G_Q corrections in the circuit theory of quantum transport

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We develop a finite-element technique that allows one to evaluate correction of the order of G_Q to various transport characteristics of arbitrary nanostructures. Common examples of such corrections are the weak-localization effect on conductance and universal conductance fluctuations. Our approach, however, is not restricted to conductance only. It allows one in the same manner to evaluate corrections to the noise characteristics, superconducting properties, strongly nonequilibrium transport, and transmission distribution. To enable such functionality, we consider Green's functions of arbitrary matrix structure. We derive a finite-element technique from Cooperon and diffuson ladders for these Green's functions. The derivation is supplemented with application examples. Those include transitions between ensembles and the Aharonov-Bohm effect.

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I. INTRODUCTION

The theoretical predictions of weak localization¹ and universal conductance fluctuations² along with experimental discoveries in this direction^{3,4} have laid the basis of the modern understanding of quantum transport—transport in nanostructures—and have stimulated considerable interest in the topic. Early studies mostly concentrated on diffusive electron transport. Both effects arise from quantum interference, which is described in the language of slow modes: Cooperons and diffusons.^{5,6} Each mode of this kind brings a quantum (fluctuating) correction of the order of $G_Q \equiv e^2/\pi\hbar$ to the classical Drude conductance G of the sample. This universal value sets an important division between classical conductors ($G \gg G_Q$) where interference effects are small and quantum ones ($G \approx G_Q$) where the transport is essentially quantum.

A complementary approach to G_Q corrections comes from the random matrix theory (RMT) of scattering.^{7,8} This approach relates the statistical properties of the scattering matrix of a nanostructure to those of a certain ensemble of random matrices. G_Q corrections are understood in terms of the fluctuations and rigidity of the spectral density of these matrices. Although the RMT approach can deal with diffusive systems, the most comprehensive setup includes the so-called quantum cavity—an element whose scattering matrix is presented by a completely random unitary matrix of a certain ensemble. The cavity can be seen as a region of space where electron motion is sufficiently chaotic (either ballistic or diffusive) and where electrons can get in and out through some constrictions.⁹ The transport is determined by the propagation in the constrictions while the random unitary matrix representing the cavity is responsible for “randomization” of the scattering. The RMT approach does not necessarily concentrate on the total conductance. One can work with the transmission distribution: the averaged density of the eigenvalues of the transmission matrix squared. This transmission distribution appears to be useful in a much broader physical context: it determines not only the conductance of nanostructures, but also the noise, full counting statistics of charge transfers, and properties of the same nanostructure with superconducting leads attached.^{8,10} It is a

modern paradigm of quantum transport that an individual nanostructure is completely characterized by a set of transmission eigenvalues while the transmission distribution describes the averaged properties of random nanostructures of the same design. This makes it relevant to study G_Q corrections and fluctuations of the transmission distribution.^{14,15} The density of the transmission eigenvalues is of the order of G/G_Q , and G_Q corrections are of the order of $\delta G/G_Q \approx 1$.

The microscopic Cooperon-diffuson description is equivalent to a proper RMT approach. This is best illustrated in the framework of a more general supersymmetric theory¹¹ that allows for a nonperturbative treatment of fluctuations in quantum scattering. The Cooperons and diffusons in this theory are fluctuations of the supersymmetric field around the saddle point. For the quantum cavity, only a single mode of these fluctuations is relevant. The integration over these modes reproduces the RMT results.^{11–13}

One can describe nanostructures in the framework of a simple finite-element approach usually termed “circuit theory.” The circuit theory has originated from attempts to find simple solutions of the Usadel equations in superconducting heterostructures.¹⁶ However, it has been quickly understood that theories of the same structure can be useful in a much broader context: one can compute the transmission distribution,^{10,18} noise, and counting statistics¹⁷ and investigate spin effects¹⁹ and nonequilibrium phenomena.²⁰ In circuit theory, a nanostructure is presented in a language similar to that of electric circuits: It consists of nodes, reservoirs, and connectors. A node is in fact a quantum cavity; a connector can be of very different types—tunnel junction, ballistic contact, and diffusive wire—and is generally characterized by a set of transmission coefficients. In circuit theory, each node is described by a matrix related to the electron Green's function. In the limit $G \gg G_Q$ circuit theory provides a set of algebraic equations that allow one to express the matrices in the nodes in terms of fixed matrices in the reservoirs.

In this article, we present a technique to evaluate G_Q corrections for arbitrary nanostructures described by circuit theory. To allow for various applications, we consider G_Q corrections to multicomponent Green's functions of arbitrary matrix structure. We are able to present G_Q corrections in the form of a determinant made of derivatives of Green's func-

tions in the nodes with respect to self-energy parts. In terms of a supersymmetric σ model, this corresponds to an expansion of the action up to quadratic terms. However, the formulation we present does not contain any anticommuting variables that complicate applications of the σ models. The determinant is just that of a finite matrix; this facilitates the computation of G_Q corrections for nanostructures of complicated design.

The structure of the article is as follows. To make it self-contained, we start with a short outline of the circuit theory of multicomponent Green's functions adjusted for the purposes of further derivations. In Sec. III we derive microscopic expressions for G_Q corrections and specify to finite elements in Sec. IV. Section V is devoted to a description of spin-orbit scattering and magnetic-field decoherence, which are used to describe transitions between different RMT ensembles. Since the Aharonov-Bohm effect plays an important role in experimental observations of G_Q corrections, we explain how to incorporate it into our scheme in a short separate section VI. We illustrate the technique with several examples (Sec. VII), concentrating on a simplest 2×2 matrix structure suitable to calculate G_Q corrections to the transmission distribution. The examples also involve the simplest circuits: one with a single node and two arbitrary connectors, a chain of tunnel junctions, and two-node four-junction circuit to demonstrate the Aharonov-Bohm effect. We summarize in Sec. VIII.

II. FROM GREEN'S FUNCTIONS TO FINITE ELEMENTS

In this article, we consider Green's functions of arbitrary additional matrix structure with N_{ch} indices. We do this for the sake of generality: This allows for a description of superconductivity, incorporating the Keldysh formalism and treating nonequilibrium and time-dependent problems. This is also extremely convenient, since most relations in use do not depend on the “physical meaning” of the structure. We use a caret symbol for operators in coordinate space and an inverted caret symbol for matrices in additional indices. The Green's function thus reads $\hat{G} \equiv \check{G}(\mathbf{x}, \mathbf{x}')$, where $\mathbf{x}$ stands for the (three-dimensional) coordinate. The general Green's function is defined as the solution of the following equation:

$$[-\check{\Sigma}(\mathbf{x}) - \hat{H}] \check{G}(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'). \quad (1)$$

All physical quantities of interest can be in principle calculated from Green's functions. We address here the quantum transport of electrons in disordered media. In this case, one can work with the common Hamiltonian

$$\hat{H} = \hat{H}_0 + u(\mathbf{x}), \quad \hat{H}_0 = -\frac{1}{2m} \nabla_{\mathbf{x}}^2 + U(\mathbf{x}). \quad (2)$$

Here $U(\mathbf{x})$ describes the *design* of the nanostructure: potential “walls” that determine its shape and form ballistic quantum point contacts, potential barriers in tunnel junctions, etc. The potential $u(\mathbf{x})$ is random: it describes the random impurity potential responsible for the diffusive motion of electrons, isotropization of the electron distribution function, and, most importantly for this article, fluctuations of the

transport properties of the nanostructure. The physics at space scales exceeding the isotropization length (which is the mean free path in the case of diffusive transport) does not depend on a concrete model of randomness of this potential. The most convenient and widely used model assumes a normal distribution of $u(\mathbf{x})$ characterized by the correlator $\langle u(\mathbf{x})u(\mathbf{x}') \rangle = w\delta(\mathbf{x} - \mathbf{x}')$. It is important for us that both $\hat{H}_0$ and $u(\mathbf{x})$ are diagonal in check indices.

Most evident choice of the self-energy matrix is $\check{\Sigma}(\mathbf{x}) = -\epsilon$, ϵ being the energy parameter of the Green's function. $\check{\Sigma}$ is more complicated in the theory of superconductivity. We will find it convenient, at least for derivation purposes, to work with arbitrary $\check{\Sigma}(\mathbf{x})$. We can also consider the more general situation with nonlocal $\check{\Sigma}(\mathbf{x}, \mathbf{x}')$.

Provided the conductance of the nanostructure is sufficiently high ($G \gg G_Q$), one can disregard quantum G_Q corrections and work with semiclassical *averaged* Green's functions. Closed equations for those are obtained with the noncrossing approximation.²¹ They include the impurity self-energy

$$[-\check{\Sigma}(\mathbf{x}) - \check{\Sigma}_{imp} - \hat{H}_0] \check{G}(\epsilon; \mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'), \quad (3)$$

$$\check{\Sigma}_{imp}(\mathbf{x}) = w \check{G}(\mathbf{x}, \mathbf{x}). \quad (4)$$

At space scales exceeding the mean free path, one can write closed equations for the Green's function in the coinciding points.^{22,23} It is convenient to change notations, introducing dimensionless $\check{G}(\mathbf{x}) \rightarrow G(\mathbf{x}, \mathbf{x})/i\pi\nu$, with ν the density of states at the Fermi energy. For purely diffusive transport, one obtains the Usadel equations

$$\nabla \cdot \check{\mathbf{j}} - i\pi\nu[\check{\Sigma}(\mathbf{x}), \check{G}(\mathbf{x})] = 0, \quad \check{\mathbf{j}} = -\pi D(\mathbf{x}) \nu \check{G} \nabla \check{G}, \quad (5)$$

D being the diffusion coefficient. The solutions of the Usadel equations are defined only if one takes into account boundary conditions at “infinity:” equilibrium Green's functions in the macroscopic leads adjacent to the nanostructure. It turns out that $\check{G}$ satisfies unity condition $\check{G}^2 = \check{1}$. For situations where the transport is not entirely diffusive, one would supplement the Usadel equation with boundary conditions of various kinds (cf. Ref. 24).

An alternative way to proceed is to notice that the Usadel equation is almost a conservation law for the matrix current $\check{\mathbf{j}}$, which allows for a finite-element approach. This conservation law is exact at a space scale smaller than the coherence length estimated as $\sqrt{D/\Sigma}$. It does not rely on an assumption of diffusivity: It occurs because the Hamiltonian $\hat{H}$ commutes with the “check” structure. It is then natural to proceed to the separation of the nanostructure into regions where $\check{G}(\mathbf{x})$ can be assumed constant. The next steps are the same as in the traditional circuit theory of electric circuits, which exploits conservation of electric current and finite discretization of the system onto regions of approximately constant voltage—*nodes*. Each node is connected by *connectors* to other nodes or reservoirs, those representing macroscopic leads.

After the discretization of the nanostructure we can write a Kirchhoff-like equation

$$\sum_c \check{I}_c + \sum_\alpha \check{I}_{l,\alpha} = 0, \quad (6)$$

where the indices c and α label the nodes and connectors of the nanostructure. The current through a connector c which connects nodes $c1$ and $c2$ reads

$$\check{I}_c = \frac{1}{4} \sum_n \frac{T_n^c [\check{G}_{c1}, \check{G}_{c2}]}{1 + T_n^c (\check{G}_{c1} \check{G}_{c2} + \check{G}_{c2} \check{G}_{c1} - 2)/4}, \quad (7)$$

$\{T_n^c\}$ being the set of transmission eigenvalues of the transmission matrix squared relative to the connector. Equation (5) shows that the current $\check{J}$ is not fully conserved; to this aim we included in Eq. (6) a *leakage* current

$$\check{I}_{l,\alpha} = -\frac{i\pi}{\delta_\alpha} [\check{\Sigma}_\alpha, \check{G}_\alpha], \quad \delta_\alpha = \frac{1}{\nu V_\alpha}, \quad (8)$$

where V_α is the volume of the node; δ_α can be easily recognized as the average level spacing in the node.

It is opportune to notice that the conservation law (6) can be obtained by requiring that the values assumed by the matrix Green's function in the nodes be such so as to minimize the following action:

$$S = \sum_c S_c + \sum_\alpha S_\alpha, \quad (9)$$

$$S_c = \frac{1}{2} \sum_n \text{Tr} \ln \left[1 + \frac{T_n}{4} (\check{G}_{c1} \check{G}_{c2} + \check{G}_{c2} \check{G}_{c1} - 2) \right], \quad (10)$$

$$S_\alpha = -\frac{i\pi}{\delta_\alpha} \text{Tr} \check{\Sigma}_\alpha \check{G}_\alpha. \quad (11)$$

The minimization of the action must be carried out provided that the Green's functions in each node satisfy the normalization condition $\check{G}_\alpha^2 = 1$; this implies that the variation of the Green's function $\delta \check{G}_\alpha$ has to anticommute with $\check{G}_\alpha$ itself. A possible way to satisfy this condition is to write the variation as

$$\delta \check{G}_\alpha = \check{G}_\alpha \delta \check{v}_\alpha - \delta \check{v}_\alpha \check{G}_\alpha, \quad (12)$$

where no restriction is imposed on $\delta \check{v}_\alpha$. The Kirchhoff equations are then rewritten as

$$\frac{\delta S}{\delta \check{v}_\alpha} = \left(\check{G}_\alpha, \frac{\delta S}{\delta \check{G}_\alpha} \right) = 0. \quad (13)$$

The same action can be used at the microscopic level too, even before averaging over $u(x)$. To do this, we note once again that $\check{\Sigma}(x)$ form, at least formally, a set of parameters of our model. This can be straightforwardly extended to nonlocal operators $\check{\Sigma}(x, x')$. To this end, we define the action in terms of the following variational formula:

$$\delta S = - \int dx dx' \text{Tr} [\check{G}(x', x) \delta \check{\Sigma}(x, x')], \quad (14)$$

which is traditional in Green's function applications.^{21,25} With making use of formal operator traces, this action can be written in terms of either $\check{\Sigma}$ or $\hat{G}$,

$$S = \text{Tr} \ln (\hat{\Sigma} + \hat{H}), \quad (15)$$

$$S = \text{Tr} [-\ln(-\hat{G}) - (\hat{\Sigma} + \hat{H})\hat{G}], \quad (16)$$

apart from a constant. Equation (1) is reproduced by varying either Eq. (15) or (16) over $\check{\Sigma}$ or $\hat{G}$, respectively. Since we are not planning to treat G_Q corrections beyond perturbation theory, we do not attempt to use the exponent of this action as an integrand in some path integral representation of a result of exact averaging over $u(x)$. This has been done in Ref. 11 for supersymmetric matrices $\check{Q}$ and in Ref. 26 for Keldysh-based two-energy matrices $\check{Q}(x, \epsilon, \epsilon')$. We note, however, that the action used in Refs. 11 and 26 is equivalent to Eq. (9) if one substitutes $\check{G} = \check{Q}$.

To this end, for averaged Green's functions the action in use is defined simply as

$$\delta S = - \int dx dx' \text{Tr} [\langle \check{G}(x', x) \rangle \delta \check{\Sigma}(x, x')]. \quad (17)$$

III. G_Q CORRECTIONS TO MULTICOMPONENT GREEN'S FUNCTIONS

In this section, we outline a microscopic approach to G_Q corrections suitable for multicomponent Green's functions. The main idea is the same as in Refs. 15 and 27 where a similar derivation has been done for the purely diffusive case and for a concrete 2×2 matrix structure. It is known^{5,6} that the diffuson and Cooperon modes responsible for G_Q corrections are presented as ladder diagrams made from averaged Green's functions. In the usual technique, such diagrams have vertices corresponding to two current operators in the Kubo formula. One does not have to work with vertices: Instead, one considers Cooperon and diffuson contributions to the action that do not have any. The part of the action that presents G_Q corrections is given by wheel diagrams. These wheels are made of either Cooperon or diffuson ladder sections in a straightforward way (see Fig. 1)

We argue that the optimal way to present this part of the action is to *double* the existing check structure and to consider Eq. (1) in a so-extended setup. Indeed, suppose we would like to address the most general application of G_Q corrections: parametric correlations.²⁸ In this case, we start with two different “worlds” corresponding to two different sets of parameters. In our case, all parameters can be in principle incorporated into $\check{\Sigma}(x)$. So we have the “white” and “black” sets $\check{\Sigma}_b$ and Σ_w ; those define the corresponding non-averaged Green's functions $\check{G}_b$ and G_w . The averaging over the random potential $u(x)$ provides correlations between the

FIG. 1. Wheel diagrams that determine G_Q corrections to the action. A single (double) line represents the Green's functions from the *black* (*white*) block while dashed lines represent the averaging over disorder. The diffuson (left) and Cooperon (right) wheels differ by mutual orientation of the lines.

two worlds and gives the rungs of the ladder diagrams that involve white and black Green's functions (Fig. 1). This is to obtain the diffuson ladder wheel. The Cooperon ladder wheel is obtained by inverting the direction of the Green's function in one of the worlds. This corresponds to using *transposed* self-energies for this world, $\check{\Sigma}_w \rightarrow (\check{\Sigma}_w)^T$. This guarantees that the corresponding Green's functions are also transposed. Fluctuations at the same values of the parameters are naturally given by diagrams where $\check{\Sigma}_b$ and $\check{\Sigma}_w$ either are the same in both worlds (diffusons) or mutually transposed (Cooperons).

Finally, we note that the doubled structure is also useful for evaluation of the weak-localization correction. In this case, the last section of the Cooperon ladder is twisted before closing the wheel.

To proceed further, let us introduce the operators $\hat{K}$ presenting a section of a corresponding ladder,

$$\hat{K}_{\text{diff}}^{ab,cd}(\mathbf{x}, \mathbf{x}') = w(\mathbf{x}) \langle G_b^{ac}(\mathbf{x}, \mathbf{x}') \rangle \langle G_w^{db}(\mathbf{x}, \mathbf{x}') \rangle, \quad (18)$$

$$\hat{K}_{\text{Cooper}}^{ab,cd}(\mathbf{x}, \mathbf{x}') = w(\mathbf{x}) \langle G_b^{ac}(\mathbf{x}, \mathbf{x}') \rangle \langle G_w^{bd}(\mathbf{x}, \mathbf{x}') \rangle, \quad (19)$$

where Latin letters represent check indices. As we have noted, the white Green's function is transposed for the Cooperon. Those are operators in the space spanned by the coordinates and the two check indices.

Summing up all diagrams, we find the formal operator expressions for contributions to the action. For fluctuations, we have

$$S_{G_Q} = \text{Tr}[\ln(1 - \hat{K}_{\text{Cooper}})] + \text{Tr}[\ln(1 - \hat{K}_{\text{diff}})], \quad (20)$$

where the diffuson and Cooperon contributions are given by $\hat{K}_{\text{diff}}$ and $\hat{K}_{\text{Cooper}}$ respectively. For the weak-localization correction, one has to account for the fact that the last ladder section is twisted. We do this by introducing the permutation operator $\hat{P}$, which exchange check indices,

$$\hat{P} \hat{K}^{ab,cd} = \hat{K}^{ba,cd}. \quad (21)$$

The contribution to the action becomes

$$S_{wl} = \frac{1}{2} \text{Tr}[\hat{P} \ln(1 - \hat{K}_{\text{Cooper}})]. \quad (22)$$

The factor of $\frac{1}{2}$ is included in the last formula to take into account that black and white Green's functions, in the case of

FIG. 2. The response of the Green's function $\delta \check{G}_{bw}$ on the self-energy $\check{\Sigma}_{bw}$ in the semiclassical noncrossing approximation is determined by the sum of ladder diagrams. This allows one to express the wheel diagrams in terms of eigenvalues of response kernels.

weak localization, are not anymore independent. We note that $\hat{K}$ for the Cooperon is symmetric with respect to index exchange, so that $\hat{K}$ and $\hat{P}$ commute. The eigenfunctions and eigenvalues of K are therefore either symmetric (K^+) or antisymmetric (K^-) with respect to permutations. We can rewrite the last expression as a sum over these eigenvalues:

$$S_{wl} = \frac{1}{2} \sum_n [\ln(1 - K_n^+) - \ln(1 - K_n^-)]. \quad (23)$$

It is clear from the previous discussion that, in order to calculate the G_Q corrections, one has to evaluate the eigenvalues of the ladder section $\check{K}$, both for Cooperons and diffusons. We introduce now a method to compute this matrix easily. The observation is that the ladders under consideration are not specific for G_Q corrections: The same ladders determine the response of *semiclassical* Green's functions upon variation of $\check{\Sigma}$.²⁹

To see this, let us go back to nonaveraged Green's functions. We keep in mind that we have doubled the check space to include *white* and *black* sectors. We add by hand a source term: the self-energy which mixes up black and white Green's functions, $\delta \check{\Sigma}_{bw}(\mathbf{x})$. This source term will give rise to a correction to the Green's function in the same black-white sector. In the first order, we have

$$\delta \check{G}_{bw}(\mathbf{x}, \mathbf{x}') = - \int d\mathbf{x}_1 d\mathbf{x}_2 \check{G}_b(\mathbf{x}, \mathbf{x}_1) \delta \check{\Sigma}_{bw}(\mathbf{x}_1, \mathbf{x}_2) \check{G}_w(\mathbf{x}_1, \mathbf{x}'), \quad (24)$$

which is best illustrated by the diagram in Fig. 2. The next step is to include the effect of the random potential $u(\mathbf{x})$. We average Eq. (24), limiting ourselves to the noncrossing approximation and obtaining a set of ladder diagrams (Fig. 2.) By summing up all the contributions we obtain the correction—taken in coinciding points—to the Green's function:

$$\langle \delta \check{G}_{bw}(\mathbf{x}, \mathbf{x}) \rangle = \frac{1}{w(\mathbf{x})} \frac{\hat{K}_{\text{diff}}}{1 - \hat{K}_{\text{diff}}} \delta \check{\Sigma}_{bw}(\mathbf{x}). \quad (25)$$

Equation (25) is very valuable: it demonstrates that the response of the Green's function to the source term $\delta \check{\Sigma}_{bw}$ is determined by the same ladder operator $\hat{K}$, which we need to compute G_Q corrections.

At the space scale of isotropization length, $\hat{K} \sim 1$. Usually one is interested in the contribution arising from the larger space scale where the Cooperon-diffuson approximation is

valid. At this scale, the eigenvalues of $\hat{K}$ are either 0 or very close to 1. To see this, we cite the results for the homogeneous case with $\check{\Sigma} = \text{const} \times (\mathbf{x})$. A convenient basis in check space is one where $\check{\Sigma}$ is diagonal, the eigenvalues being Σ_n . The Green's function is diagonal in this basis as well, $G_n = s_a \equiv \text{sgn Im}(\Sigma_a)$. Owing to homogeneity, the section operator is diagonal in the wave-vector representation, its eigenvalues being $K^{nm}(\mathbf{q})$. A direct calculation similar to that in Ref. 6 gives $K^{nm}(\mathbf{q}) = 0$ if $s_n = s_m$. If $s_n \neq s_m$,

$$1 - K^{ab}(\mathbf{q}) \approx \tau(is_b(\Sigma_a - \Sigma_b) + Dq^2) + \dots \quad (26)$$

for $\Sigma\tau, ql \ll 1$ ($\tau = 2\pi\omega\nu$ is the isotropization time). This equation makes the relation between our technique and the common technique for Cooperons and diffusons in homogeneous media. Usually, the self-energy Σ has an equal number of eigenvalues with positive and negative imaginary parts. In this case, at each $\mathbf{q}$, $\hat{K}$ has $N_{ch}^2/2$ zero and $N_{ch}^2/2$ nonzero eigenvalues.

Now we note that the zero eigenvalues contribute neither to Eq. (20) nor to Eq. (25). As to those close to 1, we may replace $\hat{K}$ by 1 in the numerator of Eq. (25). We also note that $\delta\check{G}$ can be presented as the derivative of the action [cf. Eq. (14)]. Therefore, we can write the G_Q corrections due to the diffuson modes to the action in terms of a determinant made of derivatives of the semiclassical action with respect to $\check{\Sigma}_{bw}$ and $\check{\Sigma}_{wb}$:

$$\mathcal{S}_{G_Q, \text{diff}} = -\ln \det' \left(-w(\mathbf{x}) \frac{\delta^2 \mathcal{S}}{\delta \check{\Sigma}_{wb} \delta \check{\Sigma}_{bw}} \right). \quad (27)$$

The prime on the determinant signals that the zero eigenvalues shall be excluded: $\det'$ is defined as the product of all nonzero eigenvalues. Indeed, as we have seen, some variations of self-energies do not change the Green's functions, giving rise to zero eigenvalues. We also note that the G_Q corrections are not affected by the concrete form of $w(\mathbf{x})$: Since the determinant of the matrix product is a product of their determinants, this matrix gives a constant contribution to the action which does not affect any physical quantities.

IV. METHOD

Let us now adapt the microscopic relation (27) to the finite-element approach outlined in the Sec. II.

With all previous derivations, this step is easy. We just replace the actual $\mathbf{x}$ -dependent $\check{G}$ and $\check{\Sigma}$ by constants in each node. To get the action in these terms, one integrates over the volume of each node so that the formula (14) reads

$$\delta \mathcal{S} = - \sum_{\alpha} \frac{i\pi}{\delta_{\alpha}} \text{Tr}[\check{G}_{\alpha} \delta \check{\Sigma}_{\alpha}], \quad (28)$$

where the summation is over the nodes. The discrete analog of the determinant relation (27) is now

$$\begin{aligned} \mathcal{S}_{G_Q, \text{diff}} = & -\ln \det' \left(-\frac{w_{\alpha}}{V_{\alpha}} \frac{\delta^2 \mathcal{S}}{\delta \check{\Sigma}_{wb} \delta \check{\Sigma}_{bw}} \right) \\ & -\ln \det' \left(\frac{\pi}{2\tau_{\alpha} \delta_{\alpha}} \frac{\delta^2 \mathcal{S}}{\delta \eta_{wb} \delta \eta_{bw}} \right) = \text{const} \\ & -\ln \det' \left(\frac{\delta^2 \mathcal{S}}{\delta \check{\eta}_{wb} \delta \check{\eta}_{bw}} \right) \end{aligned} \quad (29)$$

where we have introduced the dimensionless response matrix $\check{\eta}_{\alpha} \equiv i\pi \check{\Sigma}_{\alpha} / \delta_{\alpha}$ and noticed that the matrix $\propto w$ brings a constant contribution to the action. The response matrix is determined from the solution of the Kirchhoff equations at vanishing source term $\check{\eta}_{\alpha}^{bw}$. It has $N_{nodes} \times N_{ch}^2/2$ nonzero eigenvalues and the same number of zero ones. We observe that at $\Sigma_{w,b} = 0$ the eigenvalues of this matrix do not depend on the volume of the nodes; they are determined by the transmission eigenvalues of the connectors only and are of the order of G/G_Q . Since rescaling of all conductances gives only an irrelevant constant contribution to the action, the G_Q corrections depend only on the ratios of conductances of the connectors: This manifests the universality of these corrections.

The circuit theory action (9) is given in terms of $\check{G}_{\alpha}$. It is advantageous to present the answer for $\mathcal{S}_{G_Q}$ in terms of the expansion coefficients of the action around the saddle point: the solution of semiclassical circuit-theory equations—that is, to use $\delta^2 \mathcal{S} / \delta \check{G}_{wb} \delta \check{G}_{bw}$ instead $\delta^2 \mathcal{S} / \delta \check{\eta}_{wb} \delta \check{\eta}_{bw}$. If the latter matrix were invertible, we would make use of the fact that $\delta^2 \mathcal{S} / \delta \check{G}_{wb} \delta \check{G}_{bw} = (\delta^2 \mathcal{S} / \delta \check{\eta}_{wb} \delta \check{\eta}_{bw})^{-1}$. In fact, owing to the constraint $\check{G}^2 = 1$, there is a large number of zero eigenvalues in the response matrix. So the task in hand is not completely trivial.

We proceed as follows. We expand the action by replacing each $\check{G}$ in each node by

$$\check{G} = \check{G}_0 + \check{g} - \check{G}_0 \check{g}^2 / 2 + \dots \quad (30)$$

and collecting terms of the second order in $\check{g}, \check{\eta}$. The form (30) satisfies the constraint $\check{G}^2 = 1$ up to second-order terms provided $\check{g} \check{G}_0 + \check{G}_0 \check{g} = 0$. Let us work in the $(N_{ch}^2 N_{node})$ -dimensional space indexed with the barred index $\bar{a}$ composed of two check indices and one node index, $\bar{a} \equiv (a, b, \alpha)$. We present the result of the expansion as

$$\delta \mathcal{S} = g_{\bar{a}}^{wb} M_{\bar{a}\bar{b}}^{bw} - \eta_{\bar{a}} g_{\bar{a}}^{bw}. \quad (31)$$

The variation of Eq. (31) under the constraint $\check{g} \check{G}_0 + \check{G}_0 \check{g} = 0$ gives the response matrix $\delta^2 \mathcal{S} / \delta \eta_{\bar{a}} \delta \eta_{\bar{a}}$. Next we consider the matrix $\Pi_{\bar{a}\bar{b}}$ defined through the following relation:

$$\Pi_{\bar{a}\bar{b}}^{bw} \rightarrow \frac{1}{2} (\check{g} - \check{G}_0 \check{g} \check{G}_0) \rightarrow \frac{1}{2} (\check{g}^{bw} - \check{G}_b \check{g}^{bw} \check{G}_w), \quad (32)$$

the last equation making the white-black block separation explicit. We note that $\Pi_{\bar{a}\bar{b}}$ is a *projector*: It separates the barred space on two subspaces, where $\check{g}$ either commutes or anticommutes with $\check{G}_0$, and projects an arbitrary $\check{g}$ onto an-

ticommuting subspace. Applying this projector to Eq. (31), we show that the projected matrix $\Pi_{\bar{a}\bar{b}}\bar{M}_{\bar{b}\bar{c}}\Pi_{\bar{c}\bar{b}}$ is an inverse of the response matrix *within* the anticommuting subspace:

$$\mathcal{S}_{G_Q} = \ln \det' [\Pi_{\bar{a}\bar{b}}\bar{M}_{\bar{b}\bar{c}}\Pi_{\bar{c}\bar{b}}] = \ln \det(\Pi_{\bar{a}\bar{b}}\bar{M}_{\bar{b}\bar{c}}\Pi_{\bar{c}\bar{b}} + \delta_{\bar{a}\bar{b}} - \Pi_{\bar{a}\bar{b}}). \quad (33)$$

In the last equation we add the matrix $1 - \hat{\Pi}$. This procedure replaces all zero eigenvalues with 1, so one can evaluate a usual determinant.

We recall that as far as fluctuations are concerned, there are two contributions of this kind coming from diffuson and Cooperon ladders, respectively. The weak-localization correction involves a permutation operator that sorts out eigenvalues involved according to Eq. (23). With this, Eqs. (33) and (29) give the G_Q corrections in an arbitrary circuit-theory setup in the most general form.

V. DECOHERENCE AND ENSEMBLES

Until now we have assumed that the Hamiltonian commutes with the check structure and is invariant with respect to time reversal. This implies strict coherence of waves with a different check index which propagates in the disordered media described by this Hamiltonian. Even small check-dependent perturbations of the symmetric Hamiltonian give accumulating phase shifts to these waves and may significantly change their interference patterns at long distances. Due to their random nature, such phase shifts can be regarded as decoherence although this should not be confused with a real decoherence coming from interaction-driven inelastic processes.³⁰

In real experimental situations, two sources of such decoherence are usually of importance: spin-orbit scattering and magnetic fields. Already early studies of G_Q corrections^{1,4} have revealed their significant dependence on these two factors in the regime where those are too weak to affect the semiclassical transport. From the RMT point of view, these factors, upon increasing their strength, provide transitions from orthogonal ensembles (symmetric Hamiltonian) to two different ensembles: symplectic (spin-orbit interaction) and unitary (magnetic field).¹¹

We show in this section how to incorporate spin-orbit scattering and magnetic fields into our scheme. The most convenient way is to present them as perturbative corrections to the $\check{G}$ -dependent action (Fig. 3).

The spin-orbit scattering enters the Hamiltonian in the form $\hat{H}_{so} = \check{\sigma}_a H^a(\mathbf{x}, \mathbf{x}')$, $H^a(\mathbf{x}, \mathbf{x}') = -H^a(\mathbf{x}', \mathbf{x})$, $\check{\sigma}_a$ representing spin Pauli matrices in check space. In the second order in H^a the averaging gives (Fig. 3)

$$\mathcal{S}_{so} = \int d\mathbf{x} \frac{\pi\nu}{8\tau_{so}(\mathbf{x})} \text{Tr}[\check{G}(\mathbf{x})\sigma_a\check{G}(\mathbf{x})\sigma_a]. \quad (34)$$

At the level of the microscopic approach, the spin-orbit scattering takes place anywhere in the nanostructure. In the finite-element approach, it is advantageous to ascribe spin-orbit scattering to nodes rather than to connectors. This is consistent with the main idea of our scheme: Random phase

FIG. 3. Second-order diagrams in a spin-orbit interaction (left) and magnetic field (right) provide decoherence terms in the action. These terms describe transitions between pure RMT ensembles.

shifts take place in the nodes. The spin-orbit contribution in each node α is obtained by integrating Eq. (34) over the node,

$$\mathcal{S}_{so} = \frac{\eta_{so}}{4} \text{Tr}[\check{G}_\alpha \check{\sigma}_a \check{G}_\alpha \check{\sigma}_a], \quad (35)$$

where $\eta_{so} \equiv \pi/(2\tau_\alpha^{(so)}\delta_\alpha)$.

The magnetic field is incorporated into the Hamiltonian through modification of the derivative,

$$i\nabla \rightarrow i\nabla - \frac{e}{c\hbar} \check{\Sigma}_H \mathbf{A},$$

where $\mathbf{A}$ is the vector potential and $\check{\Sigma}_H$ ($\check{\Sigma}_H^2 = 1$) describes the interaction of different check waves with the magnetic field. In its simplest form, $\check{\Sigma}_H$ is the matrix in the white-black structure introduced such that $\check{\Sigma}_H^b = 1$, $\check{\Sigma}_H^w = -1$ provided we describe a Cooperon. This is consistent with the requirement that one of the Hamiltonians must be transposed to describe a Cooperon ladder. This is not the only plausible form of this matrix. For instance, in nonequilibrium superconductivity $\check{\Sigma}_H$ involves electron-hole Nambu structure.

The magnetic-field decoherence contribution can also be assigned to a node and reads

$$\mathcal{S}_H = \frac{\eta_H}{2} \text{Tr}[\check{G}_\alpha \check{\Sigma}_H \check{G}_\alpha \check{\Sigma}_H], \quad (36)$$

where $\eta_H = \pi/(\tau_H \delta_\alpha)$ and τ_H corresponds to Cooperon magnetic decoherence time in a common theory. The latter is known to depend on the geometry of the node and its characteristics.³¹ If the transport within the node is diffusive, $1/\tau_H = 4(e/\hbar)^2 D \langle \mathbf{A}^2 \rangle$, where $\langle \cdots \rangle$ denotes averaging over the volume of the node. The vector potential is taken in the gauge where it is orthogonal to the boundaries of the node. We have an order of magnitude, $1/\tau_H \delta \approx (\Phi/\Phi_0)^2 (G_{node}/G_Q)$, Φ being the magnetic flux through the node, $\Phi_0 \equiv \pi\hbar/e$ being the flux quantum, and G_{node} being a typical conductance of the node. The latter is limited by its Sharvin value in the ballistic regime where the isotropization length is of the order of the node size.

The magnetic field produces not only random but also deterministic phase shifts. This gives rise to the Aharonov-Bohm effect discussed in the next section.

To find the effect of decoherence terms (35) and (36) on the eigenvalues forming the localization correction, we expand the action as done to obtain Eq. (31). The decoherence contribution to $\hat{M}$ is diagonal in the node index and can be made diagonal in a barred index by the proper choice of basis in check space. For instance, if no external spin polarization is present in the structure, the spin-orbit contribution is diagonal in the basis made of singlets and triplets in spin space. The simple realization of Σ_H mentioned is automatically diagonal. If in addition this diagonal contribution is the same in all nodes, both decoherence effects just shift the eigenvalues of $\hat{M}$ corresponding to the symmetric Hamiltonian. This gives an extremely convenient model of decoherence effects.

The action for fluctuations is modified as follows:

$$\mathcal{S}_{\text{diff}} = \sum_n \ln(M_n) + 3 \ln(M_n + \eta_{so}), \quad (37)$$

$$\mathcal{S}_{\text{Cooper}} = \sum_n \ln(M_n + \eta_H) + 3 \ln(M_n + \eta_{so} + \eta_H), \quad (38)$$

where the summation goes over nondegenerate eigenvalues of $\hat{M}$ and the factor of 3 comes from the threefold degeneracy of the triplet. To derive the modification for the weak-localization contribution, we note that singlets and triplets are, respectively, antisymmetric and symmetric with respect to permutations. Therefore, triplet extensions of symmetric and antisymmetric eigenvalues are, respectively, symmetric and antisymmetric. The weak-localization correction thus reads

$$\mathcal{S}^{wl} = \frac{1}{2} \sum_n \ln \frac{M_n^- + \eta_H}{M_n^+ + \eta_H} + \frac{3}{2} \ln \frac{M_n^+ + \eta_{so} + \eta_H}{M_n^- + \eta_{so} + \eta_H}, \quad (39)$$

$M^{+(-)}$ being (anti)symmetric eigenvalues of $\hat{M}$.

Since eigenvalues of $\hat{M}$ are of the order of G/G_Q , the decoherence effects become important at $\eta_{so}, \eta_H \approx G/G_Q$ —that is, when inverse decoherence times match the Thouless energy $E_{th} = (G/G_Q)\delta$ of the node, $1/\tau_{so}, 1/\tau_H \approx E_{th}$.

VI. AHARONOV-BOHM EFFECT

The Aharonov-Bohm (AB) effect plays a crucial role in experimental observations and identification of G_Q corrections (see, e.g., Ref. 32). Therefore it must be incorporated into our scheme, and in this section we explain how to do this. This extends the results of Ref. 33 where the AB effect was considered in superconducting circuit theory. In the following, we do not consider any orbital effects of the magnetic field but just the topological one.

Let us suppose that the nanostructure presents a closed ring threaded by a magnetic flux Φ . As explained above, in the presence of a magnetic field the momentum operator has to be modified according to

$$i\nabla \rightarrow i\nabla - e/\hbar \tilde{\Sigma}_H \mathbf{A},$$

where $\mathbf{A}$ is the vector potential. Neglecting orbital effects, one can get rid of the vector potential in the Schrödinger

FIG. 4. Ahronov-Bohm effect: the topological phase arising from the magnetic flux Φ is incorporated into the boundary conditions that relate the wave functions on two sides of an imaginary cut. The cut can be made in any place.

equation by a gauge transformation. Let us have an ideal cut in the nanostructure that breaks the loop (Fig. 4). The topological effect of the flux can be incorporated into a boundary condition for the wave function $\psi_{L,R}$ on two sides of the cut, $\psi_L = \exp(i\phi_{AB}\tilde{\Sigma}_H)\psi_R$. The phase of the wave function therefore presents a discontinuity at the cut that is equal to $\pm\phi_{AB}$, $\phi_{AB} = \pi\Phi/\Phi_0$. Since the transformation does not explicitly depend on $\mathbf{x}$, it can be immediately extended to semiclassical Green's functions, so that those functions at two sides are related by

$$\check{G}_L = \exp(i\phi_{AB}\tilde{\Sigma}_H)\check{G}_R \exp(-i\phi_{AB}\tilde{\Sigma}_H). \quad (40)$$

This solves the problem at the microscopic level. Once the nanostructure has been discretized to finite elements, we note that the cut always occurs *between* a connector and a node. The most convenient way to deal with the gauge transformation (40) is to put it into the action of the corresponding connector. To do this, we observe that the Green's function at the right end of the connector is not $\check{G}_R$ of the node anymore: since the cut is crossed, it is eventually $\check{G}_L$ given by Eq. (40). The connector action in the presence of flux is therefore

$$\mathcal{S}_c = \frac{1}{2} \sum_n \text{Tr} \ln \left[1 + \frac{T_n}{4} (\check{G}_{c1}\check{G}_{c2}(\phi_{AB}) + \check{G}_{c2}(\phi_{AB})\check{G}_{c1} - 2) \right], \quad (41)$$

where

$$\check{G}(\phi_{AB}) = \exp(i\phi_{AB}\tilde{\Sigma}_H)\check{G} \exp(-i\phi_{AB}\tilde{\Sigma}_H). \quad (42)$$

One checks that the variation of the so-modified action reproduces the Kirchhoff laws for matrix current given in Ref. 33. Owing to global gauge invariance, it does not matter to which connector and to which end of the connector the Aharonov-Bohm phase is ascribed. If there are more loops in the nanostructure, more connector actions have to be modified in such a way.

VII. EXAMPLES

In the previous sections, we operated with general matrix check structure to keep the discussion as general as possible.

In this section, we will give a set of examples to illustrate concrete applications of the technique developed. For the sake of simplicity, we choose the simplest matrix structure that gives a sensible circuit theory. We consider 2×2 matrix Green's functions whose values in two terminals can be parametrized by a single parameter ϕ ,

$$\check{G}(\phi) = \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}. \quad (43)$$

The use of these matrix structures is that they give access to a fundamental quantity in quantum transport: the transmission distribution of transmission eigenvalues of two-terminal nanostructure. We have to explain this relation before going on to concrete examples. The averaged transmission distribution is defined as

$$\rho(T) = \sum_n \langle \delta(T - T_n) \rangle, \quad (44)$$

where the sum is done over all the transport channels and the average is, in principle, to be intended over an ensemble of nanostructures of the same design. For $G \gg G_Q$ the transmission eigenvalues are dense in the interval $[0,1]$ and self-averaging takes place.

Let us take a connector and set the Green's functions at its ends to $\check{G}(\phi_1)$ and $\check{G}(\phi_2)$. By virtue of Eq. (10) the connector action reads

$$\mathcal{S}(\phi_1 - \phi_2) = \sum_n \ln \left[1 - T_n \sin^2 \frac{\phi_1 - \phi_2}{2} \right]. \quad (45)$$

The trick is to regard the whole nanostructure as a single complex connector between left and right reservoirs and set the Green's functions in the reservoirs to $\check{G}(0)$ and $\check{G}(\phi)$. The total action (9) becomes now the connector action of the whole nanostructure and defines the transmission distribution in question,

$$\mathcal{S}(\phi) = \int dT \rho(T) \ln \left[1 - T \sin^2 \frac{\phi}{2} \right]. \quad (46)$$

If one computes the ϕ dependence of $\mathcal{S}$, the transmission distribution can be *extracted* from its analytic continuation on complex ϕ (Ref. 10):

$$\rho(T) = - \frac{1}{\pi T \sqrt{1-T}} \operatorname{Re} \left[\frac{\partial \mathcal{S}}{\partial \phi} \left(\pi + 2i \cosh^{-1} \frac{1}{\sqrt{T}} - 0^+ \right) \right]. \quad (47)$$

The circuit theory of Sec. II gives the answer in the limit $G \gg G_Q$. The weak-localization contribution $\mathcal{S}_{wl}$ gives the G_Q correction to the transmission distribution. The fluctuation contribution $\mathcal{S}_{G_Q} = \mathcal{S}_{\text{diff}} + \mathcal{S}_{\text{Cooper}}$ that depends on two parameters $\phi_{w,b}$ gives *correlations* of transmission distributions (cf. Ref. 15):

FIG. 5. The chain of tunnel junctions of the same conductance G_T . In our finite-element approach, there are two Cooperon and two diffusion eigenvalues per node. In the limit of $N \rightarrow \infty$ the chain is shown to be identical to a continuous quasi-one-dimensional diffusive wire.

$$\mathcal{S}_{G_Q}(\phi_b, \phi_w) = \int dT dT' \langle \langle \rho(T) \rho(T') \rangle \rangle \ln \left[1 - T \sin^2 \frac{\phi_b}{2} \right] \times \ln \left[1 - T' \sin^2 \frac{\phi_w}{2} \right]. \quad (48)$$

A simple application of the above formulas is the G_Q corrections to the conductance. Those are given by the derivatives of corresponding actions at $\phi_{b,w}=0$,

$$\frac{\delta G_{wl}}{G_Q} = -2 \left. \frac{\partial^2 \mathcal{S}_{wl}(\phi)}{\partial \phi^2} \right|_{\phi=0}, \quad (49)$$

$$\frac{\langle \langle G_b G_w \rangle \rangle}{G_Q^2} = 4 \left. \frac{\partial^4 \mathcal{S}_{G_Q}(\phi_b, \phi_w)}{\partial \phi_b^2 \partial \phi_w^2} \right|_{\phi_b, \phi_w=0}. \quad (50)$$

A. Junction chain

The first example is a chain of tunnel junctions. We will study G_Q corrections for a chain of N identical junctions that connects two reservoirs (Fig. 5). The connector action for a tunnel junction assumes a very simple form $\mathcal{S}_c = (G_T/4G_Q) \operatorname{Tr}(\check{G}_{1c} \check{G}_{2c})$, G_T being the conductance of the tunnel junction. Several tunnel junctions in series, however, provide a good approximate for diffusive wire. Therefore, in the limit $N \rightarrow \infty$ we can compare G_Q corrections with the known results^{14,15} for corrections to the transmission distribution of a one-dimensional diffusive conductor.

We set the Green's functions in the reservoirs on the left and on the right to $\check{G}(0)$ and $\check{G}(\phi)$, [cf. Eq. (43)], respectively. The semiclassical action for the system reads

$$\mathcal{S} = \frac{G_T}{4G_Q} \sum_{k=0}^{N-1} \operatorname{Tr} \check{G}_k \check{G}_{k+1} - \frac{i\pi}{\delta_S} \sum_{k=1}^{N-1} \operatorname{Tr} \check{\Sigma}_k \check{G}_k. \quad (51)$$

Here $k=1, \dots, N-1$ labels the nodes, while $k=0$ and $k=N$ identify left and right reservoirs, respectively. All the nodes are assumed to be identical with the same δ_S .

Since all the junctions are identical, the semiclassical solution is easy to find: the “phase” ϕ drops by the same amount at each junction, and the solution reads $\check{G}_k = \check{G}(k\phi/N)$, provided $\check{\Sigma}_k \propto \check{1}$ within each 2×2 block. This gives the optimal value of the action,

$$S = \frac{NG_T}{2G_Q} \cos\left(\frac{\phi}{N}\right), \quad (52)$$

from which one can evaluate the semiclassical transmission distribution by using relation (47). In the limit $N \rightarrow \infty$, $S = -(G_D/4G_Q)\phi^2$, $G_D \equiv G_T/N$ being the conductance of the whole chain. This gives the well-known transmission distribution for the diffusive conductor, $\rho(T) = (G_D/2G_Q)/T\sqrt{1-T}$.¹⁰

As explained above, to calculate G_Q corrections we augment the check dimension of the Green's functions by introducing the black and white structure. Consequently, the parameter ϕ gets a "color" index b or w . The semiclassical solution for resulting 4×4 matrix is nonzero in bb and ww blocks,

$$\check{G}_k^0 = \begin{pmatrix} \check{G}(k\phi_b/N) & 0 \\ 0 & \check{G}(k\phi_w/N) \end{pmatrix}. \quad (53)$$

Now we shall derive the matrix $\hat{M}$ eigenvalues which determine G_Q corrections. It is advantageous to use a parametrization of the deviations from the semiclassical solution, $\check{g}$, which automatically satisfy $\check{g}\check{G} + \check{G}\check{g} = 0$ in each node. To this end, we rewrite the action (51) in a special basis: that one where $\check{G}_k^0$ is diagonal in each node,

$$\check{G}_k^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (54)$$

Then the deviation of the form

$$\check{g}_k = \begin{pmatrix} 0 & 0 & 0 & g_{k,p}^{bw} \\ 0 & 0 & g_{k,m}^{bw} & 0 \\ 0 & g_{k,p}^{wb} & 0 & 0 \\ g_{k,m}^{wb} & 0 & 0 & 0 \end{pmatrix} \quad (55)$$

satisfies the condition mentioned.

In this basis (see the Appendix for details) the action reads

$$S = \frac{G_T}{2G_Q} \sum_{k=0}^{N-1} \text{Tr} \check{G}_k \check{L} \check{G}_{k+1} \check{L}^{-1} - \frac{i\pi}{\delta_S} \sum_{k=1}^{N-1} \text{Tr} \check{\Sigma}_k \check{G}_k, \quad (56)$$

where the bb (ww) block of $\check{L}$ is given by

$$L^{bb(ww)} = \begin{pmatrix} \cos(\phi^{b(w)}/N) & i \sin(\phi^{b(w)}/N) \\ i \sin(\phi^{b(w)}/N) & \cos(\phi^{b(w)}/N) \end{pmatrix}. \quad (57)$$

We expand the Green's matrices according to Eq. (30), write the quadratic form in terms of $\check{g}$, and diagonalize it (see the Appendix) to find the following set of eigenvalues ($l = 1, \dots, N-1$):

$$\begin{aligned} \frac{4G_Q}{G_T} M_l^\pm(\phi^b, \phi^w) &= 2 \cos \frac{\phi^b}{2N} \cos \frac{\phi^w}{2N} \cos \frac{\pi l}{N} \\ &\quad - \cos \frac{\phi^b}{N} - \cos \frac{\phi^w}{N} \\ &\quad \mp \sqrt{4 \sin^2 \frac{\phi^b}{2N} \sin^2 \frac{\phi^w}{2N} \cos^2 \frac{\pi l}{N} + \epsilon^2}, \end{aligned} \quad (58)$$

where $\epsilon \equiv 2\pi G_Q(\Sigma^b - \Sigma^w)/G_T i \delta_S$ measures the difference of the Green's function energy parameter in bb and ww blocks in units of a single-node Thouless energy. To obtain eigenvalues that determine the weak-localization contribution, we set $\phi^w = -\phi^b = \phi$, $\epsilon = 0$. This yields

$$M_{wl,l}^+(\phi) = \frac{G_T}{4G_Q} \cos\left(\frac{\phi}{N}\right) \left[\cos \frac{\pi l}{N} - 1 \right], \quad (59)$$

$$M_{wl,l}^-(\phi) = \frac{G_T}{4G_Q} \left[\cos \frac{\pi l}{N} - \cos \frac{\phi}{N} \right]. \quad (60)$$

Let us discuss the weak-localization correction first. If we neglect decoherence factors, we can sum up over l to find a compact analytical expression

$$S_{wl} = \frac{(N-2)}{2} \ln \left(\cos \frac{\phi}{N} \right) + \frac{1}{2} \ln \left(\frac{\sin \frac{2\phi}{N}}{\sin \phi} \right). \quad (61)$$

In the limit $N \rightarrow \infty$ this reproduces the known correction for a one-dimensional diffusive wire,¹⁵

$$S_{wl} = \frac{1}{2} \ln \left(\frac{\phi}{\sin \phi} \right). \quad (62)$$

It is interesting to note that the weak-localization correction is absent for $N=2$. We will see below that this is a general property of a single-node tunnel-junction system. It was observed in Ref. 15 that a part of the weak-localization correction in diffusive conductors is universal: It depends neither on the shape nor on the dimensionality of the conductor. The universal part is concentrated near transmissions close to 1 and is given by

$$\delta \rho_{wl}(T) = -\frac{1}{4} \delta(T-1), \quad (63)$$

while the nonuniversal part is a smooth function of T . The relation (61) possesses this property at any N , since the universal part comes from the divergency in Eq. (61) at $\phi = \pi$ where the eigenvalue $M_{wl,1}^-$ goes to zero. Our approach proves that this correction is universal for a large class of the nanostructures, not limited to diffusive ones, for any nanostructure where transmission eigenvalues approach 1. This is guaranteed by the logarithmic form of the action. If $M_{wl,1}^- \propto (\pi - \phi)$ at $\phi \rightarrow \pi$, the correction is given by Eq. (63) irrespective of the proportionality coefficient.

Expanding Eq. (61) at $\phi \rightarrow 0$ we find the correction to the conductance of the tunnel junction chain,

$$\frac{\delta G_{wl}}{G_Q} = -\frac{1}{3} \frac{(N-1)(N-2)}{N^2}. \quad (64)$$

This is written for an orthogonal ensemble; a well-known factor $(1-2/\beta)$ defines the correction for other pure ensembles. The effect of spin and magnetic decoherence can be taken into account by shifting the eigenvalues (59) and (60) according to Eq. (39) since the decoherence factors in each node are the same, $\zeta_{H,so} \equiv \eta_{H,so}(4G_Q/G_T)$.

The $\mathcal{S}_{wl}$ is still given by an analytical although lengthy expression (see the Appendix). The correction to the transmission distribution corresponding to this expression is plotted in Fig. 6 for different strengths of the spin-orbit coupling to illustrate the transition between the orthogonal and symplectic ensembles. The correction to the conductance is given by

$$\frac{\delta G_{wl}}{G_Q} = 3F(N, \zeta_H + \zeta_{so}) - F(N, \zeta_H), \quad (65)$$

where we define an auxiliary function $F(x, N)$:

$$F(x, N) = -\frac{(N-1)}{N^2} - \frac{1}{N^2} \frac{(2+x)^2}{x(4+x)} - \frac{1}{N\sqrt{x(4+x)}} \frac{2^{2N} + [2+x+\sqrt{x(4+x)}]^{2N}}{2^{2N} - [2+x+\sqrt{x(4+x)}]^{2N}}.$$

Let us discuss the parametric correlations. Without decoherence factors and at the same energy ($\epsilon=0$) one can still sum up over the modes to obtain an analytical expression

$$\mathcal{S}_{\text{diff}} = \mathcal{S}_{\text{Cooperon}} = (N-1) \ln \left(\cos \frac{\phi_b}{N} + \cos \frac{\phi_w}{N} \right) + \sum_{\pm} \ln \left(\frac{\sin \frac{\phi_b \pm \phi_w}{2}}{\sin \frac{\phi_b \pm \phi_w}{2N}} \right). \quad (66)$$

The fluctuation of conductance obtained with Eq. (50) reads

$$\frac{\langle\langle \delta G^2 \rangle\rangle}{G_Q^2} = \frac{2}{15} \frac{N^4 + 15N - 16}{N^4} \quad (67)$$

and converges to the known expression for a quasi-one-dimensional diffusive conductor at $N \rightarrow \infty$. We notice that this convergence is rather quick; the fluctuation at $N=5$ differs from the asymptotic value by 10% only. We see thus that the diffusive wire, which in principle contains an infinite number of Cooperon and diffuson modes, can be, with sufficient accuracy, described by the finite-element technique even at a low number of elements.

Another point to discuss concerns the correlations of the transmission eigenvalues T_n ; those can be obtained by analytic calculation of Eq. (48). It is instructive to concentrate on the relatively small eigenvalue separations, those are much smaller than 1, but still exceeding an average spacing $\simeq G_Q/G$ between the eigenvalues, $G_Q/G \ll |T-T'| \ll 1$. We observe that the correlation in this case is determined by the divergence of $\mathcal{S}$ at $\phi_b - \phi_w \rightarrow \pm 2\pi$. Indeed, M_1 approaches 0

FIG. 6. Weak-localization correction to the transmission distribution of a system of four identical junctions at different values of the spin-orbit parameter ζ_{so} . We plot here cumulate correction $X(T) \equiv \int_0^T dT' T' \delta \rho(T')$. $X(1)$ represents the universal singular part of the correction [cf. Eq. (63)] while $X(0)$ gives a correction to the conductance. The lowest curve corresponds to strictly zero ζ_{so} and therefore represents a pure orthogonal ensemble. Its negative value at $T=1$ is partially compensated for by the positive nonuniversal contribution coming from $T \simeq 1$ so that the resulting correction to the conductance, $\delta G_{wl}/G_Q = X(0) \approx 0.2$. The two higher curves correspond to relatively small values of ζ_{so} , 0.05 and 0.4. While they are close to the orthogonal ensemble result at $T \simeq 1$, their behavior at $T \simeq 1$ is quite different: the universal correction is that of a symplectic ensemble and is of positive sign. The highest curve corresponding to $\zeta_{so}=10$ is close to the cumulate correction of the pure symplectic ensemble, $X_{\text{sym}}(T) = -X_{\text{ort}}(T)/2$.

in this limit. This again suggest the universality of these correlations. Indeed, as shown in Ref. 15 for diffusive conductors, the correlations in this parameter range are determined by universal Wigner-Dyson statistics and reduce to

$$\langle\langle \rho(T) \rho(T') \rangle\rangle = -\frac{2}{\pi^2 \beta} \text{Re} \frac{1}{(T-T' + i0^+)^2}. \quad (68)$$

Since the conductance fluctuations are contributed by correlations of T_n at scale ~ 1 as well, they are not universal. We plot in Fig. 7 the correlator of conductance fluctuations as a function of the energy difference at several N .

B. AB ring

In this subsection we exemplify evaluation of the AB effect within our scheme. We concentrate on the simple circuit presented in Fig. 8. It contains four tunnel junctions and two nodes labeled A and B . The conductances of the junctions are chosen to reuse the results of the previous section for a chain of three tunnel junctions: The solution of semiclassical circuit theory equations is given by Eq. (53) for $N=3$. The action reads

FIG. 7. Energy dependence of the correlator of the conductance fluctuations $\langle\langle G(0)G(E)\rangle\rangle$ for chains with a different number of junctions N . The energy difference is normalized to the Thouless energy of the whole chain, $E_{th} \equiv \delta_s G_T / 2\pi G_Q N^2$. Note the fast convergence of the correlator to that of a diffusive wire for large N and negative correlations at large E .

$$S = \frac{G_T}{4G_Q} \text{Tr} \left\{ \check{G}_L \check{G}_A + \frac{1}{2} \check{G}_A \check{G}_B + \frac{1}{2} \check{G}_A \check{G}_B(\phi_{AB}) + \check{G}_B \check{G}_R \right\} - i \frac{\pi}{\delta} \sum_{i=A,B} \text{Tr} \check{\Sigma}_i \check{G}_i. \quad (69)$$

where $\check{G}_{L,R}$ are Green's functions in the reservoirs and $\check{G}_B(\phi_{AB})$ is modified according to Eq. (40). To study the correlation of the conductance fluctuations, we consider different Green's functions for white and black blocks and subjected to different fluxes $\phi_{AB}^b \neq \phi_{AB}^w$. For the weak-localization correction, we set $\phi_{AB}^b = -\phi_{AB}^w = \phi_{AB}$.

To calculate the matrix $\hat{M}$ we use again the basis where $G_i^{(0)}$ are diagonal and the parametrization for $\check{g}$ introduced in the previous subsection. It reads

FIG. 8. A simplest circuit to exemplify the AB effect consists of two nodes and for (tunnel) connectors. The AB phase modifies the Green's function on the right side of the lowest connector with respect to $\check{G}_B$. The cut is given by the dotted line.

$$M = \begin{pmatrix} M_d & M_{od} \\ M_{od}^* & M_d \end{pmatrix},$$

where 2×2 blocks $M_{d,od}$ are given:

$$\frac{G_T M_d}{4G_Q} = \begin{pmatrix} -\cos \varphi^b - \cos \varphi^w + \epsilon & 0 \\ 0 & -\cos \varphi^b - \cos \varphi^w - \epsilon \end{pmatrix}$$

$$\frac{G_T M_{od}}{4G_Q} = \frac{1 + e^{i(\phi_{AB}^b - \phi_{AB}^w)}}{2} \begin{pmatrix} \cos \frac{\varphi^b}{2} \cos \frac{\varphi^w}{2} & \sin \frac{\varphi^b}{2} \sin \frac{\varphi^w}{2} \\ \sin \frac{\varphi^b}{2} \sin \frac{\varphi^w}{2} & \cos \frac{\varphi^b}{2} \cos \frac{\varphi^w}{2} \end{pmatrix}.$$

The parameter ϵ which characterizes the energy difference between black and white Green's function is defined as in the previous subsection. At $\epsilon=0$ we obtain an explicit expression for the diffuson eigenvalues (the Cooperon ones are obtained by $\phi^w \rightarrow -\phi^w$ and $\phi_{AB}^w \rightarrow -\phi_{AB}^w$).

$$\frac{4G_Q}{G_T} M_{1,2}^+ = -\cos \frac{\phi_b}{3} - \cos \frac{\phi_w}{3} \pm \cos \left(\frac{\phi_{AB}^b - \phi_{AB}^w}{2} \right) \cos \left(\frac{\phi_b - \phi_w}{6} \right), \quad (70)$$

$$\frac{4G_Q}{G_T} M_{1,2}^- = -\cos \frac{\phi_b}{3} - \cos \frac{\phi_w}{3} \pm \cos \left(\frac{\phi_{AB}^b - \phi_{AB}^w}{2} \right) \cos \left(\frac{\phi_b + \phi_w}{6} \right). \quad (71)$$

The weak-localization correction to the action reads

$$\mathcal{S}_{wl}(\phi) = \frac{1}{2} \ln \left(\frac{\cos^2(\phi/3)[4 - \cos^2(\phi_{AB})]}{4 \cos^2(\phi/3) - \cos^2(\phi_{AB})} \right), \quad (72)$$

from which we calculate the correction to conductance as a function of the flux:

$$\frac{\delta G_{wl}}{G_Q} = -\frac{4 \cos^2(\phi_{AB})}{9[7 - \cos(2\phi_{AB})]}. \quad (73)$$

We see that the weak-localization correction cancels at half-integer flux $\phi_{AB} = \pi$. This is because the junctions forming the loop are taken to be identical. The flux dependence exhibits higher harmonics, indicating semiclassical orbits that encircle the flux more than once.

For the correlator of conductance fluctuations we obtain

$$\frac{\langle \delta G^2 \rangle}{G_Q^2} = \sum_{\pm} \frac{259 - 4 \cos(\phi_{AB}^w \pm \phi_{AB}^b) + \cos 2(\phi_{AB}^w \pm \phi_{AB}^b)}{81[\cos(\phi_{AB}^w \pm \phi_{AB}^b) - 7]^2}, \quad (74)$$

where plus and minus signs indicate Cooperon and diffuson contributions, respectively. The higher harmonics are present as well.

FIG. 9. The simplest possible circuit comprises one node, two connectors (L and R), and two reservoirs where the Green's functions $\check{G}_{L,R}$ are fixed.

C. Two connectors and one node

Probably the simplest system to be considered by circuit-theory methods consists of a single node and two connectors (Fig. 9). Since in this case there are only N_{ch} eigenvalues, one can straightforwardly elaborate on complicated arbitrary connectors. For this setup we are still able to find an analytical expression for Cooperon and diffusion eigenvalues. This allows us to get an expression for the weak-localization correction to the conductance which was vanishing in the case

of two tunnel junctions. Each connector is in principle characterized by the distribution of transmission coefficients $\{T_n^R\}$ and $\{T_n^L\}$ or, equivalently, by the functional form of the connector action given by Eq. (10), S_L and S_R . The action for the whole system reads

$$S = S_L(\check{G}_L, \check{G}) + S_R(\check{G}, \check{G}_R) - i \frac{\pi}{\delta_s} \text{Tr} \check{\Sigma} \check{G}. \quad (75)$$

$\check{G}$ being the Green's function of the node. We employ 2×2 matrices parametrized by Eq. (43) and set the Green's functions in the left and right reservoirs to $\check{G}(-\phi/2)$ and $\check{G}(\phi/2)$, respectively. The saddle-point value of $\check{G}$ is given by the phase χ and for a general choice of S_L and S_R does depend on ϕ , $\chi \equiv \chi(\phi)$. The total action in the saddle point is therefore $S(\phi) = S_L(\chi + \phi/2) + S_R(\chi - \phi/2)$. We expand the Green's function according to Eq. (30). The second-order correction to the action in this case reads

$$S^{(2)} = -i \frac{\pi}{\delta_s} \text{Tr} \check{\Sigma} \check{G} - \frac{1}{2} \sum_n \text{Tr} \left\{ \left[\frac{T_n^L}{4 \left[1 + \frac{T_n^L}{4} (\{\check{G}^0, \check{G}_L\} - 2) \right]} \right] \check{g}^2 \check{G}^0 \check{G}_L + \left[\frac{T_n^L}{4 \left[1 + \frac{T_n^L}{4} (\{\check{G}^0, \check{G}_L\} - 2) \right]} \right]^2 (\check{g} \check{G}_L \check{g} \check{G}_L + \check{g}^2) + (L \leftrightarrow R) \right\}. \quad (76)$$

Acting like in the previous subsections, we find the two following diffuson eigenvalues (the Cooperon ones are obtained by the substitution $\phi^w \rightarrow -\phi^w$):

$$M^\pm(\phi^b, \phi^w) = \sum_{i=b,w} I(\phi^i) \left[\cot\left(\chi(\phi^i) - \frac{\phi^i}{2}\right) - \cot\left(\chi(\phi^i) + \frac{\phi^i}{2}\right) \right] - \frac{1 - \cos\left[\chi(\phi^b) + \frac{\phi^b}{2} \mp \chi(\phi^w) \mp \frac{\phi^w}{2}\right]}{2} \sum_{i=b,w} \left\{ \frac{1}{\sin^2[\chi(\phi^i) + \phi^i/2]} \left[\frac{I'(\phi^i)}{\chi'(\phi^i) + 1/2} - I(\phi^i) \cot\left(\chi(\phi^i) + \frac{\phi^i}{2}\right) \right] \right\} + \frac{1 - \cos\left[\chi(\phi^b) - \frac{\phi^b}{2} \mp \chi(\phi^w) \pm \frac{\phi^w}{2}\right]}{2} \sum_{i=b,w} \left\{ \frac{1}{\sin^2[\chi(\phi^i) - \phi^i/2]} \left[\frac{I'(\phi^i)}{\chi'(\phi^i) - 1/2} - I(\phi^i) \cot\left(\chi(\phi^i) - \frac{\phi^i}{2}\right) \right] \right\}, \quad (77)$$

Here we introduce $I(\phi) \equiv \partial S / \partial \phi$ to characterize the derivative of the total semiclassical action. We see that M^- approaches zero in the limit $\phi_b, \phi_w \rightarrow \pm\pi, \mp\pi$ provided $I(\phi)$ stays finite. As discussed, this divergence guarantees the universality of the correlations of transmission eigenvalues.

Below we specify to three different cases.

1. Symmetric setup

If we set $S_R = S_L$, $\chi(\phi^{b(w)})$ is zero regardless of the concrete form S_L . The total action therefore reads $S(\phi) = 2S_L(\phi/2)$. The eigenvalues (77) take a simpler form. To

compute the weak-localization correction to the action we set $\phi^b = -\phi^w = \phi$ to find

$$S_{wl} = \frac{1}{2} \ln \left(\frac{I'(\phi)}{I(\phi)} \tan \frac{\phi}{2} \right) + \text{const.} \quad (78)$$

For tunnel junctions, $I(\phi) \propto \sin(\phi/2)$ and the correction disappears.

Expanding Eq. (78) in Taylor series near $\phi \rightarrow 0$ we reproduce the well-known result of Refs. 13 and 34 for the weak-localization correction to the conductance. To comply with

the notations used there, we characterize connectors with $t_p = \sum_n T_n^p$:

$$\frac{\delta G_{wl}}{G_Q} = -\frac{t_2}{4t_1}. \quad (79)$$

A similar expansion of the diffusion and Cooperon eigenvalues reproduces the result of Refs. 13 and 34 for conductance fluctuations,

$$\frac{\langle (\delta G)^2 \rangle}{G_Q} = \frac{3t_2^2 + 2t_1^2 - 2t_1(t_2 + t_3)}{8t_1^2}. \quad (80)$$

2. Diffusive connectors

It is instructive for understanding the circuit theory of G_Q corrections to specify the relation (78) to diffusive connectors. Since in this case $I(\phi) \propto \phi$, we obtain

$$\mathcal{S}_{wl,node} = \frac{1}{2} \ln[\tan(\phi/2)/\phi]. \quad (81)$$

A two-connector, single-node situation can be easily realized in a quasi-one-dimensional wire with inhomogeneous resistivity distribution along the wire. A low-resistivity region would make a node if bounded by two shorter resistive regions that would make the connectors. On the other hand, it has been proven in Ref. 27 that the weak-localization correction in inhomogeneous wires does not depend on the resistivity distribution. Therefore, it has to be universally given by Eq. (62), $\mathcal{S}_{wl,1d} = (1/2) \ln(\phi/\sin \phi) \neq \mathcal{S}_{wl,node}$. How to understand this apparent discrepancy?

This illustrates a very general point: G_Q corrections may be accumulated at various space scales ranging from the mean free path to sample size. The experimental observation of the corrections relies on the ability to separate the contributions coming from different scales—e.g., by changing the magnetic field.⁴ With our approach, we evaluate the part coming from interference at the scale of the node. The part coming from interference at a shorter scale associated with the connectors is assumed to be included in the transmission distribution of these connectors.

For our particular setup, this extra contribution comes from two identical connectors. Since only half of the phase ϕ drops at each connector, the contribution equals $2\mathcal{S}_{wl,1d}(\phi/2)$. Summing up both contributions, we obtain

$$\mathcal{S}_{wl,node} + 2\mathcal{S}_{wl,1d}(\phi/2) = \mathcal{S}_{wl,tot} = \mathcal{S}_{wl,1d}(\phi).$$

That is, the weak localization correction in this case remains universal provided the contribution of the node is augmented by the contributions of two connectors.

3. Nonideal quantum point contact

The transmission distribution of an ideal multimode quantum point contact (QPC) with conductance $G_B \gg G_Q$ is very degenerate since all $T_n = 1$ or 0. This degeneracy is lifted if the QPC is adjacent to a disordered region, even if the scattering in this region is weak. This can be modeled as a connector with conductance $G_D \gg G_B$ in series.

The weak-localization correction to the conductance was calculated a while ago.³⁵ In the relevant limit, it is parametrically small in comparison with G_Q , $\delta G_{wl} = -G_Q(G_B/G_D)$. The usual way to verify the applicability of the semiclassical approach to quantum transport is to compare the conductance of a nanostructure with the weak-localization correction to it. For generic nanostructure, this gives $G \gg G_Q$. However, for our particular example $\delta G_{wl} \ll G_B$ even for a few-channel QPC where $G_B \approx G_Q$. So the question is, is semiclassical approach really valid at $G_B \approx G_Q$?

To answer this question, we compute the weak-localization correction to the transmission distribution. Since the system is not symmetric, we make use of the full expression (77). In the limit of $G_D \gg G_B$, the relevant values of ϕ are close to π . We stress it by shifting the phase $\mu = \pi - \phi$, $|\mu| \ll 1$.

The circuit-theory analysis in the semiclassical limit gives³⁶

$$I(\mu) = G_D \left(-\frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} + R_c} \right), \quad \chi = \frac{\pi}{2} + \sqrt{\frac{\mu^2}{4} + R_c},$$

where $R_c \equiv 4G_B/G_D \ll 1$. This gives the following distribution of reflection coefficients [cf. Eq. (47)]:

$$\rho(R) = \frac{G_D}{2\pi G_Q} \Theta(R_c - R) \sqrt{\frac{R_c}{R} - 1}.$$

We use the above relations with Eq. (77) to find the Cooperon eigenvalues,

$$M^+(\mu) = -G_D \left(-\frac{1}{2} + \frac{\mu/4}{\sqrt{\frac{\mu^2}{4} + R_c}} \right) \frac{\mu^2/4 + R_c}{R_c/2}, \quad (82)$$

$$M^-(\mu) = G_D \left(-\frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} + R_c} \right) \frac{\mu}{2R_c}. \quad (83)$$

This yields the weak-localization correction to the current,

$$I_{wl}(\mu) = \frac{2R_c}{\mu(\mu^2 + 4R_c)}.$$

The resulting correction to the transmission distribution consists of two δ -functional peaks of opposite sign, those located at the edges of the semiclassical distribution,

$$\delta\rho_{wl}(R) = \frac{1}{4} [\delta(R - R_c) - \delta(R)]. \quad (84)$$

To estimate the conditions of applicability, we smooth the correction at the scale of R_c . This gives $|\delta\rho|/\rho \approx G_Q/G_B$ and the semiclassical approach does not work at $G_B \approx G_Q$. This agrees with RMT arguments given in Ref. 36. The correction to the conductance calculated with Eq.(84) agrees with the value cited above and indeed is anomalously small.

VIII. CONCLUSIONS

We present a finite-element method to evaluate quantum corrections—typically of the order of G_Q —to transport characteristics of arbitrary nanostructures. This includes universal conductance fluctuation and weak localization. We work with matrix Green's functions of arbitrary structure to treat a wider class of problems that includes superconductivity, full counting statistics, and nonequilibrium transport. At the microscopic level, the corrections are expressed in terms of diffuson and Cooperon modes of continuous Green's functions. We employ a variational method based on an action to formulate a consistent finite-element approach.

We illustrate the method with a set of simple and physically interesting examples. All examples are based on 2×2 matrices; this suffices to calculate the transmission distribution of two-terminal structures. We show how a chain of tunnel junctions approaches the diffusive wire upon increasing the number of junctions and study transitions between ideal RMT ensembles in the chain. We consider the simplest finite-element system that exhibits Ahronov-Bohm effect. We obtain general results for a single-node system with two arbitrary connectors and check their consistency, with the well-known results for quantum cavity. This allows us to improve our understanding of quantum interference in inhomogeneous diffusive wires and nonideal quantum point contacts.

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APPENDIX: JUNCTION CHAIN

In this appendix we illustrate how to find the eigenvalues of the matrix M defined in Sec. IV for the chain of tunnel junctions introduced. We consider the action in the rotated basis presented in the text

$$S = \frac{G_T}{4G_Q} \sum_{i=0}^N \text{Tr} \check{G}_i \check{L} \check{G}_{i+1} \check{L}^{-1} - \frac{i\pi}{\delta_s} \sum_{i=1}^{N-1} \text{Tr} \check{\Sigma}_i \check{G}_i. \quad (\text{A1})$$

We expand the Green's matrices according to Eq. (30)

The second-order correction to the action reads (the first order is zero because we perform the expansion around the stationary point)

$$S^{(2)} = \frac{G_T}{4G_Q} \sum_{i=0}^N \text{Tr} \left\{ \check{g}_i \check{L} \check{g}_{i+1} \check{L}^{-1} - \frac{1}{2} \check{g}_i \check{L} \check{G}_{i+1}^0 \check{L}^{-1} - \frac{1}{2} \check{G}_i^0 \check{L} \check{g}_{i+1} \check{L}^{-1} \right\} - \frac{i\pi}{\delta_s} \sum_{i=1}^{N-1} \text{Tr} \left\{ \check{\delta} \check{\Sigma}_i \check{g}_i - \frac{1}{2} \check{\Sigma}_i \check{g}_i^2 \check{G}_i^0 \right\}. \quad (\text{A2})$$

Taking the variation respect to $g_{p,k}^{wb}$ and $g_{m,k}^{wb}$ leads to the eigenvalue equations the matrix M :

$$\begin{aligned} & \sin \frac{\phi^b}{2N} \sin \frac{\phi^w}{2N} (g_{p,k+1}^{wb} + g_{p,k-1}^{wb}) + \cos \frac{\phi^b}{2N} \cos \frac{\phi^w}{2N} (g_{m,k+1}^{wb} \\ & + g_{m,k-1}^{wb}) - \left(\cos \frac{\phi^b}{N} + \cos \frac{\phi^w}{N} \right) g_{m,k}^{wb} - \frac{1}{2\xi} (\Sigma^b - \Sigma^w) g_{m,k}^{wb} \\ & - \frac{4G_Q}{G_T} M g_{m,k}^{wb} = 0, \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} & \sin \frac{\phi^b}{2N} \sin \frac{\phi^w}{2N} (g_{m,k+1}^{wb} + g_{m,k-1}^{wb}) + \cos \frac{\phi^b}{2N} \cos \frac{\phi^w}{2N} (g_{p,k+1}^{wb} \\ & + g_{p,k-1}^{wb}) - \left(\cos \frac{\phi^b}{N} + \cos \frac{\phi^w}{N} \right) g_{p,k}^{wb} + \frac{1}{2\xi} (\Sigma^b - \Sigma^w) g_{p,k}^{wb} \\ & - \frac{4G_Q}{G_T} M g_{p,k}^{wb} = 0, \end{aligned} \quad (\text{A4})$$

where $\xi = i\delta_s G_T / (4\pi G_Q)$. To solve the system of coupled equations it is convenient to write

$$g_{p,k}^{wb} = g_p^> e^{iqk} + g_p^< e^{-iqk} \quad (\text{A5})$$

and a similar equation for the m component. The coefficient q is to be determined from the boundary conditions

$$g_{p,0}^{wb} = 0,$$

$$g_{p,N}^{wb} = 0.$$

From the boundary conditions we have $g_k^> = -g_k^<$ and $q = \pi l / N$ with $l = 1, \dots, N-1$. We substitute the expressions (A5) into the equations for the eigenvalues and get

$$\begin{aligned} \frac{4G_Q}{G_T} M_l^\pm(\phi^b, \phi^w) &= 2 \cos \frac{\phi^b}{2N} \cos \frac{\phi^w}{2N} \cos \left(\frac{\pi l}{N} \right) \\ &- \cos \frac{\phi^b}{N} - \cos \frac{\phi^w}{N} \\ &\mp \sqrt{4 \sin^2 \frac{\phi^b}{2N} \sin^2 \frac{\phi^w}{2N} \cos^2 \left(\frac{\pi l}{N} \right) + \epsilon^2}, \end{aligned} \quad (\text{A6})$$

the quantity ϵ being $(\Sigma^b - \Sigma^w) / 2\xi$.

Here we report the correction to the action when decoherence and spin-orbit are present as generalization of Eq. (61). Let us define the functions

$$s_1(\phi, x, N) = \frac{1}{2} \left(2 + \frac{x}{\cos(\phi/N)} \right) + \sqrt{\frac{1}{4} \left(2 + \frac{x}{\cos(\phi/N)} \right)^2 - 1},$$

$$s_2(\phi, x, N) = \frac{1}{2} [x + 2 \cos(\phi/N)] + \sqrt{\frac{1}{4} [x + 2 \cos(\phi/N)]^2 - 1}$$

and the function

$$A(\phi, x, N) = \left[\frac{s_1^2 - s_1^{-2}}{2 - s_1^2 - s_1^{-2}} + N \frac{-s_1^{2N} + s_1^{-2N}}{2 - s_1^{2N} - s_1^{-2N}} \right] \frac{\partial}{\partial \phi} \ln[s_1(\phi, x, N)] - \left[\frac{s_2^2 - s_2^{-2}}{2 - s_2^2 - s_2^{-2}} + N \frac{-s_2^{2N} + s_2^{-2N}}{2 - s_2^{2N} - s_2^{-2N}} \right] \frac{\partial}{\partial \phi} \ln[s_2(\phi, x, N)].$$

We can finally express the correction to the action as

$$\frac{\partial}{\partial \phi} \mathcal{S}_{wl}(\phi) = A(\phi, \zeta_H, N) - 3A(\phi, \zeta_H + \zeta_{so}, N). \quad (\text{A7})$$

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