Department of Precision and Microsystems Engineering

Impulse Based Substructuring
Theory, Improvement, and its Implementation on real problems

Name: Nazgol Haghighat

Report no: EM11039
Coach: prof. dr. ir. D.J. Rixen
Professor: prof. dr. ir. D.J. Rixen
Specialisation: Engineering Mechanics
Type of report: Masters Thesis
Date: Delft, November 23, 2011
Abstract

Dynamic substructuring as a method to solve complex systems is becoming very important. Dynamic substructuring techniques are based on assembling the dynamic models of some simpler structures in order to generate the dynamic model of a complex structure. Impulse Based Substructuring (IBS) as a time domain substructuring technique computes dynamic response of linear structures by applying convolution product on the impulse responses of the subsystems.

The IBS method can be explained in 2 main steps; 1: Computing the Impulse Response Functions (IRFs) of the substructures due to a unit impact (by applying numerical time integration), 2: Assembling procedure and computing the dynamic response of the total system by applying the convolution product between the IRFs and the applied forces. The applied forces consist of both external and unknown internal forces which are made at the interfaces due to substructuring and will be found during the assembling procedure in a way to satisfy the compatibility conditions at the interface. The IBS formulations have been developed based on the 3 different impact models. Afterwards in order to reduce the computation costs for long simulations, truncation and windowing have been applied on the IRFs.

In this thesis, the 3 different formulations of the IBS method will be presented and application of the IBS method (using both normal and truncated IRFs) is illustrated by a 1D example. Finally in order to show the validity of the IBS method in real problems, a real 3D example related to an offshore wind turbine has been analyzed by IBS.
Acknowledgement

The name on the cover is me; however I could never do this thesis without people who were with me and helped me all the time. With this opportunity I would like to say how thankful I am to them.

First of all, I would like to thank my supervisor Prof D.J. Rixen for all his help, guidance and supports during my study and for always having time for my endless questions.

Next, I would like to thank my dear friends in the Netherlands for making the past 2 years the greatest experience of my life. My special thanks go to my friend Ben Schneider for all his helps in writing this thesis.

Finally, I would like to thank my family members; my sister, for helping me every time I need help, my brother for always motivating me in my work and, my parents for everything.

Delft, The Netherlands,
November 2011,
Nazgol Haghighat
Contents

Abstract i
Acknowledgement iii
Nomenclature xi

1 Introduction 1
  1.1 Impulse Based Substructuring .......................... 1
  1.2 Thesis objectives .................................. 2
  1.3 Thesis outlines .................................. 3

2 Theory 5
  2.1 Introduction ........................................ 5
  2.2 Theory of Impulse Based Substructuring ................. 5
    2.2.1 Computing the IRF ................................ 5
    2.2.2 Assembling procedure ........................... 8
  2.3 Example ............................................ 9
    2.3.1 1D bar structure .............................. 9
    2.3.2 Generating the impulse response functions (IRFs) 10
  2.4 Evaluation of the time response ......................... 11
    2.4.1 Singular Value Decomposition (SVD) ............... 12
    2.4.2 Modal Assurance Criterion (MAC) ................ 12

3 Discretized formulation of IBS 15
  3.1 Introduction ........................................ 15
  3.2 Modeling the unit impact ............................ 15
    3.2.1 Initial velocity (IV) ............................ 15
    3.2.2 Applied force at the initial time step (IF) ...... 16
    3.2.3 Applied force at the second time step (SF) ...... 19
    3.2.4 Computed IRFs of the 1D bar system ............... 22
  3.3 Discretized assembly equation of IBS ................ 23
### CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3.1</td>
<td>Initial applied velocity (IBS&lt;sub&gt;IV&lt;/sub&gt;)</td>
<td>24</td>
</tr>
<tr>
<td>3.3.2</td>
<td>Initial applied force (IBS&lt;sub&gt;IF&lt;/sub&gt;)</td>
<td>24</td>
</tr>
<tr>
<td>3.3.3</td>
<td>Applied force at the second time step (IBS&lt;sub&gt;SF&lt;/sub&gt;)</td>
<td>25</td>
</tr>
<tr>
<td>3.4</td>
<td>Dynamic response of the bar example</td>
<td>25</td>
</tr>
<tr>
<td>3.5</td>
<td>Modeling space distributed loads</td>
<td>31</td>
</tr>
<tr>
<td>4</td>
<td>Reducing the computation costs of IBS</td>
<td>33</td>
</tr>
<tr>
<td>4.1</td>
<td>Introduction</td>
<td>33</td>
</tr>
<tr>
<td>4.2</td>
<td>Truncated IBS</td>
<td>33</td>
</tr>
<tr>
<td>4.2.1</td>
<td>Non-floating substructures</td>
<td>34</td>
</tr>
<tr>
<td>4.2.2</td>
<td>Floating substructures</td>
<td>37</td>
</tr>
<tr>
<td>4.2.3</td>
<td>Results of truncated IBS and discussion</td>
<td>40</td>
</tr>
<tr>
<td>4.3</td>
<td>Down-sampling</td>
<td>47</td>
</tr>
<tr>
<td>4.3.1</td>
<td>Discretization of the external applied force</td>
<td>47</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Rearranging impulse response matrices</td>
<td>47</td>
</tr>
<tr>
<td>4.3.3</td>
<td>Results of IBS with down-sampling and discussion</td>
<td>49</td>
</tr>
<tr>
<td>5</td>
<td>Applying IBS on a 3D structure</td>
<td>51</td>
</tr>
<tr>
<td>5.1</td>
<td>Introduction</td>
<td>51</td>
</tr>
<tr>
<td>5.2</td>
<td>Problem definition</td>
<td>51</td>
</tr>
<tr>
<td>5.3</td>
<td>Dynamic response</td>
<td>52</td>
</tr>
<tr>
<td>5.3.1</td>
<td>Numerical time integration of full system</td>
<td>52</td>
</tr>
<tr>
<td>5.3.2</td>
<td>IBS method</td>
<td>53</td>
</tr>
<tr>
<td>5.4</td>
<td>Results</td>
<td>55</td>
</tr>
<tr>
<td>6</td>
<td>Conclusion</td>
<td>67</td>
</tr>
<tr>
<td>6.1</td>
<td>Conclusion</td>
<td>67</td>
</tr>
<tr>
<td>6.2</td>
<td>Future work</td>
<td>69</td>
</tr>
</tbody>
</table>

### Bibliography

<table>
<thead>
<tr>
<th>Appendix</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Computing Pure rigid body response due to unit impact</td>
<td>73</td>
</tr>
<tr>
<td>B</td>
<td>Applying truncated IBS for different types of excitations</td>
<td>75</td>
</tr>
<tr>
<td>C</td>
<td>Matrix representations of the IRFs of the wind turbine example</td>
<td>79</td>
</tr>
</tbody>
</table>
List of Figures

1.1 Modeling a continuous force with superposition of impulses [8] .... 2

2.1 1D bar example, \( f \) is an external force applied onto the system and, \( \lambda \) is the coupling force at the interface) ............... 10

2.2 Essential places to apply the unit impulse ......................... 11

2.3 Active nodes in the assembly equation ....................... 11

3.1 Impact modeled by an applied initial force ....................... 17

3.2 Discretization of the external applied force for the impact modeled by applying a unit force \( a = 0 \) ....................... 17

3.3 Physical interpretation of equation (3.7) ......................... 18

3.4 Impact modeled by an applied force at the second time step .... 20

3.5 Discretization of the external applied force for the impact modeled by applying a unit force \( a = t_1 \) ....................... 20

3.6 Modeling the external force at \( t = 0 \) \( (f_0) \) with an impact obtained by applying a unit force at the second time step ............... 21

3.7 Discretization of the applied force as described in equation (3.10) .... 21

3.8 IRFs of the substructures of the 1D bar example obtained by 3 different impact models (zoomed on the right) ......................... 22

3.9 Dynamic response of the 1D bar example under a unit impact excitation, obtained by 3 different IBS formulations ....................... 26

3.10 Dynamic response of the 1D bar example under a unit step excitation, obtained by 3 different IBS formulations ....................... 27

3.11 Dynamic response of the 1D bar example under an excitation described by equation (3.20), obtained by 3 different IBS formulations .... 28

3.12 Modeling a space distributed load .............................. 31

3.13 IRFs of the free-free bar due to space distributed impulse in the step form(3.12c) ....................................................... 32

3.14 Displacement of the interface node under a space distributed excitation 32

4.1 Different types of IRFs ................................. 35
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.2</td>
<td>Applying rectangular window on the impulse response of a non-floating structure</td>
<td>36</td>
</tr>
<tr>
<td>4.3</td>
<td>Applying cosine window on the impulse response of a non-floating structure</td>
<td>37</td>
</tr>
<tr>
<td>4.4</td>
<td>Applying exponential window on the impulse response of a non-floating structure</td>
<td>38</td>
</tr>
<tr>
<td>4.5</td>
<td>Decomposition of the impulse response of a floating substructure into pure vibrational and rigid body response</td>
<td>38</td>
</tr>
<tr>
<td>4.6</td>
<td>Dynamic response of the 1D bar example (displacement at the interface node) obtained by the truncated IBS with rectangular window</td>
<td>41</td>
</tr>
<tr>
<td>4.7</td>
<td>Dynamic response of the 1D bar example (displacement at the interface node) obtained by the truncated IBS with cosine window</td>
<td>42</td>
</tr>
<tr>
<td>4.8</td>
<td>Dynamic response of the 1D bar example (displacement at the interface node) obtained by the truncated IBS exponential window</td>
<td>43</td>
</tr>
<tr>
<td>4.9</td>
<td>Effects of changing $dt$ in discretization of the external applied force ($2 \times dt$)</td>
<td>48</td>
</tr>
<tr>
<td>4.10</td>
<td>Comparison between the impulse responses of the clamped-free bar obtained by $dt$, $2dt$</td>
<td>48</td>
</tr>
<tr>
<td>4.11</td>
<td>Dynamic response of the 1D bar example (displacement at the interface node) obtained by IBS$_{Down-Sampled}$</td>
<td>50</td>
</tr>
<tr>
<td>5.1</td>
<td>Overall sketch of the offshore wind turbine example [18]</td>
<td>52</td>
</tr>
<tr>
<td>5.2</td>
<td>Assembling the jacket and the wind mill matrices</td>
<td>53</td>
</tr>
<tr>
<td>5.3</td>
<td>IRFs of the jacket structure (obtained by 2 impact models: initial applied force, initial applied velocity)</td>
<td>56</td>
</tr>
<tr>
<td>5.4</td>
<td>IRFs of the wind mill (obtained by 2 impact models: initial applied force, initial applied velocity)</td>
<td>57</td>
</tr>
<tr>
<td>5.5</td>
<td>IRFs of the jacket structure (the impact is modeled by applying a unit force at the initial time step)</td>
<td>58</td>
</tr>
<tr>
<td>5.6</td>
<td>IRFs of the wind mill (the impact is modeled by applying a unit force at the initial time step)</td>
<td>58</td>
</tr>
<tr>
<td>5.7</td>
<td>Dynamic response of the interface point under an excitation in the form of unit step on dof$_{i}$ obtained by IBS and Newmark time integration</td>
<td>59</td>
</tr>
<tr>
<td>5.8</td>
<td>Dynamic response of the interface point under an excitation in the form of unit step on dof$_{i}$ obtained by IBS and Newmark time integration</td>
<td>60</td>
</tr>
<tr>
<td>5.9</td>
<td>The pure rigid body response of the wind mill (the impact is modeled by applying a unit force at the initial time step)</td>
<td>62</td>
</tr>
<tr>
<td>5.10</td>
<td>The vibrational response of the wind mill computed by (5.2) and (5.9)(the impact is modeled by applying a unit force at the initial time step)</td>
<td>62</td>
</tr>
<tr>
<td>5.11</td>
<td>The vibrational response of the wind mill computed by (5.13) (the impact is modeled by applying a unit force at the initial time step)</td>
<td>63</td>
</tr>
</tbody>
</table>
5.12 Dynamic response of the interface point under an excitation in the form of unit step on dof$_1$ obtained by truncated IBS (cosine window function with $\alpha = 10^{-3}$) and Newmark time integration .......................... 64

5.13 Dynamic response of the interface point under an excitation in the form of unit step on dof$_1$ obtained by truncated IBS (cosine window function with $\alpha = 10^{-3}$) and Newmark time integration .......................... 65

A.1 Pure rigid body response under the unit impact IF ....................... 74

B.1 Dynamic response of the 1D bar example (displacement at the interface node) under a periodic force obtained by the truncated IBS ($\alpha = 10^{-2}$) ............. 76

B.2 Dynamic response of the 1D bar example (displacement at the interface node) under a unit impact obtained by the truncated IBS ($\alpha = 10^{-2}$) ............. 77

C.1 IRFs of the jacket structure (obtained by 2 impact models: initial applied force (blue), initial applied velocity (green)) ....................................................... 80

C.2 IRFs of the wind mill (obtained by 2 impact models: initial applied force (blue), initial applied velocity (green)) ....................................................... 81

C.3 IRFs of the jacket structure (the impact modeled by a unit initial applied force) ............................................................... 82

C.4 IRFs of the wind mill (the impact modeled by a unit initial applied force) ............................................................... 83

C.5 Rigid body impulse responses ($H^{\text{rig}}$) obtained by equation (5.9) ....... 84

C.6 Vibrational impulse response ($H^{\text{vib}}$) obtained by equations (5.2), (5.9) .... 85

C.7 Vibrational impulse response ($H^{\text{vib}}$) obtained by equation (5.13) ....... 86
Nomenclature

$[\star]_{i}$ component $i$ of an array

$\alpha$ relative amplitude for the truncation of the IRF

$\beta, \gamma$ parameters of the Newmark time-integration scheme

$\lambda$ Lagrangian multipliers on interface

$dt$ time-step size

$N^s$ number of substructures in the system

$t_c$ cutoff time for the truncation of the IRF

$\star^{(s)}$ pertaining to substructure $s$

$\star^{(n)}$ pertaining to time-step $n$

$u$ array of degrees of freedom

$B$ signed Boolean matrix

$C$ damping matrix of a linear(ized) system

$f$ array of external forces

$K$ stiffness matrix of a linear(ized) system

$M$ mass matrix of a linear(ized) system

$R$ matrix of rigid body modes of a floating substructure

FBS Frequency Based Substructuring

FRF Frequency Response Function

IBS Impulse Based Substructuring

IRF Impulse Response Function
Chapter 1

Introduction

1.1 Impulse Based Substructuring

Dynamic substructuring (DS) as a method to do dynamic analysis of complex structures is becoming ever important in the field of structural dynamics. The main idea of dynamic substructuring is decomposing a complex and large structure into simpler structures (named substructures or subsystems) and analyzing the structure componentwise [2]. The dynamic model of the total system will be obtained by assembling the dynamic model of the pre-analyzed substructures. It should be noticed that due to decomposing the main structure, unknown forces are generated at the interfaces. These forces must be in equilibrium at the interfaces and can be obtained by satisfying compatibility condition at the interfaces [2, 5].

One of the main reasons to apply dynamic substructuring instead of solving a structure as a whole is that analyzing complex and large systems is too time consuming. For instance, in case of solving a structure with large number \(n\) of degrees of freedom (dofs) numerically, \(n\) equations must be solved at each time step. Solving these equations provides the information of all the \(n\) dofs which some of them are not important in constructing the dynamic model of the main system. By applying dynamic substructuring it is possible to eliminate the subparts which have not an important effect on the assembled system. On the other hand it is also possible to focus on the subparts which are more important. The other advantage of dynamic substructuring is that all the subsystems do not need to be solved in the same way, just analytically or experimentally or numerically, and each subsystem can be analyzed by the method which is best for it. The subsystems can be analyzed by parallel systems which also reduce the computation time. Due to all the above explained reasons, it is significant to develop dynamic substructuring techniques.[2, 6]

As explained before, in all the dynamic substructuring techniques, first the substructures must be analyzed individually. This analysis can be performed in the fre-
CHAPTER 1. INTRODUCTION

In the frequency domain, the Frequency Response Functions (FRFs) of the components are the data which are used to perform dynamic analysis of the total system. One of the difficulties in applying substructuring in the frequency domain is in simulating the impact responses of the system. In order to do the impact analysis of the system, a large frequency band of the frequency responses must be considered which can make the whole Frequency Based Substructuring (FBS) procedure expensive and badly suited. Hence, another method must be developed for the cases where FBS is incapable to provide a proper solution. One suggested method is performing the substructuring procedure in the time domain and using the impulse responses instead of the frequency responses. Since this method works with the Impulse Response Function (IRF), it is called Impulse Based Substructuring (IBS) [1].

The main idea of the IBS method is coming from the mathematical concept of applying convolution product to obtain dynamic response of the linear systems. In simple words, for a linear time invariant system [8] the external applied force can be modeled by summation of a finite number of impacts as shown in figure 1.1. Therefore, dynamic response of the linear system under any types of applied forces can be computed by knowing dynamic response of the system under the unit impact (IRFs).

![Modeling a continuous force with superposition of impulses](image)

Figure 1.1: Modeling a continuous force with superposition of impulses [8]

1.2 Thesis objectives

This thesis is constructed based on combining 2 concepts of dynamic substructuring and application of the convolution product in linear systems in order to introduce a substructuring method which can be applied in the time domain. The IBS method first was published in [1] and it is developed in more details in this thesis. The following goals are achieved during this study:

- Presenting theory of the IBS method;
- Improving the IBS method in order to reduce the computation costs;
- Implementing the IBS method on real problems;
1.3 Thesis outlines

The defined objectives of this thesis are provided in the 6 chapters as follows:

Chapter 2 starts with explaining the basic steps of the IBS method. Since the main purpose of this assignment is developing the IBS method and comparing it to the standard time integration methods, in order to have a better visualization of IBS and its applications, the formulations which will be declared in the next chapters are illustrated by a 1D example. This example will be explained later. At the end, a method for comparing the results of IBS and numerical time integration is introduced.

The main focus of chapter 3 is on deriving the discretized formulations of Duhamel integral and the IBS method. These discretized relations are defined for 3 different models of unit impact. Afterwards the dynamic response of the 1D example is computed to different types of applied forces using IBS and, the results are compared to the results of the direct Newmark time integration. Finally for the cases with space distributed loads, a special form of IBS is introduced.

The attempt made in chapter 4 is to develop some estimation in order to decrease the computation costs of IBS. The IBS method has 2 main steps. First, computing IRFs using Newmark time integration scheme. Second, assembling substructures and obtaining dynamic response of the total structure by applying the convolution product between the IRFs and applied forces. Therefore, reducing the computation costs can be achieved by applying some changes in these 2 parts. In chapter 4, 2 methods are suggested to decrease the implementation costs of IBS. The first one is truncating the IRFs and reducing the computation time in both 2 steps, this method is called truncated IBS. The second method is changing the size of the time step in the part of applying convolution product, and analyzing the problem considering less numbers of samples which is called down-sampling. Later these 2 approaches are examined on the 1D bar example.

In chapter 5, the IBS method has been applied to analyze a real 3D example. The model is provided by Siemens Wind Power which describes an offshore wind turbine. In order to compute the dynamic response of the system first the normal IBS has been applied. Afterwards, the truncated IBS is examined on the model. The obtained results are compared with the Newmark time integration of the full system. Chapter 5 also provides the relations of the truncated IBS for 3D cases.

Finally, chapter 6 provides a general conclusion of the results of previous chapters.
Chapter 2

Theory

2.1 Introduction

In this chapter the basic steps of the IBS are explained. Since the main purpose of this assignment is developing the IBS method and comparing it to the standard time integration methods, in order to have a better visualization of IBS and its applications, the formulations which will be declared in the next chapters are illustrated by a 1D example. This example will be explained later. At the end, a method for comparing the results of IBS and numerical time integration is introduced.

2.2 Theory of Impulse Based Substructuring

As explained in chapter 1 the idea of IBS method is coming from combining the application of Duhamel integral with substructuring principles. IBS method can be explained in 2 main steps. First, analyzing each subsystem individually; this analyzing includes computing the impulse response functions. The next step is assembling the subsystems. The assembling procedure obtains the dynamic response of the whole system by applying convolution product between the IRFs and the applied forces [9]. In this chapter, these 2 steps are explained in more details.

2.2.1 Computing the IRF

The impulse response functions are dynamic response of the subsystems under a unit impact. The impacts must be applied at the positions where it is expected to have forces in the total system. Before explaining different methods to compute the impulse responses, first a short description about impulse is given.
CHAPTER 2. THEORY

Unit Impact

In Mechanics, the impulse or shock can be interpreted as a step in velocity which leads to a short vary high acceleration. In order to translate this physical model into mathematics the impulse function is introduced. The impulse function is the physical meaning of mathematical concept of Dirac delta function. Dirac delta function is defined by equation (2.1).

\[ \delta(t) = \begin{cases} +\infty & t = 0 \\ 0 & t \neq 0 \end{cases} \quad (2.1) \]

\(\delta\) is a function which depends on a real parameter and is zero for all values of that parameter except for the case when the parameter is zero. Its integral over the parameter is also unit (2.2) [8].

\[ \int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (2.2) \]

Dirac delta can be used to model a unit impulse by considering time as the parameter which is changing from 0 to \(\infty\) (instead of \(-\infty\) to \(\infty\), since time is a non-negative parameter). The described model is the ideal impact function, in reality this function is modeled by applying a sudden force in a very small period of time. It can be achieved by hammer impact or numerical models.

Based on provided definitions, there are 2 methods to deliver impulse responses. One way is measuring the impulse responses due to hammer impact test in the lab. The other method is applying the numerical time integration to compute the time response of motion equation of a system under a unit impact. In the first method, the hammer impact cannot make the ideal impulse because the duration \((dt)\) is not infinity small. On the other hand, measuring the impulse responses in a lab may not be accurate enough due to the noise in the measurement instruments. These 2 main problems can make errors in the obtained impulse response functions [3]. As a consequence, this method has not been used in the future work of this report and it is focused on the results of numerical time integration method.

Numerical time integration

Numerical time integration is most of the time the only method which can be applied to obtain the dynamic response of the structures. Numerical time integration schemes are generally developed by the concept of finite time difference [6, 11]. Different methods use different formulations in time discretization. In order to integrate the linearized motion equation, having initial conditions is necessary. Since dynamic equation (2.3) is a second order differential equation therefore, 2 initial conditions \((\dot{u}_0, u_0)\) need to be specified [6].

\[ M\ddot{u} + C\dot{u} + Ku = f \quad (2.3) \]
2.2. THEORY OF IMPULSE BASED SUBSTRUCTURING

In equation (2.3) \(M, K, C\) are mass, stiffness and damping matrices which are obtained by discretizing the structure in space domain by using finite element methods [6, 11] and \(f_n\) is the discretized applied force. By discretizing the motion equation considering \(t_{n+1} = t_n + dt\) and having the initial displacement \((u_0)\) and initial velocity \((\dot{u}_0)\)

\[
M \ddot{u}_n + C \dot{u}_n + K u_n = f_n
\]

(2.4)

\[
\ddot{u}_0 = M^{-1} (f_0 - K u_0 - C \dot{u}_0)
\]

(2.5)

Since the velocity and acceleration at every time step are defined by (2.6),(2.7), the results for the next time step \((t_{n+1})\) can be calculated [6].

\[
\ddot{u}_n = \lim_{dt \to 0} \frac{\ddot{u}_n - \ddot{u}(t_n \pm dt)}{\mp dt}
\]

(2.6)

\[
\dot{u}_n = \lim_{dt \to 0} \frac{u_n - u(t_n \pm dt)}{\mp dt}
\]

(2.7)

Indeed, time integration techniques include estimating \(\ddot{u}_n, \dot{u}_n\) which make simple time marching algorithms and leads to the response of the system at every time step. Based on the different methods which approximate \(\ddot{u}_n, \dot{u}_n\), different time integration schemes are introduced. The method which is used in this assignment is Newmark time integration. The Newmark formulation is an efficient and single-step integration which was published in 1959 by N.M. Newmark. Newmark suggested the formulas (2.8) to obtain \(\dot{u}_{n+1}, u_{n+1}\) [10, 6].

\[
\begin{align*}
\dot{u}_{n+1} &= \dot{u}_n + (1 - \gamma)dt \ddot{u}_n + \gamma dt \dot{u}_{n+1} \\
u_{n+1} &= u_n + dt \dot{u}_n + dt^2 (\frac{1}{2} - \beta) \ddot{u}_n + dt^2 \beta \dot{u}_{n+1}
\end{align*}
\]

(2.8)

\(\beta, \gamma\) are factors which can make differences in the time integration scheme properties. By substituting (2.8) in (2.4), the dynamic equation (2.4) at \(t_{n+1}\) will be

\[
[M + \gamma dt C + \beta dt^2 K] \ddot{u}_{n+1} = f_{n+1} - C[\ddot{u}_n + (1 - \gamma)dt \ddot{u}_n] - K[u_n + dt \dot{u}_n + (\frac{1}{2} - \beta)dt^2 \ddot{u}_n]
\]

(2.9)

As can be realized from equation (2.9), computing \(\ddot{u}_{n+1}\) needs factorizing \(S\) instead of \(M\). Although, by decreasing the size of the time step, it converges to the mass matrix. Based on the properties of \(M, K, C, S\) is a symmetric and positive definite matrix and can be interpreted as the inertia matrix [6].

In order to simplify this Newmark algorithm, in equation (2.8), \(\dot{u}_{n+1}, u_{n+1}\) first can be predicted only by the information at time \(t_n\) (relations (2.10) [6])

\[
\begin{align*}
\dot{u}_{n+1} &= \dot{u}_n + (1 - \gamma)dt \ddot{u}_n \\
\ddot{u}_{n+1} &= u_n + dt \dot{u}_n + dt^2 (\frac{1}{2} - \beta) \ddot{u}_n
\end{align*}
\]

(2.10)
Therefore, equation (2.9) can be written in the form (2.11)

\[ S \ddot{u}_{n+1} = f_{n+1} - K \ddot{u}_{n+1} - C \dot{u}_{n+1} \]  

(2.11)

Finally relation (2.10) should be corrected

\[
\begin{align*}
    u_{n+1} &= \ddot{u}_{n+1} + dt^2 \beta \ddot{u}_{n+1} \\
    \dot{u}_{n+1} &= \dot{\ddot{u}}_{n+1} + dt \gamma \dddot{u}_{n+1}
\end{align*}
\]  

(2.12)

The explained time integration algorithm is used in next chapter to compute the impulse responses. For the further calculations of this assignment \( \gamma, \beta \) are chosen to be equal to \( \frac{1}{2}, \frac{1}{4} \) respectively which provides an implicit unconditionally stable integration scheme [6]. This type of Newmark method (2.13) which is called obtained by substituting \( \gamma, \beta \) in relation (2.8) is equivalent to the trapezoidal integration rule[6].

\[
\begin{align*}
    u_{n+1} - u_n &= \frac{dt}{2} (\ddot{u}_{n+1} + \ddot{u}_n) \\
    \dot{u}_{n+1} - \dot{u}_n &= \frac{dt}{2} (\dddot{u}_{n+1} + \dddot{u}_n)
\end{align*}
\]  

(2.13)

As mentioned before Newmark time integration scheme with average constant acceleration is stable for all the time steps \( (dt) \). Therefore, the size of the time step is chosen based on the highest frequency of the external applied force \( (f(\omega t)) \). In this case, the time step is defined to be at least 4 times bigger than the minimum period \( (\tau_{min}) \) of the external force. It can be explained by equation (2.14).

\[
dt \leq 4 \tau_{min} \quad \Rightarrow \quad dt \leq \frac{2\pi}{\omega_{max}}
\]  

(2.14)

**Generating the impulse response matrix**

Finally, the computed or measured impulse responses should be gathered in a matrix which is called \( H \) matrix. The Impulse responses are obtained due to impulses which are applied on all the interfaces nodes and all the nodes where are expected to have external applied force, separately.

These responses have to be saved at every time step \( (n) \) for all the output nodes due to all the excitations. In other words, \( [H_n]_{ij} \) is the response of the node \( j \) at time \( t_n = n \times dt \) under the unit impulse on node \( i \) at \( t = 0 \) [1].

**2.2.2 Assembling procedure**

After obtaining the impulse response functions the response of each substructure will be computed using the convolution product. In other words, the time response of each substructure is the result of applying convolution product between the applied forces
and the impulse response functions \[9\]. Therefore the applied forces should be defined for each substructure.

It should be noticed that dividing the structures into subsystems will generate interface forces. These forces should be taken into account as a part of the applied forces. Therefore, the force function includes both external forces and internal interface forces. These interface forces are unknowns which couple the interface dofs and will be found during the computations in a way to enforce compatibility at the interface nodes. Therefore, to compute the interface forces first the compatibility condition at the interfaces must be defined \[2, 1\].

In general form the compatibility condition at the interface states that the dofs on each side of the interface must be equal. Since in the substructuring problems the meshes and nodes at the interfaces are matching, Boolean matrices \((B)\) are used to localize the interface dofs. As a consequence the compatibility condition at the interface is written as (2.15) \[1\].

\[
\sum_{s=1}^{N_s} B^{(s)} u^{(s)} = 0 
\] (2.15)

Therefore, the final assembly formulation for the IBS method will be (2.16) \[1\]

\[
\begin{align*}
\mathbf{u}^{(s)}(t) &= \int_{0}^{t} \mathbf{H}^{(s)}(t - \tau) \left( \mathbf{f}^{(s)}(\tau) + \mathbf{B}^{(s)T} \mathbf{\lambda}(\tau) \right) d\tau \\
\sum_{s=1}^{N_s} \mathbf{B}^{(s)} \mathbf{u}^{(s)}(t) &= 0
\end{align*}
\] (2.16)

### 2.3 Example

The next 2 chapters of this report are devoted to develop the discretized formulations and different approximations of IBS method. In order to make all the definitions and equations more apprehensible the solving procedures have been explained by a 1D example \[1\]. Dynamic response of the total system is computed using both IBS method and Newmark time integration scheme. At the end accuracy and reliability of IBS method are discussed.

#### 2.3.1 1D bar structure

The system which is used in this assignment consists of 1 bar which is excited by a load at its end (shown in figure 2.1a). The bar is made of steel and has a uniform cross-section (radius \(r\)), table 2.1 includes the physical properties of the system.
2.3.2 Generating the impulse response functions (IRFs)

As explained before, in order to obtain the IRFs, the unit impulse must be applied at the positions where are expected to have external applied forces or the reaction forces (interface points). Since for this bar example, the external force is applied at the end of the free-free bar, therefore, in the stage of computing the IRFs, the impulse response matrices should be calculated for 3 applied impulses as shown in figure 2.2. Figures 2.2a, 2.2b are related to the reaction force at the interface and, figure 2.2c is due to the external applied force at the end of the bar.

Figure 2.3 shows the main advantage of IBS method, which is decreasing the number of active dofs. After computing IRFs, the interface dofs must be taken into account in
the assembly equation (2.16) and the internal nodes can be ignored.

![Diagram](image1)

(a) Unit impulse at the interface point (IRFs of $s^{(1)}$)

![Diagram](image2)

(b) Unit impulse at the interface point (IRFs of $s^{(2)}$)

![Diagram](image3)

(c) Unit impulse at the end point (IRFs of $s^{(2)}$)

Figure 2.2: Essential places to apply the unit impulse

![Diagram](image4)

(a) Active node of $s^{(1)}$ due to $\lambda$ in the assembly equation

![Diagram](image5)

(b) Active nodes of $s^{(2)}$ due to $\lambda$ in the assembly equation

![Diagram](image6)

(c) Active nodes of $s^{(2)}$ due to $f$ in the assembly equation

Figure 2.3: Active nodes in the assembly equation

Finally the impulse response functions will be computed by applying Newmark time integration. In this example the size of the time step is chosen based on $dt_{crit}$ which is the critical time step and is equal to $dt_{crit} = \frac{2}{\omega}$ where $\omega$ is the highest frequency of the system [1]. In this case the size of the time step has been chosen by $dt = 3 \times dt_{crit}$.

### 2.4 Evaluation of the time response

Evaluating the IBS results can be done by comparing the obtained dynamic response of the total system (using IBS) with the results of the numerical time integration. The final response are signals which are changing with time. These time responses are stored in a matrix $(n \times n_t)$ where $n$ is the number of dofs and $n_t$ is the number of time steps. In general the response matrices are rectangular matrices which include information related to time history and dofs. To be able to compare these response matrices, each matrix should be decomposed into 2 matrices carrying data for time
history and dofs individually. These separate matrices are results of the Singular Value Decomposition of the response matrix.

### 2.4.1 Singular Value Decomposition (SVD)

Singular Value Decomposition (SVD) tries to decompose a matrix into other matrices which still preserve all the information of the original matrix. In other words, SVD can be explained as extension of the concept of finding eigenvalues and eigenvectors of square matrices to rectangular matrices \[12\]. Based on this definition, a matrix \( A_{n \times m} \) can be written in form (2.17) \[13\].

\[
A = U \Sigma V^T \tag{2.17}
\]

The first step to obtain this decomposition of a rectangular matrix \( A \) of dimension \( n \times m \) is generating 2 square matrices \( AA^T \) and \( A^T A \). The first one is the outer product and leads to a matrix which is spanned by the row space of \( A \). The second one is the inner product and results in a matrix which is spanned by the column space of \( A \). \( U, V \) and \( \Sigma \) can be computed using these 2 square matrices \[12\].

\( U \) is an \( n \times n \) orthogonal matrix \( (UU^T = I) \) which is called the left singular vectors and includes the eigenvectors of \( AA^T \). On the other hand, \( V \) is an \( m \times m \) orthogonal matrix \( (VV^T = I) \) which is called right singular vectors and is made of eigenvectors \( A^T A \). \( \Sigma \) is a \( n \times m \) matrix whose off-diagonal elements are all 0 and whose diagonal entries are singular values. The singular values are nonzero square roots of eigenvalues of \( AA^T \) and \( A^T A \). They are presented in \( \Sigma \) matrix satisfying (2.18) \[13\].

\[
\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq ... \geq 0 \tag{2.18}
\]

The described procedure (SVD) can be applied on the dynamic response of the system to decompose them into \( U \) as a matrix which is carrying the spatial modes column-wisely, and \( V \) as the associate normalized time history. In this case, the diagonal components of \( \Sigma \) matrix \( (\sigma_i) \) can be interpreted as the energy of the related spatial mode \( (i) \). In this assignment the SVD function of MATLAB is used to compute the \( U, V \) and \( \Sigma \) matrices.

### 2.4.2 Modal Assurance Criterion (MAC)

In order to evaluate the results of the IBS method and compare them with the results of the Newmark time integration of the full system, the spatial modes of these 2 time responses should be extracted from the total response matrices (\( U \) matrices which are obtained by applying SVD). In fact the comparison is made between these 2 spatial modes of the dynamic responses. One possible method to compare these 2 matrix representation of the spatial modes is computing the function of the modal assurance
criterion (MAC) as described by equation (2.19). In fact the MAC function can provide a measure of consistency between 2 spatial modes. In other words, degree of linearity between the spatial modes which are obtained by the IBS and Newmark can be checked by the MAC function [4].

\[
MAC = \frac{[U_{IBS}^T U_{NI}]^2}{[U_{IBS}^T U_{IBS}][U_{NI}^T U_{NI}]} \quad (2.19)
\]

The result of relation (2.19), is an \( n \times n \) matrix where \( n \) is the number of dofs of the structure. If \([MAC]_{ii}\) has a value close to 1, it can be concluded that the spatial modes which are presented in column \( i \) of \( U_{IBS} \) and \( U_{NI} \) are consistent [4]. In this report the components of the main diagonal of the MAC matrix are called MAC numbers. It should be noticed that, it does not necessary to check the MAC numbers for all the spatial modes. In other words, it is sufficient to check the MAC numbers for the modes which have the highest energy. As mentioned before, the diagonal components of matrix \( \Sigma \) (obtained by SVD) is representing the energy of each mode and summation of the diagonal components is the total energy of the system. As a consequence, by using this definition a MAC number related to mode \( i \) can be considered as a criteria to check the degree of linearity between the computed spatial modes by IBS and Newmark integration if it satisfies condition (2.20).

\[
\sigma_i \geq 0.01 \times \sum_{i=1}^{n} \sigma_i \quad (2.20)
\]
CHAPTER 2. THEORY
Chapter 3

Discretized formulation of IBS

3.1 Introduction

In this chapter the discretized formulations of Duhamel integral and IBS method are derived. These discretized relations are defined for 3 different models of unit impact. Afterwards the dynamic response of the explained 1D example is computed to different types of applied forces using IBS and, the results are compared to the results of the direct Newmark time integration. Finally for the cases with space distributed loads, a special form of IBS is introduced.

3.2 Modeling the unit impact

As explained before, the unit impulse is generally expressed by Dirac delta. In the numerical simulation, the Delta function can be modeled by defining specified initial conditions [1]. In this section 3 different types of initial conditions are used to provide the unit impact.

3.2.1 Initial velocity (IV)

Based on the definition of Dirac Delta, the impact can be modeled with a jump in velocity. The velocity jump at time $t = 0$ due to a unit impulse can be computed by integrating the momentum equation in an infinitesimal interval $[0^{-}, 0^{+}]$ [1]

$$M(\dot{u}_{0}^{+} - \dot{u}_{0}^{-}) = \int_{0^{-}}^{0^{+}} f(t) dt \tag{3.1}$$

Since before $t = 0$ the system is at rest ($\dot{u}_{0}^{-} = 0$), the jump in velocity at the initial time step due to a unit impulse on dof j will be [1]

$$Mu_{0} = 1, \tag{3.2}$$

15
Therefore, the IRFs can be obtained by computing the time response (applying Newmark time integration) of the structure with initial conditions (3.3) [1]

\[ \begin{align*}
    u_0 &= 0 \\
    \dot{u}_0 &= M^{-1}1_j \\
    \ddot{u}_0 &= M^{-1}(-Cu_0)
\end{align*} \] (3.3)

Modeling the impulse in this form makes a reliable approximation of the Dirac delta. The results of Newmark time integration under the initial conditions (3.3) can be directly considered as the impulse response functions. The smaller time step, the better impulse response would be made.

In this case to derive the discretized formulation for the Duhamel integral, at the first time step half of the applied force should be considered. The reason of this discretization is behind the definition of impact. As described before, impact is a jump from a zero value to infinity and going back to zero in a short time step \((dt)\) therefore, the applied force must be discretized by this definition of impact. For instance, the value of the applied force at \(t_i\) \((f_i)\), will be modeled by an impact which is starts at \(t_i - \frac{dt}{2}\) from zero and go to \(f_i\) at \(t_i\) and will be back to zero at \(t_i + \frac{dt}{2}\). However, in modeling the applied force at the initial time step, the force should be modeled with half of an impact which is equal to \(f_0\) at \(t = 0\) and go to zero at \(t = \frac{dt}{2}\) (This discretization of \(f_0\) is also explained in section 3.2.3). Hence, just half of the \(f_0\) must be considered in computing the discretized form of Duhamel integral. Equation (3.4) explains how to compute dof \(u\) at \(t_n = n \, dt\) using discretized form of Duhamel integral.

\[ u_n = H_n f_0 \frac{dt}{2} + \sum_{i=1}^{n-1} H_{n-i} f_i dt \] (3.4)

### 3.2.2 Applied force at the initial time step (IF)

The impact can also be modeled by applying a unit force (or moment) at the initial time step. In this case the impact can be interpreted with a jump in the initial acceleration and, the initial conditions are described in the form (3.5) [1].

\[ \begin{align*}
    u_0 &= 0 \\
    \dot{u}_0 &= 0 \\
    f_0 &= 1_j \quad \Rightarrow \quad \ddot{u}_0 = M^{-1}1_j
\end{align*} \] (3.5)

Applying Newmark time integration to a system with initial conditions (3.5) develops the IRFs of that structure. Initial conditions (3.5) in the Newmark time integration scheme can be interpreted as a force which is unit at time \(t = 0\) and decreasing linearly to 0 at time \(t = dt\). In this case the force function makes a triangle as shown in figure
3.2. MODELING THE UNIT IMPACT

3.1. The area of the generated triangle is equal to $\frac{dt}{2}$. Since this triangle is considered as the unit impact, to obtain the IRFs for a unit impulse, the results of the time integration should be divided by $\frac{dt}{2}$.

![Figure 3.1: Impact modeled by an applied initial force](image)

As mentioned before, the total applied force can be discretized into summation of infinite number of impulses (figure 3.2a). In the case of modeling the impulse by applying a unit force at the initial time step, discretization of the total applied force would be slightly different from what is shown in figure 3.2a. Discretization of the total applied force using the triangle (3.1) as the unit impulse means the area under the curve of the applied force should be swept by triangle (3.1). This discretization, in the time integration scheme, approximates the applied force by a piece-wise linear force between time steps based on the trapezoidal rule. In other words, during a time step from $t_n$ to $t_{n+1}$ the applied force is coming one half from $f_n$ and the other half from $f_{n+1}$, as shown in figure 3.2b. The discretized form of convolution product based

![Figure 3.2: Discretization of the external applied force for the impact modeled by applying a unit force at $t = 0$](image)
on this definition of impulse will be (3.6)

\[ u_n = \sum_{i=0}^{n-1} H_{n-i} \left( \frac{f_i}{2} + \frac{f_{i+1}}{2} \right) dt \]  

(3.6)

Formulation (3.6) is the superposition of effects of \( n \) numbers of impulses. These impulses will generate the total applied force. Since the structure is linear, if the applied force is modeled by summation of impulses in the form of triangle (3.1), equation (3.6) leads to the exact solution of Newmark time integration (with average constant acceleration) of dynamic equation, under the total applied force.

Figure 3.2b and relation (3.6) express that at the end of each time step \( (i) \), the effects of 2 impulses should be added to the system. One is due to the half of the applied force at time \( t_i \) (a triangle which is 1 at \( t_i \) and goes to 0 at \( t_{i+1} \)). The other one is due to the half of the applied force at time \( t_i + 1 \) (triangle which is 0 at \( t_i \) and goes to 1 at \( t_{i+1} \)).

![Triangle Graphic](image-url)

Figure 3.3: Physical interpretation of equation (3.7)

As can be observed from figure 3.2b, 2 different types of triangles are used to discretize the applied force. These 2 triangles can be interpreted as different initial conditions. The Impact modeled by figure 3.1 is described by initial condition (3.5). The second model is equal to a force which increases linearly to 1 between \( t_0 \) and \( t_1 \) and suddenly goes back to 0 at \( t_1 \) (figure 3.3). The dynamic response of this type of applied force can be obtained by deducting the time response of the structure due to a unit impulse (initial condition (3.5)) with one step delay (\( u[^2] \) in equation (3.7)), from the computed results due to a unit applied force exactly at the second time step (\( u[^1] \))
3.2. MODELING THE UNIT IMPACT

in equation (3.7)).

\[
\begin{align*}
\mathbf{u}^{[1]}_0 &= 0 \\
\dot{\mathbf{u}}^{[1]}_0 &= 0 \\
\ddot{\mathbf{u}}^{[1]}_0 &= 0 \\
\mathbf{f}_1 &= 1_j \\
\mathbf{f}_{n\neq1} &= 0
\end{align*}
\]

(3.7)

\[
\begin{align*}
\mathbf{u}^{[2]}_0 &= 0 \\
\dot{\mathbf{u}}^{[2]}_0 &= 0 \\
\ddot{\mathbf{u}}^{[2]}_0 &= 0 \\
\mathbf{f}_1 &= 1_j \Rightarrow \ddot{\mathbf{u}}^{[2]}_1 &= M^{-1}1_j
\end{align*}
\]

Although figure 3.3, compare to figure 3.1, describes different types of applied forces and initial conditions (3.7) however, the dynamic response of the system under these 2 different initial conditions are equal. In other words, these 2 different impact models produce the same impulse response functions. Therefore it is acceptable to use figure 3.3 instead of figure 3.1 to develop the discretized model of the applied force.

3.2.3 Applied force at the second time step (SF)

The 2 previous methods make acceptable models of impact but, computing the dynamic response of the system due to a force or velocity which is applied at the first time step needs factorization of the mass matrix. Since Newmark method factorizes \( S \) matrix instead of \( M \), factorization of \( M \) matrix which is needed specifically to compute the results at the first time step, increases the computation costs. The factorization of the Mass matrix can be avoided by remodeling the impact force in a different way. In this model, the IRFs are obtained by computing the dynamic response of the system due to a unit applied force at the second time step. In this case the initial conditions can be written as (3.8) [1].

\[
\begin{align*}
\mathbf{u}_0 &= 0 \\
\dot{\mathbf{u}}_0 &= 0 \\
\ddot{\mathbf{u}}_0 &= 0 \\
\mathbf{f}_1 &= 1_j \\
\mathbf{f}_{n\neq1} &= 0
\end{align*}
\]

(3.8)

This model can be interpreted as a force increasing linearly to 1 between \( t_0 \) and \( t_1 \), then decreasing to 0 between \( t_1 \) and \( t_2 \). For this method, as shown in figure 3.4, the unit impulse is represented by a triangle with area \( dt \), therefore, the unit impulse response is obtained by dividing the computed time response by \( dt \).

Since the time response of the system is known due to a force in the form of triangle 3.4, the total applied force must be discretized by this triangle. This discretization is
CHAPTER 3. DISCRETIZED FORMULATION OF IBS

Figure 3.4: Impact modeled by an applied force at the second time step

shown in figure 3.5b. Equation (3.9) expresses the discretized formulation of Duhamel integral.

\[ u_m = \sum_{i=0}^{n-1} H_{n-i} f_i dt \]  

(3.9)

In other words, formulation (3.9) describes the application of superposition principle on time response of a linear system (IRFs) which has been computed before. Therefore, results of modeling the impact in the form of figure 3.4 and by using relation (3.9) would be completely equal to the results of Newmark time integration. Since the computed impulse responses due to a unit force at the second time step, do not supply information at the initial time step, therefore equation (3.9) will lead to the results of Newmark for the cases with \( f_0 \) is zero (\( i \) starts from 1 in equation (3.9)).

In order to apply this model for the problems with non-zero initial applied force, \( f_0 \) must be modeled by the impact in the form of figure 3.4. Discretization of \( f \) at
3.2. MODELING THE UNIT IMPACT

$t = 0$ is shown in figure 3.6. Since just half of the impact at $t = 0$ is in the acceptable time($t > 0$), therefore half of the $f_0$ will participate in the discretized formulation of Duhamel integral (similar to the case with IV impact).

Figure 3.6: Modeling the external force at $t = 0$ ($f_0$) with an impact obtained by applying a unit force at the second time step

Therefore the finalized discretized formulation of Duhamel integral for the impact which is modeled by a unit force at the second time step will be (3.10).

\[ u_n = H_nf_0 \frac{dt}{2} + \sum_{i=1}^{n-1} H_{n-i}f_idt \]  \hspace{1cm} (3.10)

It should be noticed that, the result of equation (3.10) is not exactly equal to Newmark time integration. This discretization is illustrated in figure 3.7. In fact, equation (3.9) is a special form of equation (3.10) for the cases with $f_0 = 0$.

Figure 3.7: Discretization of the applied force as described in equation (3.10)
3.2.4 Computed IRFs of the 1D bar system

The impulse response functions computed by all the 3 explained methods are shown in figure 3.8. As can be observed, 3 different impact models (IV, IF and SF) are producing similar IRFs.

Figure 3.8: IRFs of the substructures of the 1D bar example obtained by 3 different impact models (zoomed on the right)
3.3 Discretized assembly equation of IBS

As mentioned in the previous section, the IRFs are obtained using numerical time integration, therefore, similar to Duhamel integral, the IBS assembly equation (2.16) needs to be discretized. Since in order to compute the time response of each substructure, all the external applied forces and interface forces must be taken into account, the discretized form of the assembly equation must be written as equation (3.11). Equation (3.11) is discretization of the IBS in the general form, therefore it should be noticed that based on the chosen impact model, the IBS formulation should be adapted to the impact model.

\[
\begin{aligned}
\mathbf{u}^{(s)}_n &= \sum_{i=0}^{n-1} \mathbf{H}^{(s)}_{n-i} (\mathbf{f}^{(s)}_i dt + \mathbf{B}^{(s)T} \lambda_i) \\
\sum_{s=1}^{N^s} \mathbf{B}^{(s)} \mathbf{u}^{(s)} &= 0
\end{aligned}
\]  

(3.11)

In equation (3.11), \(\lambda\)s are the Lagrange multipliers related to the compatibility conditions. Since \(\lambda\)s are computed at every time step, they do not need to be discretized. In other words, \(\lambda\)s in the discretized equation can be interpreted as the interface reaction impulses. Therefore, the external applied forces are the only components which need to be written in the discretized form.

In assembly equation (3.11) in order to obtain the response at time \(t_n\), all the external forces and interface impulses from \(t = 0\) up to \(t = n - 1\) must be taken into account. As can be realized from (3.11), \(\lambda_{n-1}\) is computed by the compatibility condition at time \(t_n\). Therefore an easy method to solve the dual assembly equation is considering \(\lambda_{n-1} = 0\) and introducing the predicted displacement at \(t_n\) (equation (3.12)) using all the known external applied forces and interface impulses [1].

\[
\tilde{\mathbf{u}}^{(s)}_n = \sum_{i=0}^{n-1} \mathbf{H}^{(s)}_{n-i} \left( \mathbf{f}^{(s)}_i dt + \mathbf{B}^{(s)T} \lambda_i \right) + \mathbf{H}^{(s)}_1 \mathbf{f}^{(s)}_{n-1} dt
\]  

(3.12)

By substituting (3.12) in (3.11), the assembly equation will be

\[
\begin{aligned}
\mathbf{u}^{(s)}_n &= \tilde{\mathbf{u}}^{(s)}_n + \mathbf{H}^{(s)}_1 \mathbf{B}^{(s)T} \lambda_{n-1} \\
\sum_{s=1}^{N^s} \mathbf{B}^{(s)} \mathbf{u}^{(s)} &= 0
\end{aligned}
\]  

(3.13)

Finally, the interface impulse \(\lambda_{n-1}\) can be computed by solving the dual interface condition

\[
\left( \sum_{s=1}^{N^s} \mathbf{B}^{(s)} \mathbf{H}^{(s)}_1 \mathbf{B}^{(s)T} \right) \lambda_{n-1} = - \sum_{s=1}^{N^s} \mathbf{B}^{(s)} \tilde{\mathbf{u}}^{(s)}_n
\]  

(3.14)
CHAPTER 3. DISCRETIZED FORMULATION OF IBS

The explained stepping algorithm is the proposed way to implement the IBS method. In the previous section, based on the different definitions of the unit impact, 3 formulations were declared to discretize Duhamel integral. These relations will be used in this section to rewrite (3.11) and obtain the dynamic response of each substructure.

3.3.1 Initial applied velocity (IBS\textsubscript{IV})

For the case with IRF computed under the initial condition (3.3), which can be interpreted as a jump in the initial velocity, the assembly equation for the IBS method should be written in the form (3.15). As explained before, the external applied forces are the only terms which need to be written in the discretized form.

\[
\begin{aligned}
\mathbf{u}_n^{(s)} &= H_n^{(s)} \left( \int_0^{\Delta t} \mathbf{f}(\mathbf{s}, t) \, dt + B^{(s)^T} \mathbf{\lambda}_0 \right) + \sum_{i=1}^{n-1} H_{n-i}^{(s)} \left( \int_0^{\Delta t} \mathbf{f}(\mathbf{s}, t) \, dt + B^{(s)^T} \mathbf{\lambda}_i \right) \\
\sum_{s=1}^{N_s} B^{(s)^T} \mathbf{u}_n^{(s)} &= 0
\end{aligned}
\tag{3.15}
\]

This formulation can make an acceptable approximation of the result of the dynamic response of the full system under the total applied force. Since these two approaches, IBS with IV impact and Numerical integration, use different initial conditions, the results of these two methods are not expected to be equal. For the sake of simplicity this formulation of IBS, due to having IRFs of IV impact, is named IBS\textsubscript{IV}.

3.3.2 Initial applied force (IBS\textsubscript{IF})

The second formulation of IBS is for the case when the unit impulse is modeled by an initial applied force. Using the IRFs which are computed under the initial condition (3.5) will make equation (2.16) to be discretized as written in formulation (3.16).

\[
\begin{aligned}
\mathbf{u}_n^{(s)} &= \sum_{i=0}^{n-1} H_{n-i}^{(s)} \left( \left( \mathbf{f}_i + \mathbf{f}_{i+1} \right) \frac{dt}{2} + B^{(s)^T} \mathbf{\lambda}_i \right) \\
\sum_{s=1}^{N_s} B^{(s)^T} \mathbf{u}_n^{(s)} &= 0
\end{aligned}
\tag{3.16}
\]

Formulation (3.16) is inspired from figure 3.2 and relation (3.6). As described before this method will lead to the exact result of the direct Newmark time integration. Since the IRFs of this method are computed by having initial force this type of IBS is named IBS\textsubscript{IF}.
3.4 Dynamic response of the bar example

3.3.3 Applied force at the second time step (IBS\textsubscript{SF})

The discretized formulation of IBS method based on the third model of the unit impulse (SF) can be obtained by extending equation (3.10). Therefore, by having the IRFs the displacement of each substructure can be computed via equation (3.17). Due to applying force at the second time step to model the impact, this formulation of IBS is named IBS\textsubscript{SF}.

\[
\begin{align*}
\begin{cases}
  u^{(s)}_n &= H^{(s)}_n \left( f_0 \frac{dt}{2} + B^{(s)^T} \lambda_0 \right) + \sum_{i=1}^{n-1} H^{(s)}_{n-i} \left( f_i dt + B^{(s)^T} \lambda_i \right) \\
  \sum_{s=1}^{N_{s}} B^{(s)} u^{(s)}_n &= 0
\end{cases}
\end{align*}
\] (3.17)

Remark
- It should be noticed that $\lambda$s in equations (3.14), (3.15) and (3.17), are unknowns which are found in a way to satisfy the compatibility conditions at the interface and can be interpreted as reaction impulses. Therefore it is not necessary to write them in a format similar to discretized relations for the applied force. However, in case of using $\lambda$s to construct the interface forces, the format of discretized formulations must be taken into account.

3.4 Dynamic response of the bar example

In order to examine the reliability of the IBS, this method is used to obtain the dynamic response of the 1D bar example. In this section, all the 3 formulations of IBS -based on different definitions of impact- are applied and the computed responses are compared with the results of direct Newmark scheme. The dynamic responses of the bar example are computed due to 3 types of excitations. The external force is applied at the end point of the full structure. 2 subsystems are the clamped-free and the free-free bars as explained in chapter 2. The external excitations are described by equations (3.18), (3.19) and (3.20) which include a unit impact, a unit step and a periodic force.

\[
f(t) = \begin{cases} 
  1 & t = 0 \\
  0 & t \neq 0
\end{cases}
\] (3.18)

\[
f(t) = 1
\] (3.19)

\[
f(t) = 10\sin(300t)
\] (3.20)

In figures 3.9, 3.10, 3.11 the results of each IBS formulation are compared with the results of the Newmark time integration. In order to make a better comparison
Figure 3.9: Dynamic response of the 1D bar example under a unit impact excitation, obtained by 3 different IBS formulations.

(a) Displacement of the interface point obtained by IBS_{IV}

(b) Displacement of the interface point obtained by IBS_{IV} (zoomed on the initial time steps)

(c) Displacement of the interface point obtained by IBS_{IF}

(d) Displacement of the interface point obtained by IBS_{IF} (zoomed on the initial time steps)

(e) Displacement of the interface point obtained by IBS_{SF}

(f) Displacement of the interface point obtained by IBS_{SF} (zoomed on the initial time steps)
3.4. DYNAMIC RESPONSE OF THE BAR EXAMPLE

Figure 3.10: Dynamic response of the 1D bar example under a unit step excitation, obtained by 3 different IBS formulations.
(a) Displacement of the interface point obtained by IBS_{IV}

(b) Displacement of the interface point obtained by IBS_{IV} (zoomed on the initial time steps)

(c) Displacement of the interface point obtained by IBS_{IF}

(d) Displacement of the interface point obtained by IBS_{IF} (zoomed on the initial time steps)

(e) Displacement of the interface point obtained by IBS_{SF}

(f) Displacement of the interface point obtained by IBS_{SF} (zoomed on the initial time steps)

Figure 3.11: Dynamic response of the 1D bar example under an excitation described by equation (3.20), obtained by 3 different IBS formulations
3.4. DYNAMIC RESPONSE OF THE BAR EXAMPLE

between the results of the different IBS formulations and the numerical time integration of the total system, MAC numbers are computed (see chapter 2). The MAC numbers are presented in tables 3.1, 3.2 and 3.3 and each column shows the similarity between the obtained space modes of one type of the IBS formulations and numerical solution. Since the MAC numbers demonstrate the reliability of the IBS results by comparing it with a reference solution (Newmark time integration), they can also be used to compare the 3 different IBS formulations (related to 3 different impact models).

As explained in chapter 2, the MAC numbers are computed for the modes with highest energy\(^1\). The computed MAC numbers are shown in tables 3.1, 3.2 and 3.3.

Table 3.1: MAC numbers of the dynamic responses (obtained by different types of the IBS) due to an excitation in the form of a unit impact

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>MAC([IBS_{IV}])</th>
<th>MAC([IBS_{IF}])</th>
<th>MAC([IBS_{SF}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>2</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>3</td>
<td>9.9992e-001</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>4</td>
<td>9.9658e-001</td>
<td>1.0000e+000</td>
<td>9.9982e-001</td>
</tr>
<tr>
<td>5</td>
<td>9.4624e-001</td>
<td>1.0000e+000</td>
<td>9.9690e-001</td>
</tr>
<tr>
<td>6</td>
<td>7.0049e-001</td>
<td>1.0000e+000</td>
<td>9.8181e-001</td>
</tr>
<tr>
<td>7</td>
<td>2.5511e-001</td>
<td>1.0000e+000</td>
<td>9.3754e-001</td>
</tr>
<tr>
<td>8</td>
<td>-7.4516e-002</td>
<td>1.0000e+000</td>
<td>8.5006e-001</td>
</tr>
<tr>
<td>9</td>
<td>-1.6849e-001</td>
<td>1.0000e+000</td>
<td>7.2567e-001</td>
</tr>
<tr>
<td>10</td>
<td>-1.5357e-001</td>
<td>1.0000e+000</td>
<td>5.9684e-001</td>
</tr>
</tbody>
</table>

Figures 3.9, 3.10, 3.11 and tables 3.1, 3.2, 3.3 demonstrate that the dynamic responses which are obtained by IBS\(_{IF}\) are exactly equal to the results of Newmark integration under all types of excitations. This conclusion was also expected based on the explained theory of impact model and discretized formulation of IBS.

The other IBS formulations (IBS\(_{IV}\) and IBS\(_{SF}\)) can also make reliable approximations of the dynamic response of the system. It should be noticed that the accuracy of the obtained results using this 2 methods is changing for different type of excitations. For instance, figure 3.11 and MAC table 3.3 illustrate that, for a periodic applied force with just one frequency, the results of these 2 methods are more close to the reference solution. However, the accuracy of the results are decreasing in impact analysis when a large band of frequencies are excited.

An interesting observation is, the dynamic response of a system, under an uniform excitation, obtained by IBS\(_{IV}\) is exactly equal to the results of Newmark integration.

\(^1\)For each example of this report, the first 10 MAC numbers (related to the 10 modes with the highest energy) are presented in a table which is called MAC tables, and the MAC numbers which are chosen based on criteria (2.20) are bold
### Table 3.2: MAC numbers of the dynamic responses (obtained by different types of the IBS) due to an excitation in the form of unit step

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>MAC[IBS_{IV}]</th>
<th>MAC[IBS_{IF}]</th>
<th>MAC[IBS_{SF}]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>2</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>3</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>9.9994e-001</td>
</tr>
<tr>
<td>4</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>9.9979e-001</td>
</tr>
<tr>
<td>5</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>9.9967e-001</td>
</tr>
<tr>
<td>6</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>9.9815e-001</td>
</tr>
<tr>
<td>7</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>9.8711e-001</td>
</tr>
<tr>
<td>8</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>9.4720e-001</td>
</tr>
<tr>
<td>9</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>8.5077e-001</td>
</tr>
<tr>
<td>10</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>6.8245e-001</td>
</tr>
</tbody>
</table>

### Table 3.3: MAC numbers of the dynamic responses (obtained by different types of the IBS) due to an excitation in the form of equation (3.20)

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>MAC[IBS_{IV}]</th>
<th>MAC[IBS_{IF}]</th>
<th>MAC[IBS_{SF}]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>2</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>3</td>
<td>9.9999e-001</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>4</td>
<td>9.9994e-001</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>5</td>
<td>9.9990e-001</td>
<td>1.0000e+000</td>
<td>9.9996e-001</td>
</tr>
<tr>
<td>6</td>
<td>9.9991e-001</td>
<td>1.0000e+000</td>
<td>9.9969e-001</td>
</tr>
<tr>
<td>7</td>
<td>9.9915e-001</td>
<td>1.0000e+000</td>
<td>9.9697e-001</td>
</tr>
<tr>
<td>8</td>
<td>9.9360e-001</td>
<td>1.0000e+000</td>
<td>9.8307e-001</td>
</tr>
<tr>
<td>9</td>
<td>9.7170e-001</td>
<td>1.0000e+000</td>
<td>9.3885e-001</td>
</tr>
<tr>
<td>10</td>
<td>9.1979e-001</td>
<td>1.0000e+000</td>
<td>8.4212e-001</td>
</tr>
</tbody>
</table>
3.5 Modeling space distributed loads

In the previous sections, the formulations of IBS method are derived for point forces. This method can also be applied for the problems with space distributed loads. In these cases based on the generated mesh (using FEM) the applied force will be discretized in space and each discretized force can be considered as a point force to compute impulse responses. Therefore the IRFs must be computed for all the point forces individually which will make calculations of the assembly equation greater.

Due to these difficulties, it is tried to introduce another solution to model the space distributed loads. One suggested model is defining a space distributed impact (\(I_{sd}\)) as shown in figure 3.12. This Impact can be simplified by a space distributed profile. The assumption which is made here is that the distributed load is conserving its profile with space and is just changing with time. In this case, the response of the structure must be obtained due to this specified impact. The IRFs due to the space distributed impact (figure 3.12c) is shown in figure 3.13. For the sake of simplicity, the space distributed impact is considered to be in the step form. The impacts are modeled by applying a unit force at the initial time step (IBS\(_{IF}\)).

In the step of solving the assembly equation, the obtained IRF due to \(I_{sd}\) must be treated similar to regular IRFs and the applied force would be function of time. Figure 3.14 shows the dynamic response of the interface point of the bar structure due to a
periodic force in the form of equation (3.20). This force is exciting the last 3 nodes of the bar.

Figure 3.14 illustrates the dynamic response of the bar example under a space distributed force by computing $I_{sd}$. Since the IBS$_{IF}$ is used, the computed response is exactly equal to the results of Newmark time integration. This simple example shows the possibility of using IBS in the cases with space distributed loads.
Chapter 4

Reducing the computation costs of IBS

4.1 Introduction

In the previous chapters the IBS method was explained and 3 discretized formulations of IBS were introduced. The attempt made in this chapter is to develop some estimation in order to decrease the computation costs of IBS. As described in chapter 2, the IBS method has 2 main steps. First, computing IRFs using Newmark time integration scheme. Second, assembling substructures and obtaining dynamic response of the total structure by applying the convolution product between the IRFs and applied forces. Therefore, reducing the computation costs can be achieved by applying some changes in these 2 parts.

In this report 2 methods are suggested to decrease the implementation costs of IBS. The first one is truncating the IRFs and reducing the computation time in both 2 steps. The second method is changing the size of the time step in the part of applying convolution product, and analyzing the problem considering less numbers of samples which is called down-sampling. These 2 approaches are examined on the 1D bar example.

4.2 Truncated IBS

As mentioned before, the computation costs in the IBS method are coming from 1: computing the IRFs, 2: applying convolution product. Since the IRFs for each sub-structure can be computed once and be used for several different analysis, therefore, reducing the costs of computing IRFs will not make a noticeable difference in the total costs. Hence, the main cost in the IBS method is coming from applying convolution product. The reason of this issue is, at time step \( t_n \), all the impulses from \( t_0 \) to \( t_{n-1} \)
must be taken into account. Therefore, by increasing \( n \), the size of the calculations at each time step become greater. One solution to deal with this problem is restricting the number of calculations in computing the convolution product, by considering the last \( m \) numbers of impulses. In other words, in the assembly equation at \( t_n \) instead of considering the impulses from \( t_0 \) to \( t_{n-1} \), only impulses from \( t_{n-m} \) to \( t_{n-1} \) get involved in computing the total response. Equation (4.1) explains this approach first by splitting the discretized form of convolution product into 2 summations.

\[
\begin{align*}
    u_n^{(s)} &= \sum_{i=0}^{n-m-1} H_{n-i}^{(s)} (f_i^{(s)} dt + B^{(s)T} \lambda_i) + \sum_{i=n-m}^{n-1} H_{n-i}^{(s)} (f_i^{(s)} dt + B^{(s)T} \lambda_i) \\
    &= \sum_{i=n-m}^{n-1} H_{n-i}^{(s)} (f_i^{(s)} dt + B^{(s)T} \lambda_i) 
\end{align*}
\] (4.1)

Computations costs can be reduced by assuming IRF to be zero after \( m \) number of time steps.

\[
H_{(m+1\to n)} \cong 0 \quad (4.2)
\]

Therefore,

\[
\begin{align*}
    u_n^{(s)} &= \sum_{i=n-m}^{n-1} H_{n-i}^{(s)} (f_i^{(s)} dt + B^{(s)T} \lambda_i) 
\end{align*}
\] (4.3)

This approximation, which is made by ignoring the effect of initial impulses, is called truncating the impulse response functions. This truncation changes for different types of IRFs. A quick recap of IRFs of the 1D bar example are shown in figure 4.1. As can be observed from figure 4.1a for the clamped-free bar \((s^1)\), the impulse response converges to zero after several time steps \((m)\). The transition of the IRF demonstrates the effect of damping in the system.

On the other hand, the impulse response of the free-free bar \((s^2)\), figure 4.1b, is increasing with time. This behavior of the impulse responses is because of the fact that none of the dofs is fixed for this substructure. Therefore, the rigid body mode can be excited in the response of the substructure \((s^2)\) to the unit impulse. In this case applying truncation on the IRFs, after several time steps, will make the total response unstable. Hence, a different solution is needed for this type of IRFs.

In general, after splitting the total structure into some subsystems, some of the generated substructures may not be fixed at the boundaries which will make them floating structures. Since the IRFs of floating substructures are not similar to non-floating subsystems, therefore in the case of applying truncation on IRFs these 2 types of IRFs must be studied individually.

4.2.1 Non-floating substructures

As explained before, the IRF of a non-floating system is in the form of figure 4.1a. Figure 4.1a shows, response of an impulse which is applied at \( t_0 \) is damped after a finite number \((m)\) of time steps. Therefore, it is possible to take into account the first
4.2. TRUNCATED IBS

$m$ numbers of impulse responses and consider the rests to be equal to zero. Since the value of the impulse response is not necessarily equal to zero at the cut point, window functions can be applied to force the IRFs to be zero.

In the frequency domain, window functions are used to control the leakage of the FRFs [14] but in this assignment, the window functions are used to weight the time signals (IRFs) [15]. For the 1D bar example, 3 types of window functions are applied on the IRFs of the clamped-free bar.

**Rectangular window function**

The simplest type of truncation is cutting the IRFs after certain number of time steps. This truncation can be achieved by multiplying a rectangular window function (4.4) to the IRFs. Based on (4.4) the IRFs are cut at $t_c$ which is called the cutoff time. $t_c$ is defined as the time when the magnitude ratio (4.5) becomes less than a design variable $\alpha$. In equation (4.5), $A(t)$ is magnitude of the impulse response at peak points. Figure 4.2 shows the effect of rectangular window.

\[
W_{\text{rect}}(t) = \begin{cases} 
1 & t < t_c \\
0 & t > t_c 
\end{cases}
\quad (4.4)
\]

\[
\frac{A(t)}{A_{\text{max}}} \leq \alpha \quad \text{for all } t > t_c
\quad (4.5)
\]

Since the rectangular window forces the impulse responses to be zero from a non-zero value, it may cause noticeable errors in the dynamic response obtained by superposition.
CHAPTER 4. REDUCING THE COMPUTATION COSTS OF IBS

of IRFs. In order to deal with this problem different types of window functions are defined.

![Figure 4.2: Applying rectangular window on the impulse response of a non-floating structure](image)

**Cosine window function**

The cosine window function is applied to cut the IRFs smoothly. This window function is defined in a way to make the value of the impulse response at $t_c$ be equal to zero. Based on this definition, the cosine window is function of $t_c$ which is obtained from the magnitude ratio equation (4.5). Equation (4.6) explains the cosine window function. Effect of the cosine window function is shown in figure 4.3.

$$W_{\cos}(t) = \begin{cases} \cos \left( \frac{\pi t}{2t_c} \right) & t < t_c \\ 0 & t > t_c \end{cases}$$

As can be realized from figure 4.3, the impulse response under cosine window function is not visibly changed at the initial numbers of time steps where the value of the impulse response is high and important. In fact this is the advantage of cosine window function which reduces the impulse responses to be zero without perturbing dynamic properties of system so much.

**Exponential window function**

The other form of window function which can decrease IRFs smoothly is exponential window function. Although the exponential window cannot make the IRFs be exactly
4.2. TRUNCATED IBS

Figure 4.3: Applying cosine window on the impulse response of a non-floating structure equal to zero at $t_c$ but it is defined to become small enough at $t_c$ (equation (4.7)). The exponential window can be physically interpreted as introducing additional damping to the system. Figure 4.4 shows the impulse response under the exponential window.

$$W_{exp}(t) = \begin{cases} \exp\left(-\frac{20t}{t_c}\right) & t < t_c \\ 0 & t > t_c \end{cases}$$ (4.7)

As can be realized from figure 4.4, contrary to the cosine window, the exponential window mostly changes the impulse responses at the initial time steps. Since the initial impulse responses have more effects on the final result of the convolution product, it can be expected that applying the exponential window on the IRFs will perturb the dynamic response of the total system.

4.2.2 Floating substructures

As mentioned in the beginning of this section, the IRFs of a substructure which is not fixed at the boundaries is increasing with time (due to having rigid body mode) and, truncating these types of IRFs will make the total response unstable. In order to deal with this problem, it is suggested to decompose the impulse response of the floating substructure into superposition of pure rigid body response (figure 4.5b) and vibrational response (figure 4.5a) as shown in equation (4.8). Afterwards truncation can be applied on $H^{vb}$. The removed rigid body motion should be directly added to the result of the convolution product (4.3) at each time step.

$$H = H^{vb} + H^{vg}$$ (4.8)
Figure 4.4: Applying exponential window on the impulse response of a non-floating structure

Figure 4.5: Decomposition of the impulse response of a floating substructure into pure vibrational and rigid body response

**Pure rigid body response**

The pure rigid body response in 1D problems, under a unit impulse, can be computed by integrating dynamic equation ((4.9)).

\[ m \ddot{u} = f \]  

\[(4.9)\]
4.2. TRUNCATED IBS

\[ m \dot{u}_0^+ = \int_{0^-}^{0^+} f \]  

(4.10)

For a system starting at rest from \( t = 0 \), the initial velocity resulting from a unit impulse is

\[ m \dot{u}_0 = 1 \]  

(4.11)

Finally, the displacement of a system which is showing pure rigid body mode under a unit impulse can be obtained by equation (4.12).

\[ u = \int \frac{1}{m} dt = \frac{1}{m} t \]  

(4.12)

On the other hand, the vibrational part of impulse response can be obtained by deducing the response of the rigid body mode from the total impulse response (4.13). In the truncation of the IRFs, these 2 responses (figure 4.5b, 4.5a) are considered separately.

\[ [H^{\text{vib}}]_{ij} = [H]_{ij} - \frac{t}{m} \]  

(4.13)

Since the response of the rigid body mode is not converging to zero, it is not reasonable to truncate the IRF but it is still possible to reduce the number of calculations by making a relation between the responses at \( t_{n+1} \) and \( t_n \). In other words, at each time step the time response is calculated by using the response at the previous time step. Using the formulation of the pure rigid body response (4.12), the relations between the responses at different time steps will be

\[ u_n = \sum_{i=0}^{n-1} H_{(n-i)} f_i dt = \sum_{i=0}^{n-1} \frac{(n-i)dt}{m} f_i dt \]  

(4.14)

\[ u_{n+1} = \sum_{i=0}^{n} H_{(n+1-i)} f_i dt = \sum_{i=0}^{n} \frac{(n+1-i)dt}{m} f_i dt \]  

(4.15)

The summation in equation (4.15) can be written as

\[ u_{n+1} = \sum_{i=0}^{n-1} \frac{(n+1-i)dt^2}{m} f_i + f_n \frac{dt^2}{m} = \sum_{i=0}^{n-1} \frac{(n-i)f_i dt^2}{m} + \sum_{i=0}^{n-1} \frac{f_i dt^2}{m} + f_n \frac{dt^2}{m} \]  

\[ \sum_{i=0}^{n} \frac{f_i dt^2}{m} \]  

(4.16)

Equation (4.16) demonstrates that instead of applying the convolution product between the pure rigid body impulse response and the applied force at each time step, \( u_{n+1} \) can be computed by time stepping.

\[ u_{n+1}^{\text{rig}} = u_n + \sum_{i=0}^{n} \frac{f_i dt^2}{m} \]  

(4.17)
Pure vibrational response

As can be realized from figure 4.5a, the vibrational part of impulse response (obtained be equation (4.13)) converges to zero after a finite number of time steps. Therefore, similar to what was explained before, it is possible to truncate vibrational impulse response by applying window functions.

Dynamic response of floating substructures

As mentioned before, to obtain dynamic response of a floating system using truncated formulation, convolution product must be applied on the pure rigid body motion and vibrational motion separately. Therefore, at the final step, these separate computed responses must be combined to make the dynamic response of the floating system. Equation (4.18) expresses the truncated formulation of convolution product related to the 1D floating subsystem. In equation (4.18), $u_{n}^{rq}$ is computed by (4.17) and.

$$u_{n} \cong u_{n}^{rq} + \sum_{i=\max(0,n-m)}^{n} (H_{n-i}^{vib}f_{i})dt$$ (4.18)

Remarks

- It should be noticed that the discretized equation ((4.18)) of convolution product is written in its general form. Therefore, as explained in chapter 2, it should be adapted to the model which is used for unit impact.

- Modeling the unit impact by applying a unit force at initial or second time step (IBS_{IF}, IBS_{SF}) will make a delay in response of the rigid body mode. Therefore, in case of computing the vibrational part of response ($H^{vib}$) using equation (4.8), this delay must be taken into account (see appendix A).

4.2.3 Results of truncated IBS and discussion

In this section the truncated IBS is applied to obtain the dynamic response of the 1D bar example. The external excitation is a unit step and has been applied at the end of the bar. Since for an excitation in the form of unit step, results of both IBS_{IF} and IBS_{IV} are exactly equal to the result of the numerical integration of the total system, in this example in order to skip the delay problem in the rigid body response (which will be appeared in IBS_{IF}), IBS_{IV} has been used. The vibrational impulse responses, for both floating and non-floating substructures, are truncated under the 3 explained window functions. In order to illustrate the effect of $t_{c}$ the example has been solved for
4.2. TRUNCATED IBS

(a) $\alpha = 10^{-1}$

(b) $\alpha = 10^{-1}$ (zoomed on the last time steps)

(c) $\alpha = 10^{-2}$

(d) $\alpha = 10^{-2}$ (zoomed on the last time steps)

(e) $\alpha = 10^{-3}$

(f) $\alpha = 10^{-3}$ (zoomed on the last time steps)

Figure 4.6: Dynamic response of the 1D bar example (displacement at the interface node) obtained by the truncated IBS with rectangular window
Figure 4.7: Dynamic response of the 1D bar example (displacement at the interface node) obtained by the truncated IBS with cosine window

(a) $\alpha = 10^{-1}$

(b) $\alpha = 10^{-1}$ (zoomed on the last time steps)

(c) $\alpha = 10^{-2}$

(d) $\alpha = 10^{-2}$ (zoomed on the last time steps)

(e) $\alpha = 10^{-3}$

(f) $\alpha = 10^{-3}$ (zoomed on the last time steps)
4.2. **TRUNCATED IBS**

![Graphs showing dynamic response of the 1D bar example](image)

(a) $\alpha = 10^{-1}$  
(b) $\alpha = 10^{-1}$ (zoomed on the last time steps)

(c) $\alpha = 10^{-2}$  
(d) $\alpha = 10^{-2}$ (zoomed on the last time steps)

(e) $\alpha = 10^{-3}$  
(f) $\alpha = 10^{-3}$ (zoomed on the last time steps)

Figure 4.8: Dynamic response of the 1D bar example (displacement at the interface node) obtained by the truncated IBS exponential window
3 different design values $\alpha (10^{-3}, 10^{-2}, 10^{-1})$. Figures 4.6, 4.7 and 4.8 are respectively related to rectangular, cosine and exponential window functions.

Similar to chapter 3, the MAC numbers are also computed for the results of the truncated IBS. Tables 4.1, 4.2 and 4.3 present the MAC numbers for the truncated IBS under the rectangular, cosine and exponential window function respectively.

Table 4.1: MAC numbers of the computed dynamic responses obtained by truncated IBS (Rectangular window)

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>MAC[$\alpha = 10^{-1}$]</th>
<th>MAC[$\alpha = 10^{-2}$]</th>
<th>MAC[$\alpha = 10^{-3}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.7563e-001</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>2</td>
<td>9.0021e-001</td>
<td>9.9999e-001</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>3</td>
<td>8.8741e-001</td>
<td>9.9998e-001</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>4</td>
<td>2.6532e-001</td>
<td>9.9997e-001</td>
<td>9.9996e-001</td>
</tr>
<tr>
<td>5</td>
<td>2.4285e-001</td>
<td>9.9983e-001</td>
<td>9.9986e-001</td>
</tr>
<tr>
<td>6</td>
<td>9.3593e-001</td>
<td>9.9849e-001</td>
<td>9.9841e-001</td>
</tr>
<tr>
<td>7</td>
<td>7.617e-001</td>
<td>9.9072e-001</td>
<td>9.8999e-001</td>
</tr>
<tr>
<td>8</td>
<td>6.1840e-001</td>
<td>9.8913e-001</td>
<td>9.8868e-001</td>
</tr>
<tr>
<td>9</td>
<td>8.5509e-001</td>
<td>9.9658e-001</td>
<td>9.9628e-001</td>
</tr>
<tr>
<td>10</td>
<td>9.5513e-001</td>
<td>9.9588e-001</td>
<td>9.9597e-001</td>
</tr>
</tbody>
</table>

Table 4.2: MAC numbers of the computed dynamic responses obtained by truncated IBS (Cosine window)

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>MAC[$\alpha = 10^{-1}$]</th>
<th>MAC[$\alpha = 10^{-2}$]</th>
<th>MAC[$\alpha = 10^{-3}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.9980e-001</td>
<td>1.0000e+000</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>2</td>
<td>9.9889e-001</td>
<td>9.9999e-001</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>3</td>
<td>9.9703e-001</td>
<td>9.9997e-001</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>4</td>
<td>9.7522e-001</td>
<td>9.9997e-001</td>
<td>9.9996e-001</td>
</tr>
<tr>
<td>5</td>
<td>9.7676e-001</td>
<td>9.9984e-001</td>
<td>9.9986e-001</td>
</tr>
<tr>
<td>6</td>
<td>9.9832e-001</td>
<td>9.9845e-001</td>
<td>9.9841e-001</td>
</tr>
<tr>
<td>7</td>
<td>9.9236e-001</td>
<td>9.9044e-001</td>
<td>9.8998e-001</td>
</tr>
<tr>
<td>8</td>
<td>9.8198e-001</td>
<td>9.8900e-001</td>
<td>9.8867e-001</td>
</tr>
<tr>
<td>9</td>
<td>9.9620e-001</td>
<td>9.9645e-001</td>
<td>9.9627e-001</td>
</tr>
<tr>
<td>10</td>
<td>9.9626e-001</td>
<td>9.9590e-001</td>
<td>9.9597e-001</td>
</tr>
</tbody>
</table>

A general conclusion from all the figures 4.6, 4.7 and 4.8 is that by decreasing the value of $\alpha$ (longer time for $t_c$) the obtained dynamic response is closer to the results of the numerical integration of the total system.

As can be deduced from figure 4.6, truncating the impulse responses by applying the rectangular window function can make the final response unstable ($\alpha = 10^{-1}$).
4.2. **TRUNCATED IBS**

Table 4.3: MAC numbers of the computed dynamic responses obtained by truncated IBS (Exponential window)

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>$\text{MAC}[\alpha = 10^{-1}]$</th>
<th>$\text{MAC}[\alpha = 10^{-2}]$</th>
<th>$\text{MAC}[\alpha = 10^{-3}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.9893e-001</td>
<td>9.9993e-001</td>
<td>9.9999e-001</td>
</tr>
<tr>
<td>3</td>
<td>9.8713e-001</td>
<td>9.9874e-001</td>
<td>9.9971e-001</td>
</tr>
<tr>
<td>4</td>
<td>4.6825e-001</td>
<td>9.6823e-001</td>
<td>9.9628e-001</td>
</tr>
<tr>
<td>5</td>
<td>-2.9890e-002</td>
<td>9.4359e-001</td>
<td>9.8803e-001</td>
</tr>
<tr>
<td>6</td>
<td>8.5232e-002</td>
<td>8.4771e-001</td>
<td>9.4978e-001</td>
</tr>
<tr>
<td>7</td>
<td>-1.9111e-001</td>
<td>5.6598e-001</td>
<td>8.2151e-001</td>
</tr>
<tr>
<td>8</td>
<td>-7.0270e-001</td>
<td>5.7498e-001</td>
<td>7.9781e-001</td>
</tr>
<tr>
<td>9</td>
<td>-4.3620e-002</td>
<td>6.7945e-001</td>
<td>8.1698e-001</td>
</tr>
<tr>
<td>10</td>
<td>-1.9459e-001</td>
<td>3.9509e-001</td>
<td>7.2494e-001</td>
</tr>
</tbody>
</table>

This problem happens where the impulse responses are cut so early (small $t_c$). As mentioned before, forcing the impulse response to go to zero from a non-zero value can cause this problem. Indeed, this truncation can be interpreted as exerting an impact to the system which can make the final result unstable. However by choosing small values for $\alpha$, the value of the impulse response at the cut point is small enough, therefore the generated impact due to truncation is not too big to make the final result unstable.

The advantage of the rectangular window function is that the impulse responses are not changed under the window function (they are multiplied by 1 for $t < t_c$). Therefore for small values of $\alpha$, truncated IBS under the rectangular window function can make a reliable approximation of the dynamic response of the full system. The MAC numbers which are shown in table 4.1 also approve this conclusion.

Figure 4.7 shows the dynamic response of the interface node where the IRFs are truncated by applying the cosine window function. As mentioned before the cosine window function is defined in a way to reduce the IRFs to be zero smoothly. In other words, by applying the cosine window on the IRFs, at the cut point the impulse response is equal to zero. Hence, as can be realized from figure 4.7 the instability problem which was observed in the results of the truncated IBS with the rectangular window function does not appear here. A quick review of the results of the 3 window functions brings the conclusion that truncating the impulse responses by applying the cosine window function leads to the best approximation of the final result.

The dynamic responses of the interface point obtained by the truncated IBS, under the exponential window function, are shown in figure 4.8. As can be concluded from figure 4.8 and table 4.3, although the results of the truncated IBS with exponential window is stable, but they are not reliable. Indeed, multiplying the impulse responses by an exponential window function can be interpreted as increasing the damping in
the system. Therefore, applying the truncated IBS with rectangular window function will lead to the dynamic response of a system which is damped more.

As can be understand from figures 4.6, 4.7 and 4.8 truncating the IRFs at the early time steps is changing the equilibrium position of the bar system. This change in the equilibrium position can be specially observed in applying exponential window function and by increasing the value of $\alpha$ the difference between 2 equilibrium positions is increased too. As a consequence, the observed offset in the figures, is a result of applying truncation on IRFs. Dynamic response of the 1D bar system under other types of excitation described by equations (3.18) and (3.20) are shown in appendix B.
4.3 Down-sampling

As mentioned before, one of the advantages of IBS method is computing IRFs of (sub)structures and using them several times for different analysis and in different structures. Up to this section, in all the analyzed examples by IBS method the external applied force was discretized by the same size of the time step as the one was used in Newmark time integration to obtain impulse responses. However, there are some cases for which it is not necessary to take very small step sizes and using the IRFs computed by small $dt$ will make the assembly procedure time consuming and expensive. In this case, in order to reduce the computation costs in the part of applying superposition, it is suggested to discretize the applied force using bigger size of time step (for instance $dt_{\text{new}} = 2 \times dt_{\text{old}}$). Therefore to obtain the dynamic response at $t_n$, the assembly equation (3.17) should be solved $\frac{n}{2}$ times which will reduce the total computation costs. This new implementation of IBS method which decreases the number of considered samples is called down-sampling and is developed in this section.

In this section first the new discretization of the applied force for applying IBS with down-sampling will be explained. Afterwards, the changes which must be applied in the impulse responses will be studied. Since modeling the unit impact by applying a unit force at the initial time step ($\text{IBS IF}$) gives the best approximation of dynamic response of the system, this formulation of IBS has been used in this section.

4.3.1 Discretization of the external applied force

As explained before, by taking bigger time steps, the less number of samples will be considered in the calculations. For instance, by changing $dt$ to $2 \times dt$, discretization of the applied force would change from figure 4.9a to figure 4.9b.

As can be realized from figure 4.9, the applied force is discretized by using different impacts, therefore in the assembly equation (3.17) $dt_{\text{new}}$ must be applied to compute impact $p \times dt$. Since discretization of the applied force has been changed, the impulse response matrices need to be adapted to this change.

4.3.2 Rearranging impulse response matrices

As explained in chapter 3, dynamic response under a unit force which is applied at the initial time step must be divided by $\frac{dt}{2}$ to generate the impulse response. Therefore, changing the size of the time step in computing the dynamic response of subsystems, will not make a noticeable change in the obtained impulse response. Figure 4.10 shows the impulse responses of clamped-free bar obtained by 2 different size of time steps ($dt$, $2 \times dt$).

In fact the impulse responses which are computed by $2dt$ should be used for the force discretization figure 4.9b, but in this case the IRFs computed by $dt$ are used.
CHAPTER 4. REDUCING THE COMPUTATION COSTS OF IBS

Figure 4.9: Effects of changing $dt$ in discretization of the external applied force $(2 \times dt)$

Figure 4.10: Comparison between the impulse responses of the clamped-free bar obtained by $dt$, $2dt$
4.3. DOWN-SAMPLING

Considering \( n \) as the number of time step, the impulse response matrix of the down-sampled example must be made in the form (4.19).

\[
H(n)_{\text{Down-Sampled}} = H(2 \times n)_{\text{Normal}} \quad \text{for} \quad n = 0, 1, 2, \ldots, n_{\text{max}} \quad (4.19)
\]

4.3.3 Results of IBS with down-sampling and discussion

In order to check the validity of applying down-sampling in IBS, dynamic response of the 1D bar example is computed by changing \( dt \) to \( 2dt \) and compared with the result of Newmark time integration of the full system. It should be noticed that in applying Newmark time integration on the full structure, the down-sampled time step (\( 2dt \)) must be used. Similar to chapter 3, dynamic response of the full bar system is computed under 3 different types of applied forces. The excitation is applied at the end of the bar. The results are shown in figure 4.11.

Table 4.4: MAC numbers of the computed dynamic responses under different type of excitations obtained by IBS\(_{\text{Down-Sampled}}\)

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>MAC[unit impact]</th>
<th>MAC[unit step]</th>
<th>MAC[periodic force]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000e+000</td>
<td>9.9999e-001</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>2</td>
<td>9.9909e-001</td>
<td>9.9999e-001</td>
<td>6.5753e-001</td>
</tr>
<tr>
<td>3</td>
<td>9.8324e-001</td>
<td>9.9935e-001</td>
<td>6.4479e-001</td>
</tr>
<tr>
<td>4</td>
<td>9.0902e-001</td>
<td>9.8100e-001</td>
<td>4.6617e-001</td>
</tr>
<tr>
<td>5</td>
<td>5.5869e-001</td>
<td>7.9527e-001</td>
<td>5.8093e-001</td>
</tr>
<tr>
<td>6</td>
<td>5.1940e-001</td>
<td>5.6475e-001</td>
<td>1.8640e-001</td>
</tr>
<tr>
<td>7</td>
<td>7.7455e-001</td>
<td>3.9148e-001</td>
<td>2.5894e-001</td>
</tr>
<tr>
<td>8</td>
<td>4.2263e-001</td>
<td>3.6644e-001</td>
<td>3.6043e-001</td>
</tr>
<tr>
<td>9</td>
<td>1.5978e-001</td>
<td>8.6433e-001</td>
<td>-3.0769e-002</td>
</tr>
<tr>
<td>10</td>
<td>6.3316e-001</td>
<td>3.0683e-001</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.11 and table 4.4 demonstrate that the effect of changing \( dt \) to \( 2 \times dt \) can be different for the different types of the applied forces. The reason of this observation is that rearranging the impulse response functions in order to apply IBS with downsampling estimates the new IRFs with a small error compare to obtaining the IRFs at the beginning dy choosing \( dt = 2 \times dt \). For the cases where the external force excites a large frequency band (step forces or impacts), this error can change the final response. However for periodic forces with just one frequency, similar to the case which is shown in figures 4.11e, 4.11f, the computed dynamic response by down-sampled IBS is a reliable approximation of the Newmark results.
Figure 4.11: Dynamic response of the 1D bar example (displacement at the interface node) obtained by $\text{IBS}_{\text{Down-Sampled}}$
Chapter 5

Applying IBS on a 3D structure

5.1 Introduction

In the previous chapters, the IBS method has been developed and applied on a 1D example. In this chapter in order to show the applications of the IBS in real world, a real 3D example is analyzed by IBS. This 3D example is related to an offshore wind turbine and the time response of the structure is computed using IBS and Newmark time integration. The applied procedure is similar to 1D bar example.

5.2 Problem definition

Design of an offshore wind turbine can be split into 2 main parts. One part is design of the wind mill, this part includes design of the generator with the rotor blades, which is placed on top of a mono-pile. The other part is foundation or jacket structure, which is located on the seabed and the wind mill will be placed on the top of that. These 2 subparts are considered as the substructures in the IBS method. Therefore, in case of applying IBS to obtain dynamic response of offshore the wind mill, the impulse response functions of the wind mill and jacket structure must be computed.

The available information of this 3D problem is the linearized mass ($M$), stiffness ($K$) and damping ($C$) matrices of the jacket and wind mill which are computed by FEM\(^1\). The jacket structure has 8 nodes ($8 \times 6$ dofs) at the interface. These 8 nodes are all defined under a master node with 6 dofs. This master node in the jacket interface is attached to the bottom point of the wind mill. As a consequence, there are 6 equations at the interface. In this example, the assembly equation has been solved for the active nodes (interface nodes). The dynamic response of the system is computed under an external excitation which is applied at one of the interface dofs.

\(^1\)The example has been provided by Siemens Wind Power, for more information about the model see [16, 17]
Figure 5.1 shows an overall sketch of the problem. Based on figure 5.1, 2 different assembling procedures need to be applied to make the total system (assembling RNA (the rotating part of the wind mill) and Tower, assembling Tower and Jacket). However, it should be noticed that the mass, stiffness and damping matrices for the wind mill are generated after assembling RNA and Tower. Therefore, the interface point between Jacket and Tower is the only point which is taken into account in the IBS method.

### 5.3 Dynamic response

As mentioned before, in order to examine the validity of the IBS method in 3D systems, the dynamic response of the 3D structure which is obtained by IBS should be compared with results of Newmark time integration of the full system.

#### 5.3.1 Numerical time integration of full system

In this example, the information of the jacket structure and the wind mill are stored in 2 separated matrices. Therefore, in order to apply Newmark time integration on the full system, these matrices must be assembled. Since both of the mass (or stiffness or damping) matrices are declaring information which is in common in the interface point, assembling must be done at this point as shown in figure 5.2.
Figure 5.2 expresses that the mass (or stiffness, or damping) matrix of the full system is made by putting 2 separated mass matrices (of jacket and wind mill) in diagonal of a bigger matrix hence the data at the interface is the summation of the data of the interface point in both jacket structure and wind mill [6].

Finally the dynamic response of the full system is obtained by applying Newmark time integration while using the assembled matrices as $M$, $K$ and $C$. As mentioned in chapter 2 the time step is defined based on the highest frequency of the external force which is applied to the system.

5.3.2 IBS method

Before solving the wind turbine example using IBS, first the whole framework of the IBS method, adapted to a 3D case, is explained.

IRFs

As explained before, the external force is applied at the interface therefore the IRFs must be computed due to unit impulses which are exciting dofs at the interface. The impulse responses of each substructure are stored in a $6 \times 6$ matrix, in a form that $[H]_{ij}$ is the dynamic response of dof $j$ due to an impact on dof $i$. In this example, the impulse responses are computed by using 2 different impact models (initial applied force and initial applied velocity).

Normal IBS

Using equations (3.15), (3.16) respectively for IRF$_{IV}$ and IRF$_{IF}$, the dynamic response of the full structure at the interface due to an excitation in the form of unit step applied on dof number 1 is computed. The obtained responses are presented in the next section.
Truncated IBS

Since computation costs is one of the most crucial issues in analyzing real problems therefore it is important to examine the truncated IBS on impulse responses of the wind turbine example. As explained in chapter 3, applying truncated IBS using cosine window function \((5.1)\) makes the best approximation of the dynamic response of the system, therefore this type of window function is used in this example.

\[
W_{\text{cos}}(t) = \begin{cases} 
    \cos\left(\frac{\pi t}{2t_c}\right) & \text{if } t < t_c \\
    0 & \text{if } t > t_c 
\end{cases}
\]  

(5.1)

Similar to the 1D bar example, there is also a floating structure in the wind turbine problem. The wind mill is not fixed at the boundary and the rigid body motion is not blocked. Therefore a part of the dynamic response due to the rigid body modes must be removed from the IRFs (equation (5.2)).

\[
H_{\text{vib}} = H - H^{\text{rig}}
\]  

(5.2)

Since in this problem there are 6 rigid body modes, \(m\) in equation (4.9) is in the matrix form and equation (4.12) can not be used to compute rigid body responses. In this case, the rigid body response can be obtained by computing the null space of the stiffness matrix \((K)\) [6]. The null space of \(K\), called \(R\), is a matrix which contains the rigid body modes of the system in its column. By having \(R\) and considering \(\alpha\) as the rigid body mode amplitude, the rigid body motion can be computed by equation (5.3) [7]

\[
u = R\alpha
\]  

(5.3)

By substituting equation (5.3) in the linearized motion equation

\[
MR\ddot{\alpha} + CR\dot{\alpha} + KR\alpha = f
\]  

(5.4)

Since rigid body modes cannot produce any work on \(K\), \(KR = 0\) and by assuming that damping matrix is built based on stiffness matrix \((CR = 0)\), equation (5.4) will be

\[
MR\ddot{\alpha} = f
\]  

(5.5)

By multiplying \(R^T\) by equation (5.5) and calling \(R^T\) MR = \(M_{\text{tot}}\), \(\alpha\) can be computed by

\[
\int_0^t \int_0^t R^T MRd\alpha dt = \int_0^t \int_0^t R^T f
\]  

(5.6)

Since \(f\) is presenting a unit impulse, therefore \(\int_0^t \int_0^t f = 1t\), and \(1\) is a vector which is 1 at the excited dof and is 0 at the rest [7].

\[
\alpha = M_{\text{tot}}^{-1} R^T 1t
\]  

(5.7)
5.4. RESULTS

By obtaining $\alpha$ from equation (5.7), the rigid body motion under the unit impulse can be computed from [7]

$$u_{\text{rigid}} = RM_{tot}^{-1} R^T t$$

(5.8)

Finally, the rigid body response matrix ($H$) will be

$$H^{rig} = RM_{tot}^{-1} R^T t$$

(5.9)

By calling $RM_{tot}^{-1} R^T = \mu^{-1}$, equation (5.9) will be similar to (4.12), with a difference that in this case $\frac{1}{m}$ is matrix $\mu^{-1}$.

The method which is suggested here to compute the rigid body response, is based on integrating the motion equation under a unit impulse analytically. In this case $H^{vib}$ is computed by equation (5.2) while $H^{rig}$ is computed by equation (5.9). There is also another method to compute $H^{rig}$, $H^{vib}$ based on the fact that the rigid body modes is $M$-orthogonal to $H^{vib}$ (equation(5.10)) [7].

$$R^T M \left( H - H^{rig} \right)_{H^{vib}} = 0$$

(5.10)

By substituting equation (5.3) in equation (5.10)

$$R^T M H - R^T MR\alpha = 0$$

(5.11)

From equation (5.11), $\alpha$ can be obtained by

$$\alpha = M_{tot}^{-1} R^T M H$$

(5.12)

Finally $H^{rig}$, $H^{vib}$ will be [7]

$$H^{rig} = RM_{tot}^{-1} R^T M H$$

(5.13)

$$H^{vib} = ( I - RM_{tot}^{-1} R^T M ) H$$

After splitting $H$ into $H^{vib}$, $H^{rig}$, the truncated IBS can be applied to compute dynamic response of the full system.

5.4 Results

IRFs

Figures 5.3 and 5.4 show 4 of the computed impulse responses for the jacket structure and the wind mill respectively. All the computed IRFs are presented in the matrix form and can be found in appendix C.
As can be observed, both impact models, initial applied force and initial applied velocity, generate the main impulse responses of the jacket structure in the same way. There are some impulse responses which are different between 2 models, but they are too small compare to the main impulse responses. In fact those impulse responses are expected to be zero, and the computed response can be considered as numerical error. Therefore, it can be concluded that different models of unit impact are producing similar impulse responses for the jacket structure.

On the other hand, a brief review of figure 5.4 explains that, different impact models will make different impulse responses for the wind mill. Since the impulse responses computed by an initial velocity model is not in the expected form of the impulse response of a floating structure, it makes a conclusion that this is not a proper model to compute the impulse responses of the wind mill example. This problem can be due to having a problem in the model.

Normal IBS
Since the impulse responses obtained by initial applied velocity method are not correct, therefore this type of IBS (IBS_{IV}) is not able to solve the offshore wind turbine problem. Before applying IBS_{IF} to obtain the dynamic response, the impulse responses which are used to solve the 3D example are presented in figures 5.5 and 5.6 (the full matrix representation of the IRFs are provided in appendix C ). Afterwards the dynamic
response of the full system at the interface is computed by IBS\textsubscript{IF}. Results of the IBS and Newmark time integration of full system under a unit step are shown in figures 5.7 and 5.8. The Mac numbers of this example are presented in table 5.1. Since the problem is solved just for 6 interface dofs, there are 6 spatial modes (obtained by SVD).

Table 5.1: MAC numbers of the computed dynamic responses of the wind turbine structure obtained by normal IBS

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>MAC number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>2</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>3</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>4</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>5</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>6</td>
<td>1.0000e+000</td>
</tr>
</tbody>
</table>
Figure 5.5: IRFs of the jacket structure (the impact is modeled by applying a unit force at the initial time step)

Figure 5.6: IRFs of the wind mill (the impact is modeled by applying a unit force at the initial time step)
Figure 5.7: Dynamic response of the interface point under an excitation in the form of unit step on dof sub 1 obtained by IBS and Newmark time integration
Figure 5.8: Dynamic response of the interface point under an excitation in the form of unit step on dof$_1$ obtained by IBS and Newmark time integration
5.4. RESULTS

Truncated IBS

As explained before, in applying truncated IBS on systems with floating substructures, first the impulse responses of the floating subsystem must be split into the rigid body response and vibrational response. In this example the rigid body responses are computed by analytical solution ((5.9)) which are shown in figure 5.9, and by using the obtained rigid body responses, the vibrational responses are computed by relation (5.2) which are shown in figure 5.10. As can be observed, in figure 5.10 the vibrational impulse responses are not oscillating around zero, therefore cutting them after a finite number of time step will make the total dynamic response unstable. This unexpected behavior of the impulse responses can be due to having a problem in the finite element model of the system.

Since the obtained vibrational impulse responses (figure 5.10) can not be used in truncated IBS, the vibrational impulse responses are computed by another method which is described by relation (5.13) and are presented in figure 5.11. As can be observed from figure 5.11, computing the pure vibrational impulse responses and pure rigid body impulse responses by using the M-orthogonality quality of them will provide the vibrational impulse responses in the expected form and can be applied in the truncated IBS. In this example the computed rigid body responses by relation (5.13) are different from the analytical rigid body responses. This issue can due to the fact that the model is not defined properly therefore the rigid body responses are not produced correctly. Finally, it should be noticed that although $H^{vb}$ is computed by the rigid body response in the form of equation (5.13), in applying truncated IBS the analytical rigid body responses (figure 5.9) are added to the computed dynamic response by convolution product.

The dynamic response of the interface dofs are demonstrated in figures 5.12 and 5.13. As can be observed from these figures, the truncated IBS gives reliable and accurate approximation of the results of Newmark time integration. As a consequence, it can be concluded that the applied procedure in computing the vibrational part of the impulse response for the wind mill structure is correct. The MAC numbers of the results which are presented in table 5.2 also show the reliability of truncated IBS method.
CHAPTER 5. APPLYING IBS ON A 3D STRUCTURE

Figure 5.9: The pure rigid body response of the wind mill (the impact is modeled by applying a unit force at the initial time step)

Figure 5.10: The vibrational response of the wind mill computed by (5.2) and (5.9) (the impact is modeled by applying a unit force at the initial time step)
Figure 5.11: The vibrational response of the wind mill computed by (5.13) (the impact is modeled by applying a unit force at the initial time step)

Table 5.2: MAC numbers of the computed dynamic responses of the wind turbine structure obtained by truncated IBS (cosine window)

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>MAC number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>2</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>3</td>
<td>1.0000e+000</td>
</tr>
<tr>
<td>4</td>
<td>9.9998e-001</td>
</tr>
<tr>
<td>5</td>
<td>9.9899e-001</td>
</tr>
<tr>
<td>6</td>
<td>1.0000e+000</td>
</tr>
</tbody>
</table>
Figure 5.12: Dynamic response of the interface point under an excitation in the form of unit step on dof$_1$ obtained by truncated IBS (cosine window function with $\alpha = 10^{-3}$) and Newmark time integration.
5.4. RESULTS

Figure 5.13: Dynamic response of the interface point under an excitation in the form of unit step on dof, obtained by truncated IBS (cosine window function with $\alpha = 10^{-3}$) and Newmark time integration.
Chapter 6

Conclusion

6.1 Conclusion

This study was performed to develop the idea of introducing a substructuring method (IBS) which works in time domain [1]. The IBS method is suggested as an alternative solution which can be used, in the cases where Frequency Based Substructuring (FBS) is not capable to provide an efficient and reliable solution in order to obtain dynamic response of the systems. The existence and reliability of this method were studied in chapters 2, 3. Since the IBS method is working in the time domain, the obtained results by the IBS are compared with the results of applying numerical time integration (Newmark) on the total system.

The IBS method can be generally explained in 2 main steps; 1: Computing the time responses of the subsystems under a unit impact to generate the IRFs (computed by applying Newmark time integration), 2: Assembling procedure and computing the dynamic response of the total system by applying the convolution product between the IRFs and the applied forces. In this assignment, 3 different numerical models have been considered for the unit impact. These models are defined by different initial conditions and the IBS formulation has been derived for each of them differently in a way to discretize the applied force by the chosen impact. One of these models has been made by applying a unit force at the initial time step. The discretized formulation which is derived using this impact model provides the solution which is exactly equal to the Newmark time integration with constant average acceleration. The other 2 formulations of the IBS, which are described by modeling the impact with an initial applied velocity or applying a unit force at the second time step, are also presenting a reliable approximation of the final solution. The results of applying these 3 formulations of IBS in computing the dynamic response of a 1D bar example are presented at the end of chapter 3. (All the computations are done By MATLAB)

The formulations which are derived for the IBS method can be applied to compute
dynamic response of any linear system. However, this method is still time consuming, especially in the part of computing the convolution product when the response time is increased. In order to make the IBS method efficient, 2 solutions are suggested to reduce the computation costs.

The first one uses the effect of having damping in the system and suggests a solution which can reduce the computation costs of the IBS. In fact having damping in the system will make the vibrational part of the impulse response will be damped after a finite number of time steps. Hence, the IRFs can be cut where the value of the impulse response is decreased enough \((t = t_c)\). It should be noticed that, this truncation cannot be applied on the impulse responses of the floating substructures where the rigid body modes are also excited. In this case, truncation should be applied just on the vibrational part of the impulse response of the floating system and, the rigid body response must be computed analytically and added to the computed response at each time step. Applying truncation decreases the computation cost in both steps of the IBS; First in the step of computing IRFs, by the impulse responses must be obtained just for \(t = [0, t_c]\), afterwards in the assembly step, the convolution product will be computed for the less number of impulses.

Truncating the IRFs can be obtained applying different window functions on the impulse responses of the substructures. The results of applying truncated IBS with 3 different window functions are presented in the end of chapter 4. Applying cosine window function will cut the IRFs smoothly without adding a significant damping to the system and can make the best approximation of the final response. On the other hand applying the rectangular window function on the IRFs may make the computed response unstable in some cases. The other type of the studied window function is exponential window. Applying the exponential window function adds a significant damping to the system, therefore this type of window function is not recommended to truncate the impulse responses. The other suggested method to decrease the computation cost of the IBS is discretizing the external applied force with bigger size of the time step. This method which is called down-sampling, will also decrease the size of the calculations in the part of applying convolution product, however the part of computing the impulse responses is still time consuming and expensive.

Hence, between these 2 suggested methods to improve the IBS, truncating the IRFs (using cosine window function) decreases the computation costs more and gives better results. It should be noticed that truncation can be applied on all the 3 formulations of the IBS methods for different types of impact models.

Based on the observations of the results of these 2 improvements (truncation and down-sampling), both of them can provide better results for the periodic external forces with one eigenfrequency. However, by changing the applied force into the form of impact and step which are covering a large band of frequency, the reliability of the IBS with 2 improvements are decreased.

Finally in chapter 5, the IBS method is applied to analyze a 3D engineering structure
6.2. FUTURE WORK

(an offshore wind turbine which is made of 2 substructures: the jacket structure and the wind mill). In solving the problem with IBS, first the normal IBS with 2 different formulations for 2 different impact models (initial applied force and initial applied velocity) is used to compute the dynamic response of the total system. Based on the obtained results, the IRFs which are computed by modeling the impact with an initial applied velocity are not correct and the IBS formulation with these IRFs is unstable. This issue which was not appeared in the 1D bar example can be due to having a problem in the FEM model. Therefore, the example is solved with the IBS formulation based on the impact which is modeled by applying a unit force at the initial time step (IBS_{IF}). The results of the normal IBS are presented in chapter 5 and demonstrate that as expected based on the theory and also similar to the results of the 1D bar example, IBS_{IF} gives the result which is exactly equal to the Newmark time integration (with constant average acceleration).

In the next step, the truncated IBS is applied to compute the dynamic response of the system for a longer period of time. In chapter 5 2 different formulations are suggested to obtain the vibrational part of the impulse response. The first one is based on deducing the analytical rigid body response from the total response ((5.2)). The other one is based on projecting the vibrational response on a space which is M-orthogonal to the rigid body mode ((5.13)). The obtained results demonstrate that applying the first method will not give the proper solution and for this example equation (5.13) is used.

At the end the dynamic response of the total system under a unit step excitation which is applied at the interface point is computed by applying the truncated IBS (with cosine window function). The results which are presented at the end of chapter 5 demonstrate that the truncated IBS provides a reliable approximation of the dynamic response of the total system.

Although there are still some details which must be studied in the IBS method but the results of this thesis approve that the IBS as a method which implies the substructuring techniques and works in the time domain can be considered as an alternative solution to compute dynamic response of the linear systems where the FBS or numerical integration of the full system are meeting some difficulties to be applied.

6.2 Future work

This thesis provided the general theory of the IBS method and its application on real systems, However, there are still some issues which were not discussed during this work. Some of these issues are listed below and can be considered as the future work of this thesis.
• Applying truncated IBS on some other 3D structures\(^1\);

• Deriving IBS formulations for the IRFs which are obtained experimentally;

• Improving IBS/truncated IBS for systems with non-linear substructures\(^2\);

\(^1\)In order to check the problem which was appeared in chapter 5

\(^2\)Already has been discussed in [19]
Bibliography


[7] Rixen, D.J., Haghighat, N. : Truncating the impulse responses of substructures to speed up the Impulse-Based Substructuring. Delft University of Technology


Appendix A

Computing Pure rigid body response due to unit impact

The motion equation of a system which is showing pure rigid body mode is simplified to

\[ M\dddot{u} = f \]  

Therefore, the rigid body response of the structure under a unit impact, which is modeled by applying a unit force at the initial time step, can be obtained by (using Newmark time integration):

\[ t=0 \]

\[ u_0 = 0 \]
\[ \dot{u}_0 = 0 \]
\[ \dddot{u}_0 = M^{-1} \]

\[ t=dt \]

\[ \dddot{u}_0 = \frac{1}{4}dt^2M^{-1}1 \rightarrow u_0 = \frac{1}{3}dt^2M^{-1}1 \]
\[ \dot{u}_0 = \frac{1}{2}dtM^{-1}1 \rightarrow u_0 = \frac{1}{2}dtM^{-1}1 \]
\[ \dddot{u}_0 = 0 \]  

(A.3)

Since acceleration of the system is zero for all the other time steps, as a consequence the structure will have a constant velocity. Figure A.1 illustrates the pure rigid body response of the system under a unit impact (IF), by using the data at \( t = dt \).

As can be observed from figure A.1, the rigid body response under the unit impact starts at \( t = \frac{dt}{2} \). In other words, the pure rigid motion will be appeared with \( \frac{dt}{2} \). Similar to what was explained, modeling the impact by applying a unit force at the second
time step will produce a delay in the pure rigid body response. In this case the delay is equal to $dt$. This delay issue must be taken into account in the case of removing rigid body motion from the total response of a system.
Appendix B

Applying truncated IBS for different types of excitations
Figure B.1: Dynamic response of the 1D bar example (displacement at the interface node) under a periodic force obtained by the truncated IBS ($\alpha = 10^{-2}$)
Figure B.2: Dynamic response of the 1D bar example (displacement at the interface node) under a unit impact obtained by the truncated IBS ($\alpha = 10^{-2}$)
Appendix C

Matrix representations of the IRFs of the wind turbine example
Figure C.1: IRFs of the jacket structure (obtained by 2 impact models: initial applied force (blue), initial applied velocity (green))
Figure C.2: IRFs of the wind mill (obtained by 2 impact models: initial applied force (blue), initial applied velocity (green))
Figure C.3: IRFs of the jacket structure (the impact modeled by a unit initial applied force)
Figure C.4: IRFs of the wind mill (the impact modeled by a unit initial applied force)
Figure C.5: Rigid body impulse responses ($H^{rig}$) obtained by equation (5.9)
Figure C.6: Vibrational impulse response ($H_{\text{ vib}}^{\text{ab}}$) obtained by equations (5.2), (5.9)
Figure C.7: Vibrational impulse response ($H^{vi}(t)$) obtained by equation (5.13)