A MODEL OF TURBULENT DIFFUSION

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1. INTRODUCTION

The study of diffusion of marked fluid particles by a random, continuous velocity field has revealed a number of results, some of which are summarized here to examine the merits of a model of turbulent diffusion proposed in a subsequent section.

The variance (or "dispersion") $\sigma(t)$ of the displacements of particles released at $t = 0$ in homogeneous, isotropic turbulence is given by Taylor's expression [1],

$$\sigma(t) = \overline{y^2(t)} = 2 \overline{v^2} \int_0^t (t - t') R_L(t') dt' \quad (1)$$

in which $y = y_1, y_2$ or y_3 is a particle displacement component, and $v = v_1, v_2$ or v_3 is a particle velocity component. $R_L(t')$ is the Lagrangian auto-correlation function,

$$\overline{v^2} R_L(t') = \overline{v(t)v(t+t')}$$

The mean velocity is chosen equal to zero. The Lagrangian integral time-scale is given by

$$T_L = \int_0^\infty R_L(t') dt'$$

As regards the distribution of the mean concentration $\bar{\Gamma}$ of marked particles, which is equal to the probability density function (pdf) of the particle displacement if molecular diffusion is ignored, the following results obtained by Batchelor [2] are of interest. For diffusion times which are small with respect to T_L , we have in the case of instantaneous release at $\underline{x} = \underline{0}$, $t = 0$ in an n -dimensional situation

$$\bar{\Gamma}(\underline{x}, t) \sim \frac{1}{t^n} f_u\left(\frac{\underline{x}}{t}\right) \quad (2)$$

in which f_u is the pdf of the Eulerian velocity field. For large diffusion times ($t \gg T_L$) the application of an extended central limit theorem

[3, p. 541] shows that the distribution of $\bar{\Gamma}$ becomes normal and satisfies the classical diffusion equation

$$\frac{\partial \bar{\Gamma}}{\partial t} = \epsilon \nabla^2 \bar{\Gamma} \quad (3)$$

in which the diffusivity $\epsilon = \overline{v^2} \bar{T}_L$.

Existing models of turbulent diffusion set up to provide information also for intermediate diffusion times, are based on (i) the assumption of a normal distribution of $\bar{\Gamma}$ for all diffusion times if f_u is normal (Batchelor [2]), or (ii) on random walk analogies (Corrsin in [4], Patterson in [4]), or (iii) on an equation which is in its most general form

$$\frac{\partial \bar{\Gamma}}{\partial t} = \nabla^2 \int_0^t d\underline{x}' dt' G(\underline{x} - \underline{x}', t - t') \bar{\Gamma}(\underline{x}', t') \quad (4)$$

in which G is a weighting function. Equations of this type are considered by Roberts [5], Schönfeld [6], and Kraichnan [7]. A simpler form, in which $G(\underline{x}, t) = \delta^n(\underline{x}) R_L(t)$, is examined by Bourret [8], Saffman [9], and recently by Thiebaux [10]. Further simplification of (4) is obtained if moreover $R_L = \exp(-t/T_L)$ is assumed. (4) then reduces to the telegraph equation (Corrsin in [4], Monin and Yaglom [3, p. 676]). Several authors, e.g. Roberts, show, however, that (4) is not in agreement with (2), so that the accuracy for intermediate diffusion times can also be questioned.

The model equation given by Philip [11] does satisfy (2), but seems to be derived on a rather hypothetical basis.

In the present report a model equation for the mean concentration at all diffusion times is derived, which satisfies (1), (2) and (3). The input of the model equation is formed by the pdf of the Eulerian velocities and the Lagrangian auto-correlation function. Although the model is continuous, it shows resemblance with a certain type of random walk model. It predicts an almost normal distribution of the mean concentration for all diffusion times if f_u is normal, which is in agreement with experimental evidence [2], [12], [13]. The model approach is applied also to the concentration fluctuation problem. A mechanism is introduced which accounts for the decay of the fluctuations caused by molecular diffusion. The results

support certain assumptions for large diffusion times made in the existing literature.

2. THE MEAN CONCENTRATION

2.1. Model equation

Starting from the relationship between the velocity of a fluid particle and the particle displacements, i.e.

$$\frac{d\mathbf{y}}{dt} = \mathbf{v}$$

we obtain as a displacement of the particle from a time $t - \tau$ to a time t

$$\mathbf{y}(t) = \mathbf{y}(t - \tau) + \int_{t-\tau}^t \mathbf{v} dt'$$

We consider the particular fluid particle passing through a point $\mathbf{x}$ of the velocity field at a time t . Therefore, we write $\mathbf{v} = \mathbf{v}(t'|\mathbf{x}, t)$, in which the time t' denotes the time dependence of the velocity of the fluid particle considered. The above equation can now be written as

$$\mathbf{y}(t - \tau) = \mathbf{x} - \int_{t-\tau}^t \mathbf{v}(t'|\mathbf{x}, t) dt'$$

In the absence of molecular diffusion the concentration Γ of a scalar quantity remains constant along a particle trajectory. Consequently, we may write

$$\Gamma(\mathbf{x}, t) = \Gamma\left(\mathbf{x} - \int_{t-\tau}^t \mathbf{v}(t'|\mathbf{x}, t) dt', t - \tau\right) \quad (5)$$

The derivation of an equation for the mean concentration requires ensemble averaging of (5). The appearance of $\mathbf{v}$ as an integrand, however, makes the derivation impossible. In order to remove this difficulty we apply the mean-value theorem to the integral in (5):

$$\int_{t-\tau}^t \underline{v}(t' | \underline{x}, t) dt' = \tau \underline{\tilde{v}}$$

This expression defines the velocity $\underline{\tilde{v}}$ as a function of $\underline{x}$, t and τ .

From a geometrical view point $\tau \underline{\tilde{v}}$ is the vector connecting the position of the particle at time $t - \tau$ with that at time t , see Fig. 1a. The velocity $\underline{\tilde{v}}$ is a rather awkward variable. Little about it is known, because it depends

Fig. 1. Approximations as to particle displacements,
a. arbitrary time interval τ , b. time interval τ
determined by the tangent to the particle trajectory.

on the unspecified time interval τ . We therefore attempt to choose this time interval such, that the vector mentioned changes into the tangent to the particle trajectory at the point $\underline{x}$. We then have $\underline{\tilde{v}} = \underline{u}(\underline{x}, t)$, and (see Fig. 1b)

$$\int_{t-\tau}^t \underline{v}(t' | \underline{x}, t) dt' = \tau \underline{u}(\underline{x}, t) \quad (6)$$

in which $\underline{u}(\underline{x}, t)$ is the (Eulerian) velocity at the point $\underline{x}$ of the particle trajectory. This choice requires the tangent to intersect the particle trajectory at a point $\underline{x} - \tau \underline{u}(\underline{x}, t)$. Obviously this is true, or approxi-

mately true, only in a limited fraction of all possible realizations of the process, particularly in three-dimensional situations. Our basic assumption is that this (possibly small) subset of realizations can be seen as being representative of the random process. More specifically the assumption implies that subensemble and ensemble would have the same pdf for a random variable, like the Eulerian velocity for instance. A modification as regards the time interval τ is introduced in the next section.

The time interval τ varies in an unpredictable way from one realization to another; it is therefore likely that τ can also be conceived of as a random variable. Since a deterministic relationship between τ and $\underline{u}$ does not exist, we assume that a pdf according to

$$\text{Pr}\{\tau' < \tau < \tau' + d\tau', \underline{u}' < \underline{u} < \underline{u}' + d\underline{u}'\} = f(\tau', \underline{u}') d\tau' d\underline{u}' \quad (7)$$

may be introduced. A prime denotes a random variable in "probability space".

On substitution from (6) ensemble averaging of (5) can be carried out. Since the fluid particles are marked at $t = 0$, no marked particle is found at the point of intersection when $\tau > t$. We would then have $\Gamma(\underline{x}, t) \neq \Gamma(\underline{x} - \tau \underline{u}(\underline{x}, t), t - \tau)$, since the latter expression vanishes. The principle of causality precludes negative values of τ . Therefore, we must restrict ourselves to realizations where $0 \leq \tau \leq t$. Averaging then gives (the primes have been dropped)

$$\int_0^t \int d\tau d\underline{u} f(\tau, \underline{u}) [\Gamma(\underline{x}, t; \tau, \underline{u}) - \Gamma(\underline{x} - \tau \underline{u}, t - \tau; \tau, \underline{u})] = 0 \quad (8)$$

The integral with respect to $\underline{u}$ includes all possible values. The dependence of Γ on the realizations is represented by the random variables τ and $\underline{u}$ in the arguments. By definition, the mean concentration is given by

$$\bar{\Gamma}(\underline{x}, t) = \frac{\int_0^t \int d\tau d\underline{u} f(\tau, \underline{u}) \Gamma(\underline{x}, t; \tau, \underline{u})}{\int_0^t \int d\tau d\underline{u} f(\tau, \underline{u})}$$

The denominator is not unity, but a function of time. (8) becomes

$$\bar{\Gamma}(\underline{x}, t) = \frac{\int_0^t \int d\tau \, d\underline{u} \, f(\tau, \underline{u}) \Gamma(\underline{x} - \tau \underline{u}, t - \tau; \tau, \underline{u})}{\int_0^t d\tau \, f_{\tau}(\tau)}$$

in which f_{τ} is the pdf of τ ,

$$f_{\tau}(\tau) = \int d\underline{u} \, f(\tau, \underline{u})$$

We now write

$$\Gamma(\underline{x} - \tau \underline{u}, t - \tau; \tau, \underline{u}) = \bar{\Gamma}(\underline{x} - \tau \underline{u}, t - \tau) + \gamma(\underline{x} - \tau \underline{u}, t - \tau; \tau, \underline{u})$$

and neglect the integral

$$\int_0^t \int d\tau \, d\underline{u} \, f(\tau, \underline{u}) \gamma(\underline{x} - \tau \underline{u}, t - \tau; \tau, \underline{u})$$

This does not seem unreasonable, since γ takes on positive as well as negative values, and $\bar{\gamma} = 0$. The model equation for the mean concentration then becomes

$$\bar{\Gamma}(\underline{x}, t) = \frac{\int_0^t \int d\tau \, d\underline{u} \, f(\tau, \underline{u}) \bar{\Gamma}(\underline{x} - \tau \underline{u}, t - \tau)}{\int_0^t d\tau \, f_{\tau}(\tau)} \quad (9)$$

(9) gives $\bar{\Gamma}(\underline{x}, t)$ as a weighted average of values of $\bar{\Gamma}$ in the past and at various positions. There is some analogy with (4).

(9) should satisfy the conservation law

$$\int d\underline{x} \, \bar{\Gamma}(\underline{x}, t) = \text{constant} \quad (10)$$

Integrating (9) over all space, and introducing a dummy variable $\underline{z} = \underline{x} - \tau \underline{u}$ in the integral with respect to $\underline{x}$ on the R.H.S. of (9), yields

$$\int d\underline{x} \bar{\Gamma}(\underline{x}, t) = \int d\underline{z} \bar{\Gamma}(\underline{z}, t - \tau)$$

which shows that (10) is satisfied. Consequently, (9) describes the turbulent diffusion of a certain amount of the scalar quantity Γ released at $t = 0$. The spatial distribution at $t = 0$ is arbitrary, but only the case of local release (at $\underline{x} = \underline{0}$) is considered.

In isotropic turbulence the time interval τ does not depend (in the statistical sense) upon the direction of the tangent at point $\underline{x}$. In this case τ and $\underline{u}$ could be assumed to be independent random variables, so that $f(\tau, \underline{u}) = f_{\tau}(\tau) f_{\underline{u}}(\underline{u})$. (9) may then be written as

$$\int_0^t d\tau f_{\tau}(\tau) \left[\bar{\Gamma}(\underline{x}, t) - \int d\underline{u} f_{\underline{u}}(\underline{u}) \bar{\Gamma}(\underline{x} - \tau \underline{u}, t - \tau) \right] = 0 \quad (11)$$

If $f_{\tau}(\tau) = \delta(\tau - h)$ is substituted into (11), the model equation derived simplifies to an equation describing a random walk process (with time step h) in which the particle velocities at different time steps are independent (a Markov process). This equation is

$$\bar{\Gamma}(\underline{x}, t) = \int d\underline{u} f_{\underline{u}}(\underline{u}) \bar{\Gamma}(\underline{x} - h \underline{u}, t - h)$$

which is well known from the theory of Brownian motion. It is discussed for instance by Corrsin in [4], where further references are given.

In the next section it is shown that $f_{\tau}(\tau)$ is completely determined by (1). In turbulence f_{τ} is certainly not a delta function.

2.2. Properties of the model equation in the case of isotropic turbulence

In this section we discuss some implications of equation (11) for isotropic turbulence. Some observations regarding nonisotropy are given in Section 2.3. First we consider the function $f_{\tau}(\tau)$ occurring in (11). This equation shows that f_{τ} may contain an arbitrary constant factor. Since it is a pdf, it should satisfy the condition

$$f_{\tau}(\tau) \geq 0$$

for all $\tau \geq 0$. It is likely that time intervals $\tau \gg T_L$ have small probability, and that

$$\lim_{\tau \rightarrow \infty} f_{\tau}(\tau) = 0 \quad (12)$$

Very small time intervals will be possible if the point $\underline{x}$ is a point of inflection of the particle trajectory, or if it is a nodal point of this trajectory. The probabilities of these events are also small. We expect

$$\lim_{\tau \rightarrow 0} f_{\tau}(\tau) = 0 \quad (13)$$

Equation (11) must satisfy Taylor's relationship (1). Multiplying (11) by x^2 (where $x = x_1, x_2$ or x_3) and integrating over the whole domain gives an expression in which the second moment $\sigma(t)$ emerges, i.e.

$$\int_0^t d\tau f_{\tau}(\tau) [\sigma(t) - \sigma(t - \tau) - \overline{u^2} \tau^2] = 0 \quad (14)$$

The second and third terms in brackets arise by again putting $\underline{z} = \underline{x} - \underline{u}\tau$ in the integral with respect to $\underline{x}$ (last term in (11)), taking into account that $\underline{u} = \underline{0}$. If the Lagrangian auto-correlation function $R_L(t)$ is known, then the variance $\sigma(t)$ is determined by (1), and (14) is an integral equation (of Volterra type), the solution of which yields the unknown pdf $f_{\tau}(\tau)$ (besides the constant factor). In this way the direct derivation of the function $f_{\tau}(\tau)$ from the positions of the points of intersection in the various realizations is evaded. Essentially this means that the error introduced by the choice of the particular subset of realizations is more or less corrected by the requirement that (1) be satisfied.

Nevertheless, it can be shown that (12) and (13) are still true. For $t \gg T_L$ we have $\sigma(t) \sim 2\overline{u^2} T_L t$, so that (14) changes to

$$\int_0^{\infty} d\tau f_{\tau}(\tau) (2 T_L \tau - \tau^2) = 0 \quad (15)$$

which implies that (12) is satisfied. For a certain class of correlation functions $R_L(t)$ we find that (see App. A)

$$f_\tau(\tau) \approx \frac{C_1}{\tau^2} R_L(\tau) \quad \text{if} \quad \tau \gg T_L$$

C_1 is an arbitrary positive constant. This expression indicates that the condition $f_\tau(\tau) \geq 0$ would require R_L to take on nonnegative values for large τ . However, a closer examination shows that, because of the memory effect in (14), also oscillations of R_L about zero are allowed, provided the period of oscillation is of order T_L or less. If $t \ll T_L$, we may write

$$R_L(t) \approx 1 - a t^m$$

In general, m will be equal to two; a is a positive constant. In app. A it is shown that

$$f_\tau(\tau) \approx C_2 \exp \left[- \frac{(m+1)(m+2)}{ma \tau^m} \right] \quad \text{if} \quad \tau \ll T_L$$

Consequently, (13) is also satisfied. From the small time behaviour of $f_\tau(\tau)$ it follows that for an arbitrary function $g(\tau)$ satisfying a condition given in the Appendix, we have

$$\int_0^t d\tau f_\tau(\tau) g(\tau) \approx g(t) \int_0^t d\tau f_\tau(\tau) \quad \text{if} \quad t \ll T_L \quad (16)$$

Figs. 2 and 3 show numerical solutions of (14) for two rather arbitrary choices of $R_L(t)$. The curves denoted by $b_h(t)$ are commented on in

Fig. 2. Functions $f_\tau(t)$ and $b_4(t)$ in the case where $R_L(t) = \exp(-t/T_L)$.

Fig. 3. Functions $f_\tau(t)$ and $b_4(t)$ in the case where $R_L(t) = 1/[1 + (2t/3 T_L)^2]^{5/2}$.

Section 2.4. The exponential correlation function is suggested by the experiments of Kalinske and Pien [15]. The correlation function of Fig. 3 is perhaps more realistic; it is a crude approximation of the experimental result obtained by Shlien and Corrsin [13], and a calculation by Saffman [16]. The scales of the functions f_t in the figures are arbitrary.

We now proceed by considering the small time and large time behaviours of $\bar{\Gamma}(\underline{x}, t)$, as predicted by (11). For small times we apply (16) to (11). This gives

$$\bar{\Gamma}(\underline{x}, t) - \int d\underline{u} f_u(\underline{u}) \bar{\Gamma}(\underline{x} - t\underline{u}, 0) = 0$$

Put for an n-dimensional space

$$\bar{\Gamma}(\underline{x}, 0) = \delta^n(\underline{x})$$

We then obtain ($\underline{z} = t\underline{u}$)

$$\bar{\Gamma}(\underline{x}, t) = \frac{1}{t^n} \int d\underline{z} f_u\left(\frac{\underline{z}}{t}\right) \delta^n(\underline{x} - \underline{z}) = \frac{1}{t^n} f_u\left(\frac{\underline{x}}{t}\right)$$

which is also the exact result (2). This result could be anticipated, since the "tangent approximation" becomes exact for t tending to zero. For large diffusion times we have $\tau \ll t$. Consequently, the large time behaviour can be obtained by carrying out a Taylor expansion of $\bar{\Gamma}(\underline{x} - \tau\underline{u}, t - \tau)$ with respect to $(\underline{x}, t)$,

$$\bar{\Gamma}(\underline{x} - \tau\underline{u}, t - \tau) = \sum_{k=0}^{\infty} \frac{(-\tau)^k}{k!} \left(\frac{\partial}{\partial t} + \underline{u} \cdot \nabla \right)^k \bar{\Gamma}(\underline{x}, t)$$

Retaining only the lowest order terms of interest gives, since $\bar{\underline{u}} = \underline{0}$,

$$\int_0^{\infty} d\tau f_{\tau}(\tau) \left[\tau \frac{\partial \bar{\Gamma}}{\partial t} - \frac{1}{2} \tau^2 \bar{u}^2 \nabla^2 \bar{\Gamma} \right] = 0$$

Substituting from (15), the correct result (3) is recovered. If the above series expansion is required to converge, $R_L(t)$ should be equal to zero for t greater than some $T_0 > T_L$, see App. B. However, there is no need to

impose convergence; it is sufficient to require that the truncated series expansion be an asymptotic approximation for t tending to infinity. In App. B the weaker condition of a finite time-scale T_L is then found.

We conclude this section with a comment on a propagation property of the model equation derived. If

$$f_u(\underline{u}) = \delta^n(\underline{u} - \underline{u}_0)$$

in which $\underline{u}_0$ is constant, the mean concentration field will only translate with a velocity $\underline{u}_0$. Consequently

$$\bar{\Gamma}(\underline{x}, t) = \bar{\Gamma}(\underline{x} - \underline{u}_0 t)$$

Substituting these two relationships into (11) shows the model equation to possess this property. An arbitrary pdf $f_u(\underline{u})$ can be conceived of as a "comb" of delta functions. If $\underline{u}$ is bounded, $|\underline{u}| < u_0$, say, the above result indicates that (11) will give

$$\bar{\Gamma}(\underline{x}, t) = 0 \quad \text{if} \quad |\underline{x}| \geq u_0 t$$

as required. Philip [11] bases an error estimation on the departure of his model equation from this property, and meets with unsatisfactory behaviour at intermediate diffusion times. Roberts [5], for instance, shows that the structure of (4) implies a finite velocity of propagation $\sim \sqrt{u^2}$, irrespective of the value of u_0 .

2.3. Extension to the case of homogeneous, nonisotropic turbulence

In the case of nonisotropy, independence of the variables $\underline{u}$ and τ may no longer be assumed, so that Eq. (9) applies. For small diffusion times, however, we find, using the method of Section 2.2,

$$\bar{\Gamma}(\underline{x}, t) \approx \frac{\int_0^t \int d\tau d\underline{u} f(\tau, \underline{u}) \bar{\Gamma}(\underline{x} - \tau \underline{u}, 0)}{\int_0^t d\tau f_\tau(\tau)}$$

Since $\bar{\Gamma}(\underline{x}, 0) = \delta^n(\underline{x})$, we obtain

$$\bar{\Gamma}(\underline{x}, t) \approx \frac{1}{t^n} \frac{\int_0^t d\tau f(\tau, \underline{x}/t)}{\int_0^t d\tau f_\tau(\tau)}$$

Thus it is found that, in order to recover the exact result (2) for small diffusion times, τ and $\underline{u}$ must be assumed to be independent variables (consequently $f(\tau, \underline{x}/t) = f_\tau(\tau) f_u(\underline{x}/t)$) for τ tending to zero. This requirement is related to the fact that the particle trajectories are approximately rectilinear for small diffusion times.

The variance of the particle displacements now is a second order tensor:

$$\sigma_{ij}(t) = \frac{\int d\underline{x} x_i x_j \bar{\Gamma}(\underline{x}, t)}{\int d\underline{x} \bar{\Gamma}(\underline{x}, t)}$$

Multiplying (9) by $x_i x_j$ and integrating over all space gives the equivalent of (14),

$$\int_0^t \int d\tau d\underline{u} f(\tau, \underline{u}) \left[\sigma_{ij}(t) - \sigma_{ij}(t - \tau) - u_i u_j \tau^2 \right] = 0 \quad (17)$$

However, although Eqs. (17) impose conditions on the pdf $f(\tau, \underline{u})$, these equations do not provide a sufficient means of expressing $f(\tau, \underline{u})$ in Lagrangian properties of the (nonisotropic) turbulence. Additional assumptions as to the dependence of $f(\tau, \underline{u})$ on τ and $\underline{u}$ are required to make further progress possible.

Taylor's expression for nonisotropic turbulence reads, see for instance [12, p.309],

$$\sigma_{ij}(t) = \overline{y_i y_j} = \int_0^t d\tau (t - \tau) [\overline{v_i(t') v_j(t' + \tau)} + \overline{v_j(t') v_i(t' + \tau)}] \quad (18)$$

For large diffusion times (18) changes to

$$\sigma_{ij}(t) = 2 \varepsilon_{ij} t$$

in which the diffusion tensor

$$\varepsilon_{ij} = \frac{1}{2} \int_0^\infty d\tau [\overline{v_i(t') v_j(t' + \tau)} + \overline{v_j(t') v_i(t' + \tau)}] \quad (19)$$

For large diffusion times (17) changes to

$$\int_0^\infty d\tau \int d\underline{u} f(\tau, \underline{u}) (2 \varepsilon_{ij} \tau - u_i u_j \tau^2) = 0 \quad (20)$$

A diffusion equation for large diffusion times can be obtained by expanding $\overline{\Gamma}(\underline{x} - \underline{u} \tau, t - \tau)$ in a Taylor series, which yields

$$\int_0^\infty d\tau \int d\underline{u} f(\tau, \underline{u}) \left[\tau \frac{\partial \overline{\Gamma}}{\partial t} + \tau u_i \frac{\partial \overline{\Gamma}}{\partial x_i} - \frac{1}{2} \tau^2 u_i u_j \frac{\partial^2 \overline{\Gamma}}{\partial x_i \partial x_j} + \dots \right] = 0$$

Together with (20) and retaining only the lowest order terms we obtain, since $\underline{\overline{u}} = 0$,

$$\frac{\partial \overline{\Gamma}}{\partial t} = \varepsilon_{ij} \frac{\partial^2 \overline{\Gamma}}{\partial x_i \partial x_j}$$

which is the classical diffusion equation in the case of nonisotropy.

2.4. Two applications: normal velocity distribution, and homogeneous shear flow

In the first application we consider the case where the velocity is normally distributed, and examine the distribution of the mean concentration $\bar{\Gamma}$. For simplicity we restrict ourselves to one-dimensional diffusion. The result obtained is also applicable to continuous release from a line source if the Taylor hypothesis applies ("slender plume" approximation, see for instance [12, p.40 and p.330] and [14]). Since we expect $\bar{\Gamma}$ to be, at least approximately, normally distributed in the case under consideration [2], we expand $\bar{\Gamma}(x,t)$ in a so-called Gram-Charlier series, see for instance [17],

$$\bar{\Gamma}(x,t) = \sum_{k=0}^{\infty} b_{2k}(t) \sigma^k(t) \frac{\partial^{2k} N(x,t)}{\partial x^{2k}} \quad (21)$$

in which $b_0(t) = 1$,

$$N(x,t) = \frac{\exp [-x^2/2 \sigma(t)]}{\sqrt{2\pi \sigma(t)}}$$

and $\sigma(t) = \overline{y^2(t)}$. Using (21), the moments of $\bar{\Gamma}(x,t)$ become

$$\int_{-\infty}^{\infty} \bar{\Gamma}(x,t) dx = 1$$

$$\int_{-\infty}^{\infty} x^2 \bar{\Gamma}(x,t) dx = [1 + 2 b_2(t)] \sigma(t)$$

$$\int_{-\infty}^{\infty} x^4 \bar{\Gamma}(x,t) dx = [3 + 12 b_2(t) + 24 b_4(t)] \sigma^2(t)$$

etc. We successively determine the coefficients $b_2, b_4, \dots$ by putting the second, fourth, ... moment of the L.H.S. of (11) equal to zero. On substitution the last term on the L.H.S. gives expressions of the form ($k = 0, 1, \dots$)

$$b_{2k}(t - \tau) \sigma^k(t - \tau) \frac{\partial^{2k}}{\partial x^{2k}} \frac{\exp \left[-\frac{1}{2} x^2 / \{ \sigma(t - \tau) + \overline{u^2} \tau^2 \} \right]}{\sqrt{2\pi [\sigma(t - \tau) + \overline{u^2} \tau^2]}}$$

The second moment of (11) then becomes

$$\int_0^t d\tau f_\tau(\tau) [\{1 + 2 b_2(t)\} \sigma(t) - \sigma(t - \tau) - \overline{u^2} \tau^2 - 2 b_2(t - \tau) \cdot \sigma(t - \tau)] = 0$$

or, with (14)

$$\int_0^t d\tau f_\tau(\tau) [b_2(t) \sigma(t) - b_2(t - \tau) \sigma(t - \tau)] = 0$$

which equation is satisfied by

$$b_2(t) = 0 \quad (22)$$

Consequently, the second term in (21) vanishes. The fourth moment of (11) then is

$$\int_0^t d\tau f_\tau(\tau) [\{3 + 24 b_4(t)\} \sigma^2(t) - 3 \{ \sigma(t - \tau) + \overline{u^2} \tau^2 \}^2 + \\ - 24 b_4(t - \tau) \sigma^2(t - \tau)] = 0$$

or, again using (14)

$$\int_0^t d\tau f_\tau(\tau) [b_4(t) \sigma^2(t) - b_4(t - \tau) \sigma^2(t - \tau) + \\ - \frac{1}{8} \{ \sigma(t) - \sigma(t - \tau) - \overline{u^2} \tau^2 \}^2] = 0 \quad (23)$$

and so on for higher moments. (23) is an integral equation from which $b_4(t)$ can be determined; this coefficient does not vanish. In Figs. 2

and 3 numerical results are shown for the two auto-correlation functions mentioned. The maximum deviation from the normal distribution is shown in Fig. 4 for one of the auto-correlations. Fourth and further terms in (21) were neglected. The maximum deviation occurring at $x = 0$ amounts to 3.7 percent in the case of the exponential auto-correlation, and to 2.3 percent in the case of the transcendental function. These maxima occur at diffusion times of $3.9 T_L$ and $3.1 T_L$, respectively.

An approximately normal distribution of the mean concentration is in agreement with observations in homogeneous turbulence, in which the pdf of the velocities is practically normal [2], [12], [13], [14], [15].

Fig. 4. Maximum deviation from a normal distribution of the mean concentration.

As a second application, we consider turbulent diffusion in the simplest shear flow, i.e. a flow with a constant gradient of one of the mean velocity components, e.g. $\bar{u}_1$. The remaining components of the mean

velocity may be put equal to zero without loss of generality. Consequently, see Fig. 5.,

$$\underline{\bar{u}} = \left(\frac{d\bar{u}_1}{dx_2} x_2, 0, 0 \right) \quad (24)$$

Fig. 5. Mean velocity profile in homogeneous shear flow.

The system of coordinates has been so chosen that $\bar{u}_1(x_1, 0, x_3) = 0$. Eulerian properties of the turbulence are assumed to be homogeneous and stationary. In this case Eq. (9) can be written as

$$\int_0^t \int d\tau d\underline{u} f(\tau, \underline{u} - \underline{\bar{u}}) [\bar{\Gamma}(\underline{x}, t) - \bar{\Gamma}(\underline{x} - \underline{u}\tau, t - \tau)] = 0$$

Introducing the velocity fluctuation $\underline{u}' = \underline{u} - \underline{\bar{u}}$ yields

$$\int_0^t \int d\tau d\underline{u}' f(\tau, \underline{u}') [\bar{\Gamma}(\underline{x}, t) - \bar{\Gamma}(\underline{x} - (\underline{\bar{u}} + \underline{u}')\tau, t - \tau)] = 0 \quad (25)$$

It is easily verified that this equation satisfies the conservation law (10).

Equations for the variances $\sigma_{ij}(t)$ of the particle displacements can be obtained by multiplying (25) by $x_i x_j$ and integrating over all space. These equations are

$$\begin{aligned} \int_0^t \int \underline{d}\underline{u}' f(\tau, \underline{u}') [\sigma_{11}(t) - \sigma_{11}(t - \tau) + \left(\frac{\bar{d}u_1}{dx_2}\right)^2 \tau^2 \sigma_{22}(t) + \\ - 2 \frac{\bar{d}u_1}{dx_2} \tau \sigma_{12}(t) - u_1'^2 \tau^2] = 0 \end{aligned} \quad (26)$$

$$\int_0^t \int \underline{d}\underline{u}' f(\tau, \underline{u}') [\sigma_{12}(t) - \sigma_{12}(t - \tau) - \frac{\bar{d}u_1}{dx_2} \tau \sigma_{22}(t) - u_1' u_2' \tau^2] = 0$$

$$\int_0^t \int \underline{d}\underline{u}' f(\tau, \underline{u}') [\sigma_{22}(t) - \sigma_{22}(t - \tau) - u_2'^2 \tau^2] = 0$$

The equation for σ_{33} is trivial, furthermore $\sigma_{23} = \sigma_{13} = 0$. Eqs. (26) can be compared with results of a Lagrangian analysis, as shown in Sections 2.2 and 2.3. The Lagrangian results for the present case were derived by Corrsin, see [18] and [3], but are too complicated to reproduce here. However, tractable results are obtained for large diffusion times. Corrsin's expressions then simplify to

$$\begin{aligned} \sigma_{11}(t) &\sim \frac{2}{3} \left(\frac{\bar{d}u_1}{dx_2}\right)^2 t^3 \int_0^\infty d\tau \overline{v_2(t') v_2(t' + \tau)} \\ \sigma_{12}(t) &\sim \frac{\bar{d}u_1}{dx_2} t^2 \int_0^\infty d\tau \overline{v_2(t') v_2(t' + \tau)} \\ \sigma_{22}(t) &\sim 2t \int_0^\infty d\tau \overline{v_2(t') v_2(t' + \tau)} \end{aligned} \quad (27)$$

in which v_2 = Lagrangian velocity component in x_2 - direction. A remarkable conclusion which can be drawn from Eqs. (27) is that for large t

$$\frac{\sigma_{12}(t)}{\sqrt{\sigma_{11}(t) \sigma_{22}(t)}} = \frac{\sqrt{3}}{2}$$

Substituting (27) into (26) and retaining only the largest terms yields

$$\begin{aligned} \int_0^t \int d\tau \, d\underline{u}' \, f(\tau, \underline{u}') \left[\frac{2}{3} \left(\frac{d\bar{u}_1}{dx_2} \right)^2 3 \tau t^2 - 2 \frac{d\bar{u}_1}{dx_2} \tau \frac{d\bar{u}_1}{dx_2} t^2 \right] &= 0 \\ \int_0^t \int d\tau \, d\underline{u}' \, f(\tau, \underline{u}') \left[\frac{d\bar{u}_1}{dx_2} 2 \tau t - \frac{d\bar{u}_1}{dx_2} \tau 2t \right] &= 0 \\ \int_0^t \int d\tau \, d\underline{u}' \, f(\tau, \underline{u}') \left[2 \tau \int_0^\infty d\tau' \, \overline{v_2(t') v_2(t' + \tau')} - u_2'^2 \tau^2 \right] &= 0 \end{aligned}$$

The first and second relationships are satisfied identically. The third expression is satisfied because of (19) and (20). Thus it is found that the diffusion model discussed is in agreement with Corrsin's result (27) for large diffusion times.

Applying a Taylor expansion to the last term in (25) again leads to a diffusion equation for large diffusion times. In the case of a line source at the x_3 -axis this equation becomes, provided $|d\bar{u}_1/dx_2|$ is not too small,

$$\frac{\partial \bar{\Gamma}}{\partial t} + \frac{d\bar{u}_1}{dx_2} x_2 \frac{\partial \bar{\Gamma}}{\partial x_1} = \epsilon_{22} \frac{\partial^2 \bar{\Gamma}}{\partial x_2^2} \quad (28)$$

in which ϵ_{22} is given by (19) and (20). The second moments of (28) also yield (27).

3. THE CONCENTRATION FLUCTUATIONS

In this section we derive an equation for the variance $\overline{\gamma^2} = \overline{(\Gamma - \bar{\Gamma})^2}$ of the concentration, which is based on the approximations discussed in Section 2.1. Before doing so, a comment must be made concerning an effect of molecular diffusion. The transport equation for γ^2 reads, in isotropic turbulence with $\bar{\underline{u}} = 0$,

$$\frac{\partial \overline{\gamma^2}}{\partial t} + 2 \overline{u_i \gamma} \frac{\partial \overline{\gamma}}{\partial x_i} + \frac{\partial}{\partial x_i} (\overline{u_i \gamma^2} - \kappa \frac{\partial \overline{\gamma^2}}{\partial x_i}) + 2 \kappa \frac{\partial \overline{\gamma}}{\partial x_i} \frac{\partial \overline{\gamma}}{\partial x_i} = 0 \quad (29)$$

in which κ = molecular diffusivity. The third term in this equation represents turbulent and molecular diffusion; the last term has the character of a dissipation, since it is always positive. In contrast with the molecular diffusion term, the dissipation term has no turbulent counterpart. The order of magnitude of this term is [12, p.224]

$$2 \kappa \frac{\partial \overline{\gamma}}{\partial x_i} \frac{\partial \overline{\gamma}}{\partial x_i} \sim \frac{\kappa}{\lambda_Y^2} \overline{\gamma^2} \quad (30)$$

in which λ_Y = micro length scale of fluctuations. This micro scale tends to increase as time elapses owing to molecular diffusion, on the other hand it is decreased in consequence of the stretching of material elements caused by the turbulence [19]. Quantities on which λ_Y depends therefore are: κ , t , $\overline{u^2}$, l_e , ν , in which l_e = size of energy containing eddies [12, p. 184], and ν = viscosity. Dimensional reasoning then gives

$$\lambda_Y^2 = \frac{\kappa}{\sqrt{\overline{u^2}}} l_e F\left(\frac{\sqrt{\overline{u^2}} t}{l_e}, Re_1, Pr\right) \quad (31)$$

in which $Re_1 = \sqrt{\overline{u^2}} l_e / \nu$ and $Pr = \nu / \kappa$. If we consider, for instance, large diffusion times and large Reynolds numbers of turbulence, λ_Y will become time independent. In this case we obtain

$$\lambda_Y^2 = \frac{\kappa}{\sqrt{\overline{u^2}}} l_e F_1(Re_1, Pr)$$

and

$$2 \kappa \frac{\partial \overline{\gamma}}{\partial x_i} \frac{\partial \overline{\gamma}}{\partial x_i} \sim \overline{\gamma^2} \frac{\sqrt{\overline{u^2}}}{l_e} F_2(Re_1, Pr) \quad (32)$$

Consequently, the dissipation term depends on the ratio of diffusivity and viscosity, and not on the diffusivity separately. Saffman [9], for instance, argues that $F_2 \sim 1$ if $Pr \sim 1$, and $F_2 \sim Re_1^{1/4}$ if $Pr \gg 1$.

Molecular effects cannot be included in a derivation of an equation for the concentration fluctuations based on the approach of Section 2.1. Therefore, we first disregard these effects and afterwards modify the resulting equation to account for dissipation. According to Section 2.1 we may write

$$\Gamma^2(\underline{x}, t; \tau, \underline{u}) = \Gamma^2(\underline{x} - \underline{u}\tau, t - \tau; \tau, \underline{u})$$

Ensemble averaging yields

$$\int_0^t \int d\tau d\underline{u} f(\tau, \underline{u}) [\Gamma^2(\underline{x}, t; \tau, \underline{u}) - \Gamma^2(\underline{x} - \underline{u}\tau, t - \tau; \tau, \underline{u})] = 0 \quad (33)$$

Changing to mean values and neglecting the fluctuation of the last term in (33), as carried out when deriving (9) from (8), yields

$$\int_0^t \int d\tau d\underline{u} f(\tau, \underline{u}) [\overline{\Gamma^2}(\underline{x}, t) - \overline{\Gamma^2}(\underline{x} - \underline{u}\tau, t - \tau)] = 0$$

Since $\overline{\Gamma^2} = \overline{\Gamma}^2 + \overline{\gamma^2}$, this equation may be written as

$$\int_0^t \int d\tau d\underline{u} f(\tau, \underline{u}) \left[\overline{\gamma^2}(\underline{x}, t) - \overline{\gamma^2}(\underline{x} - \underline{u}\tau, t - \tau) + \overline{\Gamma}^2(\underline{x}, t) - \overline{\Gamma}^2(\underline{x} - \underline{u}\tau, t - \tau) \right] = 0 \quad (34)$$

As regards molecular dissipation we assume that the centre of gravity of a marked fluid element follows a trajectory of a fluid particle, along which $\overline{\gamma^2}$ decays owing to dissipation only according to

$$\overline{\gamma^2} \sim \phi(t)$$

$\phi(t)$ is a decreasing function of time, and depends on the parameters mentioned before. Obviously, this assumption leaves molecular diffusion of $\overline{\gamma^2}$ out of consideration. Introducing the assumption concerning the decay of $\overline{\gamma^2}$ in (34) yields as a model equation for the concentration fluctuations

$$\int_0^t \int d\tau d\underline{u} f(\tau, \underline{u}) \left[\overline{\gamma^2}(\underline{x}, t) - \frac{\phi(t)}{\phi(t-\tau)} \overline{\gamma^2}(\underline{x} - \underline{u}\tau, t - \tau) + \right. \\ \left. + \overline{\gamma^2}(\underline{x}, t) - \overline{\gamma^2}(\underline{x} - \underline{u}\tau, t - \tau) \right] = 0 \quad (35)$$

The third and fourth terms in (35) are known source terms, assuming the mean concentration $\overline{\gamma}$ is known.

In principle, $\overline{\gamma^2}$ could be solved from the integral equation (35). As an example of the usefulness of this equation, we consider the relatively simple case of large diffusion times ($t \gg T_L$), and isotropic turbulence. A Taylor expansion of the variables at time $t - \tau$ with respect to time t yields, in first approximation,

$$\frac{\partial \overline{\gamma^2}}{\partial t} - 2 \beta_1 \varepsilon \frac{\partial \overline{\gamma}}{\partial x_i} \frac{\partial \overline{\gamma}}{\partial x_i} - \beta_2 \varepsilon \frac{\partial^2 \overline{\gamma^2}}{\partial x_i \partial x_i} + \phi_1(t) \overline{\gamma^2} = 0 \quad (36)$$

in which $\varepsilon = \overline{u^2} T_L$, and

$$\beta_1 = \frac{\int_0^t d\tau \tau f_\tau(\tau)}{\int_0^t d\tau \tau \psi(t, \tau) f_\tau(\tau)}$$

$$\beta_2 = \frac{1}{2T_L} \frac{\int_0^t d\tau \tau^2 \psi(t, \tau) f_\tau(\tau)}{\int_0^t d\tau \tau \psi(t, \tau) f_\tau(\tau)} \quad , t \gg T_L \quad (37)$$

$$\phi_1(t) = \frac{\int_0^t d\tau [1 - \psi(t, \tau)] f_\tau(\tau)}{\int_0^t d\tau \tau \psi(t, \tau) f_\tau(\tau)}$$

$$\psi(t, \tau) = \frac{\phi(t)}{\phi(t - \tau)}$$

When deriving (36), the orders of magnitude of a number of neglected terms were determined by using (3). If dissipation would not occur, we would have a constant ϕ , so that then $\beta_1 = 1$, $\beta_2 = 1$ (using (15)) and $\phi_1(t) = 0$. Comparing (36) with (29) shows that

$$\begin{aligned} \overline{u_i \gamma} &= -\beta_1 \epsilon \frac{\partial \overline{\Gamma}}{\partial x_i} \\ \overline{u_i \gamma^2} - \kappa \frac{\partial \overline{\gamma^2}}{\partial x_i} &= -\beta_2 \epsilon \frac{\partial \overline{\gamma^2}}{\partial x_i} \quad \text{if } t \gg T_L \\ 2\kappa \frac{\partial \overline{\gamma}}{\partial x_i} \frac{\partial \overline{\gamma}}{\partial x_i} &= \phi_1(t) \overline{\gamma^2} \end{aligned} \quad (38)$$

The first two results are gradient-type approximations. It is notable that the effective turbulent diffusivities are influenced by dissipation because of the coefficients β_1 and β_2 . It can be shown that $\beta_1 > 1$ and $\beta_2 < 1$. The third result is equivalent to (30) together with (31).

As regards the magnitudes of the coefficients β_1 , β_2 and ϕ_1 we may distinguish between three different cases:

1. Decay according to a power law, $\phi(t) = 1/t^{\alpha/2}$. Following [14] an exponent $\alpha/2$ rather than α is introduced. Since $\tau \sim T_L$ we have

$$\psi(t, \tau) = 1 - \frac{\alpha}{2} \frac{\tau}{t} + O(T_L^2/t^2)$$

so that

$$\beta_1 = 1 + O(T_L/t)$$

$$\beta_2 = 1 + O(T_L/t)$$

$$\phi_1(t) = \frac{\alpha}{2t} [1 + O(T_L/t)]$$

2. Exponential decay, $\phi(t) = \exp(-\mu t)$, with $\mu \ll 1/T_L$. In this case $\phi(t, \tau) = \exp(-\mu \tau) = 1 - \mu \tau + O(\mu^2 T_L^2)$. Consequently

$$\beta_1 = 1 + O(\mu T_L)$$

$$\beta_2 = 1 + O(\mu T_L)$$

$$\phi_1 = \mu[1 + O(\mu T_L)]$$

3. Exponential decay, $\phi(t) = \exp(-\mu t)$, with μ of order $1/T_L$ or larger. In this case analytic evaluation of the integrals in (37) is not possible. In the case where $\mu \gg 1/T_L$, however, estimating the integrals yields $\beta_1 \gg \mu^2 T_L^2$, $\beta_2 \sim 1/\mu T_L$ and $\phi_1 \gg \mu^2 T_L$. Thus it is found that for very large μ (excessive dissipation) the second and fourth terms in (36) are predominant. This indicates that the production (second term) then would equal the dissipation (fourth term), which sounds plausible.

The example for large Reynolds numbers of turbulence, represented by (32), requires time independence of ϕ_1 for large diffusion times. The above results show that a constant ϕ_1 will be obtained, if $\phi(t)$ decreases exponentially (case 2 or case 3). Since in (32) $\sqrt{u^2}/l_e \sim 1/T_L$, Saffman's result for $Pr \sim 1$ will be obtained if $\mu \sim 1/T_L$. The result for $Pr \gg 1$ corresponds with the case $\mu \gg 1/T_L$. Case 1 will be appropriate, if the Reynolds number of turbulence is small; the value $\alpha = 3$ then applies for Prandtl numbers of order unity or less [12, p. 231].

Csanady [14] reports good agreement between (36), which was made suitable to stationary diffusion by using Taylor's hypothesis, and experimental evidence. The coefficients β_1 , β_2 and ϕ_1 assumed were those of case 1. A number of experiments yielded values of $\alpha \sim 3$, in other cases considerably larger values (up to about 10) were found. The latter result possibly points to a constant coefficient ϕ_1 .

4. CONCLUSIONS AND EXTENSIONS OF THE MODEL

As regards the mean concentration, the model equation derived appears to satisfactorily describe the diffusion of marked fluid particles by homogeneous turbulence. It is found to satisfy the exact relationships for small and large diffusion times (Eqs. (2) and (3) respectively), and can be brought into agreement with Taylor's expression (Eqs. (1) and (14), (17) and (18)). In the case of a normal velocity distribution, the predicted deviations from a normal distribution of the mean concentration are small at all diffusion times. The extension to the problem of diffusion in homogeneous turbulence with constant mean velocity gradient can be carried out. Apparently, the diffusion and decay of concentration fluctuations can be described also by the model discussed (Eqs. (29) and (36)).

Finally, we touch upon some possible further applications of the model proposed.

The case of diffusion in nonhomogeneous turbulence would require spatial dependence of the pdf $f = f(\tau, \underline{u}; \underline{x})$. However, this in general leads to an inconsistent model, since the conservation law (10) is violated. The application of the approach pointed out to nonhomogeneous turbulence, which is interesting also from a practical point of view, is still a point of study. Since averaging of $\Gamma(\underline{x}, t; \tau, \underline{u}) - \Gamma(\underline{x} - \underline{u}\tau, t - \tau; \tau, \underline{u})$ involves not only the coordinate $\underline{x}$, but a certain domain including $\underline{x}$, a solution to the problem may possibly be the introduction of a shift in the spatial dependence of the pdf $f(\tau, \underline{u}; \underline{x})$ so that

$$\int_0^t \int d\tau d\underline{u} f(\tau, \underline{u}; \underline{x} - \theta \underline{u}\tau) [\bar{\Gamma}(\underline{x}, t) - \bar{\Gamma}(\underline{x} - \tau \underline{u}, t - \tau)] = 0$$

in which θ is a function of time, $0 < \theta < 1$, and so chosen that (10) is satisfied for all diffusion times.

The problem mentioned does not occur in homogeneous but nonstationary turbulence. In this case time dependence of $f(\tau, \underline{u})$ must be assumed.

A further application might be the problem of two-particle analysis. In analogy to (5) and (6) we could start from

$$\Gamma(\underline{x}_a, t_a) \Gamma(\underline{x}_b, t_b) = \Gamma(\underline{x}_a - \tau_a \underline{u}_a, t_a - \tau_a) \Gamma(\underline{x}_b - \tau_b \underline{u}_b, t_b - \tau_b)$$

in which subscripts a and b indicate two different fluid particles. To this expression ensemble averaging is to be applied. Since $\underline{u}_a$, $\underline{u}_b$, τ_a and τ_b are dependent variables for not too large separations $|\underline{x}_a - \underline{x}_b|$ and $|t_a - t_b|$, much information about the structure of the velocity field is required to obtain a closed model. Moreover, a similar problem as in the case of diffusion in nonhomogeneous turbulence is found to occur.

In principle the treatment of the diffusion of discrete particles is also possible. However, the approach considered concerns only the kinematic problem; the relationship between the pdf $f_p(\tau_p, \underline{u}_p)$ pertaining to the trajectory of a discrete particle, and the statistics of the fluid motion has to be supplied by other means, including the dynamics of the fluid and the discrete particles.

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NOTATION

b_k	coefficient in Gram-Charlier expansion
f	probability density function (pdf)
f_u	probability density function of Eulerian velocity
f_τ	probability density function of time interval τ
k	index of summation
n	number of spatial dimensions
pdf	probability density function
Pr	Prandtl number
Re_1	Reynolds number of energy containing eddies
$R_L(t)$	Lagrangian auto-correlation function
t	time
T_L	Lagrangian integral time scale
u	u_1, u_2 or u_3 , Eulerian velocity components
u'	fluctuation of Eulerian velocity component
v	v_1, v_2 or v_3 , Lagrangian (particle) velocity components
x	x_1, x_2 or x_3 , Eulerian coordinates
y	y_1, y_2 or y_3 , Lagrangian coordinates (particle displacements)
β_1, β_2	coefficients related to dissipation of concentration fluctuations
Γ	concentration of scalar quantity
γ	fluctuation of concentration
$\delta^n(\underline{x})$	n -dimensional delta function ($\delta^1 = \delta$)
ϵ	turbulent diffusivity
σ	variance of particle displacements

τ	time interval related to particle motion
$\phi(t), \phi_1(t)$	functions representing molecular dissipation of concentration fluctuations
$\overline{}$	average value
$\underline{}$	vector
$\cdot\cdot_{i,j}$	subscripts of tensor notation
$\cdot\cdot_p$	discrete particle

APPENDIX A - The behaviour of the function $f_{\tau}(\tau)$ for small and large diffusion times

In this Appendix we consider the integral equation (14). In the case of large diffusion times we expand $\sigma(t - \tau)$ in a Taylor series,

$$\sigma(t - \tau) = \sigma(t) - \tau \frac{d}{dt} \sigma(t) + \dots + 2 \frac{(-\tau)^k}{k!} R_L^{(k-2)}(t) + \dots$$

$k = 2, 3, \dots$

Retaining the first two terms of the expansion, (14) changes to

$$\int_0^t d\tau f_{\tau}(\tau) \left[2\tau \int_0^t R_L(t') dt' - \tau^2 \right] \approx 0$$

Differentiating with respect to t gives

$$f_{\tau}(\tau) \left[2t \int_0^t R_L(t') dt' - t^2 \right] + 2 R_L(t) \int_0^t d\tau \tau f_{\tau}(\tau) \approx 0$$

For large t we have

$$2t \int_0^t R_L(t') dt' - t^2 \approx 2t T_L - t^2 \approx -t^2$$

and

$$2 \int_0^t d\tau \tau f_{\tau}(\tau) \approx 2 \int_0^{\infty} d\tau \tau f_{\tau}(\tau) = C_1$$

C_1 is a positive constant. With these approximations the above expression yields

$$f_{\tau}(t) \approx C_1 \frac{R_L(t)}{t^2} \quad \text{if } t \gg T_L$$

As regards the convergence of the Taylor expansion, we may apply d'Alembert's criterion, and consider

$$\lim_{k \rightarrow \infty} \left| \frac{\frac{(-\tau)^{k+1}}{(k+1)!} R_L^{(k-1)}(t)}{\frac{(-\tau)^k}{k!} R_L^{(k-2)}(t)} \right| = \lim_{k \rightarrow \infty} \frac{\tau}{k+1} \left| \frac{R_L^{(k-1)}(t)}{R_L^{(k-2)}(t)} \right|$$

The series will be convergent, if the limit is less than unity. Since t tends to infinity independently of k , and $\tau \sim T_L$, convergence requires

$$T_L \left| \frac{R_L^{(k-1)}(t)}{R_L^{(k-2)}(t)} \right| = O(1)$$

for k and t tending to infinity. This requirement is satisfied by $R_L \sim (t/T_L)^{-\alpha}$ or $R_L \sim \exp(-\alpha' t/T_L)$, for instance, but not by $R_L \sim \exp[-\alpha''(t/T_L)^2]$.

For small diffusion times ($t \ll T_L$) we put (see Section 2.2)

$$R_L(t) \sim 1 - at^m$$

which yields

$$\sigma(t) \sim \overline{u^2} \left[t^2 - \frac{2a}{(m+1)(m+2)} t^{m+2} \right]$$

(14) then becomes

$$\int_0^t d\tau f_\tau(\tau) \left[t\tau - \tau^2 - \frac{a}{(m+1)(m+2)} \left\{ t^{m+2} - (t-\tau)^{m+2} \right\} \right] \sim 0$$

Differentiating twice with respect to t gives

$$- \frac{a}{(m+1)(m+2)} t^{m+2} \frac{df_\tau}{dt} + (t - \frac{2a}{m+1} t^{m+1}) f_\tau - a \int_0^t d\tau f(\tau).$$

$$\cdot \left\{ t^m - (t-\tau)^m \right\} \sim 0$$

Since $m \geq 1$ the term $2a t^{m+1}/(m+1)$ may be neglected with respect to t . The order of magnitude of the integral at most is $a t^{m+1} f_\tau(t)$, and hence may also be neglected (with respect to the term $t f_\tau$). The resulting differential equation is

$$t^{m+1} \frac{df_\tau}{dt} - \frac{(m+1)(m+2)}{a} f_\tau = 0$$

the solution of which is

$$f_\tau(t) = C_2 \exp \left[- \frac{(m+1)(m+2)}{m a t^m} \right] \quad \text{if } t \ll T_L$$

C_2 is an arbitrary positive constant. The differential equation may be written as

$$\frac{f_\tau}{t \frac{df_\tau}{dt}} = \frac{a}{(m+1)(m+2)} t^m$$

The L.H.S. of this expression represents the ratio of the subtangent, s , and the time interval t . According to the R.H.S. this ratio will vanish if t tends to zero. This has a consequence for integrals of the form

$$\int_0^t d\tau f_\tau(\tau) g(\tau)$$

A condition the (differentiable) function $g(\tau)$ has to satisfy, is given later. Assuming

$$g(\tau) = g(t) + (t - \tau) \dot{g}(t) + \dots$$

the integral becomes

$$g(t) \int_0^t d\tau f_\tau(\tau) + \dot{g}(t) \int_0^t d\tau (t - \tau) f_\tau(\tau) + \dots$$

The maximum value of the second integrand approximately amounts to

$$\frac{a}{(m+1)(m+2)} t^{m+1} f_{\tau}(t)$$

The order of magnitude of the second integral, which is found by multiplying this maximum value by the subtangent s , is

$$\frac{a^2}{(m+1)^2(m+2)^2} t^{2m+2} f_{\tau}(t)$$

The order of magnitude of the first integral is $s f_{\tau}(t)$, or

$$\frac{a}{(m+1)(m+2)} t^{m+1} f_{\tau}(t)$$

Consequently, the integral may be approximated by

$$\int_0^t d\tau f_{\tau}(\tau)g(\tau) \approx g(t) \int_0^t d\tau f_{\tau}(\tau)$$

for small t , provided

$$\lim_{t \rightarrow 0} \frac{t^{m+1} \dot{g}(t)}{g(t)} = 0$$

Similar conditions hold good for higher order derivatives of the function $g(t)$. This result excludes, for instance, the case $g(t) = 1/f_{\tau}(t)$.

APPENDIX B - Taylor expansion of $\bar{\Gamma}(\underline{x} - \underline{r}_u, t - \tau)$ for large diffusion times.

We assume that (3) is true for large diffusion times, and examine the conditions this assumption imposes on the convergence of the Taylor expansion. According to (11), the expansion may be written as

$$\int_0^t d\tau f_{\tau}(\tau) \int d\underline{u} f_{\underline{u}}(\underline{u}) \bar{\Gamma}(\underline{x} - \underline{r}_{\underline{u}}, t - \tau) =$$

$$= \int_0^t d\tau f_\tau(\tau) \sum_{k=0}^{\infty} \frac{(-\tau)^k}{k!} \int d\underline{u} f_u(\underline{u}) \left(\frac{\partial}{\partial t} + \underline{u} \cdot \nabla \right)^k \bar{\Gamma}(\underline{x}, t)$$

As regards the differential operator we first consider two terms, viz.

$$\frac{\partial^k}{\partial t^k} \quad \text{and} \quad u^{2k} \frac{\partial^{2k}}{\partial x^{2k}}$$

in which $u\partial/\partial x = u_1\partial/\partial x_1, u_2\partial/\partial x_2$ or $u_3\partial/\partial x_3$. (3) yields

$$\frac{\partial^{2k}}{\partial x^{2k}} = \frac{1}{\epsilon^k} \frac{\partial^k}{\partial t^k} = \frac{1}{(\overline{u^2} T_L)^k} \frac{\partial^k}{\partial t^k}$$

The contribution of this term to the integral with respect to $\underline{u}$ then is

$$\frac{(-\tau)^{2k}}{(2k)!} \frac{\overline{u^{2k}}}{(\overline{u^2} T_L)^k} \frac{\partial^k \bar{\Gamma}}{\partial t^k}$$

The contribution of the term $\partial^k/\partial t^k$ is

$$\frac{(-\tau)^k}{k!} \frac{\partial^k \bar{\Gamma}}{\partial t^k}$$

Since (3) implies that

$$\frac{\partial^k \bar{\Gamma}}{\partial t^k} \sim (-1)^k k! \frac{1}{t^{k+n/2}}$$

The orders of magnitude of the terms are

$$(-1)^k \frac{k!}{(2k)!} \frac{\overline{u^{2k}}}{(\overline{u^2})^k T_L^k} \tau^{2k} \frac{1}{t^{k+n/2}}$$

and

$$\tau^k \frac{1}{t^{k+n/2}}$$

respectively. We assume $\overline{u^{2k}}/(\overline{u^2})^k < \alpha^k$. The constant α may be greater than unity: in the case $u = \hat{u} \sin \omega t$ one obtains $\alpha = 2$ for large k . Since

$$\lim_{k \rightarrow \infty} \frac{k!}{(2k)!} \alpha^k = 0$$

and $\tau \sim T_L$, we find that the contribution of the terms $u^{2k} \partial^{2k} / \partial x^{2k}$ is negligible. From considerations of symmetry it follows that the pdf $f_u(u)$ is an even function. Therefore, the contribution of the terms $u^{2k+1} \partial^{2k+1} / \partial x^{2k+1}$ vanishes. For these reasons the cross-products in the differential operator may be neglected as well. Thus, for large k the orders of magnitude of the terms in the above expansion are found to be

$$\frac{1}{t^{n/2}} \int_0^t d\tau f_\tau(\tau) \left(\frac{\tau}{t}\right)^k$$

To examine the convergence of the latter series, we apply the integral criterion; if the integral

$$\int_0^\infty dk \int_0^t d\tau f_\tau(\tau) \left(\frac{\tau}{t}\right)^k$$

exist, the series will converge, else it will diverge. Integrating with respect to k changes the integral to

$$\int_0^t d\tau \frac{f_\tau(\tau)}{\ln \frac{t}{\tau}}$$

The singularity at $\tau = t$ causes divergence, unless $f_\tau(\tau) = 0$ when τ exceeds a certain value T_0 ($T_0 \gg T_L$). The integral, and consequently the series, then is convergent when $t > T_0$. Applying a result of Appendix A, viz. $f_\tau(\tau) = C_1 R_L(\tau) / \tau^2$ if $\tau \gg T_L$, yields as a criterion for convergence of the Taylor expansion

$$R_L(\tau) = 0 \quad \text{if} \quad \tau > T_0$$

If it is only required that (3) be an asymptotic approximation when t tends to infinity, we will have the condition

$$\lim_{t \rightarrow \infty} \frac{\text{leading term neglected}}{\text{last term not neglected}} = 0$$

which in this case yields

$$\lim_{t \rightarrow \infty} \frac{\int_0^t d\tau f_\tau(\tau) \tau^2 \frac{\partial^2 \bar{\Gamma}}{\partial t^2}}{\int_0^t d\tau f_\tau(\tau) \tau \frac{\partial \bar{\Gamma}}{\partial t}} = 0$$

or, applying (15) and $\partial \bar{\Gamma} / \partial t \sim t \partial^2 \bar{\Gamma} / \partial t^2$,

$$\lim_{t \rightarrow \infty} \frac{2 T_L}{t} = 0$$

This condition is satisfied, since the Lagrangian integral time scale T_L is finite.