approximation it turns out that $A_{k0}(t_1)$ and $B_{k0}(t_1)$ now have to satisfy ($\phi = \frac{\pi^3 \nu}{c L^3}$, and $\bar{\omega}_k = k \sqrt{1 + \mu_{cr}^2 k^2}$ for $k = 2, 3$, and 5)

$$\begin{aligned}
\dot{A}_{20} &= -\frac{6\alpha}{5L\bar{\omega}_2}(\bar{\omega}_3 - \bar{\omega}_2)B_{30} - \frac{10\alpha}{21L\bar{\omega}_2}(\bar{\omega}_5 + \bar{\omega}_2)B_{50} - \frac{8\pi^4 \phi}{\bar{\omega}_2 L^4} B_{20}, \\
\dot{B}_{20} &= -\frac{6\alpha}{5L\bar{\omega}_2}(\bar{\omega}_3 - \bar{\omega}_2)A_{30} + \frac{10\alpha}{21L\bar{\omega}_2}(\bar{\omega}_5 + \bar{\omega}_2)A_{50} + \frac{8\pi^4 \phi}{\bar{\omega}_2 L^4} A_{20}, \\
\dot{A}_{30} &= -\frac{6\alpha}{5L\bar{\omega}_3}(\bar{\omega}_3 - \bar{\omega}_2)B_{20} - \frac{81\pi^4 \phi}{2\bar{\omega}_3 L^4} B_{30}, \\
\dot{B}_{30} &= -\frac{6\alpha}{5L\bar{\omega}_3}(\bar{\omega}_3 - \bar{\omega}_2)A_{20} + \frac{81\pi^4 \phi}{2\bar{\omega}_3 L^4} A_{30},
\end{aligned}$$

$$\begin{aligned}
\dot{A}_{50} &= -\frac{10\alpha}{21L\bar{\omega}_5}(\bar{\omega}_5 + \bar{\omega}_2)B_{20} - \frac{625\pi^4\phi}{2\bar{\omega}_5L^4}B_{50}, \\
\dot{B}_{50} &= \frac{10\alpha}{21L\bar{\omega}_5}(\bar{\omega}_5 + \bar{\omega}_2)A_{20} + \frac{625\pi^4\phi}{2\bar{\omega}_5L^4}A_{50},
\end{aligned} \tag{35}$$

and for $k = 1, 4, 6, 7, 8, \dots$

$$\dot{A}_{k0} = -\frac{\nu}{2\omega_k}\left(\frac{k\pi}{L}\right)^4 B_{k0}, \text{ and } \dot{B}_{k0} = \frac{\nu}{2\omega_k}\left(\frac{k\pi}{L}\right)^4 A_{k0}. \tag{36}$$

Obviously system (36) has bounded solution. To solve system (35) we have to determine the roots of the characteristic equation:

$$\begin{aligned}
\lambda^6 + (1502.1631\phi^2 + 1.8787\eta^2)\lambda^4 + (.8824\eta^4 - 1230.2972\eta^2\phi^2 + 170023.5061\phi^4)\lambda^2 \\
+ 874.7894\eta^4\phi^2 + .1876 \cdot 10^7\phi^6 - 81019.0927\eta^2\phi^4 = 0,
\end{aligned} \tag{37}$$

where $\eta = \frac{\alpha}{L}$. By putting $\lambda^2 = a$ in (37) we obtain a cubic equation for a

$$\begin{aligned}
a^3 + (1502.1631\phi^2 + 1.8787\eta^2)a^2 + (.8824\eta^4 - 1230.2972\eta^2\phi^2 + 170023.5061\phi^4)a \\
+ 874.7894\eta^4\phi^2 + .1876 \cdot 10^7\phi^6 - 81019.0927\eta^2\phi^4 = 0.
\end{aligned} \tag{38}$$

Using Cardano's formula, we can solve (38). The *radicand* R (of the reduced form of the cubic equation (38) plays an important role in the solution-structure. When the radicand R is positive the cubic equation (38) will have one real, and two complex conjugate solutions. Since $a = \lambda^2$ it follows that at least two roots of the characteristic equation (37) will have a positive real part. Consequently the solution of system (35) will be unstable. For $R < 0$ the cubic equation (38) will have three distinct real roots, and for $R = 0$ there are three real roots of which two coincide. For $R \leq 0$ it requires an additional analysis to determine whether system (35) is stable or not.

In what follows, we will present some numerical calculations as an indication of what one might expect. We take $\eta = \frac{\alpha}{L} = 0.1$, and we will vary $\phi \left(= \frac{\pi^3\nu}{cL^3} \right)$. In Figure 1 the radicand R is given as a function of ϕ for $\eta = 0.1$. From Figure 1 it can be seen that the radicand R is negative for $|\phi|$ smaller than approximately 0.00097, and for $|\phi|$ larger than approximately 0.0112. The radicand R is positive for $|\phi|$ between approximately 0.00097 and 0.0112. So, for $|\phi|$ between approximately 0.00097 and 0.0112 the belt system will be unstable. For $|\phi|$ smaller than approximately 0.00097 and for $|\phi|$ larger than approximately 0.0112 the radicand R is negative, and it turns out that the three distinct real roots of the cubic equation (38) are all negative. Since $a = \lambda^2$ it follows that roots of the characteristic equation (37) are all distinct and purely imaginary. So, for $|\phi|$ smaller than approximately 0.00097 and for $|\phi|$ larger than approximately 0.0112 the belt system will be stable.

6 Conclusions and remarks

In this paper we studied initial-boundary value problems for a beam equation which can be used as simple models to describe the vertical vibrations of a conveyor belt for which

Figure 1: Graphs of the Radicand (R) as a function of ϕ for $\eta = 0.1$. In the left graph $\phi \in [0, 0.001]$ while in the right graph $\phi \in [0, 0.012]$.

the time-varying belt velocity is small with respect to the wave speed. It is assumed that the belt velocity $V(t) = \epsilon(V_0 + \alpha \sin(\Omega t))$ where ϵ, V_0, α , and Ω are constants with $0 < \epsilon \ll 1$ and $|\alpha| < V_0$. Complicated dynamical behaviour of the belt system occurs when the frequency Ω is the sum or difference of any two natural frequencies of the system for which the belt velocity is equal to zero. For special values of the belt parameters these sum type and difference type of internal resonances can coincide giving rise to even more complicated dynamical behaviour. For both sum type and difference type of internal resonances instabilities for the belt system can occur.

In this paper we studied the following cases in detail with the following results:

- i)* $\Omega = \omega_2 - \omega_1$; interaction between the first and the second vibration modes; no instabilities for the belt system (also for the detuned case).
- ii)* $\Omega = \omega_2 + \omega_1$; interactions between the first and the second vibration modes, and for special values of the beam parameters (see Table 1) additional interactions; there will always be unstable behaviour of the belt system.
- iii)* $\Omega = 2\omega_1$; only for special values of the beam parameters (see Table 1) there will be an interaction between two different vibration modes; there are no instabilities for the belt system.
- iv)* $\Omega = \omega_2 + \omega_3$; interaction between the second and the third vibration modes, and for special values of beam parameters (see Table 1) there are additional interactions; in general there will be instabilities for the belt system. However, for special values of the beam parameters there can be stable behaviour of the belt system. When some of these beam parameters are detuned unstable behaviour can occur again (see section 5.3.1 where $\Omega = \omega_2 + \omega_3 = \omega_5 - \omega_2$ for $\mu^2 = \frac{E_b I_y \pi^2}{\rho A c^2 L^2} \approx 0.0732$).

We expect that for other values of Ω we can apply the same techniques (as presented in this paper) to determine the stability properties of the belt system.

Appendix

Appendix 1

In this appendix we will show that the equation $\Omega \pm \omega_n = \pm \omega_k$ with $\Omega = \omega_2 - \omega_1$ only has as solutions $n = 2$ and $k = 1$ if $\Omega - \omega_n = -\omega_k$, and $n = 1$ and $k = 2$ if $\Omega + \omega_n = \omega_k$. To prove this we have to consider four cases: $\omega_k = \omega_n + \omega_2 - \omega_1$, $\omega_k = -\omega_n + \omega_2 - \omega_1$, $-\omega_k = \omega_n + \omega_2 - \omega_1$, and $-\omega_k = -\omega_n + \omega_2 - \omega_1$. Note that $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$.

The case $\omega_k = \omega_n + \omega_2 - \omega_1$.

Since $\omega_k^2 = (\frac{ck\pi}{L})^2 + \delta(\frac{k\pi}{L})^4$, it follows from $\omega_k = \omega_n + \omega_2 - \omega_1$ that

$$\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} = \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} - 1, \quad (\text{A-1})$$

where $\mu^2 = \delta\frac{\pi^2}{c^2L^2}$. It can easily be shown that $f(k) = \frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}}$ is an increasing function in k , and that $k \leq f(k) < k^2$. Then it follows from (A-1) that

$$\frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} < \frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} < \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}}. \quad (\text{A-2})$$

Since $f(k)$ is increasing in k it follows from the first inequality in (A-2) that $1 \leq n < k$. From the second inequality in (A-2) it then follows that

$$\begin{aligned} \frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} &< \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} < \frac{n\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} \\ &\Rightarrow k < n + 2. \end{aligned} \quad (\text{A-3})$$

Consequently we have $k = n + 1$, and (A-1) becomes:

$$\frac{(n+1)\sqrt{1+\mu^2(n+1)^2}}{\sqrt{1+\mu^2}} - \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} = \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} - \frac{\sqrt{1+\mu^2}}{\sqrt{1+\mu^2}}.$$

Denoting the left hand side of the last equation by $g(n)$ then the right hand side of the equation is just $g(1)$. It is not too difficult to show that $g(n)$ is an increasing function, so the last equation can only be satisfied if $n = 1$. Since $k = n + 1$ it follows that the only solution in this case is $k = 2$ and $n = 1$.

The case $\omega_k = -\omega_n + \omega_2 - \omega_1$

In this case it follows from $\omega_k = -\omega_n + \omega_2 - \omega_1$ that

$$\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} = -\frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} - 1. \quad (\text{A-4})$$

The only candidate for a solution of this equation is $n = 1$ since the left hand side is always positive while the right hand side is negative for $n \geq 2$. Accordingly, by substituting $n = 1$ into (A-4) we obtain:

$$\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} = \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} - 2. \quad (\text{A-5})$$

Now it should be observed that the left-hand side of (A-5) is between k and k^2 , and that the right-hand side is between 0 and 2. So, the only candidate for a solution is $k = 1$ (and $n = 1$). Since $k = n - 2j - 1$, or $k = n + 2j + 1$, or $k = 2j + 1 - n$ with $k, n \in \mathbb{N}^+$ and $j \in \mathbb{N}$ it easily follows that there are no solutions in this case.

The case $-\omega_k = \omega_n + \omega_2 - \omega_1$

In this case it follows from $-\omega_k = \omega_n + \omega_2 - \omega_1$ that

$$-\frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} = \frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} - 1.$$

Now the left-hand side is always negative while the right-hand side is always positive. So, there are no solutions in this case.

The case $-\omega_k = -\omega_n + \omega_2 - \omega_1$

In this case it follows from $-\omega_k = -\omega_n + \omega_2 - \omega_1$ that

$$\frac{n\sqrt{1+\mu^2n^2}}{\sqrt{1+\mu^2}} = \frac{k\sqrt{1+\mu^2k^2}}{\sqrt{1+\mu^2}} + \frac{2\sqrt{1+\mu^22^2}}{\sqrt{1+\mu^2}} - 1.$$

By interchanging n and k we obtain again the first case. So, the only solution in this case is $k = 1$ and $n = 2$.

This completes the proof of the statement at the beginning of this appendix.

Appendix 2

In this appendix we will determine the solution of (21), that is, of:

$$\begin{aligned} \dot{A}_{10} &= -p \sin(\phi t_1) A_{20} + p \cos(\phi t_1) B_{20}, \\ \dot{B}_{10} &= -p \cos(\phi t_1) A_{20} - p \sin(\phi t_1) B_{20}, \\ \dot{A}_{20} &= -q \sin(\phi t_1) A_{10} - q \cos(\phi t_1) B_{10}, \\ \dot{B}_{20} &= q \cos(\phi t_1) A_{10} - q \sin(\phi t_1) B_{10}, \end{aligned} \quad (\text{A-6})$$

where p and q are given by (23).

By differentiating the first and the second equation in (A-6) we obtain

$$\begin{aligned}\ddot{A}_{10} &= -p\phi \cos(\phi t_1)A_{20} - p\sin(\phi t_1)\dot{A}_{20} - p\phi \sin(\phi t_1)B_{20} + p\cos(\phi t_1)\dot{B}_{20} \\ &= \phi[-p\cos(\phi t_1)A_{20} - p\sin(\phi t_1)] - p\sin(\phi t_1)[-q\sin(\phi t_1)A_{10} - \\ &\quad q\cos(\phi t_1)B_{10}] + p\cos(\phi t_1)[q\cos(\phi t_1)A_{10} - q\sin(\phi t_1)B_{10}] \\ &= \phi\dot{B}_{10} + pqA_{10}, \text{ and} \end{aligned} \tag{A-7}$$

$$\ddot{B}_{10} = -\phi\dot{A}_{10} + pqB_{10}. \tag{A-8}$$

By differentiating (A-7) and using (A-8), we obtain:

$$A_{10}^{(3)} - pq\dot{A}_{10} = \phi\ddot{B}_{10} = -\phi^2\dot{A}_{10} + pq\phi B_{10}, \tag{A-9}$$

and finally by differentiating (A-9) and using (A-7) we obtain

$$A_{10}^{(4)} + (\phi^2 - 2pq)\ddot{A}_{10} + (pq)^2 A_{10} = 0. \tag{A-10}$$

The characteristic equation corresponding to (A-10) is $r^4 + (\phi^2 - 2pq)r^2 + (pq)^2 = 0$, which has as solutions $r_1 = \sqrt{\frac{1}{2}[2pq - \phi^2 + \sqrt{D}]}$, $r_2 = \sqrt{\frac{1}{2}[2pq - \phi^2 - \sqrt{D}]}$, $r_3 = -r_1$, and $r_4 = -r_2$ and where $D = \phi^4 - 4pq\phi^2$. Since p and q are of opposite sign it follows that $\phi^4 - 4pq\phi^2 > 0$ and $2pq - \phi^2 < 0$. Therefore, r_2 and r_4 are purely imaginary. And, since $\phi^2 - 2pq = \sqrt{(\phi^2 - 2pq)^2} = \sqrt{\phi^4 - 4pq\phi^2 + 4p^2q^2} > \sqrt{\phi^4 - 4pq\phi^2}$ it follows that $|2pq - \phi^2| > \sqrt{\phi^4 - 4pq\phi^2}$. Accordingly r_1 and r_3 are also purely imaginary. So, all the solutions of the characteristic equation can be written in the form $r_1 = \beta_1 i$, $r_2 = \beta_2 i$, $r_3 = -r_1$, and $r_4 = -r_2$, where $\beta_1 = \sqrt{\frac{1}{2}[\phi^2 - 2pq - \sqrt{\phi^4 - 4pq\phi^2}]}$ and $\beta_2 = \sqrt{\frac{1}{2}[\phi^2 - 2pq + \sqrt{\phi^4 - 4pq\phi^2}]}$. The solution of (A-10) now becomes:

$$A_{10}(t_1) = K_1 \sin(\beta_1 t_1) + K_2 \cos(\beta_1 t_1) + K_3 \sin(\beta_2 t_1) + K_4 \cos(\beta_2 t_1),$$

where K_1, K_2, K_3 , and K_4 are constants of integration.

From (A-9) $B_{10}(t_1)$ can be derived, yielding

$$B_{10}(t_1) = \frac{1}{pq\phi}[A_{10}^{(3)} + (\phi^2 - pq)\dot{A}_{10}]; \quad \phi \neq 0.$$

From the first two equations in (A-6), A_{20} and B_{20} can now readily be determined, yielding

$$\begin{aligned}A_{20}(t_1) &= \frac{-1}{p}[\dot{A}_{10} \sin(\phi t_1) + \dot{B}_{10} \cos(\phi t_1), \\ B_{20}(t_1) &= \frac{1}{p}[\dot{A}_{10} \cos(\phi t_1) - \dot{B}_{10} \sin(\phi t_1)].\end{aligned}$$

So, the solutions of (21) has been derived.

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