Section C, Fracture Mechanics

Exact and complete Fracture Mechanics of wood
Theory extension and synthesis of all series C publications

Exact, according to boundary value- Airy stress function- and limit analysis- approach,
Complete, by derivations of softening, mixed mode, micro-crack, volume effect, etc.

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Tension perpendicular to the grain at notches and joints - C(1990 )
Evaluation of perpendicular to grain failure of beams caused by concentrated loads of joints CIB-C(2000)
Softening behaviour and correction of the fracture energy - C(2007a)
A new fracture mechanics theory for orthotropic material like wood - C(2007b)
A new fracture mechanics theory of wood - C(2011a)
Fracture mechanics of wood and wood like reinforced polymers - C(2011b)
Exact Derivation of the Geometric Correction Factor of the Center Notched Test Specimen, Based On Small
Cracks Merging As Explanation of Softening - C(2014a)
Limit analysis discussion of design methods for fracture of timber dowel joints loaded perpendicular to grain
C(2014b)

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11. Conclusions ??
1. Introduction
The flat crack singularity approach is called linear elastic fracture mechanics (LEFM) although the real behavior is non-linear. For strength problems, limit analysis always applies, and consequently the linear elastic- full-plastic approach applies and LEFM thus is a tautology. Because for wood confined plasticity is replaced by the equivalent elastic value (as for the bending strength), the linear approach in fracture mechanics applies up to failure at the crack boundary. The so called non-linear fracture mechanics approaches thus are superfluous (and questionable, see chapter 4). Because the singularity approach wrongly predicts failure when whether mode I or mode II becomes ultimate at mixed mode fracture, the exact boundary value approach for failure at a flat elliptical crack is followed in chapter 2. It then is possible to derive the mixed "mode I - II" - interaction equation, based on a new orthotropic-isotropic transformation of the Airy stress function. It follows that failure according to the modes I and II is not simply related to the dissipated stress type. The so called mode I may occur by dissipation of elastic shear stress energy only and the so called mode II, by dissipation of bending stress energy only. Determining for the strength is the stress combination at the fracture site (as also follows from the crack closure technique). Therefore, based on local failure at the crack tip, the stresses should satisfy the derived mixed mode failure criterion, which is shown to follow the critical distortional energy criterion for initial crack extension and the Coulomb criterion after hardening. It further is shown, that strain softening does not exist as material property (as assumed by cohesive zone models). This “softening” is a dynamic unloading process. At loading, in a constant strain rate test, the unloading rate due to the kinetic damage process, may become higher than the loading rate, causing unloading of the specimen. Increasing the loading rate may change this apparent softening behavior into “hardening”. “Softening” behavior therefore is not possible in a constant loading rate test. Although such test becomes a constant strain rate test at the end, where the testing machine is not able to follow the high speed of the damage process, wood still may show hardening until the sudden total failure with the speed of sound. The “softening” stress, due to crack extension, is an apparent stress, based on unnotched specimen dimensions, thus is the mean specimen stress outside the fractured area, while the real fracture stress, in the fracture plane (at the ligament), increases and remains ultimate, causing the specimen (outside this fracture plane) to unload due to the reducing intact part of the fracture plane. Apparent and real softening, (e.g. thermal softening), are fully explained by molecular deformation kinetics processes (Section B) and here by limit analysis without assuming the impossible negative dissipation, decreasing flow stress, and negative modulus of elasticity of anti-theory. The derivation of the “softening” curve of the “Griffith strength” (which is based on a constant ultimate stress) is given in § 3.3. Important is the conclusion that the Griffith stress is an apparent stress based on the intact uncracked cross section, thus is the stress outside the fracture plane, and not the real fracture stress in the fracture plane. The softening curve represents the decreasing stiffness and decreasing mean Griffith strength: \( \sigma_g = \sqrt{G_c E / \pi c} \), with the increase of the crack length \( c \). This decrease, outside the fracture plane, is necessary to maintain the constant ultimate value of the apparent surface energy, and thus constant ultimate cohesive strength for separation by crack extension, in the fracture plane. The area under the softening curve gives the total external energy, (when the deformation of the graph is the deformation of the jack on the test specimen). This total external energy is twice the fracture energy. The proof of this is given by § 3.4, and the derivation of § 2.3, showing that crack extension by any stress combination follows the Coulomb- (or Wu-) equation what implies that failure always occurs by the same ultimate uniaxial tensile strength of the crack boundary near the crack tip. Therefore always, by any combined mode I - II and any mode II failure, the opening mode occurs due to pure cohesion strength failure. In this Section C, is further discussed: the derivation of the power law; the energy method of notched beams and of joints loaded perpendicular to the grain; the explanation of the Weibull size effect in fracture mechanics, and the necessary rejection of the applied crack growth models and fictitious crack models. The high value of the fracture energy and energy release rate, with respect to the surface energy, shows that a high amount of plastic dissipation is involved in fracture. Also
the blunting at the top of the loading curve of test specimens, visible when the testing rig is stiff enough to allow the test to follow the theoretical softening curve, (the Griffith locus), shows that there is a plastic range, which is extended enough to make any stress redistribution possible. This demands the application of limit analysis, (once the basis of fracture mechanics), to obtain always possible exact solutions. Limit analysis is based on an elastic-full plastic schematization of the loading curve (see Discussion of annexes D, section D.1, for the theory). This means that in stress space, the flow criterion is a single curve and for “plastic” dissipation, the stress vector should be along (tangential to) this concave curve, and the strain vector should be perpendicular to the stress vector (normality rule) what means that the (maximum) extremum variational principle applies for “flow” and thus the virtual work equations apply and thus the theorems of limit analysis with the lower and upper bound solutions existing for any allowable equilibrium system, following as solution of the Airy-Stress function. Fracture of wood thus is a common boundary value problem of the strength at the crack boundary (or better, at the boundary of the fractured, plastic zone at the crack tip). This is derived in Chapter 2, and as mentioned, fracture occurs for any load combination by reaching the uniaxial tensile (flow) stress at the elastic–full plastic boundary around the crack tip. This uniaxial tensile failure, as measure of the cohesion strength, leads to the mixed mode Coulomb-equation, eq.(2.30), as exact failure criterion. This applies, as well for the isotropic Airy stress function of the isotropic matrix stresses, as for the orthotropic Airy stress function of the total stresses. Only for mode I loading, is crack extension collinear. For shear, mode II loading, and for combined mode I and II loading, initial oblique crack extension is determining providing the lower bound solution, as well as for the isotropic as orthotropic case. Although fracture mechanics was initially based on limit analysis, it now is always based on the “singularity” approach, thus based on the mathematical flat crack of zero thickness with singularities at the crack tips and, for wood, on collinear crack propagation behavior. This leads wrongly to \( K_I \leq K_{IIc} \) and/or \( K_{II} \leq K_{Ic} \) as fracture criterion for mixed I-II mode loading, showing this generally applied Sih solution to be not exact. Further, the singularity, with infinite fracture stress, does not exist. The center of a crack tip singularity, is an open space. There is no material that can be stressed infinitely. For predicting strength and reliability and for a physical meaning, it is necessary to leave the singularity approach, which prevents a real description of the ultimate state. Removing a singularity always leads to new theory (see B(2011): A new theory of nucleation, DOI:10.1080/01411594.2011.565187). Removing the singularity concept for black holes in Astronomy, provided important new theories. Leaving the singularity approach in fracture mechanics, provides description by the real Wu-criterion, See: C(2007a) Softening behavior and correction of the fracture energy. Theor Appl Fract Mech. vol. 48 nr 2 Oct, 2007. p. 127-139, by: van der Put T.A.C.M. also author of next two: C(2011a) A new fracture mechanics theory of wood", Nova Science Publishers, New York, 2011. C(2011b)., Adv Mech Eng Res. Vol. 2. Chap. 1: Fracture Mechanics of Wood and Wood like reinforced Polymers, Nova Science Publishers, Inc. New York, 2011

It is shown that the area under the load-displacement softening curve gives the total external work on the test specimen and not the fracture energy as wrongly is assumed. The fracture energy follows from half this area which is equal to the critical strain energy release rate at initial crack extension. For wood this correctly is applied for mode II. For mode I however, as for other materials, wrongly the total area is regarded as fracture energy, a factor 2 too high. However, this is partly compensated at “softening” by the apparent too low specific fracture energy due to a small crack merging mechanism when the ultimate state of the fracture plane is reached. Post fracture behaviour thus is shown to be different from initial macro crack extension. The derivations lead to an adaption of the energy approach for fracture of beams with square end notched and of joints loaded perpendicular to the grain, providing a simple design method. It further is shown that all fracture mechanics models applied to wood, as the Dugdale model, the fictitious crack model and the crack growth models, (which should follow from exact molecular deformation kinetics, Section B), are questionable and superfluous and should be replaced by exact theory.
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2. The boundary value problem of fracture mechanics

2.1. Basic Airy stress function

For the solution of the crack-boundary value problem of notches in wood, the orthotropic Airy stress function, is based on the spread out of the reinforcement to act as a continuum, satisfying the equilibrium, compatibility and strength conditions. This behaviour only is possible by interaction of reinforcements through the matrix. Thus also the equilibrium conditions and strength criterion of the matrix, as determining element, have to be satisfied. This also is necessary because the isotropic matrix fails earlier than the reinforcement, and determines initial “flow” behavior. It thus is necessary to solve the Airy stress function for the stresses in the isotropic matrix and then to derive the total (orthotropic) stresses from this solution. None of the applied fracture mechanics approaches, (given e.g. in chapter 2. of [1]) satisfies this requirement of equilibrium of the total stresses as well as the matrix stresses and thus are not able to derive the right, exact, mixed mode failure criterion (The Coulomb- or Wu- equation). This analysis, in total stresses, is as follows:

The stress-strain relations for the two-dimensional flat crack problem are:

\[ \varepsilon_x = c_1 \sigma_x + c_{12} \sigma_y; \quad \varepsilon_y = c_{12} \sigma_x + c_{22} \sigma_y; \quad \gamma_{xy} = c_{66} \tau_{xy}. \]  
(2.1)

This can be written:

\[ \varepsilon_x = \sigma_x / E_x - \nu \sigma_y / E_y; \quad \varepsilon_y = -\nu \sigma_x / E_x + \sigma_y / E_y; \quad \gamma_{xy} = \tau_{xy} / G_{xy}. \]  
(2.2)

The Airy function follows from: \( \sigma_x = \frac{\partial^2 U}{\partial y^2}; \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}; \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \)

satisfying the equilibrium equations: \( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0 \) and \( \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \)

(2.4)

Substitutions of eq.(2.1), using eq.(2.3): \( \varepsilon_x = c_{11} \frac{\partial^2 U}{\partial y^2} + c_{12} \frac{\partial^2 U}{\partial x^2}, \) etc., in the compatibility condition:

\[ \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \]  
(2.5)

gives:

\[ c_{22} \frac{\partial^4 U}{\partial x^4} + (c_{66} + 2c_{12}) \frac{\partial^4 U}{\partial x^2 \partial y^2} + c_{11} \frac{\partial^4 U}{\partial y^4} = 0 \]  
(2.6)

The general solution of eq.(2.6) is: \( U = \sum_{i=1}^{4} F_i (x+\mu y), \) where \( \mu \) is a root of the characteristic equation: \( c_{11} \mu^4 + (c_{66} + 2c_{12}) \mu^2 + c_{22} = 0, \) giving:

\[ \mu^2 = \frac{c_{66} + 2c_{12}}{2c_{11}} \left( -1 \pm \sqrt{1 - \frac{4c_{22}c_{11}}{(c_{66} + 2c_{12})^2}} \right), \]

thus 4 imaginary roots. Introducing the complex variables \( z_1 \) and \( z_2, \) defined by:

\( z_1 = x + \mu_1 y = x' + iy' \) and \( z_2 = x + \mu_2 y = x'' + iy'', \) the solution of eq.(2.6) assumes the form:

\[ U = F_1(z_1) + F_2(z_2) + \overline{F_1}(z_1) + \overline{F_2}(z_2), \]

where the bars denote complex conjugate values. The stresses, displacements and boundary conditions now can be written in the general form of the derivatives of these functions (as e.g. applied by Sih, Paris and Irwin). There are standard methods to solve some boundary value problems (e.g. by Fourier transforms of equations of the boundary conditions) but in principle, functions have to be guessed or chosen, for instance as polynomials, or Fourier series or power series in: \( z \) or \( z^{-1}, \) etc.

As alternative, eq.(2.6) also can be given as:
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\[
\left(\frac{\partial^2}{\partial x^2} + \alpha_1 \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2}{\partial x^2} + \alpha_2 \frac{\partial^2}{\partial y^2} \right) U = 0
\]  
(2.7)

where \( \alpha_1, \alpha_2 = c_{11} / c_{22} \) and \( \alpha_1 + \alpha_2 = (c_{16} + 2c_{12}) / c_{22}. \) Introducing 3 sets of polar coordinates for this case, \( x + iy = r e^{\theta i}, x + iy / \sqrt{\alpha_1} = r e^{\theta i}, \) \( x + iy / \sqrt{\alpha_2} = r e^{\theta i}, \) eq.(2.7) has e.g. elementary solutions as: \( r_1^{2m} \cos(m\theta_i), \) \( r_1^{2m} \sin(m\theta_i), \) \( r_2^{2m} \cos(m\theta_i), \) \( r_2^{2m} \sin(m\theta_i), \) and solutions may be chosen in the form of series of these types. For wood the elementary solution in \( r^{2m} \) are chosen e.g. in [2], what leads to:

\[
\{\sigma_x, \sigma_y, \tau_{xy}\} = \frac{K_A}{(2\pi)^q} \{f_1(0), f_2(0), f_3(0)\}
\]

(2.8)

and:

\[
\{\sigma_x, \sigma_y, \tau_{xy}\} = \frac{K_B}{(2\pi)^q} \{f_1(0), f_2(0), f_3(0)\}
\]

(2.9)

with \( q \leq s. \) The chosen solution is such that it only applies in the vicinity of the notch root as stress singularity at \( r = 0. \) Because for \( q < s, \) and \( r \) is small, the stresses of eq.(2.8) are always larger than those of eq.(2.9), the solution, eq.(2.9), should be rejected based on the boundary conditions at failure. It thus is not right to mention (by Foschi and Barrett) that there are 2 singular stress fields, only eq.(2.8) applies, as approximate solution for uniaxial stress in the main direction.

Because wood is a reinforced material where the reinforcement interacts through the matrix and also the primary cracking is in the matrix, the failure condition should be based on the strength of the matrix and first the Airy stress function of the matrix-stresses should be solved. As solution, eq.(2.8), of \( U \) of eq.(2.7), always only smaller powers than \( m = 0.5 \) (the value of the common isotropic singularity approach) are found. For instance one finite element solution did show: \( m = 0.45, \) near a rectangular notch, while another investigation of the same notch showed values of \( m = 0.45 \) for \( \sigma \) and \( m = 0.10 \) for \( \tau, \) while by the finite difference method, powers were found of \( m = 0.437 \) for the same rectangular notch of 90° and \( m = 0.363 \) and 0.327 for wider notch angles of 153° and 166°. This shows that no compatibility of the (linearly lower) stresses and strains in the isotropic wood matrix are possible and that the, for wood always applied, (isotropic) singularity approach, with always \( m = 0.5, \) is not a real solution for orthotropic wood material. Thus the singularity approach (with \( m = 0.5) \) only may apply for the stresses of the isotropic wood-matrix and not for the total applied stresses on matrix and reinforcement. Wood acts as a reinforced material because lignin is isotropic and hemicellulose and cellulose are transversely isotropic, what means that only one stiffness factor in the main direction has a n-fold higher stiffness in proportion to the higher stiffness of the reinforcement with respect to the matrix. Thus wood material can be treated to contain a shear-reinforcement and a tensile reinforcement in the main direction and eq.(2.10) applies for equilibrium of the matrix stresses:

\[
\begin{align*}
\sigma_x &= \frac{\partial^2 U}{\partial y^2}; \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}; \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \\
n_1 &= \frac{\partial^2 U}{\partial y^2}; \quad n_6 = \frac{\partial^2 U}{\partial x \partial y},
\end{align*}
\]

(2.10)

In stead of using the matrix stresses and the matrix stiffness, the orthotropic n-fold higher total stresses and n-fold higher stiffness can be used to give the same compatibility condition, (thus the same condition for the matrix and reinforcement). Inserting, in the compability equation, eq.(2.5), the total stresses, expressed in the isotropic Airy stress function \( U \) of the matrix stresses, gives:

\[
c_{22} \frac{\partial^4 U}{\partial x^4} + \left(n_6 c_{66} + (1 + n_1)c_{12}\right) \frac{\partial^4 U}{\partial x^2 \partial y^2} + n_1 c_{11} \frac{\partial^4 U}{\partial y^4} = 0
\]

For the isotropic matrix is: \( n_1 c_{11} / c_{22} = 1 \) and \( (n_6 c_{66} + (1 + n_1)c_{12}) / c_{22} = 2 \) giving:

\[
\frac{\partial^4 U}{\partial x^4} + 2 \frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = \nabla^2 (\nabla^2 U) = 0
\]

(2.11)
Thus: \[ n_1 = \frac{c_{22}}{E_x}, \quad n_6 = \left( 2 - \frac{c_{12}}{c_{22}} \right) \frac{c_{22}}{c_{11}} \frac{G_{yy}}{E_y} \] (2.12)

This new orthotropic-isotropic transformation of the Airy stress function and the calculation method based on the stresses of the matrix, is used in the following. It now is possible to use the known isotropic solutions of \( U \) to find the matrix stresses (which should be below the matrix strength) and to multiply these matrix stresses with the \( n \)-factors of eq.(2.12) to find the total, applied, orthotropic stresses of the regarded loading case. This is applied in \S 2.2 by solving first the matrix stresses.

### 2.2. The elliptical flat crack solution

As shown above, the singularity approach does not apply for the orthotropic case and only applies for uniaxial loading and thus prevents the applicability of mixed mode loading cases and thus prevents the derivation of a real failure criterion. In stead of such a criterion, critical values are assumed of e.g. the strain energy density, the \( J \)-integral, or the maximal principal stress, or a non local stress function, all at a distance away from the crack tip, thus away from the fracture site. A real failure criterion only can be based on the real ultimate stress in the material which occurs at the crack-tip boundary. A real, physical possible, crack form is the flat elliptical crack, which is the first expanded of any crack boundary form and because the crack is flat, the higher expanded terms have a negligible, in the limit zero, contribution. When “flow” occurs around the crack tip, the ultimate strain condition at the crack-boundary determines failure and the direction of crack extension. The elastic-plastic boundary (of limit analysis) then acts as an enlarged crack tip boundary, with the ultimate elastic tangential stress as “flow”-stress for the, therefore linear elastic, fracture mechanics calculation. Thus limit analysis approach incorporates linear elastic -, as well as non-linear fracture mechanics. There is no distinction possible between the two by the limit analysis approach.

#### 2.2.1. The elliptic hole in an infinite region

The classical way of analyzing the elliptic crack problem is to use complex variables and elliptic coordinates. The Airy stress function can be expressed in terms of two analytic functions [3], of the complex variable \( z (= x + iy) \) and the transformation to elliptic coordinates in Fig. 2.1, gives:

\[ z = x + iy = c \cdot \cosh(\xi + i\eta) \quad \text{or:} \quad x = c \cdot \cosh(\xi) \cdot \cos(\eta); \quad y = c \cdot \sinh(\xi) \cdot \sin(\eta). \]

For an elliptic hole, \( \xi = \xi_0 \), in an infinite region with uniaxial stress \( p \) at infinity in a direction inclined at \( \beta \) to the major axis \( OX \) of the ellipse, the Airy stress function \( U \), satisfying

\[ \nabla^2 (\nabla^2 U) = 0, \]

and satisfying the conditions at infinity and at the surface \( \xi = \xi_0 \), showing no discontinuity of displacement, thus being the solution, is: \( U = R \{ z\phi(z) + \chi(z) \} \), with [3]:

\[ 4\phi(z) = p \cdot c \cdot \exp(2\xi_0 \cdot \cos(2\beta)) \cdot \cosh(\xi_0 \cdot \cos(\eta)) \]

\[ 4\chi(z) = - p \cdot c \cdot \{ \cosh(2\xi_0) \cdot \cos(2\beta) + \exp(2\xi_0) \cdot \sinh(2\{ \xi - \xi_0 - i\beta \}) \} \cdot \cosec(\xi_0) \]

where \( \xi = \xi_0 + i\eta. \)

For the stresses at the boundary, due to a stress \( p \) at an angle \( \beta \) to the crack, is:

\[ \sigma_\eta - \sigma_\xi + 2i\tau_\xi \eta = 2[\bar{z} \phi''(z) + \chi''(z)]e^{i\delta} \quad \text{and:} \quad \sigma_\xi + \sigma_\eta = 2[\phi'(z) + \chi'(z)] = 4R\{\phi'(z)\} \]

Figure 2.1 - Elliptic hole and coordinates.
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and the tangential stress \( \sigma_t \) at the surface \( \xi = \xi_0 \) is simply known from the last equation because here \( \sigma_n = 0 \). Thus: Determining for the strength is the tangential stress \( \sigma_t \) at the crack surface \( \xi = \xi_0 \) due to a stress \( p \) at an angle \( \beta \) (of Fig. 2.3.1) to the crack. Thus:

\[
\sigma_t = 2[\varphi'(\xi_0 + i\eta) + \varphi'(\xi_0 - i\eta)] = \frac{p(\sinh(2\xi_0) + \cos(2\beta) - \exp(2\xi_0) \cdot \cos(2(\beta - \eta)))}{\cosh(2\xi_0) - \cos(2\eta)} \tag{2.13}
\]

while \( \chi'(z) \) has to vanish at: \( \xi = \xi_0 \)

Eq.(2.13) can be extended for two mutual perpendicular principal stresses \( p_1 \) and \( p_2 \) (see Fig. 2.3.1) by a simple addition leading to eq.(2.22) below.

2.2.2. The mathematical flat crack solution, explaining the singularity approach

For stresses in the wood-matrix, the results of the limit case of the elliptical notch with \( \xi_0 \) approaching zero should be comparable with the results of the mathematical flat crack of the singularity method. To derive the singularity equations, (as special case of the general exact solution), new coordinates \( X, Y \) with the origin in the focus of the ellipse are necessary (see Fig. 2.2). Thus:

\[
X = x - c = c(\xi^2 - \eta^2)/2, \quad Y = y = c\xi\eta \tag{2.14}
\]

or in polar coordinates: \( r = (X^2 + Y^2)^{0.5}, \quad X = r\cos(\theta), \quad Y = r\sin(\theta) \tag{2.15} \)

and from eq.(2.14):

\[
\xi = \sqrt{2r/c}\cdot \cos(\theta/2), \quad \eta = \sqrt{2r/c}\cdot \sin(\theta/2), \quad \eta/\xi = \tan(\theta/2) = \tan(\delta) \tag{2.17}
\]

Using these relation in the stresses \( \sigma_{\eta}, \sigma_{\xi}, \tau_{\xi\eta} \) of § 2.2.1 and applying the singularity, \( \xi_0 = 0 \) in the general solution of the elliptic Airy stress function, then the tangential stress \( \sigma_\theta \) along a crack boundary \( r_0 \), due to a stress \( p \) at infinity at an angle \( \beta \) with the notch is:

\[
(8r_0/cp^2)^{0.5} \sigma_\theta = -3\sin(\theta/2)\cos^2(\theta/2)\sin(2\beta) + 2\cos^3(\theta/2)\sin^2(\beta) \tag{2.18}
\]

for a small value of \( r_0 \), so that all terms containing the factor \( r_0^{-0.5} \) are negligible.

The other stresses are:

\[
(8r_0/cp^2)^{0.5} \sigma_\eta = \sin(\theta/2)(1-3\sin^2(\theta/2))\sin(2\beta) + 2\cos(\theta/2)(1+\sin^2(\theta/2))\sin^2(\beta) \tag{2.19}
\]

\[
(8r_0/cp^2)^{0.5} \tau_{\theta\eta} = \cos(\theta/2)(3\cos^2(\theta/2)-2)\sin(2\beta) + 2\cos^2(\theta/2)\sin(\theta/2)\sin^2(\beta) \tag{2.20}
\]

For the, for wood always applied, singularity method, the flat crack in the grain direction is supposed to propagate in that direction. Thus \( \theta = 0 \), and eq.(2.18) becomes [4]:

\[
(8r_0/cp^2)^{0.5} \sigma_\theta = 2\sin^2(\beta) \quad \text{and is:} \quad \sigma_\tau = \sigma_\theta \quad \text{and:} \quad \tau_{\theta\tau} = \sigma_\theta \cot(\beta).
\]

Mode I failure \( \sigma_\theta = \sigma_t \) occurs when \( \beta = \pi/2 \). Thus when:

![Figure 2.2 - Confocal coordinates](image-url)
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\[ p = \sigma \sqrt{2r_0/c} \]  \hspace{1cm} (2.20)

For pure shear loading, thus for superposition of \( p_1 = S \) at \( \beta = \pi/4 \) and \( p_2 = -S \) at \( \beta = 3\pi/4 \) in eq.(2.18) and in the other equations of the solution is for crack extension \( \theta = 0 \):

\[
\left( \frac{2r}{cS^2} \right)^{0.5} \tau_{\theta\theta} = \left( \cos(\theta/2) \cdot \left( 3\cos^2(\theta/2) - 2 \right) \right)_{\theta=0} = 1 \quad \text{or} \quad S = \tau_{\theta\theta} \sqrt{2r_0/c} \]  \hspace{1cm} (2.21)

with now \( \sigma_y = \sigma_\theta = 0 \), leading to an ultimate shear failure criterion (without interaction with normal stresses) although real shear failure is plastic and a real collinear mode II fracture does not exist. Eq.(2.20) and (2.21) thus are in fact maximum stress conditions for the strengths in the main planes. Fracture is predicted to occur when the tensile strength is reached perpendicular to the grain and/or when the shear strength in this plane is reached. Thus: \( K_t \leq K_{kc} \) and \( K_{II} \leq K_{kc} \) for all stress states. This also is predicted for the n-fold higher quasi orthotropic stresses and is empirically shown to be not right (see eq.(2.20) and eq.(2.21) in Fig. 2.3.4 and also shown by theory, eq.(2.30), to be not right because according to eq.(2.30) failure is always by the uniaxial maximal tangential tensile stress along the crack boundary. Thus also for the isotropic matrix, the applied singularity approach gives no right results for mixed mode failure. The right failure condition for combined stresses, eq.(2.30), is derived below. The well-known singularity equations are only applicable as limit analysis solutions for matrix stresses by a chosen equilibrium system for collinear macro-crack propagation as applied below for fracture of joints and of beams with square end-notches, wherefore, as lower bound, the mode I energy release rate is chosen as specific fracture energy.

2.3. Derivation of the mixed I- II- mode equation

A general failure criterion [5] follows from the determining ultimate tensile stress which occurs at the crack boundary or better, at the elastic-plastic boundary, as plastic stress, which is necessarily along (tangential to) this elastic full plastic boundary (of limit analysis). By an extension of eq.(2.13) (by superposition) to \( p_1 = \sigma_1 \) inclined at an angle \( \pi/2 + \beta \) to the Ox-axis and \( p_2 = \sigma_2 \) inclined at an angle \( \beta \), (see Fig. 2.3.1), eq.(2.13) turns to:

\[
\sigma_i = \frac{2\sigma_y \sinh(2\xi) + 2\tau_{\theta\theta} \left[ (1 + \sinh(2\xi)) \cdot \cot(2\beta) - \exp(2\xi) \cdot \cos(2(\beta - \eta)) \cos ec(2\beta) \right]}{\cosh(2\xi) - \cos(2\eta)}, \quad (2.22)
\]

where the stresses are given in notch coordinates with the x-axis along the crack. For small values of \( \xi \) and \( \eta \) (flat notches), this equation becomes:

\[
\sigma_i = \frac{2 \left( \xi_0 \sigma_y - \eta \tau_{xy} \right)}{\xi_0^2 + \eta^2}, \quad (2.23)
\]

The maximum (critical) value of the tangential tensile stress \( \sigma_1 \), depending on location \( \eta \), is found by: \( d\sigma_i / d\eta = 0 \), giving the critical value of \( \eta \):

![Figure 2.3.1 - Stresses in the notch plane Ox](image)

\[ \eta = \frac{\xi_0 \sigma_y}{\xi_0^2 + \eta^2} \]
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\[-2\tau_{xy}/(\xi_0^2 + \eta^2) - \left(2(\xi_0\sigma_y - \eta\tau_{xy})\right)/\left(\xi_0^2 + \eta^2\right)^2 = 0, \text{ or:}\]

\[-\tau_{xy}/(\xi_0^2 + \eta^2) = 2\eta(\xi_0\sigma_y - \eta\tau_{xy}) = \eta\sigma_i(\xi_0^2 + \eta^2)\]

(2.24)

where the second equality sign is due to the substitution of eq.(2.23).

From the first and last term follows that: \(\eta\sigma_i = -\tau_{xy}\) (2.25)

and from the first 2 terms:

\[\eta/\xi_0 = \left(\sigma_y \pm \sqrt{\sigma_y^2 + \tau_{xy}^2}\right)/\tau_{xy}\] (2.26)

Elimination of \(\eta\), from eq.(2.25) and (2.26) or from eq.(2.26) and eq.(2.23) gives:

\[\xi_0\sigma_i = \sigma_y \pm \sqrt{\sigma_y^2 + \tau_{xy}^2}\]

and this can be written:

\[1 = \frac{\sigma_y}{\xi_0\sigma_i} + \frac{\tau_{xy}^2}{\xi_0\sigma_i^2}\]

Transformation from elliptic to polar coordinates by eq.(2.17): \(\xi_0 = \sqrt{2\rho_0/\pi} \cdot \cos(\delta)\) gives:

\[1 = \frac{\sigma_y}{\tau_{xy}}\sqrt{2\rho_0/\pi} + \left(\frac{\tau_{xy}\sqrt{2\rho_0/\pi}}{\sigma_y}\right)^2 = \frac{K_i}{K_{lc} \cos(\delta)} + \left(\frac{K_{lc} \cos(\delta)}{K_{lc} \cos(\delta)}\right)^2\] (2.27)

showing that for combined (mixed mode) fracture, when \(\delta \neq 0\), the apparent stress intensity factors of Irwin, \(K_i\), \(K_{lc} \cos(\delta)\), \(K_{lc} \cos(\delta)\) are not constant. The value of \(\delta\) is stress dependent and depends on the combined loading according to:

\[tg(\delta) = \frac{\sigma_y}{\tau_{xy}} \pm \frac{\sigma_y^2}{\tau_{xy}^2} + 1\] (2.28)

for the isotropic matrix stresses. For pure mode I: \(\delta = 0\), \(\tau_{xy} = 0\), is \(K_{lc}\) equal to the Irwin value.

For pure shear loading of the isotropic matrix is \(\delta = 0\) and \(\delta = 45^0\) and is the stress intensity lower than the Irwin value, thus:

\[K_{lc} \cos(\pi/4) = K_{lc} / \sqrt{2} = 0.71 \cdot K_{lc}\] (2.29)

This is e.g. measured in: [14] according to Fig.2.3.2, for a relatively small initial crack length, in Agathis lumber, (density 480 ± 10 kg/m³; 12% m.c. 20 °C). The lumber had no defects, as knots or grain distortions so that the specimens consisted of clear wood.

Thus, according to the exact lower bond solution of limit analysis, is all combined- mode I – II and pure shear fracture a matter of oblique crack extension by failure by the maximal uniaxial tensile stress (cohesion strength) along the crack tip boundary. The oblique angle \(\delta\) of eq.(2.28) is indicated in Fig. 2.3.3. This oblique crack extension criterion applies not only for clear wood as lower bound criterion. Limit analysis is based on small displacements and small changes of geometry as principle of applying virtual work equations. Any allowable virtual displacement field can be chosen and thus any allowable virtual fracture mode and direction for initial crack extension.

Fig. 2.3.2. (see C(2011)) Fracture by pure shear loading by oblique crack extension at the uniaxial ultimate tensile stress (opening mode) near the crack tip in the asymmetric four point bending test with small center-slit. (Sketch after photo of [14]).
For approximate collinear crack extension, thus small values of \( \delta \approx 0 \), as applies for timber, eq.(2.27) becomes the Coulomb- (or Wu-) equation:

\[
\frac{K_I}{K_{IC}} + \left( \frac{K_{II}}{K_{IIc}} \right)^2 = 1
\]  

(2.30)

The fact that \( K_{IC} = \sigma_y \sqrt{\pi c} = \sigma_y r_0 / 2 \) is constant, and is therefore regarded as material property, indicates that \( r_0 \) is the radius of the elastic-plastic boundary around the fracture process zone. The size of this process zone is invariant and failure is due to the maximal tensile stress in this elastic-plastic boundary which is a measure of the uniaxial cohesion strength \( \sigma_y \). This stress is independent of the small crack dimensions, within the process zone, because failure is due to the critical small crack density, independent of the small crack dimensions. Thus special small crack behavior within the process zone determines macro crack behavior. This is discussed in § 3.6 and Chapter 10.

Fig. 2.3.3. Uniaxial tensile failure at any mixed I-II mode fracture.

The derivation of eq.(2.27) also gives the relation between \( K_{IC} \) and \( K_{IIc} \). For the stresses in the isotropic matrix this is:

\[
\frac{K_{IIc}}{K_{IC}} = (\sigma_y \sqrt{2\pi r_0} / (\sigma_y \sqrt{2\pi r_0} / 2) = 2
\]

The matrix stresses are also determining for e.g. for Balsa wood, which is highly orthotropic, but is light, thus has a low reinforcement content and shows total failure soon after matrix failure and thus shows at failure the isotropic ratio of \( K_{IIc} / K_{IC} \approx 2 \) of the isotropic matrix material, as is verified by the measurements of Wu on Balsa by: \( K_{IIc} \approx 140 \text{ psi} \cdot \text{in}^{0.5} \) and \( K_{IC} \approx 60 \text{ psi} \cdot \text{in}^{0.5} \).

Eq.(2.30) is generally applicable also when \( \sigma_y \) is a compression stress as e.g. follows from the measurements of Fig. 2.3.4. When the compression is high enough to close the small notch \( \left( \sigma_y, c \approx 2G_{xy} \xi_0 \right) \), \( \tau_{xy} \) has to be replaced by the effective shear stress: \( \tau^{*}_{xy} = \tau_{xy} + \mu \left( \sigma_y - \sigma_{y,ct} \right) \)

in eq.(2.29) or:

\[
1 = \frac{\sigma_{y,ct}}{\xi_0 \sigma_y / 2 + \frac{(\tau^*_{xy})^2}{\xi_0^2 \sigma_y^2}},
\]

what is fully able to explain fracture by compression perpendicular to the notch plane (see Fig. 2.3.4). In this equation is \( \mu \) the friction coefficient.

For species, with denser layers than those of Balsa, a much higher value of \( K_{IIc} \) than twice the value of \( K_{IC} \) is measured because due to the reinforcement, \( \eta \) is smaller than the isotropic critical value of eq.(2.26). To read the equation in applied total orthotropic stress values, the matrix stress \( \tau_{iso} \) has to be replaced by \( \tau_{ort} / n_6 \) and the maximum slope of the tangent, slope \( \delta \) in Fig. 2.2 of the location of the failure stress, is:

\[
|\tan \delta| = \eta_m / \xi_0 = K_f / K_{IC} = 1 / 2n_6
\]  

(2.32)

For small values of \( \eta = -|\eta| \), eq.(2.23) can be written, neglecting \((\eta/\xi_0)^2 \):

10
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\[
\frac{\sigma_y}{\xi_0\sigma_t / 2} = 1 + \frac{\eta^2}{\xi_0^2\sigma_t / (2|\eta|)} \approx 1 - \frac{\tau_{xy}}{\xi_0^2\sigma_t / (2|\eta|)}
\]

(2.33)

where \(|\eta|\) is the absolute value of negative \(\eta\). Thus:

\[
\frac{K_I}{K_{IIc}} \approx 1
\]

(2.34)

This is a lower bound, with: \(K_{IIc} = \left(\xi_0 / |\eta_m|\right) \cdot K_{lc}\)

(2.35)

and the maximal value of \(\eta = \eta_m\) is found by measuring \(K_{lc}\) and \(K_{IIc}\), giving e.g. a value of about \(\xi_0 / \eta_m \approx 7.7\), showing that the disregard of \((\eta / \xi_0)^2 = 0.017\) with respect to 1 is possible. Measurements between the lines eq.(2.30) and (2.34) in Fig. 2.3.4, thus indicate a strong difference between \(K_{IIc}\) and \(K_{lc}\) of the local structure that is crossed by the propagating crack. As mentioned, to obtain real orthotropic stresses, \(\tau_{iot} = \tau_{ort} / n_6\) has to be inserted in eq.(2.27):

\[
1 = \frac{\sigma_y}{\xi_0\sigma_t / 2} + \frac{\tau_{iot}^2}{\xi_0^2\sigma_t^2} = \frac{\sigma_y}{\xi_0\sigma_t / 2} + \frac{\tau_{ort}^2}{\xi_0^2\sigma_t^2 n_6^2} = \frac{K_I}{K_{lc}} + \left(\frac{K_{II}}{K_{IIc}}\right)^2
\]

(2.36)

and it follows that:

\[
\frac{K_{IIc}}{K_{lc}} = \frac{\xi_0\sigma_t n_6}{\xi_0\sigma_t / 2} = 2n_6
\]

according to eq.(2.12) is e.g. for small clear specimens:

\[
2n_6 = 2 \cdot (2 + \nu_{21} + \nu_{12}) \cdot (G_{xy} / E_y) = 2(2 + 0.57)/0.67 = 7.7 \text{ for Spruce and: } 2(2 + 0.48)/0.64 = 7.7 \text{ for Douglas Fir in TL-direction. (densities: respectively 0.37 and 0.50; moisture content of 12 %).}
\]

Thus, for \(K_{lc} \approx 265 \text{kN/m}^{1.5}\) is

\[
K_{IIc} = 7.7 \cdot 265 = 2041 \text{ kN/m}^{1.5} \text{ in the TL-direction. This agrees with measurements [1]. In RL-direction this factor is 3.3 to 4.4. Thus, when } K_{IIc} \text{ is the same as in the TL-direction, the strength in RL-direction is predicted to be a factor 1.7 to 2.3 higher with respect to the TL-direction. This however applies at high crack velocities (“elastic” failure) and is also dependent on the site of the notch. At common loading rates a factor lower than: 410/260 = 1.6 is measured [1] and at lower [7],[1] cracking speeds, this strength factor is expected to be about 1 when fracture is in the “isotropic” middle lamella. It then thus is independent of the TL and RL-direction according to the local stiffness and rigidity values. To know the mean influence, it is necessary to analyze fracture strength data dependent on the density and the elastic constants of \(n_6\). From the rate dependency of the strength follows an influence of viscous and viscoelastic processes. This has to be analyzed according to Deformation Kinetics theory [8], (Section B).

A general problem is further the possible early instability of the mode I-tests equipment. This means that small-cracks failure outside the notch-tip region may be determining as e.g. in the tests of [9]. In this case constants should be compared with the related mode II data. Empirical verification of the above derived theory equation, eq.(2.36), which is called Coulomb equation, or Wu-equation for wood, is not only obtained by precise measurements of [6], but also by tests of [10], done at the TL-system on eastern red spruce at normal climate conditions using
different kinds of test specimens. The usual finite element simulations provided the geometric correction factors, and the stress intensity factors. The lack of fit test was performed on these data, at the for wood usual variability, assuming the five different, often suggested failure equations of Table 2.1. The statistical lack of fit values in the table show, that only the Wu-failure criterion, the third equation of Table 2.1, cannot be rejected due to lack of fit. The Wu-equation is shown to fit also clear wood and timber strength data in [11] and [12], as expected from theory.

<table>
<thead>
<tr>
<th>Failure criterion</th>
<th>$p$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_i / K_k = 1$</td>
<td>0.0001</td>
</tr>
<tr>
<td>$K_i / K_k + K_{II} / K_{III} = 1$</td>
<td>0.0001</td>
</tr>
<tr>
<td>$K_i / K_k + (K_{II} / K_{III})^2 = 1$</td>
<td>0.5629</td>
</tr>
<tr>
<td>$(K_i / K_k)^2 + K_{II} / K_{III} = 1$</td>
<td>0.0784</td>
</tr>
<tr>
<td>$(K_i / K_k)^2 + (K_{II} / K_{III})^2 = 1$</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

2.4. Remarks regarding crack propagation
Because the mixed mode failure criterion shows that cracks tend to propagates in the direction perpendicular to greatest principal tangential tensile stress in the crack boundary, as shown in Fig. 2.3.2 and 2.33, the following modes occur:

![Figure 2.4.1 - a) Crazing at the crack tip and b) Possible crack extension along the fractured zone in glassy polymers](image)

![Fig. 2.4.2 Scheme of Wu, of crack extension by skipping across fibers at pure shear loading, showing “mode” II failure to be a tensile failure outside the collinear plane of pure maximal shear stress. (This also can be regarded as a zig zag, small oblique angle, tensile crack propagation according to theory).](image)

In fig. 2.4.1-b, the mixed mode crack propagation starts at an angle with its plane (due to initial matrix failure), but, (due to the reinforcement), may bend back along the fractured zone. Stage b of this crack propagation is due to small - cracks merging in the fractured zone, which propagate to the macro-crack tip. For wood, stage b occurs in a parallel crack plane as e.g. given by Fig. 2.4.2. This skipping across fibers is a form of oblique crack extension in a zigzag way, jumping when the critical crack length is reached. Real collinear shear crack extension does not exist because the tensile stress is zero and thus only plastic shear sliding is possible at a much higher shear stress. For small crack extension, is collinear crack extension possible by interference of tensile stresses, causing tensile failure in the weakest plane (along the grain) as is given by Fig. 2.4.3, by crack merging, where each small crack is propagating in the two directions towards the neighboring cracks. This is the principle of the small crack merging mechanism of [13], discussed in § 3.6.
Fig. 2.4.3. Collinear small crack merging.

Figure 2.4.4 explains why, in the mode II standard test, under shear loading, not a sliding mode II, but elastic sliding unloading after an opening mode I tensile failure, occurs.

Fig. 2.4.4. Mode II standard test loading of the single end notch beam

This “mode II” test is represented by case \( a + a'' \). If the sign of the lower reaction force \( V \) of this case is reversed and \( P = 0 \), the loading of the mode I double cantilever beam (DCB) test is obtained, identical to loading case \( c \) with \( N = 0 \). In Fig. 2.4.4, case \( a + a'' \) is split in case \( a \) and in case \( a'' \), as loading of the upper and the lower cantilever. Case \( a \) is identical to case \( a' \) which is similar to end-notched beams discussed in [13], Chapter 6. This case behaves like the mode I fracture test as can be seen by loading case \( c \). The loading near the crack tip, given by case \( a \), can be seen as the result of superposition of the stresses of cases \( b \) and \( c \), where the loading of case \( b \) is such, that the un-cracked state of the beam, case \( b' \), occurs. The loading of case \( c \) is such that the sum of cases \( b \) and \( c \) gives loading case \( a \). Case \( c \) is the real crack problem and the critical value of strain energy release rate \( G_c \) can be found by calculating the differences of elastic strain energies between case \( a' \) and \( b' \), the cracked and un-cracked system [13]. Case \( c \) shows the loading of the mode I – DCB-test by \( V \) and \( M \), combined with shear loading by \( N \) and the energy release rate thus will be somewhat smaller (by this combination with \( N \)) than the value of the puce DCB-test.

For the loading case \( a'' \), the same stresses occur as in case \( a \), however with opposite directions of \( M \) and \( V \) with respect to those of case \( c \), according to case \( c'' \), causing crack closure. To prevent that crack closure \( c'' \), and friction, dominates above crack opening \( c \), the crack slit has to be filled with a Teflon sheet. By superposition of cases \( c \) and \( c'' \), case \( c + c'' \) of shear loading of pure mode II occurs, as crack problem due to the total loading. The normal load couple of \( 2N \) is just the amount to close the horizontal shift of both beam ends with respect to each other at that loading stage. This explains the applicability of the virtual crack closure technique. Because the upper cantilever is stronger for shear than the lower cantilever, because of higher compression perpendicular and along the grain (see fig. 5.1 and 5.2, for the, with compression parabolic increasing shear strength), mechanism \( c \) will dominate above \( c'' \), when the lower cantilever start to flow in shear or fails at the support. Thus mode I, case \( c \) tensile failure occurs.

2.5. Remarks regarding the empirical confirmation

Measurements are given in Fig. 2.3.4. The points are mean values of series of 6 or 8 specimens. The theoretical line eq.(2.30) is also the mean value of the extended measurements of Wu on balsa plates. Only the Australian sawn notch data deviate from this parabolic line and lie between eq.(2.30) and the theoretical lower bound eq.(2.34). This is explained by the theory by a too high \( K_{IIc}/K_{Ic} \)-ratio, indicating a manufacturing mistake. Using general mean values of the constants, the
prediction that $K_{lk}/K_k = 2 \cdot (2 + \nu_{21} + \nu_{12}) \cdot (G_{xy}/E_y)$ agrees with the measurements. However, precise local values of the constants at the notches are not measurable and there is an influence of the loading rate and cracking speed. Thus safe lower bound values have to be used in practice. Fig. 2.3.4 shows that all measurements, including compression, are explained by the theory.

2.6. References


3. Mode I “softening" behavior and fracture energy

3.1 Introduction

The derivation of the softening behavior is discussed and it is shown that the area under the load-displacement softening curve of e.g. Fig. 3.4.1, 3.4.2, 3.6 or 3.7, divided by the total crack area, (including the initial crack) is not the fracture energy, but the total external work on the specimen. The fracture energy is half this value and is equal to the critical strain energy release rate at the top of the curve. For wood this correctly is applied for mode II. For mode I, a two times too high value is applied as done for other materials. This can be seen as follows:

When crack extension occurs in a cantilever beam loaded by a constant load $P$ at the free end, then the load gets an additional deflection $\delta$ due to this crack extension. Then the work done on the beam is $P \cdot \delta$ while the work for the elastic strain increase is: $P \cdot \delta / 2$. Then the work for crack extension thus is $P \cdot \delta - P \cdot \delta / 2 = P \cdot \delta / 2$, thus equal to the elastic work of strain increase. Therefore is the area under the load-displacement softening curve the total external work on the test specimen and not the fracture energy. The fracture energy follows from half this area what is equal to the critical strain energy release rate at the first crack increment. For wood this correctly is applied for mode II. See Fig. 3.4.3 below, where the elastic part of stored energy is subtracted from
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the total applied energy of the loading curve to get the right fracture energy. For mode I however, wrongly the total area is regarded as fracture energy. The fracture energy is a function of the Griffith strength and, as the strain energy release rate, related to the effective width of the test specimen and not to the real length of the fracture plane. The strain energy release rate is determined at the top of the softening curve as start of macro-crack extension. This top is determined by the critical small-crack density. Proceeded small-crack extension also determines the softening curve and post fracture behavior. The analysis is based on stresses in the isotropic wood matrix, accounting for mode I failure in the weak planes because of the necessary correction of the fracture energy. The analysis, according to the equilibrium method of limit analysis, then is the same as for any isotropic material.

3.2. Compliance and energy release rate.

As most materials, wood shows near failure an plastic and apparent plastic behavior and the loading curve can be approximated by equivalent elastic-plastic behaviour. Therefore limit analysis applies and linear elastic fracture mechanics can be applied based on the ultimate stress at the elastic-plastic boundary around the crack tip. The dissipation by microcracking, plastic deformation and friction within this boundary, called fracture process zone, then is regarded as part of the fracture energy of the macro crack extension. Thus the limit equilibrium method is applicable. When a specimen is loaded until just before the start of softening and then unloaded and reloaded, the behavior has become elastic-full plastic, and the real stress differs an internal equilibrium system with the linear elastic loading stresses. Because limit analysis applies, based on virtual deformations, this internal equilibrium system and other initial stresses have no influence on the value of the ultimate state and need not to be regarded. According to the limit theorems, initial stresses or deformations have no effect on the plastic limit collapse load provided the geometry is essentially unaltered and thus the calculation is based on initial dimensions. Therefore also as method of practice and of the Building Codes, calculations can be based on a reduced E-modulus up to the ultimate state and therefore also the linear elastic derivation of the softening curve of the fractured specimen, is possible, based on the compliance method, as follows:

In Fig. 3.1, a mode I, center notched test specimen is given with a length “l”, a width “b” and thickness “t”, loaded by a stress $\sigma$ showing a displacement increase $\delta$ of the loaded boundary due to a small crack extension. The work done by the constant external stress $\sigma$ on this specimen, during this crack extension is equal to:

$$\sigma \cdot b \cdot t \cdot \delta = 2W = 2(\sigma \cdot b \cdot t \cdot \delta/2)$$

(3.1)

This is twice the increase of the strain energy $W$ of the specimen. Thus the other half of the external work, equal to the amount $W$, is the fracture energy, used for crack extension. Thus the fracture energy is equal to half the applied external energy which is equal to the strain energy increase $W$ and follows, for the total crack length, from the difference of the strain energy of a body containing the crack and of the same body without a crack:

$$\frac{\sigma^2}{2E_{eff}}btl = \frac{\sigma^2}{2E}btl = W$$

(3.2)

Figure 3.1 - Specimen $b \times l$ and thickness $t$, containing a flat crack of $2c$
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The fracture energy is also equal to the strain energy decrease at fixed grips conditions when \( \delta = 0 \):

\[
W = \sigma_1 \int_{-c}^{c} vda = \pi \sigma^2 c^3 t / E
\]  

(3.3)

where the last two terms give the strain energy to open (or to close) the flat elliptical crack of length \( 2c \) and where “\( \nu \)” is the displacement of the crack surface in the direction of \( \sigma \).

From eq.(3.2) and eq.(3.3) follows that:

\[
\frac{\sigma^2}{2E_{eff}} - \frac{\sigma^2}{2E} = \pi \sigma^2 c^3 t / E
\]  

(3.4)

Thus the effective Young’s modulus of the specimen of Fig.3.1, containing a crack of \( 2c \), is:

\[
E_{eff} = \frac{E}{1 + 2\pi \sigma^2 / bl}
\]  

(3.5)

The equilibrium condition of the critical crack length is:

\[
\frac{\partial}{\partial c} \left( W - G_e 2ct \right) = 0
\]  

(3.6)

where \( G_e \) is the fracture energy for the formation of the crack surface per unit crack area. Eq.(3.6) also can be regarded as the law of energy conservation of Thermodynamics. Because:

\[
G_e = \frac{\partial W}{\partial (2ct)},
\]

it clearly also is a strain energy release rate when applied to eq.(3.3).

With \( W \) of eq.(3.2) or of eq.(3.3), eq.(3.6) becomes:

\[
\frac{\partial}{\partial c} \left[ \frac{\pi \sigma^2 c^3 t}{E} - G_e 2ct \right] = 0, \quad \text{or:} \quad \frac{\partial}{\partial c} \left[ \frac{\sigma^2 blt}{2E} \left( 1 + \frac{2\pi \sigma^2}{bl} \right) - \frac{\sigma^2 blt}{2E} - G_e 2ct \right] = 0
\]  

(3.7)

giving both the Griffith strength:

\[
\sigma_g = \sqrt{\frac{G_e}{\pi c}}
\]  

(3.8)

This stress is related to the width \( b \) of the specimen of Fig. 3.1 because the whole crack length is regarded including the initial length. The real mean stress in the determining weakest cross section (ligament) with width \( b - 2c \), where fracture occurs, and is determining for newly formed crack area according to the fictitious crack model, is:

\[
\sigma_r = \sqrt{\frac{G_e}{\pi c}} \cdot \frac{b}{b-2c} = \sqrt{\frac{G_e}{\pi b}} \cdot \frac{1}{(c/b) \cdot (1-2c/b)}
\]  

(3.9)

and:

\[
\frac{\partial \sigma_r}{\partial (c/b)} = \sqrt{\frac{G_e}{\pi b}} \cdot \frac{6c/b-1}{(c/b) \cdot (1-2c/b)^2} > 0
\]  

(3.10)

when \( c/b > 1/6 \), what always is the case for critical crack lengths. The fictitious crack model thus should account for hardening. The real stress \( \sigma_r \) increases monotonically with the increase of the crack length \( c \) and no softening behaviour exists at the critical site (as wrongly assumed by the cohesive zone – and fictitious crack models, which are mentioned to be based on this energy per new crack area, eq.(3.9)).

A first order estimation of this increase, is as follows:

For e.g. the critical crack length of \( c/b = 1/6 \), is: \( \sigma_r = \sqrt{G_e / \pi c} = \sqrt{G_e / (\pi b / 6)} \) and the real stress in the fracture plane is: \( \sigma_{r,u} = \sqrt{G_e / (\pi b / 6)} \left( b / (b - b / 3) \right) = 1.5 \sqrt{G_e / (\pi b / 6)} \), thus 1.5 times the Griffith stress. When the crack extends to twice this initial length, to: \( c/b = 1/3 \), then the Griffith stress becomes: \( \sigma_g = \sqrt{G_e / (\pi b / 3)} \), a factor \( 1 / \sqrt{2} = 0.71 \) lower and the real mean fracture stress becomes a factor: \( (1/\sqrt{2}) \left( 1/(1-2/3) \right) = 0.71 \cdot 3 = 2.1 \) higher. In the same way, when the crack is extended to 2.5 times the initial critical value, the Griffith strength \( \sigma_e \) is reduced by a factor 0.63, and is the mean fracture stress \( \sigma_{e,u} \) is 3.8 times the initial value, what flow and
failure indicates of the remaining part of the ligament which no longer follows the Griffith locus. The crack opening, in these cases in the middle of the crack length is proportional to the adjacent unloaded area, thus proportional to the St. Venant distance, and thus proportional to the crack length 2c, thus respectively 2 and 2.5 times the initial crack opening. Clearly there is at the start a strong increase of stress and hardening in the fracture zone, contrarily to the thermodynamic impossible assumption of softening by the fictitious crack models. At the same time there is a resultant unloading of the specimen according to the Griffith strength. Halfway this unloading, an other mechanism is determining as discussed in § 3.6. Softening like behavior thus only exists outside the critical cross section and is identical to a resultant elastic unloading of the specimen outside the fracture zone what is necessary to maintain equilibrium. Wrongly this mean stress in the specimen is taken to be the fracture stress in the fracture plane by the fictitious crack models, leading to impossible softening models. “Softening” thus is not a material property as is arbitrarily assumed by so called non-linear models for wood and other materials. The (softening-) unloading of the specimen also is determined by the decrease of stiffness of the specimen outside the critical cross section what is necessary to maintain equilibrium. This is discussed next in paragraph 3.3.

3.3. The “softening” curve

Softening should be described by the damage theory of Deformation Kinetics [1] but an alternative description of the so called “softening” behavior as a result of former crack propagation alone is possible by the Griffith theory. Straining the specimen of Fig.3.1 to the ultimate load at which the initial crack will grow, gives, according to eq.(3.5):

$$\varepsilon_g = \sigma_g / E_{eff} = \sigma_g \cdot (1 + 2\pi c^2 / bl) / E$$

(3.11)

Substitution of $c = G E / \pi \sigma_g^2$, according to eq.(3.8), gives:

$$\varepsilon_g = \sigma_g / E + 2G^2 E / \pi \sigma_g^3 bl$$

(3.12)

This is the equation of critical (unstable) equilibrium states, representing the “softening” curve. It is shown by the dynamics of crack propagation that velocity of crack propagation is zero at the initial critical crack length and that the Griffith relation is the condition for zero acceleration of crack extension. Thus the crack of Griffith length is in unstable equilibrium but does not propagate. The “softening” curve, eq.(3.12), is called Griffith locus and has a vertical tangent $d\varepsilon_g / d\sigma_g = 0$, occurring at a crack length of:

$$c_c = \sqrt{bl / 6\pi}.$$  

(3.13)

Smaller cracks than $2c_c$ are unstable because of the positive slope of the locus (according to eq.(3.16)). These small cracks, (near the macro-crack tip) extend during the loading stage, by the high peak stresses at the notch of the test specimen, to a stable length and only higher crack lengths than $2c_c$ are to be expected at the highest stress before softening, giving the stress-strain curve of Fig. 3.2 with $\sigma_c$ as top value.

Figure 3.2 – “Softening” curve according to eq.(3.12) for the specimen of Fig. 3.1 or 3.5.
For a distribution of small cracks, \( b \) and \( l \) in eq.(3.13) are the crack distances and the critical crack distance for extension is about 2.2 times the crack length. Because, when \( b \approx 2.2 \cdot (2c_c) \) and \( l \approx 2.2 \cdot (2c_c) \cdot \pi \approx 6 \pi c_c^2 \) according to eq.(3.13). This critical distance also is predicted by Deformation Kinetics [1] and is used in § 3.6 to explain softening by small-crack propagation in the fracture plane.

According to eq.(3.13), the softening line eq.(3.12) now can be given as:

\[
\varepsilon_g = \frac{\sigma_g}{E} \left(1 + \frac{\sigma_c^4}{3\sigma_g^4}\right),
\]

where: \( \sigma_c = \sqrt{EG_c / \pi c_c} \) (3.15) is the ultimate load with \( c_c \) according to eq.(3.13). The negative slope of the stable part of the Griffith locus, being the softening line, is:

\[
\frac{\partial \sigma_g}{\partial \varepsilon_g} = -\frac{E}{\sigma_c^2 - 1}
\]

(3.16)

Vertical yield drop occurs at the top at \( \sigma_g = \sigma_c \) and the strain then is: \( \varepsilon_{gc} = (\sigma_c / E) \cdot (1+1/3) \) and eq.(3.14) becomes:

\[
\varepsilon_{gc} = 0.75 \cdot \left(\frac{\sigma_g}{\sigma_c} + \frac{\sigma_c^3}{3\sigma_g^3}\right),
\]

(3.17)

More in general eq.(3.14) can be written, when related to a chosen stress level \( \sigma_{g1} \):

\[
\frac{\varepsilon_g}{\varepsilon_{g1}} = \frac{\sigma_g}{\sigma_{g1}} \cdot \frac{1+\sigma_c^4/3\sigma_g^4}{1+\sigma_c^4/3\sigma_{g1}^4}
\]

(3.18)

To control whether \( \sigma_c \) changes, eq.(3.18) can be written like:

\[
\frac{\sigma_c}{\sigma_{g1}} = \left(\frac{3 \cdot (\sigma_g / \sigma_{g1})^3 \cdot ((\varepsilon_g / \varepsilon_{g1}) - (\sigma_g / \sigma_{g1}) \cdot (\sigma_g / \sigma_{g1})^3)}{1 - (\varepsilon_g / \varepsilon_{g1}) \cdot (\sigma_g / \sigma_{g1})^3}\right)^{0.25}
\]

with the measured values at the right hand side of the equation. When the occurring softening curve starts to differ from the Griffith locus, \( \sigma_c \) decreases, causing a steeper decline of the curve, due to additional clear wood failure of the fracture plane (outside the macro-crack tip region). This failure by a small-crack merging mechanism is discussed in § 3.6. To measure the fracture energy as area under the softening curve, the displacement of the loading jack due to the mean deformation of the specimen has to be known. This can not be obtained by measuring the gage displacement over an crack (see Fig. 3.3) because it is not known what then is measured and this local unloading ("softening") is not related to the stress increase and later flow and hardening in the ligament.

\[\text{Fig. 3.3. Measuring nonsense data at gage 2, [7].}\]

The area under the Griffith locus (see Fig. 3.2) is mathematically equal to:
This shows that only the first part of unloading follows the Griffith locus, as also indicated earlier by determining the loading stress in the fracture plane. The behavior shows a mechanism in the fracture plane of diminishing solid area and plastic flow of the remaining intact material.

3.4. Fracture energy as area under the softening curve

The basic theory of the energy method, leading to eq.(3.1) and eq.(3.2), should be confirmed by the loading curve (Fig. 3.4.1 and 3.4.2). This will be discussed in this paragraph.

When a test specimen is mechanical conditioned, the effective stiffness is obtained given e.g. by the lines OA and OC in Fig. 3.4.1 and 3.4.2. In Fig. 3.4.1, the area OAB, written as $A_{OAB}$, is the strain energy of the specimen of Fig. 3.1 with a central crack or with two side cracks according to Fig. 3.5.

Figure 3.4.1 - Stress - displacement curve for tension, of the specimen of Fig. 3.1 or 3.5.

Figure 3.4.2 - Descending branch of the stress - displacement curve of Fig. 3.4.1.

Fig. 3.4.3. Mode II fracture energy (and similarly start mode I) Area: $OAB = CAB = ABCD/2$
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with a width “b”, length “l” and thickness “f”, loaded to the stress $\sigma$. During the quasi static crack extension from B to D in Fig. 3.4.1, the constant external load $\sigma$ does the work on the specimen of: $\sigma \cdot b \cdot f \cdot \Delta e_{bd} \cdot l = \sigma \cdot b \cdot f \cdot \delta_{bd} = A_{ABDC}$, where $\Delta e_{bd}$ is the strain increase due to the cracking and $\delta_{bd}$ the corresponding displacement. The strain energy after the crack extension is $A_{OCD}$ and the strain energy increase by the crack extension thus is in Fig. 3.4.1:

$$A_{OCD} - A_{OAB} = A_{OCD} - A_{OCB} = A_{CBD} = A_{ABDC} / 2.$$ 

Thus half of the external energy:

$$A_{ABDC} = \sigma \cdot b \cdot f \cdot \delta_{bd} / 2$$

is the amount of increase of the strain energy due to the elongation by $\delta$, and the other half thus is the fracture energy which is equal to this increase of strain energy. The same follows at unloading at yield drop. Because every point of the softening curve gives the Griffith strength, which decreases with increasing crack length, unloading is necessary to maintain equilibrium. The fracture with unloading step AC in Fig. 3.4.2 is energetic equivalent to the unloading steps AE and FC and the fracturing step EF at constant stress $EB = FD = (AB + DC)/2$. Thus $A_{ABDC} = A_{EBDF}$. Identical to the first case of Fig. 3.4.1, the increase in strain energy due to crack extension is:

$$A_{OCD} - A_{OAB} = A_{OCD} - A_{OBC} = A_{CBD} = A_{ABDC} / 2.$$ 

equal to half the work done by the external stresses during crack propagation and thus also equal to the other half, the work of crack extension. It thus is shown that half the area under the load-displacement curve represents the fracture energy. For mode II, only line OACO in Fig. 3.4.1 is measured and $A_{OAC}$ is regarded to be the fracture energy. Because $A_{OAC} = A_{BAC} = 0.5 \cdot A_{ABDC}$, thus equal to half the area under the displacement curve, the right value is measured and mode II data needs no correction. Because eq.(3.2) is based on the total crack length and the strength is a Griffith stress, the initial value $2c$ of the crack length should be accounted and $\sigma$ and $G_c$ should be related to the whole crack length, including the initial value $2c$, and thus should be related to the whole specimen width $b$ and not to the reduced width of the fracture plane: $b - 2c$ as is done now and leads to an energy dependent on the choice of the initial value of $2c$. Only for the Griffith stress, the energy method of § 6 and §7 applies, based on the energy difference of the cracked and uncracked state. This has to be corrected together with the correction by a factor 2 for the mode I fracture energy $G_i$. A third correction occurs when $\sigma_c$ of eq.(3.14) changes. The apparent decrease of $G_c$ at the end stage of the fracture process is due to an additional reduction of the intact area of the fracture plane of the specimen due to an additional clear-wood crack merging mechanism discussed in § 3.6. In [2], not $A_{ABDC}/2$ is regarded for the fracture energy the totally different amount $A_{OMCO}$ of Fig 3.4.2. This is the irreversible energy of a loading cycle by a crack increment when the specimen is regarded as one giant molecule. The elastic unloading-energies outside the fracture plane of: $A_{OEA}$ and $A_{OCF}$ are now additional measures of the bond reduction for the total molecule, representing a decrease of the apparent enthalpy and entropy terms of the activation energy. The triangle $A_{OMCO}$ thus represents the activation energy of the process [1] which is equal to the reversible work done on the system also represented by $A_{OMCO}$. This is the case because this elastic energy is given by the elastic unloading parts, which are outside the fracture plane $A_{OEA}$ and $A_{OCF}$ together with $A_{OEF}$, the strain energy increase. As discussed in [3], the measurements of [2] indicate the presence of a mechanosorptive process, acting in the whole specimen. Thus $A_{OMCO}$ gives no separate information on the fracture process at the fracture plane and should not be applied as measure of the fracture energy.

3.5. Empirical confirmation

The measurements of [4] are complete by measuring the whole loading and softening curve and using the compact tension tests as control, being a control by the different loading case. The graphs of [4], Fig. 3.6 and 3.7, are the result of tension tests on the specimen of Fig. 3.5.
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The length of the specimen was \( l = 3 \) mm, the width and thickness: \( b = t = 20 \) mm and the notch length \( 2c = 2 \times 5 = 10 \) mm with a notch width of 0.5 mm.

In figures 3.6 and 3.7, the measured stress-displacement is given together with the lines 1 and 2 according to eq.(3.17). The strain \( \varepsilon_g \) follows from the displacements at the \( x \)-axis of the figures

![Figure 3.5 - Geometry of the specimens [4]](image)

Figure 3.5 - Geometry of the specimens [4]

divided through 3 mm, the measuring length and length of the specimen. Because of the small length of 3 mm, not the whole width \( b \) of the specimen is active. Assuming a possible spreading of 1.2:1, through the thickness of 1.25 mm above and below the side notches, the working width \( b_{eff} \) is equal to the length of the fracture plane plus 2 times 1.2 \( \times \) 1.25 or \( b_{eff} = 10 + 3 = 13 \) mm.

Thus the notch lengths in Fig. 3.5 should be regarded to be 1.5 mm in stead of 5 mm. The stresses in the figures 3.6 and 3.7 of [4], are related to the length of the fracture plane and not to \( b_{eff} \), according to the Griffith stress. Thus the given stresses have to be reduced by a factor 10/13 = 0.77.

The standard compact tension tests of [4] did show a stress intensity \( K_c \) of 330 kN m\(^{-3/2}\). This result is independent on the chosen stiffness as follows from the calculation according to the series

![Figure 3.6 - Stress - displacement of specimen T 1409 of [4]](image)

Figure 3.6 - Stress - displacement of specimen T 1409 of [4].

![Figure 3.7 - Stress - displacement of specimen T 1509 of [4]](image)

Figure 3.7 - Stress - displacement of specimen T 1509 of [4]

with a finite element compliance calculation using the isotropic and the orthotropic stiffness and the
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quite different orthotropic stiffness of [5]. The corrected value of \( K_c = 330 \text{ kNm}^{-3/2} \), applies in relevant cases and also should follow from the area under the softening curve of that compact tension test. When half the area of that diagram is taken to be the fracture energy, in stead of the total area, then \( K_c \), mentioned in [4], indeed is corrected to the right value of: 
\[
467/\sqrt{2} = 330 \text{ kNm}^{-3/2},
\]
giving the first empirical verification of the theory.
Regarding the short double edge notched specimens of Fig. 3.5, the measured E-modulus should be related to the effective width of 13 mm in stead of the width of 10 mm of the fracture plane and therefore is 
\[
E = 700 \times 10/13 = 700 \times 0.77 = 539 \text{ MPa}.
\]
The critical energy release rate then is:
\[
G_c = K_c^2 / E = 330^2 / 539 = 200 \text{ N/m}.
\]
The measured value of \( G_c \) from the area under the stress-displacement curve is given in [4] to be 515 N/m. But, because half this area should have been taken and this value is wrongly related to the length of the fracture plane in stead of on \( b_{eff} \), the corrected value is:
\[
G_c = 1/2 \times 515 \times 0.77 = 200 \text{ N/m},
\]
as found above, eq.(3.20), giving again an empirical verification of the theory, now by the tests on the short double edge notched specimens.

As shown before, the softening curve of Fig. 3.6 has a vertical tangent at the top \( d\sigma_y / d\varepsilon_y = \infty \).

The critical crack length for softening is: 
\[
c_c = \sqrt{bl/6\pi}
\]
according to eq.(3.13). Thus:
\[
c_c = \sqrt{b_{eff} / 6\pi} = \sqrt{13.3 - 3/6.1} \times 10^{-3} = 1.4 \times 10^{-3} = 1.4 \text{ mm}
\]
This confirms the mentioned initial St. Venant crack length to be as small as about 1.5 mm.

In Fig. 3.6, at the Griffith maximal stress of \( (0.77) \times 7 = 5.39 \text{ MPa} \), is: 
\[
K_c = \sigma \sqrt{\pi c}
\]
or:
\[
K_c = 5.39 \times \sqrt{\pi} \times 1.4 \times 10^{-3} = 0.36 \text{ MNm}^{-3/2},
\]
thus above the mean value of 0.33 MNm\(^{-3/2}\) for this strong specimen.

Line 1 of Fig. 3.6 gives the primary crack extension, eq.(3.17), with \( \sigma_c = (0.77) \times 7 = 5.39 \text{ MPa} \) and a displacement of about 0.03 mm, (or a strain of 0.03/3 = 0.01). The strength of the fracture plane of 7 to 8 MPa is rather high and only measured 6 times of the 117 tests. The crack does not propagate in a free space, but in the limited length of the fracture plane and this area will be overloaded. Curve 1 therefore levels off from the measurements at \( \sigma = 0.77 \times 4 \text{ MPa} \). Thus:
\[
\sigma_y = \frac{EG_c}{\pi^3 c_c} = 0.57 \times (0.77 \times 7) = 0.77 \times 4 \text{ MPa}
\]
Thus this happens when the crack length has become about 3 times the initial critical value \( c_{c,0} \).

The remaining intact length of the fracture plane then is: 4.4 mm or 4.4/13 = 0.34, while the remaining intact length is 5 mm for small-crack pattern A (of § 3.6), or 5/13 = 0.38. Thus less stress is required for small-crack failure and it thus is probable that macro-crack extension is always due to small-crack propagation toward the macro-crack tip. The level above 4 (to 4.6) MPa is measured in 3 of the 10 specimens of the discussed series T1309/2309 of [4] and an example is given in Fig. 3.7. The other specimens of this series did show lower strength values than 4 MPa, indicating that this strength of the fracture plane according to crack-pattern A was determining for softening. The same applies for further softening. The transition to crack pattern B and to pattern C is according to eq.(3.18), verified by eq.(3.19), showing that in Fig. 3.6, \( \sigma_c \) is constant and equal to \( \sigma_c/0.77 = 7 \text{ MPa} \) for \( \sigma_y/0.77 = 7 \text{ down to } \sigma_y/0.77 = 4 \text{ MPa} \) and then reduces gradually to \( \sigma_c/0.77 = 4.5 \text{ at } \sigma_y/0.77 = 2 \) and further to \( \sigma_c/0.77 = 3 \text{ at } \sigma_y/0.77 = 1 \text{ MPa} \). The same applies for Fig. 3.7, \( \sigma_c/0.77 = 7 \text{ MPa above } \sigma_y/0.77 = 4 \text{ MPa} \) and then reduces in the same way. These results are given in
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Table 3.1. The departure from the Griffith theory by the gradual decrease of $\sigma_c$, below $\sigma_g/0.77 = 4$ MPa, is due to the failure of the high loaded fracture plane what is explained in the next paragraph.

3.6. Crack merging mechanism

The discussed apparent decrease of the fracture energy $G_c$ of the Griffith theory, due to reduction of intact area of the fracture plane of the specimen by small crack extensions at the fracture plane, can be explained, using the equilibrium method, by the joining of the small cracks as follows:

In [3] it is shown that the critical intermediate small crack distance of a fracture process in “clear” wood, and thus in the fracture plane, is about equal to the crack length, as given in scheme A below. In §3.3, a crack distance of 2.2 times the crack length is found, what for simplicity of the model is rounded down here to 2, giving slightly too high stresses (see Table 3.1). For these small cracks, the critical crack length according to eq.(3.13) then is:

$$c = \sqrt{\frac{6b}{6\pi}} = \sqrt{2 \cdot (2c) \cdot 2 \cdot (2c_b) / (6\pi)} = 0.92 \cdot c_0 \approx c_0,$$

for the specimen with row A.

The distance $l$ between the rows, above each other, is always two times the crack length, being the Saint-Venant distance for building up the stress again behind a crack to be able to form a new crack. Thus $l = 2 \cdot 2c$ for row A, and $l = 2 \cdot 6c = 12c$ in row B, and $2 \cdot 14c = 28c$ in row C. The critical crack length $b$ in row A is $b = 4c$, and $b = 8c$ in row B, and $16c$ in row C. Thus when crack pairs of row A join together, a crack length of $6c$ occurs, at a distance $8c$, and so on. The critical crack length thus is for row B:

$$c = \sqrt{\frac{6b}{6\pi}} = \sqrt{12 \cdot 8 \cdot c_0^2 / (6\pi)} = 2.26 \cdot c_0$$

and is

$$c = \sqrt{\frac{6b}{6\pi}} = \sqrt{28 \cdot 16 \cdot c_0^2 / (6\pi)} = 4.88 \cdot c_0$$

for row C.

The critical stress $\sigma_c$ is for row A:

$$\sigma_c = \sqrt{\frac{E G_c}{\pi 0.92 c_0^2}} = 1.04 \cdot \sqrt{\frac{E G_c}{\pi c_0^2}} = 1.04 \cdot \sigma_{cm} \approx 1.0 \cdot 0.77 \cdot 7 = 0.77 \cdot 7.0$$

MPa,

and for row B: $\sigma_c = \sigma_{cm} \cdot \sqrt{\frac{1 \cdot \sqrt{2.26}}{\pi}} = 0.67 \cdot 0.67 \cdot 0.77 \cdot 7 = 0.77 \cdot 4.6$ MPa,

and for row C: $\sigma_c = \sigma_{cm} \cdot \sqrt{\frac{1 \cdot \sqrt{4.88}}{\pi}} = 0.45 \cdot 0.45 \cdot 0.77 \cdot 7 = 0.77 \cdot 3.1$ MPa

The determining strength of the intact part of the fracture plane is:

$$\sigma_m = \sigma_u \cdot 2c / b = \sigma_u \cdot 2c / 4c = \sigma_u / 2 = 4 \cdot 0.77$$

MPa for case A; $\sigma_m = \sigma_u \cdot 2c / 8c = \sigma_u / 4 = 2 \cdot 0.77$ MPa for case B, and $\sigma_m = \sigma_u \cdot 2c / 16c = \sigma_u / 8 = 1 \cdot 0.77$ MPa for case C.

Thus the decrease of the Griffith values $\sigma_c$ and $G_c$ is fully explained by the strength of the intact part of the fracture plane $\sigma_g = \sigma_m$ as is verified by the measurements. As mentioned before, eq.(3.19) of $\sigma_c$, of the softening curve gives the measurement of Fig. 3.6 and 3.7 in the first two columns of Table 3.1, together with the prediction of a crack merging mechanism in column 5 and 6. This mechanism thus precisely explains the decrease of $\sigma_c$ of the softening curve, which also can be approximated by three equations (3.18) for the 3 critical crack densities A, B and C. The strength
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decrease by a factor 0.5 between these crack densities in column 6 causes a decrease of the top-

Table 3.1. Softening by macro crack propagation followed by fracture plane failure.

<table>
<thead>
<tr>
<th>eq. (3.19), data Fig. 3.6</th>
<th>crack merging</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_c / 0.77 )</td>
<td>( \sigma_c / 0.77 )</td>
</tr>
<tr>
<td>Chosen</td>
<td>data</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>4.6</td>
<td>2</td>
</tr>
<tr>
<td>3.0</td>
<td>1</td>
</tr>
</tbody>
</table>

value \( \sigma_c \) of eq. (3.17) of a factor 0.657 in column 1 and 5. Thus: 0.657 \times 7 = 4.6 and 0.657 \times 4.6 = 3. Thus a simple practical approximation of the mean softening curve of all specimens of the series, is possible by applying eq. (3.17) twice (or three times for the highest values), according to line 1 and 2 in Fig. 3.6 and 3.7.

The analysis above shows that in general:

\[ \Delta(2c) / (2c) = \beta_1 \cdot 2c \]

and because the strength decrease is proportional to the area decrease of the fracture plane area of the test specimen, due to the small cracks extension there, the equation becomes:

\[ \Delta(2c) / (2c) = -\beta_2 \cdot \Delta(G_c) \]

(3.26)

giving the explanation of the decrease of \( \sigma_c \).

Eq. (3.26) also can be expressed in the mean crack velocities by replacing \( c \) by \( \dot{c} \cdot t \), with \( \dot{c} \) as mean crack velocity. Thus: \( \Delta(2c) / (2c) = \Delta(\dot{c}t) / \dot{c}t = \dot{c} \Delta / \dot{c} \). Then integration of eq. (3.26) leads to:

\[ G_{c,a} = G_{c,a} - \gamma \ln(\dot{c}) \]

(3.27)

This is measured in [2] and mentioned in [6] for the irreversible work of loading cycles. It is shown in [3] that \( G \) is proportional to the activation energy and thus proportional to the driving force \( K_f \) with reversed sign and Eq. (3.27) can be written relative to a reference \( \dot{c}_m \):

\[ \frac{\sigma_c}{\sigma_{m}} = 1 + \frac{1}{n} \ln(\frac{\dot{c}}{\dot{c}_m}) = \frac{K_f}{K_{f,m}} \]

(3.28)

This semi log-plot, eq. (3.28), is given, as empirical line, in many publications from experiments on e.g. ceramics, polymers, metals and glasses and is e.g. given in [6] for wood. Because the slope is small, also the empirical double log-plot is possible, based on the power law representation of § 4.4. The kinetics shows the same behavior as for clear wood indicating that small-crack propagation is always determining. As shown in [1], two coupled processes act, showing the same time-temperature and time-stress equivalence. One process, with a very high density of sites, provides the sites of the second low site density process, as follows from a very long delay time of the second process. The notched specimen discussed here also shows the low concentration reaction by the strong softening behaviour. Probably the coupled processes are the numerous small-cracks growing towards the macro notch, providing the site for the macro crack to grow as second low
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(crack-) concentration process. This failure mechanism thus applies for every bond breaking process at any level.

3.7 References


4. Discussion of the fracture mechanics models applied to wood

4.1. Introduction

None of the applied models is exact by applying a statically admissible equilibrium system, which suffices the boundary conditions and nowhere violates the failure criterion. Therefore there exist models, which account for strength hardening by applying the R-curve, and at the same time assume strength softening for the same crack extension. The always applied singularity approach of fracture mechanics contains no physical failure criterion for the ultimate state because stresses go to infinity at the singularities and therefore additional models are applied to constitute failure behavior as e.g. energy methods or e.g. numerical crack closure techniques or J-integral to determine the initial strain energy release rate. These methods ignore the existence of singularity peak stresses. This is tried to be corrected by the fictitious crack models (Dugdale, Barenblatt, Hillerborg) by superposition of an internal equilibrium system in order to obtain finite ultimate stresses. However, these rough approximation models are superfluous, because in chapter 2 the exact limit analysis approach without singularities is presented. In this chapter, the applicability of the fictitious crack models will be discussed in comparison with the in this Section C developed limit analysis approach. The crack growth models and the critical energy criteria are discussed in § 4.3.

4.2. The fictitious crack models

The high and infinite high stresses at the crack tip, are replaced by a plastic zone in the Dugdale model following from elastic superposition of closing stresses, equal to the yield stress, on the crack tip zone of a “fictitious” enlarged crack of such a length that the stress in the elastic singularity point (end point of the enlarged crack) becomes zero. The length of that plastic zone is \( r_p \) then is:

\[
 r_p = \frac{\pi}{8} \left( \frac{K}{\sigma_f} \right)^2 = \frac{\pi^2 \sigma_c^2 c}{8 \sigma_f^2}
\]

where \( \sigma_f \) is the yield stress or is regarded to be a cohesive stress.

This leads to a maximal crack opening displacement \( \delta_c \) at the crack tip of:

\[
 \delta_c = \frac{8}{\pi E} \sigma_f r_p = \frac{K_c^2}{E \sigma_f} = \frac{\pi \sigma_c^2 c}{E \sigma_f}
\]

when \( r_p \) from eq.(4.1) is substituted. This result, based on singularity equations, was necessarily based on very small values of \( r_0 \) so that all terms containing not the factor \( r_0^{-0.5} \) were neglected at
the derivation of the equation. For finite values of \( r_0 \) this is not allowed leading to a not correct result. According to the theory, Chapter 2, applies for Mode I, at the crack tip boundary \( r_0 \), at the start of flow, the condition: \( r_0 = 2c\left(\sigma / \sigma_J\right)^2 \) according to eq.(2.29) for the elliptic crack tip and approximately \( r_0 = c\sigma^2 / 2\sigma_J^2 \) according to eq.(2.20) for the circular crack tip of the singularity approach, showing a difference by a factor 4, depending on neglected terms and form of the crack tip determining value of the tangential tensile stress along the crack-tip boundary. The Dugdale numerical factor: \( \pi^2 / 8 = 1.23 \) (based on an enlarged crack length) is between the values of 0.5 and 2, but is too far away from the elliptic value 2, which applies as highest lower bound (the closest to the real value) of limit analysis. Also the theoretical elastic elliptic crack opening displacement of \( \delta_c = (2\sigma_c) / E \) is far above the Dugdale value. The Dugdale model thus delivers a too low and thus rejectable lower bound with respect to the theoretical solution of chapter 2. The Dugdale model is based on a real, not fictive, extended crack length, because there is no need to eliminate a not existing singularity at a not existing fictive crack end, and, when the crack extension would be fictive, the singularity stresses would remain at the real crack end. The superposed compression closing stress thus is a not existing external load on the specimen. As part of an internal equilibrium system, there also should be a tensile stress field, which is in equilibrium with the compressional closing stress field. This means that then the singularity at the real crack end remains. The Dugdale type model thus calculates a real enlarged crack with a partial closed crack end by an external compressional loading, what is not comparable with the crack problem, which is not loaded perpendicular to the crack boundary by a stress depending on the crack opening, but fails independent of this, by the tangential stress in the crack boundary surface (see § 2). This strength determining stress is of higher order with respect to the regarded maximal stresses of the fictitious crack models. Because of the enlarged crack, these models deliver a too low lower bound of the strength. The same thus applies for the Hillerborg model, which is based on closing stresses, proportional to the softening curve, thus proportional to the lowering mean elastic stress far outside the fracture plane and not proportional to the increasing stress at the fracture plane. Therefore a zero tangential stress is found at the location of the highest (strength determining) tangential tensile strength stress. This of course is not right because of the increasing stress and hardening, and no softening, at the fracture plane, (see § 3.2).

For wood it is sufficient to account for empirical plasticity zones around the crack by regarding effective crack dimensions and to regard the critical state at these elastic-plastic boundaries.

### 4.3. Crack growth models

The acknowledged, in principle identical crack growth models for wood, of Williams, Nielsen and Schapery, mentioned in [1], are based on linear viscoelasticity and on the Dugdale-Barenblatt model in order to try to derive the empirical crack rate equation:

\[
\frac{da}{dt} = A \cdot K^n
\]  

(4.3)

The followed procedure is contrary to the normal one, and can not lead to a real solution, because the rate equations are constitutive and has to follow from Deformation Kinetics theory, Section B, see [2], [3], which applies for all materials and is the only way to account for time and temperature dependent behavior. Constitutive equations only can follow from theory and not from general thermodynamic considerations. In [1] is stated that Fig 4.1 of [1], represents eq.(4.3). However, eq.(4.3) is a straight line on a double log-plot, while Fig. 4.1 gives the semi-log-plot which confirms the applicability of the damage equation of Deformation Kinetics [2] in the form: \( \dot{a} \approx C \cdot \exp(\phi \sigma) \),

or: \( \ln(\dot{a}) = \ln(C) + \phi \sigma \)  

(4.4)

This equation is equal to eq.(3.28), discussed in § 3. More appropriate forms of the exact damage equations and power law forms, with the solutions as e.g. the yield drop in the constant strain rate
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test, are discussed in [2] and the meaning of the power law equation, eq.(4.3), is discussed below.

The impossibility of the derivation of the fracture rate equation from the Dugdale-Barenblatt equations follows e.g. from the derivation in [1, § 2.2] of eq.(4.5):

\[ K_k = E_0 \cdot \dot{a}^n \cdot \sqrt{E \varepsilon_y \cdot r_p^n} \]  

(4.5)

based on the relations: \( \varepsilon_y = \sigma / E \) and \( K_k = \sqrt{E \sigma_y \delta_k} \), with \( E = E_0 \cdot r^{-n} \) and \( r_p = \dot{a} \cdot t \). These four (interlinked) relations thus also can be used now to eliminate at least 4 parameters, e.g. \( K_k \), \( \varepsilon_y \), \( r_p \) and \( E_0 \) to obtain an equation in \( E \), \( t \), \( \dot{a} \), \( \sigma \) and \( \delta_c \). When this is done, eq.(4.5) turns to an identity: \( E = E \), and eq.(4.5) thus is not a new derived crack rate equation but an alternative writing of the four relations. The same follows for the other models of § 2.2 of [1] showing comparable parameter manipulations of many critical parameter values which can not be applied independently because they are part of the same failure condition. The models further are based on linear viscoelasticity which does not exist for polymers. It is shown in e.g. [2], page 97, and by the zero creep and zero relaxation tests at page 119, that a spectrum of retardation or relaxation times does not exist. The superposition integral eq.(28) or eq.(51) of [1]:

\[ \varepsilon(t) = \int_{-\infty}^{0} C(t-\tau) \frac{d\sigma(\tau)}{d\tau} d\tau \]  

(4.6)

thus has no physical meaning. This also applies for the power law models of time and power law eq.(4.3), which only apply in a limited range, making predictions and extrapolations outer the fitted range of the data impossible. It thus is necessary to apply the exact theory of Section B, for the kinetics of damage and crack growth processes.

4.4. Derivation of the power law:
The power law equation may represent any function \( f(x) \), as follows from the following derivation. It therefore also may represent, in a limited time range, a real damage equation giving then a meaning of the power \( n \) of the power law eq.(4.3).

Any function \( f(x) \) always can be written in a reduced variable \( x/x_0 \)

\[ f(x) = f_1(x/x_0) \]  

(4.7)

and can be given in the power of a function:

\[ f(x) = f_1(x/x_0) = \left( f_1(x/x_0) \right)^n \]  

and expanded into the row:

\[ f(x) = f(x_0) + \frac{x-x_0}{1!} \cdot f'(x_0) + \frac{(x-x_0)^2}{2!} \cdot f''(x_0) + \ldots \ldots \ldots \]  

giving:

\[ f(x) = \left[ \left( f_1(1) \right)^{1/n} + \frac{x-x_0}{x_0} \cdot \frac{1}{n} \cdot \left( f_1(1) \right)^{1/n-1} \cdot f'(1) + \ldots \ldots \right]^n = f_1(1) \cdot \left( \frac{x}{x_0} \right)^n \]  

(4.8)
when: \((f_1(1))^{1/n} = \frac{1}{n} (f_1(1))^{1/n-1} \cdot f'(1) \) or: \( n = f_1'(1) / f_1(1) \),

where: \( f_1(1) = (\partial f_1(x/x_0) / \partial (x/x_0))_{(x/x_0)=1} \) and \( f_1(1) = f(x_0) \)

Thus: \( f(x) = f(x_0) \left( \frac{x}{x_0} \right)^n \) with \( n = \frac{f_1'(1)}{f_1(1)} = \frac{f'(x_0)}{f(x_0)} \) \hspace{1cm} (4.9)

It is seen from this derivation of the power law, using only the first 2 expanded terms, that eq.(4.9) only applies in a limited range of \( x \) around \( x_0 \). (Using one \( x_0 \) is not limiting for strength problems).

Using this approach on the damage equation: \( \dot{\sigma} = 2C \cdot \sinh(\phi \sigma) \approx C \exp(\phi \sigma) \) gives:

\[ \dot{\sigma} = C \cdot \exp(\phi \sigma) \approx \dot{a}_0 \left( \frac{\sigma}{\sigma_0} \right)^{n \phi \sigma_0} \] \hspace{1cm} (4.10)

The power \( n = \phi \sigma_0 \) of the power law equation follows from the slope of the double log-plot:

\[ \ln(\dot{\sigma}) = n \cdot \ln(\dot{a}_0) + n \cdot \ln(\sigma / \sigma_0) \] \hspace{1cm} (4.11)

Thus: \( n = d \ln(\dot{\sigma}) / d \ln(\sigma / \sigma_0) \) and \( n = \phi \sigma_0 \) gives a meaning of \( n \) as the activation volume parameter \( \phi \sigma_0 \) of the exact equation. The values of “\( n \)” and the matching activation energies of the different creep and damage processes in wood, with the dependency on stress moisture content and temperature, are given in [2]. The constancy of the initial value of the parameter \( \phi \sigma_0 \), independent of \( \sigma_0 \), explains the time-temperature and time-stress equivalence and explains, by the physical processes, why and when at high stresses, the in [1] mentioned value of \( n + 1 \approx 60 \) is measured and at lower stresses, half this value (see [2]).

4.5 References

5. Energy theory of fracture
5.1 Introduction
The failure criterion of clear wood, i.e. wood with small defects, is the same as the failure criterion of notched wood, showing again that the small-crack is dominating and extension towards the macro-crack tip is the cause of macro-crack propagation. This small-crack failure criterion thus delivers essential information on macro-crack behavior as discussed in this chapter. The limit analysis derivation of the boundary value problem and applied Airy stress function of small crack extension are given in Chapter 10.

5.2. Critical distortional energy as fracture criterion
The failure criterion of wood consists of an anisotropic third degree tensor polynomial (see [1], Section A), which, for the same loading case, is identical to the Wu-mixed mode I-II-equation [2], eq.(5.3). The second degree polynomial part of the failure criterion, eq.(5.1), is shown to be the orthotropic critical distortional energy principle for initial yield [3] showing, also empirically, the start of energy dissipation, what is not yet, incorporated in the finite element method [4]. By this dissipation according to the incompressibility condition, the minimum energy principle is followed,
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providing therefore the exact initial yield criterion as (see Section A):

\[
\frac{\sigma_x^2}{XX'} + \frac{\sigma_y^2}{YY'} - 2F_{12} \sigma_x \sigma_y + \frac{\sigma_z^2}{YY'} + \frac{\tau_{xy}^2}{S^2} = 1
\]  

(5.1)

where \(X, Y\) are the tension strengths and \(X', Y'\) the compression strengths in the main directions and \(S\) is the shear strength and: \(2F_{12} = 1/\sqrt{XX'YY'}\)

This value of \(F_{12}\) is necessary for the elastic state which also applies at the starting point of initial stress redistribution and micro-cracking of the matrix. After further straining, \(F_{12}\) becomes zero, \(F_{12} \approx 0\), at final failure initiation. The absence of this coupling term \(F_{12}\) between the normal stresses indicates symmetry, thus (possible random oriented) initial small-cracks are extended during loading to their critical length in the weak planes, the planes of symmetry, only. Then, when these small-cracks arrive at their critical crack-density (discussed in § 3.6) and start to extend further, a type of hardening occurs because the reinforcement prevents crack extension in the matrix in the most critical direction. Then, due to hardening, \(F_{12}\) and all third degree coupling terms of the tensor polynomial become proportional to the hardening state constants [3] and therefore also dependent on the stability of the test and equipment. For the mixed I-II-loading of the crack plane by tension \(\sigma_y\) and shear \(\sigma_x\), the polynomial failure criterion reduces to:

\[
F_x \sigma_x^2 + F_{xx} \sigma_x^2 + F_{yy} \sigma_y^2 + 3F_{266} \sigma_x \sigma_y^2 = 1 \quad \text{or:} \quad \frac{\sigma_y^2}{\sigma_x^2} = 1 - \frac{1}{c} \frac{1}{Y'} \frac{(1 + \sigma_x / Y')}{1 + c \sigma_x / Y'} \]  

(5.2)

with: \(c = 3F_{266} Y' S^2 \approx 0.9\) to 0.99, depending on the stability of the test. When, due to hardening, \(c\) approaches to \(c \approx 1\), Eq.(5.2) becomes Eq.(5.3), the in § 2.3 exact, theoretically explained, Coulomb- or Wu-equation, with a cut off by the line: \(\sigma_x = Y\).

Full hardening is possible when the test rig is stiff enough to remain stable during test. The solution of the crack problem of Irwin as summation of in plane and antiplane solutions in order to use, isotropic stress functions for the orthotropic case, and to apply descriptions in the three different modes and to sum the result for a general mixed mode case is not right for wood because it misses the stress interaction terms and because the failure equation, eq.(5.2), is not orthotropic, by being not quadratic, but contains a third order term and thus does not show orthotropic symmetric. This hardening coupling term is absent in the general accepted Sih-, Paris-, Irwin solution. The stress function which leads directly to the Wu-equation, eq.(5.3), is given in § 2.3 and in [5].

\[
\left(\frac{\sigma_y^2}{S} \right)^2 + \frac{\sigma_x^2}{Y} \approx 1 \quad \text{or:} \quad \frac{K_{1c}^2}{K_{y}^2} + \frac{K_{1c}^2}{K_{1c}^2} = 1
\]  

(5.3)

Wrongly, and against the lack of fit test of Table 2.1, is for wood and other orthotropic materials, Eq.(5.2) generally replaced in literature by:

\[
\frac{\sigma_y^2}{Y^2} + \frac{\tau_{xy}^2}{S^2} = 1, \quad \text{written as:} \quad \frac{K_{1y}^2}{K_{1c}^2} + \frac{K_{1y}^2}{K_{1c}^2} = 1,
\]  

(5.4)

which surely is not a summation of energies, as is stated, but is identical to eq.(5.1) when wrongly is assumed that the compression and tension strength are equal for wood and orthotropic materials. To know the mode of failure, the stresses at the crack boundary should be known. This follows from the exact derivation in [5] and is applied by the virtual crack closure technique of finite element simulation. Contrarily this is based on a separate calculation of the energy release rates of the normal stress in the opening mode and of the shear stress in the sliding mode following the method of Sih, Paris, Irwin by giving the sum of separate solutions for the 3 modes, without interactions, what is assumed to be possible by assumed isotropic and orthotropic symmetry. This however is against eq.(5.2) because the coupling between work by normal and by shear stresses, as given by \(3F_{266} \sigma_x \sigma_y^2\) in eq.(5.2), is not present in the existing methods and “mixed mode”
5.3. Revision of the critical energy release rate equation.

Based on the failure criterion of § 5.2, adaption of the energy release equation is necessary. The Griffith strength equation, eq.(3.8) of § 3: \( \sigma_y^2 = \frac{G_c E_y}{\pi c} \) can be extended by superposition to:

\[
\sigma_y^2 + \tau_{xy}^2 = G_f E_y / \pi c \quad (5.5)
\]

This only is right, when \( G_c \) is not constant but may reach values between \( G_k \) and \( G_{llk} \) depending on \( \sigma_y / \tau_{xy} \), because \( G_c \) also has to satisfy the failure criterion eq.(5.3).

In orthotropic stresses, Eq.(5.5) is: \( \sigma_y^2 + \tau_{xy}^2 / n_6^2 = G_f E_y / \pi c \) and when \( \tau_{xy} = 0 \), is \( G_f = G_k \) and \( K_{ll} = \sqrt{E_y G_k} \). When \( \sigma_y = 0 \) is: \( \tau_{xy}^2 / \pi c = n_6^2 G_{llk} E_y = 4n_6^2 G_k E_y \), because \( K_{llk} = 2n_6 K_k \) (eq.(2.37)). Thus: \( K_{llk} = n_6 \sqrt{E_y G_{llk}} = 2n_6 \sqrt{E_y G_k} \) or: \( G_{llk} = 4G_k \) (5.6)

The failure condition Eq.(5.3) can be written in fracture energies:

\[
\frac{K_I}{K_{llk}} + \left( \frac{K_{II}}{K_{llk}} \right)^2 = 1 = \frac{\sqrt{G_I}}{G_{llk}} + \frac{\sqrt{G_{II}}}{G_{llk}} = \sqrt{\frac{G_I}{G_{llk}}} + \sqrt{\frac{G_{II}}{G_{llk}}} = \frac{1 - \gamma \cdot G_f}{G_{llk}} + \frac{(1 - \gamma) \cdot G_f}{G_{llk}} \quad (5.7)
\]

where, according to eq.(5.5): \( G_f = G_I + G_{II} = \gamma \cdot G_f + (1 - \gamma) \cdot G_f \) (5.8)

Thus:

\[
\frac{\gamma G_f}{(1 - \gamma) G_f} = \frac{K_I^2}{K_{II}^2} \quad \text{or:} \quad \gamma = \frac{1}{1 + \frac{K_{II}^2}{K_I^2}} = \frac{1}{1 + \frac{\tau_{xy}^2}{\sigma_y^2}} \quad (5.9)
\]

and \( \gamma \) depends on the stress combination \( \tau_{xy} / \sigma_y \) in the region of the macro notch-tip and thus not on the stresses of fracture energy dissipation as generally postulated by the I and II failure modes. This stress combination also may follow from a chosen stress field according to the equilibrium method of limit analysis as is applied in § 6 and 7.

With eq.(5.6): \( G_{llk} / G_k = 4 \), eq.(5.7) becomes:

\[
G_f = 4G_k / (1 + \sqrt{\gamma})^2 = G_{llk} / (1 + \sqrt{\gamma})^2 \quad (5.10)
\]

where \( \gamma \) acts as an empirical constant explaining the differences in fracture energies depending on the notch structure and shear slenderness of the beam by the different occurring \( \tau_{xy} / \sigma_y \)-values according to Eq.(5.9).

Applications of the theory with the total critical fracture energy \( G_f \) are given in § 6 and 7.

---

Fig. 5.1. Eq.(5), influence of \( 3F_{266}^2 \sigma_6 \), with the dashed parabolic limit line of eq.(6).

Fig. 5.2. Also same hardening at compression to parabolic data outside the elliptic curve (due to slip line formation).

Interactions as given by Fig. 5.1 and 5.2 can not be described by other methods, because it is not quadratic but contains a third order term and thus does not show to be orthotropic.
The theory is e.g. applied for beams with rectangular end notches as basis of the design rules of the Dutch Timber Structures Code and some other Codes and is a correction of the method of the Euro-Code. In the Euro-Code, an approximate compliance difference is used and a raised stiffness which does not apply for the applied Airy stress function. Further $G_{Ic}$ is used in stead of $G_I$ according to eq.(5.10). Important is further that the theoretical prediction $G_{IIc} = 4G_{Ic}$ is verified by measuring $G_{IIc}/G_{Ic} = 3.5 \ (R^2 = 0.64)$.

At comparing results it should be realized that there is Weibull volume effect of the strength. Further is a strong hardening possible due to compression perpendicular to grain at bending failure of small clear single-edge notched specimens, what wrongly is regarded as $G_{IIc}$ resistance increase. Eq.(5.5) is equal to eq.(5.8) and is an extension of the Griffith strength for combined loading.

5.4 References

6. Energy approach for fracture of notched beams
6.1. Introduction
The theory of total fracture energy, discussed in § 5, was initially developed to obtain simple general design rules for beams with square end-notches and edge joints, loaded perpendicular to the grain design rules of square notches and joints for the Dutch Building Code and later, as correction of the method of [1], published in [2] with the extensions for high beams. Horizontal splitting in short, high beams, loaded close to the support, causes no failure because the remaining beam is strong enough to carry the load and vertical transverse crack propagation is necessary for total failure. This is not discussed here because it is shown that also the standard strength calculation is sufficient. In [3] and [4] the theory is applied to explain behaviour, leading to the final proposal for design rules for the Eurocode, given at § 7.5, and to an always reliable simple design method. In the following, the theoretical basis and implementation of the new developments of the energy approach for fracture of notched beams are given and it is shown that the predictions of the theory are verified by the measurements. The presentation of more data can be found in [2].

6.2. Energy balance
When crack-extension occurs over the length $\Delta x$, along the grain, then the work done by the constant load $V$ is $V \cdot \Delta \delta$, where $\Delta \delta$ is the increase of the deformation at $V$. This work is twice the...
increase of strain energy of the cantilever part: \( V \cdot \Delta \delta / 2 \). Thus half of the external work done at cracking is used for crack formation being thus equal to the other half, the strain energy increase. Thus in general, when the change of the potential energy \( \Delta W = V \cdot \Delta \delta / 2 \) becomes equal to the energy of crack formation, crack propagation occurs. The energy of crack formation is: \( G_c b \Delta x = G_c b h \Delta \beta \), where \( G_c \) is the crack formation energy per unit crack area. Thus crack propagation occurs at 

\[
V = V_f \quad \text{when:} \quad \Delta W = V \Delta \delta / 2 = V^2 \Delta (\delta / V) / 2 = G_c b h \Delta \beta ,
\]

thus when:

\[
V_f = \sqrt{\frac{2G_c b h}{\partial (\delta / V) / \partial \beta}}
\]

(6.1)

and only the increase of the compliance \( \delta / V \) has to be known.

The deflection \( \delta \) can be calculated from elementary beam theory as chosen allowable equilibrium system as a lower bound of the strength. This is close to real behaviour because, according to the theory of elasticity, the deflection can be calculated from elementary beam theory while the difference from this stress distribution is an internal equilibrium system causing no deflection of the beam and also the shear distribution can be taken to be parabolic according to this elementary theory, as only component of this polynomial expansion, contributing to the deflection.

According to the Fig. 6.2, the notch can be seen as a horizontal split, case: \( a = a' \), and case “\( a \)” can be split in the superposition of case “\( b \)” and “\( c \)”, where \( b = b' \).

Case “\( c \)” now is the real crack problem by the reversed equal forces that can be analyzed for instance by a finite element method, etc. From the principle of energy balance it is also possible to find the critical value of case “\( c \)” by calculating the differences in strain energies or the differences in deflections \( \delta \) by \( V \) between case: \( b' \) and case \( a' \), thus differences in deformation of the cracked and un-cracked part to find \( \Delta (\delta / V) \) for eq.(6.1).

Deformations due to the normal stresses \( N \) of case \( c \), are of lower order in a virtual work equation and should not be accounted. It then follows that case \( c \) of Fig. 6.2 is equal to a mode I test and \( G_c = G_{hc} \). When the beam is turned upside down, or when \( V \) is reversed in direction, then \( M' \) and \( V' \) are reversed closing the crack and fracture only is possible by shear, identical to the mode II test and then \( G_c = G_{hc} \).

The change of \( \delta \) by the increase of shear deformation is, with \( h_c = \alpha \cdot h \):

\[
\delta' = \frac{1.2}{G} \left( \frac{\beta h}{ba h} - \frac{\beta h}{bh} \right) \cdot V
\]

(6.2)

Figure 6.2 - Equivalent crack problem according to superposition
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\[
\delta_v = \frac{1.2}{G} \left( \frac{\beta h}{b a h} - \frac{\beta h}{b h} \right) V
\]  
(6.2)

The change of \( \delta \) by the increase of the deflection is:

\[
\delta_m = \frac{V (\beta h)^3}{3 E b (a h)^3 / 12} - \frac{V (\beta h)^3}{3 E b h^3 / 12} = \frac{4V \beta^3}{E b} \left( \frac{1}{\alpha} - 1 \right)
\]  
(6.3)

Thus:

\[
\frac{\partial(\delta / V)}{\partial \beta} = \frac{1.2}{G b} \left( \frac{1}{\alpha} - 1 \right) + \frac{12 \beta^2}{E b} \left( \frac{1}{\alpha^2} - 1 \right)
\]  
(6.4)

The critical value of \( V \) thus is according to eq.(6.1):

\[
V_f = \sqrt{\frac{1.67 G b h^2}{\frac{1}{G} \left( \frac{1}{\alpha} - 1 \right) + \left( \frac{1}{\alpha^2} - 1 \right) \cdot 10 \beta^2}}
\]  
(6.5)

or:

\[
V_f = \frac{\alpha \sqrt{G G_E / h}}{b a h} \sqrt{0.6 (\alpha^3 - \alpha^4) + 6 \beta^2 (\alpha - \alpha^4) G / E}
\]  
(6.6)

For small values of \( \beta \) eq.(6.6) becomes:

\[
V_f = \frac{\alpha \sqrt{G G_E / h}}{b a h} \sqrt{0.6 \cdot (\alpha - \alpha^4)}
\]  
(6.7)

For high values of \( \beta \), eq.(6.6) becomes:

\[
V_f = \frac{\alpha \sqrt{E G_E / h}}{b a h} \beta \sqrt{6 (\alpha - \alpha^4)}
\]  
(6.8)

6.3. Experimental verification

A verification of the prediction of the theory for high values \( \beta \), eq.(6.8), when the work by shear is negligible, is given by Table 6.1 of an investigation of Murphy, mentioned in [1], regarding a notch starting at \( \beta = 2.5 \) and proceeding to \( \beta = 5.5 \). Further also beams were tested with a slit at a distance: \( \beta = 2.5 \). Because the exact eq.(6.6) gives a less than 1 % higher value, eq.(6.8) applies. \( \sqrt{G G_E} = 11.1 \text{ resp. } 10.9 \text{ N/mm}^{1.5} \) and: \( \sqrt{E G_E} = 48.8 \text{ N/mm}^{1.5} \).

This value is used in table 6.1 for comparison of eq.(6.8) with the measurements, showing an excellent agreement between theory and measurement. For all specimens was: \( \alpha = 0.7 \); \( \eta = L/h = 10 \) (\( L \) is distance field loading to support) and \( b = 79 \text{ mm} \). The other values are given in table 6.1.

Table 6.1. Strength of clear laminated Douglas fir

<table>
<thead>
<tr>
<th>( h ) mm</th>
<th>( \beta )</th>
<th>number of tests</th>
<th>( V/\alpha b h )</th>
<th>eq.(6.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>305</td>
<td>2.5</td>
<td>2</td>
<td>0.46</td>
<td>0.47</td>
</tr>
<tr>
<td>305</td>
<td>5.5</td>
<td>2</td>
<td>0.24</td>
<td>0.22</td>
</tr>
<tr>
<td>457</td>
<td>2.5</td>
<td>2</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
<td>457</td>
<td>5.5</td>
<td>1</td>
<td>0.16</td>
<td>0.17</td>
</tr>
</tbody>
</table>

The fracture energy is: \( G_E = \left( 48.8 \right)^2 / 14000 = 0.17 \text{ N/mm} = 170 \text{ N/m}, \) which agrees with values of the critical strain energy release rate. The value of \( K_c \) is about: \( K_c \approx \sqrt{0.17 \cdot 700} = 10.9 \text{ N/mm}^{1.5} = 345 \text{ kN/m}^{1.5}, \) as to be expected by the high density of Douglas fir.
In Table 6.2, data are given for Spruce for low values of $\beta$, to verify the then predicted theoretical behaviour according to eq. (6.7) with energy dissipation by shear stresses only. It appears for these data that the difference between the mean values according to eq. (6.7) and eq. (6.6) are 10% and thus not negligible small and also the values of eq. (6.6) are given to obtain a possible correction factor. It follows from Table 6.2 for Spruce that: $\sqrt{GG_f} = 6.8 \text{ N/mm}^{1.5}$ or:

$$G_f = \frac{6.8^2}{500} = 0.092 \text{ N/mm}$$

For Spruce is $K_f \approx 6.3$ to 7.6 according to [5], depending on the grain orientation and then also applies: $E_2 \approx G$ and: $K_f = \sqrt{E_2G_c} = 6.8 \text{ N/mm}^{1.5}$.

Although the fracture energy is shear-stress energy, failure still is by mode I (of Fig. 6.2) and not by the shear mode II, as is supposed by other models. Thus the total work contributes to failure, whether it is bending stress energy (Table 6.1) or shear stress energy (Table 6.2) and $\gamma = 1$ (eq. (5.9)) for failure of this type of notch by the high tensile stress perpendicular to the grain at the notch root.

In [2] more data are given regarding the strength of square notches. The size influence, or the influence of the height of the notched beam on the strength, is tested on beams with notch parameters $\alpha = 0.5$ and 0.75; $\beta = 0.5$ and heights $h = 50, 100$ and $200 \text{ mm}$ with $b = 45 \text{ mm}$ at moisture contents of 12, 15 and 18%. The strength $\sqrt{GG_f}$ appeared to be independent of the beam depth as to be expected for macro crack extension along an always sufficient long fracture plane.

<table>
<thead>
<tr>
<th>$h$ mm</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\eta/\alpha$</th>
<th>$b$ mm</th>
<th>$n$</th>
<th>$V/bah$ N/mm$^2$</th>
<th>var. coeff. %</th>
<th>$\sqrt{GG_f}$ eq.(6.6)</th>
<th>$\sqrt{GG_f}$ eq.(6.7)</th>
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</thead>
<tbody>
<tr>
<td>120</td>
<td>.917</td>
<td>.25</td>
<td>3.4</td>
<td>32</td>
<td>6</td>
<td>2.36</td>
<td>11</td>
<td>(5.8)</td>
<td>(5.5)</td>
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<td>.833</td>
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<td></td>
<td></td>
<td>6.8</td>
<td>6.2</td>
</tr>
</tbody>
</table>

Testing time more than 1 min., m.c. 11%, $\rho = 510 \text{ kg/m}^3$.

The value of $\sqrt{GG_f}$ at moisture contents of 12, 15 and 18% was resp.: 6.7; 7.7 and 8.0 Nmm$^{1.5}$. Higher values of $\sqrt{GG_f}$ of Spruce, given in [2], are possible for loads close to the support. Then horizontal splitting does not cause failure because the remaining beam is strong enough to carry the total load and the derivation is given by regarding vertical crack propagation necessary for total failure (bending failure of the remaining beam). For this mode I,

$$\sqrt{GG_m} = 57.5 \text{ N/mm}^{1.5} = 1818 \text{ kN/m}^{1.5}$$

( comparable with 1890 kN/m$^{1.5}$ of [5])

For still higher values of $\alpha$, above $\alpha = 0.875$, compression with shear failure is determining by direct force transmission to the support.

In [3] is shown that Foschi’s finite element prediction and graphs, given in [5] can be explained and are identical to eq.(6.8).

### 6.4 References

Section C, Fracture Mechanics


7. Energy approach for fracture of joints loaded perpendicular to the grain.

7.1 Introduction

It was for the first time shown in [2] that fracture mechanics applies for these type of joints. As for square end-notches, the analysis can be based on the compliance change by an infinitesimal crack increase. Because measurements show no difference in strength and fracture energy between joints at the end of a beam (Series G6.1 and G6.2 of [1]) and joints in the middle of the beam (the other G-series), and also the calculated clamping effect difference by crack extension is of lower order, this clamping effect of the fractured beam at the joint in the middle of a beam, has to be disregarded as necessity of the virtual energy equation of fracture disregard lower order terms. This is according to the limit state analysis which is based on the virtual work equations. For end-joints, the split off part is unloaded and there is no normal force and no vierendeel-girder action at all and the situation and fracture equations are the same as for the notched beams of § 6. For joints in the middle of the beam, splitting goes in the direction of lower moments and is stable until the total splitting of the beam. The analysis in [1] and [2] shows this stable crack propagation because the terms in the denominator become smaller at crack length increase, until the shear term remains, giving the maximal value of V according to eq.(7.6), the same value as for end-joints. It thus is not true, as is stated in the CIB/W18-discussion of [1], that the analysis and theory are incorrect when virtual lower order terms are omitted in the analysis and that splitting of joints analysis is not comparable to splitting of notched beam analysis. The proof that this neglecting of the vierendeel-action is right is given (outer the empirical proof by the measurements) by the complete analysis for this case in [3], where also the influence on the strain of normal stresses is accounted, leading to eq.(7.5) containing the negligible clamping effect term in the denominator, (based on the assumption that not total splitting of the beam is the end state).

7.2. Energy balance

For a simple calculation of the compliance difference of the cracked and un-cracked state, (maintaining the clamping action in the end state) half a beam is regarded, as given in Fig. 7.1, loaded by a constant load V. At the start of cracking, the deflection at V increases with \( \delta \) (see Fig. 7.2) and the work done by the force V is: \( 2\Delta W = V'\delta \), which is twice the increase of the strain energy \( \Delta W = V'\delta/2 \) of the beam and therefore the amount \( \Delta W \) is used to increase the strain energy and the other equal amount of \( \Delta W \) is used as fracture energy. Because \( \delta \) is the difference of the cracked and "un-cracked" state, only the deformation of the cracked part \( \beta h \) minus the deformation of that same part \( \beta h \) in the un-cracked state, need to be calculated, because the deformation of all other parts of the beam by load V are the same in cracked and un-cracked state. As discussed at 6.2, the deformation \( \delta \) can be calculated from elementary beam theory of elasticity. It thus is not right to regard an additional deformation \( \delta' \), as is done, due to the non-linearity and clamping effect of the cantilevers \( \beta h \) formed by the crack. The clamping effect change is of lower order at an infinitesimal crack extension. If this effect would have an influence, there should be a difference in notched beams in the splitting force for a real square notch of length \( \beta h \) and a vertical saw cut at a distance \( \beta h \) from the support, because that slit has at least twice that clamping effect (see Fig. 6.2).

For a connection at the middle of a beam the following applies after splitting (see Fig. 7.1). The part
above the crack (stiffness $I_2 = b(1-\alpha)^3h^3/12$) carries a moment $M_2$ and normal force $N$ and the part below the crack (stiffness $I_1 = b\alpha^3h^3/12$) carries a moment $M_1$, normal force $N$ and a shear force $V$. and at the end of the crack a negative moment of about: $M_2 \approx -M_1$. Further is $M_2 = M_1 - V\lambda$, thus $M_1 = V\lambda/2$.

The deformation of beam 2 of the cracked part $\beta h$ is equal to the un-cracked deformation $\delta_{un}$ of that part and the deformation of beam 1 is $\delta_{un}$ plus the crack opening $\delta$ (see Fig. 7.1 and 7.2) and $\delta$ is:

$$\delta = \frac{1}{2} V \frac{\lambda^2}{EI} - \frac{1}{2} \frac{1}{EI} + \frac{1}{4} \frac{1}{EI} = \frac{V \lambda^3}{12 EI} = \frac{V \beta^3}{bE\alpha^3}$$

(7.1)

The deflection difference of the cracked and un-cracked state is total:

$$\delta = \frac{1.2}{G} \left( \frac{\beta h}{b\alpha h} - \frac{\beta h}{bh} \right) \cdot V + \frac{V \beta^3}{bE\alpha^3}$$

(7.2)

The condition of equilibrium at crack length $\beta$ is:

$$\partial(V \cdot \delta/2 - G_c b \beta h) / \partial \beta = 0 \quad \text{or:} \quad \{\partial(\delta/V) / \partial \beta\} \cdot V^2 / 2 = G_c b h \quad \text{or:}$$

$$V_i = \frac{2G_c bh}{\partial(\delta/V)}$$

(7.3)

where $G_c$ is the fracture energy. It follows from eq.(7.2) that:

![Figure 7.1 - Beam with crack by the dowel force of a joint and bending moment](image)

$$\frac{\partial(\delta/V)}{\partial \beta} = \frac{1.2}{bG} \left( \frac{1}{\alpha} - 1 \right) + \frac{3\beta^2}{Eb\alpha^3}$$

(7.4)

and eq.(7.3) becomes:

$$V_f = bah \sqrt{\frac{GG_c/h}{0.6(1-\alpha)\alpha + 1.5\beta^2 G/\alpha E}}$$

(7.5)

giving, for the always relatively small values of $\beta$, the previous found eq.(6.7):

$$V_f = \frac{\sqrt{GG_c/h}}{bah \sqrt{0.6 \cdot (1-\alpha) \cdot \alpha}}$$

(7.6)

which thus also applies for notched beams and for end-joints and verifies the lower bound of the
strength, predicted by the theory of [1]. This also indicates that only work by shear stresses contributes to fracture. The fit of the equation with vierendeel action, eq.(7.5), to the data is not better than the fit by eq.(7.6) what shows that the term $1.5\beta^2 G/\alpha E$ is small with respect to $0.6(1 - \alpha)/\alpha$ and also that $\beta$ is about proportional to $\alpha$ and is of the same order. Comparison of eq.(7.5) and eq.(6.6) shows that the higher value of the end joint is determining for this definition of the strength and the same design rules as for notches are possible for joints when not the joint but splitting is determining. However design should be based on “flow” of the joint before splitting of the beam and the interaction of joint failure and beam splitting has to be regarded as follows.

When crack extension starts of a cantilever beam loaded by a constant load $V$, giving a deflection increase of $\delta$ at $V$, then the applied energy to the beam is $V\delta$. The energy balance equation then is:

$$V\delta = V\delta/2 + E_c$$

(7.7)

where $E_c = V\delta/2$ is the increase of the elastic energy and $E_e$ the energy of crack extension.

Thus $E_c = V\delta/2$ (7.8)

Thus the energy of crack extension is equal to the increase of elastic energy. Eq.(7.8) also can be written with de incremental deflection $\delta = du$:

$$E_c = V^2 d(u/V)/2 = G_f b h d(\beta)$$

(7.9)

where $G_f$ is the fracture energy per unit crack surface and “$bh d(\beta)$” the crack surface increase with “$b$” as width and “$h$” the height of the beam with a crack length $l = \beta h$.

When the load on the cantilever beam, mentioned above, is prevented to move, the energy balance, eq.(7.7) becomes:

$$0 = E_c + E_e$$

or: $E_c = -E_e = -V\delta/2$ (7.10)

for the same crack length and now the energy of crack extension is equal to the decrease of elastic energy in the beam.

When the joint at load $V$ becomes determining and just start to flow at $\delta_1$ when splitting of the beam occurs, then eq.(7.7) becomes:

$$V\delta = (V\delta_1)/2 + V(\delta - \delta_1) + E_c$$

(7.11)

where again $V\delta_1/2$ is the increase of the elastic energy and $V(\delta - \delta_1)$ the plastic energy of the flow of the joint. From eq.(7.11) then follows:

$$E_c = V\delta_1/2$$

(7.12)

the same as eq.(7.8), despite of the plastic deformation.

For connections, plastic deformation in the last case will not yet occur because it is coupled with crack extension. When the dowels of the joint are pressed into the wood, the crack opening increases and thus also crack extension. It can be seen in eq.(7.11), that when flow occurs, the total applied energy $V\delta$ is used for plastic deformation. This is a comparable situation as given by eq.(7.11), and the at the plastic flow coupled crack extension will cause a decrease of the elastic energy. Eq.(7.11) thus for joints is:

$$V\delta = (V\delta_1 - \delta_2)/2 + V(\delta - \delta_1) + E_s$$

(7.13)

where $V\delta_1/2$ is the decrease of the elastic energy by the part of crack extension due to the plastic deformation. From eq.(7.13) now follows:

$$E_s = V(\delta_1 - \delta_2)/2$$

(7.14)

and eq.(7.9) becomes:

$$V = \sqrt{\frac{2G_f b h}{\bar{\epsilon}(u_1 + u_2)/V}}$$

(7.15)
From eq. (7.12) and (7.14) follows that \( V_c \delta_c = V(\delta_1 + \delta_2) \), where \( V_c \delta_c \) is the amount when the connection is as strong as the beam. Thus:

\[
\frac{\delta_1 + \delta_2}{\delta_c} = \frac{V_c}{V} = \frac{n V_u}{n V_u} = \frac{n}{n_c} \tag{7.16}
\]

where \( V_u \) is the ultimate load of the dowel at flow and \( n \) the number of dowels.

Substitution of eq. (7.16) into eq. (7.15) gives:

\[
V = \sqrt{\frac{2G_f bh}{\partial(u_c/V)/\partial \beta}} \cdot \frac{n}{n_c} \tag{7.17}
\]

what is equal to \( \sqrt{n/n_c} \) times the strength according to eq. (7.9) for \( u = u_c \), thus \( \sqrt{n/n_c} \) times the splitting strength of the beam as is applied in [1].

According to eq. (7.13), the theoretical lower bound of \( V \) according to eq. (7.17) occurs at \( \delta_1 = \delta_2 \), thus when \( n/n_c = 1/2 \). In [1], the empirical value of 0.5 to 0.4 is mentioned according to the data giving:

\[
V = \sqrt{\frac{2G_f bh}{\partial(u_c/V)/\partial \beta}} \cdot \sqrt{0.45} = \sqrt{\frac{2G_f bh}{\partial(u_c/V)/\partial \beta}} \cdot 0.67 \tag{7.18}
\]

This requirement for “flow” of the joint at failure: \( \sqrt{G G_f} = 0.67 \cdot 18 = 12 \text{ Nmm}^{-1.5} \) is included in the Eurocode (see § 7.5).

The condition \( \delta_1 = \delta_2 \) means that there is sufficient elastic energy for total unloading and thus full crack extension with sufficient external work for plastic dissipation by the joints. According to eq. (7.13) is for that case:

\[
E_c = V \delta_1 \tag{7.19}
\]

### 7.3. Experimental verification

The value of \( E_c \) of eq. (7.19) is 12 Nmm\(^{-1.5}\) as follows from the test data given in [1]. In [1], first test-results of 50 beams of [4] with one or two dowel connections are given of beams of 40x100 and 40x200 mm with \( \alpha - \) values between 0.1 and 0.7 and dowel diameters of 10 and 24 mm. In all cases \( n \leq 0.5 \cdot n_c \) and not splitting but flow of the connection is determining for failure reaching the in [1] theoretical explained high embedding strength by hardening as to be expected for the always sufficient high spreading possibility of one- (or two-) dowel joints. The same applies for the 1 and 2 dowel joints of the Karlsruhe investigation. Splitting then is not the cause of failure but the result of post-failure behaviour due to continued extension by the testing device.

Table 7.1 of [1] shows that for series B, splitting of the beam is determining. Whether there are 10, 15, 20 or 25 nails per shear plane, the strength is the same: \( \sqrt{G G_f} = 16.7 \text{ Nmm}^{-1.5} \). This is confirmed by the too low value of the embedding strength of the nails \( f_e \) of series B. A more precise value of \( \sqrt{G G_f} \) follows from the mean value of 17.1 Nmm\(^{-1.5}\) of series B2 to B4. Then the value for 10 nails of series B1 is a factor 15.5/17.1 = 0.9 lower.

Thus \( \sqrt{n/n_c} = \sqrt{10/n_c} = 0.9 \). Thus \( n_c = 12 \) for series B. This means that the number of 5 nails of series A is below \( n_c / 2 = 6 \) and the measured apparent value of \( \sqrt{G G_f} \) is the minimal value of \( \sqrt{G G_f} \cdot 0.5 n_c / n_c = 17.1 \cdot 0.5 = 12.1 \text{ Nmm}^{-1.5} \). The same value should have been measured for series C because the number of 3 nails also is below \( n_c / 2 = 6 \). Measured is 11.7 Nmm\(^{-1.5}\). For the 53 beams of all the series G of [1] this is 12.0 Nmm\(^{-1.5}\). As mentioned a mean value of 12 is now the Eurocode requirement.
The value of \(0.5 \cdot n_c\), depends on dimensioning of the joint and thus on amount of hardening by the spreading effect of embedding strength. Thin, long nails at larger distances in thick wood members are less dangerous for splitting and show a high value of \(n_c\). For series G, with \(b = 100\) mm, \(n_c / 2\) is at least below 8 nails. For series V of [1] with dowels of 16 mm, \(n_c = 8.6\). For design, \(n_c\) need not to be known. But dimensioning of the joint to meet also the requirement of \(\sqrt{G G_c} = 12\) Nmm\(^{-1.5}\), will lead to the number of nails of \(n_c / 2\). This dimensioning also determines the value of \(f_c\). The value of \(f_c = 4.4\) MPa of series A is lower than \(f_c = 6.2\) MPa of series C, in proportion to the square root of the spreading lengths per nail as expected from theory [1].

### 7.4. Design equation of the Eurocode 5

As discussed in [1], the shear capacity is (for \(h_e \leq 0.7\) \(h\))

\[
\frac{V_u}{b a h} = 10.3 \sqrt{\frac{\alpha}{1 - \alpha}} = 10.3 \sqrt{\frac{h_e}{(h - h_e)}}
\]

where \(10.3 = (2/3)\sqrt{(G G_c / 0.6)}\) is the characteristic value.

This can be replaced by the tangent line through this curve at point \(\alpha = 0.5\) giving:

\[
\frac{V_u}{b a h} = 1.7 \sqrt{G G_c} = 1.7 \cdot (2/3) \cdot 12 = 13.6\ \text{Nmm}^{-1.5}.
\]

### 7.5. References


8. Conclusions §1 to §7

- Important conclusions are that 1) the orthotropic solution and singularity approach don’t apply for wood, but the solution for the isotropic matrix is determining; that 2) softening is elastic unloading outside the fracture zone and thus is not a material property; that 3) failure is always by tension and is not coupled to the postulated failure modes I and II; that 4) the softening curve is determined by a crack merging mechanism and all points of this curve are fully explained by the ultimate state of the intact part of the fracture plane; that 5) the fracture energy as area under the softening curve should be based on half this area for mode I, as is already applied for mode II. The stress should be Griffith stress related to the whole width of the specimen and not to the intact part of the fracture plane and the initial crack length (2c) should be summed up to the total crack length to make the energy method possible based on the energy difference of the cracked and the fully un-cracked state of the specimen.

- Fracture mechanics of wood and comparable materials appears to be determined by small-crack propagation towards the macro-crack tip. This follows e.g. from the same failure criterion for “clear” wood and for macro-crack extension. The presence of small-crack propagation is noticeable by the Weibull volume effect of timber strength. There is no influence on macro-crack propagation of the geometry of notches and sharpness of the macro crack-tip in wood (against orthotropic theory). Thus orthotropic fracture mechanics is not determining. This also follows from the nearly same fracture toughness and energy release rate for wide and slit notches and the minor influence of rounding the notch (also against orthotropic theory). Determining thus is the influence of small cracks in the isotropic matrix for the total behavior, having the same influence at the tip of wide as well as slit notches.

- The orthotropic boundary value limit state solution does not apply for fracture mechanics of reinforced materials because of lack of equilibrium of the matrix stresses. The equilibrium method of Limit design thus has to be based on the isotropic matrix stresses.

- The always, for all materials, applied singularity approach appears to be not valid for orthotropic materials and only may apply for the isotropic matrix. The method, based on collinear crack propagation, does not predict the right strength for combined stresses. Instead therefore, the complete solution of the Airy stress function, based on the flat elliptical crack, has to be applied.

- The empirical mixed I-II-mode fracture criterion is explained by the elliptical small-crack approach, providing the exact theoretical basis of this criterion. This criterion is the consequence of the ultimate uniaxial cohesive strength along the micro-crack boundary. The theory therefore also explains the relations between \( K_{ic} \) and \( K_{IIIc} \) in TL- and in RL-direction and the relations between the related fracture energies. This leads to one overall apparent critical energy release rate which may be different for different structures but is independent of the stress combinations of the dissipated strain energy of fracture. Whether, for a square end-notch, work is done by only bending or by only shear deformation, failure is in mode I and not in mode II in the last case as predicted by other models.

- The orthotropic approach, based on equilibrium of the homogenized reinforcement in wood gives incorrect results, because the matrix is not in equilibrium and does not satisfy the strength criterion. It therefore is necessary to start with equilibrium, compatibility and strength requirements of the
isotropic matrix stresses providing a simple orthotropic-isotropic transformation of the Airy-stress function. for the total solution.

- Based on this approach is: $K_{Ic} = \sqrt{E_y G_{lc}}$, $K_{IIc} = n_b \sqrt{E_y G_{IIc}}$ and $G_{IIc} = 4G_{lc}$

$$G_I = 4G_{lc} \frac{1}{(1 + \sqrt{\gamma})^2} = G_{IIc} \frac{1}{(1 + \sqrt{\gamma})^2} \quad \text{with} \quad \gamma = 1/(1 + \frac{\tau_{xy}}{\sigma_y^2})$$

and:

$$n_b = (2 + v_{21} + v_{12}) \left( G_{xy} / E_y \right)$$

- The theoretical value of $G_{IIc} = 4G_{lc}$ is verified by reported measurements where ratio 3.5 is found ($R^2 = 0.64$) instead of 4. This lower measured ratio is due to the applied too high value of $G_{lc}$ which should be corrected to be equal to the energy release rate.

- It is shown, that the models applied to wood, (as necessary replacement of the infinite fracture stresses of the singularity approach), as e.g. the Dugdale model, fictitious crack model and crack growth models are incorrect and have to be replaced by the general theory.

- A derivation of the softening curve is given based on small-crack extensions. The softening curve follows at the start the “stable” part of the Griffith locus. This means that every point of the softening curve gives the Griffith strength. This curve depends on only one parameter, the maximal critical Griffith stress $\sigma_c$ and therefore depends on the critical crack density. This applies until half way of unloading. The fracture energy is down to this point equal to the critical energy release rate. After that, the strength of the fracture plane of the test specimen becomes determining due to a crack merging mechanism, changing the crack density and intact area of the fracture plane and therefore causing a decrease of $\sigma_c$ and an apparent decrease of the fracture energy. The strength at every point of the softening curve is fully determined by the strength of the intact area of the fracture plane. Softening thus is a matter of elastic unloading of the specimen outside the fracture zone and softening thus is not a material property.

- The fracture energy for mode I is stated in literature to be equal to the area under the softening curve divided through the crack length. This is not right. It is half this area when the fracture plane is not limiting. This is applied and accepted for mode II in wood.

- It also is stated that the area of a loading cycle at softening, divided by the area of the crack increment, is equal to the fracture energy. This also is not right. It is shown that this energy is proportional to the apparent activation energy of all processes in the whole test specimen.

- A revision is necessary of all published mode I data of the fracture energy, based on the area of the softening curve, because of the dissimilar behavior of post fracture behaviour giving no right prediction of the fracture energy. Therefore this area method should not be used anymore. A right simple description follows from the derived apparent energy release rate adapted to the measured strength data.

- The theory shows that the Eurocode design rules for beams with rectangular end notches or joints should be corrected to the right real compliance difference and the right measured uniaxial stiffness.

- The verification of the derived theory by measurements shows the excellent agreement. The method provides an exact solution and is shown to be generally applicable also for joints and provides as simple design equations as wanted.


Because the Weibull size effect is normally not regarded as a fracture mechanics subject, this influence is discussed in a separate chapter.

9.1. Overview

A new explanation is given of the strength of wide angled notched timber beams by accounting for a Weibull type size effect in fracture mechanics. The strength of wood is described by the probability of critical initial small crack lengths. This effect is opposed by toughening by the probability of having a less critical crack tip curvature. The toughening effect dominates at the
different wide angle notched beams showing different high stressed areas by the different angles and thus different influences of the volume effect. This is shown to explain the other power of the depth in Eq.(9.18) and (9.19) than applies for the sharp notch value of 0.5 of Eq.(9.17). It further is shown to explain why for very small dimensions, also for sharp notches, the volume effect applies. The explanation by the Weibull effect implies that the strength depends on small crack extension, in the neighbourhood of the macro crack tip. This initial crack population can be different for full scale members indicating that correction of the applied data is necessary and that additional toughness tests have to be done on full scale (or semi full scale) test specimens. Small cracks fracture mechanics is discussed in chapter 10.

9.2. Introduction

Fracture mechanics of wood is normally restricted to fracture along the grain. It is e.g. not possible to have shear crack propagation across the grain. Also the mixed mode crack follows the weak material axes and only may periodically jump to the next growth layer at a weak spot. Thus the direction of the collinear crack propagation is known. As shown in § 2, the singularity approach gives no right results in this case and the analysis has to be based on linear elastic flat elliptic crack extension by the maximal stress at the elastic-plastic boundary around the small crack. This response at randomized stress raisers near weak spots is indicated by the volume effect of the strength. There also is no clear influence on macro-crack propagation of the crack geometry and notch form and sharpness of the macro crack tip, showing orthotropic fracture mechanics to be not decisive. This also is indicated by the not orthotropic, but isotropic relation between mode I stress intensity and strain energy release rate of wood. The determining small crack behaviour also follows from the failure criterion of common un-notched wood being of the same form as the theoretical explained fracture mechanics criterion for notched wood. The matrix is determining for initial failure and not the reinforcement. The failure criterion of unnotched wood shows no coupling term between the reinforcements in the main directions confirming the orthotropic strength schematization to be not determining. The determining small crack dimension follows from the Weibull size effect. The here treated strength of wide angle notched beams is an example of a determining size effect in fracture mechanics.

The strength analysis of [1] of wide angle notched beams, given in Fig. 9.1, was based on the orthotropic Airy stress function. However, despite of the dominant mode I loading, none of the solutions of this function are close enough to the measurements to be a real solution. The reason of this is the absence of the Weibull size effect in the equations as will be shown in this paragraph.

The in [1] chosen solutions of the biharmonic Airy stress function are:

\[
\begin{align*}
 r_1^{2n} \cos(n\theta), \quad r_1^{2n} \sin(n\theta), \quad r_2^{2n} \cos(n\theta_1), \quad r_2^{2n} \cos(n\theta_2)
\end{align*}
\]

resulting in:

\[
\begin{align*}
 \{\sigma_r, \sigma_\theta, \sigma_r\theta\} = \frac{K_A}{(2\pi r)^n} \left\{ f_1(n\theta), f_2(n\theta), f_3(n\theta) \right\}
\end{align*}
\]

where \(K_A\) is the stress intensity factor and “\(r\)” the distance from the notch root. In the direction of crack extension, along the g rain (\(\theta = 0\)), the tensile strength perpendicular to the grain \(\sigma_\theta\) is determining for fracture. The boundary conditions for the different notch angles \(a/g\) provide
different values of the power “n” and thus different slopes of the lines in Fig. 9.2. However, it is theoretically not possible that these lines intersect through one point, as is measured, because the different boundary conditions by different notch angles cannot be satisfied at the same time and the chosen mathematical solution of [1] thus has to be rejected. The fact that these lines cross one point, at the elementary volume, proves the existence of a volume effect of the strength. This is introduced in the fracture mechanics energy method calculation in § 9.4. In § 9.3, the derivation of the size effect is given to show the equivalent derivation of the toughening size effect in § 9.4.

9.3. Size effect

Due to the initial small crack distribution, clear wood shows a brittle like failure for tension and shear. According to the Weibull model, the probability of rupture, due to propagation of the biggest crack in an elementary volume $V_0$ is equal to $1 - P_s(\sigma)$, when $P_0$ is the probability of survival. For a volume $V$ containing $N = V / V_0$ elementary volumes the failure probability is:

$$1 - P_s = (1 - P_0)(1 - P_0)(1 - P_0)\cdots = (1 - P_0)^N.$$  

Thus $\ln(1 - P_s) = N \ln(1 - P_0) \approx -NP_0$ because $P_0 << 1$. Thus the probability of survival of a specimen with volume $V$, loaded by a constant tensile stress $\sigma$, as in the standard tensile test, is given by:

$$P_s(V) = \exp(-NP_0) = \exp\left(-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^k\right)$$  \hspace{1cm} (9.2)

where $P_0(\sigma) = (\sigma / \sigma_0)^k$ is chosen, because the power law of $\sigma$ may represent any function of $\sigma$ around a chosen stress value as e.g. the mean failure stress (see § 4.4 for the proof). For a stress distribution, Eq.(9.2) becomes:

$$P_s(V) = \exp\left[-\int\left(\frac{\sigma(x, y, z)}{\sigma_0}\right)^k \, dx \cdot dy \cdot dz / V_0\right]$$  \hspace{1cm} (9.3)

This specimen has an equal probability of survival as the standard test specimen Eq.(9.2), when the exponents are equal thus when:

$$\int\left(\frac{\sigma(x, y, z)}{\sigma_0}\right)^k \, dV = \left(\frac{\sigma_s}{\sigma_0}\right)^k V_s$$  \hspace{1cm} (9.4)

For a constant stress $\sigma(x, y, z) = \sigma$, the specimen strength thus will decrease with its volume $V$ according to:

$$\sigma = \sigma_s \cdot \left(\frac{V_s}{V}\right)^{1/k}$$  \hspace{1cm} (9.5)
where $\sigma_s$ is the mean strength of the specimen with volume $V_s$. The power $k$ depends on the coefficient of variation $s/\sigma$ according to:

$$\left(\frac{s}{\sigma}\right)^2 = \frac{\Gamma(1+2/k)}{\Gamma^2(1+1/k)} - 1$$

(9.6)

From the row-expansion of the Gamma-functions it can be seen that:

$$k \cdot \frac{s}{\sigma} = f\left(\frac{s}{\sigma}\right) \approx 1.2$$

(9.7)

where $f(s/\sigma)$ is normally a little varying function. Thus: $1/k = s/(1.2 \cdot \sigma)$

For a stress distribution, Eq.(9.4) becomes:

$$\int \left(\frac{\sigma_m}{\sigma_0}\right)^k \left(\frac{\sigma(x,y,z)}{\sigma_m}\right)^k \, dx\,dy\,dz = \left(\frac{\sigma_m}{\sigma_0}\right)^k V_{ch} = \left(\frac{\sigma_s}{\sigma_0}\right)^k V_s$$

(9.8)

where $\sigma_m$ is the determining maximal stress in volume $V$ and $V_{ch} = \int \left(\sigma/\sigma_m\right)^k \, dV$, a characteristic volume. Eq.(9.8) thus becomes:

$$\sigma_m = \sigma_s \left(\frac{V}{V_{ch}}\right)^{1/k} = \sigma_s \left(\frac{V_s}{V_{ch}}\right)^{1/1.2\sigma}$$

(9.9)

This applies for the strength of common unnotched specimens.

This strength also is determined by fracture mechanics. The tensile strength is e.g.:

$$f_t = \frac{K_c}{\sqrt{\pi c}} \quad \text{or} \quad f_t = f_{ts} \sqrt{\frac{c_t}{c}}.$$ 

(9.10)

where $K_c$ is the stress intensity factor.

Substitution of the strength according to Eq.(9.5) (or Eq.(9.9)) leads to:

$$f_t = f_{ts} \sqrt{\frac{c_t}{c}} = f_{ts} \left(\frac{c_t}{c}\right)^{0.5} \left(\frac{V_s}{V}\right)^{1/k}.$$ 

(9.11)

This equation gives the probability of a critical Griffith crack length $c$ leading to fracture. Also in this case, a crack toughening mechanism is thinkable, discussed in § 9.4, leading to the opposite volume effect with a negative value of the exponent $1/k$. This can not be distinguished and the resultant value of $1/k$ then is given by Eq.(9.11). Because for every type of wood material the value of $c$ is specific, determining the specimen strength, Eq.(9.9), as shortcut of Eq.(9.11), is applied in practice.

According to Eq.(2.29), the stress intensity factor of Eq.(9.10) is: $K_c = \sigma_s \sqrt{\pi r/2}$ where $\sigma_s$ is the equivalent cohesion strength at the crack tip boundary and $r$ is the radius of the elastic-plastic boundary of the crack tip zone. A constant stress intensity factor $K_c$ means that $\sigma_s \sqrt{r}$ is constant and only the crack length $c$ is a variable as for brittle fracture. Toughening means an increase of the plastic zone, thus of $r$ of the small cracks, within the characteristic volume. This influence is visible at the different wide angle notches as discussed in § 9.4.

Because fracture across the grain is tough and the lengths of applied beams don’t vary much, the size effect of the length dimension is small and the volume effect for bending is replaced by a height effect of the beam only. It is postulated that this absence of a width effect is explained by the constant widths of $2b'$ of 2 planes of weakness adjacent to the sides of the beam due to the cutting action at manufacturing. Then: $(V_s/V_{ch})^{1/k} = (2b'hl/2b'hl)^{1/k} = (h_j/h)^{1/k}$, becomes the height factor of the Codes. This width effect is applied in § 9.4.
9.4. Size effect of wide notched beams

The analysis of the strength of the notched beams can be based on the energy method where the critical fracture energy is found from the difference of the work done by the constant force due to its displacement by a small crack extension minus the increase of the strain energy due to this displacement. According to this approach of [3], [4], and § 6, the bending stress $\sigma_m$ at the end of the notched beam at $l = \beta D$ in Fig. 9.1 is:

$$\sigma_m = \frac{6V_0 \beta D}{b(aD)^2} \approx \sqrt{\frac{6EG_c / D}{\sqrt{(\alpha - \alpha^2)}}}$$ (9.12)

when the notch is not close to the support. In [1] is chosen: $\alpha = d/D = 0.5$, what means that $d = a$. Further the length is $l = 2D$ when $g/a = 0$ and 2, while $l = 4D$ for $g/a = 4$ in Fig. 9.1. $E$ is the modulus of elasticity and $G_c$ the critical energy release rate, given in [3]. Eq.(9.12) applies for the rectangular notch $(g = 0)$. For wide notch angles a more complicated expression applies because of the changing stiffness over length $\Delta l$ of the crack extension. However, for given dimensions and loading, the basic form of the equation is the same as Eq.(9.12), thus:

$$\sigma_m = B \sqrt{EG_c / D}$$ (9.13)

where $B$ is a constant depending on dimensions and notch angle. According to §2 and [3] is, as mentioned, $\sqrt{EG_c} \approx K_c \approx \sigma_c \sqrt{\pi \lambda}$, where $\sigma_c$ is the equivalent cohesion strength and the crack tip radius $r$ is the only parameter of the notch strength. The volume effect depending on the stress follows from § 9.3 and the analysis thus can be based on the flow stress and the characteristic volume around the notch tip. For the probability of a critical value of $r$, of the small initial cracks within the high stressed characteristic volume around the notch tip, the probabilistic reasoning of § 9.3 can be repeated as follows. The probability of having a critical flaw curvature $1/r$ in an elementary volume $V_0$ is equal to $1 - P_0(1/r)$, when $P_0$ is the survival probability. For a volume $V$ containing $N = V / V_0$ elementary volumes the survival probability is in the same way:

$$P_s(V) = \exp(-NP_0) = \exp \left(- \frac{V}{V_0} \left( \frac{r}{r_0} \right)^{-k} \right)$$ (9.14)

where $P_0(1/r) = (r_0 / r)^k$, because the power law may represent any function in $1/r$. At “flow”, this probability is not a function of $\sigma$, but of the flow strain, given by a critical $r$

Equal exponents for the same probability of failure in two cases now lead to:

$$r = r_c \left( \frac{V}{V_0} \right)^{1/k}$$ (9.15)

and Eq.(9.13) becomes:

$$\sigma_m = B' \sigma_s \sqrt{r_c / D} \left( \frac{V}{V_s} \right)^{1/2k} \quad \text{or:} \quad \sigma_m = \sigma_{m0} \left( \frac{D}{D_0} \right)^{-0.5} \left( \frac{V}{V_0} \right)^{1/2k}$$ (9.16)

For the notch angle of $90^{\circ}$, $(g = 0$ in Fig. 9.1), or smaller angles, the high stressed elastic region around the crack tip is, as the fracture process zone itself, independent of the beam dimensions. Thus in characteristic dimensions $V = b'l'h' = V_0$ and Eq.(9.16) becomes:

$$\sigma_m = \sigma_{m0} \left( \frac{D}{D_0} \right)^{-0.5}$$ (9.17)

independent of a volume effect. For the widest notch angle of $166^{\circ}$ $(g/a = 4)$, there is a small stress gradient over a large area and $V$ is proportional to the beam dimensions. Thus: $V(\cdot) b'dl = \gamma D \delta D \beta D = \gamma \beta \delta D^3$ and: $V/V_0 = (\gamma \beta D^3 / \gamma \delta D_0^3) = (D / D_0)^3$. Thus is, with $1/k = 0.18$: 45
Section C, Fracture Mechanics

\[ \sigma_m = \sigma_{m0} \left( \frac{D}{D_0} \right)^{-0.5+3/(2k)} = \sigma_{m0} \left( \frac{D}{D_0} \right)^{-0.23} \]  \hspace{1cm} (9.18)

For the angle of 153.40°, \((g/a = 2)\), the high stressed region dimensions becomes proportional to the dimensions \(b\) and \(D\) and:

\[ V/V_0 = \frac{b d l}{(b_0 d_0 l)} = \left( \frac{\gamma \delta D^2}{\gamma \delta D_0^2} \right) = \left( \frac{D^2}{D_0^2} \right) \]  \hspace{1cm} (9.18)

It follows from Fig. 9.2, that the values of exponents of \(-0.5\), \(-0.32\), and \(-0.23\) are the same as measured. The coefficient of variation of the tests must have been: 1.2 × 0.18 = 0.22, as common for wood. According to the incomplete solution of [1], discussed in the Introduction, these values of the exponents were respectively -0.437, -0.363 and -0.327, thus too far away from the measurements. The explanation of no volume effect of sharp notches due to the invariant characteristic volume, independent of the beam dimensions, explains also why for very small beams, also for sharp notches, there is a volume effect because then the beam dimensions are restrictive for the characteristic volume. As shown above, the exponent may change from \(-0.5\) to \(-0.23\) with decrease of the beam dimensions. This is measured and e.g. discussed at pg. 85 of [2] and it now is shown that toughening (and not nonlinear behaviour) is the explanation of this volume effect.

The lines in Fig. 9.2 intersect at the elementary Weibull volume wherefore the depth dimension is \(10^{0.6} = 4\) mm with a material bending strength of 147 MPa.

9.5. Conclusions regarding the size effect
- A explanation is given of the strength of wide angled notched beams of [1] by introducing the Weibull type size effect in fracture mechanics based on the critical curvature of the initial small cracks near the high stressed notch tip zone.
- For sharp notch angles, up to \(90°\), there is no volume effect due to the constant volume of the characteristic volume, containing the fracture process zone. For wider notch angles, the peak stresses and stress gradients become lower and are divided over a larger region and influenced by the dimensions and thus a volume effect correction applies.
- The intersect of the three lines in Fig. 9.2, with different values of “n” of Eq.(9.1), due to different boundary conditions by the different notch angles, can not be explained by the boundary value analysis. This intersect only can be explained to be due to the volume effect of the strength indicating failure by small crack extension within the high stressed region at the notch tip.
- Using the Energy approach and the volume effect correction according to Eq.(9.16), the measured values of the powers of the depths (or the slopes of the lines of Fig. 9.2) are precisely explained.

9.6. References

10. Small crack fracture mechanics
10.1. Introduction.
Because small crack behavior is a new subject, it is discussed in a separate chapter. Determining small crack extension is e.g. indicated by the volume effect of the strength and by the
no clear influence on macro-crack propagation of the crack geometry and sharpness of the crack-tip of notches in wood. Also the only possible explanation of “softening” and the dynamics of crack propagation by small crack merging shows this behavior. The failure criterion of clear wood and of timber [1], [2], and the failure criterion by a single macro notch [3], [4], are the same, showing that small-crack extension towards the macro-crack tip is the cause of macro-crack extension. This is confirmed by the fact that the stress intensity factor is the same independent on the macro-form and dimensions of the notch. It also is confirmed by molecular deformation kinetics, showing the same processes in clear- and in notched wood (see discussion Annexes B). Also the exact solutions given in [4] and below of the geometric correction factor and of [5] of the strength behavior of long post-critical crack lengths is totally based on small crack behavior. The small-crack merging mechanism explains, in [3], precisely the mode I softening curves of [6]. The failure criterion [1], shows no coupling term between the normal stresses at “flow”, and thus shows no dowel action of the reinforcements and there only is a direct interaction of the reinforcement with the matrix and the matrix stresses determine the stresses in the reinforcements. Because the initial small cracks in wood are in the matrix and start to extend in the matrix, the stress equilibrium condition of the isotropic matrix by the matrix-stresses has to be regarded. The isotropic solution of the matrix stresses thus has to be regarded in the end state. The total stresses, due to the reinforcement, then follow by multiplication of an elastic constants factor (e.g. derived in § 2 of [3]).

In [4], and below in § 10.2, the exact derivation of the geometric correction factor of the center notched test specimen is given, based on small cracks merging. As known, this geometric correction factor accounts for the difference of finite specimen dimensions with respect to the notch in an infinite plate.

Because unloading (called softening) by step wise multiple small crack merging needs a lower mean stress than is necessary for a single macro-crack extension, it can be postulated that small crack merging always takes place in the high loaded zone near the macro-crack tip and that macro-crack extension is always due to small crack extension towards the macro-crack tip.

### 10.2. Derivation of the geometric correction factor of the center notched specimen

Because all calculations are based on the singularity approach, (called LEFM) it is for comparison of results, necessary, to also give, for small cracks, a mathematical flat crack (singularity) solution of the Airy stress function. Started is here with a crack in an infinite plate, which is loaded by a tensile stress $\sigma$. The stress distribution along the fracture plane, line AB of Fig. 10.2, then is:

$$
\sigma_{y: x} = \frac{\sigma}{\sqrt{1 - (a/x)^2}} \quad x > a \tag{10.2.1}
$$

where $2a$ is the crack length and $x$ is the distance from the center of the crack. This stress distribution is according to the solution of the Airy stress function of [7]. Such solution satisfies the equilibrium, compatibility and boundary conditions and thus is an exact solution.

To obtain the ultimate state of the specimen given in Fig. 10.1, we may cut out the specimen dimensions from the infinite plate, as is given in Fig. 10.2. Next we may multiply the stress $\sigma_{y: x}$ by a (by definition stress independent) factor $Y$ with such magnitude that the resultant shear loading $2R$ in the planes AD and BC of Fig. 10.2 becomes zero. There remains an equilibrium system in those vertical planes giving an internal equilibrium system in the cut-out specimen which, as such, has no influence on the strength. Because limit analysis applies with virtual deformations there is no effect of initial stresses or deformations on the plastic limit collapse load. As condition for zero values of $R$, the sum of the normal stresses in the upper plane AB should be equal and opposite to the normal stresses in the bottom plane CD. Thus the total sum of the (vertical) normal stresses on the cut out specimen should be zero giving:

$$
\sigma W = 2 \int_{a}^{W/2} \sigma_{y} dx = 2 \int_{a}^{W/2} \left( Y \sigma / \sqrt{1 - (a/x)^2} \right) dx = Y \sigma W \sqrt{1 - (2a/W)^2} \right), \tag{10.2.2}
$$
and the stress multiplication factor thus is:

\[ Y = 1 / \sqrt{1 - (2a/W)^2} \]  

(10.2.3)

The stress intensity factor \( K_y \) due to the critical small crack concentration follows from:

\[ K_y = \sigma_y \sqrt{\pi a} = \sigma_y \sqrt{\pi 2(x-a)} = \left( Y\sigma / \sqrt{1-(a/x)^2} \right) \sqrt{\pi 2(x-a)} = Y\sigma \sqrt{2\pi / (x+a)} \]  

(10.2.4)

As shown § 3.6 and in [3], the small crack merging towards the macro-crack tip causes the macro-crack extension. When the nearest, determining small crack tip is situated at a distance \( x \), then the one sided small crack merging distance to the macro-crack tip is \( x - a \), which is equal to half the small crack length \( c \) of row A of fig. 3.8. Thus: \( c = (x-a) \), and total \( 2(x-a) \) applies, of both sides of two sided macro-crack extension of the initial crack length of \( 2a \). This also applies when the macro crack-tip has become sharp enough to take part in the crack merging process. Then all active crack tips extend over a distance \( c \), which is equal to \( c = (x-a) \) in the analysis.

For \( x \rightarrow a \), the lowest, thus first occurring, initial flow value for \( K_y \) of eq.(10.2.4), becomes:

\[ K_y = Y\sigma \sqrt{\pi a} \]  

(10.2.5)

This is identical to the results of other methods, showing the mathematical flat crack, singularity solution, to apply for the smallest initial small crack system and to represent an estimate of the start of crack extension (see also § 3.6). For initial small cracks smaller than \( c_c \) of eq.(3.13), instable crack propagation occurs to a new equilibrium crack-length, which is on the Griffith locus, eq.(3.12). When different small crack-lengths are supposed to act, a mean value factor \( c_1 \) has to be inserted for a mean length value. Then the geometric correction factor with respect to the stress in the infinite plate is: \( Y \sqrt{c_1} \). But, by comparing this result with other exact geometric correction factors, given in Tada et al [8], the value \( c_1 = 1 \) is found. Thus the derived geometric correction factor \( Y \) is comparable to the other solutions of Tada, Feddersen, Koiter, Isida and Irwin [8]. The value \( c_1 = 1 \) thus means that the first small crack merging process, row A of fig. 3.8, solely act at the start of further crack extension, as predicted by theory.

The value \( Y \), according to eq.(10.2.3): \( Y = 1 / \sqrt{1 - (2a/W)^2} \), can be seen as a mean value, intermediate between the, in [8] given, values of Koiter et al. and Feddersen around \( Y = \sec(\pi a/W) \) and the solution of Irwin: \( \sqrt{(W/\pi a) \cdot \tan(\pi a/W)} \). In Table 10.1, eq.(10.2.3) is compared with the solution of Irwin and the usual applied Feddersen equation. The precise
description by the exact derivation shows that small crack merging does not only explains

<table>
<thead>
<tr>
<th>(2a/W)</th>
<th>(Y = \sqrt{\sec(\pi a/W)})</th>
<th>(Y = 1/\sqrt{1-(2a/W)^2})</th>
<th>((W/\pi a)\cdot \tan(\pi a/W))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.006</td>
<td>1.005</td>
<td>1.004</td>
</tr>
<tr>
<td>0.2</td>
<td>1.025</td>
<td>1.021</td>
<td>1.016</td>
</tr>
<tr>
<td>0.3</td>
<td>1.059</td>
<td>1.048</td>
<td>1.040</td>
</tr>
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<td>0.4</td>
<td>1.112</td>
<td>1.091</td>
<td>1.075</td>
</tr>
<tr>
<td>0.5</td>
<td>1.189</td>
<td>1.155</td>
<td>1.128</td>
</tr>
<tr>
<td>0.6</td>
<td>1.304</td>
<td>1.250</td>
<td>1.208</td>
</tr>
<tr>
<td>0.8</td>
<td>1.799</td>
<td>1.667</td>
<td>1.565</td>
</tr>
<tr>
<td>0.9</td>
<td>2.528</td>
<td>2.294</td>
<td>2.113</td>
</tr>
<tr>
<td>0.95</td>
<td>3.570</td>
<td>3.203</td>
<td>2.918</td>
</tr>
<tr>
<td>Feddersen</td>
<td>Crack merging, Eq.(10.2.3),</td>
<td></td>
<td>Irwin</td>
</tr>
</tbody>
</table>

Softening, but is the basic mechanism of all crack extension. This is discussed further in chapter 3. The possibilities of the singularity approach are very limited. For instance extension of the theory for crack bridging with a plastic zone of length \(a_0 - a\) at the initial crack tip of the elongated crack length \(a_0\), changes eq.(10.2.2) to:

\[
\sigma W - 2\sigma_t (a_0 - a) = 2 \int_{a_0}^{W/2} \sigma_t dx = 2 \int_{a_0}^{W/2} \left( \frac{Y\sigma}{\sqrt{1-(a_0/x)^2}} \right) dx = Y\sigma W \sqrt{1-(2a_0/W)^2} \tag{10.2.6}
\]

where \(\sigma_t\) is the ultimate (plastic) tensile stress of the bridging area. The stress multiplication factor then is:

\[
Y = \left(1 - 2(\sigma_t/\sigma)(a_0 - a)/W \right)/\sqrt{1-(2a_0/W)^2} \tag{10.2.7}
\]

showing this factor to be stress dependent and thus showing that by maintaining the singularity solution, the approach is not applicable for theory extensions. The general accepted replacement of surface energy by strain energy release rate, which is equal to bond-breaking and plastic dissipated energy (while maintaining the old stress independent geometric correction factor) thus is not exact. New exact theory extensions (necessary e.g. for combined loading) only may follow from limit analysis as applied in this Section C. The small crack limit behavior is discussed next.

10.3. Small crack limit strength behavior

10.3.1. Small crack limit dimensions

The singularity approximation is called LEFM (linear elastic fracture mechanics) and by superposition of a flow stress field in such a way that the singularity of this field neutralizes the singularity due to the external loading, the so called nonlinear fracture approach is created. Both approximations are not general enough (see § 4.2). A general approach is possible by limit analysis, as followed in this Section C, based on a nonsingular Airy stress function. Because for wood, contained plastic flow is described by the equivalent linear elastic ultimate stress behavior, a linear elastic approach is possible, up to the elastic-plastic boundary (of limit analysis) around the crack. The full-plastic zone is given by a single curve in stress space as shown in Fig. 10.3. In this figure is \(d/d_0\), the ratio of specimen size to the fracture process zone size. But, because the volume effect is tested, is the initial crack length proportional to the specimen length. Thus, \(d/d_0\) also can be regarded to be the ratio initial open crack length to the process zone size. Then, for small values of \(d\), this \(d/d_0\) ratio also may represent the critical small crack density in a macro specimen (\(d\) also is
crack interspace). The curved line of Fig. 10.3, follows the equation:

\[ \ln \sigma = \ln \sigma_0 - 0.5 \ln (1 + d/d_0) \]  

(10.3.1)

Fig. 10.3. from [9], showing wrong interpretations about small crack strength

This can be written:

\[ \ln \left( \frac{\sigma}{\sigma_0} \right) = \ln \left( \frac{d_0 + d}{d_0} \right)^{-0.5} = \ln \left( \frac{d_0}{d_0 + d} \right)^{0.5} \]

(10.3.2)

or: \[ \sigma \sqrt{d_0 + d} = \sigma_0 \sqrt{d_0} = K_c / \sqrt{\pi} \]  

(10.3.3)

This confirms that the curve to represent the stress intensity as ultimate state with \( K_c \) as critical stress intensity factor as should be for values of \( d/d_0 \gg 1 \). For these higher values the curved line approaches the drawn straight tangent line \( \ln \sigma = \ln \sigma_0 - 0.5 \cdot \ln (1 + d/d_0) \approx \ln \sigma_0 - 0.5 \ln (d/d_0) \) with the necessary slope of the curve \( \frac{\partial \ln (\sigma / \sigma_0)}{\partial \ln (d/d_0)} \approx -0.5 \) as limit. The real slope however is:

\[ \frac{\partial \ln \sigma}{\partial \ln (d/d_0)} = \frac{\partial \ln (\sigma / \sigma_0)}{(d_0 / d) \partial (d/d_0)} = \frac{d}{d_0} \frac{\partial (\ln (1 + d/d_0))^{-0.5}}{\partial (d/d_0)} = \frac{d}{d_0} \frac{-0.5}{1 + d/d_0} = \frac{-0.5}{1 + d_0/d} \]

(10.3.4)

This slope is: \(-0.5\) for \( d \gg d_0 \) and this slope is zero when \( d = 0 \).

This shows that for the whole curve LEFM applies and it is an indication that, at zero dimensions, thus zero: \( d = 0 \), the strength theory still follows LEFM, because it applies also for initial length \( d_0 \).

The strength theory applies for ultimate high loaded areas and represents the small crack merging mechanism, when the critical small crack density is reached. This means that the small crack spacing is in the order of the small crack length (see [3] or § 3.6) and this applies for any initial small crack length \( d + d_0 \). However, when \( d_0 \) is of the order of \( d \), the crack length, and thus of the crack spacing, there only is plastic flow in the intact ligament material. For wood this is determining for very small specimens, or for small loaded areas. For instance, the maximal bending tension stress occurs at one point and thus maximal bending tension occurs at a small area and will be plastic. For wood loaded in bending and compression therefor is the ratio bending tension to bending compression \( s = f_{mt} / f_{c} \) constant (and thus both plastic), as shown in [10], for any load combination of bending with compression, indicating also that there always is failure by the ultimate bending tensile strength. A volume effect by stress distribution thus needs not to be regarded. The volume effect thus now is caused by the volume alone due to decreasing quality by volume increase (For that reason also the compression strength may show a volume effect). This does not apply when the shear strength is determining, which shows a strong volume effect.

10.3.2. Small crack failure criterion

Because the isotropic matrix “flows” before the reinforcement, limit analysis has to be applied for the isotropic stresses in the isotropic matrix. This is not followed by all other approaches, which
therefore don’t satisfy the failure criterion and are not able to give the right exact mixed mode fracture criterion. At initial flow of the matrix, the stresses of the still elastic reinforcement follow in proportion the matrix stresses. That the matrix is first determining follows e.g. for Balsa wood, which is highly orthotropic, but is light, thus has a low reinforcement content and shows total failure soon after matrix failure and thus shows at failure the isotropic ratio of \( \frac{K_p}{K_c} \approx 2 \) of the isotropic matrix material. But also for strong clear wood which is failing by shear by single oblique crack extension according to Fig. 2.3.2, it appears that the start of crack extension shows the isotropic oblique angle, showing the matrix to be determining for initial failure. The truss action, at bending failure of a beam, causes a negative contraction coefficient in the bending tension zone. This shows that the reinforcement holds, even after flow in compression and stress redistribution, with increased tension in the reinforcement. It is therefore a requirement for an exact orthotropic solution of the total applied stress, applicable to wood, to also satisfy the isotropic flow solution of the matrix-stresses.

As discussed at Section A and at § 5.2, the (small crack) failure criterion for shear with tension is: eq.(5.1) or eq.(5.2), which becomes, as limit behavior, equal to the Wu-equation when due to full hardening \( c \to 1 \) in eq.(5.2). Full hardening is possible when the test rig is stiff enough to remain stable during test. The solution of the crack problem of Irwin as summation of in plane and antiplane solutions in order to use (with minor adaptions) isotropic stress functions for the orthotropic case, and to apply descriptions in the three characteristic modes and to sum the result for the general mixed mode case is not right for wood. It misses the interaction terms and the failure equation, eq.(5.1), is not orthotropic, because it is not quadratic but contains a third degree term and thus does not show orthotropic symmetric. This coupling term is absent in the general accepted Sih-Paris-, Irwin solution. The stress function which leads directly to the Wu-equation is given in § 2.3 and in [3]. Necessary are the stresses at the crack boundary to know the mode of failure. This follows from the exact derivation in chapter 2 and is applied by numerical simulation by the virtual crack closure technique of the finite element method, and thus can not be based on a separate calculation of the energy release rates of the normal stress in the opening mode and of the shear stress in the sliding mode according to the method of Sih, Paris, Irwin by giving the sum of separate solutions for the 3 modes, without interactions, (as e.g. \( 3F_{iso}\sigma_1\sigma_2^2 \)) what is assumed to be possible by assumed isotropic and orthotropic symmetry. Thus the, not orthotropic, “mixed mode”, interactions, as given by Fig. 5.1 and 5.2, can not be described by other methods.

### 10.4. Softening by the crack merging mechanism

#### 10.4.1. Mode I tension tests of Chapter 3.

For description of “softening”, which occurs when the rate of crack extension of a test specimen, surpasses the loading rate, the damage theory of Deformation Kinetics [11], also has to be applied at measuring compliance changes. An alternative derivation of softening behavior (at standard climatic conditions and neglecting plasticity) is given in chapter 3 by the energy method of the Griffith theory, based on a constant \( G_c \), (the critical energy release rate or specific fracture energy).

For a distribution of small cracks, \( b \) and \( l \) in eq.(3.13) are the St. Venant crack distances as critical crack distances, because for lower distances, (thus higher crack concentrations) the stress in the intact material will be lower than the ultimate value and unloading occurs. This critical initial distance, of about 2 to 2.2 times the crack length, also is predicted by Molecular Deformation Kinetics theory[11].

When the intact part of the fracture plane is halved by macro crack extension, it becomes overloaded, causing unloading by small crack extension everywhere in the fracture plane also outside the crack tip region. At that point there is not enough strain energy any more for single macro-crack extension and unloading takes place by small crack merging while the macro-crack extends by small crack propagation towards the macro-crack-tip. The departure of the softening behavior from the Griffith locus, by the apparent gradual decrease of \( G_c \) (of the macro crack), thus
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is due to clear-wood failure of the high loaded fracture plane, which takes lower mean stress for local reactions to small crack rows B and C of fig. 3.8, than for a single macro-crack extension. This clear wood fracture plane is equally strong everywhere and thus contains a critical small crack density over the whole intact fracture plane at the start of softening. At the initial critical small crack density, the crack distance is about equal to the crack length, as given by row A below.

\[
\begin{align*}
\text{c} & \quad \text{---2c---} \quad 2c \quad \text{---2c---} \quad 2c \quad \text{---2c---} \quad 2c \quad \text{---2c---} \quad 2c \quad \text{---2c---} \\
\text{c} & \quad \text{-----------} \quad 6c \quad \text{-----------} \quad 6c \quad \text{-----------} \quad 6c \quad \text{-----------} \\
\text{c} & \quad \text{-----------} \quad 14c \quad \text{-----------} \quad 2c \quad \text{-----------} \\
\end{align*}
\]

Fig. 3.8. Crack rows of successive processes.

When 2 cracks merge, a crack length of 6c occurs, given by row B which is the site for the next merging reaction, giving a crack length of 14c, as shown by row C. The area of the intact fracture plane of row A then is twice the intact area of row B and four times the area of row C. Because this system of crack merging requires lower mean stresses than at macro-crack propagation by one macro-crack alone, it provides a more probable solution. As shown in chapter 3, the decrease of the Griffith values \( \sigma_g \) and \( G_c \), then is fully explained by the strength of the intact part of the fracture plane, which remains at the ultimate state \( \sigma_g = \sigma_m \), as is verified by the measurements, eq.(3.12), in the form of eq.(3.19), of the softening curve, is according the measurement of [6] by Figure 3.6 and 3.7 and precisely explains the decrease of \( \sigma_c \), the top-value of the softening curve, which also can be approximated by three equations (3.12) for the 3 critical crack densities of rows A, B and C. The strength decrease by a factor 0.5 between these crack densities and causes a decrease of the top-value \( \sigma_c \) of eq.(3.17) by a factor 0.657. Thus: \( 0.657 \times 7 = 4.6 \) and \( 0.657 \times 4.6 = 3 \). Thus a simple practical approximation of the mean softening curve of all specimens of the series, is possible by applying eq.(3.12) twice (or three times for the highest values), according to line 1 and 2 in Figure 3.6 and 3.7. Analogues behavior is to be expected and is shown to exist for mode II. This is discussed in the next § 10.4.2.

10.4.2. Mode II shear tests

In Yoshihara [12], [13], results of mode II tests, called asymmetric four point bending (AFPB) tests, are given, applied on very long sub-critical initial crack lengths, which clearly represent the end softening stage at first loading, because the measured \( K_{lc} \)-values were a factor 3 to 4 lower than those of the control tests on standard end-notched flexure (ENF) test-specimens. Of standard equation for this case, eq.(10.4.1), the chosen value of \( a \) and measured strength value \( \tau_0 \) are known and \( K_{lc} \) is determined by measuring the energy release rate \( G_{II} \) by the compliance method at different crack lengths and, as control, by the finite element virtual crack closure method.

\[
K_{lc} = \tau_0 \sqrt{\pi a} \cdot Y \left( \frac{a}{W} \right) \tag{10.4.1}
\]

The geometric correction factor \( Y \left( \frac{a}{W} \right) \) is not determined theoretically, but is simply found as equating value: \( Y \left( \frac{a}{W} \right) = K_{lc} / (\tau_0 \sqrt{\pi a}) \) \tag{10.4.2}

There thus is no control on applying right parameter values and it is e.g. not noticed that a wrong, not measured, stiffness factor is used. However, this factor is the same for all cases and has no influence on the in the following presented relative values and on the control of the crack merging theory:

The data of Yoshihara [12], [13] are purely empirical, based on measurements and empirical methods as the compliance measurement and finite element calculation. Therefore it is empirically shown, that the real mean shear strength on the intact part of the fracture plane is determining and
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not the measured apparent \( K_{IIc} \) value, which is too low for macro-crack extension. For macro-crack extension applies that according to eq.(1) of [12], for the top of the loading curve, at \( a/W = 0.7 \):

\[
K_{II,0} = \frac{3P}{4BW} \sqrt{\pi a_c} = \tau_u \sqrt{\pi a_c} \quad \text{(10.4.3)}
\]

because then \( f(a/W) = 1 \) and the nominal Griffith shear stress is: \( \tau_u = 3P/4BW \) \( \text{(10.4.4)} \)

As follows from [2], eq.(10.4.11) is the critical crack length: \( a_c = \sqrt{bl/6\pi} \) for mode I, according to fig. 3.1. For the AFPB-test is, \( b = W \) and \( l = c_W \), proportional to \( W \), as a Saint-Venant distance. A similar relation applies for mode II, following from the Griffith locus derivation for compression and shear. Thus \( a_c = \sqrt{Wc_W/6\pi} = c_W \) \( \text{(10.4.5)} \)

For higher values of \( a \), when \( a/W > 0.7 \) is, as applied in Yoshihara [12], [13]:

\[
K_{II} = \tau_u \sqrt{\pi a} \cdot f(a/W) \quad \text{(10.4.6)}
\]

It is shown in [2], § 3.6, that at a far stage of softening, the small crack merging mechanism shows that at any moment the strength of the intact fracture area is determining for the strength. This can be interpreted that not the nominal stress but the ultimate real stress \( f_i \) in the fracture plane is determining. Thus eq.(10.4.3) becomes:

\[
f_i = K_{II,0} f(a/W) = K_{II,0} l/(\sqrt{\pi a_c}(1-a_c/W)) = K_{II,0} l/(\sqrt{\pi c_W}(1-c_2)) \quad \text{(10.4.7)}
\]

and eq.(10.4.6 3.8): \( f_i = K_{II} l/(\sqrt{\pi a}(1-a/W) \cdot f(a/W)) \) \( \text{(10.4.8)} \)

and from eq.(10.4.7 3.9) and (10.4.8 3.10) follows:

\[
\frac{K_{II}}{\sqrt{\pi a}(1-a/W) \cdot f(a/W)} = \frac{K_{II,0}}{\sqrt{\pi c_W}(1-c_2)} \quad \text{or:}
\]

\[
\frac{K_{II}}{\sqrt{a/W}(1-a/W) \cdot f(a/W)} = \frac{K_{II,0}}{\sqrt{c_W}(1-c_2)} = c_4 \quad \text{(constant)}
\]

According to fig.12 of Yoshihara [13], there is no difference (by volume effect) between the data for \( W = 40 \) mm and \( W = 20 \) mm, thus mean values of both can be regarded.

For \( a/W = 0.7 \), \( K_{II} = 0.79 \) MPa \( \sqrt{m} \) thus: \( c_4 = 3.15 \) \( \text{(10.4.10)} \)

For \( a/W = 0.8 \), \( K_{II} = 0.71 \) MPa \( \sqrt{m} \) thus: \( c_4 = 3.3 \) \( \text{(10.4.11)} \)

For \( a/W = 0.9 \), \( K_{II} = 0.52 \) MPa \( \sqrt{m} \) thus: \( c_4 = 3.28 \) \( \text{(10.4.12)} \)

\( f(a/W) = 1.0 \) for \( a/W = 0.7 \) and is 1.2 for \( a/W = 0.8 \) and is 1.67 for \( a/W = 0.9 \), as measured and calculated by Yoshihara according to eq.(10.4.2), based on the crack closure technique. Thus the mean value of \( c_4 \) is: \( c_4 = 3.24 \). Numerically this calculation is:

\[
0.79/(\sqrt{0.7}(0.3)1) = 3.15 \quad \text{and:} \quad 0.71/(\sqrt{0.8}(0.2)1.2) = 3.31 \quad \text{and:} \quad 0.52/(\sqrt{0.9}(0.1)1.67) = 3.28
\]

It thus is confirmed by the data of Yoshihara [12], [13], that the real mean shear strength of the intact part of the fracture plane is determining and not the apparent \( K_{II} \) value, which is too low for macro-crack extension. Therefore, the value of the nominal stresses are determined by the small crack extension as follows from eq.(4.7) of § 4.2 and from the following:

\[
f(a/W) = \frac{K_{II}}{\tau u \sqrt{\pi a}} = \frac{K_{IIc}}{\tau u \sqrt{\pi a_c}} , \quad \text{or:} \quad \sqrt{\pi a_c} \cdot f(a/W) = \frac{K_{IIc}}{\tau u} = c_5 \quad \text{(constant)}
\]

Now the for the critical small crack concentration equivalent macro-crack concentration is:

\[
a_c = \sqrt{bl/6\pi} = \sqrt{(c_W(1-a/W)\cdot W(1-a/W)/6\pi)} = c_6(1-a/W)
\]

because \( l = W(1-a/W) \) and \( b = c_W(1-a/W) \) as St. Venant distance. Thus:
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For \(a/W = 0.7\), is: \[\sqrt{a_c} \cdot f(a/W) = c_6 \sqrt{1-a/W} \cdot f(a/W) = c_6 \cdot \sqrt{0.3} \cdot 1.0 = 0.55c_6 \quad (10.4.15)\]

For \(a/W = 0.8\), is: \[\sqrt{a_c} \cdot f(a/W) = c_6 \sqrt{1-a/W} \cdot f(a/W) = c_6 \cdot \sqrt{0.2} \cdot 1.2 = 0.54c_6 \quad (10.4.16)\]

For \(a/W = 0.9\), is: \[\sqrt{a_c} \cdot f(a/W) = c_6 \sqrt{1-a/W} \cdot f(a/W) = c_6 \cdot \sqrt{0.1} \cdot 1.67 = 0.53c_6, \quad (10.4.17)\]

giving the explanation and exact control of the \(f(a/W)\) values of Yoshihara [12], [13]. It thus is shown that, also for mode II, small crack merging and extension towards the macro-crack tip is determining for fracture. This of course is evident because all fracture, for any stress combination, is due to reaching the ultimate tensile strength at the crack-boundary. This, (with the mentioned oblique crack extension in e.g. glasses) applies in principle for all structural materials and is e.g. derived in principal stresses in [14] for critical oriented small cracks in concrete.

10.5. Conclusions regarding small crack fracture mechanics
- Part of the conclusions are given in Chapter 8.
- An exact explanation is given of mode II softening behavior by small crack merging mechanism. The data of [12], [13] on long over-critical initial crack lengths, follow the theory precisely. This small crack merging mechanism determines always the ultimate strength of timber.
- The properties, following from exact strength theory of wood, should be accounted in a right fracture mechanics theory. It therefore is necessary to regard:
  - that limit analysis applies, with elastic-full plastic behavior and thus elastic up to fracture,
  - that wood behaves as a reinforced material, and the solutions of the isotropic Airy stress function of the matrix stresses as well as the orthotropic Airy stress function of the total stresses are needed,
  - that reaction kinetics and the general applicable failure criterion indicate that, small-crack processes are always determining for fracture.
  - that for high loaded fracture planes of long post-critical initial cracks, the strength of the intact part of the fracture plane determines softening by the maximal possible loading state. The intact area is determined by the small crack merging mechanism which e.g. explains the factor 2.5 to 3.9 too low stress intensity factors of [12], [13].
  - The theory predicts that for mode II and for combined loading, the right fracture energy is found by the finite element virtual crack closure technique applied on oblique crack extension in the right critical direction. This is not the case for an assumed collinear crack.
  - There is no difference between linear elastic- and non-linear fracture mechanics because for both approaches linear elastic behavior is regarded up to failure and plastic flow. This is possible because by the virtual work approach at the ultimate state there is no influence on the strength of the loading path followed and of initial stresses and internal equilibrium systems. The critical energy release rate is in both cases determined by plastic behavior. In fact always the linear – full plastic approach of limit analysis applies for the boundary value approach and ultimate state at the crack-tip boundary.
  - The start of softening by small crack extension can be given by a critical macro-crack length for the space were the critical small crack density occurs. Further empirical investigation is necessary to find which processes act together providing the constants.
  - Because the macro-crack kinetics is the same as for clear wood, this small-crack behavior is always determining (see last part of Section 3.6 of [2]).
  - For long sub critical initial cracks as in [13], the strength of the intact part of the fracture plane is always determining and explains the measured too low apparent stress intensity.
  - Small-crack merging explains precisely the softening curve (of [11]) by the strength (or plastic flow stress) of the intact part of the fracture plane, which is always in the ultimate state and is most probable because it requires a lower stress than single macro-crack propagation ([2], Section 3.5 and 3.6) and shows in rate form the necessary molecular deformation kinetics equation of this damage process. (see [11]).
  - Because at the crack boundary, or at the elastic-plastic boundary at the crack-tip, the elastic solution as well as the ultimate strength solution applies, according to limit analysis, it is necessary
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to solve the elastic solution of the isotropic matrix stresses and then calculate the stresses in the reinforcement in proportion to the modulus of elasticity of matrix and reinforcement. This way of calculation is necessary for wood because of the single, direct interaction of the reinforcement and matrix and the start of crack extension in the matrix.

10.6. References

11. Conclusions
In chapter 8, conclusions are given regarding chapters 1 to 7.
Conclusions of chapter 9, regarding the size effect, are given in § 9.5
Conclusions of chapter 10 regarding small crack fracture are given in § 10.5.