Reproduction of velocity and shear stress profiles in estuaries by some one-dimensional mathematical models (interim report)

R. Booy
Intern Rapport no. 2-81
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R. Booy

Technische Hogeschool Delft
Afdeling Civiele Techniek
Vakgroep Vloeiystofmechanica
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Summary

In this investigation the reproduction of tidal flow in channels of estuaries by means of several simple mathematical models is considered. Attention is mainly paid to velocity and shear stress profiles and to hysteresis effects of the shear stresses with respect to the surface velocity. Only one-dimensional models obtained by neglect of convective derivatives of the longitudinal velocity are considered. The models are of the eddy viscosity type, where the eddy viscosity is a prescribed function of depth and flow velocity or is dependent on the turbulence energy.

Until now the prescribed eddy viscosity got most attention, but it is shown that eddy viscosities which are dependent on the turbulence energy, will not give very different results. Models with an eddy viscosity being a quadratic function of the depth and proportional to a varying velocity (e.g. the depth-averaged velocity) reproduce the velocity and the shear stress profiles in estuaries quite well. The hysteresis effect of the shear stresses, however, is much smaller than found in the few prototype measurements.
1. **Introduction**

In many mathematical investigations of the velocity and shear stress profiles of tidal flow in estuaries, width-averaged Reynolds equations for unsteady flow are used. To get a system of equations that is mathematically closed, it is necessary then to make an assumption, relating the Reynolds stress term to the flow field.

In many models the Reynolds stress is equated to a product of the velocity gradient and a coefficient, the so-called eddy viscosity (or effective viscosity). These eddy viscosity models can be divided in two kinds of models. In the first kind the eddy viscosity is assumed to depend only on the velocity field. Some of these models are the mixing length model of Prandtl and relatively simple models where the eddy viscosity is assumed to be a particular function of the velocity field. (In this report called pure eddy viscosity models). In the second kind of models the eddy viscosity may also depend on one or more other turbulence quantities, such as the mean turbulence energy (k-model), or a combination of the mean turbulence energy and its dissipation rate (k-ε-model). This second kind of models has come to the fore recently in connection with the growing computing power.

The value of a model depends on its ability to reproduce some of the properties of the tidal flow in estuaries. In this connection attention is primarily paid to velocity and shear stress profiles and to differences between the shear stresses in the accelerating and the decelerating phase of the tide.

Most measurements in estuaries are executed at the higher current velocities. In general the velocity profiles show an almost logarithmical behaviour. In many models using an eddy viscosity that varies over depth such velocity profiles can easily be produced.

Not much experimental evidence about shear stress profiles in tidal flow is available. The few measurements executed seem to show a dependence of the shear stress profile on the sign of the acceleration. In the decelerating phase of the tide the maximum of the shear stress sometimes lies above the bottom, whereas in the accelerating phase the shear stress increases with depth (7). The shear stresses as measured in the decelerating phase are larger than the shear stresses in the accelerating phase (4,13). This 'hysteresis effect' is shown clearly in the hysteresis diagram obtained
by Gordon (12) by plotting the shear stress measurements against the current velocity.

Gordon attributes this dependence of the shear stress on the sign of the acceleration to the 'bursting' phenomenon (12). Measurements in estuaries reveal the importance of 'bursting' to shear stresses (10,14,16). Moreover the properties of bursting seem to be dependent on the direction of the pressure gradient (33) and hence on the sign of the acceleration.

The bursting phenomenon is supposed to be more pronounced at larger Reynolds numbers (27). Therefore bursting may be responsible for the large scatter in and poor reliability of many measurements in flows in estuaries (17).

In this report various models using an eddy viscosity to close the equations are used to calculate velocity and shear stress profiles. The ability of the different models to produce the desired properties of tidal flows is considered. Much attention is paid to hysteresis effects and connected time and phase lags of the shear stresses with respect to the current velocity.

Three pure eddy viscosity models are considered:
The used eddy viscosities are:
a) constant eddy viscosity
b) quadratic eddy viscosity
c) an eddy viscosity that is quadratic with depth and proportional to the depth-averaged velocity.

Until now less attention has been paid to the $k$-model and the $k-\epsilon$-model. The influence of the use of the turbulence energy in the expression for the eddy viscosity on time and phase lags has been investigated more thoroughly, however.
2. Mathematical Description

2.1 One-Dimensional Models

In this report a long-wave, small amplitude motion in a straight open channel of constant width and depth is considered. In the absence of Coriolis accelerations and transverse oscillations, the motion will be essentially two-dimensional. To describe this motion a rectangular coordinate system Ox, Oz is used, where Ox is situated on the bottom and directed along the channel and Oz is positive upwards (see fig.1).

Following Proudman (30) the shallow-water equations are

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{1}
\]

\[
\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} \int_0^h \zeta zdz = 0 \tag{2}
\]

\[
\frac{Du}{Dt} = -g \frac{\partial \zeta}{\partial x} - \frac{\partial \tau}{\partial z} \tag{3}
\]

In formula (3)

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z} \tag{4}
\]

is the Stokes derivative, \(u\) and \(w\) represent the ensemble-averaged velocities in the \(x\) and \(z\) directions respectively, \(\tau\) is the horizontal kinematic Reynolds shear stress, \(h\) is the mean free surface level and \(\zeta(x,t)\) its displacement.

The term \(-g(\partial \zeta / \partial x)\) in equation (3) represents the pressure gradient, which is connected to the surface gradient.

Replacing the Stokes derivative by the time derivative in equation (3) gives equation (5)
where $S(t)$ is the kinematic pressure gradient. The error introduced by the neglect of the terms

$$u \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial z}$$

of equation (3) is not large in most tidal flows.

In a tidal channel with a depth of 10 meters or more the tidal wave celerity is at least 10 m/s. If the velocity $u(z)$ does not exceed 1 m/s, the first neglected term amounts to less than about a tenth of the maximum value of $\frac{\partial u}{\partial t}$. Lower values of $u(z)$, as for instance encountered near the bottom, bring about proportionally smaller errors. The error introduced by the neglect of the second term is much smaller. Near the surface this error is about a tenth of the error by the neglect of the first term. The errors are of the same order directly above the bottom. But the error by neglecting the first term is already small in the bottom layer.

Assuming that the Reynolds shear stress, $\tau$, can be related to the local mean velocity gradient by means of an effective viscosity or eddy viscosity $\nu_t(x,z,t)$ defined by

$$\tau = - \nu_t \frac{\partial u}{\partial z},$$

equation (5) becomes

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial z} (\nu_t \frac{\partial u}{\partial z}) = - S \quad (7)$$

To solve equation (7) a hypothesis about the variation of $\nu_t$ with $z$ and $t$ is needed. If $\nu_t$ can be expressed in variables at the same $x$, then equations (1) and (2) are redundant for calculating velocity profiles and shear stress profiles. In this report only models with this property, one-dimensional models, are considered.
2.2 Various Eddy Viscosity Models

Models that use an eddy viscosity to close the system of equations are called eddy viscosity models. Some of the models using different hypotheses about the variation of \( \nu_t \) with \( z \) and \( t \) encountered in the literature about boundary layers are introduced below (23).

2.2.1 Pure Eddy Viscosity Models

In this class of models the eddy viscosity is represented by

\[
\nu_t = u_e z f(z/h)
\]

where \( u_e \) and \( z_e \) are a characteristic global velocity scale and a global length scale of the flow respectively. \( f(z/h) \) is a prescribed function of \( z/h \) and may e.g. be a constant (18,22), a pair of constants (2), or a quadratic function (18,19,20,26,37).

Concerning the velocity scale, most investigators use a constant velocity e.g. the maximum of the depth-averaged velocity, \( u_{av}^{max} \), or the maximum of the friction velocity, \( u_x^{max} \), over a tidal period (see page 14). In this type of problem it seems more useful to use a varying velocity scale. A tidal flow in an estuary is a slowly changing boundary layer flow, resembling a stationary boundary layer flow at most times (see page 24). The depth-averaged velocity, \( u_{av} \), and the friction velocity, \( u_x \), may be used as velocity scales for stationary boundary layers flows. Use of one of these velocities as a varying velocity scale in a model of a tidal flow will then give satisfactory results at most tidal phases.

2.2.2 Mixing-Length Model

The expression for \( \nu_t \) in Prandtl's mixing length hypothesis is

\[
\nu_t = \frac{1}{m} V_t \text{ with } V_t = \frac{1}{m} \left| \frac{3u}{\partial z} \right|
\]

This mixing length model is closely related to the eddy viscosity models. \( l_m \), the mixing length, must be prescribed as a function of \( z/h \) (21). \( l_m \) can be written as a product of a global length scale \( h \) and a function of \( z/h \). On the other hand \( V_t \), a velocity that is presumed to be characteristic for the turbulence intensity, is a local velocity scale in this model.
Some defects of the mixing-length model and of eddy viscosity models are

1) The eddy viscosity behaves unrealistic at some points in the flow. In the mixing length model the eddy viscosity vanishes where the velocity gradient is zero. In the eddy viscosity models the use of a global velocity causes the eddy viscosity to vanish over the whole depth when this global velocity is zero, and to remain too large when the velocity gradient is small at non-zero global velocities. These circumstances are to be expected when dealing with long waves.

2) The local turbulence level, which influences the eddy viscosity, is determined not only by events at the place and the time in question (or on average velocities), but also by events at other times (and places).

In the following two models these shortcomings do not occur (23.31).

2.2.3 $k$-Model

When the characteristic velocity for the turbulence intensity in the mixing length model, $V_t$, is taken proportional to the root of the ensemble-averaged turbulence energy $k$,

$$V_t = C_v \sqrt{k} \quad ,$$

the eddy viscosity is described by

$$\nu_t = C_v L \sqrt{k} \quad ,$$

where $L$ like $l_m$ is a characteristic length scale of the turbulent motion. $L$ is closely connected to $l_m$ and, in fact, is often identified with it. $C_v$ is a constant which equals unity for suitable chosen $L$ (23).

The turbulence energy, $k$, is determined by a transport equation (23)

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial z} \left( \frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial z} \right) + \nu_t \left( \frac{\partial u}{\partial z} \right)^2 - C_p k^{3/2} \frac{1}{L} \quad ,$$

where the terms on the right side account for diffusion, production and dissipation of the turbulence energy respectively. The length scale distribution $L(z)$ and the constants $\sigma_k$, $C_v$ and $C_p$ have to be prescribed (See page 33).
In equation (12), strictly speaking, a Stokes derivative is replaced by a time derivative as in equation (5). This replacement is based on analogous considerations.

A shortcoming of this model is, that the transport of the turbulence length scale is not accounted for. By using a second property of turbulence and its corresponding transport equation, the model can be improved in this respect. Various second properties are in use (23). Some attention will be given to one specific model, the k-\(\varepsilon\)-model, where the dissipation rate, \(\varepsilon\), of the turbulence energy is used as the second transported property. (See page 36).

2.2.4 k-\(\varepsilon\)-Model

Writing \(\varepsilon\) for the dissipation rate of the turbulence energy instead of \(C_D k^{3/2}/L\) in equation (12) gives \((31,35)\)

\[
\frac{\partial k}{\partial t} = C_1 \frac{k^2}{\varepsilon} \left( \frac{\partial u}{\partial z} \right)^2 + C_2 \frac{\partial}{\partial z} \left( \frac{\partial k}{\partial z} \frac{\partial k}{\partial z} \right) - \varepsilon \tag{13}
\]

where \(C_1 = C_\nu C_D\).

The transport equation for the dissipation rate closes the set of equations

\[
\frac{\partial \varepsilon}{\partial t} = C_3 k \left( \frac{\partial u}{\partial z} \right)^2 + C_2 \frac{\partial}{\partial z} \left( \frac{\partial \varepsilon}{\partial z} \frac{\partial \varepsilon}{\partial z} \right) - C_2 \frac{\varepsilon^2}{k} \tag{14}
\]

Again the constants \(C_1, C_2, C_3, \sigma_\varepsilon, \text{ and } \sigma_k\) have to be prescribed.
3. Pure Eddy Viscosity Models

3.1. Comparison with Measurements in Tidal Flows

In an eddy viscosity model \( v_t \) in equation (7) is replaced, as seen above, by an expression according to equation (8). Solving equation (7) yields the profiles of velocity and shear stress, etc. In expression (8) the choices of \( f(z/h) \) and the used global velocity scale \( u_e \) and length scale \( z_e \) have to be realistic. They will be considered first by taking account of some measurements done in tidal channels.

Most measurements in tidal channels show almost logarithmic velocity profiles \((8, 9, 24, 25, 36)\). These measurements are generally executed at near maximum velocities, Gordon \((11)\) finds a profile apparently indicative of a boundary layer not fully developed up to the surface. As an underdeveloped boundary layer is not expected (see 4), this phenomenon is probably caused by channel topography, density gradients, or the like. At tidal phases where the velocities are smaller, deviations of the logarithmic behaviour may be expected and have indeed been found by Bowden et al. \((7)\). These last measurements, however, are not very conclusive, as only one series of profiles has been measured, during half a tidal cycle, and in this series an inexplicable change between two succeeding profiles is reported. Anyhow, the measurements by Bowden et al. must be used with care, when discriminating between the performances of the various models.

Values of \( u/\bar{u}_{100} \) where \( u_{100} \) is the velocity at 1 meter above the bed, are of the order of 1/15 to 1/17 \((8, 9, 15, 25)\). If a drag coefficient \( C_{100} \) is defined as

\[
C_{100} = \frac{u^2_z}{u^2_{100}}
\]  

then this value corresponds to \( C_{100} = 4.4 \times 10^{-3} \) to \( 3.5 \times 10^{-3} \).
This is comparable to the value $C_{100} = 2.2 \times 10^{-3}$ to $4.0 \times 10^{-3}$ as found by Sternberg (36) derived from extensive measurements in various tidal channels. Sternberg computed the shear stress from the velocity profile.

The logarithmic velocity profiles above a rough bed in an uniform and steady flow can be described by

$$u(z) = \frac{u_\infty}{\kappa} \ln\left(\frac{z+z_0}{z_0}\right),$$  \hspace{1cm} (16)

where $z_0$ is the characteristic roughness height and $\kappa$ is Von Karman's constant. When $z_0$ is neglected compared to $z_*$, division of equation (16) by $u_\infty$ gives

$$\frac{u(z)}{u_\infty} = \frac{1}{\kappa} \ln\left(\frac{z}{z_0}\right).$$  \hspace{1cm} (17)

Writing $z_{rel}^o$ for the relative characteristic roughness height $z_0/h$ it yields

$$\frac{u(z/h)}{u_\infty} = \frac{1}{\kappa} \ln\left(\frac{z/h}{z_{rel}^o}\right).$$  \hspace{1cm} (18)

The velocity profiles following from equation (18) with $z_{rel}^o = 1.1 \times 10^{-4}$ are given in fig. 2. This value of $z_{rel}^o$ corresponds to e.g. a depth of 10 m combined with a roughness height of 1.1 mm. (see page 14).

The value of $u_\infty/u_{100}$ in this case is $1/17$ and so the value of $C_{100}$ is $3.5 \times 10^{-3}$. $z_0 = 1.1$ mm corresponds to an equivalent sand-roughness $k_s$ of $30 z_0 = 3.3$ cm. As the value of $u_{av}/u_\infty$ is about 20.2 in this case (see fig. 2), the value of the Chézy resistance coefficient $C$, defined by

$$C = \frac{u_{av}}{u_\infty} g^{1/2},$$  \hspace{1cm} (19)

is about $63 \text{ m}^{1/2}/\text{s}$. 
McCave (25) mentions non-constant values of $z_o$ in tidal flow. The bed seems to be transitional between smooth and rough for a flow with $Re < 1.5 \times 10^{-5}$ where $Re = \frac{u_100 z}{v}$ and $z = 1$ meter (36). This may influence the value $z_o$ in a part of the tidal cycle.

Following the mixing length hypothesis of Prandtl (29)

$$v_t = \frac{1}{2} \frac{3u}{\partial z}$$

as in equation (9), and assuming the mixing length to be

$$l_m = \kappa z$$

(21)

the velocity profile in a steady and uniform boundary layer shows the same logarithmic behaviour (see equation 17). Substitution of equation (21) and the derivative of equation (17)

$$\frac{3u}{\partial z} = \frac{1}{\kappa z} |u_x|$$

(22)

into equation (9) gives

$$v_t = \kappa |u_x| z.$$  

(23)

Equation (21), however, applies only in the region near the channel bed, where $z$ is smaller than $h/5$. According to Townsend (38), assumption of a constant eddy viscosity in large regions of the flow gives correct velocity profiles in many types of turbulent flow. In open channel flow he proposes a constant eddy viscosity above $z = h/5$, with the value

$$v_t = \kappa |u_x| \frac{h}{5}.$$  

(24)
Inhibition of large turbulent motions near the free surface, however, reduces the mixing length and the eddy viscosity in the surface layer in open channel flow.

A harmonic kinematic pressure gradient is considered in this report

\[ S = B \cos \omega t \]  

(25)

Using the non-dimensional variables

\[ u^+ = \frac{\omega}{B} u \]

(26)

\[ t^+ = \frac{t}{T} \]

\[ z^+ = \frac{z}{h} \]

\[ v_+ = \frac{v}{h} \]

\[ v_t = \frac{v}{h^3} \]

where \( T \) is the period of the variation of the pressure gradient \( 2\pi/\omega \), equation (7) becomes

\[ \frac{1}{2\pi} \frac{\partial u^+}{\partial t^+} + \frac{\partial}{\partial z^+} (v_t^+ \frac{\partial u^+}{\partial z^+}) = \cos (2\pi t^+) \]

(27)

Written in this way the amplitude of \( u^+ \) is 1 if the shear stress term is unimportant. Flows with equal \( v_t^+ (z^+, t^+) \) are similar.

Substitution of equation (26) in equation (23) and (24) gives the non-dimensional eddy viscosity

\[ v_t^+ = \frac{\kappa |u_\infty|}{h \omega} f \left( \frac{z}{h} \right) \]

(28)

where \( f(z/h) \) is \( z/h \) and \( 1/5 \) in the case of equation (23) and (24), respectively. \( u_\infty \) can be related to the pressure gradient. When the acceleration of the fluid is negligible, equilibrium of momentum requires
\[ |\tau_0| = u^2 = Bh \cos^2 \omega t , \]

where \( \tau \) is the bed shear stress. \hspace{1cm} (29)

In this report the following situation is considered:
The tidal channel has a depth of 10 m and a characteristic roughness height of 1.1 mm. The tide is semidiurnal. The values of the amplitude of the pressure gradient and the period and the frequency of the tide are respectively

\[ B = 1.454 \times 10^{-4} \text{ m/s}^2 \]

\[ T = 4.32 \times 10^4 \text{ s} \] \hspace{1cm} (30)

\[ \omega = 1.454 \times 10^{-4} \text{ /s} \] .

The eddy viscosity and the non-dimensional eddy viscosity in this case are, according to Townsend, respectively,

\[ \nu_t = 0.1525 \times f \left( \frac{z}{h} \right) \text{ m}^2/\text{s} \]

\[ \nu^*_t = 10.49 \times f \left( \frac{z}{h} \right) , \] \hspace{1cm} (31)

where \( f(z/h) \) of equation (28) is used.

3.2. Constant Eddy Viscosity

If \( \nu_t \) is constant, equation (7) allows an analytical solution (32)

\[ u(z,t) = \text{Re} \left\{ \frac{Be^{i\omega t}}{i\omega} \left\{ -1 + \frac{\cosh \left( \frac{\omega}{2\nu_t} (1+i)(z-h) \right)}{\cosh \left( \frac{\omega}{2\nu_t} (1+i)h \right)} \right\} \right\} \]

(32)

A reasonable value for \( \nu_t \) seems to be \( \kappa U_x^{\max} h/5 \) (see page 12), corresponding in the situation of this page to the viscosity and non-dimensional viscosity, respectively,
\[ \nu = 0.03051 \, m^2/s \]

\[ \nu^+ = 2.098 \]  

The profiles of the velocity \( u(z,t) \) during a tidal cycle are presented in fig. 3A and the profiles of the shear stress in fig. 3B. (For all results in this report non-dimensional quantities are used).

The velocity profiles at higher flow velocities differ very little from parabolas. At lower flow velocities fluid at a distance from the bed shows a small phase lag with respect to the fluid near the bed, as the time needed for the pressure gradient to alter the smaller velocities near the bed will be shorter than the time needed to alter the larger velocities nearer to the surface. This phase lag results in a small hysteresis effect as can be seen by plotting the shear stress at fixed depths against the surface velocity \( u_s \). (See the hysteresis diagram in fig. 3E). The velocities and the shear stresses vary harmonically with time because of the linearity of equation (7) in this case (See fig. 3C and 3D respectively), so the hysteresis diagram consists of exact ellipses.

The model with constant eddy viscosity fails in reproducing the appropriate velocity profiles. The value \( k\nu_{\text{max}} h/5 \), which is close to the values encountered over most of the depth at maximum velocity, gives values of \( u_{\text{avg}} \) that lie much below 20.2 \( u_s \) (compare fig. 2). The model yields a small hysteresis effect, but the curves in the hysteresis diagram do not resemble the hysteresis curves measured (see fig. 7a).

### 3.3 Quadratic Eddy Viscosity

Near the bed \( (z/h \leq 1/5) \), a quadratic eddy viscosity as given by (see fig. 4).

\[ \nu = k\nu_{\text{max}} (z - \frac{z^2}{h}) \] 

is a good approximation of the eddy viscosity distribution proposed on page 12 (at maximum velocity).
Above $z/h = 1/5$ and near the free surface the deviation is not very large either. The eddy viscosity and the non-dimensional eddy viscosity in the situation of page 14 have the values of expression (31), where

$$f\left(\frac{z}{h}\right) = \frac{z}{h} - \left(\frac{z}{h}\right)^2.$$  \hfill (35)

Equation (7) with $\nu_t$ as given by expression (34) also allows an analytical solution (26). The solutions of the velocity profiles are confluent hypergeometric functions (1), which at higher flow velocities approximate logarithmic functions (26). The velocity profiles and the shear stress profiles during a tidal cycle are shown in fig. 5A and 5B, respectively. The corresponding hysteresis diagram is shown in fig. 5E. Again, it consists of ellipses, in accordance with the harmonical relation of velocities and shear stresses with time (see fig. 5C and 5D).

The quadratic eddy viscosity model is superior to the constant eddy viscosity model in the reproduction of almost logarithmical profiles and in the value of $u_{av}^+$, which is nearer to $20.2 \ u_x^+$, but the hysteresis curves are not improved.

3.4. Time-Dependent Quadratic Eddy Viscosity

Considering the argument on page 7, it seems reasonable to use a time-dependent eddy viscosity, instead of eddy viscosities that do not change with time as in 3.2 and 3.3. Using $|u_{av}/20.2|$ as the global velocity scale, $h$ as the global length scale and $f(z/h)$ as given by expression (35) in equation (8) gives an eddy viscosity

$$\nu_t = \kappa \left| \frac{u_{av}}{20.2} \right| (z - \frac{z^2}{h}).$$  \hfill (36)

The dependence on depth of the eddy viscosities in expressions (36) and (34) is the same. The choice of the global length scale $|u_{av}/20.2|$ instead of $|u_x|$ is in accordance with fig. 2. In the situation of page 14 the non-dimensional viscosity has the value,

$$\nu_t^+ = 13.62 \ |u_{av}^+| f\left(\frac{z}{h}\right).$$  \hfill (37)
Equation (7) with $v_t$ as given by expression (36) is solved numerically by means of an implicit finite difference method, based on the Crank-Nicholson scheme. The velocity profiles at the larger mean velocities are almost logarithmic (see fig. 6A), and are equal to the profiles of fig. 2. The corresponding shear stress profiles are nearly linear (see fig. 6B). The phase lag of the velocities is larger than in the foregoing models, as the velocities are higher, due to the smaller eddy viscosities. Again the phase lag is highest near the surface for the same reason as in the constant viscosity model (see page 15).

To compare the hysteresis curves of fig. 6E with curves, as found by Gordon by measurements in an estuary (12), and by Anwar by measurements in a flume (3), allowance has to be made for the different circumstances. For this comparison in fig. 7A, the relative shear stress
\[
\frac{\tau_{o}^{\text{max}} \{(h-z)/h\}}{\tau_{s}^{\text{max}} \left(\frac{h-z}{h}\right)} \text{ is plotted against the relative surface velocity } \frac{u_s}{u_s^{\text{max}}}
\]
where $\tau_{o}^{\text{max}}$ and $u_s^{\text{max}}$ are the maximum bed shear stress and the maximum surface velocity during a tidal cycle, respectively. $\tau_{o}^{\text{max}} \{(h-z)/h\}$ would be the maximum shear stress at level $z$ when the shear stresses were exactly linear with depth.

The hysteresis effect computed with this model at $z/h = 0.25$ is much smaller than the effect shown by the curve obtained from Gordon's measurements at $z/h = 0.28$ (see fig. 7A). As no bed shear stress is known, because not sufficient information about the velocity profile is given, the reduction of his hysteresis measurement into fig. 7A is executed by means of the maximum shear stress on this level instead of $\tau_{o}^{\text{max}} \{(h-z)/h\}$.

The agreement between the curve computed with this model at $z/h = 0.75$ and Anwar's measurements at $z/h = 0.72$ (and the acceleration period of 200 s) is bad. This is not surprising as Anwar mentions extremely high shear stresses in connection with his hysteresis diagram. Even in the accelerating phase the shear stress exceeds the shear stress that fits a linear stress profile by a factor 2. The measurements of Anwar show, however, again a much larger hysteresis effect than this model does.
Another curve in fig. 7A is a hysteresis curve derived by the author (5), also based on the measurements of Anwar (3). The anomalous shear stresses do not appear here. The hysteresis effect, however, is still considerable.

Except for the hysteresis effect the curves in fig 7A obtained from the measurements of Gordon and of Anwar and those obtained by the computations in this model show the same quadratic relation between $u_s$ and $\tau$. In this respect the time-dependent global velocity is a considerable improvement in comparison to the constant global velocity.

In fig. 6C and fig. 6D, the variation of the velocities and the shear stresses at various depths during a tidal cycle are presented. The velocity variations in fig. 6C are slightly asymmetric. Such an asymmetry, but less pronounced, is measured in the Humber off Grimsby (19), and is calculated in a k-model of the same estuary (34). The shear stress variations in fig. 6D correspond to calculations in the same k-model and to measurements (7).

From a supposed analogy between laminar and turbulent flow, Yalin and Russell (39) conclude to the following expression for the bed shear stress under long waves

$$\tau_o = \alpha u_s^2 + \beta gh \frac{\partial \zeta}{\partial x}. \quad (38)$$

Measurements of the time variation of $\tau_o$ in a flume, the surface velocity $u_s$ and the gradient of the surface elevation $\partial \zeta / \partial x$ lead to the values $\alpha = (2.5 \ln(11h/k_s))^{-2}$ and $\beta = 0.018$. In the model of a tidal channel as used here, with $h = 10$ m, $z_o = 1.1$ mm (thus $k_s = 33$ mm) $\alpha$ gets the value $2.43 \times 10^{-3}$. This combination of $\alpha$ and $\beta$ leads to a small hysteresis effect, comparable to the effect computed with this model (see fig. 7B). Only the bed shear stress lags behind the surface velocity, whereas in this model the bed shear stress leads the surface velocity (see fig. 7B). Farther from the bed in this model the shear stresses lag behind the surface velocity too. (see fig. 7A).
In a steady (or very slowly varying) boundary layer flow the logarithmic profile of fig. 2 agrees with
\[ \tau_o = \left( \frac{1}{22.76} \right)^2 u_s^2 = 1.93 \times 10^{-3} u_s^2 \] (39)

Larger values of \( \beta \) lead to larger hysteresis effects.

Using \( \alpha = 2.43 \times 10^{-3} \) and \( \beta = 0.2 \) (see fig. 7B) gives a hysteresis effect, comparable to the effects measured by Gordon and by Anwar except for the behaviour of the shear stress when the surface velocities are small. (The value \( \beta = 0.018 \) was derived from measurements at small surface velocities).

The relative shear stresses for the expression of Yalin and Russell in fig. 7a and 7b are taken with respect to
\[ \tau_{o \text{max}} = 1.93 \times 10^{-3} (u_{s \text{max}})^2 \] (40)

To obtain a relation between \( \tau \) and \( u_s \) the relation between \( \partial \tau / \partial x \) and \( u_s \) must be known. For this the computations in this eddy viscosity model were used.

This eddy viscosity model combines the reproduction of almost logarithmic velocity profiles and almost linear shear stress profiles of the quadratic non-varying eddy viscosity model with a quadratic relation between \( u_s \) and \( \tau \). This relation shows a much too small hysteresis effect, however, compared to the measurements by Gordon in a tidal flow and by Anwar in a flume. Extrapolation of the formula of Yalin and Russell to tidal flows does not seem to be advisable.

In this eddy viscosity model and in the eddy viscosity model of 3.3 the bed roughness is introduced in the following way. In the finite difference scheme used the exactitude of the approximation of the velocity gradient close to the wall depends on the relation between the depth step and the characteristic roughness height, \( z_0 \). To compensate for the error in this approximation a small deviation of relations (34) and (36) at the meshpoints most close to the bed is used. This correction depends on \( z_0 \).
3.5. *A Hysteresis Parameter*

Flows with the same parameter

\[ P = \frac{h^2 \omega}{v_t} \]  

behave exactly similar in the constant eddy viscosity model (3.2) (see equation (27)). The non-dimensional velocities \( u^+ \) in the different flows are exactly equal at the same relative distance above the bed \( z/h \) and at the same tidal phase. This holds too for the non-dimensional shear stresses and so for the phase lags of \( \tau \) and \( u \). Consequently the parameter \( P \) is decisive for the hysteresis effect in this model.

The phase lags are proportional to \( P \) in the case of flows with a small phase lag. If no phase lag were present, then the relation between the relative shear stress and the relative surface velocity would be a linear one at each depth, corresponding to a straight line in the hysteresis diagram. If \( P \) has a value that implies an important phase lag at a certain depth, than the straight line in the hysteresis diagram becomes an ellips. (see fig. 3E).

In eddy viscosity models with non-constant eddy viscosities (33 and 34) flows with similar eddy viscosity profiles will again be similar if \( P \) is equal. So if different flows under long waves exhibit logarithmic velocity profiles, corresponding to similar linear eddy viscosity profiles for \( z/h \leq 1/5 \) and almost constant eddy viscosities for \( z/h \geq 1/5 \), then the same parameter \( P \) will be important. To compare the hysteresis effects of such flows, \( v_t \) in \( P \) has to be taken at equal relative depths.

The values of \( P \) referring to fig. 7a are, using the eddy viscosity model of 3.4, and taking \( v_t \) to be the maximum eddy viscosity at \( z/h = 0.2 \):

- \( P \) (Anwar) = 0.63 (at the acceleration period of 200 s)
- \( P \) (Gordon) = 0.56 (a rough guess)
- \( P \) (this report) = 0.54 (the tidal situation as used in this report (see page 14)).
In this light not much difference between the phase lags and the hysteresis effects in these three cases has to be expected. Only the larger $P$ in Anwar's measurements may bring about slightly larger phase lags and hysteresis effects. Without a phase lag the relation between the relative shear stresses at different depths and the relative surface velocity would be a quadratic one, using the eddy viscosity model of 3.4. The phase lags implied by a non-zero value of $P$ widens the curves in the hysteresis diagram (see fig. 7).

3.6. Future Developments

Some changes in the model might be interesting

1) Using $u_\times$ instead of $u_{av}/20.2$ for the global velocity scale in expression (36). This accounts in a better way for the generation of the turbulence, which determines the eddy viscosity, as this generation of turbulence depends largely on the conditions in the undermost centimeters. This change will not have much consequence for the velocity profiles close to maximum flow velocity, as there the flow profiles are almost logarithmic with $u_\times = u_{av}/20.2$. The changes at lower flow velocities and the changes in the hysteresis curves may be appreciable.

2) An attempt may be made to simulate the "bursting" phenomenon in boundary layer flow in an eddy viscosity model. A problem is the lack of knowledge of the bursting process, in particular in flows of a large Reynolds number. It is doubtful, whether an eddy viscosity model is attractive for this purpose. Possibilities are an eddy viscosity that is larger in the decelerating phase than in the accelerating phase, and use of a time delay with respect to $u_{av}$ or $u_\times$ to take account of the transport of turbulent fluid in coherent structures from the bed region to the surface region of the flow. As these structures have an upward velocity of about one tenth of the mean velocity (28), the time lag connected with this transport is of the order of 200 s in tidal flow, so no dramatic effects are to be expected. Only the flow at low mean velocities will be more seriously affected.
3) Calculation of turbulence energy profiles may make possible an extra check by measurements. It may also serve as a connection and a comparison with the k-model and the k-ε-model.

4) The mixing length model of 2.2.2 requires roughly the same computational method as eddy viscosity models. Only, points with small \( \partial u / \partial z \) may give extra problems. The model is not trustworthy around these points, because of the vanishing eddy viscosity. At high mean velocities the profile is almost logarithmic, so the mixing length model is not expected to give results, deviating much from the eddy viscosity model of 3.4.
4. Phase Lag of Velocity and Turbulence Energy

4.1 Phase Lag in Periodic Flow

In the preceding section phase lags of velocity and shear stress with respect to the pressure gradient (and other varying properties) appear. The phase lags are related to the acceleration of the flow under the influence of the pressure gradient. These accelerations arise when the pressure gradient and the derivative of the shear stress $\frac{\partial \tau}{\partial z}$ do not balance each other everywhere, as they do in steady flow. In a varying flow, consequently, the shear stress profile and, with it, the velocity profile lag behind the equilibrium profiles, that constitute the steady state solution matching the pressure gradient at the same moment. In a periodic flow this time lag related to the period of the flow gives a phase lag.

Larger accelerations are connected with profiles that deviate more from equilibrium profiles so they are combined with larger phase lags. These larger accelerations may be brought about by a higher frequency or a lower viscosity (causing higher velocities) (see 3.5).

Generally the phase lag is dependent on depth and time. The smaller velocities near the bed mean smaller accelerations and smaller phase lags.

In eddy viscosity models where the eddy viscosity is determined by the turbulence energy $k$, the phase lag picture is more complicated. The $k$-value at each depth will not match the velocity profile, but will lag behind. If the phase lag of the turbulence energy with respect to the velocities is small, then the $k$-profiles and hence the eddy viscosity profiles almost match the velocity profiles. Use of these eddy viscosity profiles in an eddy viscosity model will then give the same solution as the $k$-model. In that case the only use of the $k$-model may lie in the computation of the eddy viscosity profiles. If the phase lag of the turbulence energy with respect to the velocity is important, then the dependence of the eddy viscosity on the turbulence energy (33) can not be expressed in a dependence on the velocity. The eddy viscosity model is clearly insufficient in that case.
In the k-ε-model an analogous consideration is possible. The condition of a small phase lag of $k$ will presumably imply a small phase lag of $\varepsilon$.

### 4.2. Phase Lag of the Velocity in Eddy Viscosity Models

The values of the phase lags below, calculated in various eddy viscosity models (3.2 to 3.5) are all based upon the tidal situation of page 14.

In the model with a constant eddy viscosity (3.2) the velocity is harmonic with time at every depth, so the phase lag of the velocity depends only on the depth. This depth-dependency is small, however. At the surface a phase lag of 11 degrees and at 10 cm of the bottom a phase lag of $8.5^\circ$ is calculated. This corresponds to a time lag of 22 minutes at the surface and of 17 minutes at 10 cm above the channel bed.

In the model with a quadratic eddy viscosity (3.3) the phase lag again is only dependent on the depth, but very weakly. At the surface the phase lag is $32^\circ$ corresponding with a time lag of 64 minutes.

Even larger phase lags are calculated in the eddy viscosity model where the eddy viscosity has a quadratic dependence on $z$ and is proportional to $|u_{av}|$ (see 3.4). In that case the phase lag is not only dependent on depth, but also on time. At high flow velocities the dependence on depth is small. The phase lag is about $36^\circ$ in that case, corresponding to a time lag of 72 minutes. At low flow velocities the phase lags calculated are much larger: from $51^\circ$ at the surface to about $46^\circ$ at 10 cm above the bed, corresponding to time lags of 1 hour and 42 minutes at the surface to 1 hour and 32 minutes near the bottom.

Few mention is made of phase or time lags in estuaries. When measurements are reported (18,34,37), time lags of the velocity appear to be between about half an hour and one hour. The values of the last model seem a bit high, but time lag data in tidal flows are not easily measured exactly, especially at low flow velocities.
4.3 Estimate of the Time Lag of the Turbulence Energy near the Bed

A rough estimate of the time lag of $k$ with respect to the velocity in a tidal channel can be made near the bed. There the terms concerning production and dissipation in the transport equation of turbulence energy (equation 12) are much larger than the term for the turbulent diffusion. Following Townsend (38), at $z/h = 1/10$ the ratio between the three terms is about 7:8:1 respectively (in pipe flow). Therefore the diffusion-term may now be neglected. When the velocity profile is logarithmic with $z$, then the production-term in equation (12) is, using equations (22) and 23),

$$\frac{\partial k}{\partial t} \text{prod} = \nu \left( \frac{\partial u}{\partial z} \right)^2 = \left| \frac{u^3}{kz} \right|$$

If the boundary layer were steady, the dissipation rate $\varepsilon$ would be equal to the production as diffusion is neglected. If, as expected in non-steady flow, the turbulence energy is not exactly adjusted to the velocity profile, then both the production and the dissipation of turbulence energy differ from the matching value (see equations (11) and (12)). As the production is proportional to $k^{1/2}$ and the dissipation to $k^{3/2}$ a net production (or dissipation) of turbulence energy will result.

If the turbulence energy deviates a small amount $\Delta k$ from the matching value $k$, then the production rate is multiplied by a factor $(1 + \frac{1}{2} \frac{\Delta k}{k})$ as

$$(k + \Delta k) = (k^{1/2} + \frac{1}{2} \frac{\Delta k}{k^{1/2}})^2$$

In the same way the dissipation rate is multiplied by a factor $(1 + 3/2 \frac{\Delta k}{k})$. As equation (42) gives the production rate and the (equal) dissipation rate of the turbulence energy when $k$ has the matching value, the value of $\frac{\partial k}{\partial t}$ becomes in case of a small mismatching $\Delta k$

$$\frac{\partial k}{\partial t} = \frac{1}{2} \frac{\Delta k}{k} \left( \frac{\partial k}{\partial t} \right) \text{prod} - \frac{3}{2} \frac{\Delta k}{k} \left( \frac{\partial k}{\partial t} \right) \text{prod} = -\frac{\Delta k}{k} \left( \frac{\partial k}{\partial t} \right) \text{prod} = -\frac{\Delta k}{k} \left| \frac{u^3}{kz} \right|$$

(44)
The time lag is the time that would be needed for the rate of change of the turbulence energy (see expression 44) to make up for the original deviation $\Delta k$. So the time lag can be estimated by

$$\text{time lag} = -\frac{\Delta k}{(\partial k/\partial t)_{\text{prod}}} = \frac{k}{(\partial k/\partial t)_{\text{prod}}} = k \left| \frac{kz}{u_x^3} \right|$$ (45)

To estimate the turbulence energy $k$ in equation (45) a relation with the shear stress can be used (23,38)

$$\frac{k}{t} = 3.54$$ (46)

This relation is valid only near the channel bed. Assumption of a linear stress distribution relates the turbulence energy to the bed shear stress, so near the channel bed the following expression gives an estimation of the turbulence energy

$$k = 3.54 u_x^2 \left( \frac{h-z}{h} \right)$$ (47)

Substitution in equation (45) gives for the time lag

$$\text{time lag} = 3.54 \left( \frac{h-z}{h} \right) \frac{kz}{u_x}$$ (48)

In a slowly varying flow the bed shear stress and the pressure gradient are almost in equilibrium, so in that case

$$u_x^2 = h |S(t)|$$ (49)

Using equation (49), expression (48) becomes

$$\text{time lag} = 3.54 \frac{(h-z)z}{h^{3/2}} \frac{k}{|S(t)|^{1/2}}$$ (50)

Using the values of $S$ and $h$ pertaining to the tidal situation simulated in this report (see page 14) equation (50) becomes
time lag = \( 3.7 \times 10^2 \frac{Z}{h} (1 - \frac{Z}{h}) \frac{1}{|\cos \omega t|} \) seconds \( (51) \)

At \( z/h = 1/10 \) and near maximum flow velocities a time lag of about 33 seconds results. Nearer to the bottom the time lags are smaller, more or less proportionally to \( z \). At small velocities the time lags become more important. A \( k \)-model may then give appreciably better results.

4.4. Estimate of the Time Lag of the Turbulence Energy at all Depths

Equation (46) holds approximately throughout a large part of the depth, as the relation between \( \tau \) and depth is almost linear and that between turbulence energy and depth does not deviate far from a linear relation except for the surface layer \( (38) \). Neglect of the diffusion of turbulence energy then would lead to a maximum time lag at half-depth of about 93 seconds. Diffusion outside the near-bed layer is transporting turbulence energy upwards. This transport from a region with smaller time lags will generally reduce the time lags farther from the bed too.

The time lag of the turbulence energy with respect to the velocity is quite small throughout the flow depth, except at low flow velocities. Because of this small time lag much more accurate solutions of velocity profiles and other properties by using a \( k \)-model, instead of an eddy viscosity model with an appropriate eddy viscosity distribution, are not to be expected. (At low flow velocities the solutions may be improved.) As logarithmic velocity profiles are not very critical with respect to the exact eddy viscosity distribution, in particular near the surface, a \( k \)-model can perhaps be used to decide on the eddy viscosity profiles.

4.5. An Analytical Solution of the Variation of the Turbulence Energy near the Bed

For the near-bed layer \( (z/h \leq 1/10) \), where diffusion is negligible, an analytical solution of the transport equation of the turbulence energy (equation (12)) is possible. In contradistinction to the approximation method of 4.3 this method gives the possibility to compute the turbulence energy at low flow velocities.
By substitution of expression (11) and neglect of the diffusion term

equation (12) becomes

\[ \frac{\partial k}{\partial t} = C_v L \sqrt{k} \left( \frac{\partial u}{\partial z} \right)^2 - C_D \frac{k^{3/2}}{L} \] (52)

Assuming \( \frac{\partial u}{\partial z} \) to vary harmonically with time near the bed

\[ \frac{\partial u}{\partial z} = \gamma \cos \omega t \] (53)

equation (52) becomes

\[ \frac{\partial q}{\partial t} + \frac{C_D}{2L} q^2 = \frac{C_v L \gamma^2}{2} \cos^2 \omega t, \] (54)

where \( q \) is written for \( \sqrt{k} \)

Using the non-dimensional variables

\[ t^+ = \frac{t}{T_p} \] (55)

\[ q^+ = \frac{\omega}{B} q \]

equation (54) becomes

\[ \frac{\partial q^+}{\partial t^+} + (q^+)^2 = a^2 \cos^2 2\pi \] (56)

where

\[ \delta = \frac{2L \omega}{C_D B T_p} \]

\[ a = \sqrt{\frac{C_v L \gamma^2 \omega^2}{C_D B^2}} \] (57)

When \( q^+ \) has a positive initial value equation (56) only permits positive solutions.
Except for regions of $t^+$ with small $\cos 2\pi t^+$, equation (56) can be solved using an asymptotical method. In regions of $t^+$ where $\cos 2\pi t^+$ is small, equation (56) can be approximated by

$$\delta \frac{3q^+}{\delta t^+} + q^+ = a^2 (2\pi t^+)^2$$

(58)

where

$$t^+ = t^+ + \frac{1}{4} - \frac{n}{2} \ ; \ \frac{1}{4} \leq t^+ < \frac{1}{4}$$

(59)

Here $n$ is an integer.

If regions exist where both methods are applicable, then the solutions should give the same results.

In the asymptotical method it is assumed, that the solution $q^+(t^+)$ of equation (56) can be written as

$$q^+(t^+) = q_0^+(t^+) + \delta q_1^+(t^+) + \delta q_2^+(t^+) + \ldots$$

(60)

where each term on the right side is an order smaller than the preceding term. For each power of $\delta$ equation (56) can be solved.

**0-order:** The 0-order part of equation (56) is

$$\langle q_0^+ \rangle^2 a^2 \cos^2 2\pi t^+$$

(61)

so the 0-order solution is

$$q_0^+(t^+) = a |\cos 2\pi t^+|$$

(62)

The negative solution in each half-cycle can be excluded as

$$q^+(t^+) = q_0^+(t^+) \times (1\text{-order } \delta) \geq 0$$

(63)

**1-order:** The 1-order part of equation (56) is
\[
\delta \frac{\delta q_0}{\partial t^+} = -2\delta q_0 - q_1^+
\]  \hspace{1cm} (64)

giving as 1-order solution
\[
\delta q_1^+ (t^+) = \delta \pi \tan 2\pi t^+
\]  \hspace{1cm} (65)

2-order: The 2-order part of equation (56) is
\[
\delta^2 q_2^+ = -\delta^2 (q_1^+)^2 - 2\delta^2 q_0 q_2^+
\]  \hspace{1cm} (66)

The 2-order solution is
\[
\delta^2 q_2^+ (t^+) = \frac{\delta^2 \pi^2}{2a|\cos 2\pi t^+|} \left( 1 - \frac{3}{\cos^2 2\pi t^+} \right)
\]  \hspace{1cm} (67)

In fig. 8a the 0-order, 1-order and 2-order solutions at \( z/h = 1/10 \) are plotted, together with their sum as an approximation for \( q^+ (t^+) \). The values of \( a \) and \( \delta \) pertain to the tidal situation in this report (see page 14). \( a \) and \( \delta \) are evaluated at page 34. Clearly the asymptotical method can not be used for \( |t'| < 0.03 \).

At small values of \( t' \) solutions of equation (58) are used as approximations to solutions of equation (56).

Taking \( \phi (t') \) to be a function with the property
\[
q^+ (t') = \frac{\delta}{\phi(t')} \frac{d\phi(t')}{dt'}
\]  \hspace{1cm} (68)

equation (58) can be rewritten as
\[
\frac{1}{\phi} \frac{d^2 \phi}{d(t')^2} = \delta^2 a^2 (2\pi t')^2
\]  \hspace{1cm} (69)
The solution to this equation is a parabolic cylinder function (1):

\[ \phi(t') = \sum_{n=0}^{\infty} a_n (\sqrt{2\pi n} a t')^n \]

\[ \begin{align*}
    a_n &= \frac{1}{n(n-1)} a_{n-4} \\
    a_2 &= 0 \\
    a_3 &= 0
\end{align*} \tag{70} \]

The parameter \( a_1/a_0 \) specifies a series of solutions for \( q^+(t') \) of equation (58) obtained by expression (68) from the solutions (70) of equation (69). A good transition from solution (70) to the solution as found by using the asymptotical method can be achieved by a proper choice of \( a_1/a_0 \).

In fig. 8b the solutions of \( q^+(t') \) at \( z/h = 1/10 \) as found by means of both methods is plotted. As the approximation in equation (58) is satisfactory for \( |t'| < 0.055 \), a sufficient time-region, where both methods apply, exists.

The strange behaviour of \( q^+(t^+) \) in fig. 8b as computed by means of the approximation method near \( t^+ = 0.20 \) is a by-product of the calculation method. It is caused by the very critical behaviour of \( q^+(t^+) \) with regard to the value of \( a_1/a_0 \) for negative values of \( t' \).

The turbulence energy follows from \( q^+(t^+) \) with

\[ k(t^+) = \frac{B^2}{\omega^2} (q^+(t^+))^2 \tag{71} \]

In fig. 8c the variation of the turbulence energy at \( z/h = 1/10 \) over the tidal cycle is shown.

Because of the relation of \( k \) and \( \tau \) (see equation 46) \( k \) is made non-dimensional by means of a division by the same factor \( Bh \) as was used in the case of \( \tau \).
5. Turbulence Energy Models

5.1 Introduction

In the last few years many boundary layer problems have been attacked numerically with k-models and k-ε-models. In these models use is made of the turbulence energy $k$, as a property which determines the eddy viscosity, and of $k$ and the dissipation rate $\varepsilon$ respectively. For each turbulence property used a transport equation has to be solved. (See pages 8 and 9).

To obtain realistic models the constants in equations (11) to (14) have to be chosen carefully. Calibration of the models by optimizing the constants, as is done by Smith and Takhar (35) is dangerous. It is possible to generate the wanted almost logarithmic velocity profiles with different combinations of the constants. The constants, so found, and hence the whole physics of the model may be completely wrong.

5.2 The k-Model: Choice of the Constants

If a k-model has to make sense, the constants and the distribution of $L$ must be chosen equal to the constants in a k-model for appropriate steady boundary layer flows. An appropriate steady boundary layer flow is the steady flow, corresponding to the almost logarithmic velocity profiles as found in an estuary at high flow velocities, as the velocity distribution and the turbulence energy distribution are almost stationary then.

$L$ in equations (11) and (12) is a length scale that characterizes the large turbulent motions (31), as is $l_m$, the mixing length. $L$ is chosen equal to $l_m$ here (32).

In a steady boundary layer, diffusion of turbulence energy is negligible near the wall, so production and dissipation are in equilibrium there (see equation 12).

$$v_t \left(\frac{\partial}{\partial z}\right)^2 - C_k \frac{k^{3/2}}{D L} = 0$$

Multiplication of this equation by $v_t$ and substitution of equations (6) and (11) yields
The ratio of $k$ and $\tau$, as derived from various measurements, is near the wall, given by equation (46). This supplies for $C_vC_D$ the value

$$C_vC_D = 0.08 \quad (74)$$

Combination of equations (11) and (23) gives near the bed

$$C_v = \frac{|u_x|}{\sqrt{k} L(z)} \quad (75)$$

$k$ can be eliminated by equation (47) near the bed

$$k = 3.54 u_x^2 \quad (76)$$

As $L$ is chosen equal to $l_m = \kappa z$, it follows that

$$C_v = 0.53 \quad \text{and} \quad C_D = 0.15 \quad (77)$$

The diffusion coefficient of the turbulence energy is about as large as the diffusion coefficient for momentum $v_t$. A value of 1.0 or 1.1 is hence generally used for $\sigma_k$.

Smith and Takhar (35) use $C_vC_D^2 = 0.09$ instead of 0.012 in their simulation of tidal flow in an estuary. They use an $L$-profile in which the $L$ remains large up to the surface (see fig.10). Such an $L$-profile fits in with boundary layer flow without a free surface. In open channel flow the length scale declines again near the surface, as the surface inhibits large turbulent motions.

Boundary layer flows are dependent on the roughness of the bed. As no dependence of the roughness is included in the equations of the $k$-model or in the constants the only possibility to introduce the roughness of the bed left is in the boundary conditions.
The constants \(a\) and \(\delta\) in 4.5 can be calculated from \(C_D, C_V\) and \(L\). Using the values for \(C_V\) and \(C_D\) of expression (77), \(L = 0.4\) m at \(z = 1\) m, and reckoning with the phase lag of \(u\) (see page 24),

\[
\gamma^2 = \frac{u^2}{k z} = \frac{B h}{k z} \cos^2 36^\circ ,
\]

see equations (22) and (29), the values of \(a\) and \(\delta\) at \(z = 1\) m are

\[
a = 5.80 \times 10^{-2} \\
\delta = 1.235 \times 10^{-4}
\]

5.3 The \(k\)-model: A First Calculation

A first calculation with a \(k\)-model simulating the tidal situation of page 14 is performed. As boundary conditions are used:

At the bed: \(u = 0\) and \(k = |\tau_o|/(C_V C_D)^{1/2}\), corresponding with equation (73).

The bed-roughness is introduced by means of the choice of \(\tau_o\) in the boundary condition of \(k\).

\[
\tau_o = \frac{k^2 u(z) |u(z)|}{(\ln(z/z_o))^2}
\]

for small \(z\), in conformity with the law of the wall.

At the surface: \(\partial u/\partial z = 0\) and \(\partial k/\partial z = 0\) or absence of momentum and energy transport.

The length scale distribution of Smith and Takhar (see page 33) is used for this calculation. It has to be changed to account for the inhibition of the large turbulent motions near the surface.

The equations are solved numerically by means of an implicit finite difference method, based on the Crank-Nicholson scheme. The errors introduced by the finite difference scheme at the meshpoints near the bed are corrected for.

The depth-step, \(\Delta z\), is, for the time being, chosen quite large (\(\Delta z = 0.25\) m). The time-step, \(\Delta t\), depends mainly on the time lag between \(k\) and \(u\) at the lowest meshpoint in the flow, and has to be very short in order to keep the turbulence energy positive. Here \(\frac{1}{2400}\) of a tidal cycle or 18 s has been used.
Choosing the first mesh point in the flow nearer to the bed requires in this model a smaller time-step as can be concluded from equation (51). As a much finer depth mesh, in particular near the bed, is desired, a scheme with different time-steps near the bed may be considered.

The differences between the velocity profiles computed using the k-model (see fig. 9A) and the time-dependent quadratic eddy viscosity model (3.4) (see fig. 6A) are small. A deviation of the logarithmic shape is connected with the L-profile used. This deviation is too small to be measured in a prototype. The large depth-step influences the profile close to the bed. The shear stress profile (see fig. 9B), the variation of velocity and shear stress with time (see fig. 9D and 9E, respectively), and the hysteresis curves of the shear stress (fig. 9G) correspond closely to their counterparts in the eddy viscosity model of 3.4. The phase lag of the velocity in the k-model deviates only some tenths of degrees from the corresponding phase lag computed using the eddy viscosity model.

The turbulence energy profiles are shown in fig. 6C. The shape of the profiles near the bed has to be improved. The value of k near the bed determines the eddy viscosity there and hence influences the amplitude at all depths. The values of k in the surface region are quite high. This may be due to the L-profile used. The hysteresis curves of the turbulence energy at various depths are given in fig. 9H. The variation of k with time is shown in fig. 9F. The agreement with the analytical solution in fig. 8C is satisfactory. Only the value of k in the maximum phase is somewhat higher than it was computed analytically, and the asymmetry is stronger. The first difference may be due to the neglect of the diffusion in the analytical solution or to the large depth-step in the k-model. The second difference is caused by the assumption of a harmonic variation of $\partial u/\partial z$ in the case of the analytical solution.
5.4 k-ε-Model

In the future use of a k-ε-model (see page 9) is planned. The same attention as in the k-model has to be given to the constants. The length-scale distribution $L$ can be calculated using the solution of $k$ and $\varepsilon$ by

$$L = C_D \frac{k^{3/2}}{\varepsilon}$$

Smith and Takhar (35) compute a length-scale distribution in their k-ε-model analogous to the distribution which they use in the k-model. As the length-scale does not decline near the surface, this distribution was rejected on page 34. In the k-ε-model this questionable distribution is possibly caused by wrong boundary conditions at the surface.
6. Conclusions

In spite of the very simple one-dimensional mathematical models of the tidal channel used in this investigation, many aspects of turbulent flow as measured in estuaries and in flumes under long waves are reproducible. Velocity profiles, shear stress profiles, time lags, hysteresis effects of shear stresses against surface velocity etc. are considered.

The few measurements available show much scatter in the data and many aspects of tidal flows in estuaries have not been measured at all. This is a handicap concerning the calibration of the models and concerning the determination of the ability of the various models to reproduce the tidal flow. For that reason more measurements in tidal flows are needed. Measurements in flumes, though less suitable because of scale effects, are important, as measurements in tidal flows are costly, difficult and mostly imprecise. Density differences, channel topography, bed roughness and cooperating tidal components all act together. Their respective influences on the flow are not always easily separable.

In this investigation computations are made in various eddy viscosity models and in the k-model until now. The difference between the results in the used eddy viscosity model with varying eddy viscosity (3.4) and in the k-model is small, as is expected because of the small phase lag between velocity and turbulence energy.

The choice of the formulas used for the eddy viscosity depends on the physical factors which are supposed to play a part and on the desired results. Taking into account the "bursting" phenomenon requires a knowledge of this phenomenon that is hardly available at this moment. For instance, the choice of the global velocity scale $u_e$ (in equation 8) has to be made. As the turbulence is generated near the bed, $u_\ast$ seems to be the appropriate velocity, but as the "bursting" phenomenon seems to depend on the near-surface velocity in some aspects, $u_s$ or $u_{av}$ may be a better choice. For the time being in the best pure eddy viscosity model the eddy viscosity is quadratic in $z$, and proportional to a varying velocity such as $u_{av}$ or $u_\ast$. A model of this kind will reproduce many of the aspects of the flow in tidal channels sufficiently.
One aspect did not reproduce until now. The computed hysteresis effect is much smaller than the effect measured by Gordon (12). Presumably a model has to be devised, that takes into account the enhancement of the generation of turbulence near the wall in decelerating flow and the displacement of fluid from the bottom to higher layers in the flow through coherent structures connected with the "bursting" phenomenon. The time lag of the turbulence energy is apparently too small to cause the hysteresis effect as measured by Gordon.

Smith and Takhar (35) have used a k-model to compute the flow in a tidal channel. They claim to have obtained good results in the reproduction of velocity profiles in the Humber, using, however, completely unrealistic constants. They compare, however, only velocity profiles at near maximum current velocities. As these velocity profiles are almost logarithmic a correct k-model will give equally good results when a proper bed roughness is used.
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Notation

- a constant (p 28)
- a, a, a1 constants (p 31)
- B amplitude of the pressure gradient (p 13)
- C, C drag coefficient (p 10)
- C1, C2, C3 constants (p 9)
- C100 drag coefficient (p 10)
- C resistance coefficient of Chézy (p 11)
- f(\frac{z}{h}) function of z/h (p 7)
- g acceleration of gravity (p 5)
- h mean free surface level (p 5)
- i \sqrt{-1} (p 14)
- k turbulence energy (p 8)
- \Delta k small change in turbulence energy (p 25)
- \frac{\partial k}{\partial t}_{prod} production rate of turbulence energy by the shear stress (p 25)
- k_s equivalent sand roughness (p 11)
- l_m mixing length (p 7)
- L characteristic length for the turbulence intensity (p 8)
- n integer (p 29)
- \Omega_x, \Omega_z coordinate axes (p 5)
- P hysteresis parameter (p 20)
- q root of the turbulence energy (p 28)
- q non-dimensional root of the turbulence energy (p 28)
- q_i, q_0, q_1 ... i-th, 0-th, first, ... order term of q (p 29)
- Re Reynolds number (p 12)
- S kinematic pressure gradient (p 6)
- t time (p 5)
- t non-dimensional time (p 13)
- t non-dimensional time with displaced zero (p 29)
- \Delta t timestep (p 34)
- T period (p 13)
- u longitudinal velocity (p 5)
- u non-dimensional velocity (p 13)
- u100 longitudinal velocity at z = 1 m (p 10)
- u maximum longitudinal friction velocity (p 7)
- u maximum longitudinal friction velocity (p 7)
- u depth-averaged longitudinal velocity (p 7)
- u non-dimensional u (p 16)
<table>
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<td>( u_{\text{av}} )</td>
<td>maximum depth-averaged velocity (p 7)</td>
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<tr>
<td>( u_s )</td>
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</tr>
<tr>
<td>( u_{s \text{max}} )</td>
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<td>( u_e )</td>
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<td>( \gamma )</td>
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FIG. 1 COORDINATE SYSTEM

FIG. 2 LOGARITHMIC VELOCITY PROFILES (CORRESPONDING TO $Z_0^{REL} = 0.00011$)

$U_{AV} = 20.2 \times U_*$

$U_* = 22.8 \times U_*$

$U_{100} = 17.0 \times U_*$
ONE TIME-STEP IS 1/48 PERIOD
O IS THE PHASE OF MAXIMUM PRESSURE GRADIENT

--- ACCELERATION OF THE FLOW
----- DECELERATION OF THE FLOW

FIG. 3A CONSTANT EDDY VISCOSITY VELOCITY PROFILES

FIG. 3B CONSTANT EDDY VISCOSITY SHEAR STRESS PROFILES
FIG. 3C CONSTANT EDDY VISCOSITY
VARIATION OF VELOCITY WITH TIME

FIG. 3D CONSTANT EDDY VISCOSITY
VARIATION OF SHEAR STRESS WITH TIME
FIG. 3E CONSTANT EDDY VISCOSITY Hysteresis Diagram

FIG. 4 EDDY VISCOSITY PROFILES
ONE TIME-STEP IS 1/48 PERIOD

CO IS THE PHASE OF MAXIMUM PRESSURE GRADIENT

--- ACCELERATION OF THE FLOW
------ DECELERATION OF THE FLOW

FIG. 5A QUADRATIC EDDY VISCOITY
VELOCITY PROFILES

FIG. 5B QUADRATIC EDDY VISCOITY
SHEAR STRESS PROFILES
FIG. 5C QUADRATIC EDDY VISCOSITY VARIATION OF VELOCITY WITH TIME

FIG. 5D QUADRATIC EDDY VISCOSITY VARIATION OF SHEAR STRESS WITH TIME
FIG. 5E QUADRATIC EDDY VISCOSITY HYSSTERESIS DIAGRAM

ONE TIME-STEP IS 1/48 PERIOD
O IS THE PHASE OF MAXIMUM PRESSURE GRADIENT

- ACCELERATION OF THE FLOW
----- DECELERATION OF THE FLOW

FIG. 6A TIME DEPENDENT QUADRATIC EDDY VISCOSITY VELOCITY PROFILES
ONE TIME-STEP IS 1/48 PERIOD
0 IS THE PHASE OF MAXIMUM PRESSURE GRADIENT

--- INCREASE OF THE SHEAR STRESS
--- DECREASE OF THE SHEAR STRESS

FIG. 6B TIME DEPENDENT QUADRATIC EDDY VISCOSITY
SHEAR STRESS PROFILES

FIG. 6C TIME DEPENDENT QUADRATIC EDDY VISCOSITY
VARIATION OF VELOCITY WITH TIME
FIG. 6D TIME DEPENDENT QUADRATIC EDDY VISCOSITY VARIATION OF SHEAR STRESS WITH TIME

FIG. 6E TIME DEPENDENT QUADRATIC EDDY VISCOSITY HYSTERESIS DIAGRAM
FIG. 7B HYSTERESIS DIAGRAM
COMPARISON OF BED SHEAR STRESS

FIG. 7A HYSTERESIS DIAGRAM
COMPARISON WITH MEASUREMENTS

Gordon \( \frac{z}{h} = 0.28 \)

Anwar \( \frac{z}{h} = 0.72 \)

Booj (Anwar) \( \frac{z}{h} = 0.72 \)

Yalin–Russell \( \frac{z}{h} = 0 \)

(\( \alpha = 2.43 \times 10^{-3}, \beta = 0.2 \))

eddy visc. model (34)

\( \frac{z}{h} = 0.25 \)

\( \frac{z}{h} = 0.75 \)
FIG. 8A ANALYTICAL SOLUTION OF TURBULENCE ENERGY AT Z/H=0.1
ASYMPTOTICAL SOLUTION OF Q^+

FIG. 8B ANALYTICAL SOLUTION OF TURBULENCE ENERGY AT Z/H=0.1
VARIATION OF Q^' WITH TIME
FIG. 8C ANALYTICAL SOLUTION OF TURBULENCE ENERGY AT Z/H = 0.1
VARIATION OF TURBULENCE ENERGY WITH TIME

ONE TIME-STEP IS 1/48 PERIOD
0 IS THE PHASE OF MAXIMUM PRESSURE GRADIENT

FIG. 9A K-MODEL
VELOCITY PROFILES
FIG. 9B K-MODEL
SHEAR STRESS PROFILES

FIG. 9C K-MODEL
TURBULENCE ENERGY PROFILES

ONE TIME-STEP IS 1/48 PERIOD
0 IS THE PHASE OF MAXIMUM PRESSURE GRADIENT

—— INCREASE OF THE SHEAR STRESS
——— DECREASE OF THE SHEAR STRESS

—— INCREASE OF THE TURB. ENERGY
——— DECREASE OF THE TURB. ENERGY
FIG. 9D K-MODEL
VARIATION OF VELOCITY WITH TIME

FIG. 9E K-MODEL
VARIATION OF SHEAR STRESS WITH TIME
FIG. 9F K-MODEL
VARIATION OF TURBULENCE ENERGY WITH TIME

FIG. 9G K-MODEL
HYSTERESIS DIAGRAM (SHEAR STRESS)
FIG. 9H K-MODEL
HYSTERESIS DIAGRAM (TURB. ENERGY)

FIG. 10 L-DISTRIBUTION