THE ISOCHRONOUS CYCLOTRON

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAP AAN DE TECHNISCHE HOGESCHOOL TE DELFT OP GEZAG VAN DE RECTOR MAGNIFICUS DR. R. KRONIG, HOOGLEERAAR IN DE AFDELING DER TECHNISCHE NATUURKUNDE, VOOR EEN COMMISSIE UIT DE SENAAT TE VERDEDIGEN OP WOENSDAG 20 JANUARI 1960 DES NAMIDDAGS TE 4 UUR

DOOR

KHOE KONG TAT
ELEKTROTECHNISCH INGENIEUR
GEBOREN TE SEMARANG
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I. INTRODUCTION

Conventional fixed-frequency cyclotrons are limited to low energies since the radial decrease in the magnetic field that is necessary to achieve axial focusing is inconsistent with the radial increase required to compensate the relativistic mass increase. Therefore the circulation frequency of the particle changes monotonically with respect to the constant frequency of the accelerating voltage. This leads to a phase shift, so that the number of revolutions and consequently the maximum energy obtainable is limited. In the synchrocyclotron the frequency of the accelerating voltage is varied (modulated) in such a way as to compensate for the deviation from resonance as the particle energy increases. During the modulation cycle only a group of particles is accelerated so that the average current is restricted.

The magnetic field averaged over the equilibrium orbit should increase with radius in order to make the particles revolve at constant frequency. To avoid the resultant axial defocusing, an azimuthal variation of the magnetic field is introduced. The magnet structure consists of N identical sections, each composed of two sectors with different value of the magnetic field. There are a number of possible field configuration for such a sectorial cyclotron. The first such cyclotron was proposed by Thomas. The Thomas cyclotron has radial sectors with a sinusoidal azimuthal field variation. Two electron models have been successfully constructed at the University of California which accelerate electrons up to half the speed of light.

A considerable amount of theoretical and experimental work on the spiral ridge cyclotron has been carried out at A.E.R.E., Harwell. In this cyclotron the magnetic field consists of a radially increasing azimuthally constant field (the guide field) on which is superimposed a radially increasing azimuthally periodic field (the flutter field). The position of the maxima and minima of the periodic field spirals outward. The particles crossing the field ridges (maxima) experience alternating focusing forces which overcome the defocusing effect of the guide field.

Another configuration of the magnetic field is a.o. proposed by Moroz and Rabinovich. The magnet is divided into 2N sectors with alternating low and high uniform (or almost uniform) field. Axial stability of the motion is obtained by the edge focusing effect of the sectors. Constant angular velocity of the particles is obtained by increasing the azimuthal extent of the high field sectors with radius.
Preliminary field computations and measurements have been made which indicated that it is possible to design a magnet having negligible even harmonics in the Fourier series expansion of the flutter field. Towards the centre of the cyclotron the azimuthal variation of the magnetic field is almost sinusoidal. The pole-faces of this magnet structure are much easier to machine than those of the above mentioned cyclotrons.

In section 1 a brief account of particle dynamic in an azimuthally periodic magnetic field is given. The condition for isochronism is determined in section 2. Section 3 gives methods for the calculations of the frequency of the horizontal and vertical betatron oscillations. The energy limit set by the stability boundary of the radial motion, \( \nu_x = \frac{1}{2} N \), has also been calculated. An analysis of the design of the magnet is given in section 4. Section 5 discusses the measurements of the magnetic field in the horizontal plane of symmetry of the magnet and gives estimates of the practical tolerances of the field and the frequency of the accelerating voltage in order to avoid serious phase-slip. The effects of perturbations and nonlinear terms in the equations of motion are treated in section 6 and 7. In section 8 and 9 parameters computed for a 12 Mev cyclotron and, by way of illustration, for a 200 Mev cyclotron are given. The former cyclotron has been successfully constructed.
II. THEORY

1. Basic formulas

The Lagrangian of a positive charged particle in a general electromagnetic field is\(^{[11]}\)

\[
L = m_0 c^2 \left(1 - \sqrt{1 - (\beta)^2}\right) + ev \cdot A - e\phi, \quad \beta = \frac{v}{c}. \quad (1.1)
\]

where \(e\) is the charge of the particle, \(c\) the velocity of light, \(v\) the velocity and \(m_0\) the rest mass of the particle. The electric field strength \(E\) and the flux density vector \(B\) are given in terms of the vector potential \(A\) and the scalar potential \(\phi\) by

\[
E = -\frac{\partial A}{\partial t} - \text{grad. } \phi,
\]

\[
B = \text{curl } A. \quad (1.2)
\]

The canonical momentum is defined by

\[
p = \frac{\partial L}{\partial v} = mv + eA. \quad (1.3)
\]

where

\[
m = \frac{m_0}{\left(1 - (\beta)^2\right)^{1/2}}.
\]

The Hamiltonian function is

\[
H = p \cdot v - L = \sum_{i=1}^{3} p_i q_i - L, \quad (1.4)
\]

where the \(q_i\)'s stand for the components of \(v\).

Differentiation of \(H\) with respect to time and making use of the Lagrange equations

\[
\frac{d}{dt} \left(\frac{\partial L}{\partial q_i}\right) = \frac{\partial L}{\partial q_i}, \quad i = 1, 2, 3,
\]

gives
Ignoring the electric field will give a Lagrangian which does not contain the time explicitly and then the Hamiltonian \( H \) is a constant of motion; consequently

\[
\delta H = \delta (p, y) - \delta L = 0.
\]

Substitution of this in

\[
\delta \int_{t_1}^{t_2} L dt = 0
\]

leads to the principle of least action

\[
\delta \int_{t_1}^{t_2} p_y \ dt = \frac{\partial}{\partial t} \int_{A}^{B} p_y dl.
\]

where \( A \) and \( B \) are fixed terminal points of the trajectory and \( \underline{l} \) is a vector. Elimination of \( p \) by means of (1.3) gives

\[
\delta \int_{A}^{B} (mv + eA) \ dl = 0.
\]

We now define an orthogonal coordinate system \((x, z, s)\) as follows (see fig. 1)
x is in the plane of symmetry of the magnetic field and measures the distance normally outward from a closed curve.

z measures the displacement perpendicular to the plane.

s is the arc length of the closed curve, measured from some reference point.

Let \( \mathbf{r} \) be the position vector of any point on the orbit curve then we may write

\[
\mathbf{r} = \mathbf{r}_0 + x\mathbf{n} + z\mathbf{b},
\]

where \( \mathbf{r}_0 \) is the position vector of the corresponding point on the closed curve, \( \mathbf{n} \) the unit principal normal and \( \mathbf{b} \) the unit binormal. The unit tangent \( \mathbf{t} \) is given by

\[
\mathbf{t} = \frac{d\mathbf{r}_0}{ds}.
\]

The unit vectors \( \mathbf{t}, \mathbf{n} \) and \( \mathbf{b} \) satisfy the formulas

\[
\frac{d\mathbf{t}}{ds} = -\frac{n}{\rho},
\]

\[
\frac{d\mathbf{n}}{ds} = \frac{t}{\rho},
\]

\[
\frac{d\mathbf{b}}{ds} = 0,
\]

where \( \rho (s) \) is the radius of curvature of the closed curve.

Differentiation of (1.7) with respect to \( s \) gives

\[
\frac{d\mathbf{r}}{ds} = \left(1 + x\right)\frac{t}{\rho} + \frac{dx}{ds} n + \frac{dz}{ds} b.
\]

Thus the metric of the system is given by

\[
(d\mathbf{r})^2 = (dx)^2 + (dz)^2 + \left(1 + \frac{x}{\rho}\right)^2 (ds)^2.
\]

In terms of the coordinate system \((x, z, s)\) (1.6) may be written in the form

\[
\delta \int_A Fds = 0,
\]

where
\[ F = \rho \left( x'^2 + z'^2 + \left( \frac{x}{\rho} \right)^2 \right)^{\frac{3}{2}} + \rho \left( x'^2 + z'^2 \right) \]

\[ + (1 + \frac{x}{\rho}) A_x \cdot A_z \]  

Equation (1.11)

Here \( \rho \) is the kinetic momentum of the particle and primes denote derivatives with respect to \( s \).

However, information about the magnetic field obtained from measurements are expressed in terms of the quantity \( B = \text{curl } A \). It is therefore necessary to express \( A_x', A_z \) and \( A_s \) in terms of the components of the vector \( B \). Since we have assumed that the vector \( B \) is everywhere normal to the plane \( z = 0 \), the flux density at this plane may conveniently be expanded as a power series in \( x \)

\[ B(x,s) = B(s) + \left( \frac{\partial B}{\partial x} \right)_0 x + \frac{1}{2} \left( \frac{\partial^2 B}{\partial x^2} \right)_0 x^2 + \frac{1}{3!} \left( \frac{\partial^3 B}{\partial x^3} \right)_0 x^3 + \ldots . \]  

Equation (1.12)

Expanding \( A_x', A_z \) and \( A_s \) as a power series in \( x \) and \( z \) and using the relations \( B = \text{curl } A \), \( \text{div } B = 0 \) and \( \text{curl } B = 0 \), we find from (1.11)

\[ F = \rho + \rho \left( x'^2 + z'^2 \right)^{\frac{3}{2}} + \rho \left( x'^2 + z'^2 \right) - \rho \frac{x}{\rho} \left( x'^2 + z'^2 \right) - \frac{1}{8} \rho \left( x'^2 + z'^2 \right)^2 + \frac{1}{8} \rho \left( \frac{x}{\rho} \right)^2 \left( x'^2 + z'^2 \right) + e \left[ A_s - Bx \right] - \]

\[ \frac{1}{8} \left( \frac{1}{\rho^2} \left( \frac{\partial B}{\partial x} \right)_0 \right) x^2 + \frac{1}{2} \left( \frac{\partial^2 B}{\partial x^2} \right)_0 x z^2 + \frac{1}{3} \left( \frac{\partial^3 B}{\partial x^3} \right)_0 x z^2 - \]

\[ \frac{1}{31} \left( \frac{2}{\rho} \left( \frac{\partial B}{\partial x} \right)_0 + \left( \frac{\partial^2 B}{\partial x^2} \right)_0 \right) x^3 + \frac{1}{3!} \left( B'' + \frac{3}{\rho} \left( \frac{\partial B}{\partial x} \right)_0 + 3 \left( \frac{\partial^2 B}{\partial x^2} \right)_0 \right) x z^2 - \]

\[ \frac{1}{4} \left( \frac{1}{\rho} \left( \frac{\partial B}{\partial x} \right)_0 - \left( \frac{\partial^2 B}{\partial x^2} \right)_0 \right) x z(xz'-x'z) - \frac{1}{4!} \left( \frac{3}{\rho^2} \left( \frac{\partial B}{\partial x} \right)_0 + \frac{3}{\rho} \left( \frac{\partial^2 B}{\partial x^2} \right)_0 \right) x^4 + \]

\[ \frac{1}{8} \left( \frac{1}{\rho^2} B'' + \frac{\rho'}{\rho^2} B' + \left( \frac{\partial^2 B''}{\partial x^2} \right)_0 + \frac{4}{\rho} \left( \frac{\partial B}{\partial x} \right)_0 + 2 \left( \frac{\partial^2 B}{\partial x^2} \right)_0 \right) x^2 z^2 + \]

\[ \frac{1}{4!} \left( \frac{2}{\rho^2} B'' + \frac{\rho'}{\rho^2} B' - \left( \frac{\partial B''}{\partial x} \right)_0 + \frac{1}{\rho^2} \left( \frac{\partial B}{\partial x} \right)_0 - \frac{1}{\rho} \left( \frac{\partial^2 B}{\partial x^2} \right)_0 - \frac{\partial^3 B}{\partial x^3} \right)_0 \right) z^4 + \]

Equation (1.13)
From the variational equation (1.10) we may derive the Euler-Lagrange equations
\[
\frac{d}{ds} \frac{\partial F}{\partial x'} - \frac{\partial F}{\partial x} = 0,
\]
\[
\frac{d}{ds} \frac{\partial F}{\partial z'} - \frac{\partial F}{\partial z} = 0.
\] (1.14)

2. The equilibrium orbit

Substituting (1.13) in (1.14) we get for
\[
x = z = x' = z' = 0,
\] (2.1)
\[
\frac{mv}{\rho} = eB.
\] (2.2)

The closed curve which satisfies equation (2.2) throughout its length we call the equilibrium orbit. This orbit is specified by its equivalent radius \( R \) defined by
\[
R = \frac{1}{2\pi} \int \phi \, ds.
\] (2.3)

Integrating (2.2) and using the relation
\[
\frac{\phi \, ds}{\rho} = \frac{2\pi}{m}
\] (2.4)
we obtain
\[
\frac{mv}{R} = e < B >,
\] (2.5)

where
\[
<B> = \frac{1}{2\pi R} \int \phi B \, ds.
\] (2.6)

Introducing the mean angular velocity of the particle \(<\omega> = \frac{V}{R}\) (2.5) yields
\[
<\omega> = \frac{e < B >}{m}.
\] (2.7)

It follows from (2.7) that for a constant mean angular velocity of the particles, the magnetic field in the median plane, averaged over an
equilibrium orbit, must satisfy the relation

$$\frac{e < B >}{m} = \frac{e B_0}{m_0}.$$  \hspace{0.5cm} (2.8)

To save writing we adopt as unit of length

$$\frac{c m_0}{e B_0} = \frac{c}{< \omega >}.$$  

In this unit $\beta = R$ and we may write instead of (2.8)

$$< B > = \frac{B_0}{(1 - \beta^2)^{1/2}} = \frac{B_0}{(1 - R^2)^{1/2}},$$  \hspace{0.5cm} (2.9)

where $B_0$ is the flux density in the centre. However, this magnetic field would be axially defocusing since the field index

$$n = - \frac{R dB}{B dR} = - \frac{R^2}{1 - R^2}.$$  \hspace{0.5cm} (2.10)

is negative.

As was stated in the introduction this axial defocusing can be overcome by an azimuthal variation of the magnetic field. Introducing the orbit coordinates $(R, \theta)$, where $R$ is defined by (2.3) and the azimuthal coordinate $\theta$ by

$$R d\theta = ds.$$  \hspace{0.5cm} (2.11)

(2.6) reduces to

$$< B > = \frac{1}{2\pi} \int_0^{2\pi} B(R, \theta) d\theta.$$  \hspace{0.5cm} (2.12)

Then it is evident that in the orbit coordinates $(R, \theta)$ the magnetic field in the median plane can be written in the form

$$B(R, \theta) = \frac{B_0}{\sqrt{1 - R^2}} \{1 + \mu_o^2(R, \theta)\}, \quad \int_0^{2\pi} \mu_o d\theta = 0.$$  \hspace{0.5cm} (2.13)

It will be shown in section 3 that the axial focusing of the flutter field is proportional to $< \mu_o^2 >$. Since this focusing action of $\mu_o$ must compensate the defocusing effect of the mean field, we impose the requirement, that $\mu_o$ is of the form
\[ \mu_0 (R, \theta) = c \frac{(R)}{(1-R^2)^{1/2}} f(\theta) , \]  

where \( c \) is a constant and \( f(\theta) \) has zero mean value.

In view of the magnet design and the field measurements it is necessary to express the magnetic field in terms of the more familiar polar coordinates \((r, \phi)\). If \( \mu_0 (R, \theta) \) is known, relations between the orbit coordinates \((R, \theta)\) and the polar coordinates \((r, \phi)\) can be derived. On such basis, then it is in principle possible to obtain numerical relations between the desired median plane field and the polar coordinates.\(^{19}\) In actuality, however, the determination of the iron configurations to achieve the required field is extremely difficult to effect. A more rational and, as it turns out, a more fruitful procedure consists in straightforward computation of the conditions for isochronism from the equilibrium orbit equation (2.2).

Examination of (2.13) and (2.14) suggests that the median plane field may be written in the form

\[ B(r, \phi) = B_0 \left[ 1 + a r^2 + b r^4 + \ldots. + \right. \]

\[ \left. \{ A r + B r^3 + \ldots. \} g(r, \phi) \right] , \]  

(2.15)

where \( g(r, \phi) \) is a periodic function of \( \phi \) with zero mean value.

Preliminary field computations and measurements indicated that it is possible to design a magnet with almost rectangular varying sectors of which \( g(r, \phi) \) is, to a good approximation, given by

\[ g(r, \phi) = \sum_{k=0}^{\infty} \alpha_k \cos (2k+1) N\phi , \]  

(2.16)

where \( \alpha_k \) can be approximated by

\[ \alpha_k = \frac{(-1)^k \sinh N \frac{r_m}{F}}{\sinh (2k+1)N \frac{r_m}{F}} . \]  

(2.17)

(see section 4).

Finally, substitution of (2.16) in (2.15) gives the expression for the median plane field

\[ B(r, \phi) = B_0 \left[ 1 + a r^2 + b r^4 + \ldots. + \right. \]

\[ \left. \{ A r + B r^3 + \ldots. \} \sum_{k=0}^{\infty} \alpha_k \cos (2k+1) N\phi \right] . \]  

(2.18)
In the expression (2.18) for $B(r, \phi)$ it has been assumed that all even harmonics are zero. This is very convenient in view of the determination of the condition for isochronism and the field measurements. Substituting (2.18) in (2.2) and expressing the radius of curvature $\rho$ in polar coordinates we obtain

$$1 + 2\left(\frac{1}{r} \frac{\partial r}{\partial \phi}\right)^2 - \frac{1}{r} \frac{\partial^2 r}{\partial \phi^2} = \left(1 + \left(\frac{1}{r} \frac{\partial r}{\partial \phi}\right)^2\right)^{3/2} \left(1 - \frac{r^2}{R}\right)\frac{1}{R} \left[1 + a r^2 + b r^4 + \ldots \right) + \left(\sum_{k=0}^{\infty} \alpha_k \cos(2k+1)N\phi\right].$$

(2.19)

The equilibrium orbit is that periodic solution of (2.21) with period $2\pi/N$. It is clear that (2.19) is not exactly soluble. If the kinetic energy of the particle is much smaller than its rest energy, a sufficiently accurate solution can be obtained by expanding $r$ in powers of $R$ and retaining only a few terms, thus

$$r = R \left[1 + R x_1(\phi) + R^2 x_2(\phi) + R^3 x_3(\phi) + \ldots \right],$$

(2.20)

with

$$x_1(\phi + 2\pi/N) = x_1(\phi).$$

Inserting (2.20) in (2.19) and equating the coefficients of like powers of $R$ to zero we obtain

$$\frac{d^2 x_1}{d\phi^2} + x_1 = -A \sum_{k=0}^{\infty} \alpha_k \cos(2k+1)N\phi,$$

$$\frac{d^2 x_2}{d\phi^2} + x_2 = -x_1^2 + \frac{1}{2} \frac{dx_1}{d\phi} + \frac{1}{2} - a - 3x_1 A \sum_{k=0}^{\infty} \alpha_k \cos(2k+1)N\phi,$$

$$\frac{d^2 x_3}{d\phi^2} + x_3 = \left(\frac{dx_1}{d\phi}\right)^2 - 2x_1 x_2 - 2x_1 \frac{dx_1}{d\phi}^2 + (1-4a)x_1 - \left(A(3x_1^2 + \frac{3}{2} \frac{dx_1}{d\phi}^2 + 3x_2 - \frac{1}{2}) + B \right) \sum_{k=0}^{\infty} \alpha_k \cos(2k+1)N\phi.$$

(2.21)

The solutions of (2.21) which has period $2\pi/N$ are given by
\[ x_1 = A \sum_{k=0}^{\infty} \beta_k \cos(2k+1)N\phi, \]

\[ x_2 = -\frac{1}{4} A^2 \sum_{k=0}^{\infty} \gamma_k^2 + \ldots, \]

\[ x_3 = (B - \frac{1}{2}A) \sum_{k=0}^{\infty} \beta_k \cos(2k+1)N\phi + \ldots, \]

where

\[ \beta_k = \frac{\alpha_k}{(2k+1)^2N^2-1}, \]

\[ \gamma_k = \frac{(2k+1)N \alpha_k}{(2k+1)^2N^2-1}. \]

In \( x_2 \) and \( x_3 \) terms of order \( 1/N^4 \) and of higher orders are ignored. Substitution of (2.22) in (2.20) gives

\[ r = R \left[ 1 + A R \sum_{k=0}^{\infty} \beta_k \cos(2k+1)N\phi - \frac{1}{4} A^2 R^2 \sum_{k=0}^{\infty} \gamma_k^2 + \right. \]

\[ \left. (B - \frac{1}{2}A) R^3 \sum_{k=0}^{\infty} \beta_k \cos(2k+1)N\phi \right]. \]

It follows from (2.11) that

\[ \theta = \frac{1}{R} \int \phi \frac{ds}{R} = \frac{1}{R} \int_{0}^{\phi} \left\{ r^2 + \left( \frac{dr}{d\phi} \right)^2 \right\} \frac{1}{2} d\phi, \]

where the origin of \( \theta \) is chosen at \( \phi = \gamma \). Making use of (2.24) we obtain

\[ \theta = \phi + AR \sum_{k=0}^{\infty} \frac{\beta_k}{(2k+1)N} \sin(2k+1)N\phi - \]

\[ \frac{1}{4} A^2 R^2 \sum_{n=1}^{\infty} \left\{ \gamma_1 \gamma_{n-1} - \sum_{i=0}^{\infty} \gamma_1 \gamma_{i+n} + \sum_{i=0}^{\infty} \gamma_i \gamma_{i+n} \right\} \frac{\sin 2 nN\phi}{2 n N} + \]

\[ (B - \frac{1}{2}A) R^3 \sum_{k=0}^{\infty} \frac{\beta_k}{(2k+1)N} \sin(2k+1)N\phi + \ldots. \]

This equation is now solved for \( \phi \) by successive approximation. The zero order approximation is
\[
\phi = \theta .
\]

The first order approximation is

\[
\phi = \theta - A R \sum_{k=0}^{\infty} \frac{\beta_k}{(2k+1)N} \sin(2k+1)N\theta + \frac{1}{4} A^2 R^2 \sum_{n=1}^{\infty} \left\{ \sum_{i=0}^{n-1} \gamma_i \gamma_{n-i-1} \right\} + \sum_{i=n}^{\infty} \gamma_i \gamma_{1-n} - \sum_{i=0}^{\infty} \gamma_i \gamma_{i+n} \frac{\sin 2nN\theta}{2nN}.
\]

\[
(B - \frac{3}{2} A) R^3 \sum_{k=0}^{\infty} \frac{\beta_k}{(2k+1)N} \sin(2k+1)N\theta.
\]

(2.27)

The expression for \( r \) becomes

\[
r_r = R \left[ 1 + A R \sum_{k=0}^{\infty} \beta_k \cos(2k+1)N\theta - \frac{1}{4} A^2 R^2 \sum_{k=0}^{\infty} \gamma_k^2 + \cdots \right] + (B - \frac{3}{2} A) R^3 \sum_{k=0}^{\infty} \beta_k \cos(2k+1)N\theta.
\]

(2.28)

It is clear from (2.24) and (2.25) that the first order approximation gives already the required degree of accuracy.

Finally, substitution of (2.27) and (2.28) in (2.18) gives

\[
B(R, \theta) = B_0 \left[ 1 + (a + A^2 \sum_{k=0}^{\infty} \alpha_k \beta_k) R^2 + \left\{ b + (3AB - \frac{3}{2}(1+a)A^2) \sum_{k=0}^{\infty} \alpha_k \beta_k \right\} R^4 + \cdots \right] + (A R + B R^3 + \cdots) \sum_{k=0}^{\infty} \alpha_k \cos(2k+1)N\theta.
\]

(2.29)

Comparison of (2.29) with (2.13) gives

\[
a = \frac{1}{2} - A^2 \sum_{k=0}^{\infty} \alpha_k \beta_k ,
\]

(2.30)

\[
b = \frac{3}{8} - (3AB - \frac{3}{2}(1+a)A^2) \sum_{k=0}^{\infty} \alpha_k \beta_k + \cdots ,
\]

(2.31)

etc.

\[
\mu_0(R, \theta) = \{ A R + (B - \frac{3}{2} A) R^3 \} \sum_{k=0}^{\infty} \alpha_k \cos(2k+1)N\theta + \cdots .
\]

(2.32)

Equation (2.30) gives an exact relation between \( a \) and \( A \), and (2.31) gives \( b \) correctly to \( 1/N^2 \) terms. In (2.32) terms of order \( 1/N^2 \) and of
higher orders are neglected. It is obvious that more terms in the series expansion of the median plane field in powers of \( r \) must be retained if the final energy of the particles is higher. The computation of the parameters will then be very tedious. However, it is not difficult to see by examination of (2.27) and (2.28) that, if the median plane field is written in the general form

\[
B = B_0 \left[ 1 + f(r) + F(r) \sum_{k=0}^{\infty} \alpha_k \cos (2k+1)n\phi \right], \tag{2.33}
\]

the transformation between the orbit coordinates \((r, \phi)\) and the polar coordinates \((R, \theta)\) are to the same approximation given by

\[
r = R \left[ 1 + F(R)(1 - R^2)^{1/2} \sum_{k=0}^{\infty} \beta_k \cos (2k+1)n\theta - \frac{1}{4} F^2 (1 - R^2) \sum_{k=0}^{\infty} \gamma_k^2 \right], \tag{2.34}
\]

\[
\phi = \theta - F(R)(1 - R^2)^{1/2} \sum_{k=0}^{\infty} \frac{\beta_k}{(2k+1)N} \sin (2k+1)N\theta - \frac{1}{4} F^2 (1 - R^2) \sum_{n=1}^{\infty} \sum_{n=0}^{n-1} \gamma_{i+n}^2 \sum_{i=n}^{\infty} \gamma_{i-n} \sin \frac{2nN\theta}{2N}. \tag{2.35}
\]

Substitution of (2.34) and (2.35) in (2.33), expansion of \( f(r) \) and \( F(r) \) in Taylor series and comparison with (2.13) gives

\[
< \frac{B}{B_0} > = 1/(1 - R^2)^{1/2} = 1 + f(R) - \frac{1}{4} Rf'F^2 (1 - R^2) \sum_{k=0}^{\infty} \gamma_k^2 + \frac{1}{4} F^2 (1 - R^2)^{1/2} (1 + RF'/F) \sum_{k=0}^{\infty} \alpha_k \beta_k + \ldots \ldots, \tag{2.36}
\]

\[
\mu_0(R, \theta) = F(R)(1 - R^2)^{1/2} \sum_{k=0}^{\infty} \alpha_k \cos (2k+1)N\theta + \ldots \ldots. \tag{2.37}
\]

The expression (2.36) for \( < B >/B_0 \) is of great value in checking the properties of a magnetic field configuration, since the numerical values of \( f, f', F \) and \( F' \) can directly be obtained from the field measurements (see section 5).
If in view of the design of the magnet it is convenient to express \( f(r) \) and \( F(R) \) in power series

\[
f(r) = a r^2 + b r^4 + c r^6 + d r^8 + \ldots \quad (2.38)
\]

\[
F(r) = A r + B r^3 + C r^5 + D r^7 + \ldots
\]

the following recurrence formulas can readily be verified by insertion of the series (2.38) in (2.36).

\[
\frac{1}{2} = a + A^2 \sum_{k=0}^{\infty} \alpha_k \beta_k,
\]

\[
\frac{3}{8} = b - \frac{1}{2} aA \sum_{k=0}^{\infty} \alpha_k \beta_k + (3AB - \frac{1}{2}A^2) \sum_{k=0}^{\infty} \alpha_k \beta_k,
\]

\[
\frac{5}{16} = c - (aAB + bA^2 - \frac{1}{2}aA^2) \sum_{k=0}^{\infty} \alpha_k \beta_k + \left(2B^2 + 4AC - \frac{3}{2}AB - \frac{1}{2}A^2\right) \sum_{k=0}^{\infty} \alpha_k \beta_k,
\]

\[
\frac{35}{128} = d - \left(\frac{1}{2}aB^2 + aAC + \frac{3}{2}cA^2 - aAB - bA^2 + 2bAB\right) \sum_{k=0}^{\infty} \alpha_k \beta_k + \left(5AD + 5BC - B^2 - 2AC - \frac{3}{8}AB - \frac{1}{16}A^2\right) \sum_{k=0}^{\infty} \alpha_k \beta_k.
\]

The relation (2.36) gives the condition for isochronism. The flutter amplitude function \( F(r) \) is determined by the focusing requirements which will be discussed in the next section.
3. The linear betatron oscillations

Substituting (1.13) in (1.14), retaining only the first order terms and using the equilibrium orbit equation (2.2), we obtain

\[ \frac{d^2 x}{ds^2} + \frac{1}{\rho^2} (1-n) x = 0 \]  
and

\[ \frac{d^2 z}{ds^2} + \frac{1}{\rho^2} nz = 0 , \]

where

\[ n = - \frac{\rho}{B} \left( \frac{\partial B}{\partial x} \right)_0 . \]

Since we have assumed that the equilibrium orbit is closed, the quantities \( \rho \) and \( B \) are periodic functions of \( s \) with period \( 2\pi R/N \). The general properties of these second order equations with periodic coefficients have been extensively discussed in the mathematical literature (Floquet's theorem). Various approaches to the calculation of the characteristic exponent \( \nu \), i.e. the number of betatron oscillations per revolution, have been described (1).

Making use of the relation (2.11), both equations (3.1) and (3.2) can be written in the form

\[ \frac{d^2 x}{d\theta^2} + g_x(R, \theta) x = 0 \]  
and

\[ \frac{d^2 z}{d\theta^2} + g_z(R, \theta) z = 0 , \]

where

\[ g_x(R, \theta) = \left( \frac{R}{\rho} \right)^2 (1-n) \]  
and

\[ g_z(R, \theta) = \left( \frac{R}{\rho} \right)^2 n . \]
Defining the parameter $\mu$ by

$$\mu = \frac{R}{\rho} = 1 + \mu_{\ast}(R, \theta) \quad (\text{see also (2.13)})$$

the formulas (1.8) can be rewritten in the form

$$\frac{dt}{d\theta} = -\mu t,$$

$$\frac{dn}{d\theta} = \mu n,$$

with the unit tangent

$$t = \frac{1}{R} \frac{\partial R}{\partial \theta}.$$

To express $n$ in the orbit coordinates $(R, \theta)$ we introduce the parameters $\eta (R, \theta)$ and $\epsilon (R, \theta)$ defined by

$$\frac{\partial R}{dR} = \eta t - \epsilon n.$$

From fig. 2 we see that $\eta$ and $\epsilon$ satisfy the relations

$$\eta = \frac{\partial x}{\partial R},$$

$$\epsilon = R \frac{\partial \theta}{\partial R}.$$
Differentiating (3.11) with respect to $\theta$, we obtain by means of (3.9) and (3.10)

$$\dot{t} - n R \int \frac{\partial \mu}{\partial R} d\theta = t (\mu - \epsilon) + n \left( \frac{\partial \eta}{\partial \theta} + \epsilon \mu \right),$$

where we have chosen the integration constant equal to zero. Equating the coefficients of $\dot{t}$ and $n$ to zero we get

$$\frac{\partial \eta}{\partial \theta} = - \epsilon \mu - R \int \frac{\partial \mu}{\partial R} d\theta$$

and

$$\frac{\partial \epsilon}{\partial \theta} = \mu \eta - 1.$$  \hspace{1cm} (3.15)

Differentiating (3.14) and using (3.15) we obtain

$$\frac{\partial^2 \eta}{\partial \theta^2} = - \epsilon \frac{\partial \mu}{\partial \theta} - R \frac{\partial \mu}{\partial R} - \mu (\mu \eta - 1).$$  \hspace{1cm} (3.16)

Making use of (3.8) and (3.12), we find from (3.3) that

$$n = - \frac{1}{\mu \eta} \left( \frac{\partial \ln<\beta>}{\partial \ln R} \right) = - \frac{1}{\mu^2 \eta^2} \left( \mu \frac{\partial \ln<\beta>}{\partial \ln R} + \epsilon \frac{\partial \mu}{\partial \theta} + R \frac{\partial \mu}{\partial R} \right)$$

and the substitution of (3.16) yields

$$n = 1 - \frac{1}{\mu^2 \eta^2} \left( \alpha \mu - \frac{\partial^2 \eta}{\partial \theta^2} \right),$$  \hspace{1cm} (3.17)

where the momentum compaction factor $\alpha$ is given by

$$\alpha = 1 + \frac{\partial \ln<\beta>}{\partial \ln R} = \frac{\partial \ln p}{\partial \ln R}.$$  \hspace{1cm} (3.18)

For an exactly isochronous machine we have from (2.9)

$$\alpha = 1/(1-R^2) = 1/(1-R^2).$$  \hspace{1cm} (3.19)

The linearized equations for betatron oscillations (3.4) and (3.5) may now be written as

$$\frac{d^2 x}{d\theta^2} + \frac{\mu}{\eta} - \frac{1}{\eta} \frac{\partial^2 \eta}{\partial \theta^2} x = 0.$$  \hspace{1cm} (3.20)
and

\[ \frac{\partial^2 z}{\partial \vartheta^2} + \left( \mu^2 - \frac{\mu}{\eta} + \frac{1}{\eta} \frac{\partial^2 \eta}{\partial \vartheta^2} \right) z = 0 . \]  

(3.21)

To obtain an expression for \( \eta \) we eliminate \( \varepsilon \) from (3.14) and find

\[ \frac{\partial}{\partial \vartheta} \left( \frac{1}{\mu} \frac{\partial \eta}{\partial \vartheta} + \frac{R}{\mu} \int \frac{\partial}{\partial \varepsilon} \mathrm{d} \vartheta \right) = 1 - \mu \eta . \]  

(3.22)

Introducing the notations

\[ f_0 = f - \langle f \rangle , \]

\[ f_1 = \int f_0 \mathrm{d} \vartheta , \quad \langle f_1 \rangle = 0 , \]

\[ f_{n+1} = \int f_n \mathrm{d} \vartheta , \quad \langle f_{n+1} \rangle = 0 , \]

where angular brackets denote values averaged over \( \vartheta \), we obtain the formal solution of (3.22)

\[ \eta = \langle \eta \rangle - R \frac{\partial \mu}{\partial \varepsilon} + \{ (1 - \mu \eta) \} , \]

(3.23)

where the integration constant \( \langle \eta \rangle \) is to be chosen so that \( 1 - \mu \eta \) has zero mean value. Equation (3.23) can be solved by successive approximation. Since \( \langle 1 - \mu \eta \rangle = 0 \) we may take for the zero-order approximation

\[ \eta = 1 . \]

Substituting this in the right-hand side of (3.23) we obtain the first-order approximation

\[ \eta = 1 - R \frac{\partial \mu}{\partial \varepsilon} - \mu_2 - \frac{1}{2} \langle \mu_1^2 \rangle - \langle \mu_1^2 \rangle - \langle \frac{\partial \mu}{\partial \varepsilon} \rangle . \]  

(3.24)

To obtain the second-order approximation we substitute this first-order approximation in the right-hand side of (3.23), and so on. It is interesting to note that the first-order approximation yields terms of order \( 1/N^2 \) and in general, the n-th approximation yields terms of order \( 1/N^{2n} \). If we now substitute (2.37) in (3.24) and assume that

\[ \frac{\partial \alpha_k}{\partial \varepsilon} = 0 , \]  

i.e. neglecting the gradient focusing of the higher harmonics, we find

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\[ \eta = 1 - \frac{1}{2} \mathcal{A}^2 (1+ \frac{R}{\mathcal{A}} \frac{d\mathcal{A}}{dR}) \sum_{k=0}^{\infty} \frac{\alpha_k^2}{(2k+1)^2 N^2} \]

\[ + \mathcal{A} (1+ \frac{R}{\mathcal{A}} \frac{d\mathcal{A}}{dR}) \sum_{k=0}^{\infty} \frac{\alpha_k}{(2k+1)^2 N^2} \cos (2k+1) N \theta \]

\[ + \mathcal{A}^2 \left\lbrace \frac{\alpha_0^2 - 2/3 \alpha_0 \alpha_1}{(2N)^2} \cos 2N \theta + \frac{8 \alpha_0^2}{3 (4N)^2} \alpha_0 \alpha_1 \cos 4N \theta + \ldots \right\} , \quad (3.25) \]

where

\[ \mathcal{A}(R) = (1-R^2)^{1/2} F(R) . \quad (3.26) \]

Inserting (3.25) and (2.37) into (3.20) and (3.21) we obtain

\[ \frac{d^2 x}{d\phi^2} + (\sum_{n=0}^{\infty} \gamma_{xn} \cos nN \theta) x = 0 \]

(3.27)

and

\[ \frac{d^2 z}{d\phi^2} + (\sum_{n=0}^{\infty} \gamma_{zn} \cos nN \theta) z = 0 , \]

(3.28)

where

\[ \gamma_{x0} = \alpha - \frac{\mathcal{A}^2 (1+R \frac{d\mathcal{A}}{dR})^2}{2N^2} \sum_{k=0}^{\infty} \frac{\alpha_k^2}{(2k+1)^2} - \frac{1}{2} \mathcal{A}^4 \left( \alpha_0^2 - 2/3 \alpha_0 \alpha_1 + \ldots \right) \]

\[ \gamma_{x1} = \alpha \mathcal{A}^2 (\alpha + 1 + \frac{R}{\mathcal{A}} \frac{d\mathcal{A}}{dR}) , \]

\[ \gamma_{x2} = \frac{\mathcal{A}^2}{2N^2} (\alpha_0^2 - 2/3 \alpha_0 \alpha_1 + \ldots) , \]

\[ \gamma_{x3} = \mathcal{A} \alpha_1 (\alpha + 1 + \frac{R}{\mathcal{A}} \frac{d\mathcal{A}}{dR}) . \]

\[ \gamma_{z0} = 1 - \alpha + \frac{1}{2} \mathcal{A}^2 \sum_{k=0}^{\infty} \frac{\alpha_k^2}{(2k+1)^2} + \frac{\mathcal{A}^2}{2N^2} (1+R \frac{d\mathcal{A}}{dR})^2 \sum_{k=0}^{\infty} \frac{\alpha_k^2}{(2k+1)^2} \]

\[ + \frac{\mathcal{A}^4}{(2N)^2} \left( \alpha_0^2 - 2/3 \alpha_0 \alpha_1 + \ldots \right) , \]
It is readily verified by means of (2.17) that \( \sum_{n=1}^{\infty} g_{xn} \) and \( \sum_{n=1}^{\infty} g_{zn} \) converge absolutely. The characteristic exponent \( \nu \) can then be obtained to a good approximation from the relation (21)

\[
\sin^2 \frac{\nu \pi}{N} = \Delta (0) \sin^2 \sqrt{\frac{g_0}{N}} \frac{\pi}{N}, \tag{3.29}
\]

where \( \Delta (0) \) is the infinite Hill's determinant

\[
\Delta (0) = \begin{vmatrix}
1 & -\frac{1}{2} g_1 & -\frac{1}{2} g_2 & -\frac{1}{2} g_3 & -\frac{1}{2} g_4 \\
\frac{1}{2} g_1 & 1 & -\frac{1}{2} g_1 & -\frac{1}{2} g_2 & -\frac{1}{2} g_3 \\
\frac{1}{2} g_2 & \frac{1}{2} g_1 & 1 & -\frac{1}{2} g_1 & \frac{1}{2} g_2 \\
\frac{1}{2} g_3 & \frac{1}{2} g_2 & \frac{1}{2} g_1 & 1 & -\frac{1}{2} g_1 \\
(2N)^2 - g_0 & (2N)^2 - g_0 & (2N)^2 - g_0 & (2N)^2 - g_0 & 1
\end{vmatrix}
\]
If in addition we assume that $g_1, g_2, \ldots$ are small enough, then it can be shown with the aid of the expansion of $\cot \pi x$

$$\pi \cot \pi x = 1/x + 2x/x^2 - 1 + 2x/x^2 - 4 \ldots$$

that

$$\Delta(0) = 1 + \frac{\pi \cot \frac{\pi}{N} \sqrt{g_0}}{2N \sqrt{g_0}} \left[ \frac{g_1^2}{N^2 - 4g_0} + \frac{g_2^2}{(2N)^2 - 4g_0} \right. + \left. \frac{g_3^2}{(3N)^2 - 4g_0} + \ldots \right]. \quad (3.30)$$

It follows at once from (3.29) and (3.30) that

$$\cos \frac{2\pi}{N} \nu_x = \cos \frac{2\pi}{N} \sqrt{g_{x0}} - \frac{\pi}{2N} \frac{\sin \frac{2\pi}{N} \sqrt{g_{x0}}}{\sqrt{g_{x0}}} \left( \frac{g_{x1}}{N^2 - 4g_{x0}} \right)$$

$$+ \frac{g_{x2}}{(2N)^2 - 4g_{x0}} + \frac{g_{x3}}{(3N)^2 - 4g_{x0}} + \ldots, \quad (3.31)$$

$$\cos \frac{2\pi}{N} \nu_z = \cos \frac{2\pi}{N} \sqrt{g_{z0}} - \frac{\pi}{2N} \frac{\sin \frac{2\pi}{N} \sqrt{g_{z0}}}{\sqrt{g_{z0}}} \left( -\frac{g_{z1}}{N^2 - 4g_{z0}} \right)$$

$$+ \frac{g_{z2}}{(2N)^2 - 4g_{z0}} + \frac{g_{z3}}{(3N)^2 - 4g_{z0}} + \ldots. \quad (3.32)$$

Since $\nu_z^2 \ll 1$ and $g_{z0} \ll 1$ equation (3.32) reduces to

$$\nu_z^2 = 1 + \frac{\nu_z^2}{2} \sum_{k=0}^{\infty} \alpha_k^2 + \frac{\nu_z^2}{2N^2} \left[ \alpha_0^2 \right.$$

$$- (2 - 2 \frac{R}{A} \frac{dA}{dR} \alpha + 2 + 2 \frac{R}{A} \frac{dA}{dR} \alpha^2) \sum_{k=0}^{\infty} \frac{\alpha_k^2}{(2k+1)^2}$$

$$+ \frac{5}{32N^2} \alpha_0^4 \left. + \ldots \right]. \quad (3.33)$$
In the same way we may also reduce equation (3.31) and obtain

\[ \nu_z^2 = \alpha + \frac{4}{2N^2} \left[ \alpha^2 + 2\alpha \left( 1 + \frac{R}{\beta} \frac{d\beta}{dR} \right) \right] \sum_{k=0}^{\infty} \frac{\alpha_k^2}{(2k+1)^2}, \]  

(3.34)

However, the value of \( \nu_x \) obtained from (3.34) is a rough approximation since \( g_{x0} \approx 1 \) and \( \nu_x \gg 1 \).

\[ \begin{align*}
F_{\text{min}}(R) &\approx \frac{E}{E_0} \\
\text{Fig. 3}
\end{align*} \]

Equating the right-hand side of (3.33) to zero we may determine the lower limit of \( F(R) = \mathcal{A}(R) \alpha \). Fig. 3 shows a plot of \( F_{\text{min}} \) as a function of \( \alpha = (E/E_0) \), for various values of \( N \) and a parameter \( p = \left( \frac{R}{\beta} \frac{d\beta}{dR} \right) \).

This figure demonstrates that large values of \( F \) are required if the final energy of the particle is high. Accordingly, the available gap height in the hill sectors is small in that case, and the mean field \( < B > \) is also small, so that the magnet will be larger than the magnet of a conventional cyclotron of the same energy. This serious disadvantage can be overcome by the use of spiral sectors. In this design the sectors spiral outward at a angle \( \zeta (r) \) to radii. The flutter field then becomes

\[ B_0 F(r) \sum_{k=0}^{\infty} \alpha_k \cos (2k+1)N(\theta - \psi(r)) \],

where \( \psi(r) \) is defined by
\[ \psi(r) = \int \tan \zeta(r) \frac{dr}{r}. \] (3.35)

The parameter \( \mu(R, \theta) \) is, to the same order of approximation as equation (2.37), given by

\[ \mu_0(R, \theta) = F(R) \left(1 - R^2\right)^{1/4} \sum_{k=0}^{\infty} \alpha_k \cos \left(2k+1\right)N(\theta - \psi(r)) . \] (3.36)

Substituting (3.36) in (3.24), making use of (3.35) and taking the most important term \( \tan \zeta \sum_{k=0}^{\infty} \frac{\alpha_k}{(2k+1)^2} \sin \left(2k+1\right)N(\theta - \psi(r)) \) of the second-order approximation into account, we obtain \( \eta \) correctly to \( 1/N^3 \) terms

\[ \eta = 1 + \left(1 + \frac{R}{\mathcal{A}} \right) \sum_{k=0}^{\infty} \frac{\alpha_k}{(2k+1)^2} \cos \left(2k+1\right)N(\theta - \psi(r)) \]
\[ + \mathcal{A} \sum_{k=0}^{\infty} \alpha_k \frac{\sin \left(2k+1\right)N(\theta - \psi(r))}{(2k+1)^3N^3} \]
\[ - \frac{\mathcal{A}^2}{2N^2} \left(1 + \frac{R}{\mathcal{A}} \right) \sum_{k=0}^{\infty} \left(\frac{\alpha_k}{2k+1}\right)^2 + \mathcal{A}^2 \sum_{k=0}^{\infty} \frac{\alpha_0^2 - 2/3 \alpha_0 \alpha_1 + \cdots}{(2N)^2} \cos 2N(\theta - \psi(r)) \]
\[ + \frac{3}{4N^2} \alpha_0 \alpha_1 + \cdots \cos 4N(\theta - \psi(r)) + \right) , \] (3.37)

where \( \mathcal{A}(R) \) is given by (3.26). In the same way as we have established (3.27) and (3.28), we may now show that

\[ \frac{d^2x}{d\theta^2} + \left( \sum_{n=-\infty}^{\infty} \xi_n e^{i n N(\theta - \psi)} \right) x = 0 \] (3.38)

and

\[ \frac{d^2z}{d\theta^2} + \left( \sum_{n=-\infty}^{\infty} \xi_n e^{i n N(\theta - \psi)} \right) z = 0 , \] (3.39)

where

\[ \sum_{n=-\infty}^{\infty} \xi_n e^{i n N(\theta - \psi)} = \alpha - \mathcal{A}(\tan \zeta) \sum_{k=0}^{\infty} \frac{\alpha_k^2}{2} \]
\[ x \left\{ \frac{1}{(2k+1)^2N^2} - \frac{\alpha}{(2k+1)^2N^2} \right\} - \frac{\mathcal{A}^2}{2N^2} \left( 1 + \frac{R \frac{d\mathcal{A}}{dR}}{2N} \right) \sum_{k=0}^{\infty} \left( \frac{\alpha_k}{2k+1} \right)^2 \]

\[ \mathcal{A} (\alpha + 1 + R \frac{d\mathcal{A}}{dR} \sum_{k=0}^{\infty} \alpha_k \cos (2k+1) \mathcal{N} (\theta - \psi) \]

\[ \mathcal{A} \tan \zeta \sum_{k=0}^{\infty} \alpha_k \left\{ \frac{(2k+1)N + \frac{1-\alpha}{(2k+1)N}}{2N^2} \right\} \sin (2k+1)N(\theta - \psi) + \ldots \ldots \quad (3.40) \]

\[ \sum_{n=-\infty}^{\infty} g_{zn} \sin N(\theta - \psi) = 1 - \alpha + \frac{\mathcal{A}^2}{2} \left\{ 1 + \left( \tan \zeta \right)^2 \right\} \sum_{k=1}^{\infty} \alpha_k^2 \]

\[ \frac{\left( \mathcal{A} \tan \zeta \right)^2}{2N^2} \sum_{k=0}^{\infty} \left( \frac{\alpha_k}{2k+1} \right)^2 + \frac{\mathcal{A}^2}{2N^2} \left( 1 + \frac{R \frac{d\mathcal{A}}{dR}}{2N} \right) \sum_{k=0}^{\infty} \left( \frac{\alpha_k}{2k+1} \right)^2 \]

\[ \mathcal{A} (1 - R \frac{d\mathcal{A}}{dR} - \alpha) \sum_{k=0}^{\infty} \alpha_k \cos (2k+1) \mathcal{N} (\theta - \psi) \]

\[ - \mathcal{A} \tan \zeta \sum_{k=0}^{\infty} \alpha_k \left\{ \frac{(2k+1)N + \frac{1-\alpha}{(2k+1)N}}{2N^2} \right\} \sin (2k+1)N(\theta - \psi) + \ldots \ldots \]

We assume that \( \mathcal{A} \) is small enough so that terms of higher than second order in \( \mathcal{A} \) can be neglected. Then, to second order in \( \mathcal{A} \),

\[ \cos \frac{2\pi}{N} \nu_x = \cos \frac{2\pi}{N} \sqrt{g_{x0}} - \frac{\pi}{N} \sum_{k=0}^{\infty} \frac{g_{x_k} x_{k-1}}{\sqrt{g_{x0}}} \]

\[ \cos \frac{2\pi}{N} \nu_z = \cos \frac{2\pi}{N} \sqrt{g_{z0}} - \frac{\pi}{N} \sum_{k=0}^{\infty} \frac{g_{z_k} z_{k-1}}{\sqrt{g_{z0}}} \]

where

\[ g_{x0} = \alpha - \frac{\left( \mathcal{A} \tan \zeta \right)^2}{2} \sum_{k=0}^{\infty} \alpha_k^2 \left\{ 1 + \frac{1}{(2k+N^2N^2)} \right\} - \frac{\alpha}{(2k+1)N} \]

\[ - \frac{\mathcal{A}^2}{2N^2} \left( 1 + \frac{R \frac{d\mathcal{A}}{dR}}{2N} \right) \sum_{k=0}^{\infty} \left( \frac{\alpha_k}{2k+1} \right)^2 \]
We now obtain from (3.41) and (3.42), in place of (3.33) and (3.34)

\[ \nu_x^2 = \alpha + 2N^2 \left( \alpha^2 + (2+2 \frac{R \partial A}{A} \tan \zeta \alpha + 3 \tan \zeta \alpha) \right) \sum_{k=0}^{\infty} \frac{\alpha_k^2}{2k+1} \]  

(3.44)

and

\[ \nu_z^2 = (1-\alpha) + \frac{2N^2}{2} \left( 1+2 \tan \zeta \right) \sum_{k=0}^{\infty} \frac{\alpha_k^2}{2k+1} \]

\[- (2-2 \frac{R \partial A}{A} + 7 \tan \zeta \alpha + 2 \frac{R \partial A}{A} \tan \zeta \alpha \]

\[ + 8 \tan \zeta \left( \sum_{k=0}^{\infty} \frac{\alpha_k^2}{2k+1} \right) \]  

(3.45)

We see from (3.45) that the spiral-sector magnet has the advantage of increasing the vertical focusing action of the field. Accordingly, the flutter amplitude can be reduced, resulting in a larger available gap and a higher mean field.

We shall now consider the maximum energy obtainable for a given number of sectors N. The investigation on the A.G. synchrotron have shown that the motion of a particle in a periodic field is in linear
Approximation stable if

\[ 2 \nu_x \neq \text{integer}, \]
\[ 2 \nu_z \neq \text{integer}, \]
\[ \nu_x + \nu_z \neq \text{integer}. \]

However, there is an important difference between the A.G. synchrotron and the isochronous cyclotron. In an ideal A.G. synchrotron the working point in the \((\nu_x, \nu_z)\)-plane is fixed. In an isochronous cyclotron, on the other hand, the working point moves during the acceleration. Several resonances may have to be crossed. The spiral angle \(\zeta(r)\) and the flutter amplitude \(F(r)\) must be chosen so that the number of resonances crossed is made as small as possible.

It is readily verified from (3.33) or (3.44) that \(\nu_x\) is a monotonically increasing function of \(\alpha\). The condition for a stable radial motion is

\[ |\cos \frac{2\pi}{N} \nu_x| < 1 \]

so that the boundary of the stable region is reached if \(\nu_x = \frac{1}{2}N\) or \(\cos \frac{2\pi}{N} \nu_x = -1\). Since at the centre of the cyclotron, the flutter amplitude is inevitably zero, as we shall show in section 4, \(\nu_x\) is equal to unity there, so that the choice \(N=2\) will give an unstable radial motion. One may also see this by noting that for \(N=2\) and small values of \(R\), (3.27) can be considered as an extended Mathieu equation with \(g_{x_0} \sim 1\) and \(1 \gg g_{x_1} \gg g_{x_2} \ldots\). In this case it is readily verified by examination of the stability chart for Mathieu functions that the radial oscillations are unstable.

The result of the above discussion can be summarized as follows: In an isochronous cyclotron the maximum energy obtainable is in linear approximation determined by the condition

\[ \nu_x = \frac{1}{2}N \text{ or } \cos \frac{2\pi}{N} \nu_x = -1 \]

and

\[ \nu_z = 0 \text{ or } \cos \frac{2\pi}{N} \nu_z = 1. \]
In order to obtain a rough estimate of this energy limit we eliminate $\hat{A}$ from (3.44) and (3.45) by equating the right-hand side of (3.45) to zero. In this way we obtain $\alpha$ as a function of $N$, $p = \left( \frac{R}{\hat{A}} \right) d\hat{A}/dR$, $\tan \xi$, and the "form factor" $q = \left( \frac{\sum_{k=0}^{\infty} \alpha_k^2}{\sum_{k=0}^{\infty} \frac{\alpha_k}{2k+1}} \right)^{1/2}$.

It turns out that $\alpha_{\text{max}}$ is almost independent of $p$, $\xi$ and $q$. Fig. 4 shows a plot of $E/E_0 = \alpha_{\text{max}}^{1/2}$ as a function of $N$. 

![Graph showing the relationship between $E/E_0$ and $N$.](image)
III. THE MAGNETIC FIELD

4. Pole-face design

The magnetic field is characterized by the vector \( \mathbf{B} \), which is in general not irrotational. If, however, a closed path \( C \) is chosen which does not encircle the coils, the line integral along \( C \) does vanish. We can thus use the scalar potential concept in the gap between the two poles. We can then introduce the magnetostatic potential \( V \) defined by

\[
\mathbf{B} = -\nabla V,
\]

(4.1)

where \( V \) satisfies the three-dimensional Laplace's equation which in cylindrical coordinates \( (r, \phi, z) \) is written as

\[
\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0.
\]

(4.2)

We first take up the question of the solution of (4.2) for the case that the median plane field is given by (2.20). Then, within a region \( r < r_1 \), the boundary conditions are of the form

\[
V = 0,
\]

when \( z = 0 \)

and

\[
\frac{\partial V}{\partial \phi} = 0
\]

(4.4)

when \( \phi = 0 \) and \( \phi = \pi/N \).

Since the potential function \( V \) is analytic in the neighbourhood of the plane \( z = 0 \), we can obtain the solution of (4.2) which satisfies the boundary conditions (4.3) and (4.4) by means of the power series expansion

\[
V(r, \phi, z) = \sum_{k=0}^{\infty} V_{2k+1}(r, \phi) \frac{z^{2k+1}}{(2k+1)!},
\]

(4.5)
where only odd powers of z can appear because \( B_z = -\frac{\partial V}{\partial z} \) is an even function of z. Substitution of (4.5) in (4.2) yields the following relations

\[
\sum_{k=0}^{\infty} \left( \frac{1}{r} \frac{\partial}{\partial r} \frac{r^{2k+1}}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + V_{2k+3} \right) \frac{z^{2k+1}}{(2k+1)!} = 0 ,
\]

which is satisfied only if the coefficients of all powers of z vanish; we have therefore the following recursion relation

\[
\frac{1}{r} \frac{\partial}{\partial r} \frac{r^{2k+1}}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + V_{2k+3} = 0 .
\]

Noting that \( V_1(r, \phi) = -B(r, \phi) \) we may rewrite (4.5) in the form

\[
V(r, \phi, z) = - \left[ B(r, \phi) z - \left( \frac{1}{r} \frac{\partial}{\partial r} \frac{r \partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) B(r, \phi) \frac{z^3}{3!} \right.
\]

\[
\left. + \left( \frac{1}{r} \frac{\partial}{\partial r} \frac{r \partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right)^2 B(r, \phi) \frac{z^5}{5!} + \ldots \right] .
\]

Using the expression (4.3) and carrying out the differentiation, we obtain after some manipulations

\[
V(r, \phi, z) = B_0 z \left[ 1 + a r^2 + b r^4 - \frac{2}{3} a z^2 - \frac{8}{3} b (z r)^2 + \frac{8}{15} b z^4 + \sum_{k=0}^{\infty} \left\{ \frac{A r a_k \left( z/r \right)}{}\right. \right.
\]

\[
\left. + B r^3 \left( z/r \right) \left\{ \alpha_k \cos(2k+1) \phi \right. \right. \}
\]

where

\[
a_k \left( z/r \right) = 1 + \frac{(2k+1)^2 N_2 - 1}{3!} \left( z/r \right)^2 + \sum_{k=0}^{\infty} \frac{(2k+1)^2 N_2 - 1}{5!} \left( z/r \right)^4 \]

\[
+ \frac{(2k+1)^2 N_2 - 1}{7!} \left( z/r \right)^6 \]

\[
+ \frac{(2k+1)^2 N_2 - 1}{9!} \left( z/r \right)^8 + \ldots
\]

and
\[
\begin{align*}
b_k(z/r) &= 1 + \frac{(2k+1)^2 N^2 - 3^2}{3!} \left(\frac{z}{r}\right)^2 + \frac{(2k+1)^2 N^2 - 1}{5!} \left(\frac{z}{r}\right)^4 \left(\frac{z}{r}\right)^6 \\
&\quad + \frac{(2k+1)^2 N^2 - 1}{7!} \left(\frac{z}{r}\right)^8 + \ldots.
\end{align*}
\]

It is easy to show from (4.10) that \(a_k(z/r)\) and \(b_k(z/r)\) are polynomials when \(N\) is odd and infinite series when \(N\) is even. We now introduce a parameter \(\sim = z_{m}/r\), where \(2z_{m}\) is the gap height in the so called valley sectors (sectors where the gap height is large; see fig. 15). Then (4.9) may be written in the form

\[
V(r, \phi, z) = B_0 z \left[ 1 + a r^2 + b r^4 - \frac{2}{3} a z^2 - \frac{8}{15} b(zr)^2 + \ldots \right] \cos(2k+1)N\phi \right].
\]

If we now choose

\[
\begin{align*}
\alpha_{k_1} &= (-1)^k \frac{1}{2k+1} \frac{a_0(\lambda)}{a_k(\lambda)} \\
\alpha_{k_3} &= (-1)^k \frac{1}{2k+1} \frac{b_0(\lambda)}{b_k(\lambda)}
\end{align*}
\]

we find

\[
V(r, \phi, z) = B_0 z \left[ 1 + a r^2 + b r^4 - \frac{2}{3} a z^2 - \frac{8}{15} b(zr)^2 \\
+ \frac{8}{15} b z^4 + \sum_{k=0}^{\infty} \left\{ A r a_0(\lambda) \frac{a_k(\frac{Z}{Z_m})}{a_k(\lambda)} \right\}
\right].
\]
Making use of the relation

\[
\sum_{k=0}^{\infty} (-1)^k \frac{\cos((2k+1)N\phi)}{2k+1} = \frac{\pi}{4}, \quad \frac{\pi}{2N} < \phi < \frac{3\pi}{2N}
\]

we have

\[
V(r, \phi, z_m) = B_0 z_m \left[ 1 + a r^2 + b_{13} z_m^2 + \frac{8}{3} b (rz_m)^2 + \frac{8}{15} b z_m^4 + \ldots \right.
\]

\[
- \frac{\pi}{4} \left\{ a_0(\lambda) A r + b_0(\lambda) B r^3 + \ldots \right\},
\]

when

\[
\frac{\pi}{2N} < \phi < \frac{3\pi}{2N}.
\]

Consequently, the gap height in the valley sectors is independent of azimuth as illustrated in fig. 5.

![Fig. 5](image)

It should be emphasized that (4.13) gives an exact solution for the fundamental wave only and is merely an approximation for the higher harmonics as we have chosen the \(\alpha_k\)'s and \(\alpha_{k3}\)'s \((k=1,2,\ldots)\) functions of \(\lambda\), i.e. functions of \(r\). But the approximation will be exceedingly good since it can be easily verified that
\[ \frac{r}{\alpha_k} \frac{d\alpha_k}{dr} \ll (2k+1)^2 N^2, \quad k = 1, 2, \ldots \]

\[ \frac{r^2}{\alpha_k} \frac{d^2\alpha_k}{dr^2} \ll (2k+1)^2 N^2, \quad i = 1, 3, \ldots \] (4.15)

for small values of \( \lambda \). For large values of \( \lambda \) (near the centre of the magnet) the pole-profile is mainly determined by the fundamental of the flutter field.

Numerical calculations indicate that for small value of \( \lambda, \alpha_k \) and \( \alpha_k \) can be approximated by

\[ (-1)^k \frac{\sinh N \lambda}{\sinh (2k+1)N \lambda} = \alpha_k', \] (4.16)

It is evident from (4.10) that \( a_0(\lambda) \) and \( b_0(\lambda) \) are infinite at \( r = 0 \) (\( \lambda = \infty \)). Therefore a magnetic field as given by (4.3) can not be realized in the region near the centre of the magnet. Let us now assume that the flutter field is given by

\[ \sum_{k=0}^{\infty} F_k(r) \cos (2k+1)N \phi, \] (4.17)

where \( F_k \) is an even function of \( r \) if \( N \) is even and an odd function of \( r \) if \( N \) is odd. If (4.17) is substituted in (4.8), the \((2k+1)\)th harmonic of the potential function \( V \) may be written in the form

\[ \sum_{n=0}^{\infty} \left\{ \frac{(2k+1)^2 N^2}{r^2} - \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} \right\}^n F_k(r) \frac{z^{2n+1}}{(2n+1)!} \cos (2k+1)N \phi. \] (4.18)

For a finite value of the amplitude of this harmonic at \( r = 0 \) and \( z \neq 0 \), \( F_k \) must be a power series in \( r \) which starts with \( r^{(2k+1)N} \). Hence, it is obvious that for small values of \( r \) all higher harmonics can be ignored, and we may consider only the fundamental of the flutter field.

The amplitude of this fundamental is a power series in \( r \), starting with \( r^N \).

Therefore we may assume that along the axis of the magnet (4.15) reduces to

\[ V(0, \phi, z) = B_0 z (1 - \frac{2}{3} a z^2 + \frac{8}{15} b z^4 + \ldots ). \] (4.19)

Accordingly, \( z_m(r) \) can be calculated by means of the relation
where 2d is the gap height at the centre of the magnet. For a given value of \( \lambda \), (4.20) is a polynomial in \( z_m \) of degree \( (2l+1) \), where \( 2l \) is the highest power of \( r \) in (4.20). In general, the expressions for the exact roots, provided that they can be found at all, are complicated. However, since we are interested only in one real root, the location of which is not difficult, we may use approximation methods to obtain \( z_m \) to any desired degree of accuracy.

Plotting \( z_m \) against \( r \) gives the pole-profile of a valley. The azimuthal variation of the gap height for a constant value of \( r \) can be determined by means of (4.11), where \( \lambda \) is now a known constant. We may compute \( z \) for several values of \( \phi \). It is obvious that an exact solution of (4.11) can not be found. A graphical method of solution has been used. The solution of (4.11) is equivalent to the simultaneous solution of the system

\[
X = d \left[ 1 - \frac{2}{3} a d^2 + \frac{8}{15} b d^4 \right] - z \left[ 1 + a r^2 + b r^4 \right] - \frac{2}{3} a z^2 - \frac{8}{3} b (zr)^2 + \frac{8}{15} b z^4 \right] , \tag{4.21}
\]

\[
Y = z \sum_{k=0}^{\infty} \left\{ a_0(\lambda) A r \frac{a_k(\lambda z_m)}{a_k(\lambda)} + b_0(\lambda) B r^3 \frac{b_k(\lambda z_m)}{b_k(\lambda)} \right\}
\]

\[
X (-1)^k \cos \left( \frac{(2k+1)N\phi}{2k+1} \right). \tag{4.22}
\]

The intersection of these two curves (4.21) and (4.22), yields an approximate solution of (4.11). In practice we restrict this tedious and time consuming process to \( \phi = 0 \) and assume that the gap height in hill sectors is independent of azimuth. As a result of this the amplitude of the higher harmonics increases. In order to estimate the amplitude of these higher harmonics we define a modified potential function

\[
G(x,y) = V(r,\phi,z) \tag{4.23}
\]
which is a function only of the new variables

\[ x = N \phi \]
\[ y = N \sinh^{-1} \left( \frac{Z}{r} \right) , \]  
\[ (4.24) \]

Introducing (4.23) and (4.24) into (4.2) yields the two dimensional Laplace's equation

\[ \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0 \]  
\[ (4.25) \]

with the boundary conditions

\[ G = 0 , \text{ when } y = 0 , \]
\[ \frac{\partial G}{\partial x} = 0 , \text{ when } x = 0 \text{ and } x = \pi/N . \]  
\[ (4.26) \]

The problem is a simple conformal mapping problem if \( y_2 \ll \pi \). (fig.6) and can be solved directly. In what follows we shall give a general solution of (4.25). The solution satisfying the boundary conditions (4.26) can be written in the form

\[ G(x,y) = C \left[ y + \sum_{m=1}^{\infty} a_m \sinh m y \cos mx \right] . \]  
\[ (4.27) \]

Inserting the boundary condition \( G(x,y) = V_0 \), when

\[ y = y_o - \frac{4}{\pi} \Delta y \sum_{k=0}^{\infty} (-1)^k \cos \frac{(2k+1)x}{2k+1} . \]  
\[ (4.28) \]
where
\[ y_0 = \frac{1}{2}(y_1 + y_2) \text{ and } \Delta y = \frac{1}{2}(y_2 - y_1) \]

and noting that
\[
\cosh n \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\cos (2k+1)x}{2k+1} = \cosh n \Delta y \\
\sinh n \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\cos (2k+1)x}{2k+1} = \left( \sinh n \Delta y \right) \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\cos (2k+1)x}{2k+1}
\]

we find
\[
V_n = C \left[ y_0 - \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\cosh (2k+1)y_0 \sinh (2k+1) \Delta y}{2k+1} a_{2k+1} \right]
\]

\[
\frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\cos (2k+1)x}{2k+1} + \sum_{m=1}^{\infty} \left\{ a_m \sinh m y_0 \cosh m \Delta y - \frac{4}{\pi} b_m \right\} \cos m x \right],
\]

where
\[
b_{2n} = \sum_{k=0}^{\infty} (-1)^{n+k} \frac{2k+1}{(2k+1)^2 - 4n^2} a_{2k+1} \cosh (2k+1)y_0 \sinh (2k+1) \Delta y
\]

\[
b_{2n+1} = \sum_{k=1}^{\infty} (-1)^{n+k} \frac{2n+1}{(2n+1)^2 - 4k^2} a_{2k} \cosh 2ky_0 \sinh 2k \Delta y
\]

Using the orthogonal properties of the trigonometric functions, we obtain
\[
a_{2n+1} = \frac{4}{\pi} \left\{ \sinh (2n+1)y_0 \cosh (2n+1) \Delta y \right\}^{-1} (-1)^n \frac{\Delta y}{2n+1}
\]

\[
+ \sum_{k=1}^{\infty} (-1)^{n+k} \frac{2n+1}{(2n+1)^2 - 4k^2} a_{2k} \cosh 2ky_0 \sinh 2k \Delta y
\]

\[
a_{2n} = \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^{n+k} \frac{2k+1}{(2k+1)^2 - 4n^2} a_{2k+1} \cosh (2k+1)y_0 \sinh (2k+1) \Delta y
\]

\[
\sinh 2ny_0 \cosh 2n \Delta y
\]
Substituting (4.34) in (4.33) we obtain
\[
a_{2n+1} = \frac{(-1)^n \frac{4}{\pi} \frac{\Delta y}{2n+1} + \frac{\infty}{\pi} \frac{\infty}{k=0} \frac{\infty}{p=0} \sum \sum g_{nkp} a_{2p+1}}{\sinh (2n+1) y_o \cosh (2n+1) \Delta y},
\]
(4.36)
where
\[
g_{nkp} = (-1)^p \frac{4}{\pi} \frac{\tanh 2k \Delta y (2n+1)(2p+1) \cosh (2p+1) y_o \sinh (2p+1) \Delta y}{\tanh 2k y_o} \frac{(2n+1)^2}{(2n+1)^2 - 4k^2} \frac{(2p+1)^2}{(2p+1)^2 - 4k^2}.
\]
(4.37)
The particular form of (4.36) suggest to solve this equation by repeated substitution. Writing
\[
e_{n1} = \left(\frac{4}{\pi^2}\right)^2 \frac{\infty}{k=1} \frac{\infty}{p=0} \sum \sum \frac{\tanh 2k \Delta y \tanh (2p+1) \Delta y}{\tanh 2k y_o \tanh (2p+1) y_o} \frac{(2n+1)^2}{(2n+1)^2 - 4k^2} \frac{(2p+1)^2}{(2p+1)^2 - 4k^2},
\]
for short, we find the following expression
\[
a_{2n+1} = \frac{(-1)^n \frac{4}{\pi} \frac{\Delta y}{2n+1} \frac{(2n+1)^2}{(2n+1)^2 - 4k^2} \frac{(2p+1)^2}{(2p+1)^2 - 4k^2}}{\sinh (2n+1) y_o \cosh (2n+1) \Delta y},
\]
(4.39)
where \(e_{nq}\) can be obtained from the recursion formula
\[
e_{nq+1} = \left(\frac{4}{\pi^2}\right)^2 e_{nq} \frac{\infty}{k=1} \frac{\infty}{p=0} \sum \sum \frac{\tanh 2k \Delta y \tanh (2p+1) \Delta y}{\tanh 2k y_o \tanh (2p+1) y_o} \frac{(2n+1)^2}{(2n+1)^2 - 4k^2} \frac{(2p+1)^2}{(2p+1)^2 - 4k^2}.
\]
(4.40)
At this point it is interesting to note that the labour involved in obtaining numerical results is shortened considerably if we substract
\[
S_1 = \left(\frac{4}{\pi^2}\right)^2 \frac{\infty}{k=1} \frac{\infty}{p=0} \sum \sum \frac{(2n+1)^2}{(2n+1)^2 - 4k^2} \frac{(2p+1)^2}{(2p+1)^2 - 4k^2}.
\]
from (4.38). This equation then becomes

\[
\begin{align*}
e_{n1} & = s_1 + \frac{4}{\pi} \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} \frac{\tanh 2k\sqrt{y}}{\tanh 2ky_0} \frac{\tanh (2p+1)\sqrt{y}}{\tanh (2p+1)y_0} \\
x & = \frac{(2n+1)^2}{\left\{ (2n+1)^2 - 4k^2 \right\} \left\{ 2p+1 \right\}^2 - 4k^2}.
\end{align*}
\]

(4.41)

By the aid of the expansion of $\cot \pi x$

\[
\pi \cot \pi x = \sum_{k=-\infty}^{\infty} \frac{x}{x^2 - k^2}
\]

it can be shown that

\[
S_1 = \frac{4}{\pi} \sum_{p=0}^{\infty} \left[ \frac{\pi}{8} \frac{(2n+1)^2}{(2p+1)^2 - (2n+1)^2} \right]
\]

\[= \frac{4}{\pi^2} \left[ \frac{\pi^2}{16} - \frac{1}{2} \sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} \right] = 0
\]

so that (4.41) reduces to

\[
\begin{align*}
e_{n1} & = \frac{4}{\pi} \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} \frac{\tanh 2k\sqrt{y}}{\tanh 2ky_0} \frac{\tanh (2p+1)\sqrt{y}}{\tanh (2p+1)y_0} \\
x & = \frac{(2n+1)^2}{\left\{ (2n+1)^2 - 4k^2 \right\} \left\{ (2p+1)^2 - 4k^2 \right\}}.
\end{align*}
\]

(4.42)

It is not difficult to confirm that (4.42) converges rapidly. Similar transformations of series may be applied to simplify the calculation of $e_{n2}$, $e_{n3}$, etc. If the values of $a_{2n+1}$ are found, $a_{2n}$ and $C$ can be calculated by means of (4.34) and (4.35). Next we modify the steel configuration of the hill sector so that

\[-a_2 = \frac{\pi}{\pi} \int y \cos 2x \, dx \quad \text{and} \quad -a_4 = \frac{\pi}{\pi} \int y \cos 4x \, dx.\]
Then the first two even harmonics are equal to zero to a first approximation.

An example of the procedure discussed in this section will be given in section 8 in connection with the design of the 12 Mev cyclotron.

In the treatment given here it has been tacitly assumed that the permeability of the steel is infinite. This is justified if the steel is not saturated. Towards the edge of the poles the treatment given in this section breaks down due to saturation and fringing fields.

To correct for these effects it is preferable to use empirical methods. There are two possibilities to correct the radial fall-off of the magnetic field: decreasing the gap height or increasing the angular width of the hill-sectors. The latter method has the advantage that it does not decrease the available gap height. However, the checking of the condition for isochronism, will be very difficult.

5. Field tolerances and measurements

The condition for a particle to be accelerated is, that its phase with respect to the rf voltage remains always between -90° and 90°. Assuming that the frequency of the rf voltage is constant, errors in the magnetic field will cause phase slip. The phase shifting errors may conveniently be divided into two classes:

(a) An uniform fractional error in the magnetic field, so that

\[ B_{\text{actual}} = B_{\text{ideal}}(1+h), \]  

where \( h \) is a small constant. The time variation of the magnetic field is the main source of this error. An error in the value of the central field \( B_0 \) can readily be corrected by adjusting the exciting current. The phase slip due to errors of this type may be computed by means of the relations shown in section 2:

\[ mv = e < B > R \]  

or

\[ \left( \frac{E}{E_0} \right)^2 = 1 + \left( \frac{< B >}{B_0} \right)^2 R^2 \]  

and according to (2.9)

\[ < B > = \frac{B_0}{(1-R^2)^{1/2}}(1+h) \]

or

\[ \left( \frac{< B >}{B_0} \right)^2 = 1 + \left( \frac{< B >}{B_0} \right)^2 R^2 + 2h + h^2. \]
where it should be remembered that the unit of length is \( \frac{m_o c}{eB_o} \).

Substitution of (5.2) in (5.3) gives
\[
\left( \frac{B}{B_o} \right)^2 = \left( \frac{E}{E_o} \right)^2 + 2h + h^2.
\]

The mean angular velocity can now be written
\[
\langle \omega \rangle = \frac{e \langle B \rangle}{m} = \frac{E_o}{E} \left( \frac{E}{E_o} \right)^2 + 2h + h^2 \right)^{1/2} \frac{eB_o}{m_o}
\]

and for the phase change per revolution we find
\[
\frac{dx}{d\nu} = 2\pi \frac{E_o}{E} \left[ 1 - \frac{E}{E_o} \left( \frac{E}{E_o} \right)^2 + 2h + h^2 \right]^{1/2}.
\] (5.5)

The energy gain per turn is
\[
\frac{dE}{d\nu} = 2eV_o \cos x,
\] (5.6)

where \( V_o \) is the dee to dee voltage and \( x \) is the phase of the particle with respect to the rf voltage. Making use of (5.6) we obtain from (5.5)
\[
\cos x dx = \frac{\pi E_o}{eV_o} \left[ 1 - \frac{E}{E_o} \left( \frac{E}{E_o} \right)^2 + 2h + h^2 \right]^{1/2} dE.
\] (5.7)

Integration of (5.7) with respect to \( E \) yields
\[
\sin x - \sin x_i = \frac{\pi E_o}{eV_o} \left[ \frac{E_m}{E_o} + h - \left( \frac{E_m}{E_o} \right)^2 + 2h + h^2 \right]^{1/2}
\]

\[
- (2h + h^2)^{1/2} \ln \left[ \frac{E_m}{E_o} \left( \frac{E_m}{E_o} \right)^2 + 2h + h^2 \right]^{1/2} + (2h + h^2)^{1/2}
\] (5.8)

where \( E_m \) is the final energy of the particle and \( x_i \) is the phase of the particle with respect to the rf voltage after a few turns. From (5.8) we may calculate the requirements as to the stability in time of the magnetic field. Ignoring all higher order terms in \( h \) we obtain
Assuming the phase grouping theory of Bohm and Foldy\(^{(3)}\) valid, the condition for a phase slip of less than 90° becomes

\[
| h | < \frac{eV_o E_m}{\pi E_o (E_m - E_o)} .
\]

This yields for a 12 MeV cyclotron with \(V_o = 25\) kV: \(| h | < 6.6 \times 10^{-4}\)
and for a 200 MeV cyclotron with \(V_o = 200\) kV: \(| h | < 4 \times 10^{-4}\).

Equation (5.10) gives also the stability requirements for the frequency of the rf voltage.

(b) Imperfections in the magnetic field. In this case we may write

\[
\langle B \rangle_{\text{actual}} = \langle B \rangle_{\text{ideal}} (1 + h(R)) .
\]

Substituting this in (5.5) and using (5.6) we obtain

\[
\cos x \frac{dx}{dE} = \frac{\pi}{eV_o} \frac{\langle B \rangle_{\text{actual}} - \langle B \rangle_{\text{ideal}}}{\langle B \rangle_{\text{ideal}}}
\]

or

\[
\cos x \frac{dx}{dE} = \frac{\pi E_o}{eV_o} h(R) \frac{\beta d\beta}{(1 - \beta^2)^{3/2}} ,
\]

where we have made use of the relation

\[
\frac{d\langle E \rangle}{E_o} = \frac{\beta d\beta}{(1 - \beta^2)^{3/2}} .
\]

Remembering that in the unit of length adopted by us \(\beta = R\), we obtain by integration

\[
\sin x - \sin x_1 = \frac{\pi E_o}{eV_o} \int_0^{R_{\text{max}}} h(R) R dR = \frac{\pi E_o}{eV_o} I(R) .
\]

The condition for a phase slip of less than 90° then becomes

\[
| I(R) | < \frac{eV_o}{\pi E_o} | (1 - \sin x_1) | .
\]
At this point it is worth noticing that the quantity $h(R)$ must also satisfy the condition that $\Delta \frac{R d\langle R \rangle}{dR} = R \frac{dh}{dR}$ is small enough for a stable vertical motion.

As an example of an error of this type we consider the phase slip due to the approximation of $(1 - R^2)^{-\frac{1}{2}}$ by a polynomial of degree $2n$ in $R$. The Maclaurin expansion of $(1 - R^2)^{-\frac{1}{2}}$ is

$$(1 - R^2)^{-\frac{1}{2}} = 1 + \frac{1}{2} R^2 + \frac{3}{8} R^4 + \cdots + \frac{(2n-1)!}{2^{2n-1} n! (n-1)!} \frac{R^{2n}}{(1-fR^2)^{n+\frac{1}{2}}}$$

with $0 < f < 1$. The error made in using only $n$ terms is then

$$h(R) = \frac{(2n-1)!}{2^{2n-1} n! (n-1)!} \frac{R^{2n}}{(1-fR^2)^{n+\frac{1}{2}}}$$

Hence we may write

$$\frac{(2n-1)!}{2^{2n-1} n! (n-1)!} \frac{R^{2n}}{(1-R^2)^{n}} > h(R) > \frac{(2n-1)!}{2^{2n-1} n! (n-1)!} \frac{R^{2n}}{(1-fR^2)^{n+\frac{1}{2}}}.$$  

Substitution of this in (5.13) gives

$$\frac{\pi E_0}{eV_0} I_1 > \sin x - \sin x_i > \frac{\pi E_0}{eV_0} I_2,$$

where

$$I_1(R) = \frac{(2n-1)!}{2^{2n-1} n! (n-1)!} \int_0^{R_{\text{max}}} \frac{R^{2n+1} dR}{(1-R^2)^{n+\frac{1}{2}}},$$

$$I_2(R) = \frac{(2n-1)!}{2^{2n-1} n! (n-1)!} \int_0^{R_{\text{max}}} \frac{R^{2n+1} dR}{1-R^2}.$$
In fig. 7 $I_1$ and $I_2$ are plotted against $R^2$ for various values of $n$. From these curves we see that to accelerate protons to 12 MeV with $V_0 = 25$ kV, the phase slip due to the approximation of $(1 - R^2)^{-1/2}$ by $1 + 1/2 R^2$ is negligible. However, to accelerate protons to 200 MeV ($R^2 = 0.32$) we must retain in the series expansion at least five terms even with $V_0 = 300$ kV. It is clear from this consideration that the expansion of the median plane field in powers of $r$ is not suitable for the determination of the condition for isochronism for high energy cyclotrons (see also section 3).

In order to check the performance of the magnet three types of measurements are required.

a. Measurements of the median plane field.
b. Determination of the magnetic centre.
c. Location of the median plane.

a. The measurements of the median plane field can be done by moving a coil radially or azimuthally at uniform velocity. The coil may be connected directly to a recording potentiometer, which necessitates graphical integration, or to an integrator followed by the recording potentiometer. The coil is connected in series with another one, kept at the centre of the magnet, in order to compensate for time variation of the magnetic field.

In practice data obtained from the azimuthal measurements at various radii are used to check the data obtained from the radial measurements at various azimuth, and vice versa. An additional advantage of
the azimuthal measurements is, that it enables one to measure the
dangerous subharmonics \( \cos \phi \) and \( \cos 2\phi \). Two coils with the same area
are kept at the same distance from the centre, but \( 2\pi/N \) in azimuth
apart. The voltage outputs of these coils are made to oppose each other.
In a perfectly periodic field with period \( 2\pi/N \) the resulting output
will be equal to zero. Accordingly, if we put the signal into re-
cording potentiometer we obtain a curve, which is a plot of

\[
2A \sin \frac{\pi}{N} \cos \left( \phi - \frac{\pi}{2N} \right) + 4B \sin \frac{\pi}{N} \cos (2\phi + p),
\]

(5.15)

where \( A \) and \( B \) are proportional to the amplitude of the harmonics and \( p \)
is the phase between these harmonics. At small radii we may increase
the speed of the rotating coils without causing damage so that the
output can be measured with a wave analyser, whose frequency range has
a limit.

It is interesting to note that radial measurements at azimuth \( \phi = 0 + k\pi/N \)
give already complete information about the median plane field
if only odd harmonics in the Fourier series expansion of the flutter
field are present.

b. Determination of the magnetic centre. A thin wire loop is placed in
the gap between the two poles. A current \( i \) is passed in a direction
opposite of that of the beam. Assuming that the wire is perfectly
flexible and perpendicular to the magnetic field lines, we have at any
point along the wire

\[
B i = T/\rho,
\]

or

\[
B \rho = T/i,
\]

(5.16)

where \( \rho \) is the radius of curvature and \( T \) is the tension in the wire.
Comparing (5.16) with (2.2) we see that the wire loop takes the form
of an equilibrium orbit curve, provided the effect of gravity on the
wire can be neglected. It is not difficult to verify that the horizont-
al position of the wire loop is stable if \( \frac{d<B>}{dr} < 0 \) and unstable if \( \frac{d<B>}{dr} > 0. \)

Accordingly, in an isochronous cyclotron the loop must be supported in
two points in order to give it some "stabilizing" forces. An increase
in the current would cause the loop centre to move away from the mag-
netic centre along a line connecting these centres. This line can be
obtained by plotting the position of the loop centre for several wire
currents.
Repeating this in the direction perpendicular to the first one gives another line. The intersection of the two lines gives the position of the magnetic centre.

c. The median plane is defined as the plane at which the horizontal components of the magnetic field are zero. In a perfect magnet this plane is the plane of symmetry of the gap between the two poles. However, due to constructional errors and difference in the permeability of the steel we may expect that the magnetic field is asymmetrical with respect to the geometrical median plane. In fact, the position of the median plane may vary not only with radius but also with azimuth. Accordingly, we define the "median plane" as the plane towards which particles are focused, i.e. the plane in which the equilibrium orbit lies. It will be shown in the next section that this plane cannot be appreciably titled, although its position may vary in height at different radii. It is therefore sufficient to locate the vertical position. This position of the median plane is readily determined using the wire loop described above (23).
IV. IMPERFECTIONS AND NONLINEAR EFFECTS

6. The effects of linear perturbations

In section 3 we have assumed that the equilibrium orbit is closed. This approximation is only valid if

$$\frac{eV_0}{E-E_0} << 1,$$

where $V_0$ is the amplitude of the rf voltage, and $E$ and $E_0$ the total and the rest energy of the particle respectively. Towards the centre of the cyclotron where the energy of the particle is comparatively small, the spacing between successive orbits cannot be neglected in comparison with the radius. Accordingly, the crossing of resonance lines by the betatron frequency is not necessarily disastrous, provided the transit time is sufficiently small.

In an actual machine the magnetic field will differ somewhat from the ideal field. The periodicity $2\pi/N$ is not exact, but small errors of periodicity $2\pi$ is always present.

In the discussion to follow we shall neglect the nonlinear terms in the equations of motion. We begin by proving the following theorem (2): If the position and the momentum of a particle is given by the column vector $Y$ with components $Y_i$, then the dynamical transformation

$$(Y)_2 = (Y)_1 + (C), \quad (6.1)$$

characterizing the motion of the particle through an appropriate interval $s_2 - s_1$ (e.g. one revolution), can be made independent of the perturbation to first order in perturbation which may be represented by the parameter $\epsilon$.

Let us assume that the transformation, which is not necessarily linear, is defined by a general functional dependence of $Y_{i2}$ on $Y_{i1}$ and $\epsilon$

$$Y_{i2} = f_i(Y_{j1}, \epsilon) \quad (6.2)$$

If $Y_{j1}$ and $\epsilon$ are increased by a small amount we have

$$dY_{i2} = \frac{\partial f_i}{\partial \epsilon} d\epsilon + \sum_j \frac{\partial f_i}{\partial Y_{j1}} dY_{j1}.$$
Hence
\[ d(Y_{i2} - Y_{i1}) = \frac{\partial f_i}{\partial \epsilon} d\epsilon + \sum_j \left( \frac{\partial f_i}{\partial Y_{j1}} - \delta_{ij} \right) dY_{j1}, \]  
\text{(6.3)}

where \( \delta_{ij} \) is the Kronecker's delta.

Thus, if \( Y_{i1} \) is a solution of the inhomogeneous equation
\[ -dE + L \cdot \delta_{ij} = 0 \] 
\text{(6.4)}

then we have
\[ Y_{i2}(\epsilon) - Y_{i1}(\epsilon) = Y_{i2}(0) - Y_{i1}(0) = C_1. \] 
\text{(6.5)}

Equation (6.4) is only soluble if the matrix
\[ \left( \frac{\partial f_i}{\partial Y_{j1}} - \delta_{ij} \right) \] 
\text{(6.6)}
is non singular, i.e. if none of its eigenvalues is zero. It is not difficult to see that these eigenvalues are
\[ \lambda_i - 1, \ i = 1, 2, 3, 4 \]
if \( \lambda_i \) is the eigenvalue of the matrix
\[ \left( \frac{\partial f_i}{\partial Y_{j1}} \right). \]

Thus we have proved that for sufficiently small values of the perturbation, there exists a displaced equilibrium orbit so long as none of the characteristic roots of \( \left( \frac{\partial f_i}{\partial Y_{j1}} \right) \) is equal to unity.

To determine the displacement of the equilibrium orbit quantitatively we consider the inhomogeneous equation
\[ \frac{d^2y}{ds^2} + (f(s) y = \epsilon(s) / \rho, \] 
\text{(6.7)}

where \( y \) is now the displacement, vertical or horizontal, from the ideal closed equilibrium orbit, and \( f(s) = (1 - n)/\rho, \epsilon(s) = = B_z/B \) for the horizontal motion, and \( f(s) = n/\rho, \epsilon(s) = B_x/B \) for the vertical motion.
\( \Delta B_z \) and \( \Delta B_x \) are respectively the deviations of the vertical and horizontal components of the magnetic field on the ideal orbit from their ideal values.

The steady solution of (6.7) can be expressed in terms of the solutions of the homogeneous equation

\[
\frac{d^2 y}{ds^2} + f(s) Y = 0.
\]  

(6.8)

Following Courant(5) we write the solutions of (6.8) in the form

\[
y = F(s) e^{\pm i\nu \psi(s)},
\]  

(6.9)

where \( F(s) \) is periodic in \( s \) with period \( S/N \) and where the \( \nu \)'s can be obtained by the aid of the method described in section 3.

Substituting of (6.9) in (6.8) yields

\[
\frac{d^2 F}{ds^2} - \frac{\nu^2}{F^3} + f(s) F = 0,
\]  

(6.10)

\[
\frac{d\psi}{ds} = \frac{1}{F^2}.
\]  

(6.11)

Introducing the new variables

\[
\psi = \int \frac{ds}{F^2},
\]

\[
\zeta = Y/F
\]  

(6.12)

equation (6.7) then reduces to

\[
\frac{d^2 \zeta}{d\psi^2} + \nu^2 \zeta = \frac{\epsilon}{\rho} F^3.
\]  

(6.13)

From the Floquet theory

\[
y(s + S/N) = e^{i\nu \frac{2\pi}{N}} y(s)
\]

it follows that

\[
\psi(s + S/N) - \psi(s) = \int_{s}^{s+S/N} \frac{ds}{F^2} = \frac{2\pi}{N}
\]

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Thus, if we neglect the ripple in the amplitude function \( F(s) \), we have \( \psi = \Theta \) and (6.13) becomes

\[
\frac{d^2y}{d\Theta^2} + \nu^2 y = R\epsilon(R, \Theta)\mu(R, \Theta). \tag{6.15}
\]

The steady solution of (6.15) is in terms of Fourier components

\[
y = -R \sum_{k=-\infty}^{\infty} \frac{a_k e^{ik\Theta}}{k^2 - \nu^2}, \tag{6.16}
\]

where

\[
a_k = \frac{1}{2\pi} \int_{0}^{2\pi} \epsilon(R, \Theta) \mu(R, \Theta) e^{-ik\Theta} d\Theta.
\]

Thus we have found the same resonance properties in the solution as in the general discussion given above. It is clear from (6.16) that the orbit is very sensitive to the Fourier component of the perturbation whose order is close to the betatron oscillation frequency. We may also see from (6.16) that, since in practice \( \nu^2 z \ll 1 \), it is not likely that the equilibrium orbit tips much, although it may vary in height at different radii.

It has been assumed that the disturbances are magnetic. However, towards the centre of the cyclotron the energy of the particle is relatively low so that the effect of the electric field cannot be ignored. In this region it is thus necessary to avoid asymmetries in the construction.

So far our treatment has been based on the assumption that the number of betatron oscillations per revolution is constant. However, as has been observed in section 3, the frequency of the radial betatron oscillations changes during the acceleration. We may anticipate that, if the crossing of the resonances is rapid enough, the losses of beam intensity will be small. In order to solve this problem we replace in (6.15) \( y \) and \( \nu \) by \( x \) and \( \nu_x \) respectively. Retaining the dangerous harmonic only we obtain

\[
\frac{d^2x}{d\Theta^2} + \nu_x^2 x = 2Ra_k \cos k\Theta, \tag{6.17}
\]
where we have assumed that \( a_k = a_{-k} \). Now \( \nu_x \) is a slowly varying function of \( \theta \). Introducing the new variables

\[
\phi = \int_0^\theta \nu_x \, d\theta ,
\]

\[
V = \sqrt[4]{\nu_x} ,
\]

we obtain

\[
\frac{d^2V}{d\phi^2} + \left[ 1 + \frac{3}{4} \left( \frac{1}{\nu_x^2} \frac{d\nu_x}{d\phi} \right)^2 - \frac{1}{\nu_x^3} \frac{d^2\nu_x}{d\phi^2} \right] V = \frac{2Ra_k \cos k\theta}{\nu_x^{3/2}} .
\]

The increase of the amplitude occurs in a narrow region in the vicinity of \( \nu_x = k \). Therefore, we may expand \( \nu_x(\theta) \) in the Taylor series

\[
\nu_x = k + k_1 \theta + \frac{1}{2} k_2 \theta^2
\]

where

\[
k_1 = \frac{d\nu_x}{d\theta} , \quad \nu_x = k
\]

\[
k_2 = \frac{d^2\nu_x}{d\theta^2} , \quad \nu_x = k
\]

Since \( N^2 \) is a large integer it can be shown by means of (3.34) or (3.44) that

\[
\frac{d\nu_x}{d\theta} = \frac{eV}{ne_0} \frac{d\nu_x}{d(E/E_o)} \ll 1
\]

where \( 2eV \) is the energy gain per turn, so that the bracketed term in (6.19) differs but little from unity. The particular solution of (6.19), ignoring terms of order \( k_1^2 \) and \( k_2 \) and noting that to this approximation \( \phi = k\theta + \frac{1}{2} k_1 \theta^2 \), can be written as

\[
Y_p = \frac{2Ra_k}{\sqrt[4]{\nu_x}} \left[ A(\theta) \cos \left( k\theta + \frac{1}{2} k_1 \theta^2 \right) + B(\theta) \sin \left( k\theta + \frac{1}{2} k_1 \theta^2 \right) \right] ,
\]

where \( A(\theta) \) and \( B(\theta) \) are given by
\[ A(\theta) = - \int \cos \frac{k \theta}{\sqrt{\nu_x}} \sin (k \theta + \frac{1}{2} \theta_1^2) \, d\theta \]

\[ = - \frac{1}{2} \int \{ \sin (2k \theta + \frac{1}{2} \theta_1^2) + \sin \frac{1}{2} \theta_1^2 \} \frac{d\theta}{\sqrt{\nu_x}}, \quad (6.22) \]

\[ B(\theta) = \int \frac{\cos \frac{k \theta}{\sqrt{\nu_x}}}{\cos (k \theta + \frac{1}{2} \theta_1^2)} \, d\theta \]

\[ = \frac{1}{2} \int \{ \cos (2k \theta + \frac{1}{2} \theta_1^2) + \cos \frac{1}{2} \theta_1^2 \} \frac{d\theta}{\sqrt{\nu_x}} \, . \quad (6.23) \]

It is obvious that the second terms give the largest contribution to the integral, and since \( \nu_x(\theta) \) changes very slowly with \( \theta \), the dominant parts of (6.22) and (6.23) are

\[ A = - \frac{1}{2} \frac{1}{k} \int_{\theta_1}^{\theta_2} \sin \frac{1}{2} \theta_1^2 \, d\theta, \]

\[ B = \frac{1}{2} \frac{1}{k} \int_{\theta_1}^{\theta_2} \cos \frac{1}{2} \theta_1^2 \, d\theta. \quad (6.24) \]

The amplitude of the oscillation is then given by

\[ x = \frac{2Ra_k}{\sqrt{\nu_x}} \left[ A^2 + B^2 \right]^{\frac{1}{2}} \quad (6.25) \]

In order to get an estimate of the largest amplitude that can reasonable be expected we choose

\[ \frac{1}{2} \theta_1^2 = \frac{1}{2} \theta_2^2 = \frac{\pi}{2}. \]

The maximum amplitude can then be expressed in terms of Fresnel integrals

\[ (x)_{\text{max}} = 2 \frac{Ra_k}{k} \frac{\sqrt{\pi}}{k_1} \left[ S(1) + C^2(1) \right]^{\frac{1}{2}}, \quad (6.26) \]
where
\[ S(1) = \int_0^1 \sin \frac{\pi}{2} t^2 dt \approx 0.44 \]

\[ C(1) = \int_0^1 \cos \frac{\pi}{2} t^2 dt \approx 0.78 \]

so that
\[ (x)_{\text{max}} \approx 1.8 \frac{\text{Ra}_k}{k^2} \sqrt{\frac{\gamma}{k_1}} \quad (6.27) \]

We now turn our attention to field gradient errors. It can be shown that these imperfections may produce instabilities when the number of betatron oscillations per revolutions, \( \nu \), takes integral or half-integral values. Integral values of \( \nu \), as we have seen, may already be harmful because they may cause a large displacement of the equilibrium orbit.

To consider the effect of the gradient errors we write the equation of motion in the form
\[
\frac{d^2 x}{ds^2} + \left\{ f(s) + \frac{1}{\rho^2(s)} \right\} x = 0, \quad (6.28)
\]

where \( \frac{\epsilon(s)}{\rho^2} \) is the perturbation of \( f(s) \) in consequence of the gradient errors.

We shall first give a brief account of the frequency shift \( \Delta \nu_x \) and the width of the stop band \( \Delta \nu_x \) for the static case (constant value of \( \nu_x \)). The equation analogous to (6.15) is readily found to be
\[
\frac{d^2 x}{d\theta^2} + (\nu_x^2 + \mu^2 \epsilon) x = 0, \quad (6.29)
\]

If we expand \( \mu^2 \epsilon \) in a Fourier series
\[
\mu^2 \epsilon = \sum_{k=0}^{\infty} \left\{ a_k \cos k\theta + b_k \sin k\theta \right\}
\]

and use (3.41) or (3.42) to determine \( \cos(\nu_x + \delta \nu_x)2\pi \), we find
\[
\cos(\nu_x + \delta \nu_x)2\pi = \cos\sqrt{\frac{\nu_x^2}{2} + a_0^{2\pi}} - \sum_{k=1}^{\infty} \frac{a_k^{2} + b_k^{2}}{k^2 - 4(\nu_x^2 + a_0^2)} \]

\[ \approx \frac{\text{Ra}_k}{k^2} \sqrt{\frac{\gamma}{k_1}} \quad (6.30) \]
or since \( \cos(\nu_x + \delta \nu_x) 2\pi \sim \cos \nu_x 2\pi - \delta \nu_x 2\pi \sin \nu_x 2\pi \) we have

\[
\delta \nu_x = \frac{a_0}{2\nu_x} + \sum_{k=1}^{\infty} \frac{a_k^2 + b_k^2}{4\nu_x \{k^2 - (2\nu_x)^2\}}, \quad 2\nu_x \neq k. \tag{6.31}
\]

We next determine the width of the stop band \( \Delta \nu_x \) which is done by substituting \((\nu_x + \Delta \nu_x) 2\pi = k\pi (k = \text{integer})\) in the left-hand side of (6.30) and approximating \( \cos \sqrt{\nu_x^2 + a_0^2} \) and \( \sin \sqrt{\nu_x^2 + a_0^2} \) by

\[
\cos \sqrt{\nu_x^2 + a_0^2} = (-1)^k \frac{a_0}{2\nu_x} - \Delta \nu_x^2 2\pi^2,
\]

\[
\sin \sqrt{\nu_x^2 + a_0^2} = (-1)^k \frac{a_0}{2\nu_x} - \Delta \nu_x 2\pi.
\]

The width of the stop band is therefore

\[
2 \Delta \nu_x = \frac{1}{2\nu_x} \sqrt{a_0^2 + b_0^2} \tag{6.32}
\]

We now discuss the beat phenomena outside the stop band. We write the solution of (6.28) in the form

\[
x = F(s) x^+ + \frac{i\nu}{\chi} \psi^{(s)} = (F_0 + \Delta F) e^{+i\nu/\chi} \psi^{(s)}, \tag{6.33}
\]

where \( F_0 \) is the amplitude function corresponding to the unperturbed system. Substitution of (6.33) in (6.28) and making use of (6.10) gives

\[
\frac{d^2 \Delta F}{ds^2} + \frac{3\nu_x^2}{F} \left[ 1 + \frac{\Delta F}{F_0} + \frac{1}{3} \left( \frac{\Delta F}{F_0} \right)^2 \right] \frac{\Delta F}{F_0} + f(s) \Delta F = -\frac{\epsilon}{\rho^2} (F_0 + \Delta F).
\]

Introducing the new variable

\[
\psi = \int \frac{1}{F_0^2} ds
\]

and neglecting terms of higher order in \( \Delta F \) and \( \epsilon \) we obtain

\[
\frac{d^2}{dy^2} \frac{(\Delta F)}{F_0} + (2\nu_x)^2 \frac{(\Delta F)}{F_0} = -\frac{\epsilon}{\rho^2} F_0^4 - \mu^2 \epsilon, \tag{6.34}
\]
where \( \nu_x \) is now a slowly varying function of \( \theta \). Equation (6.34) is similar in form to (6.15) so that we may write

\[
\frac{\Delta F}{F_0 \max} = \frac{\Delta x}{x \max} \sim 0.9 \sqrt{\frac{a^2 + b^2}{2\nu_x^2}} \sqrt{\frac{2\pi}{2k_1}} ,
\]

(6.35)

where

\[
a_k = \frac{1}{\pi} \int_0^2 \mu^2 \epsilon \cos k\theta \, d\theta ,
\]

\[
b_k = \frac{1}{\pi} \int_0^2 \mu^2 \epsilon \sin k\theta \, d\theta , \quad k = 2\nu_x .
\]

We see from (6.32) and (6.34) that the width of the stop band and the beat phenomena outside the stop band are mainly determined by the Fourier component of \( \mu^2 \epsilon \), whose order is nearest to \( 2\nu \). However, we shall now show that the motion is in general sensitive to Fourier components of order \( 2\nu/n = k \) (\( n \) and \( k \) integers). To this end we retain in the Fourier series expansion of \( \mu^2 \epsilon \) only the component

\[a_k \cos k\theta + b_k \sin k\theta .\]

Introducing the variable

\[t = \frac{1}{2}(k\theta - \tan^{-1} \frac{b_k}{a_k})\]

we obtain from (6.19) Mathieu's equation in standard form

\[
\frac{d^2x}{dt^2} + (a + 16 q \cos 2t) x = 0 ,
\]

(6.36)

where

\[a = (\frac{2\nu}{k})^2 = n^2 \]

\[q = \frac{1}{4k^2} \sqrt{a^2 + b^2} \]

(6.37)

For \( n \) is an integer the solution of (6.36) are unstable if \( q \) is small but not zero. For small values of \( q \), the width of the unstable region is approximately proportional to \( q^n \). It is therefore not likely that the Fourier component of \( \mu^2 \epsilon \) whose order is close to \( 2\nu/n \) (\( n = 2,3,... \))
should cause serious build up, since the width of the stop band is not only much smaller, but the beat phenomena is also negligible.

We now consider the exponential increase of the oscillations amplitude when the unstable region is traversed at a certain rate. Following Whittaker we write the quasi-periodic solution of (6.36) in the form

\[ y = e^{\mu t} \phi(t). \]

For the first unstable region \((n = 1)\) we have

\[ \phi(t) = \sin(t - \tau) + a_3 \cos(3t - \tau) + b_3 \sin(3t - \tau) + \ldots \]

We can now express \(\mu\) and \(a\) in terms of \(q\) and \(\tau\).

\[ \mu = 4q \sin 2\tau - 12 q^3 \sin 2\tau + \ldots \]

\[ a = 1 + 8 q \cos 2\tau + q^2 (-16 + 8 \cos 4\tau) + \ldots \quad (6.38) \]

Neglecting in (6.38) terms of second and higher orders in \(q\) and expanding \(\nu_x\) in the Taylor series (see (6.20))

\[ \nu_x = \frac{1}{2}k + k_1\theta = \frac{1}{2}k + 2 \frac{k_1}{k}t \]

we obtain

\[ \mu = \frac{1}{k^2} (a_k^2 + b_k^2) \frac{\nu_x^2}{2} \left(1 - \frac{(4k_1t)^2}{a_k^2 + b_k^2} \right). \quad (6.39) \]

The ratio between the oscillations amplitudes before and after passing the stop band is therefore

\[ \exp \left[ \frac{\pi}{8} \frac{a_k^2 + b_k^2}{k_1k^2} \right], \quad k = 2\nu_x. \quad (6.40) \]

The total increase of the amplitude of oscillation is then given by

\[ \left\{ 1 + \frac{0.9}{2\nu_x} \left(\frac{a_{2\nu_x}^2 + b_{2\nu_x}^2}{2k_1} \right)^{\frac{1}{2}} \right\} \exp \frac{\pi}{8} \frac{a_{2\nu_x}^2 + b_{2\nu_x}^2}{4k_1\nu_x^2}. \quad (6.41) \]

The present of a component of the magnetic field along the s-axis introduces linear coupling between the horizontal and vertical oscillations (18). The linearized equations of motion are then found to be
\[ x'' + \frac{1-n}{\rho^2} x = \frac{1}{\rho B} B_z z' \]
\[ z'' + \frac{n}{\rho^2} z = -\frac{1}{\rho B} B_x x' \]  \hspace{1cm} (6.42)

Making use of the relations (6.8) to (6.15) the equation (6.42) is transformed to

\[ \frac{d^2 x}{d \theta^2} + \nu_x x = \frac{B_x}{B} \frac{dz}{d \theta} \]  \hspace{1cm} (6.43)
\[ \frac{d^2 z}{d \theta^2} + \nu_z z = -\frac{B_x}{B} \frac{dx}{d \theta} \]  \hspace{1cm} (6.44)

where \( B_x \) is the s-component of the magnetic field on the equilibrium orbit. The solutions of (6.43) and (6.44) can be obtained by iteration. We may start the iteration with the solution of (6.43) ignoring the right-hand side. Expanding \( \nu_x \) in the Taylor series \( \nu_x = \nu_{x_0} + k_x \theta \) we obtain by means of (6.18) and (6.19) the approximate solution

\[ x = A \cos \left( \nu_{x_0} \theta + \frac{1}{2} k_x \theta^2 \right) + B \sin \left( \nu_{x_0} \theta + \frac{1}{2} k_x \theta^2 \right), \]  \hspace{1cm} (6.45)

where \( A \) and \( B \) are constants. If we insert (6.45) in (6.43), expand the right-hand side in a Fourier series and retain the dangerous harmonic only, then we get

\[ \frac{d^2 z}{d \theta^2} + \nu_z z = \epsilon \left( A^2 + B^2 \right)^{1/2} \left( \nu_{x_0} + k_x \theta \right) \cos \left\{ \frac{1}{2} \nu_{x_0} \theta \pm \frac{1}{2} k_x \theta^2 \right\}, \]  \hspace{1cm} (6.46)

where for convenience we have ignored a phase term on the right-hand side of (6.46). The quantity \( \epsilon \) is given by

\[ \epsilon^2 = \frac{1}{\pi^2} \int_0^{2\pi} \mu B_x e^{i1\theta} d\theta \int_0^{2\pi} \mu B_x e^{-i1\theta} d\theta \]

Expanding \( \nu_z \) in the Taylor series \( \nu_z = \nu_{z_0} + k_z \theta \) we can, in the same way as we have established (6.27), show that
\[
\left( \frac{z}{(A^2 + B^2)^{1/2}} \right)_{\text{max}} = \left\{ \begin{array}{ll}
\frac{c^2 z}{(A^2 + B^2)^{1/2}} & \text{if } \nu_{x_0} + \nu_{z_0} = \text{integer}, \\
\nu_{x_0} \in \sqrt{\frac{\pi}{k_1}} & \text{if } \nu_{x_0} \in \sqrt{\frac{\pi}{k_1}}, (6.47)
\end{array} \right.
\]

where

\[k_1 = \left| k_x + k_y \right| \text{ if } \nu_{x_0} + \nu_{z_0} = \text{integer}, \]

\[k_1 = \left| k_x - k_y \right| \text{ if } \nu_{x_0} - \nu_{z_0} = \text{integer}. \] (6.48)

The difference resonance ($\nu_x - \nu_z = \text{integer}$) cannot lead to instability. However, in a cyclotron we have generally $\nu_x > \nu_z$ so that the axial amplitude may become very large.

In section 8, in connection with the design of the 12 Mev cyclotron, numerical results of the passage through the resonances will be given.
7. Nonlinear effects

Nonlinear effects were extensively studied in connection with the Ag synchrotron by Hagedorn, Moser and Sturrock. Their results may be summarized as follows:

Instabilities arise due to quadratic terms in the equations of motion if 
\[ 3\nu_x + 2\nu_z \] or 
\[ 3\nu_z \] or 
\[ 2\nu_x + \nu_z = \text{integer}. \]

If 
\[ 4\nu_x + 2\nu_z \] or 
\[ 4\nu_z \] or 
\[ \nu_x + 3\nu_z \] or 
\[ 3\nu_x + \nu_z = \text{integer}, \]
cubic terms may produce instabilities or not depending on the relative magnitude of certain coefficients. Fourth and higher orders terms in the equations of motion are harmless. The isochronous cyclotron is a strongly nonlinear machine unlike the AG synchrotron, which is in principle designed to be linear. On the other hand, however, the resonances are crossed only once during the acceleration.

We shall for the time being ignore the nonlinear coupling and restrict our attention to the passage through the subharmonic resonances 
\[ \nu_x = N/3, N/4, \ldots \]. Neglecting in (1.13) the coupling terms we obtain by means of the Euler-Lagrange equation (1.14)

\[
x'' + \frac{1-n}{\rho^2} x = -(1-2n+\frac{1}{2}n_1) \frac{x^2}{\rho^3} + \left(n-n_1 + \frac{1}{6} n_2 \right) \frac{x^3}{\rho^4} + \frac{x'}{2\rho} \left(1 - \frac{x}{\rho} \right) - \frac{\rho'}{\rho^2} xx' \left(1 - \frac{x}{\rho} \right) - \frac{3}{2}(1-n) \frac{xx'^2}{\rho^2},
\]  

(7.1)

where

\[ n = - \frac{\rho}{B} \left( \frac{\partial B}{\partial x} \right)_0, n_1 = \frac{\rho^2}{B} \left( \frac{\partial^2 B}{\partial x^2} \right)_0, n_2 = - \frac{\rho^3}{B} \left( \frac{\partial^3 B}{\partial x^3} \right)_0. \]

Making use of the relations (6.9) to (6.14) and introducing the dimensionless variable \( \varsigma = \frac{x}{r} \), equation (7.1) transforms to

\[
\frac{d^2\varsigma}{d\theta^2} + \nu_x^2 \varsigma = -(1 - 2n + \frac{1}{2}n_1) \mu^3 \varsigma^2 + \left(n-n_1 + \frac{1}{6} n_2 \right) \mu^4 \varsigma^3
\]

\[
+ \frac{\nu_x}{\mu} \left(1 - \mu \varsigma \right) \left( \frac{d\varsigma}{d\theta} \right)^2 + \frac{\nu_x}{\mu} \left(1 - \mu \varsigma \right) \varsigma \left( \frac{d\varsigma}{d\theta} \right) \left( \frac{d\varsigma}{d\theta} \right)^2 + \frac{3}{2}(1-n) \mu^2 \varsigma \left( \frac{d\varsigma}{d\theta} \right)^2 ,
\]

(7.2)

To consider the differential equation (7.2) it is convenient to use the polar variables A and \( \psi \) defined by
The equivalent first order equations of (7.2) can then be written in the form

\[
\frac{dA^2}{d\theta} = \frac{A^3}{2} \left[ - \left( \frac{1-2n+n^2}{\nu_x} \right) \mu^3 - \left( \frac{3}{2} \nu_x \right) \sin^3 \psi - \frac{1}{2} \left( \frac{2n-n_1+n_2}{\nu_x} \right) \mu^3 \right] \cos \psi
\]

\[
- \frac{d\mu}{d\theta} \left( \cos 3\psi - \cos \psi \right) + \frac{A^4}{4 \nu_x} \left[ \left( \frac{n-n_1+n_2}{\nu_x} \right) \mu^3 + \frac{1}{2} \mu^2 \nu_x \left( 2 - \frac{3}{2} n \right) \right] \sin 4\psi
\]

\[
+ \left( \frac{1-2n+n^2}{\nu_x} \right) \mu^4 - \left( 4 - 3n \right) \mu^2 \nu_x \right] \sin 2\psi - \mu \frac{d\mu}{d\theta} \left( 1 - \cos 4\psi \right), \quad (7.3)
\]

\[
\frac{d\psi}{d\theta} = -\nu_x \left[ \frac{1-2n+n^2}{\nu_x} \right] \mu^3 + \frac{3\mu^3}{2} - \nu_x \nu_x \left( \frac{2n-n_1+n_2}{\nu_x} \right) \mu^3 \cos \psi
\]

\[
+ \frac{d\mu}{d\theta} \left( \sin 3\psi + \sin \psi \right) + \frac{A^2}{4 \nu_x} \left[ \left( \frac{n-n_1+n_2}{\nu_x} \right) \mu^4 + \frac{1}{2} \mu^2 \nu_x \left( 2 - \frac{3}{2} n \right) \right] \cos 4\psi
\]

\[
+ \frac{4n-n_1+n_2}{\nu_x} \mu^4 \cos 2\psi - \left[ \frac{3n-n_1+n_2}{\nu_x} \right] \mu^4 \left( 2 \frac{3}{2} n \right) \mu^2 \nu_x \right] \cos 4\psi
\]

\[
- \mu \frac{d\mu}{d\theta} \left( \sin 4\psi + 2 \sin 2\psi \right) \right], \quad (7.4)
\]

Since \( \mu, n, n_1 \) and \( n_2 \) are periodic in \( \theta \), the right-hand side of (7.3) and (7.4) can be expanded in the form

\[
\frac{dA^2}{d\theta} = \frac{A^3}{4} \sum_{k=-\infty}^{\infty} \left\{ \alpha_k \sin(3\psi + kn\theta + \alpha_k) + C_k \sin(\psi + kn\theta + \gamma_k) \right\}
\]

\[
+ \frac{A^4}{8} \sum_{k=-\infty}^{\infty} \left\{ b_k \sin(4\psi + kn\theta + \beta_k) + d_k \sin(2\psi + kn\theta + \delta_k) \right\}
\]

\[
- \frac{A^4}{4} \sum_{k=1}^{\infty} \epsilon_k \sin(kn\theta + \epsilon_k), \quad (7.5)
\]

\[
\frac{d\psi}{d\theta} = -\nu_x \sin(3\psi + kn\theta + \alpha_k) \right\} + C_k \cos(\psi + kn\theta + \gamma_k) \right\}
\]

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\[ + \frac{1}{16} A^2 \sum_{k=-\infty}^{\infty} \{ b_k \cos(4\psi + kn\theta + \beta_k) + d_k \cos(2\psi + kn\theta + \delta_k) \} \]

\[ - \frac{1}{8} A^2 \sum_{k=0}^{\infty} - e_k \cos(kn\theta + \epsilon_k) \]  

(7.6)

We see from (7.5) that the amplitude will increase considerably if

\[ \psi = \text{close to } \frac{kN\theta}{3} \text{ or } kN\theta \text{ or } \frac{kN\theta}{2} \text{ or } \frac{kN\theta}{4} \text{ (} k = \pm 1, \pm 2, \ldots \). \]

Let us now assume that \( \psi \) is close to \(-N\theta/3\). If we consider a large number of revolutions the mean contribution of the non-resonance term is zero. Neglecting these terms and introducing the new angular variable \( \psi_1 = \psi + \frac{N\theta}{3} + \frac{v}{3} \) we obtain the simple equations

\[ \frac{dA^2}{d\theta} = \frac{1}{4} a_1 A^3 \sin 3\psi_1 \]  

(7.7)

\[ \frac{d\psi_1}{d\theta} = \frac{N}{3} - \nu_x + \frac{1}{8} a_1 A \cos 3\psi_1 - \frac{1}{8} e_0 \cos \epsilon_0 A^2 \]  

(7.8)

These equation yield the Hamiltonian

\[ H(A^2, \psi_1) = \left( \nu_x - \frac{N}{3} \right) A^2 - \frac{1}{12} a_1 A^3 \cos 3\psi_1 + \frac{1}{16} e_0 \cos \epsilon_0 A^4 \]  

(7.9)

It follows from (7.7) that the amplitude \( A \) has its extreme values for

\[ \sin 3\psi_1 = 0 \text{ or } \cos 3\psi_1 = \pm 1. \]

In fig. 8 a to f we have plotted the quantity

\[ \frac{16H}{e_0 \cos \epsilon_0} = A^4 + 4 \frac{a_1}{3 e_0 \cos \epsilon_0} A^3 + \frac{16(\nu_x - \frac{N}{3}) A^2}{e_0 \cos \epsilon_0} \]  

(7.10)

as a function of \( A \) for various values of the parameter

\[ \lambda = \frac{16(\nu_x - \frac{N}{3})}{e_0 \cos \epsilon_0} \]  

(7.11)
Fig. 8
Fig. 9
The corresponding phase-plane diagrams are shown in fig. 9a to f. It should be noted that the flow pattern of the configuration points near the origin reverses if \( \nu_x - \frac{N}{3} \) changes sign. It is clear from the figures that the coefficient \( e_0 \) has a stabilizing effect.

In an isochronous cyclotron \( \nu_x \) increases with the energy of the particle and it is therefore interesting to consider the build-up of the oscillations amplitude if \( \nu_x \) is swept through the resonance.

In order to simplify the discussion we assume that \( e_0 = 0 \). The equations (7.7) to (7.9) then reduce to

\[
\frac{dA^2}{d\theta} = \frac{1}{4} a_1 A^3 \sin 3\psi ,
\]

\[
\frac{d\psi}{d\theta} = \left( \frac{N}{3} - \nu_x \right) + \frac{1}{8} a_1 A \cos 3\psi ,
\]

\[
H = \left( \nu_x - \frac{N}{3} \right) A^2 - \frac{1}{12} a_1 A^3 \cos 3\psi ,
\]

where we have dropped the suffix 1.

It is not difficult to see from fig. 8 e and f that for fixed value of \( \nu_x \), the motion in the vicinity of the resonance \( N/3 \) is always stable if \( A < A_m \), always unstable if \( A > A_m \). If \( A_m \leq A \leq A_M \) the phase determines whether the motion is stable, or unstable.

The quantities \( A_m \) and \( A_M \) are given by

\[
A_m = 4 \left| \frac{\nu_x - N/3}{a_1} \right|
\]

\[
A_M = 8 \left| \frac{\nu_x - N/3}{a_1} \right|
\]

We now assume that \( \nu_x \) increases linearly with \( \theta \) i.e.

\[
\nu_x = \frac{N}{3} + k_1 \theta ,
\]

where \( k_1 \ll 1 \) (see (6.20)).
Fig. 10 shows the boundary of the stable and unstable region. In order to obtain a conservative estimate of the largest amplitude that can be tolerated for a stable passage through the subharmonic resonance \( N/3 \) we replace in (7.13) the sine by unity and obtain on integration

\[
A_i = \frac{A_i}{1 - \frac{(a_1A_i)^2}{32k_1} - \frac{a_1A_i}{8} \theta}, \quad (7.16)
\]

where terms of second and higher orders in \( k_1 \) are neglected and where \( A_i \) is the oscillation amplitude at the boundary between the region of certain stability and possible stability. It is obvious from fig. 10 that on passing the subharmonics resonance \( N/3 \) the motion remains stable if \( A = A(\theta) \) as defined by (7.16) intersects the line \( A = \frac{4k_1\theta}{a_1} \).

Thus for a stable passage through the resonance \( \nu_x = N/3 \) we have the requirement

\[
\left(\frac{4}{a_1A_i} - \frac{a_1A_i}{8k_1}\right)^2 \geq \frac{2}{k_1}
\]

which yields for \( A_i \)

\[
A_i \leq \frac{4(2 - \sqrt{2})}{a_1} \sqrt{k_1} \approx 2.4 \frac{\sqrt{k_1}}{a_1}. \quad (7.17)
\]

In the case that the motion is such that it starts with \( \nu_x = N/3 \) (7.16)
reduces to

\[ A = \frac{A_1}{1 - \frac{a_1 A_1}{8} \theta} \]

and the motion of the particle will become stable if

\[ A_1 \leq \frac{2 \sqrt{2} k_1}{a_1} \]

(7.18)

We next consider the subharmonic resonance of order N/4. Using the same method as above, we arrive at the following equations

\[ \frac{dA^2}{d\theta} = \frac{1}{8} b_1 A^4 \sin 4\psi_1 \]

(7.19)

\[ \frac{d\psi}{d\theta} = N - \nu x - \frac{1}{16} (2e_0 \cos \epsilon_0 b_1 \cos 4\psi_1) A^2 \]

(7.20)

\[ H = (\nu x - \frac{N}{4} A^2 + \frac{1}{32} (2e_0 \cos \epsilon_0 b_1 \cos 4\psi_1) A^4) \]

(7.21)
In fig. 11 we plotted $H$ as a function of $A$. Fig. 11a for the case $\nu_x < N/4$ and fig. 11b for the case $\nu_x > N/4$. It is clear from these figures that, if $2\epsilon_0 \cos \phi_0 > |b_1|$, the motion remains stable even if the passage through the resonance is very slow.

We shall now consider the case that $2\epsilon_0 \cos \phi_0 < |b_1|$ and to simplify the discussion we assume $\epsilon_0 = 0$. It is then not difficult to verify that in the case $\nu_x$ is constant the subharmonic resonance $N/4$ is always stable if

$$A^2 < \frac{16}{1 + \sqrt{2}} \left| \frac{\nu_x - N/4}{b_1} \right| .$$

Replacing in (7.19) $\sin 4\psi_1$ by unity we find in the same way as we have established (7.16)

$$A^2 = \frac{A_1^2}{1 - \frac{1 + \sqrt{2}}{128} \frac{(b_1 A_1^2)^2}{k_1} - \frac{b_1 A_1^2}{8} \theta} .$$

Thus for harmless crossing of the subharmonic resonance $N/4$ we have the requirement

$$\left( \frac{8}{b_1 A_1^2} - \frac{1 + \sqrt{2}}{16 k_1} b_1 A_1^2 \right)^2 \geq \frac{2(1 + \sqrt{2})}{k_1}$$

which yields the relation

$$A_1^2 \leq \frac{8(2 - \sqrt{2}) \sqrt{k_1}}{\sqrt{1 + \sqrt{2}} b_1} \simeq 3 \frac{\sqrt{k_1}}{b_1} . \quad (7.22)$$

In the case that the motion starts with $\nu_x = N/4$ we have

$$A_1^2 \leq \frac{4 \sqrt{2 k_1}}{b_1 \sqrt{1 + \sqrt{2}}} \simeq 3.6 \frac{\sqrt{k_1}}{b_1} . \quad (7.23)$$

Since $k_1$ is a small number ($\sim 10^{-4}$, see (6.20)), we see by comparison of (7.17) and (7.22) that the permissible amplitude for passage through the subharmonic resonance $N/4$ is considerably larger than that for the passage through the resonance $N/3$. This leads to the important conclusion that the permissible amplitude for stable passage through subharmonic resonances increases with the order of the corresponding nonlinear term.

The expansion (1.13) of the variational function contains also terms
involving both 'off-axis' coordinates. It is well-known that these terms may cause nonlinear coupling resonance if (5,7,14)

\[ k \nu_x + l \nu_z = p, \quad (7.24) \]

where \( k, l \) and \( p \) are integers. Because of the symmetry of the magnetic field about the median plane all terms are of even order in \( z \). Moreover, in a cyclotron we have generally \( \nu_z < 0.5 \). Examination of the resonance lines on the \((\nu_x, \nu_z)\)-plane\(^6\) then shows that we may confine our attention to the coupling resonance

\[ 2\nu_x + 2\nu_z = N \quad (N = 3,4) . \quad (7.25) \]

Introducing the amplitude variables

\[ A_x^2 = \nu_x \left\{ z^2 + \left( \frac{1}{\nu_x} \frac{d \zeta}{d \theta} \right)^2 \right\} \]

\[ A_z^2 = \nu_z \left\{ z^2 + \left( \frac{1}{\nu_z} \frac{d \zeta}{d \theta} \right)^2 \right\} \]

and the angular variables

\[ \psi_x = \tan^{-1} \frac{1}{\nu_x} \frac{d \zeta}{d \theta} + \nu_x \theta + \gamma_1 \]

\[ \psi_z = \tan^{-1} \frac{1}{\nu_z} \frac{d \zeta}{d \theta} + \nu_z \theta + \gamma_2 , \]

where \( \gamma_1 \) and \( \gamma_2 \) are constant phase angles, we may show that

\[ \frac{dA_x^2}{d\theta} = c \frac{\nu_x}{8} A_x^2 A_z^2 \sin 2 (\psi_x + \psi_z) , \quad (7.26) \]

\[ \frac{dA_z^2}{d\theta} = c \frac{\nu_z}{8} A_x^2 A_z^2 \sin 2 (\psi_x + \psi_z) , \quad (7.27) \]

\[ \frac{d\psi_x}{d\theta} = \nu_x \theta_0 - \nu_x + \frac{1}{c} A_z^2 \cos 2 (\psi_x + \psi_z) , \quad (7.28) \]

\[ \frac{d\psi_z}{d\theta} = \nu_z \theta_0 - \nu_z + \frac{1}{c} A_x^2 \cos 2 (\psi_x + \psi_z) , \quad (7.29) \]
\[ H = (\nu_x - \nu_{x0})A_x^2 + (\nu_z - \nu_{z0})A_z^2 - \frac{c}{16}A_x^2A_z^2 \cos 2(\psi_x + \psi_z), \]  \hspace{1cm} (7.30)

It follows from (7.26) and (7.27) that

\[ E = A_x^2 - A_z^2 \]  \hspace{1cm} (7.31)

is an invariant. In principle we may, by eliminating either \( A_x \) or \( A_z \), reduce the two-dimensional problem to an one-dimensional problem with \( E \) as a parameter. In the simple case that \( E = 0 \) we obtain for a save passing through the resonance

\[ A_{x1}^2 = A_{z1}^2 \leq \frac{8(\sqrt{2} - 1)}{\sqrt{1 + \sqrt{2}}} \frac{\sqrt{k_1}}{c} \approx 2.1 \frac{\sqrt{k_1}}{c}, \]  \hspace{1cm} (7.32)

where \( k_1 = | k_x + k_z | \) (see (6.47)).

Finally we consider the coupling resonance

\[ \nu_x - 2\nu_z = 0 \]  \hspace{1cm} (7.33)

which in conventional cyclotrons and synchro-cyclotrons causes the blow up of the beam. Since for small values of the energy, \( \nu_x \) is very close to unity, the coupling resonance (7.33) may cause trouble if \( \nu_z \approx 0.5 \). The easiest way to treat this problem is to write the axial equation of motion in the form of a Mathieu equation. Substitution of (1.13) in the second equation of (1.14) and retaining only the most important terms, we obtain

\[ \frac{d^2Z}{ds^2} + nz = \frac{-2n+n_1}{\rho^3} xz. \]  \hspace{1cm} (7.34)

Making use of the relations (6.9) to (6.14) we may rewrite (7.34) in the form

\[ \frac{d^2\zeta}{d\theta^2} + \nu_z^2 \zeta = -a(\theta) \xi, \]  \hspace{1cm} (7.35)

where \( a(\theta) \) is now

\[ a(\theta) = \mu^3 (2n-n_1). \]

Writing the solution of the linearized equation for \( \xi \) in the form
\[ \xi = \xi_0 \cos (\nu_x \theta - \alpha) \]

and retaining in the Fourier expansion of \( a(\theta) \) only the term which is independent of \( \theta \), we obtain

\[ \frac{d^2 \xi}{d\theta^2} + \{ \nu_z^2 + a_0 \xi_0 \cos (\nu_x \theta - \alpha) \} \cdot \xi = 0. \quad (7.36) \]

Equation (7.36) can be transformed into the standard form

\[ \frac{d^2 \gamma}{dt^2} + (p + 16q \cos 2t) \cdot \xi = 0, \quad (7.37) \]

where \( p = \left( \frac{2\nu_z}{\nu_x} \right)^2 \) and \( q = \left( \frac{1}{2\nu_x} \right)^2 a_0 \xi_0 \).

If \( p \) is the square of an integer and if \( q \) is small but not zero the solution of (7.37) is not stable. For \( p = 1 \) and \( \nu_x = 1 \) we find from (6.32) that the width of the unstable region is given by

\[ |2\nu_z - \nu_x| = a_0 \xi_0, \quad (7.38) \]

We see from this relation that the motion in the vicinity of the coupling resonance (7.33) is stable as long as

\[ \xi_0 < \left| \frac{2\nu_z - \nu_x}{a_0} \right|, \quad (7.39) \]

where

\[ a_0 = \frac{1}{2\pi} \int_0^{2\pi} \mu^3 (2n - n_1) d\theta. \]
V. APPLICATIONS

8. The Delft 12 MeV cyclotron

By way of illustration we shall give a brief account of the design of a 12 MeV radial-sector machine, which has been built at the Technological University of Delft.

In section 3 is shown that for a stable vertical motion the flutter function $F(R)$ must initially increase as $R$. However, we have shown in section 4 that $F(R)$ increase initially as $R^N$ so that there is always a central region where we have vertical defocusing. There are two possibilities to obtain vertical focusing at small radii: (a) electric focusing of the beam; (b) abandonment of isochronism.

(a) Electric focusing can be achieved by the use of focusing grids as has been done in the proton linear accelerator before the advent of quadrupole focusing. This method is only applicable if the spacing between successive orbits is large enough so that the beam does not strike the grid. Since in our cyclotron the rf voltage is relatively low we must discard this method.

(b) The only possibility left for obtaining a stable vertical motion is the abandonment of isochronism at small radii. In our first approach the mean field in the central region was made to decrease slightly with increasing radius. As a result large displacements of the equilibrium orbit were observed. It is not difficult to verify by means of (3.34) that for $<B>$ almost constant, the radial betatron oscillation frequency is practically equal to unity and we have shown in section 6 that subharmonic errors in $B$ may cause large deviations of the equilibrium orbit. Furthermore resonant build-up may be introduced by subharmonic gradient errors. As a result of these investigations it becomes clear that it is necessary that $\nu_x$ differs somewhat from unity. In order to achieve this we made the magnetic field at the centre of the cyclotron considerably larger than the isochronous mean field. As a result of this $\nu_x$ differs enough from unity to prevent large deviations of the equilibrium orbit. We keep the slope of the curve $<B(R)>$ non isochronous until the edge focusing is sufficiently strong to render the vertical motion stable. It is obvious that the integral resonance $\nu_x = 1$ must then be traversed. In section 6 it has been shown that, if the resonance is crossed fast enough, the build-up is not serious.

It is clear from (3.33) that in view of the axial motion it is advantageous to choose $N$ as small as possible. However, in the neighbourhood of the integral resonance $\nu_x = 1$ the radial oscillation amplitude may be comparatively large so that the nonlinear effects cannot be neglected. Since for a three sector machine the subharmonic
resonance is of lower order \( \nu_x = N/3 \) than for a four sector machine \( \nu_x = N/4 \) we have chosen \( N = 4 \). At this point it may be useful to observe that if the energy gain per turn is large enough so that the particles spend only a short time (a few turns) in the axial defocusing region, the choice \( N = 3 \) is to be preferred.

From fig. 7 we see that to accelerate protons up to 12 MeV terms of higher orders than the second can be neglected. The median plane field can then be written in the form

\[
B(r, \phi) = B_0 \left[ 1 + ar^2 + Ar \sum_{k=0}^{\infty} a_k \cos(2k+1)N \phi \right]
\]  

(8.1)

A good vertical focusing of the beam requires that \( A \) must be as large as possible as long as \( \nu_z \) is less than \( N/2 \). However, a large value of \( A \) gives a small useful gap height. A suitable compromise is obtained by choosing \( A = 1.5 \). As mentioned above it is impossible to keep \( a_0 = 1 \) at small radii. Preliminary calculations indicated that a constant value of \( a_0 \) can be achieved for \( r \geq 2z_m \). The parameter \( a \) can then be calculated from

\[
a = \frac{1}{2} - A^2 \sum_{k=0}^{\infty} a_k \beta_k \geq \frac{1}{2} - \frac{A^2}{N^2 - 1} = 0.35
\]

and the equation determining the height of the gap in the valleys reduces to

\[
d \left\{ 1 - \frac{0.7}{3} d^2 \right\} = z_m \left\{ 1 + 0.35 d^2 + \frac{0.7}{3} z_m^2 - \frac{\pi}{4} 1.5 a_0 (\lambda) r \right\}
\]

(8.2)

The height of the gap at the hill \( (\phi = 0) \) can be obtained by solving

\[
X = d \left[ 1 - \frac{0.7}{3} d^2 \right] - z \left[ 1 + 0.35 r^2 - \frac{0.7}{3} z^2 \right],
\]

\[
Y = z \sum_{k=0}^{\infty} 1.5 a_0 (\lambda) \frac{a_k(\lambda, z_m)}{a_k(\lambda)} r (-1)^k \frac{1}{2k+1}
\]

(8.3)

The results of these calculations are plotted in fig. 12. The height of the gap at the hill is then assumed to be independent of azimuth. Noting that \( B = - \left( \frac{\partial V}{\partial z} \right)_{z=0} = \frac{1}{r} \left( \frac{\partial G}{\partial y} \right)_{y=0} \), the magnitude of the fundamental and the higher harmonics of the flutter field can be calculated by means of (4.39).

\[
F(r) = \frac{\frac{4}{\pi} \Delta y (1 + \sum_{k=1}^{\infty} e_{ok})}{\sinh y_o \cosh \Delta y}
\]

(8.4)
\[ a_{2n+1} = (-1)^n \frac{\sinh y_0 \cosh \Delta y \left(1 + \sum_{k=1}^{\infty} e^{\kappa k} \right)}{\sinh (2n+1)y_0 \cosh(2n+1) \Delta y \left(1 + \sum_{k=1}^{\infty} e^{\kappa k} \right)} \]  

(8.5)

where \( y_0 = \frac{1}{2} N (\sinh^{-1} \frac{d_1}{r} + \sinh^{-1} \frac{d_2}{r}) \) and

\[ \Delta y = \frac{1}{2} N (\sinh^{-1} \frac{d_2}{r} - \sinh^{-1} \frac{d_1}{r}) \]

\( 2d_1 = \) height of the gap at the hill and \( 2d_2 = \) height of the gap at valley.

Fig. 13 shows a plot of the calculated and measured values of \( F(r) \) as a function of \( r \). The agreement is seen to be fairly good. For comparison the required value of \( F(r) \) for vertical motion that is just stable,
obtained by equating the right-hand side of (3.33) to zero, is included. This is a straight line, the slope of which is proportional to the central field \( B_0 = 1.4 \text{ Wb/m}^2 \). From this figure we see that in the case of an isochronous field the radius of the central region of defocusing is as large as 25 cm.

In fig. 14 the calculated and measured values of the third and fifth harmonics are plotted as a function of \( r \). A plot of the second harmonic after correction of the hill sector to reduce the even harmonics (see fig. 15) is included.
Fig. 16 shows a plot of $\frac{\langle B \rangle - B_{0i}}{B_{0i}}$ as a function of $R$, where $B_{0i}$ is the ideal (isochronous) central field. For comparison an isochronous curve of $\frac{\langle B \rangle - B_{0i}}{B_{0i}}$ is included. From this figure we see that the phase is first shifted negative and then back through zero to positive. To calculate the phase slip we plot in fig. 17 $\frac{\langle B \rangle - B_{0i}}{B_{0i}}$ as a function of $R^2$. The isochronous curve is then a straight line. We see from this figure that deviation of the magnetic field from the isochronous value at small radii is not serious. The maximum negative phase can be ob-
tained from (5.13).

\[
\sin \alpha_p = \sin \alpha_1 + \frac{\tau E_o}{2eV_0} \int_0^{R_p} \frac{<B> - B_{oi}}{B_{oi}} dR^2,
\]

(8.6)

where it should be remembered that in (8.6) \( R = \beta \).

The condition for the acceleration of the particles are

\[
\sin \alpha_p = \sin \alpha_1 + \frac{\tau E_o}{2eV_0} \int_0^{R_p} \frac{<B> - B_{oi}}{B_{oi}} dR^2 < 1
\]

and

\[
\sin \alpha_e = \sin \alpha_p + \frac{\tau E_o}{2eV_0} \int_{R_p}^{R_{max}} \frac{<B> - B_{oi}}{B_{oi}} dR^2 < 1.
\]

Fig. 17

Assuming that the initial phase \( \alpha_1 = 0 \) the rf voltage is a minimum (the threshold voltage) when \( 0_1 = \frac{4}{5}0_2 \) (see fig. 17). In connection with the deflection of the beam it is advantageous that the spacing between successive orbits is a maximum, which can be achieved by tuning the rf voltage so that \( 0_1 = 0_2 \). The required rf voltage is then nearly twice the threshold voltage.

Fig. 18 shows a plot of \( \nu_x \) as a function of \( R \). We see from this figure that \( \nu_x \) increases almost linearly with \( R \) at resonance.
With $eV = 25 \text{ keV}$ we have at $R = 15 \text{ cm}$

$$k_1 = \frac{d\nu_x}{d\theta} \left| \nu_x = 1 \right| = \frac{eV}{\pi E} \frac{E_0^2}{E^2 - E_0^2} R \frac{d\nu_x}{dR} \approx 2 \cdot 10^{-5}.$$  \hfill (8.7)

We are now in a position to consider the deviations of the equilibrium orbit and the build-up of the radial oscillation amplitude when the betatron oscillation frequency is swept through the integral resonance $\nu_x = 1$.

In section 6 we have shown that due to the subharmonic error in the magnetic field given by

$$\epsilon \cos \theta = \frac{\Delta B}{B} \cos \theta,$$

the maximum displacement of the equilibrium orbit, that can be expected, is

$$x_{\text{max}} = \frac{0.9 \epsilon}{\nu_x} R \sqrt{\frac{\pi}{k_1}}.$$

With $\epsilon = 10^{-3}, R = 15 \text{ cm}$ and $k_1$ given by (8.7) we get

$$x_{\text{max}} = 5.4 \text{ cm}.$$

It is thus apparent that $\frac{\Delta B}{B}$ must be reduced to considerable less than $10^{-3}$. The method of the rotating coils connected to a wave analyzer makes it possible to measure the relative amplitude of the first harmonic to less than $3 \times 10^{-4}$.

A subharmonic gradient error of the form $\epsilon x \cos 2\theta$, as we have shown
in section 6, increases the oscillation amplitude by a factor

\[ K = (1 + \frac{0.9\varepsilon}{\lambda^x} \exp \frac{\pi}{8k_1} \exp \frac{(\varepsilon)^2}{2i_x}) \]

With \( \varepsilon = 10^{-3} \) we get

\[ K \approx 1.1 \]

which is thus negligible.

The passage through the integral resonance \( \nu_x = 1 \) is also a passage through the subharmonic resonance of order \( N/4 \). Equation (7.22) gives for a stable traversal of the resonance the requirement

\[ A^2 \leq 3 \frac{\sqrt{b_1}}{b_1} \approx 1.3 b_1^{-1} 10^{-2} \]

From the field measurements (see fig. 19) we see that \( n_1 \approx 0 \) and \( n_2 \approx 0 \) at \( r = 15 \) cm. Using (2.37), (3.17), (3.25) and (3.26) we then obtain from (7.3) \( b_1 \approx 0.25 \) so that the traversal of the subharmonic resonance is no trouble at all as long as the radial oscillation amplitude is less than 3.5 cm.
9. Design of a 200 Mev spiral-ridge cyclotron

As a second example of the application of the theory which has been given in the preceding sections, we outline a preliminary design of a spiral-ridge cyclotron for the acceleration of protons up to 200 Mev. This is about the maximum energy obtainable with $N = 3$.

Fig. 20 shows a plot of the relation (8.4) for several values of $d_1$ and $d_2$: (a) for $N = 3$ and (b) for $N = 4$. Here $2d_1$ is the minimum gap height (hill sector) and $2d_2$ is the gap height in a valley. The dotted curves in these figures are the required minimum values of $F(r)$ for stable vertical motion, corresponding to a central field $B_0 = 1.4$ Wb/m$^2$. We see from these figures that the central region of defocusing is much smaller for a three sector machine than for a four sector machine. Furthermore we see that an increase of the gap height in the valleys has no effect on the flutter amplitude at small radii. It turns out that the only effective way to increase the flutter amplitude at small radii is by reducing the mean gap height keeping the difference between the gap height in the hills and valleys constant. However, this results in a decrease of the available gap height, so that some compromise must be sought.

We shall now derive a relation between the build-up of the axial oscillation amplitude and the extent of the central region of defocusing. We express this extent in terms of the particle energy $T_c$ at the boundary of it.

Substitution of (3.26) in (3.33) gives at small radii ($F^2 \ll \alpha - 1$)

$$\nu_z^2 \geq 1 - \alpha \approx -\frac{2T}{E_0}$$

so that a particle leaving the ion source with a vertical displacement $z_0$ reaches the boundary of the central region of defocusing with an amplitude

$$Z \leq z_0 \exp \int_0^\theta \sqrt{2T} \ d\theta = z_0 \exp \frac{2\pi T C}{3eV} \left(\frac{2T c}{E_0}\right)^{\frac{1}{2}}$$  \hspace{0.5cm} (9.1)

where $2eV$ is the energy gain per turn.

From fig. 20a we see that for $d_1 = \frac{1}{3}d_2 = 10$ cm and $N = 3$ the central region of defocusing extends to $r \approx 11.5$ cm ($T_c \approx 1.5$ Mev). With $V = 150$ kV, the amplitude increases to $3.3 z_0$. On the other hand, for $d_1 = \frac{1}{2}d_2 = 7.5$ cm the defocusing region extends only to $r \approx 7$ cm ($T_c \approx 0.55$ Mev).
fig. 20a

- $N = 3$
- $F(r)$
- $r(cm)$

- (1): $d_1 = 10\text{ cm}, d_2 = 15\text{ cm}$
- (2): $d_1 = 10\text{ cm}, d_2 = 20\text{ cm}$
- (3): $d_1 = 7.5\text{ cm}, d_2 = 15\text{ cm}$

fig. 20b

- $N = 4$
- $F(r)$
- $r(cm)$

- (1): $d_1 = 10\text{ cm}, d_2 = 15\text{ cm}$
- (2): $d_1 = 10\text{ cm}, d_2 = 20\text{ cm}$
- (3): $d_1 = 7.5\text{ cm}, d_2 = 15\text{ cm}$
With $V = 150$ kv the amplitude increases to $1.3 z_0$ only. In this way we may find the optimum gap height and the corresponding dee voltage. However, there are other requirements such as field tolerances and sufficient initial energy gain to clear the ion source, which render undesirable to choose a low energy gain per turn.

Returning to fig. 20a curve 3 we see that the axial focusing in the case of radial sectors becomes inadequate at large radii ($r > 65$ cm). Choosing the spiral angle $\zeta$ so that the relation

$$\tan \zeta = \frac{1}{F} \left( \frac{18\alpha(a-1)}{26 - 7\alpha} \right)^{1/2}$$

is satisfied, we find by inserting (3.26) in (3.45)

$$\nu_z^2 = \frac{F^2}{2\alpha} \left[ \sum_{k=0}^{\infty} \alpha_k^2 + \left\{ \frac{2 + \frac{RF'}{F}}{3} \right\}^2 \sum_{k=0}^{\infty} \left( \frac{\alpha_k}{2k+1} \right)^2 \right]$$

$$+ \frac{18(a-1)}{26-7\alpha} \left( \sum_{k=0}^{\infty} \alpha_k^2 - 1 \right).$$

To evaluate $\nu_x$ we substitute (9.2) in (3.41) or (3.44).

It turns out (from numerical calculations) that equation (3.44), which is merely a first approximation, gives too low values of $\nu_x$. The results of these calculations are shown in fig. 21.

![Graph](image)

The quantity $\psi(r)$ defined by (3.36) can be obtained by numerical integration of (9.2) which can be written
\[
\frac{d\psi}{dR} = \frac{\sqrt{18}}{FR} \left\{ \frac{a(a-1)}{26-7a} \right\}^{\frac{1}{2}}
\]

or
\[
\frac{d\psi}{dr} = \frac{\sqrt{9}}{13} \frac{1}{F} \sqrt{\left\{ \frac{r^2}{r^2+2} \right\}^{\frac{1}{2}} - \frac{19}{26} \left( r^2 \right)^{\frac{1}{2}}}.
\] (9.3)

This equation is not valid at \( r = 0 \) since we must have \( \frac{d\psi}{dr} \to 0 \) as \( r \to 0 \).

However, since spiralling of the sectors is only necessary at large radii, where \( F \) is approximately constant (see fig. 20a curve 3), we may overcome this difficulty by integrating (9.3) over the range \( r \geq 50 \text{ cm} \). The quantity \( \psi(r) \) can then be expressed in terms of elliptic integrals of the first kind
\[
\psi(r) = \frac{\sqrt{-9}}{13} \frac{1}{F} \left[ \bar{\Phi}(k, r) - \bar{\Phi}(k, 0.262) \right].
\] (9.4)

where \( \bar{\Phi}(k, x) = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad (k^2 = \frac{19}{16}) \).

Fig. 22 shows a plot of \( \psi \) as a function of \( r \). We see from fig. 21 that the working line crosses the difference resonance \( \nu_x - \nu_z = 1 \) which is not serious if the first order subharmonic error in the longitudinal component of the magnetic field is small enough.

The radial betatron oscillation frequency \( \nu_x \) is initially unity.
Nonlinear terms in the equation of motion may introduce the subharmonic resonance \( N/3 \). Since the particle starts the motion with \( \nu_x = N/3 \) formula (7.18) gives for a stable motion

\[
A = \frac{2\sqrt{2k_1}}{a_1}
\]  

(7.18)

From (3.34) we get

\[
k_1 \simeq \frac{eV}{\tau E_0} + \frac{eV}{\tau(E-E_0)} \frac{F}{2N} \left\{ \frac{3RdF}{F} + \left(\frac{R}{F}\right) \frac{dF}{dR} + \left(\frac{R}{F}\right)^2 \frac{d^2F}{dR^2} \right\}
\]

Using (2.37), (3.17), (3.25) and (3.26) and neglecting terms of higher than the first order in \( F \) we obtain from (7.3)

\[
a_1 \simeq -(3.5 - N) \frac{F - 2R}{dF/dR} - \frac{1}{2} R^2 \frac{d^2F}{dR^2}.
\]

Substituting these results in (7.18) and using the values of \( F \), \( \frac{dF}{dR} \) and \( \frac{d^2F}{dR^2} \) at \( R = 10 \text{ cm} \) we find that the radial motion is stable if the oscillation amplitude is less than 1 cm.

It has been assumed that the gap height in the hills and valleys is constant. Up to \( r \approx 70 \text{ cm} \) (48 Mev) the angular width of the hill and valley sectors can be taken approximately equal. Equations (8.4) and (8.5) give sufficient information about the flutter field. The required increase of the mean field in this region can be achieved using auxiliary coils.

At larger radii (\( r > 70 \text{ cm} \)) the required increase of the mean field can be obtained by increasing the angular width of the hill sectors. In this region \( r \gg d_2 \) and the problem of the field design can, to a sufficient degree of accuracy be reduced to a two dimensional problem. The Schwarz-Christoffel transformation can then be used to estimate the relative widths of the sectors. Since the labour involved in the determination of \( <B> \) is considerably reduced if the magnetic field varies discontinuously on the equilibrium orbit (hard-edge approximation), we shall consider a pole-profile of great 'hardness' (fig. 23). An additional advantage of this steel configuration is that a part of the auxiliary coils can be placed in the crevices.

Since the derivation of the Schwarz-Christoffel transformation is given at length in numerous sources, it suffices here to set forth the results without proof(22). With the dimensions of fig. 23 we find the parametric equations
Fig. 23

\[ x = b - \frac{d_1}{\pi} \ln \frac{(u+q)^{\frac{1}{2}} + (u+1)^{\frac{1}{2}}}{(u+q)^{\frac{1}{2}} - (u+1)^{\frac{1}{2}}} + \frac{d_2}{\pi} \ln \frac{(qu+q)^{\frac{1}{2}} + (u+q)^{\frac{1}{2}}}{(qu+q)^{\frac{1}{2}} - (u+q)^{\frac{1}{2}}} \]  

\[ - \frac{2b}{\pi} \tan^{-1} \left( \frac{q-p}{p-1} \right)^{\frac{1}{2}}, \quad 0 \leq u \leq \infty \]  

\[ B_z = B \frac{u+p}{(u+1)^{\frac{1}{2}} (u+q)^{\frac{1}{2}}} \]  

where \( x \) is defined by

\[ x = r \phi \cos \zeta, \]  

\( B \) is the flux density in the hill sectors, \( \zeta \) the spiral angle and

\[ p = m + \left( m^2 - \frac{(d_1)^2}{d_2} \right)^{\frac{1}{2}} \]  

\[ q = \left( \frac{d_2}{d_1} \right)^2 \]  

\[ m = \frac{1}{2}(1 + \frac{d_1^2 + b^2}{d_2^2}) \]  

Requiring that \( B_z \), be a monotonically decreasing function as \( x \) goes from \( -\infty \) to \( +\infty \), we obtain the relation

\[ 2q - qp - 1 = 0 \]  

Substituting in this equation the value of \( q \) from (9.9), solving for \( p \)
and comparing the result with the first equation of (9.9), we find

\[ b^2 = d_2^2 - d_1^2 \]

With \( d_2 = 2d_1 = 15 \text{ cm} \), we get \( b = d_1 \sqrt{3} \approx 13 \text{ cm} \).

Fig. 24 shows a plot of \( \frac{B_z}{B} \) as a function of \( x \). By means of (9.8) we may plot \( \frac{B_z}{B} \) as a function of \( \phi \) for several values of \( r \). We see from fig. 24 that the actual angular width of the hill sector must be smaller than the magnetic width (including fringing field).

In practice circular pole-face windings of a few thousands ampere-turns will be necessary to provide the final correction of the mean field and to adjust the magnetic median plane.

The table below lists the proposed design data:

<table>
<thead>
<tr>
<th>Number of sector pairs</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proton energy</td>
<td>200 Mev</td>
</tr>
<tr>
<td>Maximum particle mean radius</td>
<td>125 cm</td>
</tr>
<tr>
<td>Central flux density</td>
<td>1.4 wb/m²</td>
</tr>
<tr>
<td>Flux density in hill sector</td>
<td>2.0 wb/m²</td>
</tr>
<tr>
<td>Gap height in hills</td>
<td>15 cm</td>
</tr>
<tr>
<td>Gap height in valleys</td>
<td>30 cm</td>
</tr>
</tbody>
</table>

To complete this section, we consider what happens when the exciting current of the main coils is increased. The effect of saturation in the hill sectors becomes stronger. As a consequence of which the
relative increase of the mean field at large radii is smaller than the relative increase of the central field. It is thus in principle possible to adjust the exciting current so that \( \frac{\langle B \rangle}{B_0} \) satisfies roughly the condition for isochronism, for the acceleration of a particle with a smaller \( \frac{e}{m} \) ratio than that of a proton. The final isochronous field can then be obtained by the use of properly placed sector coils and the circular trimming coils. In this way the number of ampere turns in auxiliary coils required to increase the versatility of the machine, is very much reduced.
De omlooptijd van een deeltje, dat in een conventioneel cyclotron versneld wordt, zal om twee redenen toenemen:

1e. Het flux dichtheid $B$ neemt in radiale richting af. Dit is nodig voor een stabiele axiale beweging.

2e. Het toenemen van de snelheid van het deeltje gaat gepaard met een toeneming van de massa $m$.

Het deeltje raakt dus 'uit de pas' met het versnellende elektrische veld en kan slechts een beperkt aantal malen versneld worden, waardoor de bereikbare energie begrensd is. Teneinde aan dit bezwaar tegemoet te komen, kan men de frequentie van de wisselspanning op de versnellingselectroden moduleren, zodat het deeltje 'synchroon' met het elektrische veld kan rondlopen. In de zo ontstane machine, die synchrocyclotron wordt genoemd, worden de deeltjes in groepen versneld, waardoor de gemiddelde stroomsterkte vergeleken met die van het gewone cyclotron laag is. Een andere methode om aan het bezwaar van het 'uit de pas' raken tegemoet te komen zonder dat het nadeel van een lagere stroomsterkte optreedt, is reeds in 1938 door Thomas (20) aangegeven. Hij stelde voor om het magnetveld niet meer rotatie-symmetrisch te maken, maar om het azimuthaal sinusvormig te varieren. Hierdoor kan de beweging van de deeltjes axiaal stabiel worden gemaakt, terwijl toch de langs de banen gemiddelde waarde van de flux dichtheid in radiale richting toeneemt, zodat omlooptijd constant blijft. Machines van dit type noemt men isochrone cyclotrons.

In hoofdstuk II van het proefschrift wordt nagegaan welke eigenschappen een azimuthaal periodiek magnetcveld (met oneven hogere harmonischen) moet hebben om aan de eisen van constante omlooptijd van de deeltjes en een stabiele beweging in radiale en axiale richting te voldoen. Hoofdstuk III behandelt het ontwerpen en nameten van een magnetveld, dat aan bovengenoemde eisen voldoet. Verder wordt de invloed van afwijkingen van het magnetveld van de ideale waarde op de omlooptoegenaag aangegeven. Resonanties tengevolge van fouten in het magnetveld en niet-lineaire termen in de bewegingsvergelijkingen worden in hoofdstuk IV behandeld. In hoofdstuk V wordt de theorie toegepast op een machine van het radiaal-sector type, welke in Delft werd gebouwd. Ter verdere illustratie wordt een ontwerp voor een 200 Mev cyclotron van het spiraal-sector type besproken.
REFERENCES

STELLINGEN

I

Men kan experimenteel aantonen, dat de suggestie van Ganger, dat de elektrische doorslag tussen twee elektroden in een ruimte waarin een zeer lage druk heerst, veroorzaakt kan worden door Rontgenstraling, niet juist is.

B. Ganger, Der elektrische Durchschlag von Gassen.
Springer-Verlag, Berlin/Göttingen/Heidelberg (1953) 236.

II

De hypothese van Cranberg over het ontstaan van een ontlading in een ruimte, waarin een zeer lage gasdruk heerst en hoge elektrische spanningen, is niet juist.


III

Wanneer men in staat is om de radiale 'acceptance' van een proton synchrotron zowel met 'multiturn' als met 'single turn' injectie met de zelfde dichtheid-verdeling te vullen, dan verdient 'multiturn' injectie de voorkeur.

IV

Past men in een proton synchrotron 'multiturn' injectie toe, dan wordt de effectieve hoogte van de vacuumdoos hoofdzakelijk bepaald door het toelaatbare verlies t.g.v. botsingen van de protonen met de overblijvende gasdeeltjes.

V

De door Symon en Sessler gevonden kanonieke vergelijkingen voor de fase oscillaties van deeltjes in een versnellingsmachine kunnen direct uit de functie van Lagrange voor een geladen deeltje in een elektromagnetisch veld afgeleid worden.

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VI

Bij een magneet met symmetrisch juk kan men, door een geschikte keuze van de poollengte, het verloop van het veld in de luchtspleet nagenoeg onafhankelijk maken van de verzadiging van het ijzer.
Het zwaarder isoleren van de eindspoelen van de hoogspanningswikkeling van een transformator heeft alleen zin indien de verhouding van de seriecapaciteit tot de aardcapaciteit hierdoor niet kleiner wordt.

Een berekening van de verstrooing-verliezen door het resterende gas in een versnellingsmachine m.b.v. de eenvoudige verstrooilingsformule van Rutherford met goed gekozen maximum en minimum afwijkingshoek geeft resultaten, die beter met de gemeten waarden overeenkomen dan de formule van Courant.


De bewering van Cheng dat het kriterium van Routh-Hurwitz geen indicatie geeft over de mate van stabiliteit van een systeem kan tot een verkeerde interpretatie aanleiding geven.


Het bestaan van de wet van Parkinson kan grotendeels verklaard worden uit de 'ziekte van Parkinson: jalonbequitis'.


De ingenieur moet de wiskunde zonodig durven gebruiken zoals een verstandige huisvrouw haar elektrische huishoudelijke apparaten gebruikt.