THEORETICAL ANALYSIS OF LAMINAR
INCOMPRESSIBLE FLOW IN
SLENDER CHANNELS

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Delft/Rijswijk, The Netherlands
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Abstract

In this report incompressible flow in a channel with mass injection is studied analytically. As the channels considered are axisymmetric, introducing a Stokes streamfunction into the Navier Stokes Equations yields one non-linear partial differential equation for this streamfunction. When non-dimensional coordinates and a non-dimensional streamfunction are introduced the partial differential equation is rewritten in a non-dimensional form. The resulting equation is then simplified by assuming changes in the axial direction to be small compared to changes in the transverse direction (slender flow) and taking the cross-sectional area to be constant. The resulting parameter $\lambda$ in this equation is the Reynolds number of the injected flow. Exact solutions are calculated for extremely small and large values of this parameter. For intermediate values of $\lambda$ the dimensionless streamfunction is expanded in a power series of the non-dimensional transverse coordinate, yielding two types of solutions: Poiseuille like flow and a solution with a reversed flow region. Subsequently, the influence of sine-like wall disturbances is studied. The magnitude of these disturbances is such that the flow in these channels can be supposed to be a small perturbation of the flow in a channel with a constant cross section. The non-dimensional perturbation streamfunction is separated into two functions, one depending on the transverse coordinate only and one depending on the axial coordinate only. Again the function depending on the transverse coordinate is expanded in a power series of this coordinate. For rapidly varying wall undulations there appears to be a small phase difference between the wall disturbances and the pressure field.
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0 terms of differential equation; order
P pressure; transverse coordinate parameter; index in power series
\textbf{P} unknown perturbation vector
\Delta p pressure difference
Re Reynolds number
R_o undisturbed channel radius
R_d disturbed channel radius
R \text{wall disturbance function}
r cylindrical coordinate
S_1, S_2 terms of power series
T temperature
t additional coordinate
u arbitrary variable
v velocity
x additional coordinate
z cylindrical coordinate
\alpha mass flow injection parameter per unit of length
\bar{\alpha} mass flow injection parameter
\beta wall surface angle
\beta_b additional wave parameter
\delta dimensionless wall disturbance parameter
\epsilon heat flux
\eta non-dimensional transverse coordinate
\theta cylindrical coordinate
\lambda injection Reynolds number
\lambda_b critical injection Reynolds number
\mu dynamic viscosity
\nu kinematic viscosity
\zeta non-dimensional axial coordinate
\rho density
\tau shear stress
x arbitrary property
\varphi phase difference
\psi Stokes streamfunction
indices
c center

-5-
i  injection
o  inlet, undisturbed
r  r-component; r-direction
w  wall
z  z-component; z-direction
θ  θ-component; θ-direction
l  disturbed

superscripts

\text{t}  \quad \text{transformed}
-  \quad \text{dimensionless}
I. Introduction.

The thesis work described in this report was performed within the framework of the Solid Fuel Combustion Chamber (SFCC) project. This project aims at obtaining a thorough understanding of the flow and combustion processes in an SFCC. A solid fuel combustion chamber consists of a solid fuel grain with an inner bore (see Figure I.1).

![Figure I.1. Channel geometry.](image)

The main flow consists of air which is fed into this bore. At the interface between the air and the fuel, the fuel pyrolizes and combustion takes place. The pyrolizing fuel causes a secondary mass flow (Figure I.1). An SFCC can be applied in such diverse areas as waste combustion, coal gasification and aerospace propulsion systems.

- The aim of this report is to give a theoretical background for the phenomena as observed in experiments. Recently, a computer program was developed [7,8], which describes the steady 2-dimensional turbulent flow with and without combustion and heat and mass transfer in an SFCC. Although the flow fields calculated with this computer program are in good agreement with the experiments, the complexity of the underlying theoretical model hampers a good insight into the correlation between model and physics. In this report, laminar flow in an SFCC is treated analytically. Velocity and pressure profiles are determined. A method to find temperature distributions is pointed out. These temperature profiles will be calculated in the near future.
- The flow in an SFCC is modeled as a fluid in an axisymmetric pipe with variable cross sections and with mass injection through the wall. The flow is assumed to be laminar and incompressible. Furthermore, the fluid is thought to be homogeneous. Under these constraints it is possible to find an analytical solution of the flow for the fully developed case. Flow fields in channels with varying cross sections and no mass injection are studied in [11] and [12].

- Chapter II lists the equations of motion. The assumptions underlying of the flow model developed are presented and used to simplify the general equations of motion. A streamfunction $\psi$ is defined, and introduced in the governing equations, yielding one partial differential equation for $\psi$.

- Chapter III deals with the flow through a channel with constant radius and constant mass injection. The partial differential equation for $\psi$ is rewritten in non-dimensional coordinates and parameters representing mass injection and change of diameter are introduced. The unknown streamfunction is expanded in a power series the non-dimensional coordinates.

- Chapter IV treats the flow through a channel with variable cross sections and with constant mass injection. The variation in the radius of the channel is assumed to be small compared to the undisturbed radius $R_0$, and the corresponding flow pattern is supposed to deviate little from that of a channel with constant cross sections. An approximate solution is hence found in the form of a perturbation of the streamfunction, as found for a channel with constant radius.
II. Conservation equations.

In this section the general conservation equations as given in Appendix I are simplified in order to be able to derive an analytical solution for the flow field. For an incompressible axisymmetric fluid it is possible to define a streamfunction $\psi$ satisfying the continuity equation. Introducing of this streamfunction in the momentum equation results in one non-linear differential equation for $\psi$. For the sake of completeness the energy equation is also given.

II.1. Assumptions and their implications.

In order to make an analytical approach not only possible but also successful, the problem of determining the flow field in an SFCC has to be simplified. The following assumptions are made:

a) The flow is laminar, i.e. at every point in the fluid the velocity has a fixed value and direction.

b) The main flow as well as the injected fluid are homogenous.

c) The fluid is incompressible, i.e. the density and the viscosity of the fluid are constant. Close examination of Equations (A.I.3), (A.I.5) and (A.I.6) shows that the continuity equation and the momentum transport equation are uncoupled from the energy equation.

d) The flow in the channels considered is axisymmetric, implying that the tangential velocity component $v_\theta$ equals zero. Furthermore, all flow properties are independent of the $\theta$-coordinate, so that

$$v_\theta = 0$$

$$\frac{\partial}{\partial \theta} = 0$$

(II.1)

e) As the flow is assumed to be fully developed, the time dependence disappears, so

$$\frac{\partial}{\partial t} = 0$$

(II.2)

f) Gravitational, electromagnetic and body forces etc. are negligibly small. The body force term in the momentum transport Equation (A.I.5) is

$$f = 0$$

(II.3)
II.2. Governing equations.

Introducing the assumptions made in Section II.1 in the equations of motion yields the general form of the equations for incompressible axisymmetric flow. The continuity Equation (A.I.1) becomes

\[(\nabla \cdot \mathbf{v}) = 0\]  \hspace{2cm} (II.4)

The momentum Equation (A.I.5) becomes

\[\rho(\nabla \cdot \mathbf{v})\mathbf{v} = -\nabla p + \mu(\nabla^2 \mathbf{v})\]  \hspace{2cm} (II.5)

The energy Equation (A.I.6) can be written as

\[\rho(\nabla \cdot \mathbf{v})h_t = \nabla \cdot (\mathbb{T} \cdot \mathbf{v}) + k \nabla^2 T\]  \hspace{2cm} (II.6)

where \(\mathbb{T}\) given by Equation (A.I.4) simplifies to

\[\mathbb{T} = \mu \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T\right)\]  \hspace{2cm} (II.7)

II.3. Definition of the streamfunction \(\psi\).

Stokes [9] showed that it is possible to define a streamfunction \(\psi\) for incompressible axisymmetric flows. Using the continuity Equation (II.4) the streamfunction is given by

\[v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}\] \hspace{2cm} (II.8)

\[v_z = \frac{1}{r} \frac{\partial \psi}{\partial r}\]

Instead of two velocity components, now only one function \(\psi\) remains to be determined.

II.4. A differential equation for \(\psi\).

Introducing \(\psi\) and taking the curl of the momentum Equation (II.5) a partial differential equation for \(\psi\) is found. Since the flow is axisymmetric only one
component of the momentum equation in \( \theta \)-direction remains. The curl of Equation (II.5) yields, using \( \nabla \times \text{grad} \ p = 0 \):

\[
\nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) = \nabla \times (\nabla^2 \mathbf{v}) \tag{II.9}
\]

where \( \mathbf{v} \) denotes the kinematic viscosity \( \mu/\rho \).

The left-hand side of Equation (II.9) becomes, when Equations (A.I.4) and (II.1) are introduced:

\[
\nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) = 0 \ \mathbf{e}_r \ + \ \frac{r^2}{\partial z} \left( \frac{\partial v_r}{\partial r} \frac{\partial}{\partial r} + \frac{v_z}{\partial z} \frac{\partial v_r}{\partial z} \right) - \frac{\partial}{\partial r} \left( \frac{\partial v_z}{\partial r} + \frac{v_z}{\partial z} \frac{\partial v_z}{\partial z} \right) \mathbf{e}_\theta \ + \ 0 \ \mathbf{e}_z \tag{II.10}
\]

and when Equations (A.I.4) and (II.1) are introduced, the right-hand side of Equation (II.9) results in:

\[
\nabla \times (\nabla^2 \mathbf{v}) = 0 \ \mathbf{e}_r \ + \ \frac{1}{r^2} \left( \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} + \frac{v_z}{r^2} \frac{\partial^2 v_r}{\partial z^2} \right) - \frac{\partial}{\partial r} \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{v_z}{r^2} \frac{\partial^2 v_z}{\partial z^2} \right) \mathbf{e}_\theta \ + \ 0 \ \mathbf{e}_z \tag{II.11}
\]

Expressions in \( \psi \) for the partial derivatives of \( \mathbf{v} \) with respect to \( r \) and \( z \) can be found by using the defining Equations (II.8). Upon introduction of these derivatives, as listed in Appendix IV, Equation (II.10) yields

\[
[\nabla \times (\mathbf{v} \cdot \nabla) \psi]_\theta = \frac{1}{r^2} \left( \frac{\partial^2 \psi}{\partial z^2} \left( \frac{\partial^3 \psi}{\partial r \partial z^2} + \frac{\partial^3 \psi}{\partial r^3} \right) - \frac{\partial \psi}{\partial r} \left( \frac{\partial^3 \psi}{\partial z^3} + \frac{\partial^3 \psi}{\partial r^2 \partial z} \right) \right) + \frac{1}{r^3} \left( -2 \frac{\partial^2 \psi}{\partial z \partial z} - 2 \frac{\partial \psi}{\partial r} \frac{\partial^2 \psi}{\partial r^2} \frac{\partial \psi}{\partial z} + \frac{\partial \psi}{\partial r} \frac{\partial^2 \psi}{\partial r \partial z} \right) + \frac{3}{4} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial r} \tag{II.12}
\]

and Equation (II.11) becomes

\[
[\nabla \times \nabla^2 \psi]_\theta = -\frac{1}{r} \left( \frac{\partial^4 \psi}{\partial r^4} + 2 \frac{\partial^4 \psi}{\partial r^2 \partial z^2} + \frac{\partial^4 \psi}{\partial z^4} \right) + \frac{4}{r^4} \frac{\partial \psi}{\partial r} \tag{-11-}
\]
\[ + \frac{2}{r^2} \left( -\frac{a^2 \psi}{r^2} + \frac{a^3 \psi}{a r^3} \right) - \frac{3}{r^3} \frac{a^2 \psi}{a r^2} + \frac{3}{r^4} \frac{a \psi}{a r} \]  

Equation (II.13)

Hence, a resulting equation in \( \psi \) can be obtained by combining Equations (II.9), (II.12) and (II.13)

\[
\frac{1}{r^2} \left( \frac{\partial^2 \psi}{\partial r \partial z} - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial r} \right) D^2 \psi + \frac{1}{r^3} \left( 2 \frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial z^2} + 3 \frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial r^2} - \frac{\partial \psi}{\partial r} \frac{\partial^2 \psi}{\partial r \partial z} \right) 
\]

\[ - \frac{3}{4} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial r} = \nu \left[ \frac{1}{r} D^4 \psi + \frac{a}{r^2} \frac{\partial}{\partial r} \left( D^2 \psi \right) \right] + \]

\[ + \frac{3}{r^3} \frac{a^2 \psi}{a r^2} + \frac{3}{r^4} \frac{a \psi}{a r} \]  

Equation (II.14)

where \( D^2 \) is a shorthand notation for the operator \( \frac{\partial^2}{\partial r^2} + \frac{1}{a^2} \frac{\partial^2}{\partial z^2} \).

The solutions of Equation (II.14) are also solutions of the continuity Equation. Substitution of a solution for \( \psi \) in Equation (II.8) gives the corresponding velocity field. With Equation (II.5) the pressure distribution can subsequently be calculated. Once the velocity field is known, Equation (II.6) can be used to find the temperature profile. However, in this report only velocity and pressure distributions will be calculated. Temperature profiles will be determined in the near future.

II.5 Pressure gradients.

In this section the momentum Equation (II.5) is used to find expressions for the axial and the radial pressure gradients in terms of the streamfunction \( \psi \). The momentum Equation (II.5) reads in the axial direction

\[
[p(vv)] \cdot e_z = -\nu p \cdot e_z + [\mu (v^2 v)] \cdot e_z \]  

Equation (II.15)

With the assumptions made in Section II.1, Equation (II.15) becomes in cylindrical coordinates

\[
v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_r}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[ \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right] \]  

Equation (II.16)
where (II.15) is divided by $\rho$ and $v = \mu/\rho$. Using the expressions for the partial derivatives of the velocity components as given in Appendix IV, Equation (II.16) becomes

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = \left( \frac{1}{r} \frac{\partial v}{\partial r} \right) - \frac{1}{r} \frac{\partial v}{\partial r} \frac{\partial^2 v}{\partial r^2}$$

$$v \left[ \frac{1}{r^3} \frac{\partial^3 v}{\partial r^3} - \frac{1}{r^2} \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial^2 v}{\partial r \partial z} - \frac{1}{r} \frac{\partial^2 v}{\partial z^2} \right]$$

Equation (II.17)

The momentum Equation (II.5) in the radial direction is

$$[\rho (v^2 v)] \cdot \mathbf{e}_r = -v_p \cdot \mathbf{e}_r + [\mu (v^2 v)] \cdot \mathbf{e}_r$$

Equation (II.18)

With the assumptions made in Section II.1 this equation can be written as

$$\frac{\partial v}{\partial r} + v_z \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + v \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{\partial v}{\partial r} \frac{\partial v}{\partial z} \right)$$

Equation (II.19)

where $v = \mu/\rho$. In terms of $\psi$ this equation becomes, upon substitution of the expression for the partial derivatives of $v_r$ and $v_z$ with respect to $r$ and $z$ as given in Appendix IV:

$$\frac{1}{\rho} \frac{\partial p}{\partial r} = \left( \frac{1}{r} \frac{\partial v}{\partial r} \right) - \frac{1}{r} \frac{\partial v}{\partial r} \frac{\partial^2 v}{\partial r^2}$$

$$v \left[ \frac{1}{r^3} \frac{\partial^3 v}{\partial r^3} - \frac{1}{r^2} \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial^2 v}{\partial r \partial z} - \frac{1}{r} \frac{\partial^2 v}{\partial z^2} \right]$$

Equation (II.20)
III. A channel with constant cross sections.

In this chapter the flow through a channel with constant cross sections is studied. As the mass injection through the wall is supposed to be constant, the local mass flow rate in the channel is a linear function of the axial coordinate. The partial differential equation for $\psi$, as derived in Chapter II, is rewritten in a non-dimensional form, in order to be able to estimate the relative importance of the various terms. A Reynolds injection number $\lambda$ appears in the non-dimensional equation for the streamfunction. This parameter $\lambda$ is a measure for the relative importance of inertial forces and viscous forces: for small values of $\lambda$ the viscous forces dominate (Poiseuille flow), while for large values of $\lambda$ the inertial forces dominate. For both cases an exact solution is found. For moderate values of $\lambda$ an approximate solution is found by expanding the non-dimensional streamfunction in a power series of the non-dimensional coordinates.

III.1. Dimensionless form of the governing equations.

Equation (II.14) for $\psi$ as derived in Chapter II is a non-linear partial differential equation (PDE) of order 4. Since there is no general analytical means of solving non-linear PDE's with an order higher than 2, further approximations have to be made. As the injected mass is supposed to be constant, the mass flow rate can be written as

$$M(z) = M_0 (1 + \alpha z)$$  \hspace{1cm} (III.1)

where $M_0$ is the mass flow rate at the entrance of the channel, and $\alpha M_0$ is the injected mass flow rate per unit of length. For the flows being considered, $\alpha$ is a small parameter, say of order $10^{-2}$.

A dimensionless streamfunction $F = \rho \psi / M$ is introduced, where $M$ is the local mass flow rate and $\rho$ the density. Partial derivation of $\psi(r,z) = M_v(z) F(r,z)$ yields

$$\frac{\partial^n \psi}{\partial r^n} = M_v \frac{\partial^n F}{\partial r^n}$$  \hspace{1cm} (III.2)

and

$$\frac{\partial^n \psi}{\partial z^n} = M_v \frac{\partial^n F}{\partial z^n} + \frac{1}{\rho} \frac{\partial M_v}{\partial z} \frac{\partial^{n-1} F}{\partial z^{n-1}}$$  \hspace{1cm} (III.3)
where \( \frac{\partial M_v}{\partial z} = \frac{\rho}{\rho} M_0 \). Expressions for \( \frac{\partial^m}{\partial z^m} \left( \frac{\partial^n}{\partial r^n} \right) \) are found by substituting (III.2) in (III.3)

\[
\frac{\partial^{m+n}}{\partial z^m \partial r^n} (\psi) = M_v \frac{\partial^{m+n}}{\partial z^m \partial r^n} (F) + m \frac{\partial M_v}{\partial z} \frac{\partial^{m+n-1}}{\partial z^{m-1} \partial r^n} (F)
\]

(III.4)

which also holds for \( m=0 \) and \( n=0 \) if by definition \( \frac{\partial^0}{\partial z^0} = 1 \) and \( \frac{\partial^0}{\partial r^0} = 1 \).

The coordinates \( r \) and \( z \) are made dimensionless by introducing characteristic scales in respectively \( r \)-and \( z \)-directions. For a channel of length \( L \) and of radius \( R_0 \), the dimensionless coordinates are defined as

\[
\eta = \frac{r}{R_0}, \quad \zeta = \frac{z}{L}
\]

(III.5)

Introduction of (III.5) yields for the partial differential operators of the general type (with \( m,n \) integer):

\[
\frac{\partial^{m+n}}{\partial z^m \partial r^n} = \frac{1}{L^m R_0^n} \frac{\partial^{m+n}}{\partial \eta^m \partial \zeta^n}
\]

(III.6)

After \( F \) and the non-dimensional coordinates \( \eta \) and \( \zeta \) are introduced, and the result is ordered in powers of \( \frac{R_0}{L} \), Equation (II.14) for \( F \) reads

\[
\frac{\alpha M_v M_0}{\rho R_0^5} \left( \frac{1}{n^2} \frac{\partial F}{\partial \eta} + \frac{3}{n^3} \frac{\partial^2 F}{\partial \eta^2} \right) + \frac{\nu M_v}{R_0^5} \left( \frac{1}{n} \frac{\partial^2 F}{\partial \eta^2} + \frac{1}{n^2} \frac{\partial^3 F}{\partial \eta^3} \right) + \frac{\alpha^2 M_v^2}{\rho^2 R_0^4} \left( \frac{1}{n^2} \frac{\partial^2 F}{\partial \eta^2} + \frac{1}{n^3} \frac{\partial^3 F}{\partial \eta^3} \right)
\]
\[
\frac{\nu M_o}{4} \left[ - \frac{4}{n} \frac{a^3 F}{\alpha \eta n^2} + \frac{4}{n} \frac{a^2 F}{\alpha \eta n} \right] + 
\]

\[
\frac{R_o}{L}^2 \left[ \frac{a M V_o}{\rho R_o} \left( \frac{3}{n} \frac{a F}{\eta n^2} - \frac{1}{n} \frac{a^3 F}{\eta \alpha \zeta^2} - \frac{2}{n} \frac{a F}{\eta \alpha \zeta} \right) \right] + 
\]

\[
\frac{\nu M V}{5} \left[ - \frac{2}{n} \frac{a^4 F}{\eta \alpha^2 \eta n^2} + \frac{2}{n^2} \frac{a^3 F}{\eta \alpha \zeta^2} \right] + 
\]

\[
\frac{R_o}{L}^3 \left[ \frac{M^2}{R_o^5} \left( \frac{1}{n} \frac{a F}{\eta n^3} + \frac{1}{n^2} \frac{a^3 F}{\eta \alpha \zeta^2} + \frac{2}{n^2} \frac{a F}{\eta \alpha \zeta} - \frac{\nu M}{4} \frac{a^3 F}{\eta n \alpha \zeta^3} \right) \right] = 0 \quad \text{(III.7)}
\]

If \( M_1 \) represents the total injected mass flow rate, the total injected fraction of the mass flow rate \( \bar{\alpha} \) per unit of length can be written as

\[
\bar{\alpha} = \frac{M_1}{M_o} \quad \text{(III.8)}
\]

Since \( \alpha \) is the fraction of the mass flow rate that is injected, \( \bar{\alpha} \) equals \( \alpha L \). If Equation (III.7) is divided by \( \frac{\nu M V}{5} \) the first factor on the LHS of this equation can be written as \( \frac{\alpha M_o}{\nu \rho R_o} \). With the Reynolds number \( \text{Re} = \frac{M_o}{\nu V L} \), this factor equals \( \bar{\alpha} \text{Re} \). By definition \( \bar{\alpha} \text{Re} \) is the Reynolds number \( \text{Re}_1 \) of all injected mass after \( L \) meters, and a parameter \( \Lambda \) can be defined as

\[
\Lambda = \text{Re}_1 = \bar{\alpha} \text{Re} \quad \text{(III.9)}
\]

The flow through a pipe of length \( L \) and radius \( R_o \) is supposed to be a slender flow. This means that changes in the axial direction occur at a much larger scale than changes in the radial direction; hence the number \( \frac{R_o}{L} \) is a small number and terms of power one and higher in \( \frac{R_o}{L} \) can be neglected, unlike terms of power 0. Equation (III.7) yields

\[
\Lambda \left[ \frac{1}{n^2} \frac{a F}{\eta n^2} \left( \frac{a^2 F}{\eta n^2} - \frac{1}{n} \frac{a F}{\eta n^2} \right) + \frac{1}{n} \frac{a^3 F}{\eta \alpha \zeta^2} \right] + \frac{1}{n} \frac{a F}{\eta \alpha \zeta} \left[ - \frac{a F}{\eta n^3} + \frac{3}{n} \frac{a^2 F}{\eta n^2} - \frac{3}{n^2} \frac{a F}{\eta n^2} \right] =
\]

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\[
\left[ \frac{1}{\eta} \frac{\alpha F}{4} - \frac{2}{\eta^2} \frac{\alpha^2 F}{6} + \frac{3}{\eta^3} \frac{\alpha^2 F}{2} - \frac{3}{\eta^4} \frac{\alpha F}{4} \right]
\]  \quad (III.10)

Since \( \lambda \) is independent of \( \zeta \), solutions for \( F \) can be found independent of \( \zeta \). Equation (III.10) is solved exactly for two limiting cases \( \lambda \).

For moderate values of \( \lambda \) an approximation of a solution for \( F \) is found by substituting a power series in \( \eta \) for \( F \).

For \( \lambda \to 0 \) there is no mass injection through the wall and the resulting flow is the well-known Poiseuille flow.

For \( \lambda \to \infty \) the viscous terms in Equation (III.10) can be neglected and an exact solution can be found.

III.2 Boundary conditions.

The boundary conditions for flow through a channel of constant cross section and constant mass injection are:

1) The axial velocity at the wall equals zero.

2) The radial velocity at the wall equals the velocity of the injected mass.

3) The axial velocity at the center of the tube is finite. At the entrance of the tube it equals by definition \( v_c \).

4) For symmetry reasons the radial velocity at the center equals zero.

Since the radius and the injected mass per unit of length are constant the velocity of injection \( v_i \) is given by

\[
v_i = -\frac{\alpha M_0}{2 \rho n R_0^2}
\]  \quad (III.11)

The velocity components \( v_r \) and \( v_z \) are given by (II.8). In non-dimensional terms these read

\[
v_r = -\frac{1}{\rho n R_0^2} \left[ \frac{M}{L} \frac{\partial F}{\partial \zeta} + \alpha M_0 F \right]
\]  \quad (III.12)

\[
v_z = \frac{1}{\rho n R_0^2} M \frac{\partial F}{\partial \eta}
\]

The first boundary condition \( v_z = 0 \) at \( \eta = 1 \) yields, with (III.12):

\[
\frac{\partial F}{\partial \eta} (1) = 0
\]  \quad (III.13)
The second boundary condition at \( n=1 \) following from Equation (III.12) is, since \( F \) is supposed to be independent of \( \zeta \),

\[
- \frac{1}{R_o} \frac{M_o}{\rho} F(1) = - \frac{1}{\rho} \frac{M_o}{2\pi R_o} \tag{III.14}
\]

or

\[
F(1) = \frac{1}{2\pi} \tag{III.15}
\]

The third boundary condition becomes, with Equation (III.12):

\[
\lim_{n \to 0} \left( \frac{1}{n R_o} \frac{\partial^2 F}{\partial n^2} \right) = v_c < \infty \tag{III.16}
\]

This means that the lowest power of \( n \) in a series expansion of \( F \) is \( n^k \) with \( k \geq 2 \), in which case the fourth boundary condition is also fulfilled.

III.3 Limiting cases of \( \lambda \).

There are two values of \( \lambda \) for which Equation (III.10) can be solved exactly, namely

1) \( \lambda \) is small, so \( \lambda \to 0 \). This type of flow is called a type I flow.

2) \( \lambda \) is very large, so \( \lambda \to \infty \). This type of flow is called a type II flow.

If \( \lambda \) tends to zero the left-hand side of (III.10) can be neglected and the equation to be obtained becomes, after integrating once with respect to \( n \),

\[
\frac{1}{n^3} \frac{\partial F}{\partial n} - \frac{1}{n^2} \frac{\partial^2 F}{\partial n^2} + \frac{1}{n} \frac{\partial^3 F}{\partial n^3} = C_1 \tag{III.17}
\]

The introduction of a new function \( G = \frac{1}{n} \frac{\partial F}{\partial n} \) in Equation (III.17) yields

\[
\frac{\partial^2 G}{\partial n^2} + \frac{1}{n} \frac{\partial G}{\partial n} = C_1 \tag{III.18}
\]

Solutions for \( G \) are \( G = \frac{C_1}{4} n^2 + C_2 \ln n + C_3 \). The corresponding solutions for \( F \) can be written as

\[
F = \frac{C_1}{16} n^4 + \frac{C_2}{2} n^2 \ln n + \frac{1}{2}(C_3 - \frac{1}{2}C_2) n^2 + C_4 \tag{III.19}
\]
By substituting the boundary conditions, one gets $C_1 = -\frac{8}{\pi}$, $C_2 = 0$, $C_3 = \frac{2}{\pi}$ and $C_4 = 0$. Therefore $P = \frac{1}{\pi} \left(-\frac{1}{2} \eta^4 + \eta^2\right)$. This solution is the well-known Poiseuille flow. The corresponding velocity profile is given in Figure III.1.

![Figure III.1. Axial velocity profile for Poiseuille flow.](image)

The axial velocity at $\eta = 0$ is

$$v_c = \frac{2}{\pi} \frac{1}{\rho} \frac{M_0}{R_o^2}$$  \hspace{1cm} (III.20)

In Figure III.1 the dimensionless velocity $\bar{v}_z = v_z / v_c$ is drawn. For Poiseuille flow the radial velocity and the radial pressure gradient are both equal to zero. The axial pressure gradient can be calculated from Equation (II.17). In dimensionless coordinates the axial pressure gradient is

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\[
\frac{\partial p}{\partial t} = - \frac{8}{\pi} \frac{\nu M O L}{R_o^4} \tag{III.21}
\]

Since the pressure gradient according to (III.21) is a constant, the pressure is a linear function of \(z\), and the pressure profile is a straight line. The pressure drop along the channel is

\[
\Delta p_c = - \frac{8}{\pi} \frac{\nu M O}{R_o^4} L \tag{III.22}
\]

For large values of \(\lambda\) the left-hand side of (III.10) becomes the dominant term. Upon integration with respect to \(\eta\) the equation for \(F\) becomes

\[- \frac{1}{\eta^2} \int F \frac{\partial^2 F}{\partial \eta^2} + \frac{1}{\eta^3} \int F \frac{\partial F}{\partial \eta} + \frac{1}{\eta^2} \left(\frac{\partial F}{\partial \eta}\right)^2 = C_1 \tag{III.23}\]

where \(C_1\) is a constant. This equation can be simplified by introducing a new variable \(t = \eta^2\). With \(\frac{\partial}{\partial \eta} = \frac{\partial}{\partial t} \frac{\partial}{\partial \eta} = 2\eta \frac{\partial}{\partial t}\) and \(\frac{\partial^2}{\partial \eta^2} = 2 \frac{\partial}{\partial t} + 4\eta^2 \frac{\partial^2}{\partial t^2}\), Equation (III.23) becomes

\[- F \frac{\partial^2 F}{\partial t^2} + \left(\frac{\partial F}{\partial t}\right)^2 = \frac{C_1}{4} \tag{III.24}\]

This equation can also be solved analytically. As the solution \(F\) is a function of \(t\) only, \(F = F(t)\) and \(t = t(F)\). If \(G\) is defined by \(G(F) = \frac{\partial F}{\partial t}\), Equation (III.24) yields

\[\frac{\partial^2 F}{\partial t^2} = \frac{\partial}{\partial t}(G) = \frac{\partial G}{\partial F} \frac{\partial F}{\partial t} = F \frac{\partial G}{\partial F} \tag{III.25}\]

The introduction of Equation (III.25) in Equation (III.24) results in a differential equation for \(G\), written as

\[- F \frac{\partial G}{\partial F} + G^2 = \frac{C_1}{4} \tag{III.26}\]

If \(x\) and \(u\) are defined by \(x = F\) and \(u = G^2\) the resulting equation is

\[- x \frac{1}{2} \frac{\partial u}{\partial x} + u = \frac{C_1}{4} \tag{III.27}\]
Separation of variables yields a solution for \( u \) given by \( u = C_2 x^2 + \frac{C_1}{4} \). Hence

\[
G^2 = C_2^2 F^2 + \frac{C_1}{4} \quad \text{(III.28)}
\]

where \( C_1 \) and \( C_2 \) are constants of integration. With (III.28) Equation (III.24) becomes

\[
F \left( -\frac{\partial^2 F}{\partial t^2} + C_2^2 F \right) = 0 \quad \text{(III.29)}
\]

The non-trivial solutions for \( F \) are given by \( (C_2 \neq 0, \quad C_2 = 0 \text{ yields potential flow}) \)

\[
F = C_3 e^{-C_2 t} + C_4 e^{C_2 t} \quad \text{(III.30)}
\]

where \( C_2, \quad C_3 \) and \( C_4 \) are constants of integration to be determined from the boundary conditions. A value for \( C_2 \) can be obtained by using \( v_r = 0 \) at \( \eta = 0 \). \( C_1 \) can be expressed in \( C_2, \quad C_3 \) and \( C_4 \) by substituting (III.30) in (III.24). The boundary conditions yield

\[
\begin{align*}
& v_r = 0 \text{ at } t = 0 : \quad C_3 = -C_4 \\
& v_z = 0 \text{ at } t = 1 : \quad C_2 C_4 (e^{-C_2} + e^{C_2}) = 0 \quad \text{(III.31)} \\
& F = \frac{1}{2\pi} \text{ at } t = 1 : \quad C_4 (-e^{-C_2} + e^{C_2}) = \frac{1}{2\pi}
\end{align*}
\]

Equations (III.31) yield \( C_2 = \frac{n}{2} \), \( C_3 = \frac{1}{4\pi} i \) and \( C_4 = \frac{1}{4\pi} i \). Hence \( C_1 \) becomes

\[
C_1 = -16 C_3 C_4 C_2^2 = \frac{1}{4} \quad \text{(III.32)}
\]

The function \( F \) that fulfills Equation (III.24) and the boundary conditions is then

\[
F = \frac{1}{2\pi} \sin \frac{n}{2} \frac{\eta^2}{2} \quad \text{(III.33)}
\]

The axial and radial velocities resulting from Equations (III.12) are
\[ v_z = \frac{M_v}{2R_o} \cos \frac{n^2}{2} \]  
\[ v_r = -\frac{\alpha M_o}{\rho 2\pi R_o} \frac{\sin \frac{n^2}{2}}{n} \]  
\[ (III.34) \]

For large values of \( \lambda \) the axial pressure difference \( \Delta p_z \) between the entrance and a point \( z \) downstream can be calculated from the Navier Stokes Equation in the axial direction which equals

\[ \frac{\partial p}{\partial z} = \frac{\nu M_o}{R_o^4} \lambda \left( \frac{1}{n^3} F \left[ \frac{2}{\partial n^2} - \frac{3}{\partial n^3} \right] \right)^2 - \eta \left[ \frac{\partial F}{\partial n} \right]^2 \]  
\[ (III.35) \]

This expression for the axial pressure gradient can be integrated once with respect to \( z \), yielding, with the aid of Equation (III.23):

\[ \Delta p_z = -\frac{\nu M_o}{4 R_o^2} \lambda z (1 + \frac{a}{2}) \]  
\[ (III.36) \]

If a mean velocity \( \bar{v} \) is defined by \( \bar{v} = \int_0^\infty \frac{v}{2 R} \frac{r v}{r} \frac{dr}{r^2} \) it can easily be shown that \( M_o \) is proportional to \( \rho \bar{v} \) and that \( \Delta p_z \) can be written in a more conventional form as \( \Delta p_z = -c_f(z) \rho \bar{v}^2 \).

The radial pressure difference \( \Delta p_\eta \) between the center and a point \( \eta \) can be calculated from the equation of motion in the radial direction which equals

\[ \frac{\partial p}{\partial \eta} = \frac{\nu a M_o}{R_o^2} \lambda F \left( \frac{1}{\eta^3} F - \frac{1}{\eta^2} \frac{\partial F}{\partial \eta} \right) \]  
\[ (III.37) \]

Integration once with respect to \( \eta \) yields for the pressure difference in radial direction from the center to a point \( \eta \):

\[ \Delta p_\eta = -\frac{\nu a M_o}{R_o^2} \lambda \frac{1}{2\eta^2} F^2 \]  
\[ (III.38) \]

Upon substitution of \( F \) according to (III.33) \( \Delta p_\eta \) becomes

\[ \Delta p_\eta = -\frac{\nu a M_o}{8 R_o^2 n^2} \lambda \frac{1}{2} \sin^2 \frac{\pi n^2}{2} \]  
\[ (III.39) \]
Velocity and pressure profiles are presented in Figures III.2a through to III.2d. The velocities plotted in these figures are dimensionless velocities $\bar{v}$ with $\bar{v} = \frac{v}{v_c}$ and where $v_c$ is the center velocity for Poiseuille flow. The pressure differences $\Delta p$ in Figures 2c and 2d are dimensionless too, where $\bar{\Delta p} = \frac{\Delta p}{\Delta p_c}$ and $\Delta p_c$ is the axial pressure drop in a channel for Poiseuille flow.
Figure III.2a Axial velocity profile for type II flow.

Figure III.2b Radial velocity profile for type II flow.
\[ M_0 = 0 \]
\[ \frac{1}{M_0} \frac{\partial \bar{M}}{\partial \zeta} = 0.245 \]
\[ \lambda = 245 \]
profile at \( \eta = 0 \)

Figure III.2c Axial pressure profile for type II flow.

\[ M_0 = 0 \]
\[ \frac{1}{M_0} \frac{\partial \bar{M}}{\partial \zeta} = 0.245 \]
\[ \lambda = 245 \]
profile at \( \zeta = 0 \)

Figure III.2d Radial pressure profile for type II flow.
III.4 Moderate values of the injection parameter $\lambda$.

III.4.1 Approximate solutions of the equation of motion.

In order to find approximate solutions of Equation (III.10) $F$ is expanded in a power series of $n$. If $n$ is defined as the highest power of $n$ in this series, $F$ can be written as

$$F = \sum_{i=0}^{n} a_{i} n^{i}$$  \hspace{1cm} (III.40)

In this section values of the coefficients $a_{2i}$ are determined for $n=6, 8, 10$ and 12. For $n=6$ Equation (III.10) is met by equating terms of equal power $n^{m}$ for all $m$. This implies that the order of the factors of $n^{m}$ on the left-hand side equals the order of the factors of $n^{m}$ on the right-hand side. If terms of powers $n^{m}$ on the right-hand side are neglected, the left-hand side should also be truncated after powers of $n^{m}$. Therefore, the approximate solution $F$ of Equation (III.10) holds only for powers of $n^{m}$ when $m<=n$. Since it is not possible to find a solution for $n=6$ a criterion which gives an impression of the convergence of the power series substituted for $F$ has to be formulated. A convergence number $C_{v}$ is defined by

$$C_{v} = \left| \frac{a_{n}}{\sum_{i=0}^{n-1} a_{i}} \right|$$  \hspace{1cm} (III.41)

The criterion used is whether $C_{v} < 1$. Values of $C_{v}$ can be calculated once the values of the coefficients $a_{i}$ are known.

Equation (III.10) can be integrated once with respect to $n$, yielding, after multiplication with $n^{2}$:

$$\lambda[F(n) \frac{\partial F}{\partial n^{2}} - \frac{\partial F}{\partial n}] - \frac{\partial^{2} F}{\partial n^{2}} + \left[ \frac{\partial F}{\partial n} - \frac{\partial^{2} F}{\partial n^{2}} + \frac{n}{2} \frac{\partial^{2} F}{\partial n^{2}} \right] = Cn^{3}$$  \hspace{1cm} (III.43)

where $C$ is a constant of integration. Expanding $F$ in a series for $n$ according to (III.40) yields for terms up to powers of $n^{n}$:

$$n \frac{\partial F}{\partial n^{2}} - \frac{\partial F}{\partial n} = \sum_{i=0}^{n-2} (i+1) (i+2) a_{i+2} n^{i+1} - \sum_{i=0}^{n-1} (i+1) a_{i+1} n^{i}$$
\[= - a_1 + \sum_{i=0}^{n-2} i(i+2) a_{i+2} \eta^{i+1}\]  \hspace{1cm} (III.44)

and

\[
\frac{\partial F}{\partial \eta} \eta \frac{\partial^2 F}{\partial \eta^2} + \eta^2 \frac{\partial^3 F}{\partial \eta^3} = a_1 - \sum_{i=0}^{n-2} i(i+2) a_{i+2} \eta^{i+1} + \\
\sum_{i=0}^{n-3} (i+3)(i+2)(i+1) a_{i+3} \eta^{i+2} = \\
= a_1 - \sum_{i=0}^{n-3} (i+1)^2(i+3) a_{i+3} \eta^{i+2}\]  \hspace{1cm} (III.45)

The third boundary condition (III.16) states that the lowest power \(n\) in \(F\) is 2, so \(a_0 = a_1 = 0\). With Equation (III.44) and the general expression (A.3.9) for the product of two series, \(F (\eta \frac{\partial^2 F}{\partial \eta^2} - \frac{\partial F}{\partial \eta})\) becomes

\[
F (\eta \frac{\partial^2 F}{\partial \eta^2} - \frac{\partial F}{\partial \eta}) = \sum_{m=1}^{2n-1} \min(n,m-1) \sum_{j=\max(0,m-2)}^{m-j+1} a_j a_{m-j+1} \eta^m
\]  \hspace{1cm} (III.46)

In the same way the product \(\eta (\frac{\partial F}{\partial \eta})^2\) becomes

\[
\eta (\frac{\partial F}{\partial \eta})^2 = \sum_{m=1}^{2n-1} \min(n-1,m-1) \sum_{j=\max(0,m-n)}^{m-j+1} a_j a_{m-j} \eta^m
\]  \hspace{1cm} (III.47)

Since \(a_0\) and \(a_1\) are equal to zero, terms of power \(n^1\) in Equations (III.46) and (III.47) are zero. Changing the index in Equation (III.45) yields an expression for the coefficients \(a_i\) given by

\[
\lambda \sum_{m=2}^{2n-1} \min(n,m-1) \sum_{j=\max(0,m-n+1)}^{m-j+1} a_j a_{m-j+1} + \\
- \sum_{j=\max(0,m-n)}^{m-j+1} (j+1) a_{j+1} a_{m-j} \eta^m + \\
+ \sum_{m=2}^{n-1} (m-1)^2(m+1) a_{m+1} \eta^m = C \eta^3
\]  \hspace{1cm} (III.48)
As was reasoned in the beginning of this section, terms of powers of \( n \) of order \( n \) and higher are neglected in Equation (III.48). For values \( 2 \leq m \leq n-1 \) the equation becomes

\[
\sum_{j=1}^{m-1} \sum_{j=1}^{m-1} \begin{array}{l}
a_j a_{m-j+1} - (j+1) (m-j) a_{j+1} a_{m-j} + \\
+ (m-1)^2 (m+1) a_{m+1} \end{array} n^m = c n^3
\]  

(III.49)

Then an expression for \( a_{m+1} \) can be obtained from Equation (III.49) for \( m=3 \):

\[
a_{m+1} = \frac{\lambda}{(m-1)^2 (m+1)} \sum_{j=1}^{m-1} [(j+1)(m-j) a_{j+1} a_{m-j} + \\
- (m-j-1) (m-j+1) a_j a_{m-j+1}] \]

(III.50)

while for \( m=3 \) Equation (III.49) yields

\[
c = 16 a_4 - 4 a_2^2 \lambda \]

(III.51)

The flow is supposed to be axisymmetric. This implies that values \( a_m \) for odd \( m \) are equal to zero. This can easily be seen by replacing \( n \) by \(-n\) in (III.43). Since the equation remains the same it follows directly that \( F(n) = F(-n) \). Therefore \( a_{2j+1} = 0 \) for all \( j \). With Equation (III.50) the same result can be derived by putting \( m=2j \) and \( a_1=0 \). The values \( a_m \) for even \( m \) follow from Equation (III.50). When the index \( j \) is replaced by \( j=2\ell-1 \) in the first series in (III.50) and by \( j=2\ell \) in the second series, Equation (III.50) simplifies to

\[
a_{2n} = \frac{\lambda}{2(n-1)^2} \begin{array}{l}
[ (4-n) (n-1) a_{2n-2} a_2 + \\
+ \sum_{\ell=2}^{n-2} a_{2\ell} a_{2n-2\ell} (n-\ell) (2\ell+1-n) ]
\end{array}
\]

(III.52)

Now that expressions can be found for the coefficients \( a_1 \) in Equation (III.40) \( a_2 \) and \( a_4 \) appear to be independent coefficients and \( a_{2n+1}=0 \). The coefficients \( a_n \) for \( n=6,8,10 \) and 12 are

\[
a_6 = \frac{1}{12} \lambda a_4
\]
\[ a_8 = \frac{1}{36} \lambda a_4^2 \]  

(III.53)

\[ a_{10} = \frac{1}{40} \lambda (-a_2 a_8 + a_4 a_6) \]

\[ a_{12} = \frac{1}{300} \lambda (-10 a_2 a_{10} + 2a_4 a_8 + 3a_6^2) \]

Equation (III.53) is a system of 4 equations with 6 unknown variables. Two more equations result from the boundary conditions given by Equations (III.13), (III.15) and (III.16). The last condition is already fulfilled since \( a_0 = a_1 = 0 \). For \( F \) given by (III.40) Condition (III.13) becomes

\[ \sum_{i=2}^{n+1} a_i = 0 \]  

(III.54)

and (III.15) becomes

\[ \sum_{i=2}^{n} a_i = \frac{1}{2\pi} \]  

(III.55)

The system of Equations (III.53), (III.54) and (III.55) can be solved for several values of \( n \) in \( F = \sum_{i=0}^{n} a_i n^i \). The values for \( n \) to be studied are \( n = 6, 8, 10 \) and 12. For values of \( n > 12 \) the complexity of the resulting system of equations does not allow a simple analytical approach. Consequently, values of \( n > 12 \) are not considered. For \( n = 4 \) Equations (III.54) and (III.55) immediately yield a solution for \( F \) corresponding to Poiseuille flow. To solve the system of equations all coefficients are expressed as functions of \( a_2 \) and \( a_4 \).

If Equations (III.54) and (III.55) are used to eliminate the coefficient \( a_i \) with the highest index, the resulting equation gives, upon substitution of (III.53), an expression in \( a_2 \) and \( a_4 \). Rearranging this equation gives

\[ a_2 = \frac{\frac{3}{\pi} - 4a_2 - e_8 \frac{\lambda}{18} a_4^2}{5 + \frac{\lambda}{4} a_4 + e_{10} \frac{\lambda^2}{720} a_4^2} \]  

(III.56)

where \( e_8 = 0 \) for \( n = 6 \) and \( e_8 = 1 \) for \( n \) equal to 8, 10 or 12; \( e_{10} = 0 \) for \( n \) equal to 6 or 8 and \( e_{10} = 1 \) for \( n \) equal to 10 or 12. Substituting \( a_2 \) according to (III.56) in Equations (III.53) and (III.54) yields the following equations in \( a_2 \) for all values of \( n \):
n=6: \[-\frac{\Lambda}{2} a_4^2 + (6 + \frac{3}{4} \frac{\Lambda}{n}) a_4 + \frac{3}{n} = 0\] (III.57)

n=8: \[\frac{\Lambda^2}{72} a_4^3 + (6 + \frac{3}{4} \frac{\Lambda}{n}) a_4 + \frac{3}{n} = 0\] (III.58)

n=10: \[-\frac{\Lambda^3}{6} a_4^4 - 8 \Lambda^2 a_4^3 + \frac{15}{n} \Lambda^2 a_4^2 + (4320 + 540 \frac{\Lambda}{n}) a_4 + \frac{2160}{n} = 0\] (III.59)

n=12: \[\frac{\Lambda^6}{100} a_4^7 + \frac{41}{10} \Lambda^5 a_4^6 + 342 \Lambda^4 a_4^5 + 27 \Lambda^3 (184 - 3 \frac{\Lambda}{n}) a_4^4 + \]
\[+ 4536 \Lambda^2 (-20 + \frac{3}{4} \frac{\Lambda}{n}) a_4^3 + 1944 \Lambda (3600 + 710 \frac{\Lambda}{n} + 3 \frac{\Lambda^2}{n^2}) a_4^2 + \]
\[+ 36^3 \cdot 150 (20 + \frac{3}{4} \frac{\Lambda}{n}) a_4 + 36^3 \frac{150}{n} = 0\] (III.60)

With the aid of these equations one obtains a value for \(a_4\) (numerically or analytically) for each value of \(\Lambda\). Subsequently, Equations (III.56) and (III.53) yield values for the remaining coefficients \(a_i\). In the following sections the velocity profiles and pressure gradients for several values of \(n\) in \(F = \sum_{i=2}^{n} a_i n^i\) are calculated.

### III.4.2 The axial velocity \(v_z\).

The axial velocity \(v_z\) calculated from Equation (III.12) yields

\[v_z = \frac{M_v}{R_o^2} \frac{1}{n} \frac{\partial F}{\partial n}\] (III.61)

If \(F\) is a power series in \(n\) according to (III.40) \(v_z\) becomes

\[v_z = \frac{M_v}{R_o^2} \sum_{i=2}^{n} a_i n^{i-2}\] (III.62)

For several values of \(\Lambda\), \(v_z\) profiles are presented in graphs a and b of Figures III.3 up to III.10. \(v_z\) is the dimensionless velocity \(v_z/v_c\), where \(v_c\) is the axial velocity at \(n=0\) for Poiseuille flow. A discussion of these profiles is postponed until Section III.4.8.
III.4.3 The radial velocity $v_r$.

The radial velocity $v_r$ can be written as

$$v_r = -\frac{1}{\rho n R_o} \left( \frac{M}{L} \frac{\partial F}{\partial \zeta} + \alpha M_o F \right) \quad \text{(III.63)}$$

Since $F$ is independent of $\zeta$ this expression reduces to

$$v_r = -\frac{\alpha M_o n}{\rho R_o} \sum_{i=2}^{\infty} a_i n^{i-1} \quad \text{(III.64)}$$

Figures c and d of Figures III.3 to III.10 present the dimensionless velocity $\bar{v}_r$ for several values of $\lambda$. The plotted velocity is $\bar{v}_r = \frac{v_r}{v_c}$ at $n = 0.5$, where $v_c$ is the axial velocity at $n=0$ for Poiseuille flow. These profiles will be discussed in Section III.4.8.

III.4.4 The axial pressure difference $\Delta p_z$.

The axial pressure difference $\Delta p_z$ is found from the momentum equation in the axial direction. In terms of the dimensionless streamfunction $F = \psi / M_v$ the momentum Equation (II.17) becomes

$$\frac{1}{r^2} \left[ \frac{M}{M_v} \frac{\partial F}{\partial z} + \alpha \frac{M_o}{\rho} F \right] \left[ \frac{1}{r} \frac{M}{M_v} \frac{\partial F}{\partial r} - \frac{M}{M_v} \frac{\partial^2 F}{\partial r^2} \right] +$$

$$\frac{1}{r^2} \frac{M}{M_v} \frac{\partial F}{\partial r} \left[ \frac{M}{M_v} \frac{\partial^2 F}{\partial r \partial z} + \frac{M_o}{\rho} \frac{\partial F}{\partial r} \right] = -\frac{1}{\rho} \frac{\partial p}{\partial z} +$$

$$\frac{M}{r} \left[ \frac{1}{r^2} \frac{\partial F}{\partial r} - \frac{3}{r} \frac{\partial^2 F}{\partial r^2} + \frac{3}{r^2} \frac{\partial^3 F}{\partial r^3} + \frac{\partial^3 F}{\partial r \partial z^2} \right] + 2 \frac{\nu a M_o}{\rho r} \frac{\partial^2 F}{\partial r \partial z} \quad \text{(III.65)}$$

By introducing the non-dimensional coordinates $(\eta, \zeta)$ according to Definition (III.5) this equation becomes

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = \frac{\alpha M M_o}{\rho R_o} \left[ -\frac{1}{n} \frac{\partial F}{\partial \eta} + \frac{1}{n^2} \frac{\partial^2 F}{\partial \eta^2} - \frac{1}{n^2} \left( \frac{\partial F}{\partial \eta} \right)^2 \right] +$$
\[
\begin{align*}
M_v^2 & \left[ \frac{1}{R_0} \left( \frac{1}{\eta} \frac{\alpha F}{\eta} \right) + \frac{1}{2} \left( \frac{\alpha F}{\eta} \right)^2 \right] \nonumber \\
& + \frac{\nu M_v}{R_0} \left[ \frac{1}{\eta} \frac{\alpha F}{\eta} + \frac{1}{2} \left( \frac{\alpha F}{\eta} \right)^2 \right] + \frac{1}{\eta} \frac{\alpha F}{\eta} + \frac{1}{2} \left( \frac{\alpha F}{\eta} \right)^2 \right] \nonumber \\
& + 2 \frac{\nu a M_o}{\rho R_0} \frac{1}{2} \frac{1}{L} \frac{1}{\eta} \frac{\alpha F}{\eta} \nonumber \\
\end{align*}
\]

(III.66)

Since \( F \) is independent of \( \zeta \) and \( \lambda = \frac{aM_o}{\rho \nu} \) the axial pressure gradient can be written as

\[
\frac{\partial p}{\partial z} = \frac{\nu M_v}{R_0} \frac{\lambda}{\eta} \left[ F \left( \eta \frac{\partial^2 F}{\partial \eta^2} - \eta \frac{\partial F}{\partial \eta} \right) - \eta \left( \frac{\partial F}{\partial \eta} \right)^2 \right] \right] + \frac{1}{3} \left( \frac{\partial F}{\partial \eta} \right)^2 \right] \right] \nonumber \\
\end{align*}
\]

(III.67)

Close examination of Equations (III.67) and (III.43) shows that

\[
\frac{\partial p}{\partial z} = \frac{\nu M_v}{R_0} C \nonumber \\
\]

(III.68)

or if \( C \) is substituted according to (III.51) the axial pressure gradient becomes

\[
\frac{\partial p}{\partial z} = \frac{\nu M_v}{R_0} \left( 16a_4 - 4a_2^2 \lambda \right) \nonumber \\
\]

(III.69)

Consequently, the pressure difference between the inlet to a point \( z=a \) downstream equals

\[
\Delta p_z = p_z(n,a) - p_0(n,0) = \int_0^a \left( 16a_4 - 4a_2^2 \lambda \right) \frac{\nu M_v}{R_0} d\eta \nonumber \\
\]

(III.70)

The local pressure at a point \( z \) is then

\[
p_z(n,z) = p_0(n,0) + \left( 16a_4 - 4a_2^2 \lambda \right) \frac{\nu M_v}{R_0} \eta \nonumber \\
\]

(III.71)

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where \( p_0(n,0) \) is the radial pressure distribution at the inlet.

Graphs f of Figures III.3 to III.10 show the axial pressure difference \( \Delta p = \Delta p_z \) according to Equation (III.70) for several values of \( \lambda \). \( \Delta p_z \) is the axial pressure drop for Poiseuille flow from the inlet to the outlet. These profiles are discussed in Section III.4.8.

For small values of \( \lambda (\lambda \leq 1) \) the amount of injected mass is very small and the value of \( a_4 \) differs little from \( \frac{1}{2n} \) for all values of \( n \). For \( a_4 = \frac{1}{2n} \) the pressure drop in the channel as can be calculated from Equation (III.71) equals the pressure drop for Poiseuille flow (III.22) exactly.

For large values of \( \lambda (\lambda \geq 100) \) \( a_2 \) appears to approach 0.25. The pressure drop from the inlet to a point \( z \) downstream as calculated from Equation (III.69) is approximated by

\[
\Delta p_z = -4 \frac{vM}{R_o^2} a_2^2 \lambda z (1 + \frac{a}{2})
\]

(III.72)

Since \( a_2^2 = 1/16 \) the pressure drop as calculated in Section III.3 for large values of \( \lambda \) given by (III.36) equals (III.69).

III.4.5 The radial pressure difference \( \Delta p_r \).

The radial pressure difference \( \Delta p_r \) follows from the momentum equation in the radial direction. The result of introducing the dimensionless streamfunction \( \psi = \psi / M_v \) in the momentum Equation (II.20) is

\[
\frac{1}{r} \left[ \frac{M_v}{r} \frac{\partial F}{\partial r} + \frac{M_o}{\rho} \frac{\partial F}{\partial z} \right] \left(- \frac{M_v}{r^2} \frac{\partial F}{\partial z} - \frac{\alpha M_o}{\rho r} F + \frac{M_v}{r} \frac{\partial^2 F}{\partial r \partial z} + \frac{\alpha M_o}{\rho r} \frac{\partial F}{\partial r} \right) +
\]

\[
- \frac{M_v}{r^2} \frac{\partial F}{\partial r} \left( \frac{\partial^2 F}{\partial z^2} + 2 \frac{\alpha M_o}{\rho} \frac{\partial F}{\partial z} \right) = - \frac{1}{\rho} \frac{\partial \Delta p}{\partial z} +
\]

\[
\frac{v}{r} \left[ \frac{1}{M_v} \frac{\partial^3 F}{\partial r \partial z^2} + \frac{\alpha M_o}{\rho r} \frac{\partial F}{\partial r} - \frac{M_v}{r} \frac{\partial^3 F}{\partial r^2 \partial z} - \frac{\alpha M_o}{\rho r^2} \frac{\partial^2 F}{\partial r^2} \right] - \frac{\alpha M_v}{r^3} \frac{\partial^2 F}{\partial z^3} - \frac{3\alpha M_o}{\rho} \frac{\partial^2 F}{\partial z^2} - \frac{M_o}{r^2} \frac{\partial^2 F}{\partial z^2}
\]

(III.73)

while transforming this into the non-dimensional coordinates \((\eta, \xi)\) as defined by (III.5) one gets
\[
\frac{\alpha M_o \nu}{\rho n R_o^2} \frac{1}{3} \left[ - \frac{2}{n} F \frac{\partial F}{\partial \zeta} + F \frac{\partial^2 F}{\partial \eta \partial \zeta} - \frac{\partial F}{\partial \eta} \frac{\partial F}{\partial \zeta} \right] + \\
\frac{M^2}{n^2 R_o^2} \frac{1}{3} \left[ - \frac{1}{n} F^2 + \frac{F}{\eta} \frac{\partial F}{\partial \eta} - \frac{\partial F}{\partial \zeta} \frac{\partial F}{\partial \zeta} \right] + \\
\frac{\alpha^2 M^2_o}{R_o^2} \frac{1}{3} \left[ - \frac{1}{n} F^2 + \frac{F}{\eta} \frac{\partial F}{\partial \eta} \right] = - \frac{1}{\rho} \frac{\partial P}{\partial r} + \\
\frac{\nu M_o}{n R_o^2} \frac{1}{3} \left[ \frac{1}{n} \frac{\partial^2 F}{\partial \eta \partial \zeta} - \frac{\partial^2 F}{\partial \eta^2} - \frac{R_o^2}{L} \frac{\partial^2 F}{\partial \zeta^2} \right] + \\
\frac{\nu M_o \alpha}{\rho n R_o^3} \frac{1}{3} \left[ \frac{1}{n} \frac{\partial F}{\partial \eta} - \frac{\partial^2 F}{\partial \eta^2} - \frac{R_o^2}{L} \frac{\partial^2 F}{\partial \zeta^2} \right]
\]

(III.74)

Since \( F \) is independent of \( \zeta \) and \( \lambda = \frac{\alpha M_o}{\rho \nu} \) the radial pressure gradient is

\[
\frac{\partial P}{\partial \eta} = \frac{\nu \alpha M_o}{R_o^2} \left[ \lambda \left( F \left( \frac{1}{3} F - \frac{1}{2} \frac{\partial F}{\partial \eta} \right) \right) + \left[ \frac{1}{n} \frac{\partial F}{\partial \eta} - \frac{1}{n} \frac{\partial^2 F}{\partial \eta^2} \right] \right]
\]

(III.75)

Integrating this equation once with respect to \( \eta \), one obtains

\[
p_n(n, \zeta) - p_o(0, \zeta) = - \frac{\nu \alpha M_o}{R_o^2} \left[ \lambda \frac{1}{2n^2} F^2 + \frac{1}{n} \frac{\partial F}{\partial \eta} \right]
\]

(III.76)

where \( p_o(0, \zeta) \) is the axial pressure distribution at the center. An expression for \( \frac{1}{n} F^2 \) is obtained by using the general expression for the product of two series (A.III.9). If terms of powers of \( n^m \) are neglected for \( m > n - 2 \) the radial pressure difference \( \Delta p_n \) becomes

\[
\Delta p_n = - \frac{\nu \alpha M_o}{R_o^2} \left[ \sum_{m=0}^{n-2} (m+2) a_{m+2} + \sum_{j=2}^{m+2} \frac{1}{2} a_j a_{m+2-j} \right] n^m - 2a_2
\]

(III.77)

An expression for the pressure at a point \((n, \zeta)\) derived from Equations (III.71) and (III.77) becomes
\[ p = p_o + \frac{\nu M_o}{4 R_o} \left( \frac{a}{z} \right) (1 + \frac{a}{z}) (16a_4 - 4a_2^2 \lambda) + \]

\[ - \frac{\nu a M_o}{R_o^2} \left\{ \frac{n-2}{\eta} \left[ \sum_{m=0}^{n-2} a_{m+2} + \lambda \sum_{j=2}^{m+2} \frac{1}{2} a_j a_{m+2-j} \right] \eta^{m-2a_2} \right\} \text{(III.78)} \]

where \( p_o \) is the pressure at \( \eta = \zeta = 0 \).

Graphs e of Figures III.3 to III.10 present the dimensionless radial pressure difference \( \Delta p = \frac{\Delta p^n}{\Delta p_c} \) at the inlet. \( \Delta p_c \) is the axial pressure drop for Poiseuille flow according to (III.22). These profiles will be discussed in Section III.4.8.
Figure III.3a Radial profiles of axial velocity at \( \xi = 0 \) for several values of the injection Reynolds number \( \lambda \).

- \( M_0 = 10 \)
- \( n = 12 \)
- \( \text{Re} = 10^3 \)
- a: profile for \( \lambda = 0.1 \)
- b: profile for \( \lambda = 1 \)
- c: profile for \( \lambda = 10 \)
- Type A flow

Figure III.3b Axial profiles of axial velocity at \( \eta = 0 \) for several values of the injection Reynolds number \( \lambda \).

- \( M_0 = 10 \)
- \( n = 12 \)
- \( \text{Re} = 10^3 \)
- a: profile for \( \lambda = 0.1 \)
- b: profile for \( \lambda = 1 \)
- c: profile for \( \lambda = 10 \)
- Type A flow
Figure III.3c Radial profiles of radial velocity at ζ = 0 for several values of the injection Reynolds number λ.

Figure III.3d Axial profiles of radial velocity at η = 1/2 for several values of the injection Reynolds number λ.
Figure III.3e Radial pressure profile at $\zeta=0$ for several values of the injection Reynolds number $\lambda$.

Figure III.3f Axial pressure profile at $n=0$ for several values of the injection Reynolds number $\lambda$. 

$M_0 = 10$
$n = 12$
$Re = 10^3$

a: profile for $\lambda = 0.1$
b: profile for $\lambda = 1$
c: profile for $\lambda = 10$

Type A flow
\( M_0 = 10 \)
\( n = 12 \)
\( Re = 10^3 \)

a: profile for \( \lambda = 10 \)

b: profile for \( \lambda = 20 \)

c: profile for \( \lambda = 100 \)

d: profile for \( \lambda = 245 \)

Type A flow.

Figure III.4a Radial profiles of axial velocity at \( \zeta=0 \) for several values of the injection Reynolds number \( \lambda \).

\( M_0 = 10 \)
\( n = 12 \)
\( Re = 10^3 \)

a: profile for \( \lambda = 10 \)

b: profile for \( \lambda = 20 \)

c: profile for \( \lambda = 100 \)

d: profile for \( \lambda = 245 \)

Type A flow.

Figure III.4b Axial profiles of axial velocity at \( \eta=0 \) for several values of the injection Reynolds number \( \lambda \).
Figure III.4c Radial profiles of radial velocity at $\zeta=0$ for several values of the injection Reynolds number $\lambda$.

Figure III.4d Axial profiles of radial velocity at $\eta=\frac{1}{2}$ for several values of the injection Reynolds number $\lambda$. 

$M_0 = 10$
$n = 12$
$Re = 10^3$

a: profile for $\lambda = 10$
b: profile for $\lambda = 20$
c: profile for $\lambda = 100$
d: profile for $\lambda = 245$

Type A flow.
Figure III.4e Radial pressure profile at $\zeta=0$ for several values of the injection Reynolds number $\lambda$.

$M_0 = 10$
$n = 12$
$Re = 10^3$

a: profile for $\lambda = 10$
b: profile for $\lambda = 20$
c: profile for $\lambda = 100$
d: profile for $\lambda = 245$

Type A flow.

Figure III.4f Axial pressure profile at $\eta=0$ for several values of the injection Reynolds number $\lambda$.

$M_0 = 10$
$n = 12$
$Re = 10^3$

a: profile for $\lambda = 10$
b: profile for $\lambda = 20$
c: profile for $\lambda = 100$
d: profile for $\lambda = 245$

Type A flow.
Figure III.5a Radial profiles of axial velocity at $z=0$ for several values of the injection Reynolds number $\lambda$.

Figure III.5b Axial profiles of axial velocity at $\eta=0$ for several values of the injection Reynolds number $\lambda$. 

$M_0 = 10$
$n = 12$
$Re = 10^3$

a: profile for $\lambda = 0.1$
b: profile for $\lambda = 1$
c: profile for $\lambda = 10$
type B flow
Figure III.5c Radial profiles of axial velocity at $\zeta=0$ for several values of the injection Reynolds number $\lambda$.

Figure III.5d Axial profiles of radial velocity at $\eta=\frac{1}{2}$ for several values of the injection Reynolds number $\lambda$. 

$M_0 = 10$
$n = 12$
$Re = 10^3$

a: profile for $\lambda = 0.1$
b: profile for $\lambda = 1$
c: profile for $\lambda = 10$
type B flow
\( M_0 = 10 \)
\( n = 12 \)
\( \text{Re} = 10^3 \)

a: profile for \( \lambda = 0 \)
b: profile for \( \lambda = 1 \)
c: profile for \( \lambda = 1 \)

**Figure III.5e** Radial pressure profiles at \( \zeta = 0 \) for several values of the injection Reynolds number \( \lambda \).

\( M_0 = 10 \)
\( n = 12 \)
\( \text{Re} = 10^3 \)

a: profile for \( \lambda = 0 \)
b: profile for \( \lambda = 1 \)
c: profile for \( \lambda = 10 \)

type B flow.

**Figure III.5f** Axial pressure profile at \( \eta = 0 \) for several values of the injection Reynolds number \( \lambda \).
Figure III.6a Radial profiles of axial velocity at $\xi=0$ for several values of the injection Reynolds number $\lambda$.

- $M_0 = 10$
- $n = 12$
- $Re = 10^3$
- a: profile for $\lambda = 10$
- b: profile for $\lambda = 20$
- c: profile for $\lambda = 100$
- d: profile for $\lambda = 245$

Type B flow

Figure III.6b Axial profiles of axial velocity at $n=0$ for several values of the injection Reynolds number $\lambda$.
Figure III.6c Radial profiles of radial velocity at $\zeta=0$ for several values of the injection Reynolds number $\lambda$.

Figure III.6d Axial profiles of radial velocity at $\eta=\frac{1}{2}$ for several values of the injection Reynolds number $\lambda$. 
Figure III.6e Radial pressure profiles at \( \zeta = 0 \) for several values of the injection Reynolds number \( \lambda \).

\[
M_0 = 10 \\
n = 12 \\
Re = 10^3 \\
a: \text{profile for } \lambda = 10 \\
b: \text{profile for } \lambda = 20 \\
c: \text{profile for } \lambda = 100 \\
d: \text{profile for } \lambda = 245 \\
\text{Type B flow.}
\]

Figure III.6f Axial pressure profiles at \( n = 0 \) for several values of the injection Reynolds number \( \lambda \).

\[
M_0 = 10 \\
n = 12 \\
Re = 10^3 \\
a: \text{profile for } \lambda = 10 \\
b: \text{profile for } \lambda = 20 \\
c: \text{profile for } \lambda = 100 \\
d: \text{profile for } \lambda = 245 \\
\text{Type B flow.}
\]
Figure III.7a Radial profiles of axial velocity at $\zeta=0$ for several values of $n$.

Figure III.7b Axial profiles of axial velocity at $\eta=0$ for several values of $n$. 

$M_0 = 10$
$\lambda = 10$
$Re = 10^3$

a: Number of terms $n = 6$
b: Number of terms $n = 8$
c: Number of terms $n = 10$
d: Number of terms $n = 12$

Type A flow.
Figure III.7c Radial profiles of radial velocity at $\zeta=0$ for several values of $n$.

Figure III.7d Axial profiles of radial velocity at $n=\frac{1}{3}$ for several values of $n$. 

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Figure III.7e Radial pressure profiles at $\zeta=0$ for several values of $n$.

$M_0 = 10$
$\lambda = 10$
$Re = 10^3$

- $a$: number of terms $n = 6$
- $b$: number of terms $n = 8$
- $c$: number of terms $n = 10$
- $d$: number of terms $n = 12$

Type A flow

Figure III.7f Axial pressure profiles at $\eta=0$ for several values of $n$.

$M_0 = 10$
$\lambda = 10$
$Re = 10^3$

- $a$: number of terms $n = 6$
- $b$: number of terms $n = 8$
- $c$: number of terms $n = 10$
- $d$: number of terms $n = 12$

Type A flow.
Figure III.8a Radial profiles of axial velocity at \( \zeta = 0 \) for several values of \( n \).

Figure III.8b Axial profiles of axial velocity at \( \eta = 0 \) for several values of \( n \).
Figure III.8c Radial profiles of radial velocity at ζ=0 for several values of n.

Figure III.8d Axial profiles of radial velocity at η=½ for several values of n.
Figure III.8e Radial pressure profiles at $\zeta=0$ for several values of $n$.

Figure III.8f Axial pressure profiles at $\eta=0$ for several values of $n$. 

$M_0 = 10$
$\lambda = 100$
$Re = 10^3$

a: number of terms $n = 6$
b: number of terms $n = 10$
c: number of terms $n = 12$

Type A flow.
Figure III.9a Radial profiles of axial velocity at $\zeta=0$ for several values of $n$.

Figure III.9b Axial profiles of axial velocity at $\eta=0$ for several values of $n$. 

$M_0 = 10$
$\lambda = 245$
$Re = 10^3$

- $a$: number of terms $n = 12$
- $b$: number of terms $n = 10$
- $c$: number of terms $n = 6$

Type A flow.
$M_0 = 10$
$\lambda = 245$
$Re = 10^3$

a: number of terms  
$n = 12$

b: number of terms  
$n = 10$

c: number of terms  
$n = 6$

Type A flow

Figure III.9c Radial profiles of radial velocity at $\zeta=0$ for several values of $n$.

$M_0 = 10$
$\lambda = 245$
$Re = 10^3$

a: number of terms  
$n = 12$

b: number of terms  
$n = 10$

c: number of terms  
$n = 6$

Type A flow.

Figure III.9d Axial profiles of radial velocity at $n=\frac{1}{2}$ for several values of $n$.  

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Figure III.9e Radial pressure profiles at $\zeta=0$ for several values of $n$.

$M_0 = 10$
$\lambda = 245$
$Re = 10^3$

a: number of terms $n = 12$
b: number of terms $n = 10$
c: number of terms $n = 6$

Type A flow

Figure III.9f Axial pressure profiles at $\eta=0$ for several values of $n$.

$M_0 = 10$
$\lambda = 245$
$Re = 10^3$

a: number of terms $n = 12$
b: number of terms $n = 10$
c: number of terms $n = 6$

Type A flow.
Figure III.10a Radial profiles of axial velocity at $z=0$ for large values of $\lambda$ for type A and B flow.

Figure III.10b Axial profiles of axial velocity at $\eta=0$ for large values of $\lambda$ for type A and B flow.
Figure III.10c Radial profiles of radial velocity at $\zeta=0$ for large values of $\lambda$ for type A and B flow.

Figure III.10d Axial profiles of radial velocity at $\eta=\frac{1}{2}$ for large values of $\lambda$ for type A and B flow.
$M_0 = 10$
$n = 12$
$Re = 10^3$

$a: \lambda = 240; \text{ type B flow}$
$b: \lambda = 240; \text{ type A flow}$
$c: \lambda = 248; \text{ type B flow}$
$d: \lambda = 248; \text{ type A flow}$

$M_0 = 10$
$n = 12$
$Re = 10^3$

$a: \lambda = 240; \text{ type B flow}$
$b: \lambda = 240; \text{ type A flow}$
$c: \lambda = 248; \text{ type B flow}$
$d: \lambda = 248; \text{ type A flow}$

**Figure III.10e** Radial pressure profiles at $\zeta = 0$ for large values of $\lambda$ for type A and B flow.

**Figure III.10f** Axial pressure profiles at $\eta = 0$ for large values of $\lambda$ for type A and B flow.
III.4.6 The truncation error.

The equation for $F$ as derived in Section III.4.1 is

$$\lambda[F \left( n \frac{\partial^2 F}{\partial n^2} - \frac{\partial F}{\partial n} \right) - n \left( \frac{\partial F}{\partial n} \right)^2] + \left[ \frac{\partial F}{\partial n} - n \frac{\partial^2 F}{\partial n^2} + n^2 \frac{\partial^3 F}{\partial n^3} \right] = C n^3$$  \hspace{1cm} (III.43)

As was discussed in that section solutions $F$ of Equation (III.43) hold only for terms up to powers of $n^{n-1}$. Equation (III.48) gives the exact expression if a power series in $n$ is substituted for $F$ in Equation (III.43). Equation (III.48) can be written as

$$\lambda D_0(n,n) + \lambda E(n,n) + D_1(n,n) - C n^3 = 0$$  \hspace{1cm} (III.79)

where $D_0(n,n)$ represent terms of powers of $n$ of order up to $n-1$ of

$$F \left( n \frac{\partial^2 F}{\partial n^2} - \frac{\partial F}{\partial n} \right) - n \left( \frac{\partial F}{\partial n} \right)^2$$

$E(n,n)$ represent the remaining terms of powers of $n$ of

$$F \left( n \frac{\partial^2 F}{\partial n^2} - \frac{\partial F}{\partial n} \right) - n \left( \frac{\partial F}{\partial n} \right)^2$$

$D_1(n,n)$ represent all terms of

$$\left[ \frac{\partial F}{\partial n} - n \frac{\partial^2 F}{\partial n^2} + n^2 \frac{\partial^3 F}{\partial n^3} \right]$$

$C$ is given by Equation (III.51) and equals $16a_4 - 4a_2^2 \xi$.

The assumption made in Section III.4.1 in order to find values of the coefficients of the power series for $F$ is that the value of $E(n,n)$ in (III.79) is negligible compared to the other values. An expression for $E(n,n)$ is found with the aid of Equation (III.48). If $S_1$ and $S_2$ are defined as

$$S_1 = (m-j-1) (m-j+1) \ a_j a_{m-j+1} \ n^m$$

$$S_2 = (j+1) (m-j) \ a_{j+1} a_{m-j} \ n^m$$  \hspace{1cm} (III.80)

then $E(n,n)$ can be written as

$$E(n,n) = \left[ \sum_{j=1}^{n-1} S_1 - \sum_{j=0}^{n-1} S_2 \right]$$

$$+ \sum_{m=n+1}^{2n-1} \left[ \sum_{j=m-n+1}^{n-1} S_1 - \sum_{j=m-n}^{n-1} S_2 \right]$$  \hspace{1cm} (III.81)
Combining terms and changing the index \(j\) for \(S_1\) in \(j-1\) one obtains with \(a_1 = 0\)

\[
E(n,n) = \sum_{j=1}^{n-2} (n-2j-3)(n-j) a_{j+1} a_{n-j} n^j \\
+ \sum_{m=n+1}^{2n-1} \sum_{j=m-n}^{n-1} (m-2j-3)(m-j) a_{j+1} a_{m-j} m^j
\]

(III.82)

Since only even \(n\) are considered the first term in (III.82) equals zero and for \(m=2p-1\), the second term becomes when \(j\) is replaced by \(2\ell-1\):

\[
E(n,n) = \sum_{p=\frac{n+2}{2}}^{n/2} \sum_{\ell=\frac{2p-n}{2}}^{4(p-2\ell-1)(p-\ell) a_{2\ell} a_{2p-2\ell}} n^{2p-1}
\]

(III.83)

The dominating part in the first term of (III.48) is

\[
D_0(n,n) = \sum_{m=2}^{n-1} \sum_{j=1}^{m-2} (m-2j-3)(m-j) a_{j+1} a_{m-j} m^j
\]

(III.84)

Alternatively, since all \(a_j\) are zero for odd \(j\), this becomes, when \(m\) is replaced by \(2p-1\) and \(j\) by \(2\ell-1\):

\[
D_0(n,n) = \sum_{p=2}^{n/2} \sum_{\ell=1}^{4(p-2\ell-1)(p-\ell) a_{2\ell} a_{2p-2\ell}} n^{2p-1}
\]

(III.85)

Equation (III.48) yields for \(D_1(n,n)\):

\[
D_1(n,n) = \sum_{m=2}^{n-1} (m-1)^2 (m+1) a_{m+1} m^m
\]

(III.86)

The error made by neglecting \(\lambda E(n,n)\) in Equation (III.79) is defined as

\[
\text{Error} = \frac{\lambda E(n,n)}{O(n,n)}
\]

(III.87)

where \(O(n,n)\) stands for the maximum of \(|\lambda D_0(n,n)|, |D_1(n,n)|\) and \(|Cn^3|\). For several values of \(n\) and \(\lambda\) Table III.1 lists the errors according to (III.87). Since the truncation error is believed to increase with increasing values of \(n\) this table gives values of the errors at \(n=1\). These errors will be discussed in Sections III.4.9 and III.4.10.
Table III.1 Truncation errors for several values of $\lambda$ and $n$ (absolute values)

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$n$</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12 type A</th>
<th>12 type B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>6</td>
<td>$2.0.10^{-3}$</td>
<td>$6.05.10^{-6}$</td>
<td>$8.57.10^{-7}$</td>
<td>$3.45.10^{-9}$</td>
<td>0.348</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>$1.71.10^{-2}$</td>
<td>$5.19.10^{-4}$</td>
<td>$6.82.10^{-5}$</td>
<td>$2.91.10^{-6}$</td>
<td>0.351</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>$7.04.10^{-2}$</td>
<td>$1.781.10^{-2}$</td>
<td>$9.68.10^{-4}$</td>
<td>$9.12.10^{-4}$</td>
<td>0.385</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>$8.10.10^{-2}$</td>
<td>$3.40.10^{-2}$</td>
<td>$4.0.10^{-5}$</td>
<td>$3.40.10^{-3}$</td>
<td>0.422</td>
</tr>
<tr>
<td>100</td>
<td>6</td>
<td>$8.41.10^{-2}$</td>
<td>-</td>
<td>$1.16.10^{-2}$</td>
<td>$4.26.10^{-2}$</td>
<td>0.710</td>
</tr>
<tr>
<td>240</td>
<td>6</td>
<td>$8.36.10^{-2}$</td>
<td>-</td>
<td>$1.84.10^{-2}$</td>
<td>-</td>
<td>0.399</td>
</tr>
<tr>
<td>245</td>
<td>6</td>
<td>$8.36.10^{-2}$</td>
<td>-</td>
<td>$1.86.10^{-2}$</td>
<td>-</td>
<td>0.365</td>
</tr>
<tr>
<td>248</td>
<td>6</td>
<td>$8.36.10^{-2}$</td>
<td>-</td>
<td>$1.86.10^{-2}$</td>
<td>-</td>
<td>0.331</td>
</tr>
</tbody>
</table>
III.4.7 Convergence of F.

Solutions F of Equation (III.43) are acceptable if the power series for F converges for all n. The only value of \( \eta \) for which the convergence of F is checked is for \( \eta = 1 \), since the contributions of terms of powers of \( \eta \) are maximum at \( \eta = 1 \). At this value F equals

\[
F(1) = \sum_{i=2}^{n} a_i
\]

(III.88)

where \( a_{2i+1} = 0 \). This series is convergent if \( C_v \) defined by (III.41) is less than 1. For several values of \( n \) Table III.2 list the values of \( C_v \)

\[
C_v = \left| \frac{a_n}{\sum_{i=2}^{n-1} a_i} \right|
\]

(III.41)

In order to get an impression of the improvement obtained by extending the number of terms in the series expansion for F, Table III.3 lists values of

\[
C_v^* = \left| \frac{a_{n+2}}{\sum_{i=2}^{n} a_i} \right|
\]

(III.89)

where \( a_i \) are coefficients for F if F is expanded in powers of \( \eta \) up to \( n+2 \) and \( a_i \) are coefficients for F if F is expanded in powers of \( \eta \) up to \( n \). The conclusions that can be drawn from Tables III.2 and III.3 are discussed in the upcoming sections.
Table III.2 Convergence of $F$. The numbers listed are values of $\left| \frac{a_n}{\sum_{i=2}^{n-2} a_i} \right|$ for

$F_0 = \sum_{i=2}^{n} a_i \lambda^n$ (value of $C_v$ according to (III.41))

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12 type A</th>
<th>12 type B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.63.10^{-3}</td>
<td>4.39.10^{-4}</td>
<td>6.97.10^{-7}</td>
<td>4.60.10^{-8}</td>
<td>1.00</td>
</tr>
<tr>
<td>1</td>
<td>2.43.10^{-2}</td>
<td>4.11.10^{-3}</td>
<td>6.46.10^{-3}</td>
<td>3.84.10^{-6}</td>
<td>1.02</td>
</tr>
<tr>
<td>10</td>
<td>0.141</td>
<td>2.31.10^{-2}</td>
<td>3.46.10^{-3}</td>
<td>5.54.10^{-6}</td>
<td>1.21</td>
</tr>
<tr>
<td>20</td>
<td>0.194</td>
<td>2.80.10^{-2}</td>
<td>8.38.10^{-3}</td>
<td>4.50.10^{-4}</td>
<td>1.51</td>
</tr>
<tr>
<td>100</td>
<td>0.288</td>
<td>-</td>
<td>2.85.10^{-2}</td>
<td>1.38.10^{-2}</td>
<td>2.31</td>
</tr>
<tr>
<td>240</td>
<td>0.312</td>
<td>-</td>
<td>3.76.10^{-2}</td>
<td>9.84.10^{-2}</td>
<td>0.221</td>
</tr>
<tr>
<td>245</td>
<td>0.312</td>
<td>-</td>
<td>3.77.10^{-2}</td>
<td>0.114</td>
<td>0.191</td>
</tr>
<tr>
<td>248</td>
<td>0.313</td>
<td>-</td>
<td>3.78.10^{-2}</td>
<td>0.132</td>
<td>0.164</td>
</tr>
</tbody>
</table>
Table III.3 Convergence of $F$. The numbers listed are values of $\left| \frac{a_n^{n+2}}{\sum_{i=1}^{n} a_i} \right|$ for $F_0 = \sum_{i=2}^{n} a_i n^i$ and $F_1 = \sum_{j=2}^{n} a_j n^j$ for a type A flow where $\sum_{i=2}^{n} a_i = \frac{1}{2\pi} (\text{value of } C_v^o)$ according to (III.89))

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>20</th>
<th>100</th>
<th>240</th>
<th>245</th>
<th>248</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n=6$</td>
<td>2.64 \times 10^{-3}</td>
<td>2.49 \times 10^{-2}</td>
<td>0.164</td>
<td>0.241</td>
<td>0.404</td>
<td>0.454</td>
<td>0.455</td>
<td>0.455</td>
</tr>
<tr>
<td>$n=8$</td>
<td>4.40 \times 10^{-4}</td>
<td>4.09 \times 10^{-3}</td>
<td>2.25 \times 10^{-2}</td>
<td>2.72 \times 10^{-2}</td>
<td>-</td>
<td>3.62 \times 10^{-2}</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$n=10$</td>
<td>6.97 \times 10^{-7}</td>
<td>6.46 \times 10^{-5}</td>
<td>3.45 \times 10^{-3}</td>
<td>8.31 \times 10^{-3}</td>
<td>2.77 \times 10^{-2}</td>
<td>-</td>
<td>3.64 \times 10^{-2}</td>
<td>-</td>
</tr>
<tr>
<td>$n=12$</td>
<td>4.60 \times 10^{-8}</td>
<td>3.84 \times 10^{-6}</td>
<td>5.54 \times 10^{-6}</td>
<td>4.5 \times 10^{-4}</td>
<td>1.36 \times 10^{-2}</td>
<td>8.96 \times 10^{-2}</td>
<td>0.102</td>
<td>0.117</td>
</tr>
</tbody>
</table>
III.4.8. Resulting solutions for F.

Figures III.3 to III.10 present velocity and pressure profiles for several values of the injection Reynolds number \( \lambda \) and for several values of \( n \), where \( n \) is the value of the highest power in the series expansion for F. For each \( n \) one of the non-linear Equations (III.57) to (III.60) has to be solved, leading to several values of \( a_1 \). Some of the resulting values are complex numbers and some of these values lead to divergent series \( a_1 \).

The profiles considered in this section are profiles for which the truncation error according to (III.87) is less than 100% and for which \( C_v < 2 \). These values for the truncation error and \( C_v \) are rather arbitrary and they result in solutions for F which do not seem to be completely realistic. For small values of the injection parameter \( \alpha (\alpha < 10^{-3}) \), the injection Reynolds number \( \lambda \) is relatively small compared to the Reynolds number of the main flow and the resulting flow field is a small disturbance of the Poiseuille flow. The axial velocity profiles and the axial pressure profiles resemble the Poiseuille profile for all values of \( n \) (see Figures III.3a and III.3f). Tables III.1 and III.3 show that for values of \( \lambda \leq 10 \) the truncation error as defined in Section III.4.6 and \( C_v \) as defined in Section III.4.7 decrease for increasing values of \( n \) (where \( n \) is the value of the highest power of \( n \) in F). As was reasoned in Section III.4.1 values of \( n > 12 \) are not considered. This is why for small values of \( \lambda \) only profiles for \( n=12 \) are drawn.

For values of \( \alpha (\alpha < 1) \) the total percentage of the injected mass is small. This yields velocities \( v_F \ll v_z \) (see e.g. Figures III.3c and III.3d).

![Estimated streamlines on injected particles for small values of \( \lambda \).]

For small values of \( \alpha \), Figure III.11 shows the streamlines which are believed to occur for the injected particles. Only a few particles penetrate to the central
region of the flow. For small values of $\alpha$ the viscous forces dominate and these forces decelerate the radially moving particles. Since the number of particles increases near the wall the pressure near the wall is relatively high (see Figure III.3e).

As the axial velocity is small near the wall, the injected particles move in the radial direction. Due to the geometry (Figure III.12) the radial velocity increases (principle of the conservation of mass) to a maximum value.

![Figure III.12. Effect of the curvature of the wall.](image)

Graphs c of Figures III.3 to III.10 show the radial velocity profiles. Near the central region the axial movement becomes dominant and at the center the radial velocity equals zero. Since $F$ is supposed to be independent of $z$, $\bar{v}_r$ profiles with $\zeta$ have to be straight lines if $\alpha$ is independent of $z$ (see pictures d of Figures III.3 to III.10). Applying the principle of the conservation of mass to a ring-shaped element shows that the axial velocity increases near the wall compared to the corresponding axial velocity for the Poiseuille flow. However, the total mass flow rate at the inlet equals by definition $M_0$ and can be calculated from

$$M_0 = \int_0^R \rho v_z 2\pi r \, dr$$

(III.90)

If $v_z$ increases near the wall, Equation (III.90) states that $v_z$ decreases in the central region of the flow. Figure III.3a shows the axial velocity profile for
small values of $\alpha$. Due to the increase of $M$ in the axial direction according to
(III.1) $v_z$ increases slowly with $\zeta$ (see e.g. Figure III.3b).

For increasing values of $\lambda$ the absolute value of the axial pressure difference
increases (see Figure III.3f) since the injected mass so to speak blocks the
channel, which means that the pressure at the inlet has to be increased to blow
the same mass flow rate $M_0$ through the channel. For all values of $\lambda$ the value of
the pressure level at the inlet $p_0$ equals zero, so that the pressure at $\zeta=1$ and
$n=0$ decreases with increasing values of $\lambda$ (Figures III.3f and III.4f).

For values of $\lambda>10$ Figure III.4 yields the velocity and pressure profiles. For
increasing values of $\lambda$ the radial velocity at the wall increases (Figure III.4c)
and the injected mass penetrates further into the flow. The streamlines which
are believed to occur for large values of $\lambda$ are given in Figure III.13.

Figure III.13. Estimated streamlines of injected particles for
large values of $\lambda$.

For symmetry reasons the radial velocity at the center equals zero, which means
that radially moving particles have to be decelerated. For large values of $\lambda$ the
viscous forces become relatively small and the particles are decelerated by the
high pressure at the center of the flow (Figure III.4e). Near the wall, however,
the pressure decreases in the direction of the center due to the increase of
particles near the wall. Figure III.4c shows that for increasing values of $\lambda$ the
radial velocity profile is qualitatively the same. The point of minimum radial
velocity (defined by $n=n_1$) moves to the center line for increasing values of $\lambda$.
The reason for the change in position is the fact that for increasing values of
$\lambda$ the point of minimum pressure (defined by $n=n_2$) also moves in the direction of
the center (Figure III.4e) and the value of the pressure difference between the
wall and $n=n_2$ increases.
Figure III.4a shows that for values of \( \lambda \) up to 100 the velocity near the wall increases and the velocity at \( n=0 \) decreases which is the same effect that was observed for values of \( \lambda \) up to 10. For velocity profiles as a function of \( \zeta \) (Figures III.4b and III.4d) for values \( 10<\lambda<100 \) the same reasoning applies as for the corresponding profiles for \( \lambda<10 \).

Values of \( \lambda \) near 250 yield for \( n=12 \) axial velocity profiles which differ considerably from the profiles which could be expected for large values of \( \lambda \) (compare Figures III.2a and III.4a). The pressure profile for \( \lambda=245 \) (Figure III.4e) shows a considerable difference in the pressure profile for type II flow (type II flow means no viscous forces) as plotted in Figure III.2d.

Close examination of Figures III.4a and III.10a shows that the gradient of \( v_z \) with \( n \) near the wall tends to zero for values of \( \lambda=250 \) and \( n=12 \). So, for a certain value of \( \lambda \), say \( \lambda=\lambda_b \), the gradient \( \frac{\partial v_z}{\partial n} \) equals zero at the wall

\[
\frac{\partial v_z}{\partial n} (\lambda_b, n_w) = 0
\]  

(III.91)

This means that at the wall the particles are not accelerated in the axial direction. The axial pressure forces near the wall equal the viscous forces in the axial direction.

It would be interesting to increase the value of \( \lambda \) beyond (\( \lambda_b=250 \)) for \( n=12 \). The question which arises is whether \( \frac{\partial v_z}{\partial n} \) would become negative (reversed flow). Calculation shows that for values of \( \lambda>\lambda_b \) Equation (III.60) yields complex values of \( a_q \) (and values of \( a_q \) which yield divergent series \( a_i \)). For \( \lambda=\lambda_b \) (III.60) gives two equal real values of \( a_q \) (convergent). For \( \lambda<\lambda_b \) two real solutions for \( a_q \) exist. One of these values results in profiles as given in Figures III.3 and III.4 (this flow is called a type A flow). The other value of \( a_q \) (for which a convergent series \( a_i \) results) gives a solution with reversed flow (Figure III.5; this flow is called a type B flow). The axial velocity is, however extremely high for values of \( \lambda \) up to 20. Therefore the solutions of Figures III.5 and III.6 can hardly represent physical solutions. For values of \( 100<\lambda<\lambda_b \) Figures III.6 and III.10 depict solutions for \( \lambda \) which yield acceptable values of the axial velocity. In considering these graphs one might conclude that for \( \lambda<\lambda_b \) Equation (III.43) allows two different solutions. For \( \lambda=\lambda_b \) these two solutions coincide.

The reason that there are two different solutions for a certain value of \( \lambda \) might be that no assumptions about how the flow is initiated were made in the analysis. Starting the flow by first injecting mass through the wall and then
starting the main flow is clearly different from starting the main flow and then starting transpiration through the wall.
Since there is a zone of reversed flow, there is a pressure increase near the wall for a type B flow (Figures III.10e and III.6e).

III.4.9 Accuracy and convergence of the solutions.

In this section the accuracy of the solutions found is discussed. Tables III.1 to III.3 give values of the error made in Equation (III.43) by substituting a power series, and values for the convergence of the series. These tables show that for values of \( \lambda \leq 10 \) both the error in Equation (III.43) according to (III.86) and the convergence of \( F = \sum_{i=2}^{n} a_i n^i \) is best for \( n=12 \) (type A flow).

Figure III.7 shows that for \( \lambda=10 \) the solutions \( F \) for several values of \( n \) resemble each other quite well. For \( n=10 \) and 12 the profiles coincide almost exactly, as might be expected from Table III.3 (for smaller values of \( \lambda \) this improves even more). Since the truncation errors (Table III.1) for \( \lambda=10 \) are far better for \( n=10 \) and 12 than for \( n=6 \) and 8 the solutions for the former are believed to be more accurate than those for the latter.

For \( \lambda>100 \) there is no real convergent solution for \( n=8 \). Table III.1 shows that the truncation error is smallest for \( n=10 \) and that convergence for \( n=6 \) is poor. Figure III.8 presents profiles for \( \lambda=100 \). The differences in velocity profiles are rather small, but the differences in the radial pressure profile are considerable, especially for \( n=6 \).

Figure III.9 shows the profiles for \( n=6, 10 \) and 12 for \( \lambda=245 \). Table III.1 shows that the error in Equation (III.43) becomes rather big for \( n=12 \) and that the convergence for \( n=12 \) and \( n=6 \) is poor. Both the truncation error and the convergence are the best for \( n=10 \).

If the viscous terms are neglected the solution \( F \) is given by (III.33). Figure III.2 shows the profiles for \( \lambda=245 \) of this solution. Comparing Figure III.2 and the solution for \( n=10 \) in Figure III.9 shows that these solutions resemble each other quite well.

The error made by neglecting the viscous part in Equation (III.43) can be calculated exactly. In terms of \( t=n^2 \) Equation (III.43) reads

\[
4\lambda \left[ \frac{\partial^2 F}{\partial t^2} - \left( \frac{\partial F}{\partial t} \right)^2 \right] + 8 \left( \frac{\partial^2 F}{\partial t^2} \right) + t \left( \frac{\partial^3 F}{\partial t^3} \right) = 0 \quad \text{(III.92)}
\]
As was seen in Section III.3 the solution for F is, if the viscous part given by 
\[ 8 \left( \frac{a^2 F}{at^2} + t \frac{a^3 F}{at^3} \right) \] is neglected,
\[ F = \frac{1}{2n} \sin \frac{n}{2} n^2 \]  
(III.33)
and \( C = -\frac{1}{4} \lambda \). For F given by (III.33) the viscous term in Equation (III.92) becomes
\[ 8 \left( \frac{a^2 F}{at^2} + t \frac{a^3 F}{at^3} \right) - \frac{1}{n} \sin \frac{n}{2} t - \frac{t}{2n^2} \cos \left( \frac{n}{2} t \right) \]  
(III.93)
Defining a truncation error similar to Definition (III.87), one obtains
\[ \text{Error} = \frac{\frac{1}{n} \sin \left( \frac{n}{2} t \right) + \frac{t}{2n^2} \cos \left( \frac{n}{2} t \right)}{\frac{1}{4} \lambda} \]  
(III.94)
For \( t=0.909 \) the error according to (III.94) reaches its uppermost value
\[ \text{Error} = \frac{1.287}{\lambda} \]  
(III.95)
For \( \lambda=245 \) expression (III.95) gives a value of \( 5.25 \times 10^{-3} \) for the truncation error. Close examination of Table III.1 shows that this error is of the same order as the truncation error for \( \lambda=245 \) and \( n=10 \). For \( n=12 \) and \( n=6 \) the truncation error for a series expansion in \( n \) for F is rather big.

### III.4.10 Conclusions.

The conclusions to be drawn from the discussion in the previous sections are that for values of \( \lambda<100 \) a power series \( F = \sum_{i=2}^{n} a_i \lambda^i \) yields acceptable solutions for a type A flow. The accuracy increases for increasing values of \( n \) (Table III.1), while the convergence (Tables III.2 and III.3) also increases with \( n \). For increasing values of \( \lambda \) the accuracy and the convergence decrease.

For values of \( \lambda>100 \) a solution for \( F = \frac{1}{2n} \sin \left( \frac{n}{2} n^2 \right) \) gives a better result than a power series.

Type A flow (Poiseuille like flow) seems to be the only physical solution for F since the convergence and the accuracy are both poor for all values of \( \lambda \) for a type B flow (flow with reversed flow regions).
IV. Channels with varying cross sections

In this chapter slender flow through a channel with mass injection and wall disturbances is studied. The wall disturbances are assumed to be small in comparison with the undisturbed radius \( R_0 \). The channel is supposed to be axisymmetric and the same assumptions are made as listed in Section II.1. This implies that Equation (II.14) holds for the streamfunction \( \psi \). Furthermore, since the wall disturbances are small the resulting flow fields will be a small perturbation of the flow field found in Chapter III for a channel with a constant cross section.

- In Section IV.1, Equation (II.14) is rewritten for the streamfunction \( \psi \) in a dimensionless form; a characteristic transverse length scale is introduced and the length scale of the disturbances is taken as the characteristic axial scale. The dimensionless streamfunction \( F \) is assumed to be a perturbation of the order of the wall disturbances of the undisturbed solution \( F_0 \) for a channel with a constant cross section. Writing the equation for the stream function in a non-dimensional form makes it possible to estimate the relative importance of the various terms and simplify the equation for \( F_1 \) considerably for several combinations of the wall disturbance parameters.

- In Section IV.2, the boundary conditions for the flow considered are translated into conditions for \( F_1 \). Because the characteristic transverse length scale is either the undisturbed channel radius \( R_0 \) or the disturbed local channel radius \( R_1 \), the boundary conditions are derived for both cases by introducing a parameter \( p \) which equals by definition 0 and 1, respectively, if \( R_0 \) and \( R_1 \), respectively, are taken as the characteristic transverse length scales.

- In Section IV.3, a channel with a slowly varying cross section is considered. This means that the change of the radius, and therefore the change of the flow field in the axial direction is slow compared to the change of the flow field in the transverse direction. The undisturbed flow is still assumed to be independent of the axial coordinate \( \zeta \). The wall profiles considered are slowly varying sine functions. For this type of channel the local radius \( R_1 \) is taken as the characteristic transverse length scale. The partial differential equation for \( F_1 \) is simplified by neglecting small terms. Under the boundary conditions derived in Section IV.2, a solution for the disturbance streamfunction \( F_1 \) is found by separating \( F_1 \) into two functions \( F_a \) and \( F_b \), depending on respectively the dimensionless transverse coordinate and the dimensionless axial coordinate. This separation means that the disturbances of
the flow field are similar in the axial direction. Once the disturbance function $F_1$ is known, the velocity components and the pressure distributions can be calculated.

In Section IV.4, a channel with a rapidly varying cross section is considered. For the channel considered, the disturbances of the flow field in the axial direction are assumed to be of the same order as the changes of the flow field in the radial direction. However, the undisturbed flow is still assumed to be independent of the axial coordinate $\zeta$. The wall profiles considered are rapidly varying sine functions. For this type of channel the undisturbed radius $R_o$ is used to introduce a non-dimensional transverse coordinate. Physically this means that the disturbed channel is transformed into a channel with a constant cross section, and the results obtained in this way have to be transformed back to values for a disturbed channel. The differential equation for $F_1$ is simplified by neglecting terms which are relatively small for this combination of channel parameters. Again a solution for the disturbance streamfunction $F_1$ is found by separating of $F_1$ into two functions $F_a$ and $F_b$, each depending on one of the dimensionless variables $n$ and $\zeta$. Under the boundary conditions a solution is obtained by expanding the $n$-dependent function $F_a$ in a power series of $n$. The flow field (velocity and pressure distributions) can be calculated with the solution for $F_b$ from the boundary conditions.

IV.1. Dimensionless form of the governing equations

As was mentioned earlier two types of wall disturbances are studied, namely disturbances with a long length scale and a short one. For both types of disturbances the local radius $R_1$ is defined as

$$R_1 = R_o (1 + \delta R^*(z)) \quad (IV.1)$$

where $\delta$ is a small dimensionless parameter. This implies that the wall disturbances are small compared to the undisturbed radius $R_o$. In this chapter channels are considered for which the equation of motion permits sine functions for the wall disturbance function $R^*(z)$. Figure IV.1 shows a characteristic wall profile that will be analyzed. For this type of channel $R^*$ is given by

$$R^* = \sin(\zeta) \quad (IV.2)$$
where the dimensionless axial coordinate $\zeta$ is defined as

$$\zeta = \frac{2\pi z}{L_\ast}$$  \hspace{1cm} (IV.3)

where $L_\ast$ is the wave length as depicted in Figure IV.1.

![Figure IV.1. Wall profile. Disturbances are strongly enlarged.](image)

The governing equation for axisymmetric incompressible flow as derived in Chapter II is given by

$$\frac{1}{r} \left( \frac{\partial^2 \psi}{\partial r \partial z} - \frac{\partial^2 \psi}{\partial z \partial r} \right) \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^3} \left( 2 \frac{\partial^2 \psi}{\partial z^2} + 3 \frac{\partial^2 \psi}{\partial r^2} - \frac{\partial^2 \psi}{\partial r \partial z} \right) +$$

$$- \frac{3}{4} \frac{\partial^2 \psi}{\partial z \partial r} = \nu \left[ \frac{1}{r} D^2 \psi - \frac{2}{r^2} \frac{\partial}{\partial r} \left( D^2 \psi \right) + \frac{3}{r^3} \frac{\partial^2 \psi}{\partial r^2} + \frac{3}{4} \frac{\partial^2 \psi}{\partial z^2} \right]$$  \hspace{1cm} (II.14)

where $D^2$ is a shorthand notation for the operator $\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial z^2}$.

In order to be able to estimate the relative importance of the various terms, this equation is written in a non-dimensional form. To obtain the dimensionless form of the equation of motion the coordinates $r$ and $z$ have to be transformed into dimensionless coordinates. For large values of $L_\ast$ (Figure IV.1) the flow field changes very slowly with $z$. For small values of $L_\ast$, however, in the neighborhood of the wall, there is a considerable change in the disturbance of the flow field with $z$. A "natural" choice for the dimensionless axial coordinate is therefore
\[ \zeta = \frac{2\pi \ast z}{L_g} \]

The transverse coordinate can be made dimensionless either by \( R_0 \) or the local radius \( R_1 \). Then \( \eta \) is defined as

\[ \eta = \frac{r}{R_0(1 + p\delta R^*)} \]

where \( p \) is a parameter which equals either 0 or 1. For \( p=0 \) the characteristic length in the radial direction is the undisturbed channel radius \( R_0 \) and for \( p=1 \) this characteristic length is the local radius \( R_1 \). Appendix II gives relations for transformation from the coordinates \( r \) and \( z \) to the dimensionless coordinates \( \eta \) and \( \zeta \).

The injected mass flow rate \( (\alpha) \) through the wall is supposed to be constant, in which case the total mass flow rate is given by Equation (III.1)

\[ M(z) = M_0(1 + \alpha z) \]

where \( M_0 \) is the mass flow rate at the entrance of the channel.

With the relations in Appendix II a dimensionless equation for \( F = \psi/M_v \) is obtained similar to Equation (III.7) for a channel with a constant cross section. \( M_v \) is the volumetric mass flow rate \( M/\rho \). Because the bulk flow in the channels considered in this chapter is supposed to be a small disturbance of the flow in a channel with a constant cross section, a solution \( F \) for this type of flow can be written as

\[ F = F_0(\eta) + \delta F_1(\eta, \zeta) \]

where \( F_0(\eta) \) is the solution found in Chapter III for a channel with a constant cross section and a corresponding value of the injection Reynolds number \( \lambda \), and \( \delta F_1(\eta, \zeta) \) is the contribution caused by the wall disturbances. Because series in \( \eta \) for \( F_0 \) of powers up to \( \eta^{10} \) and \( \eta^{12} \) yielded the best results only those series are considered in this chapter.

A differential equation for \( F_1 \) can be obtained by writing Equation (II.14) in dimensionless coordinates \( (\eta, \zeta) \) and by introducing the dimensionless streamfunction \( F = \psi/M_v \). If quadratic and higher order terms of \( \delta \) are neglected, tedious calculation leads to an equation of 59 terms. These terms can be ordered
in terms of factors containing $\alpha$ and $\delta$. By dividing the equation by $\frac{v M_v}{n R_1^2}$ and introducing the injection parameter $\lambda = \frac{a M_o}{p v}$ the left-hand side of the differential equation for $F_1$ becomes

$$\lambda \left[ \frac{1}{n} \frac{a F_o}{\alpha n} \frac{a^2 F_o}{\alpha n^2} - \frac{1}{n} \frac{F_o}{\alpha n} \frac{a^3 F_o}{\alpha n^3} + \frac{3}{n^2} \frac{F_o}{\alpha n} \frac{a^2 F_o}{\alpha n^2} - \frac{1}{n^2} \left( \frac{a F_o}{\alpha n} \right)^2 + \frac{3}{n^3} \frac{a F_o}{\alpha n} \right]$$

$$\left[ \frac{M_v}{v L_g} \right]$$

$$\frac{\delta}{\alpha L} \lambda \left[ \frac{1}{n} \frac{a F_o}{\alpha \zeta n} \frac{a^3 F_1}{\alpha \zeta n^2} - \frac{1}{n} \frac{a F_1}{\alpha \zeta n} \frac{a^3 F_o}{\alpha \zeta n^3} + \frac{3}{n^2} \frac{a F_1}{\alpha \zeta n} \frac{a^2 F_o}{\alpha \zeta n^2} - \frac{1}{n^2} \frac{a F_o}{\alpha \zeta n} \frac{a^2 F_1}{\alpha \zeta n^2} + \frac{3}{n^3} \frac{a F_1}{\alpha \zeta n} + \frac{R_o}{R_1} \frac{a R}{\alpha \zeta n} \frac{a F_o}{\alpha \zeta n} \left( - \frac{a^2 F_o}{\alpha \zeta n^2} - \frac{1}{n} \frac{a F_o}{\alpha \zeta n} \right) \right]$$

$$\left[ \frac{R_1}{L_g} \frac{1}{n} \frac{a F_1}{\alpha \zeta n} \frac{a^3 F_1}{\alpha \zeta n^2} - \frac{R_o}{R_1} \frac{a R}{\alpha \zeta n} \frac{a^3 F_1}{\alpha \zeta n^2} \right]$$

$$\left[ \frac{1}{n} \frac{a F_o}{\alpha n} \frac{a^2 F_o}{\alpha n^2} + \frac{1}{n} \frac{a F_1}{\alpha n} \frac{a^2 F_1}{\alpha n^2} - \frac{1}{n} \frac{F_o}{\alpha n} \frac{a^3 F_1}{\alpha n^3} - \frac{1}{n} \frac{F_1}{\alpha n} \frac{a^3 F_o}{\alpha n^3} + \frac{3}{n^2} \frac{F_o}{\alpha n} \frac{a^2 F_1}{\alpha n^2} + \frac{3}{n^2} \frac{F_1}{\alpha n} \frac{a^2 F_o}{\alpha n^2} - \frac{2}{n^2} \frac{F_o}{\alpha n} \frac{a F_1}{\alpha n} - \frac{3}{n^3} \frac{F_o}{\alpha n} \frac{a F_1}{\alpha n} + \frac{R_1}{L_g} \left( \frac{a F_o}{\alpha n} \frac{a^2 F_1}{\alpha n^2} \right) - \frac{R_o}{R_1} \frac{a R}{\alpha n} \frac{a^2 F_1}{\alpha n^2} + \frac{2}{n^2} \frac{F_o}{\alpha n} \frac{a^2 F_1}{\alpha n^2} + \frac{R_o}{R_1} \frac{a R}{\alpha n} \frac{a^2 F_1}{\alpha n^2} \left( - \frac{1}{n} \frac{a F_o}{\alpha n} - \frac{3}{n} \frac{a F_o}{\alpha n} \right) + \frac{a^2 F_1}{\alpha n^2} \right]$$

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\[
\lambda \frac{a}{L} \frac{M_o}{M} \begin{bmatrix} 0 \\ \end{bmatrix} + \]
\[
\frac{\delta}{\alpha L} \frac{\lambda a^2 M_o}{M R_1^2} \left[ \begin{array}{c}
-\frac{2}{n} \frac{a^2 F_1}{a \eta \alpha} + \frac{4}{n^2} \frac{a F_1}{a \xi} + p \frac{R_o a R^*}{R_1 a \xi} \\
+ 2 \frac{a^2 F_o}{a \eta^2}
\end{array} \right]
\]

(IV.7a)

The right-hand side of the equation for \( F_1 \) becomes
\[
\begin{bmatrix}
\frac{a^4 F_o}{a \eta^4} \\
- \frac{2}{n} \frac{a^3 F_0}{a \eta^3} \\
+ \frac{3}{n^2} \frac{a^2 F_1}{a \eta^2}
\end{bmatrix}
\]
\[
\frac{\alpha}{L} \frac{M_o}{M} \begin{bmatrix} 0 \\ \end{bmatrix} + \]
\[
\delta \frac{a^4 F_1}{a \eta^4} - \frac{2}{n} \frac{a^3 F_1}{a \eta^3} + \frac{3}{n^2} \frac{a^2 F_1}{a \eta^2} - \frac{3}{n^3} \frac{a F_1}{a \eta} \]
\[
+ \left( \frac{R_1}{L} \right)^2 \left[ p \frac{R_o a^2 R^*}{R_1 a \xi} \left( \frac{2}{n} \frac{a F_0}{a \eta} - \frac{2}{a \eta^2} - 2n \frac{a^3 F_o}{a \eta^3} \right) \right. \]
\[
+ 2 \frac{a^4 F_1}{a \eta^2 a \xi^2} - \frac{2}{n} \frac{a^3 F_1}{a \eta a \xi} \left. \right] + \frac{R_1}{L} \frac{a^4 F_1}{a \xi^4} \left( \frac{4}{a \xi^4} \right)
\]
\[
- p \frac{R_o a^4 R^*}{R_1 a \xi} \frac{a F_0}{a \eta} \end{bmatrix}
\]

(IV.7b)
The leading factors of the terms on the left-hand side and the right-hand side of Equation (IV.7) are used to identify these terms. When term numbers I to X are defined by

\[ I : A \]

\[ III: \frac{1}{aLg} \delta \lambda \frac{M}{M_0} \]

IV : \( \delta \lambda \)

VI : \( \alpha^2 \frac{1}{aLg} \delta \lambda \frac{M_0}{M} R_1^2 \) \hspace{1cm} (IV.8)

VII : 1

IX : \( \delta \)

\[ X : \alpha^2 \frac{1}{aLg} \delta \lambda \frac{M_0}{M} R_1^2 \]

Terms II, V, and VIII are equal to zero.

Equating terms I and VIII of (IV.7) yields the equation for \( F_0 \) as was analyzed in Chapter III.

Before solutions \( F_1 \) of Equation (IV.7) can be found for several values of \((aLg)\) the boundary conditions have to be determined.

**IV.2 Boundary conditions**

The boundary conditions for flow through a channel having a variable cross section and constant mass injection are

1) There is no slip at the wall; the tangential velocity at the wall is zero.
2) The radial velocity at the wall equals the radial component of the injection velocity.
3) The axial velocity at the center of the tube is finite.
4) The radial velocity at the center of the tube is equal to zero.
These boundary conditions are translated in this section into conditions for $F_1$. The boundary conditions at the wall yield conditions for $F_1$ by expanding $F_o, F_1$ and their derivatives at $n=1$ into a Taylor series of $\delta$. Furthermore, the axial and radial components of the injection velocity have to equal the velocity components as calculated from the general expressions (II.8) (see Figure IV.2). Since the resulting flow field is supposed to be a small perturbation of the flow fields as calculated in Chapter III, the boundary conditions as listed in that chapter hold for $F_o$, so that $\frac{\partial F_o(1)}{\partial n} = 0$ and $F_o(1) = \frac{1}{2\pi}$.

![Figure IV.2. Wall velocities.](image)

Using Definition (II.8), the velocity components $v_z$ and $v_r$ become in dimensionless coordinates and in terms of $F_o$ and $F_1$:  
\[
v_z = \frac{1}{\rho} \frac{M}{R_1^2} \left( \frac{\partial F_o}{\partial n} \right) + \delta \frac{\partial F_o}{\partial n} 
\]

\[
v_r = -\frac{1}{n R_1} \frac{1}{\rho} \left( \frac{\partial M}{\partial \zeta} \right) \left[ -\frac{R_o}{R_1} \frac{\partial R}{\partial \zeta} \frac{\partial F_o}{\partial n} \frac{\partial F_1}{\partial \zeta} \right] + \alpha M_o \left[ F_o + \delta F_1 \right]
\]

where quadratic and higher order terms of $\delta$ are neglected.

At the wall $n$ equals $\eta_w$ given by

\[
\eta_w = \frac{R_1}{R_o (1 + p\delta R^*)}
\]

If $\eta_w$ is expanded into a Taylor series of $\delta$ the first-order approximation of $\eta_w$ is
\[ \eta_w = 1 + (1-p) \delta R^* \]  

(IV.11)

Values of \( F \) and its derivatives at the wall are found by expanding \( F \) and its derivatives into a Taylor series of \( \delta \). At the wall \( F_0 \) and \( F_1 \) equal (where the index \( w \) denotes the wall)

\[
F_{o_w} = F_0(1) + (1-p) \delta R \frac{\partial F_0}{\partial \eta} 
\]

(IV.12)

\[
F_{1_w} = F_1(1) + (1-p) \delta R \frac{\partial F_1}{\partial \eta} 
\]

The derivatives of \( F_0 \) and \( F_1 \) with respect to \( \eta \) and \( \zeta \) at the wall are

\[
\frac{\partial F_0}{\partial \eta} = \frac{\partial F_0(1)}{\partial \eta} + (1-p) \delta R \frac{\partial^2 F_0}{\partial \eta^2} 
\]

(IV.13)

\[
\frac{\partial F_1}{\partial \eta} = \frac{\partial F_1(1)}{\partial \eta} + (1-p) \delta R \frac{\partial^2 F_1}{\partial \eta^2} 
\]

\[
\frac{\partial F_1}{\partial \zeta} = \frac{\partial F_1(1)}{\partial \zeta} + (1-p) \delta R \frac{\partial^2 F_1}{\partial \eta \partial \zeta} 
\]

As mentioned earlier \( F_0 \) is the solution for flow in a channel with a constant cross section as studied in Chapter III, so that \( \frac{\partial F_0}{\partial \eta} = 0 \) and \( F_0(1) = \frac{1}{2\pi} \).

At the wall the velocity \( v_i \) of the injected mass equals

\[
v_i = -\frac{\alpha M_0}{2\pi \rho R_1} \cos\beta - \frac{\alpha M_0}{2\pi \rho R_1} 
\]

(IV.14)

and the velocity components at the wall are (see Figure IV.2)

\[
v_{z_w} = v_i \sin\beta 
\]

(IV.15)

\[
v_{r_w} = v_i \cos\beta 
\]

Expressions for \( \sin\beta \) and \( \cos\beta \) are found with the aid of Figure IV.2. For small values of \( \delta \) one gets
\[
\sin\beta = \frac{3R}{az} \left[ 1 + \left( \frac{\partial R}{\partial z} \right)^2 \right] - \frac{1}{2} \delta \frac{R}{Lg} \frac{\partial R}{\partial \zeta}^* \\
\cos\beta = \left[ 1 + \left( \frac{\partial R}{\partial z} \right)^2 \right] - \frac{1}{2} \delta \frac{R}{Lg} \left( \frac{\partial R}{\partial \zeta} \right)^2 \\
\tan\beta = \frac{3R}{az} - \delta \frac{R}{Lg} \frac{\partial R}{\partial \zeta}^* \\
\]

With \( v_1 \) according to (IV.14) one now obtains

\[
\begin{align*}
v_{z_w} &= -\frac{\alpha M}{2\pi \rho R_1^{\frac{3}{2}}} \frac{R}{Lg} \frac{\partial R}{\partial \zeta}^* \delta \\
v_{r_w} &= -\frac{\alpha M}{2\pi \rho R_1^{\frac{3}{2}}} \left[ 1 - \delta^2 \frac{R}{Lg} \left( \frac{\partial R}{\partial \zeta} \right)^2 \right] 
\end{align*}
\]

The axial and the radial velocity at the wall must equal the velocities following from Equations (IV.9) at the wall. Expanding the expressions as found from Equation (IV.9) at the wall in powers of \( \delta \) and substituting the Taylor series (IV.12) and (IV.13) for \( F \) and its derivatives one obtains, if quadratic and higher order terms of \( \delta \) are neglected,

\[
\begin{align*}
v_{z_w} &= \frac{M}{\rho R_1^2 \eta_w} \left[ (1-p) R \frac{\partial^2 F}{\partial \eta^2} (1) \frac{\partial F_1}{\partial \eta} \right] \\
v_{r_w} &= -\frac{1}{\eta_w \rho R_1^{\frac{3}{2}}} \left[ \frac{\alpha M}{2\pi} + \delta \left( \alpha M \frac{F_1}{F} (1) + \frac{M}{Lg} \frac{\partial F_1}{\partial \zeta} \right) \right]
\end{align*}
\]

An expression for \( \eta_w R_1 \) is found by expanding \( \eta_w R_1 \) in terms of \( \delta \). Neglecting quadratic terms of \( \delta \) yields

\[
\eta_w R_1 = R_1 \left[ 1 - (2-p) \delta R^* \right] 
\]

The axial velocity given by (IV.17a) has to equal the axial velocity given by (IV.18a). Equating these two expressions yields with (IV.19)

\[
\frac{\partial F_1}{\partial \eta} = -\alpha \frac{M}{R_1^{\frac{3}{2}}} \frac{R}{M} \frac{1}{2\pi} \frac{\partial R}{\partial \zeta}^* + (p-1) R^* \frac{\partial^2 F}{\partial \eta^2} (1) 
\]
Since \( F_0 \) was already determined in Chapter III, Equation (IV.20) represents a boundary condition for \( F_1 \).

Equating the radial velocities given by Equations (IV.17b) and (IV.18b) yields a second boundary condition for \( F_1 \). Expanding \( F_1 \) and its derivatives with respect to \( \eta \) and \( \zeta \) at the wall into a Taylor series (Equations (IV.12) and (IV.13)) and neglecting quadratic and higher order terms of \( \delta \) one obtains a second boundary condition for \( F_1 \):

\[
\alpha \frac{M}{g M} F_1^{(1)} + \frac{\alpha_f^1}{\zeta} = \alpha \frac{M}{g M} \frac{1}{2 \pi} (1 - p) R^* \tag{IV.21}
\]

At the center of the flow the axial velocity \( v_{z_c} \) is finite. Using the general expression (IV.9) for \( v_z \) the velocity at the center becomes

\[
v_{z_c} = \frac{M}{\rho R_1^2} \lim_{\eta \to 0} \left( \frac{1}{\eta} \frac{\partial F_0}{\partial \eta} + \delta \frac{1}{\eta} \frac{\partial F_1}{\partial \eta} \right) \tag{IV.22}
\]

As was shown in Chapter III the value of \( \frac{1}{\eta} \frac{\partial F_0}{\partial \eta} \) is finite for \( \eta = 0 \). The third boundary condition for \( F_1 \) becomes with (IV.22)

\[
\lim_{\eta \to 0} \frac{1}{\eta} \frac{\partial F_1}{\partial \eta} \text{ is finite} \tag{IV.23}
\]

and so the third condition states that \( F_1 \) is of order \( \eta^2 \).

The radial velocity at the center, \( v_{r_c} \), is obtained from the general expression (IV.9) for \( v_r \). As was shown in Chapter III, the lowest power of \( \eta \) in a series expansion for \( F_0 \) is two; hence \( v_{r_c} \) becomes

\[
v_{r_c} = - \frac{1}{\rho R_1^2} \lim_{\eta \to 0} \left[ \frac{\delta M}{\eta} \frac{1}{\eta} \frac{\partial F_1}{\partial \eta} + \frac{\delta M}{\eta} \frac{1}{\eta} \partial F_1 \right] = 0 \tag{IV.24}
\]

Therefore, the third and the fourth boundary conditions result in the lowest power of \( \eta \) in a series expansion for \( F_1 \) also being two, or

\[
F_1 = 0(\eta^2) \tag{IV.25}
\]

The boundary conditions obtained for \( F_1 \) are
\[ \frac{\partial F_o}{\partial n} = - \frac{\alpha R_o^2}{L_g} \frac{M_o}{M} \frac{1}{2\pi} \frac{\partial R}{\partial \zeta} + (p-1) R \frac{\partial^2 F_o}{\partial n^2} \]  \hspace{1cm} (IV.20)

\[ \alpha L_g M_o F_1 = \frac{\partial F_o}{\partial n} + \frac{\partial F_1}{\partial \zeta} = \alpha L_g M_o \frac{1}{2\pi} (1-p) R \]  \hspace{1cm} (IV.21)

\[ F_1 = 0(n^2) \]  \hspace{1cm} (IV.25)

The boundary conditions for \( F_o \) as presented in Chapter III are

\[ \frac{\partial F_o}{\partial n} = 0 \]  \hspace{1cm} (III.13)

\[ F_o(1) = \frac{1}{2n} \]  \hspace{1cm} (III.15)

\[ F_o = 0(n^2) \]  \hspace{1cm} (III.16)

### IV.3 Solutions for \( F_1 \) for slowly varying cross sections

In this section channels are considered with slowly varying cross sections. For this type of channels the local radius \( R_1 \) is given by (IV.1)

\[ R_1 = R_o (1 + \delta R (\zeta)) \]  \hspace{1cm} (IV.1)

where \( \zeta = \frac{Z}{L_g} \). For the channels considered in this section the ratio \( R_1/L_g \) is small, so

\[ \frac{R_1}{L_g} \ll 1 \]  \hspace{1cm} (IV.26)

For small values of the ratio \( R_1/L_g \), several terms of Equation (IV.7) can be neglected. With the aid of (IV.8) the relative importance of the various terms in Equation (IV.7) can be estimated. When relatively small terms are neglected an equation for \( F_1 \) results. Separation of variables then yields an equation for the \( \zeta \) dependent and the \( n \) dependent part of \( F_1 \). An exact solution is obtained for the \( \zeta \) dependent part. The \( n \) dependent part has to be approximated by a series expansion in \( n \). With the boundary conditions a solution for \( F_1 \), and the velocity and pressure distributions can then be calculated.
IV.3.1 The resulting equation for $F_1$

The governing equation for $F_1$ for flow through a channel with varying cross sections is given by Equation (IV.7). If the wall disturbance function $R^*$ in Equation (IV.1) is a slowly varying function the following assumptions can be made

1) The ratio $\frac{R_1}{L_g}$ is small, where $L_g$ is the wave length and $R_1$ the local radius.

2) The mass flow rate of the injected mass $\tilde{\alpha}$ per unit of length is small. Furthermore the ratio of the local radius and the channel length $R_1/L$ is small. Therefore the product $\tilde{\alpha}R_1 = \tilde{\alpha}\left(\frac{R_1}{L}\right)$ is a small number.

3) The wave length $L_g$ is large. The value of $L_g$ is such that $\frac{1}{\alpha L_g}$ is a small number.

In order to determine the dominant terms in Equation (IV.7) the relative importance of the factors (IV.8) is estimated. As only values of $\alpha \leq 0.25$ are studied, the order of the ratio $M/M_0$ is 1. For the solutions $F_0$ as determined in Chapter III, terms I and VII as defined by (IV.8) are equal. With the above assumptions it is easy to see that term IV is the dominant term on the left-hand side of Equation (IV.7) and term IX is the dominant term on the right-hand side. The equation for $F_1$ now becomes

$$\lambda \left[ F_1 (-\frac{1}{n} \frac{a^2 F_0}{a \eta} + \frac{3}{n^2} \frac{a^2 F_0}{a \eta} - \frac{3}{n^3} \frac{a^2 F_0}{a \eta}) + \frac{a F_1}{\eta} (\frac{1}{n} \frac{a^2 F_0}{a \eta} + \frac{2}{n^2} \frac{a F_0}{a \eta} - \frac{3}{n^3} F_0) \right] + \frac{a^2 F_1}{\eta^2} (\frac{1}{n} \frac{a F_0}{a \eta} + \frac{3}{n^2} F_0) - \frac{1}{n} \frac{a^3 F_1}{a \eta^3} F_0 \right] =$$

$$\frac{a^4 F_1}{\eta^4} - \frac{2}{n} \frac{a^3 F_1}{a \eta^3} + \frac{3}{n^2} \frac{a^2 F_1}{a \eta^2} - \frac{3}{n^3} \frac{a F_1}{a \eta} \quad \text{(IV.28)}$$

Equation (IV.28) can be integrated once with respect to $n$. Multiplication by $n^3$ yields
\[ \lambda \left[ F_1 \left( \frac{\partial^2 F_1}{\partial \eta^2} - \eta \frac{\partial^2 F_1}{\partial \eta^2} \right) + \frac{\partial F_1}{\partial \eta} \left( F_0 + 2n \frac{\partial F_0}{\partial \eta} - \eta \frac{\partial^2 F_1}{\partial \eta^2} F_0 \right) \right] + \]
\[ - \frac{\partial F_1}{\partial \eta} + n \frac{\partial^2 F_1}{\partial \eta^2} - \eta^2 \frac{\partial^3 F_1}{\partial \eta^3} = C_2 \eta^3 \]  

where \( C_2 \) is a constant of integration which is not necessarily independent of \( \zeta \).

If \( F_1 \) is separated into an \( \eta \) and a \( \zeta \) dependent part according to \( F_1(\eta, \zeta) = F_a(\eta) F_b(\zeta) \), and if the radial coordinate is made dimensionless with the local radius \( R_1 \) (p equals 1) the first boundary condition (IV.20) yields

\[ F_b(\zeta) = (\alpha R O M L / 2\pi \eta) \frac{\partial R^*}{\partial \zeta} \]  

With \( p \) equal to 1, the second boundary condition (IV.21) yields

\[ F_a(1) = 0 \]  

If \( C_2 \) is defined by \( C_2 = C_2 \) \( F_b(\zeta) \) Equation (IV.29) becomes, upon division by \( F_b(\zeta) \),

\[ \lambda \left[ \frac{1}{n} F_a \left( n \frac{\partial F_a}{\partial \eta} - \eta \frac{\partial^2 F_a}{\partial \eta^2} \right) + \frac{\partial F_a}{\partial \eta} \left( F_0 + 2n \frac{\partial F_0}{\partial \eta} - \eta \frac{\partial^2 F_a}{\partial \eta^2} F_0 \right) \right] + \]
\[ - \frac{\partial F_a}{\partial \eta} + n \frac{\partial^2 F_a}{\partial \eta^2} - \eta^2 \frac{\partial^3 F_a}{\partial \eta^3} = C_2 \eta^3 \]  

where \( F_0 \) is the solution for a channel with a constant cross section and mass injection as found in Chapter III.

In order to find solutions \( F_a \) of Equation (IV.32) with boundary conditions (IV.30) and (IV.31), \( F_a \) is expanded into a power series of \( \eta \), according to

\[ F_a = \sum_{j=2}^{n} b_j \eta^j \]  

where all coefficients \( b_j \) are zero for odd indices \( j \) since Equation (IV.32) is invariant when \( \eta \) is replaced by \(-\eta\). Because \( n=10 \) and \( n=12 \) yield the best results
for $F_0$, the coefficients $b_j$ in (IV.33) are determined for $n=10$ and $n=12$. $F_0$ is defined similar to $F_0$, and is given by

$$F_0 = \sum_{i=0}^{n} a_i n^i$$

Hence

$$n \frac{\partial F_0}{\partial n} - n^2 \frac{\partial^2 F_0}{\partial n^2} = -\sum_{i=0}^{n-2} i(i+2) a_{i+2} n^{i+2}$$

and

$$(F_0 + 2n \frac{\partial F_0}{\partial n}) = \sum_{i=0}^{n-2} (2i+5) a_{i+2} n^{i+2}$$

With the aid of the general expression (A.III.9) for the product of two series, one obtains expressions for the non-linear terms in Equation (IV.32). If these expressions are truncated after powers of $n^{-1}$, Equation (IV.32) becomes

$$\sum_{m=3}^{n-1} \sum_{j=0}^{m-3} \left[ \sum_{m=3}^{n-1} (1-m)(m-7) + 4j (m-3-j) \right] b_{j+2} a_{m-j+1} n^m +$$

$$- \sum_{m=3}^{n-1} (m-1)^2(m+1) b_{m+1} n^m = \bar{c}_2 n^3$$

For $n=10$ and $n=12$ the resulting system of Equations from (IV.36), made complete by the boundary equations (IV.30) and (IV.31), is presented in matrix form in Figure IV.3. For both values of $n$ this system of equations can be generalized by

$$A \bar{P} = \bar{c}_2$$

where the matrix $A$ and the vector $\bar{P}$ and $\bar{c}_2$ are defined in Figure IV.3. Solving this system of equations yields the coefficients $b_1$ of the series expansion for $F_1$. 

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Figure IV.3 Perturbation equation for large values of \(g\). For \(n=10\) the last column and the last row of \(A\) have to be omitted as well as the last row of \(P\) and \(C_2\). For \(n=12\) \(A\), \(P\) and \(C_2\) hold as listed above.
IV.3.2. The axial velocity \( v_z \)

When \( F_o \) and \( F_1 \) are known the axial velocity \( v_z \) can be calculated from Equation (IV.9). The resulting expression for \( v_z \) is

\[
v_z = \frac{M}{\rho R_1^2} \sum_{i=2}^{n} \left[ a_i + \delta b_i F_b(\eta) \right] \eta^{i-2}
\]  

(IV.38)

Graphs a of Figures IV.4 and IV.5 present the axial velocity profile for two values of \( (aL_g) \) for \( n=12 \). For \( n=10 \) Figure IV.6a shows the axial velocity profile for the same value of \( (aL_g) \) as Figure IV.5a does. In this particular case \( \frac{1}{R_1^2} \) is approximated by \( \frac{1}{R_0^2} (1-2\delta R^* \) and quadratic and higher order terms of \( \delta \) are neglected. The plotted velocity is the dimensionless velocity \( \frac{v_z}{v_c} \), where \( v_c \) is the central Poiseuille velocity at the inlet, as given in Section III.3.

\[
v_c = \frac{2 \frac{1}{n} \frac{M_o}{\rho R_0^2}}
\]  

(III.20)

These graphs show that the profiles plotted resemble the corresponding profiles for flow in a channel with a constant cross section quite well (Figures III.3 to III.10). The increase in the axial velocity as shown in Figures IV.4a and IV.5a is entirely due to the development of the flow caused by the injected mass. Figure IV.4c shows the disturbance of the axial velocity caused by the disturbance of the local radius. At the inlet the local radius equals \( R_o \); hence the disturbance of the axial velocity is zero. For large values of \( L_g \) the disturbance function \( R^* \) can be approximated by \( R^* = \sin \zeta = \frac{z}{L_g} \). Figure IV.4d for \( v_z \) shows that the axial velocity difference due to the disturbance of the local radius decreases linearly with \( z \). Furthermore, the ratio of the velocity disturbance and the undisturbed velocity is independent of \( n \) (see Figures IV.4a and IV.4c).

The differences between Figure IV.5a for \( n=12 \) and Figure IV.6a for \( n=10 \) are caused by the inaccuracy of \( F_o \) (see Chapter III). As was shown in Chapter III, \( n=10 \) yields a better approach for \( F_o \) for this particular value of \( a \).
**Figure IV.4a Radial profiles of axial velocity for a slowly varying channel.**

- $M_0 = 10$
- $\lambda = 100$
- $n = 12$

Size of disturbances is 1% of radius

Length scale disturbances is $600 \times$ channel length.

- a: undisturbed velocity profile (type A flow)
- b: disturbed velocity profile at $\zeta=0$.
- c: disturbed velocity profile at $\zeta=0.5$
- d: disturbed velocity profile at $\zeta=1$

**Figure IV.4b Axial profiles of axial velocity for a slowly varying channel.**

- $M_0 = 10$
- $\lambda = 100$
- $n = 12$

Size of disturbances is 1% of radius

Length scale disturbances is $600 \times$ channel length.

- a: undisturbed velocity profile at $\eta=0$ (type A flow)
- b: disturbed velocity profile at $\eta=0$. 

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$M_0 = 10$
$\lambda = 100$
$n = 12$
$Re = 10^3$

Size of disturbances is 1% of radius.
Length scale of disturbances is 600 x channel length

a: disturbances at $\zeta = 0$
b: disturbances at $\zeta = 0.5$
c: disturbances at $\zeta = 1$

Figure IV.4c Radial profile of axial velocity disturbances for a slowly varying channel.

$M_0 = 10$
$\lambda = 100$
$n = 12$
$Re = 10^3$

Size of disturbances is 1% of radius.
Length scale of disturbances is 600 x channel length

a: disturbances at $\eta = 0$

Figure IV.4d Axial profile of axial velocity disturbances for a slowly varying channel.
Figure IV.5a Radial profiles of axial velocity for a slowly varying channel.

Figure IV.5b Axial profiles of axial velocity for a slowly varying channel.
Figure IV.5c Radial profiles of axial velocity disturbances for a slowly varying channel.

$M_0 = 10$
$\lambda = 245$
$n = 12$
$Re = 10^3$

Size of disturbances is 1% of radius.
Length scale of disturbances is 600 x channel length.

a: disturbances at $\zeta=0$
b: disturbances at $\zeta=0.5$
c: disturbances at $\zeta=1$

Figure IV.5d Axial profile of axial velocity disturbances for a slowly varying channel.

$M_0 = 10$
$\lambda = 245$
$n = 12$
$Re = 10^3$

Size of disturbances is 1% of radius.
Length scale of disturbances is 600 x channel length.

a: disturbances at $\eta=0$
Figure IV.6a Radial profiles of axial velocity for a slowly varying channel.

Figure IV.6b Axial profiles of axial velocity for a slowly varying channel.

\( \frac{v_z}{z} \)

\( \eta \)

\( \zeta \)

\( M_0 = 10 \)

\( \lambda = 245 \)

\( n = 10 \)

\( \text{Re} = 10^3 \)

Size of disturbances is 1% of radius.
Length scale of disturbances is 600 x channel length.

a: undisturbed profile at \( \zeta = 0 \) (type A flow)
b: disturbed profile at \( \zeta = 0 \)
c: disturbed profile at \( \zeta = \frac{1}{2} \)
d: disturbed profile at \( \zeta = 1 \)
\( M_0 = 10 \)
\( \lambda = 245 \)
\( n = 10 \)
\( \text{Re} = 10^3 \)

Size of disturbances is 1% of radius.

Length scale of disturbances is 600 x channel length.

a: disturbances at \( \zeta = 0 \)
b: disturbances at \( \zeta = 0.5 \)
c: disturbances at \( \zeta = 1 \)

Figure IV.6c Radial profiles of axial velocity disturbances for a slowly varying channel.

\( M_0 = 10 \)
\( \lambda = 245 \)
\( n = 10 \)
\( \text{Re} = 10^3 \)
a: disturbances at \( n = 0 \)

Figure IV.6d Axial profile of axial velocity disturbances for a slowly varying channel.
With \( F_b(\zeta) \) according to (IV.30) Equation (IV.38) becomes

\[
v_z = \frac{M}{\rho R_o^2} \sum_{i=2}^{n} \left(i a_i + \delta (-2R^* \frac{\partial}{\partial \zeta} a_i + i b_i \beta_b \frac{\partial R^*}{\partial \zeta})\right) n^{-2} \tag{IV.39}
\]

where \( \beta_b \) is a dimensionless parameter defined as

\[
\beta_b = \alpha R_o \frac{R_o}{M} \frac{1}{g} \frac{1}{2\pi} \tag{IV.40}
\]

For long waves it is clear that \( \beta_b \) is a small number. Approximation of \( R^* \) by \( \frac{Z}{L} \) and \( \frac{\partial R^*}{\partial \zeta} \) by 1 shows with Equation (IV.40) that for values of \( z > R_o \) the value of \( \beta_b \) is far less than the value of \( \frac{Z}{L} \), so that \( v_z \) can be approximated by

\[
v_z = \frac{M}{\rho R_o^2} \sum_{i=2}^{n} i a_i (1-2\delta R^*) n^{-2} \tag{IV.41}
\]

Equation (IV.41) states that the axial velocity at a cross section downstream with radius \( R_i \) in a channel with a slowly varying cross section can be approximated by the corresponding axial velocity in a channel with a constant radius \( R \), where \( R \) equals the local radius \( R_i \).

**IV.3.3 The radial velocity \( v_r \)**

Substituting \( F_o \) and \( F_i \) in Equation (IV.9) yields an expression for the radial velocity. With \( p=1 \) this becomes

\[
v_r = \frac{\alpha M_o}{\rho R_i} \sum_{i=2}^{n} (a_i + \delta b_i F_b(\zeta)) n^{-1} + \frac{\delta M}{\rho L g} \sum_{i=2}^{n} \left[ \frac{R_o}{R_i} \frac{\partial R^*}{\partial \zeta} a_i - \frac{\partial F_b}{\partial \zeta} b_i \right] n^{-1} \tag{IV.42}
\]

If \( \frac{1}{R_i} \) and \( \frac{1}{R_i^2} \) are expanded in terms of \( \delta \) the expression for \( v_r \) becomes, neglecting quadratic and higher order terms in \( \delta \) and substituting \( F_b \) according to (IV.30),
\[ v_r = - \frac{\alpha M}{\rho R_0} \sum_{i=2}^{n} \left[ a_i (1 - \delta R) + \delta b_i \beta_b \frac{\partial R}{\partial \zeta} \right] n^{i-1} + \]
\[ + \frac{\delta M}{\rho \eta \gamma R_0} \sum_{i=2}^{n} \left[ \frac{\partial R}{\partial \zeta} a_i - b_i \left( \beta_b \frac{\partial^2 R}{\partial \zeta^2} - \frac{M_0}{M} (\alpha L_g \frac{\partial R}{\partial \zeta}) \right) \right] n^{i-1} \] (IV.43)

where \( \beta_b \) is defined by Equation (IV.40).

Figures IV.7a and IV.8a present the dimensionless radial velocity \( v_r / v_c \) for two values of \( (\alpha L_g) \) and \( n=12 \). Figure IV.9a shows the radial velocity profile for \( n=10 \) for the same value of \( (\alpha L_g) \) as Figure IV.8a. The differences for \( n=10 \) and \( n=12 \) are caused by the differences in \( F_0 \) for \( n=10 \) and \( n=12 \) (see Chapter III).

\( v_c \) is the central axial velocity for the Poiseuille flow according to Equation (III.21).

Figure IV.7c and IV.7d present the velocity difference between the radial velocity in a slowly varying channel and the radial velocity in a channel with a constant cross section. Because the injected mass is assumed to be independent of \( \zeta \) the absolute value of the radial velocity decreases at the wall if the radius increases (see Figure IV.7c). For long waves the channel being studied is a diverging channel because \( R = \sin \zeta \) can be approximated by \( R = \frac{Z}{L_g} \). The diverging channel causes a relative movement in the direction of the wall. This means that the absolute radial velocity decreases. For flow without mass injection the radial velocity at the center equals zero and the radial velocity at the wall equals \( v_{r_w} = \tan \beta v_{z_w} \), where \( \beta \) is the angle of divergence (Figure IV.2). When going in the direction of the center the effect of the diverging channel diminishes. However, the radial velocity still depends on the axial velocity; due to the increase of the axial velocity the radial velocity also increases to a maximum value and then drops to zero at the center.

A similar effect occurs for flow in a diverging channel with mass injection (Figure IV.7c). In the axial direction the radial disturbance velocity increases linearly (Figure IV.7d). For large values of \( L_g \) the following approximations can be made:

\[ R^* \sim \zeta = \frac{Z}{L_g} \]

\[ \frac{\partial R^*}{\partial \zeta} \sim 1 \]  

\[ \frac{\partial^2 R^*}{\partial \zeta^2} \sim \frac{Z}{L_g} \]  

(IV.44)

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\[ \beta_b \ll 1 \]

Now \( v_r \) according to (IV.43) this can be approximated by

\[
v_r = -\frac{\Delta M}{\rho R^o} \sum_{i=2}^{n} a_i (1-\delta R^*) n^{i-1} + \frac{\delta M}{\rho L} \sum_{i=2}^{n} \frac{\delta R^*}{\delta R^o} ia_i n^{i-1} \quad (IV.45)
\]

As is seen from Equation (IV.45) the coefficients \( b_i \) do not occur in this equation. This means that the radial velocity at a cross section downstream is completely determined by \( P^o \). For flow without mass injection (\( \alpha = 0 \)) Equation (IV.45) becomes, with Equation (IV.41) for the axial velocity and \( \tan \beta \) according to (IV.16),

\[
v_r = (\tan \beta v_z) n \quad (IV.46)
\]

Equation (IV.46) states that when going from the center to the wall the effect of the diverging channel on \( v_r \) increases linearly.
Figure IV.7a Radial profiles of radial velocity for a slowly varying channel.

Figure IV.7b Axial profiles of radial velocity for a slowly varying channel.
$M_0 = 10$
$\lambda = 100$
$n = 12$
$Re = 10^3$

Size of disturbances is 1% of radius.

Length scale of disturbances is 600 x channel length.

a: disturbances at $\zeta=0$

b: disturbances at $\zeta=\frac{1}{3}$

c: disturbances at $\zeta=1$

Figure IV.7c Radial profile of radial velocity disturbances for a slowly varying channel.

$M_0 = 10$
$\lambda = 100$
$n = 12$
$Re = 10^3$

Size of disturbances is 1% of radius.

Length scale of disturbances is 600 x channel length.

a: disturbances at $\eta=\frac{1}{3}$

Figure IV.7d Axial profile of radial velocity disturbances for a slowly varying channel.
Figure IV.3a Radial profiles of radial velocity for a slowly varying channel.

Figure IV.3b Axial profiles of radial velocity for a slowly varying channel.
Figure IV.8c Radial profile of radial velocity disturbances for a slowly varying channel.

Figure IV.8d Axial profile of radial velocity disturbances for a slowly varying channel.
**Figure IV.9a** Radial profiles of radial velocity for a slowly varying channel.

- $M_0 = 10$
- $\lambda = 245$
- $n = 10$
- $Re = 10^3$

Size of disturbances is 1% of radius.

Length scale of disturbances is $600 \times$ channel length.

- a: undisturbed profile at $\zeta = 0$ (type A flow)
- b: disturbed profile at $\zeta = 0$
- c: disturbed profile at $\zeta = 0.5$
- d: disturbed profile at $\zeta = 1$

**Figure IV.9b** Axial profiles of radial velocity for a slowly varying channel.

- $M_0 = 10$
- $\lambda = 245$
- $n = 10$
- $Re = 10^3$

Size of disturbances is 1% of radius.

Length scale of disturbances is $600 \times$ channel length.

- a: undisturbed profile at $\eta = \frac{1}{2}$
- b: disturbed profile at $\eta = \frac{1}{2}$
Figure IV.9c Radial profile of radial velocity disturbances for a slowly varying channel.

\[
\begin{align*}
M_0 &= 10 \\
\lambda &= 245 \\
n &= 10 \\
Re &= 10^3 \\
\text{Size of disturbances is} & \text{1% of radius.} \\
\text{Length scale of disturbances is} & \text{600 x channel length.} \\
a: & \text{disturbances at } \zeta=0 \\
b: & \text{disturbances at } \zeta=0.5 \\
c: & \text{disturbances at } \zeta=1
\end{align*}
\]

Figure IV.9d Axial profile of radial velocity disturbances for a slowly varying channel.

\[
\begin{align*}
M_0 &= 10 \\
\lambda &= 245 \\
n &= 10 \\
Re &= 10^3 \\
\text{Size of disturbances is} & \text{1% of radius.} \\
\text{Length scale of disturbances is} & \text{600 x channel length.} \\
a: & \text{disturbances at } \eta=\frac{1}{2}
\end{align*}
\]
In order to find an expression for the axial pressure gradient $\frac{\partial p}{\partial z}$, the axial component of the momentum Equation (III.68) has to be transformed to dimensionless coordinates. When the dimensionless coordinates $\eta, \zeta$ according to Definitions (IV.4) and (IV.5) are introduced, the equation for the axial pressure gradient becomes after multiplication by $\frac{nR_1^4}{\nu M}$ and by the injection Reynolds number $\lambda = \frac{\alpha M_o}{\rho v}$

$$\frac{\partial p}{\partial z} = \frac{\mu v}{nR_1^4} \left[ \lambda \left( -\frac{1}{n^2} F_0 \frac{\partial F_o}{\partial \eta} + \frac{1}{n} F_0 \frac{\partial^2 F_o}{\partial \eta^2} + \frac{1}{n} \left( \frac{\partial F_o}{\partial \eta} \right)^2 \right) + \right.$$\n
$$\delta \eta \lambda \left( -\frac{1}{n^2} \frac{\partial F_1}{\partial \zeta} \frac{\partial F_o}{\partial \eta} + \frac{1}{n} \frac{\partial F_1}{\partial \zeta} \frac{\partial^2 F_o}{\partial \eta^2} \right) + \frac{R_o}{R_1} \frac{\partial R_1}{\partial \zeta} \left( \frac{\partial F_o}{\partial \eta} \right)^2 + \right.$$\n
$$\delta \lambda \left[ \left( -\frac{1}{n^2} F_0 \frac{\partial F_o}{\partial \eta} - \frac{1}{n} F_0 \frac{\partial^2 F_o}{\partial \eta^2} + \frac{1}{n} F_1 \frac{\partial^2 F_1}{\partial \eta^2} + \frac{1}{n} \left( \frac{\partial F_1}{\partial \eta} \right)^2 \right) + \frac{2}{n} \frac{\partial F_o}{\partial \eta} \frac{\partial F_1}{\partial \eta} \right]$$

$$\left[ \frac{1}{n^2} \frac{\partial F_1}{\partial \eta} - \frac{1}{n} \frac{\partial^2 F_1}{\partial \eta^2} + \frac{1}{n} \frac{\partial^2 F_o}{\partial \eta^2} \right] +$$

$$\delta \left[ -\frac{1}{n^2} \frac{\partial F_1}{\partial \eta} - \frac{1}{n} \frac{\partial^2 F_1}{\partial \eta^2} + \frac{1}{n} \frac{\partial^2 F_o}{\partial \eta^2} \right]$$

$$+ \left( \frac{R_1}{L} \right)^2 \left[ -\frac{R_o}{R_1} \frac{\partial R_1}{\partial \zeta} \left( \frac{\partial F_o}{\partial \eta} \right)^2 + \frac{\partial^2 F_1}{\partial \eta^2} \right]$$

$$\left( \frac{R_1}{R} \right)^2 \left[ -\frac{R_o}{R_1} \frac{\partial R_1}{\partial \zeta} \left( \frac{\partial F_o}{\partial \eta} \right)^2 + \frac{\partial^2 F_1}{\partial \eta^2} \right]$$

$$\delta \frac{\partial M_o}{\partial \eta} \left[ -2 \frac{R_o}{R_1} \frac{\partial R_1}{\partial \zeta} \left( \frac{\partial F_o}{\partial \eta} \right)^2 + \frac{\partial^2 F_1}{\partial \eta^2} \right]$$

(IV.47)

where quadratic and higher order terms of $\delta$ are neglected. As was seen in Sections IV.3.2 and IV.3.3 the disturbances of the axial and the radial velocity
caused by the disturbance function $F_1$ are very small compared to the disturbances due to $F_\sigma$. With $F_1 = F_a(n) F_b(\xi) = \beta_b \frac{\partial R}{\partial \xi} F_a(n)$ and with the assumptions made in Section IV.3.1, analysis of Equation (IV.47) shows that the axial pressure gradient for a channel with a slowly varying cross section also depends on $F_\sigma$ only,

$$\frac{\partial P}{\partial z} = \frac{\nu M}{R_\sigma^4} C(1-4\delta R^*) + \frac{\partial}{\partial L_g} \lambda \frac{\nu M^2}{R_\sigma^4 M_g} \left( 2 \frac{\partial R^*}{\partial \xi} \left( \frac{1}{n} \frac{\partial F_\sigma}{\partial n} \right)^2 \right)$$  \hspace{1cm} (IV.48)

A value for $C$ can be obtained from Equation (III.51).

The local pressure at a point $z$ can be found by integrating (IV.48) once with respect to $z$. For large values of $L_g$, the local radius $R^*$ and its derivative with respect to $\xi$ can be expanded into a power series of $\frac{z}{L_g}$. For the pressure difference $\Delta p_z$, if terms of $\left(\frac{z}{L_g}\right)^k$ are neglected for $k > 3$, this yields

$$\Delta p_z = p_z(n, \xi) - p_o(n, o) = \frac{\nu M}{R_\sigma^4} \left[ C \frac{z}{R_\sigma} \left( 1 + \frac{2}{2} z \right) - \frac{2\delta}{R_\sigma} C \frac{z}{L_g} \left( 1 + \frac{2}{3} \frac{z}{L_g} \right) + \frac{2\delta}{L_g} \lambda \frac{z}{L_g} \left( 1 + \frac{z}{L_g} \right) \left( \frac{1}{n} \frac{\partial F_\sigma}{\partial n} \right)^2 \right]$$  \hspace{1cm} (IV.49)

where $p_o(n, o)$ is the radial pressure distribution at the inlet.

Figures IV.10b and IV.11b show the axial pressure profiles at the center for two values of $(aL_g)$ and $n=12$ for flow in a slowly varying channel. By definition the local pressure at the center of the inlet is zero. The pressure difference plotted is the dimensionless pressure difference $\frac{P_z - p_o}{\Delta p_c}$, where $\Delta p_c$ is the pressure drop along the channel for the Poiseuille flow. Figure IV.12b shows the axial pressure profile for the same value of $(aL_g)$ as in Figure IV.11a for $n=10$. The differences between these figures are caused by the differences in $F_\sigma$ for $n=10$ and $n=12$ (see Chapter III).

Figures IV.10d, IV.11d and IV.12d show the disturbance of the pressure profile due to the wall disturbances. As can be expected for a diverging channel, the axial pressure gradient decreases (it is relatively easier to blow the same mass through a diverging channel).
Figure IV.10a Radial pressure profile for a slowly varying channel.

Figure IV.10b Axial pressure profile for a slowly varying channel.
Figure IV.10c Radial profile of pressure disturbances for a slowly varying channel.

Figure IV.10d Axial profile of pressure disturbances for a slowly varying channel.
Figure IV.11a Radial pressure profiles for a slowly varying channel.

Figure IV.11b Axial pressure profiles for a slowly varying channel.
Figure IV.11c Radial profiles of pressure disturbances for a slowly varying channel.

Figure IV.11d Axial profile of pressure disturbances for a slowly varying channel.
Figure IV.12a Radial pressure profiles for a slowly varying channel.

Figure IV.12b Axial pressure profiles for a slowly varying channel.
Figure IV.12c Radial profiles of pressure disturbances for a slowly varying channel.

Figure IV.12d Axial profile of pressure disturbances for a slowly varying channel.
IV.3.5 The radial pressure difference $\Delta p_r$

An expression for the radial pressure gradient $\frac{\partial p}{\partial r}$ can be derived from the equation of motion in the radial direction (III.81). In dimensionless coordinates the equation for $\frac{\partial p}{\partial \eta}$ becomes, with $\lambda = \frac{\alpha M}{\rho v}$ and after multiplication by $R_1$,

$$\frac{\partial p}{\partial \eta} = \frac{\nu \alpha M}{R_1^2} \left[ \lambda \left( \frac{1}{n^2} F_0 \frac{aF_1}{\partial \eta} - \frac{1}{n^2} F_0 \frac{aF_0}{\partial \eta} \right) + \right. $$

$$+ \delta \lambda \left( \frac{2}{n^2} F_0 \frac{aF}{\partial \eta} - \frac{1}{n^2} F_0 \frac{aF_1}{\partial \eta} \right) + \right.$$  

$$+ \frac{\lambda}{M_0} \frac{1}{\alpha L_g \eta} \left( \frac{- \rho}{R_1} \frac{\partial R}{\partial \zeta} \left( - \frac{1}{n^2} F_0 \frac{aF_0}{\partial \eta} + \frac{1}{n^2} F_0 \frac{a^2 F_0}{\partial \eta^2} \right) \right. $$

$$+ \frac{1}{n^2} \left( \frac{aF_0}{\partial \eta} - \frac{1}{n} \frac{a^2 F_0}{\partial \eta^2} \right) + \right.$$  

$$+ \delta \left( \frac{1}{n^2} \frac{aF_0}{\partial \eta} - \frac{1}{n^2} \frac{a^2 F_0}{\partial \eta^2} + \left( \frac{R_1}{L_g} \right)^2 \left( \frac{3}{n^2} \frac{a^2 F_0}{\partial \eta^2} \right) \right. $$

$$+ \frac{\lambda}{M_0} \frac{1}{\alpha L_g \eta} \left( \frac{- \rho}{R_1} \frac{\partial R}{\partial \zeta} \left( - \frac{1}{n^2} \frac{aF_0}{\partial \eta} + \frac{1}{n^2} \frac{a^2 F_0}{\partial \eta^2} \right) \right. $$

$$+ \frac{1}{n^2} \left( \frac{a^2 F_0}{\partial \eta} - \frac{1}{n} \frac{a^3 F_0}{\partial \eta^3} + \left( \frac{R_1}{L_g} \right)^2 \left( \frac{- \rho}{R_1} \frac{\partial R}{\partial \zeta} \left( - \frac{1}{n} \frac{a^3 F_0}{\partial \eta^3} \right) \right) \right) \) \) (IV.50)

where quadratic and higher order terms of $\delta$ are neglected. With $F_1 = F_a(\eta)$, $F_b(\zeta)$ =

$\frac{\partial R}{\partial \zeta} F_a(\eta)$ analysis of Equation (IV.50) shows that the disturbances caused by $F_1$ can be neglected compared to the disturbances due to $F_0$. The pressure gradient $\frac{\partial p}{\partial \eta}$ can now be approximated by

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\[ \frac{\Delta p}{\Delta \eta} = \frac{\nu \omega M}{R_o^2} \left[ (1-2\delta R^*) \left( \lambda \left( \frac{1}{n^2} F_0^2 - \frac{1}{n^2} F_0 \frac{a F_0}{\Delta \eta} \right) + \left( \frac{1}{n^2} \frac{a F_0}{\Delta \eta} - \frac{1}{n} \frac{a^2 F_0}{\Delta \eta^2} \right) \right) + \left( \alpha M \frac{1}{n^2} \frac{a R}{\Delta \zeta} \left( \lambda \left[ - \frac{1}{n^2} F_0 \frac{a F_0}{\Delta \eta} + \frac{1}{n} F_0 \frac{a^2 F_0}{\Delta \eta} - \frac{1}{n} \frac{a F_0}{\Delta \eta} \right] \right) \right) + \left( \alpha M \frac{1}{n^2} \frac{a R}{\Delta \zeta} \left( \lambda \left[ - \frac{1}{n^2} F_o \frac{a F_0}{\Delta \eta} + \frac{1}{n} F_0 \frac{a^2 F_0}{\Delta \eta} - \frac{1}{n} \frac{a F_0}{\Delta \eta} \right] \right) \right) \right] \] 

With the aid of the differential Equation (III.43) for \( F_o \), Equation (IV.51) for the radial pressure gradient can be integrated once with respect to \( \eta \), yielding

\[ p_n(\eta, \zeta) - p_o(\eta, \zeta) = \frac{\nu \omega M}{R_o^2} \left[ (1-2\delta R^*) \left( \lambda \left( \frac{1}{n^2} F_0^2 \right) + \frac{1}{n} \frac{a F_0}{\Delta \eta} - \frac{1}{n} \frac{a F_0}{\Delta \eta} \right) \right] \]

\[ - \delta \frac{M}{M_o} \frac{1}{n^2} \frac{a R}{\Delta \zeta} \left( \frac{C}{2} \eta^2 + \frac{2}{n} \frac{a F_0}{\Delta \eta} - 2 \left( \frac{1}{n} \frac{a F_0}{\Delta \eta} \right) \right) \] (IV.52)

where \( p_o(\eta, \zeta) \) is the axial pressure distribution according to (IV.49). If \( F_o \) is expanded into a power series of \( \eta \), according to \( F_o = \sum_{i=0}^{\infty} a_i \eta^i \) the radial pressure difference \( \Delta p_n \) between the center and a point \( \eta \) equals

\[ \Delta p_n = p(\eta, \zeta) - p_o(\eta, \zeta) = \]

\[ \frac{\nu \omega M}{R_o^2} \left[ (1-2\delta R^*) \left( \lambda \left( \frac{1}{n^2} F_0^2 \right) + \frac{1}{n} \frac{a F_0}{\Delta \eta} - \frac{1}{n} \frac{a F_0}{\Delta \eta} \right) \right] \]

\[ + \lambda \sum_{m=2}^{m+2} \frac{1}{2} a_m a_{m-j+2} \eta^m - 2a_2 \left( \frac{C}{2} \eta^2 + \sum_{m=0}^{n-2} 2(2m+2) a_{m+2} - 4a_2 \right) \] (IV.53)

In graphs a of Figures IV.10 up to IV.12 the radial pressure differences for several values of \( (a R) \) and for \( n=12 \) and \( n=10 \) are drawn. The plotted pressure difference is the dimensionless pressure difference \( \Delta p_n \), where \( \Delta p_c \) is the pressure drop along a channel for the Poiseuille flow. The differences between
Figures IV.11a and IV.12a, which are both plotted for the same value of \((aL_g)\), are caused by the differences in \(F_0\) for \(n=10\) and \(n=12\) (see Chapter III). Figures IV.10c and IV.11c present the disturbance of the radial pressure profile caused by the wall disturbances. Two effects are observed from these figures. The first effect is the decrease of the pressure near the wall due to the diverging wall. This effect is the one which occurs in flow in a diverging channel without mass injection; the relatively low pressure near the wall causes mass to flow into the direction of the wall. The second effect is the slight increase in absolute value of the pressure disturbance near the wall when moving downstream. Since the local radius and the local mass flow increase downstream, the pressure drop from the center to the wall increases in the direction of the outlet.
IV.4 Solutions for $F_1$ for rapidly changing cross sections

In this section channels with rapidly varying cross sections are considered. For this type of channel the local radius $R_1$ is again given by (IV.1):

$$R_1 = R_0 (1 + \delta R_L (\zeta))$$  \hspace{1cm} (IV.1)

where $\zeta$ is a dimensionless axial coordinate $\zeta = \frac{z}{L_g}$. In this section the discussion is confined to channels for which the disturbance function $R_L^*$ equals $\sin(\zeta)$ (see Figure IV.1). The characteristic length $L_g$ is in this case the wave length. For the channels considered the ratio $R_1/L_g$ is supposed to be of order one. This implies that the number of undulations in the wall geometry of the channel is considerable. For values of order 1 of the ratio $R_1/L_g$ several terms of Equation (IV.7) can be neglected. With the aid of (IV.8) the relative importance of the various terms in Equation (IV.7) can be estimated. When relatively small terms are neglected, a non-linear partial differential equation for $F_1$ results. Separation of variables then yields separate equations for the $\zeta$ and $n$ dependent parts of $F_1$. An approximate solution for $F_1$ is obtained by a series expansion of the $n$ dependent part. With the solution $F_1$ the velocity and pressure distributions can be calculated.

IV.4.1 The governing equation for $F_1$

The governing equation for $F_1$ for flow through a channel with varying cross sections is given by Equation (IV.7). If the wall disturbance function $R_L^*$ is a rapidly varying function the following assumptions can be made:

1) Because the ratio $\frac{R_1}{L_g}$ is taken to be of order 1, and since the ratio of the channel is small compared to the length of the channel the value of $\frac{L}{L_g}$ is small.

2) For small values of $\frac{L}{L_g}$ the values of $\frac{1}{\alpha L_g}$ are large, since the percentage of the injected mass flow rate, $\alpha$, through the wall is relative small.

3) The values of $\delta$ are such that the values of $\frac{\delta}{\alpha L_g}$ are small.

4) The mass flow rate of the injected mass $\bar{\alpha}$ per unit of length is small. For slender channels the product $\alpha R_1 = \bar{\alpha}(\frac{R_1}{L})$ is therefore a small number.
With these assumptions it is possible to estimate the order of the various terms in Equation (IV.7). With the aid of (IV.8) it is clear that the term labeled as term III is the dominant term on the left-hand side. For values of \( \frac{\lambda}{aL_g} \gg 1 \) the entire right-hand side in Equation (IV.7) is negligibly small compared to the left-hand side.

In this section the discussion is limited to functions \( F_1 \) which can be separated into \( n \) and \( \zeta \) dependent parts according to

\[
F_1(n, \zeta) = F_a(n) F_b(\zeta) \tag{IV.54}
\]

This implies that disturbances of the flow field are similar, and hence that radial and axial profiles of disturbances are similar. The equation for \( F_a \) and \( F_b \) becomes, upon substitution of (IV.54) in the equation for \( F_1 \),

\[
\begin{align*}
\frac{aF_b}{\zeta} & \left[ \frac{aF_o}{n} \frac{a^2F_a}{\zeta^2} - \frac{1}{n} \frac{a^3F_a}{\zeta^3} + \frac{3}{2} \frac{a^2F_o}{\zeta} - \frac{1}{2} \frac{aF_o}{\zeta} \right] F_a(n) + \\
& - \frac{3}{n} \frac{aF_b}{\zeta} \left[ \frac{aF_o}{n} \right] + \frac{R_1}{L_g} \frac{a^3F_b}{\zeta^3} \frac{aF_o}{n} + \\
& + \frac{R_o}{R_1} \frac{aR_1}{\zeta} \frac{aF_0}{n} \left[ \frac{a^2F_o}{\zeta} + \frac{1}{n} \frac{aF_o}{\zeta} \right] - \frac{R_1}{L_g} \frac{R_o}{R_1} \frac{a^3R_1}{\zeta^3} \left[ \frac{aF_o}{n} \right]^2 = 0 \tag{IV.55}
\end{align*}
\]

In Section IV.2 general expressions were derived for the boundary conditions in a channel with varying cross sections. For small values of \( (aL_g) \) the no-slip condition in the axial direction (IV.20) becomes

\[
\frac{aF_1(n)(1)}{\zeta} = -R_o \frac{M_o R_o}{L_g} \frac{1}{2\pi} \frac{\partial^* a}{\zeta} + (p-1) R \frac{a^2F_1(n)(1)}{\zeta^2} \tag{IV.56}
\]

The second boundary condition (IV.21) is simplified by assuming that the radial velocity at the wall is not changed by the wall disturbances. This additional assumption is necessary to be able to find solutions \( F_1 \) for non-exponentially varying cross sections. The second boundary condition is

\[
\frac{aF_b}{\zeta} \left( F_1(n)(1) \right) = 0 \tag{IV.57}
\]

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As was seen in Section IV.2 $F_1$ is of order $\eta^2$, which implies that a third boundary condition for $F_a$ is

$$F_a = O(\eta^2) \quad (IV.58)$$

Analysis of Equation (IV.56) shows that for $p = 1$, $F_b(\zeta)$ is proportional to $\frac{\partial R}{\partial \zeta}$. In that case Equation (IV.55) does not allow wall disturbances according to $R = \sin \zeta$. For $p = 0$, however, Equation (IV.55) simplifies to an equation which yields solutions $F_a(\eta)$ and $F_b(\zeta)$ for $R = \sin \zeta$. For $p=0$ the channel with varying cross sections is transformed into a channel with constant sections. The values of properties as found for a channel with constant cross sections have to be transformed back to the corresponding values for a channel with varying cross sections. A property $x_1$ at $\eta_1$ in a channel with a constant cross section yields the value of $x_2$ at $\eta_2$ in a channel with wall disturbances from

$$x_2(\eta_2) = x_1(\eta_1) + \eta_1 \frac{\partial x_1}{\partial \eta}(\eta_1) \quad (IV.59)$$

with $\eta_2 = \eta_1(1+\delta R)$. For $p=0$ and $R = \sin \zeta$ boundary condition (IV.56) becomes

$$F_b \frac{\partial F_a}{\partial \eta} = - \alpha R_o \frac{M_o}{M} \frac{R_o}{L} \frac{1}{2\pi} \cos \zeta - \frac{\partial^2 F_0}{\partial \eta^2}(1) \quad (IV.60)$$

As the local mass flow rate $M$ increases linearly with $z$, $M$ can be written as

$$M(\zeta) = M_o (1+(\alpha L_g)\zeta) \quad (IV.61)$$

Equation (IV.60) shows that the change of $M$ with $\zeta$ is slow compared to the change of $R$ with $\zeta$. In order to be able to find solutions for $F$, the dependance $\frac{\partial F_a}{\partial \eta}$ of $M$ on $\zeta$ is neglected. Taking $\frac{\partial F_a}{\partial \eta} = 1$ (the value of $\frac{\partial R}{\partial \eta}$ is arbitrary since $F_1=F_a F_b$) Equation (IV.60) yields an equation for $F_b$:

$$F_b(\zeta) = A_0 \cos \zeta + A_1 \sin \zeta \quad (IV.62)$$

where $A_0$ and $A_1$ are constants defined as

$$A_0 = - \alpha R_o \frac{M_o}{M} \frac{R_o}{L} \frac{1}{2\pi} \quad (IV.63)$$
\[ A_1 = - \frac{a^2 F_0}{\alpha n^2} \]

For \( F_b(\zeta) \) according to (IV.59) it is clear that \( \frac{a F_b}{\alpha \zeta} = - \frac{a^3 F_b}{\alpha \zeta^3} \) and therefore with \( p=0 \) and an approximation of \( \left( \frac{R_1}{L} \right)^2 \) by \( \left( \frac{R_0}{L} \right)^2 \) Equation (IV.54) becomes

\[
\frac{1}{n^2} F_a \left( - \frac{3 a F_0}{n \alpha n} - \left( \frac{R_0}{L} \right)^2 n \frac{a F_0}{\alpha n} + 3 \frac{a^2 F_0}{\alpha n^2} - n \frac{a^3 F_0}{\alpha n^3} \right) +
\]

\[
- \frac{1}{n} \frac{a F_a}{\alpha n} \left( \frac{1}{n} \frac{a F_0}{\alpha n} \right) + \frac{a^2 F_a}{\alpha n^2} \left( \frac{1}{n} \frac{a F_0}{\alpha n} \right) = 0 \tag{IV.64}
\]

The problem of determining the flow field in a channel with varying cross sections is then reduced to the problem of finding solutions \( F_a \) of Equation (IV.64) with boundary conditions

\[
F_a(1) = 0 \tag{IV.65}
\]

\[
\frac{a F_a(1)}{\alpha n} = 1
\]

If \( F_a \) is expanded into a power series of \( n \), \( F_a \) is taken to be given by

\[
F_a = \sum_{j=2}^{n} c_j n^j \tag{IV.66}
\]

and if \( F_0 \) is the solution for a channel with a constant cross section, Equation (IV.60) yields the following expression for the coefficients \( c_j \):

\[
\sum_{m=2}^{n-2} \sum_{j=0}^{m} \left[ \left( \frac{R_0}{L} \right)^2 \left( \frac{m}{L} \right) c_{j+2} a_{m-j+2} \right] n^m +
\]

\[
- \left( \frac{R_0}{L} \right)^2 \sum_{m=2}^{n-2} \sum_{j=0}^{m-2} \left( m-j \right) c_{j+2} a_{m-j} n^m +
\]

\[
\sum_{j=2}^{n-2} \left[ n(n-j+2) (2-n+2j) c_{j+2} a_{n-j+2} \right] n^n +
\]

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\[
\frac{R^2}{L_g} \left[ \sum_{j=0}^{n-2} (n-j) c_{j+2} a_{n-j} \right] \eta^n + \\
+ \sum_{m=n+2}^{2n-4} \sum_{j=m-n+2}^{n-2} (m-j) c_{j+2} a_{m-j+2} + \\
- \left( \frac{R^2}{L_g} \right) \sum_{j=m-n}^{n-2} (m-j) c_{j+2} a_{m-j} \eta^m \\
+ n c_n a_n \eta^{2n-2}
\]  

(IV.67)

Figure IV.13 presents the resulting system of equations for \( n=10 \) and \( n=12 \) in matrix form. This system is the system of equations that follows if Equation (IV.67) is truncated after powers of \( \eta^{n-4} \). The system of equations is made complete by the boundary conditions (IV.65) for \( F_a \). Once the values of \( c_i \) are determined, an expression for \( F_1 \) is found with the aid of Equations (IV.54), (IV.62) and (IV.66).
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 \\
-2t_{a_2} & 16a_2 & 0 & 0 & 0 & 0 \\
-48a_6t_{a_4} & 32a_4 & 2t_{a_2} & 48a_2 & 0 & 0 \\
-192a_8t_{a_6} & 4t_{a_4} & 96a_4 & 2t_{a_2} & 96a_2 & 0 \\
\end{bmatrix}
\begin{bmatrix}
c_2 \\
c_4 \\
c_6 \\
c_8 \\
c_{10} \\
c_{12} \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
\frac{1}{2} \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

A

P

c_2

\[
\begin{align*}
\text{with } t &= \left(\frac{R_0}{L_g}\right)^2 \\
\end{align*}
\]

Figure IV.12 Pertubation equation for small values of \(dL_g\). For \(n=10\) the last column and the last row of \(A\) have to be omitted as well as the last row of \(P\) and \(C_2\). For \(n=12\) \(A\), \(P\) and \(C_2\) hold as listed above.
IV.4.2 The axial velocity $v_z$

Once $F_o$ and $F_1$ are known the axial velocity $v_z^t$ in the disturbed channel can be calculated with the aid of Equation (IV.9) and the transformation relation (IV.59). The axial velocity becomes, if quadratic and higher order terms of $\delta$ are neglected,

$$v_z^t = \frac{M}{\rho R_o^2} \left[ \frac{1}{n} \frac{aF_o}{a\eta} + \delta R \left[ -\frac{1}{n} \frac{aF_o}{a\eta} + \frac{a^2F_o}{a\eta^2} \right] + \delta F_b \frac{1}{n} \frac{aF}{a\eta} \right] \quad (IV.68)$$

where $F_a$ and $F_b$ are given by (IV.66) and (IV.62), respectively. Graphs a and b of Figure IV.14 represent the axial velocity profiles for one value of $(aL_g)$. The plotted velocity is the dimensionless velocity $v_z^t/v_c$ where $v_c$ is the corresponding axial velocity at the center for the Poiseuille flow in a channel with radius $R_o$.

![Figure IV.14a Radial profile of axial velocity for rapidly varying channels.](image-url)

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$M_a = 10$
$\lambda = 100$
$n = 12$
$Re = 10^3$

Size of the wall disturbances is $\frac{1}{3}$% of the radius.
Length scale of disturbances is $0.1\pi$ x channel length.

a: undisturbed profile at $\eta = 0$ (type A flow)
b: disturbed profile at $\eta = 0$

Figure IV.14b Axial profile of axial velocity for rapidly varying channels.
Figure IV.14c Radial profiles of axial velocity disturbances for rapidly varying channels.

$M_0 = 10$
$\lambda = 100$
$n = 12$
$Re = 10^3$

Size of wall disturbances is $\frac{1}{8}$% of the radius.

Length scale of disturbances is $0.1\pi$ channel length.

- **a**: disturbances at $\zeta=0$
- **b**: disturbances at $\zeta=0.079$
- **c**: disturbances at $\zeta=0.157$
- **d**: disturbances at $\zeta=0.236$
Figure IV.14d Axial profile of axial velocity disturbances for rapidly varying channels.

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Figure IV.14e Axial profile of axial velocity disturbances for rapidly varying channels.
Figure IV.14c shows the axial velocity differences due to the wall disturbances at several locations downstream. Figure IV.15 shows the channel geometry. Examination of these figures shows that at cross sections with a radius larger than \( R_0 \) the axial velocity is less than the axial velocity in a channel with constant cross sections of radius \( R_0 \). Figure IV.14e shows that the phase difference between the wall disturbances and the velocity disturbance at the center is 180 degrees. This effect is in agreement with the principle of continuity.

However, Figure IV.14d shows that the absolute values of the velocity disturbances at the wall are very small compared to the velocity disturbances at the center and that the phase difference between wall disturbances and axial velocity disturbances is 90 degrees at the wall. Both effects are caused by the no-slip condition at the wall.

\[ \begin{align*}
\zeta &= 0.08 \\
\zeta &= 0.16 \\
\zeta &= 0.24 \\
\zeta &= 1
\end{align*} \]

Figure IV.15 Channel geometry. Wall disturbances are strongly enlarged.

IV.4.3 The radial velocity \( v_r \)

Substitution of \( F_0 \) and \( F_1 \) in Equation (IV.9) yields an expression for the radial velocity \( v_r \) in a channel with constant cross sections (radius \( R_0 \)). Using the transformation Equation (IV.59) the radial velocity \( v_r \) in a channel with wall disturbances becomes

\[
v_r = - \frac{1}{g \rho_o} \left[ \alpha_o \frac{1}{n} \frac{\partial F_0}{\partial \zeta} + \delta \left( \alpha_o R^* \left[ - \frac{1}{n} F_o + \frac{\partial F_0}{\partial \eta} \right] + \frac{1}{n} \frac{\delta F_r}{\delta \eta} \right) + \frac{1}{n} F_a \left( \frac{M}{L} \frac{\delta F_r}{\delta \zeta} + \alpha_o F_b \right) \right]
\]

(IV.69)
Again quadratic and higher order terms of $\delta$ have been neglected. $F_a$ and $F_b$ are given by (IV.66) and (IV.62), respectively. Velocity profiles according to (IV.69) are plotted in Figure IV.16. The plotted velocity is a dimensionless velocity $v_r^t/v_c$, where $v_c$ is the axial velocity for the Poiseuille flow at the center of the inlet of a channel with a constant radius $R_o$. Figure IV.16a shows $v_r^t/v_c$ versus $n$ for several locations downstream.

Examination of Figure IV.15 makes it clear that the disturbances of the radial velocity are proportional to the wall gradient (see Figure IV.16e). For profiles denoted by b and d in Figure IV.16c the change of the radius with the axial coordinate is very small and the velocity disturbances are caused by the deviation of the local radius from $R_o$. For these cross sections the velocity disturbances are of order $\delta$ of the undisturbed radial velocity. Similar to the reasoning presented in Section IV.3.3 the disturbances of the radial velocity are also proportional to the axial velocity $v_z$ and the non-dimensional transverse coordinate $n$. Therefore, at cross sections where changes of $R^*$ are of order 1, the disturbances of the radial velocity are of order $\delta$ of the undisturbed axial velocity $v_z$. This explains the relatively large disturbances $v_r^t$ at the locations denoted by a and c in figures IV.16c and IV.16a. At the wall, however, the axial velocity is very small, and the disturbances of the radial velocity are entirely determined by the deviation of the local radius $R_i$ from the undisturbed radius $R_o$ (see figures IV.16d and IV.16b).
Figure IV.16a Radial profiles of radial velocity for rapidly varying channels

- $M_0 = 10$
- $\lambda = 100$
- $n = 12$
- $Re = 10^3$

Size of the wall disturbances is $\frac{1}{3}$% of the radius.

Length scale of disturbances is $0.1\pi x$ channel length.

a: undisturbed profile at $\zeta=0$ (type A flow)
b: disturbed profile at $\zeta=0$
c: disturbed profile at $\zeta=0.079$
d: disturbed profile at $\zeta=0.157$
e: disturbed profile at $\zeta=0.236$
Figure IV.16b Axial profiles of radial velocity for rapidly varying channels.
Figure IV.16c Radial profiles of radial velocity disturbances for rapidly varying channels.

$M_0 = 10$
$\lambda = 100$
$n = 12$
$Re = 10^3$

Size of wall disturbances is $\frac{1}{3}$% of the radius.

Length scale of wall disturbances is $0.1\pi \times \text{channel length}$

a: disturbances at $\zeta = 0$
b: disturbances at $\zeta = 0.079$
c: disturbances at $\zeta = 0.157$
d: disturbances at $\zeta = 0.236$
Figure IV.16d Axial profile of radial velocity disturbances for rapidly varying channels.
Figure IV.16e Axial profile of radial velocity disturbances for rapidly varying channels.

$M_0 = 10$
$\lambda = 100$
$n = 12$
$Re = 10^3$

Size of wall disturbances is $\frac{1}{4}$% of the radius.
Length scale of disturbances is $0.1\pi x$ channel length

a: profile at $n=0.5$
The axial pressure difference $\Delta p$ can be calculated by means of Equation (III.47). With the approximations made in Section IV.4.1 the axial pressure gradient becomes with $p=0$ and $F_1 = F_a(n) F_b(\zeta)$, and if the channel with wall disturbances is transformed into a channel with constant cross sections:

$$\frac{\Delta p}{\Delta z} = \frac{\nu M}{n R_0} \left[ \lambda \left( -\frac{1}{n^2} \frac{\partial F}{\partial n} + \frac{1}{n} \frac{\partial^2 F}{\partial n^2} + \frac{1}{n} \left( \frac{\partial F}{\partial n} \right)^2 \right) + \right.$$

$$+ \left( \frac{1}{n^2} \frac{\partial F}{\partial n} - \frac{1}{n} \frac{\partial^2 F}{\partial n^2} + \frac{3}{n^3} \right) +$$

$$+ \frac{1}{a L_g} \delta \delta A M \frac{\partial F}{\partial \zeta} \left[ -\frac{1}{n^2} \frac{\partial F}{\partial n} + \frac{1}{n} \frac{\partial^2 F}{\partial n^2} - \frac{1}{n} \frac{\partial F}{\partial n} \frac{\partial F}{\partial n} \right] \right]

(IV.70)

With the expression for the axial pressure gradient in a channel with constant cross sections (III.71) the axial pressure gradient (IV.70) becomes

$$\frac{\Delta p}{\Delta z} = \frac{\nu M}{R_0} \left[ C + \frac{\delta A M}{\alpha M_0} \frac{\partial F}{\partial z} \left[ -\frac{1}{n^3} \frac{\partial F}{\partial n} + \frac{1}{n^2} \frac{\partial^2 F}{\partial n^2} - \frac{1}{n^2} \frac{\partial F}{\partial n} \frac{\partial F}{\partial n} \right] \right]

(IV.71)

where quadratic and higher order terms of $\delta$ are neglected.

This equation can be integrated once with respect to $z$. With $F_0$ and $F_a$ independent of $z$ one obtains, assuming $\left( \frac{M}{M_0} \right)^2$ to be constant,

$$p_z(n,z) - p_0(n,0) = \frac{\nu M}{R_0} \left[ Cz (1 + \frac{a}{2z}) + \right.$$

$$\frac{\delta A}{\alpha} D(n) \left[ \left( \frac{M}{M_0} \right)^2 (A_0 \cos \frac{z}{L_g} + A_1 \sin \frac{z}{L_g}) - A_0 \right] \right]

(IV.72)

where $p_0(n,0)$ is the local radial pressure at the inlet, $A_0$ and $A_1$ are defined by (IV.63) and $C$ is given by (III.51). Furthermore $D(n)$ is a shorthand notation for
\[ D(n) = \frac{1}{n} F_a \left( -\frac{1}{2} \frac{\partial F}{\partial n} + \frac{1}{n} \frac{\partial^2 F}{\partial n^2} \right) - \frac{1}{n} \frac{\partial F}{\partial n} \frac{\partial F}{\partial n} \]  \hspace{1cm} (IV.73)

Upon introduction of \( F_a \) and \( F_o \) expanded into a power series of \( n \) in (IV.73) and truncation of the exact expression after \( n^{-4} \), \( D(n) \) becomes

\[ D(n) = \sum_{m=0}^{n-4} \sum_{j=0}^{m} \frac{a_{m-j+2}}{a_{m-j+2}} \eta^m \]  \hspace{1cm} (IV.74)

The pressure difference \( \Delta p_z^t \) in a channel with varying cross sections becomes, using the transformations Equation (IV.59) and again neglecting quadratic and higher order terms of \( \delta \),

\[ \Delta p_z^t = \frac{\nu M}{R_o} \left[ C z (1 + \frac{a}{2 z}) + \frac{\delta \lambda}{a} D(n) \left( \frac{M}{M_o} \right)^2 \left( A_o \cos \frac{z}{L_g} + A_1 \sin \frac{z}{L_g} \right) - A_o \right] \]  \hspace{1cm} (IV.75)

Figure IV.17b presents the axial pressure difference \( \Delta p_z^t/\Delta p_c \) according to (IV.75) as a function of the axial coordinate. \( \Delta p_c \) is the pressure drop for the Poiseuille flow in a channel of length \( L \). Figures IV.17d and IV.17e give the pressure disturbances at the wall and at the center. These profiles will be discussed in the next section.

### IV.4.5 The radial pressure difference \( \Delta p_r \)

The radial pressure difference can be calculated from Equation (IV.52). When the assumptions made in Section IV.4.1 are introduced, the equation for the radial pressure difference becomes, if the channel with wall disturbances is transformed to a channel with constant cross sections,

\[ \frac{\partial p}{\partial n} = \frac{\nu a M}{R_o} \left[ \frac{1}{n^3} F_o^2 - \frac{1}{n^2} F_o \frac{\partial F}{\partial n} \right] + \frac{1}{n^2} \frac{\partial F}{\partial n} - \frac{1}{n} \frac{\partial^2 F}{\partial n^2} \] 
\[ + \frac{\delta \lambda}{a} \left( \frac{M}{M_o} \right)^2 \left( \frac{1}{a L_g} \right)^2 \frac{1}{n^2} \frac{\partial F}{\partial n} \frac{\partial^2 F}{\partial n^2} \]  \hspace{1cm} (IV.76)

This equation can be simplified by using Equation (IV.64) for \( F_a \). Introducing of \( D(n) \) according to (IV.73) in Equation (IV.64) yields

-134-
\[
\frac{1}{n^2} F_o \frac{\partial F_o}{\partial n} = - \left( \frac{L_g}{R^2} \right)^2 \frac{\partial D(n)}{\partial n} \tag{IV.77}
\]

whereupon the equation for the radial pressure gradient can be integrated once with respect to \( n \). This yields

\[
p_n(n, \zeta) - p_o(p, \zeta) = - \frac{\nu a M}{R_o^2} \left[ \left( \frac{A}{2n^2} F_o^2 + \frac{1}{n} \frac{\partial F_o}{\partial n} - 2a_2 \right) + \right.
\]

\[
+ \delta \left( \frac{M_{c2}}{M_o} \right)^2 \left( \frac{1}{\alpha R_o} \right)^2 \left( \frac{L_g}{M_o} \right)^2 \frac{\partial^2 b}{\partial \zeta^2} \left[ D(n) + 4a_2 c_2 \right] \tag{IV.78}
\]

\( p_o(o, \zeta) \) is the local pressure at the center and \( D(n) \) is defined by (IV.73).

The pressure difference \( \Delta p_n \) in a channel with wall disturbances can be calculated with the aid of the transformation Equation (IV.59). If quadratic and higher order terms of \( \delta \) are neglected the radial pressure difference \( \Delta p_n \) becomes

\[
\Delta p_n = - \frac{\nu a M}{R_o^2} \left[ \frac{A}{2n^2} F_o^2 + \frac{1}{n} \frac{\partial F_o}{\partial n} - 2a_2 \right] + \right.
\]

\[
+ \delta R \left( \frac{A}{n} F_o \left( \frac{1}{n} F_o + \frac{\partial F_o}{\partial n} \right) - \frac{1}{n} \frac{\partial F_o}{\partial n} + \frac{\partial^2 F_o}{\partial n^2} \right) + \right.
\]

\[
+ \delta \left( \frac{M_{c2}}{M_o} \right)^2 \left( \frac{1}{\alpha R_o} \right)^2 \frac{\partial^2 b}{\partial \zeta^2} \left[ D(n) + 4a_2 c_2 \right] \tag{IV.79}
\]

With the assumptions made in Section IV.4.1 it is clear that \( \lambda \left( \frac{M_{c2}}{M_o} \right)^2 \left( \frac{1}{\alpha R_o} \right)^2 > 1 \), and hence the pressure difference \( \Delta p_n \) can be approximated by

\[
\Delta p_n = - \frac{\nu a M}{R_o^2} \left[ \frac{A}{2n^2} F_o^2 + \frac{1}{n} \frac{\partial F_o}{\partial n} - 2a_2 \right] + \right.
\]

\[
+ \delta \left( \frac{M_{c2}}{M_o} \right)^2 \left( \frac{1}{\alpha R_o} \right)^2 \frac{\partial^2 b}{\partial \zeta^2} \left[ D(n) + 4a_2 c_2 \right] \tag{IV.80}
\]

Figures IV.17a and IV.17c present the pressure differences \( \Delta p_n / \Delta p_c \) as a function of \( n \). Figure IV.17a gives the total pressure difference and Figure
IV.17c presents the pressure disturbances due to the wall disturbances. The pressure difference $\Delta p_c$ is the total pressure drop in a channel of length $L$ for the Poiseuille flow. Examination of Equation (IV.80), with the aid of Equations (IV.62) and (IV.63), shows that the radial pressure disturbance depends on the change of the wall gradient $\frac{\partial R}{\partial \zeta}$ with the axial coordinate $\zeta$. In other words, the wall curvature determines the pressure difference. Near the wall the curvature of the streamlines is believed to resemble the wall curvature, while near the center of the flow the streamlines are straight lines (for reasons of symmetry). This means that the radial pressure disturbances near the center are relatively small compared to the radial pressure disturbances near the wall (Figure IV.17c).

Figures IV.17d and IV.17e present the axial profiles of the pressure disturbances at the wall and at the center. These figures show that the pressure disturbances resemble the wall curvature quite well. The mass injection causes the axial pressure disturbance to increase. Examination of Equations (IV.72) and (IV.62) shows that the axial pressure disturbance is proportional to $F_b(\zeta)$. Using Equation (IV.63) $F_b(\zeta)$ can be written as

$$F_b(\zeta) = A_2 \sin(\zeta + \phi) \quad (\text{IV.81})$$

with

$$\tan \phi = \left(\frac{dL}{g}\right) \frac{R_c}{L} \frac{R_o}{g} \frac{1}{m^2} \frac{\partial^2 F_0(1)}{\partial \zeta^2}$$

$$A_2 = -\frac{\partial^2 F_0(1)}{\partial \zeta^2} / \cos \phi \quad (\text{IV.82})$$

Because $\frac{\partial^2 F_0(1)}{\partial \zeta^2}$ is negative for all values of $\lambda$, the phase difference $\phi$ between the wall disturbances and the pressure disturbances is negative.
\[ \Delta p \times 10^3 \]

-3
-2
-1
0
1
2
3
0
0.2
0.4
0.6
0.8
1

\[ M_0 = 10 \]
\[ \lambda = 100 \]
\[ n = 12 \]
\[ Re = 10^3 \]
Size of wall disturbances is \( \frac{1}{8} \% \) of the radius.
Length scale of disturbances is \( 0.1\pi \times \) channel length.

- a: undisturbed profile at \( \zeta = 0 \) (type A flow)
- b: disturbed profile at \( \zeta = 0 \)
- c: disturbed profile at \( \zeta = 0.079 \)
- d: disturbed profile at \( \zeta = 0.157 \)
- e: disturbed profile at \( \zeta = 0.236 \)

Figure IV.17a Radial pressure profiles for rapidly varying channels.
Figure IV.17b Axial pressure profiles for rapidly varying channels.
\[ M_0 = 10 \]
\[ \lambda = 100 \]
\[ n = 12 \]
\[ Re = 10^3 \]

Size of wall disturbances is \( \frac{1}{10} \% \) of the radius.

Length scale of wall disturbances is \( 0.1 \theta_x \) channel length.

a: disturbances at \( \zeta = 0 \)
b: disturbances at \( \zeta = 0.079 \)
c: disturbances at \( \zeta = 0.159 \)
d: disturbances at \( \zeta = 0.236 \)

Figure IV.17c Radial profiles of pressure disturbances for rapidly varying channels.
Figure IV.17d Axial profile of pressure disturbances for rapidly varying channels.
Figure IV.17e Axial profile of pressure disturbances for rapidly varying channels.
V. Summary of calculation results

For a channel with a constant cross-sectional area, the following results were obtained in Chapter III:

- Expanding the Stokes streamfunction into a power series of the non-dimensional transverse coordinate \( \eta \) yields realistic solutions for the flow field.

- The number of solutions found for the dimensionless streamfunction, and hence for the flow field, depends on the highest power of \( \eta \) considered in the series expansion.

- For small values of the injection Reynolds number \( \lambda \) only one physically feasible solution remains, independent of the number of terms considered in the power expansion. The resulting flow is a Poiseuille-like flow.

- For very large values of the injection parameter \( \lambda \), a solution was also calculated analytically. The solution found using a series expansion for the dimensionless streamfunction with terms up to \( \eta^{10} \) is to a high degree the same as the analytical solution.

- For moderate values of the injection Reynolds number \( \lambda \) two solutions for the flow field were found:
  - the type A flow qualitatively resembles a Poiseuille flow,
  - the type B flow is characterized by a region of reversed flow near the wall.

For a channel with a variable cross-sectional area the following results were obtained in Section IV:

For channels with slowly varying wall geometry (long waves) the channel is a sinusoidally diverging channel.

- The axial velocity profile equals the axial velocity profile for the Poiseuille flow in a channel with a constant cross-sectional area and with a radius equal to the local radius.

- The radial velocity disturbances are proportional to the transverse coordinate, the local axial velocity and the wall surface gradient.

- For these types of channels the total pressure drop in the axial direction along the channel decreases. The radial pressure disturbance remains very small and is proportional to the wall surface gradient.
For channels with rapidly varying wall geometry (short waves) the following results were obtained:

- The main disturbance of the axial velocity is caused by the deviation of the local radius from $R_0$. Therefore, the disturbances of the axial velocity and the disturbance of the radius keep pace at all locations where the undisturbed axial velocity is not equal to zero. At the wall, however, the wall surface gradient determines the disturbance of the axial velocity. The effect of the wall disturbances increases in the direction of the wall.

- The disturbances of the radial velocity depend on the undisturbed axial velocity, the wall surface gradient and the transverse coordinate. The phase difference between the wall disturbances and disturbances of the radial velocity is therefore about $90^\circ$ at all transverse locations except for the wall. At the wall the deviation of the local radius from $R_0$ determines the disturbance of the radial velocity.

- The pressure disturbances are proportional to the wall surface curvature except for a small phase difference.
VI. Conclusions

- Expanding of the dimensionless streamfunction into a power series of the dimensionless coordinates yields acceptable results for the flow field in channels with mass injection and with a constant or a variable cross-sectional area.

- The number of solutions obtained for the flow field depends on the number of terms considered in the power series expansion and the value of the injection Reynolds number \( \Lambda \). However, the only physically realistic solution is believed to be the Poiseuille-like flow; this is the only solution for which the convergence of the power series is acceptable. It might be useful to extend the number of terms in the power series for the dimensionless streamfunction and to check the stability of the solutions found.

- Although viscous forces are implicitly neglected for high values of the injection Reynolds number \( \Lambda \), the flow field still resembles viscous flow fields. It might therefore be interesting to study this type of flow in more detail.

- For channels with a slowly varying wall geometry the axial velocity profile and the axial pressure profile can be found by calculating the corresponding profiles for the Poiseuille flow in a channel with a cross-sectional area equal to the local cross-sectional area of the diverging channel. The radial profiles depend on the wall surface gradient and have to be calculated independently.

- For channels with a rapidly sinusoidally varying wall geometry the axial velocity disturbances keep pace with the wall disturbances, the radial velocity disturbances keep pace with the wall surface gradient and the pressure disturbances keep pace with the wall surface curvature, except for a small phase difference. For a solid fuel the burning rate is proportional to the pressure [18, 22] so that, the phase difference between pressure and wall disturbances causes the wall disturbances to propagate in the direction of the outlet.

- In order to enhance the understanding of the flow fields which were calculated, it is believed to be useful to calculate temperature profiles.
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Appendix I Conservation equations

A.I.1 General remarks

There are three fundamental physical principles upon which fluid dynamics is based:

1) the conservation of mass
2) Newton's second law (F=ma)
3) the conservation of energy

Mathematical relations between the physical quantities describing the fluid are derived from these principles. Four approximations are possible:

a) Consider a fluid particle moving with the fluid:
   1) Imagine a finite control volume $V$ such that the control volume always contains the same particles. The resulting equations will be in the integral form (conservation form).
   2) Take an infinitesimal fluid element $\delta V$. This approach leads to the partial differential form of the equations.

b) Alternatively, consider a fixed fluid particle:
   1) Again consider a finite controle volume $V$ with the fluid moving through it. Integral equations again result (transport form).
   2) Imagine an infinitesimal fluid particle $\delta V$. This leads to partial differential equations in Eulerian form.

In this appendix the equations presented will be partial differential equations. They can be derived by using method a-2 (see e.g. [1], [2] and [3]).

A.I.2 The equations of motion

In this section the equations of motion are presented. For a derivation of these equations the reader is referred [1], [2] and [3].

The principle of the conservation of mass leads to the continuity equation, which in vector notation reads

$$\rho(\vec{v}.\vec{v}) + \frac{D\rho}{Dt} = 0 \quad \text{(A.I.1)}$$

where $\frac{D}{Dt}$ is the so-called substantial derivative, given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v} \cdot \vec{v}) \quad \text{(A.I.2)}$$
Figure A.I.1 shows both Cartesian and cylindrical coordinate systems. When \( \mathbf{V} \) and \( \mathbf{v} \) are written in cylindrical coordinates the continuity equation becomes

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{\partial \theta} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0 \tag{A.I.3}
\]

From Newton's second law, \( F = m a \), the equation of momentum transport follows. Using Stokes' relation [3] between stresses and the rate of deformation,

\[
\tau_{ij} = -\mu \frac{\partial v_k}{\partial x_k} \delta_{ij} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{A.I.4}
\]

where \( \mu \) is the coefficient of absolute viscosity and \( \mu' \) is the second viscosity coefficient (or volume viscosity coefficient), the momentum equation becomes

\[
\rho \frac{D \mathbf{V}}{Dt} = -\nabla p + \mu (V^2 \mathbf{V}) + (\mu - \mu') \nabla \cdot (\mathbf{V} \cdot \mathbf{V}) + \rho \mathbf{f} \tag{A.I.5}
\]

where \( \mathbf{f} \) represents body forces. Figure A.I.2 presents the components of the equation of motion in cylindrical coordinates when \( \mathbf{f} = 0 \).

The principle of the conservation of energy yields the energy equation. With the convention that a positive force performs positive work for positive displacements, the energy equation becomes

\[
\rho \frac{D h}{Dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{V} + \frac{v_r}{r} \frac{\partial v_r}{\partial r} + \frac{v_\theta}{\partial \theta} + \frac{v_z}{\partial z} = \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{V} + \frac{\partial (\rho \mathbf{V})}{\partial r} + \frac{\partial (\rho \mathbf{V})}{\partial \theta} + \frac{\partial (\rho \mathbf{V})}{\partial z} +
\]

\[
- \mu' \left( \frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \right)^2 +
\]

\[
+ 2 \mu \left[ \left( \frac{\partial v_r}{\partial r} \right)^2 + \left( \frac{\partial v_\theta}{\partial \theta} \right)^2 + \left( \frac{\partial v_z}{\partial z} \right)^2 \right] +
\]

\[
+ \mu \left[ \left( \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)^2 + \left( \frac{\partial v_r}{\partial \theta} \right)^2 + \left( \frac{\partial v_z}{\partial z} \right)^2 \right] +
\]

\[
(\frac{\partial v_z}{\partial r} + \frac{\partial v_\theta}{\partial z})^2 \right] +
\]

\[-2-\]
\[- \left[ \frac{1}{r} \frac{\partial c_r}{\partial r} + \frac{1}{r} \frac{\partial c_\theta}{\partial \theta} + \frac{c_r}{r} + \frac{c_z}{z} \right] \]  

(A.I.7)
Fig. A.I.1. Cartesian and cylindrical coordinate systems.
\[ \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_r^2}{r} \right) = \right. \\
- \frac{\partial p}{\partial r} + \mu \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{v_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} \right) \right. \\
(\mu - \mu') \left( \frac{\partial^2 v_r}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial^2 v_r}{\partial \theta \partial z} - \frac{v_r}{r} \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial r \partial z} \right) \\
\left. \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = \right. \\
- \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left( \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{v_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} \right) \right. \\
(\mu - \mu') \left( \frac{\partial^2 v_\theta}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial \theta \partial z} \right) \\
\left. \rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \right. \\
(\mu - \mu') \left( \frac{\partial^2 v_z}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{1}{r} \frac{\partial v_r}{\partial z} + \frac{\partial^2 v_z}{\partial z^2} \right) \right. \\
\right. \\
Figure A.I.2: Momentum equation in cylindrical coordinate system
Appendix II Coordinate transformations

To study the flow through an axial symmetric channel of a variable or constant cross section it is convenient to introduce non-dimensional variables. The radius of the cross section is defined as

\[ R = R_0 (1 + \delta R (\zeta)) \]  
\[ \text{(A.II.1)} \]

where \( \zeta \) stands for a non-dimensional coordinate in the axial direction. If \( L \) is a characteristic length in the axial direction then \( \zeta \) is defined as

\[ \zeta = \frac{Z}{L} \]  
\[ \text{(A.II.2)} \]

Examination of (A.II.1) shows that the shapes considered are cylinders whose cross section is disturbed by an order \( \delta \). The order of \( \delta \) is small, say \( 10^{-2} \) or less, so only small errors are made if quadratic or higher order terms of \( \delta \) are neglected. The non-dimensional coordinates are defined as

\[ \eta = \frac{r}{R_1} \]  
\[ \zeta = \frac{Z}{L} \]  
\[ \text{(A.II.3)} \]

where the characteristic radius \( R_1 = R_0 (1 + p\delta R (\zeta)) \). For \( p=0 \) the characteristic radius is equal to the undisturbed radius \( R_0 \), while for \( p=1 \) the local radius \( R \) is used to define \( \eta \). With Definition (A.II.3) it is possible to find expressions for the partial differential operators with respect to \( \eta \) and \( \zeta \). The first-order differential operators become

\[ \frac{\partial}{\partial z} = \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial z} \frac{\partial}{\partial \eta} \]  
\[ \frac{\partial}{\partial r} = \frac{\partial \zeta}{\partial r} \frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial r} \frac{\partial}{\partial \eta} \]  
\[ \text{(A.II.4)} \]

The partial derivatives of \( \eta \) and \( \zeta \) with respect to \( r,z \) are

\[ \frac{\partial \zeta}{\partial r} = 0 \]  
\[ \frac{\partial \zeta}{\partial z} = \frac{1}{L} \]  
\[ \frac{\partial \eta}{\partial z} = \frac{1}{L} \]
\[
\frac{\partial n}{\partial r} = \frac{1}{R_1}
\]

\[
\frac{\partial n}{\partial z} = -p\delta \frac{1}{L_g} \frac{R_0}{R_1} \frac{\partial R}{\partial \zeta}
\]

Substituting the partial derivatives according to (A.II.5) in (A.II.4) yields for the first-order differential operators

\[
\frac{\partial}{\partial z} = \frac{1}{L_g} \left( \frac{\partial}{\partial \zeta} - p\delta n \frac{R_0}{R_1} \frac{\partial R}{\partial \zeta} \frac{\partial}{\partial n} \right)
\]

\[
\frac{\partial}{\partial r} = \frac{1}{R_1} \frac{\partial}{\partial n}
\]

The higher order differential operators with respect to \( r \) are

\[
\frac{\partial^n}{\partial r^n} = \frac{1}{R_1^n} \frac{\partial^n}{\partial n^n}
\]

The remaining second-order differential operators are

\[
\frac{\partial^2}{\partial r \partial z} = \frac{1}{R_1} \frac{\partial}{\partial n} \left( \frac{\partial}{\partial z} \right) = \frac{1}{R_1 L_g} \left[ \frac{\partial^2}{\partial n \partial \zeta} - p\delta n \frac{R_0}{R_1} \frac{\partial R}{\partial \zeta} \frac{\partial^2}{\partial n^2} \left( \frac{\partial}{\partial n} + n \frac{\partial^2}{\partial n^2} \right) \right]
\]

and

\[
\frac{\partial^2}{\partial z^2} = \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \right) = \frac{1}{L_g^2} \left( \frac{\partial^2}{\partial \zeta^2} - p\delta n \frac{R_0}{R_1} \left[ 2 \frac{\partial R}{\partial \zeta} \frac{\partial^2}{\partial n \partial \zeta} + \frac{\partial^2 R}{\partial \zeta^2} \frac{\partial}{\partial n} \right] + \right)
\]

\[
+ (p\delta n \frac{R_0}{R_1} \frac{\partial R}{\partial \zeta})^2 \left[ 2 \frac{\partial}{\partial n} + \frac{\partial^2}{\partial n^2} \right]
\]

Neglecting quadratic terms of \( \delta \) in (A.II.9), \( \frac{\partial^2}{\partial z^2} \) becomes

\[
\frac{\partial^2}{\partial z^2} = \frac{1}{L_g^2} \left( \frac{\partial^2}{\partial \zeta^2} - p\delta n \frac{R_0}{R_1} \left[ 2 \frac{\partial R}{\partial \zeta} \frac{\partial^2}{\partial n \partial \zeta} + \frac{\partial^2 R}{\partial \zeta^2} \frac{\partial}{\partial n} \right] \right)
\]
In a similar way expressions for the higher order differential operators can be found. If quadratic and higher order terms of $\delta$ are neglected these differential operators are

\[
\frac{a^3}{az^3} = \frac{a}{az} \left( \frac{a^2}{az^2} \right) = \frac{1}{L^3} \left( \frac{a^3}{a\zeta^3} - p\delta \frac{R_0}{R_1} \left[ 3 \frac{aR^*}{a\zeta} \frac{a^3}{a\eta a\zeta^2} + 3 \frac{a^2 R^*}{a\zeta^2} \frac{a^2}{a\eta} + \frac{a^3 R^*}{a\zeta^3} \frac{a}{a\eta} \right] \right)
\]

(A.II.11)

\[
\frac{a^3}{az^3} = \frac{a}{az} \left( \frac{a^2}{az^2} \right) = \frac{1}{R_1^2 L} \left( \frac{a^3}{a\eta a\zeta} - p\delta \frac{R_o}{R_1} \frac{aR^*}{a\zeta} \left[ 2 \frac{a^2}{a\eta} + n \frac{a^3}{a\eta^3} \right] \right)
\]

(A.II.12)

\[
\frac{a^3}{az^3} = \frac{a}{az} \left( \frac{a^2}{az^2} \right) = \frac{1}{R_1 L^2} \left( \frac{a^3}{a\eta a\zeta} - p\delta \frac{R_0^*}{R_1^2} \left[ 2 \frac{a^2}{a\eta} + n \frac{a^3}{a\eta^3} \right] \right)
\]

(A.II.13)

and $\frac{a^4}{az^4}$ can be approximated by

\[
\frac{a^4}{az^4} = \frac{1}{L^4} \left( \frac{a^4}{a\zeta^4} - p\delta \frac{R_0}{R_1} \frac{aR^*}{a\zeta^3 a\eta} + 6 \frac{a^2 R^*}{a\zeta^2} \frac{a^4}{a\eta a\zeta^2} + 4 \frac{a R^*}{a\zeta} \frac{a^2}{a\eta} \right)
\]

(A.II.14)

and $\frac{a^4}{az^4}$ becomes

\[
\frac{a^4}{az^4} = \frac{1}{R_1^2 L^2} \left( \frac{a^4}{a\eta a\zeta^2} - p\delta \frac{R_0}{R_1} \left[ 2 \frac{a^2 R^*}{a\eta} \left( 2 \frac{a^3}{a\eta a\zeta^2} + n \frac{a^4}{a\eta^3} \right) + \frac{a^2 R^*}{a\zeta^2} \frac{a^3}{a\eta a\zeta} \right] \right)
\]

(A.II.15)
Appendix III Products of series

In this appendix a general expression for the product of two series of type R defined as

\[ R = \sum_{j=0}^{J_1} s(j) a_{j_1+j_2} n^{j_1+j_2} \quad (A.III.1) \]

is derived. \( s(j) \) is a function of index \( j \); \( j_1, j_2 \) and \( j \) are constants. The product \( P \) of two series of type R can be written as

\[ P = \sum_{j=0}^{J_1} \left[ (s(j) a_{j_1+j_2} n^{j_1+j_2}) \cdot \sum_{i=0}^{I_1} (t(i) b_{i_1+i_2} n^{i_1+i_2}) \right] \quad (A.III.2) \]

The series \( P \) can also be written as \( P = \sum_{m=m_0}^{m_1} A_m n^m \), where \( A_m \) depends on the indices \( m, i \) and \( j \) and on coefficients \( a_j \) and \( b_i \). Examination of (A.III.2) yields an expression for \( m \),

\[ m = j + j_2 + i + i_2 \quad (A.III.3) \]

or

\[ i = m - (j + i_1 + j_1) \quad (A.III.4) \]

Since \( 0 \leq i \leq i_1 \) the boundaries for \( j \) are \( j \geq m-j_2-i_1-i_2 \) and \( j \leq m-j_2-i_2 \). From the original series the boundaries for \( j \) are \( 0 \leq j \leq j_1 \). Now an expression for the contribution of (A.III.2) to \( n^m \) can be obtained:

\[ \sum_{j=j_1}^{j_2} (s(j) t(m-j-j_2-i_2) a_{j_1+j_2} b_{m+i_2-j-j_2-i_1}) n^m \quad (A.III.5) \]

If the maximum of two numbers \( x, y \) is denoted as \( \max(x,y) \) and the minimum by \( \min(x,y) \), then \( j_1 \) can be written as

\[ j_1 = \max(0, m-j_2-i_2-i_1) \quad (A.III.6) \]

and \( j_2 \) as

\[ j_2 = \min(j_1, m-j_2-i_1) \quad (A.III.7) \]
The powers $m$ of $n^m$ in $P$ vary between $m_o$ and $m_1$, where
\[ m_o = i_o + j_o \]  
\[ m_1 = j_1 + j_o + i_1 + i_o \]  
(A.III.8)

$P$ then becomes
\[ P = \sum_{m=m_o}^{m_1} \left[ \sum_{j=j_o}^{j_1} (s(j) t(m-j-o-i_o) a_{j+j_o} b_{m+j-o-j-o-i_o}) \right] n^m \]  
(A.III.9)

with $j_o$, $j_1$, $m_o$ and $m_1$ according to (A.III.6), (A.III.7) and (A.III.8).
Appendix IV Expressions for partial derivatives of $v_z$ and $v_r$.

In this appendix expressions in $\psi$ for the partial derivatives of $v_z$ and $v_r$ with respect to $r$ and $z$ are given. Furthermore, the $\theta$-components of $V_x(V^2 v)$ and $V_x(v \cdot V)_\theta$ are expressed in terms of $\psi$.

The first-order derivatives of the velocity components with respect to $r$ and $z$ become, if $\psi$ is defined by (II.8),

\[
\frac{\partial v_r}{\partial r} = \frac{1}{r^2} \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta \partial z} \\
\frac{\partial v_r}{\partial z} = -\frac{1}{r} \frac{\partial^2 \psi}{\partial z^2} \\
\frac{\partial v_z}{\partial r} = -\frac{1}{r^2} \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} \\
\frac{\partial v_z}{\partial z} = \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta \partial z} 
\]

(A.IV.1)

The second-order derivatives of the components of $v$ are

\[
\frac{\partial^2 v_r}{\partial r^2} = -\frac{2}{r^3} \frac{\partial \psi}{\partial z} + \frac{2}{r^2} \frac{\partial^2 \psi}{\partial \theta \partial z} - \frac{1}{r} \frac{\partial^3 \psi}{\partial \theta \partial z^2} \\
\frac{\partial^2 v_r}{\partial r \partial \theta} = \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{1}{r} \frac{\partial^3 \psi}{\partial \theta^2 \partial z} \\
\frac{\partial^2 v_r}{\partial z^2} = -\frac{1}{r} \frac{\partial^3 \psi}{\partial z^3} \\
\frac{\partial^2 v_z}{\partial r^2} = \frac{2}{r^3} \frac{\partial \psi}{\partial r} - \frac{2}{r^2} \frac{\partial^2 \psi}{\partial \theta \partial r} + \frac{1}{r} \frac{\partial^3 \psi}{\partial \theta \partial r^2} \\
\frac{\partial^2 v_z}{\partial r \partial \theta} = -\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta \partial r} + \frac{1}{r} \frac{\partial^3 \psi}{\partial \theta^2 \partial r} 
\]

(A.IV.2)

The higher order derivatives are (only relevant derivatives are listed)
\[
\frac{a^3 v_r}{ra^2 az} = -2 \frac{a^2 \psi}{r^3 a^2 z^2} + 2 \frac{a^3 \psi}{r^2 a^3 rz^2} - \frac{1}{r} \frac{a^4 u}{a^2 z^2}
\]

\[
\frac{a^3 v_r}{az^3} = -\frac{1}{r} \frac{a^4 u}{az^4}
\]

\[
\frac{a^3 v_z}{a^2 rz^2} = -6 \frac{a \psi}{r^4 az} + 6 \frac{a^2 \psi}{r^3 a^2 rz^2} - 3 \frac{a^3 \psi}{r^2 a^3 rz^2} + 1 \frac{a^4 u}{a^2 rz^2}
\]

\[
\frac{a^3 v_z}{az^2 ar} = -\frac{1}{r^2} \frac{a^3 \psi}{ar^2 az^2} + 1 \frac{a^4 u}{ar^2 az^2}
\]

(A.IV.3)

Substitution of the above partial derivatives in (II.10) results in an expression for the \( \theta \)-component of \( \mathbf{V} \times (\mathbf{v} \cdot \mathbf{V}) \mathbf{v} \):

\[
[\mathbf{V} \times (\mathbf{v} \cdot \mathbf{V}) \mathbf{v}]_\theta = -\frac{1}{r} \frac{a^2 \psi}{az^2} \left( \frac{1}{r^2} \frac{a \psi}{az} - \frac{1}{r} \frac{a^2 \psi}{ar az} \right) - \frac{1}{r} \frac{\partial \psi}{az} \left( \frac{1}{r^2} \frac{a^2 \psi}{az^2} - \frac{1}{r} \frac{a^3 \psi}{ar az^2} \right) +
\]

\[
+ \frac{1}{r} \frac{a^2 \psi}{ar az} \left( -\frac{1}{r} \frac{a^2 \psi}{az^2} + \frac{1}{r} \frac{\partial \psi}{ar} \left( -\frac{1}{r} \frac{a^3 \psi}{az^2} \right) +
\]

\[
- \left( \frac{1}{r^2} \frac{a \psi}{az} - \frac{1}{r} \frac{a^2 \psi}{ar az} \right) \left( -\frac{1}{r^2} \frac{a \psi}{az^2} + \frac{1}{r} \frac{a^2 \psi}{ar^2} \right) +
\]

\[
+ \frac{1}{r} \frac{\partial \psi}{az} \left( \frac{2}{r^3} \frac{a \psi}{ar} - \frac{2}{r^2} \frac{a^2 \psi}{ar} + \frac{1}{r} \frac{a^3 \psi}{ar^3} \right) +
\]

\[
- \left( -\frac{1}{r^2} \frac{a \psi}{ar} + \frac{1}{r} \frac{a^2 \psi}{ar^2} \right) \left( \frac{1}{r} \frac{a^2 \psi}{ar az} \right) - \frac{1}{r} \frac{\partial \psi}{ar} \left( -\frac{1}{r^2} \frac{a^2 \psi}{ar az} + \frac{1}{r} \frac{a^3 \psi}{ar^2 az} \right)
\]

(A.IV.4)

This equation can be simplified to

\[
[\mathbf{V} \times (\mathbf{v} \cdot \mathbf{V}) \mathbf{v}]_\theta = \frac{1}{r^2} \left( \frac{\partial \psi}{az} \left( \frac{a^3 \psi}{ar^2 az^2} + \frac{a^3 \psi}{ar^3} \right) - \frac{\partial \psi}{ar} \left( \frac{a^3 \psi}{ar^2 az} + \frac{a^3 \psi}{ar^2 az^2} \right) \right) +
\]

\[
+ \frac{1}{r^3} \left( -2 \frac{a \psi}{az} \frac{2 a^2 \psi}{az^2} - 3 \frac{\partial \psi}{az} \frac{a^2 \psi}{ar} + \frac{\partial \psi}{ar} \frac{a^2 \psi}{ar az} \right) + \frac{3}{r^4} \frac{a \psi}{az} \frac{a \psi}{ar}
\]

(A.IV.5)

Upon substitution of the partial derivatives of \( \mathbf{v} \) the \( \theta \)-component of \( \mathbf{V} \times (\mathbf{v}^2 \mathbf{v}) \) becomes:
\[ [v \times (v^2v)]_\theta = -\frac{2}{r^3} \frac{a^2 \psi}{az^2} + \frac{2}{r^2} \frac{a^3 \psi}{ar az^2} - \frac{1}{r} \frac{a^4 \psi}{ar^2 az^2} + \]

\[ \quad + \frac{1}{r^3} \frac{a^2 \psi}{az^2} - \frac{1}{r^2} \frac{a^3 \psi}{ar az^2} + \frac{1}{r^3} \frac{a^2 \psi}{az^2} - \frac{1}{r^2} \frac{a^4 \psi}{az^2} + \]

\[ - \left( -\frac{6}{r^4} \frac{a \psi}{ar} + \frac{6}{r^3} \frac{a^2 \psi}{ar^2} - \frac{3}{r^2} \frac{a^3 \psi}{ar^3} + \frac{1}{r} \frac{a^4 \psi}{ar^4} + \right) + \]

\[ \quad + \frac{1}{r^4} \frac{a \psi}{ar} - \frac{1}{r^3} \frac{a^2 \psi}{ar^2} + \frac{2}{r^4} \frac{a \psi}{ar^2} - \frac{2}{r^3} \frac{a^2 \psi}{ar^3} + \frac{1}{r^2} \frac{a^3 \psi}{ar^3} + \]

\[ - \frac{1}{r^2} \frac{a^3 \psi}{ar az^2} + \frac{1}{r} \frac{a^4 \psi}{ar^2 az^2} \]  

(A.IV.6)

which can be simplified to

\[ [v \times (v^2v)]_\theta = -\frac{1}{r} \left( \frac{a^4 \psi}{ar^4} + 2 \frac{a^4 \psi}{ar^2 az^2} + \frac{a^4 \psi}{az^4} \right) + \]

\[ + \frac{2}{r^2} \left( -\frac{a^3 \psi}{az^2 ar} + \frac{a^3 \psi}{ar^3} - \frac{3}{r^3} \frac{a^2 \psi}{ar^2} + \frac{3}{r^4} \frac{a \psi}{ar} \right) \]  

(A.IV.7)