The pricing of an Asian option on a basket of futures

(Nederlandse titel: “Het prijzen van Aziatische opties op baskets van futures”)
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“The pricing of an Asian option on a basket of futures”
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Preface

This thesis is the result of my Bachelor of Science research project at the Delft University of Technology, The Netherlands and it has been carried out at the faculty of Applied Mathematics (EEMCS) in Delft and at the Institute for Mathematics and Computer science (CWI) in Amsterdam. The goal of this project is to price Asian options using both numerical methods as well as approximating methods.

I would like to thank my supervisor professor Kees Oosterlee for the guidance he provided me with during the last half year. He gave me the chance to do my research at the CWI in Amsterdam, where I could enjoy an overall great working environment. Furthermore, I want to thank his colleague at the CWI, Lech Grzelak, who was prepared to answer any questions I had and help me with the Matlab codes.

Lastly, I hope you enjoy reading my thesis!

Anastasia Borovykh, June 2013
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1 Introduction

The financial market is a place where one can trade financial securities, commodities and many other financial products. It is the driving force for economic growth. A derivative is a product whose value is derived from another asset i.e., a stock or commodity. The total world derivatives market has been estimated at about $791 trillion face or nominal value\[7\], which is more than our total world economy. Since the earliest recorded transaction in history, markets have been extremely important, fast-moving and have always generated new types derivatives, to decrease risk or overcome other difficulties. It is necessary to have accurate pricing algorithms for all the derivatives traded on the stock market.

There are many types of non-standard options traded on the stock market. The focus in this thesis lies on the pricing of the Asian option on a basket of commodity futures with a fixed strike and a floating strike. Both are very popular types of exotic options. This thesis starts with an introduction into stocks, options and futures in chapter two. Next in chapter three, we explain the mathematical models used for simulating the stock and futures price.

The Asian option owes its name to its inventors, who were in Tokyo on business when they developed the first commercially used pricing formula for options linked to the average price of crude oil. Because they were in Asia, they called the option an Asian option. The main characteristic of the Asian option is its averaging feature (the price is averaged over a time interval using an arithmetic average), which results in less risk and a lower price compared to European and American options. Traders of commodities like oil or gold tend to be exposed to average prices over time, so Asian options may appeal to them. Furthermore, futures are also embraced by traders, since a future has a minimum investment and can minimize risk and maximize profit. Thus, an Asian option on a future is an extremely popular derivative, and therefore it is very interesting to look at different pricing algorithms.

One of the main formulas used for pricing an option on a future has always been the Black 76 formula, which is explained in chapter four. However, one important assumption made by this formula is that the underlying is driven by a lognormally distributed stochastic process. An assumption we make is that one future is lognormally distributed, however it is a known fact that the sum of more than one future (as is the case in an Asian option on a basket: the payoff depends on the arithmetic average of futures) does not follow a lognormal distribution. So, one way to price the Asian option on a basket of futures is to use Monte Carlo simulation, as is explained in chapter five. However, the Monte Carlo method can be slow, since it can take a large number of simulations to obtain a high-precision value. A solution to this problem can be to use approximating methods. These will work fast, but since they are approximating methods it is important to know under which circumstances the methods provide exact results and when they fail to do this. In chapter six we will explain and test four approximating methods, namely the Sophis method\[1\], the Gentle method\[1\], the Levy method\[9\] and the Curran method\[8\]. The Sophis method approximates the sum of lognormally distributed futures with a lognormally distributed variable and with a geometric average. Gentle approximates the arithmetic average by a geometric average, since the geometric average of lognormally distributed futures will remain lognormally distributed. Levy uses a moment matching method to match the lognormal distribution with the distribution of the arithmetic average. And finally, Curran separates the price of the option in two parts by conditioning. The first part can be evaluated exactly, and the second part is determined using the Levy approximation and numerical integration. Furthermore, we will find mathematical explanations for when these methods work well and when they fail.
so that in the future even more exact methods can be developed.

There are of course many more types of Asian options, than just the Asian option with a fixed strike. In chapter nine we choose to also take a look at an option with a floating strike, namely the Average Rate/Average Strike option. Using the previous approximating methods we developed a new approximating method to obtain an accurate price for this option in a fast way.

Finally, all simulations and implementations of the methods are done with Matlab, and the used .m-files are added in the appendix.
2 Stocks, Options and futures

This chapter will give an introduction into some products that are traded on the financial market: stocks, options and futures.

2.1 Stocks and bonds

A well known financial product is a stock. In simple terms, one could define a stock as a share of ownership in a company. The holder of a stock typically has voting rights that can be exercised in corporate decisions and he can receive a certain percentage of the profit the company has made, called a dividend payment. Stockholders may want to trade their stocks and this has led to the establishment of stock exchanges: marketplaces for trading shares and other financial products.

One can imagine that the price of a stock fluctuates due to the theory of supply and demand. When prospective buyers outnumber sellers, the price rises. Eventually, sellers attracted to the high selling price enter the market and buyers leave, achieving equilibrium between buyers and sellers. When sellers outnumber buyers, the price falls and the buyers and sellers leave, again achieving equilibrium. The price of the stock today is known, however this price will fluctuate in the future. It is understandable that a stockholder wants to buy a stock at a low price and try to sell it for a higher price, thus making a profit. We denote the stock price at a time \( t \) with \( S(t) \).

With a bond the issuer owes the holders a debt and, depending on the terms of the bond, is obliged to pay them interest (the coupon) or to repay the principal at a later date, termed the maturity. A government bond is a bond issued by the government. Government bonds are usually referred to as risk-free bonds, because the government can raise taxes or create additional currency to repay the loan. The rate of return of this bond would be the risk-free interest rate (theoretical rate of return of an investment with no risk of financial loss).

2.2 Vanilla options

One way to make money is to buy a stock when it is at a low price, and then to try and sell it at a higher price. Another way to make profit is by buying or selling derivatives. As the name suggests this is a contract whose value is derived from the value of underlying market factors, such as stocks or indices, interest rates, currency exchange rates, and commodities. Derivatives have become very popular in the past few decades, as they can be a very effective and efficient tool for managing risk (hedging), if used correctly. In practice, it is a contract between two parties that specifies conditions. One of the most common derivatives is an option.

A regular option is called a vanilla option and it is a financial contract giving the owner the right, but not the obligation to buy or sell an underlying at a pre-determined price (the strike price) at a pre-determined time (the maturity). The underlying can be for example a stock, an index, a commodity or a future. The option can be a call option, which gives the right to buy the underlying at the strike price, or a put option, which gives the right to sell an underlying at again the strike price.

An option buyer can choose to exercise his or her right and take a position in the underlying. So, a call buyer might choose to buy the underlying, whereas a put buyer may choose to sell the underlying. An option seller, unlike a buyer, is obligated to take the opposite position if the buyer decides to exercise his/her right. In return for the extra risk, the option seller receives a premium from the buyer. When one is expecting the price of a stock to go down, one might...
buy a put option. Whereas when one expects the price of the stock to rise, a good idea would be
to buy a call option. Furthermore, an option may be used to construct a hedge. When people
decide to hedge, they are insuring themselves against a negative event. This doesn’t prevent a
negative event from happening, but if it does happen and you’re properly hedged, the impact
of the event is reduced. There are two main Vanilla options: Europeans and Americans. The
European option can only be exercised at maturity, while the American option can be exercised
any time before maturity. Therefore the price of an European option is typically less than or
equal to that of an American.

2.3 Asian Options

Exotic options are basically options that have a more complex financial structure than vanilla
options. An Asian option is an example of a path-depending exotic option. The pay-off of an
Asian option is determined by the average underlying price over some pre-determined period of
time. The main difference with a European option is therefore that the pay-off does not depend
only on the price of the underlying at exercise, but also on the price the underlying had during
a period of time.

There are a few different types of Asian options. We can have, of course, a put option and
a call option. However we can also have different strikes:

- A fixed strike (underlying settlement price determined by the average price of underlying).
The pay-off of a fixed strike call is thus:

\[ V(T) = (A(T) - K)^+ := \max(A(T) - K, 0) \]

with \( A(T) \) the average of the underlying price and \( K \) the strike. The pay-off of a fixed
strike put is then:

\[ V(T) = (K - A(T))^+ \]

- A floating strike (strike determined by underlying average).
The pay-off of a floating strike call is thus:

\[ V(T) = (S(T) - A(T))^+ \]

with \( A(T) \) the average of the underlying price (the strike) and \( S(T) \) the underlying price
at time \( T \). The pay-off of a floating strike put is then:

\[ V(T) = (A(T) - S(T))^+ \]

There are different types of averaging we may use to compute \( A(T) \): we can use the arithmetic
mean or the geometric mean. We will focus on the arithmetic discrete average:

\[ A(T) = \frac{1}{n} \sum_{i=1}^{n} S(t_i) \]

where \( n \) is the number of time steps, and the average is taken on times \( t_1, ..., t_n \), with \( 0 = t_0 <
t_1, ..., t_n = T \). Asian options are attractive to buyers because they tend to be less expensive than
comparable vanilla calls or puts, since the volatility in the average value of the underlying is
usually lower than the volatility of the value of the underlying. Furthermore the Asian option
offers more protection, because it is more difficult to manipulate an average value of the under-
lying at maturity.
2.4 Futures

A future is a type of contract between two parties to sell or buy a specified underlying in the future for a certain price. The underlying of the future may be a commodity, but it can also be any other asset, like currencies, stocks and bonds, or even indices and interest rates. We will from here on focus on the future contract on a commodity e.g. on gold, oil, silver.

One of the first future trades was written about in Aristotle’s work Politics. It tells the story of Thales, a poor philosopher who used his skills in forecasting and predicted that the olive harvest would be exceptionally good the next autumn. He then made agreements with the local olive-press owners to guarantee him exclusive use of their olive presses when the harvest was ready for a certain price he would pay them. The price was low, since no one knew whether the harvest would be good or bad and olive-press owners were willing to hedge against the possibility of a poor yield. The harvest turned out to be good, and the demand for the olive-presses increased, giving Thales the possibility of selling his right to use the olive-press at a very high price, and thus making a profit.

Originally, futures markets were set up to meet the needs of farmers who wanted to lock in advance a fixed price for their harvests. The first trading of futures may even be traced back centuries ago in Europe and Japan. The first modern future exchanges began in the 18th century in Japan, and later on in the 19th century over 1500 exchanges were established across the United States. The 1970s saw the development of the financial futures contracts, which allowed trading in the future value of interest rates. Today, the futures markets are used for more than just agricultural purposes. [4]

When entering a futures contract one agrees to buy or sell a certain underlying for a price agreed upon today: the futures price. One agrees upon a date at which the contract will be settled and the amount of underlying that will be traded. One party agrees to deliver an underlying at some time in the future (the seller, short position), and the other party pays a price agreed upon today (the buyer, long position). The buyer expects that the asset price is going to increase (so he will gain when the underlying price in the future turns out to be higher than the agreed upon buy price), the seller expects the price to decrease (so he will gain when the buyer pays him more than the cost of the underlying at that time). An advantage for both the seller and the buyer is that they can lock in the price of that commodity in the future, so their risk is reduced.

After the agreement is made, the underlying will, of course, change in price. This can result in gains to one party, while in losses for the other. To trade in futures one needs to set up a margin account, and every day, the profit or loss is calculated on the futures position, and added to/subtracted from the futures account. Therefore, the price of the futures contract depends on the price of the underlying. An example would be, suppose we have a farmer who produces coffee beans, and a coffee maker who used beans to produce his product (coffee). Both want to lock in the price that they will receive (the farmer) and pay (the coffee maker) in the future. They work out a futures contract which guarantees the delivery of beans at a fixed time and price. The contract (and not the coffee beans themselves) is then traded on the futures market. So, say they agree for a price of 2 euros per kilogram of beans, and they trade 1000 kg of beans. Now, suppose that the next day, the price per kilogram of beans increases to 3 euros. This results in a loss of 1 euro per kilogram for the short position (the farmer), since on the market he could have sold the product for 3 euros while he did so for 2 euros. The holder of the
long position (the coffee maker) however has gained 1 euro per kilogram. If now, the contract was settled at the time the price was 3 euro, the farmer lost 1 euro per kilogram, while the coffee maker made 1 euro. After this settlement, the coffee maker needs to buy the beans for the current cash market value, which is 3 euro, however, because of the profit he made, he ends up paying 2 euros, which is the agreed upon futures price. After closing the contract, for the farmer can now sell it for 3 euros, but due to his loss, he still in the end sells it for the 2 euro price.

The main difference between options, stocks and futures are:

- Whereas an option gives the buyer the right, but not the obligation to buy (or sell) a certain asset at a specific price at any time during the life of the contract, a futures contract gives the buyer the obligation to purchase a specific asset, and the seller to deliver that asset at a specific future date, unless the holder’s position is closed prior to expiration.

- It costs nothing to enter a futures contract (aside from setting up the margin account) whereas buying an option does require the payment of a premium. We may therefore see the option premium as the fee paid for the privilege of not being obliged to buy the underlying.

- The difference between stocks and futures is that with stocks the gains and losses are not calculated daily. You only get a profit or loss when you sell the stock, unlike with futures where your gains and losses are calculated daily.

We have two different types of settlements with a futures contract, a cash settlement and a physical settlement:

- The cash settlement means that both parties receive or pay a gain or loss in cash when the contract expires, so there is basically no item being delivered. This type of settlement is usually used by speculators, who just want to make a profit by selling the contract for a higher price than the one they bought it at and are not interested in the actual commodity. So, when a speculator shorts a contract he anticipates a price decline, when he goes long he anticipates a price increase. Suppose we have a bank which currently has no exposure to the price of fuel. If this bank takes a short or long position in a futures contract on oil, it creates exposure to the oil price. Whether the bank goes short or long on the contract is determined by the view that the bank has on the future movements of the oil price. It is in fact betting that this price will go up or will go down and when it does it hopes to gain a profit.

- A physical delivery means that a specified amount of the underlying asset (such as a commodity or a bond) is delivered by the seller and the buyer pays a price for it. This is used by hedgers, who want to secure a price, and reduce their risk associated with price volatility when purchasing the commodity. When a hedger shorts the futures contract, he tries to hedge himself against a price decline, and when he goes long he hedges himself against a price rise in the future. Suppose that it is January and an airline company knows that it will have to buy in June (date $T$) of the same year one million tons of fuel to provide transport for the people going on summer holidays. In order to hedge against the possible increase in fuel prices between January and June, the airline company will buy a futures contract written on fuel, maturity June and in an amount corresponding to the necessary quantity of fuel. By doing so, the airline company has in January locked the price it will pay in June with no payment in January.
The futures contract itself can also be traded (usually the cash settled futures contract is traded) until a particular date (the last trading date). When the contract closes, the holder of the long position pays the price of the initial contract (which is equal to the current price minus/plus the profit/loss) and receives the underlying, the holder of the short position sells the underlying for again the initial price (equal to the current price plus/minus the profit/loss). However, futures contracts are often traded without trading the actual underlying, so one has just made a profit or a loss.

Determining the futures price is basically trying to predict the value in the future by using all available information (political, social and of course the supply and demand of the underlying).

So, in short: there exists a future price at time \( t \) for the delivery of some underlying at delivery time \( T \) which is quoted in the market, this price is \( F(t,T) \). The price for entering the futures contract is zero (since when the agreement is made no price is paid, it is just an agreement to pay a price in the future). The holder of the contract receives payments \( F(t_2,T) - F(t_1,T) \), so equal to the price change during the period from \( t_1 \) to \( t_2 \). Then, at the delivery time \( T \), the holder pays \( F(T,T) = S(T) \) and receives the underlying. However, since we calculated the loss or profit every time step, and subtracted or added this to the margin account, in the end the holder still pays the agreed upon \( F(t,T) \).

With a last example: suppose in August we buy one November futures contract of gold at 350 euro per kg, for a total of 1000 kg or 350000 euro. By buying in August, we are going long and we expect that the price of gold will rise by the time the contract expires. Now, in October, the price of gold has increased to 352 euro per kg, and we decide to sell our contract. The 1000 kg is now worth 352000 euro and we have made a profit of 2000 euro. On the other hand, if we would have expected the price to fall, we could have shorted the contract.

### 2.5 Different types of underlyings

There are many types of other derivatives in modern financial and commodity markets. One thing they have in common is that they cannot exist without the underlying asset and their price is derived from the price of that underlying asset. As mentioned, this underlying can be a stock, basket of stocks, index, future, commodity, or even another derivative. The price of the underlying is the main factor that determines prices of derivative securities. The underlying we will focus on in this thesis will be a commodity. Commodities are basically raw or primary products. They surround us everywhere in our daily life. Our cars drive on gas or oil, metals and woods are used to make these cars, we use electricity and oil as energy resources. Commodities have major impact on economies of developed and developing countries. As a result, the worldwide commodity markets have grown explosively in the past several decades.

The modern commodity markets have their roots in the trading of agricultural products. While wheat and corn, cattle and pigs, were widely traded in the 19th century in the United States, other basic foodstuffs such as soybeans were only added quite recently in most markets. In the Netherlands, commodity markets began in the 16th century. Future contracts on tulip bulbs have been traded in the Netherlands since the early 1600s. Currently almost all trading in commodity markets takes place in the trading of commodity derivatives (especially the above mentioned futures contracts) and not commodities themselves. As an example, in the case of oil, trading volumes in derivatives markets are nine times larger than those occurring in trading of actual real oil.
2.6 Options on futures

Options on futures can be used to protect against price moves in commodity, interest rate and foreign exchange. They can be used both by hedgers and speculators, however the reduced risk makes them increasingly popular with hedgers. Some advantages of options on futures with respect to futures contracts are:

- While a futures position exposes a trader to a theoretically unlimited risk of gain or loss, this is not true for the buyer of a futures option.
- Options on futures also generally require less investment than options on the physical good itself.

We will start with the basics. An option on a future is just like an option on a stock or other underlying. It is the right, but not the obligation to buy or sell a particular futures contract for a specific price (the strike) before expiration. An option on a future is basically an option with expiration date $T_1$ and an underlying is a futures contract with some delivery date $T_2$, so that $T_1 \leq T_2$. Changes in the price of the future will affect the option value. The strike price is the specified futures price at which the future is traded at time $T_1$ if the option is exercised. So, for example, a call on a futures contract would give the holder the right to receive the amount by which the futures price exceeds the strike price, and if the option is exercised, the holder at time $T_1$ also obtains a long position in the futures contract with delivery date $T_2$.

The buyer is under no obligation to exercise an option on a futures contract. As a matter of fact, many traders choose to offset their position prior to expiration, so they can get their profits earlier or limit the losses. A buyer may offset his position by selling the option, whereas a seller can buy back the option.

When looking at the payoff at the expiration time $T_1$ of an option on a future with expiration $T_2$ we have as usual:

For a call: $V(T_1) = (G(F(t, T_2)) - K)^+$
For a put: $V(T_1) = (K - G(F(t, T_2)))^+$

where $G(F(t, T_2))$ is a function of the futures price at times $t$. For example with a fixed strike Asian call option we could have $A(T)$ as function of the futures price at times $t_1, \ldots, t_n = T_1$ (meaning the arithmetic average of the futures price), so that we have $(A(T_1) - K)^+$ as payoff function.

If we want to find the value of the option at time $t_i < T_1$ we have to discount the payoff we would have gotten at time $T_1$ by growth rate $e^{r(T_1-t_i)}$. Furthermore, at time $t_i$ we only know the futures prices until time $t_i$. We do not know how the futures price will behave at further times, so we take the expectation of $G(F(t, T_2))$ under the risk-neutral measure given $F(t_i)$ (the information until $t_i$) to determine the futures prices at all times $t$. So we get:

For a call: $V(t) = e^{-r(T-t)}E[(G(F(t, T_2)) - K)^+|F(t_i)]$
For a put: $V(t) = e^{-r(T-t)}E[(K - G(F(t, T_2)))^+|F(t_i)]$

This discounting is done because of the following assumption: We assume money that we put away in a bank account grows with rate $e^{rt}$, therefore, to calculate the amount of money we had at time zero, when we know the amount we had at time $t$, we multiply by the discount factor $e^{-rt}$. The option holder has the right but not the obligation to buy or sell the underlying future contract. For this right the buyer has to pay a premium to the option seller. The seller must fulfill the obligation of the contract if the option is exercised, so for this obligation he gets to keep the premium. When determining the option premium we must look at different factors:
• The volatility of the underlying futures market is important. If the market is highly volatile (so there is a greater chance of price change in the underlying) an option buyer is willing to pay a larger amount for the protection that the option provides, whereas the option seller demands a larger amount for the risk he is taking when selling the option. If the volatility is low the option premium will be lower.

• The exercise price compared to the underlying futures price also influences the option premium.

• The time until expiration. If expiration nears, the price of the option will go down, since there is less time for the underlying future to move in price.
Models for the stock and future prices

3.1 Market assumptions

One important assumption we make is that we have an efficient and complete market. This means that the market prices right now reflect all publicly available information about the history of the prices and that prices instantly change to reflect new information. There is a market for every good: it is possible to exchange every good, without transaction costs. An asset can be sold without causing a significant movement in the price and with minimum loss of value. Furthermore there are no arbitrage opportunities (opportunities to make a sure profit from zero initial money). Lastly, we assume that the interest rate is deterministic (the interest rate equals the constant \( r \)).

3.2 The risk-neutral measure

The price of an asset (e.g. a stock, option) depends on its risk as investors typically demand more profit when faced with more uncertainty. To price assets, consequently, the calculated expected values need to be adjusted for an investor’s risk preferences. Unfortunately, this means that prices would vary between investors and an individual’s risk preference is difficult to quantify. However, in a complete market with no arbitrage opportunities (as we assume) one can instead of first taking the expectation and then adjusting for an investor’s risk preference, adjust, once and for all, the probabilities of future outcomes such that they incorporate all investors’ risk preferences, and then take the expectation under this new probability distribution. This new probability distribution is called the risk-neutral measure. Using this distribution one can price every asset by simply taking its expected payoff [14].

This is basically what the Fundamental Theorem of Asset Pricing states: An arbitrage-free market \((S, B)\) consisting of a collection of stocks \(S\) and a risk-free bond \(B\) is complete if and only if there exists a unique risk-neutral measure \(Q\) that is equivalent to the original (real) probability measure \(P\), and has numeraire \(B\). In our case the numeraire is used for discounting the future/option price, so we usually have numeraire \(e^{-rt}\). The risk neutral approach amounts to a replacement of the real probability structure of the stock model by an alternative measure \(Q\), so that the resulting framework describes the world in which the expected growth of the stock equals the risk-free rate of interest (as we show in section 3.3.2) and the price of an option can be obtained by simply taking its expected payoff and discounting it over the remaining lifetime.

Furthermore, we will need the following definitions:

- The stochastic process \(Y(t)\) is a martingale with respect to a \(\mathcal{F}(s)\) (all known values of \(Y(t)\) until time \(s\)), if for every \(s \leq t\) we have

\[
E(Y(t) - \mathcal{F}(s)) = Y(s).
\]

This expresses the property that the conditional expectation of a stochastic process at time \(t\), given all the observations up to time \(s\), is equal to the observation at time \(s\). As we will see under the risk neutral measure \(Q\), the discounted stock price is martingale (section 3.3.2).

- The stochastic process \(Y(t)\) possesses the Markov property with respect to \(\mathcal{F}(t)\) if:

\[
Pr(Y(t)\mid \mathcal{F}(s)) = Pr(Y(t)\mid Y(s)),
\]
where $Pr$ refers to the probability of that event. This means that if we make a prediction for the future $Y(t)$ based on the known past until time $s$, $\mathcal{F}(s)$, it will be just as good as the prediction for $Y(t)$ based on only the stock price at the last known time step $Y(s)$. This corresponds with our assumption about the market: the price right now reflects all the past information on the price.

3.3 Stock price

3.3.1 Standard, Arithmetic and Geometric Brownian motion

Before explaining the movement of the stock price, we adress the standard, arithmetic and geometric Brownian motions.

**Definition 1.** (Standard Brownian Motion) A standard Brownian motion $W(t)$ has the following properties:

- $W(t)$ is a stochastic process with $W(0)=0$;
- $W(t)$ has independent increments, so any set of increments $W(t_j + h_j) - W(t_j)$ for $j = 1,2,\ldots,n$ is independent;
- for all $t \geq 0$ and $h>0$, the increment $W(t + h) - W(t)$ is a normal random variable with mean 0 and variance $h$.
- The function $t \rightarrow W(t)$ is almost surely everywhere continuous

Note furthermore the following properties of a standard brownian motion: the expectation is zero, the variance is $\text{Var}(W_t) = E[W_t^2] - E^2[W_t] = 0 = E[W_t^2] = t$ and the covariance is between $W_s$ and $W_t$ is equal to $\min(s,t)$.

A standard Brownian motion always has mean 0 but has linearly growing variance equal to $t$. Arithmetic Brownian motion can have a linearly growing mean and the variance can be proportional to $t$ rather than simply equal to $t$.

**Definition 2.** (Arithmetic Brownian Motion) Arithmetic Brownian motion $A(t)$ with drift parameter $\mu$, volatility parameter $\sigma$, and initial value $A_0$ is a stochastic process $A(t)$ with

$$A(t) = A_0 + \mu t + \sigma B(t)$$

where $B$ is standard Brownian motion.

Note that the increments of an arithmetic Brownian motion $A(t)$ (as defined in the definition) has the following properties: $A(0) = A_0$ and $A$ again has independent increments $(A(t+h) - A(t))$ which are normally distributed with mean $\mu h$ and variance $\sigma^2 h$.

Lastly, the geometric Brownian motion is simply the exponent of an arithmetic Brownian motion $A$. Note that $G_0 = G(0) = e^{A(0)} = e^{A_0}$, so

$$G(t) = e^{A(t)} = e^{A_0 + \mu t + \sigma W(t)} = e^{\ln(G_0) + \mu t + \sigma W(t)} = G_0 e^{\mu t + \sigma W(t)}$$

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Definition 3. (Geometric Brownian Motion) Geometric Brownian motion $G(t)$ with drift parameter $\mu$, volatility parameter $\sigma$, and initial value $G_0$ is a stochastic process $G(t)$ with

$$G(t) = G_0e^{\mu t + \sigma W(t)}$$

where $W(t)$ is standard Brownian motion.

Note that the growth factor $\frac{G(t+h)}{G(t)}$ is a lognormal random variable: $e^{N(\mu h, \sigma^2 h)}$ (since taking the logarithm yields a normal distribution: the arithmetic Brownian motion). Furthermore, we have for $G(t)$ that the mean is $E[G(t)] = G_0e^{(\mu + \frac{1}{2} \sigma^2)t}$ and the variance is $\text{Var}(G(t)) = G_0^2e^{2\mu t}(e^{\sigma^2 t} - 1)$. Knowing these definitions we can continue to the derivation of the movement of the stock price.

3.3.2 Movement of the Stock Price

For a stock price we assume that it is following a geometric Brownian motion (equivalent with the lognormal distribution), so that

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \quad (1)$$

where $W(t)$ is a standard Brownian motion, $\mu$ is the drift (the change of the average value of a stochastic (random) process) and $S(t)$ the stock price at time $t$. The reason we choose this model is that it coincides greatly with the way stock prices move in the real world. A plausible model for the price $S(t)$ of a stock at time $t$ is that log($S(t)$) should be equal to log($S(0)$) plus a normal random variable with mean and variance both proportional to $t$ [13]:

$$\log(S(t)) = \log(S(0)) + N(\mu t, \sigma^2 t)$$

where $N$ denotes the normal distribution.

![Lognormal distribution compared to the normal distribution](image)

Figure 1: An illustration of the lognormal distribution compared to the normal distribution (with mean 0, variance 1)
We can explain equation (1) by the following. When looking at the change of \( S(t) \) over a time period \( dt \), this change would be \( S(t+dt) - S(t) \), defined as \( dS(t) \), so the relative change in the stock price is then

\[
\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t)
\]

This is then equal to the sum of:

- A predictable, deterministic part, which is the expected return that we would have earned if the money would be put into a bank account for a time period \( dt \), where the return had zero risk, so it would result in

\[
\mu dt
\]

where \( \mu \) is the average growth of an asset price.

- A stochastic part, the random change in the asset price caused by external effects. We model it as:

\[
\sigma dW(t)
\]

with \( \sigma \) the volatility of the asset (a constant). The variable \( W(t) \) is a standard Brownian motion. We define \( W(t) = \sqrt{t}Z \), so that \( dW(t) = Z\sqrt{dt} \), where \( Z \) is standard normally distributed.

So we have

\[
\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t)
\]

We can solve this equation. To do that we will use Ito’s lemma:

If \( S = S(t) \) follows an Ito drift-diffusion process, \( dS = \mu S dt + \sigma S dW(t) \) and if we have a function \( h(S,t) \) with continuous \( \frac{\partial h}{\partial S}, \frac{\partial^2 h}{\partial S^2} \) and \( \frac{\partial h}{\partial t} \) (so it is twice differentiable), then \( h := h(S,t) \) also follows an Ito process with the same \( W(t) \), and

\[
dh(t) = \left( \mu S \frac{\partial h}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 h}{\partial S^2} + \frac{\partial h}{\partial t} \right) dt + \sigma S \frac{\partial h}{\partial S} dW(t)
\]

When solving the equation we let \( h(S) = \ln(S) \), so that \( S = \exp(h) \), using Ito’s lemma we get:

\[
dh(t) = \left( \mu S \frac{1}{S} - \frac{1}{2} \sigma^2 S^2 \frac{1}{S^2} + 0 \right) dt + \sigma S \frac{1}{S} dW(t)
\]

so

\[
dh(t) = \left( \mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW(t)
\]

And finally, integrating gives us:

\[
h(t) = h(0) + \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma (W(t) - W(0))
\]

We know \( W(0) = 0 \), since \( W \) follows a Wiener process, so when using \( S = \exp(h) \) we arrive at:

\[
S(t) = S(0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right)
\]
Now we want to look at the behaviour of $S(t)$ under the risk-neutral measure. We assume
the stock price $S(t)$ follows a geometric Brownian motion with drift $\mu$, volatility $\sigma$ and Brownian motion $W(t)$ with respect to the real probability measure $\mathcal{P}$.
We define
$$\widetilde{W}(t) = W(t) + \frac{\mu - r}{\sigma}t$$
where $r$ is the risk-free interest rate and $\frac{\mu - r}{\sigma}$ is the market price of risk.
There exists a theorem (the Girsanov theorem, for more information see other literature e.g. [3])
that states that there exists a measure $\mathcal{Q}$ under which $\widetilde{W}(t)$ is a Brownian motion. Rewriting
gives:
$$dW(t) = d\widetilde{W}(t) - \frac{\mu - r}{\sigma}dt$$
Inserting this back into the equation of our stock price
$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$
$$= \mu S(t)dt + \sigma S(t)d\widetilde{W}(t) - \mu S(t)dt + rS(t)dt$$
Thus, under the risk-neutral measure the stock has drift $r$. From this moment on we will be
working in the risk-neutral world, therefore our stock will have drift $r$ and the Brownian motion
$W(t)$ will be the Brownian motion under the risk-neutral measure. So:
$$S(t) = S(0) \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right)$$
equivalently,
$$S(t_1) = S(t) \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) (t_1 - t) + \sigma (W(t_1) - W(t)) \right)$$
This price process is a Markov process. If we let $X(t) = (r - \frac{1}{2}\sigma^2)t + \sigma W(T)$
$$S(t + h) = S(0) \exp(X(t + h))$$
$$= S(0) \exp(X(t) + X(t + h) - X(t))$$
$$= S(0) \exp(X(t)) \exp(X(t + h) - X(t))$$
$$= S(t) \exp(X(t + h) - X(t))$$
We see that the future price $S(t + h)$ only depends on the current known price $S(t)$ and not on
any of the other past prices. However, this price process is not martingale. Only the discounted
stock price is martingale under the risk neutral measure:
$$E[\exp(-r(T - t))S(T)|\mathcal{F}(t)] = E[\exp(-r(T - t))(S(t) \exp((r - \frac{1}{2}\sigma^2)(T - t) + \sigma W(T - t))|\mathcal{F}(t))]$$
$$= S(t) \exp(-\frac{1}{2}\sigma^2(T - t))E[\exp(\sigma W(T - t))|\mathcal{F}(t)]$$
$$= S(t) \exp(-\frac{1}{2}\sigma^2(T - t)) \exp(\frac{1}{2}\sigma^2(T - t))$$
$$= S(t)$$
### 3.4 Futures price

We have denoted the futures price at time $t$ for the delivery of some underlying at delivery time $T$ as $F(t, T)$. At time $t$ this price can be observed in the market. Using the no arbitrage arguments we say that $F(T, T) = S(T)$, since buying the commodity at the spot price is the same as going long on a futures contract maturing immediately.

Now we would like to know the following: Given that the spot price at time $T$ is a random quantity $S(T)$ viewed from date $t$, is the Futures price at time $t$ the best representation of this quantity? For a futures price we can think of a few different possibilities:

- $F(t, T) > E(S(T)|F(t))$: this means that the futures price is upward biased. The market participants are willing to pay more than the expected spot price in order to secure access to the underlying commodity, usually when the commodity is scarce.

- $F(t, T) < E(S(T)|F(t))$: the futures price is downward biased. This usually happens when there is an excess of supply.

- $F(t, T) = E(S(T)|F(t))$: the futures price is a non biased predictor for the unknown future spot price. When we have a market which is liquid and arbitrage-free (as is the case in our assumptions) this equality holds.

From now on we will assume that the equality $F(t, T) = E(S(T)|F(t))$ holds. We will always assume that the commodity market is arbitrage-free, in the sense already mentioned: if we have no initial wealth and take no risk, then our terminal wealth will be surely zero. Under this assumption, we are going to show that the future price for maturity $T$, denoted by $F(t, T)$ is related to the spot price at date $t$ by the fundamental relationship:

$$F(t, T) = S(t) \exp(r(T - t)).$$  

(2)

This relation arises from the no arbitrage assumption. Let’s look at the proof. Suppose at time $t$ we do three things: we buy an asset $S$ at the price $S(t)$, to finance this we borrow $S(t)$ from the bank and lastly we short a futures contract for underlying $S$ and delivery time $T$ (when shorting this contract we receive 0 at time $t$), so the total value of our position will then be $S(t) - S(t) + 0 = 0$. Then if we do nothing until time $T$, at time $T$ we will have to deliver our asset $S$ for which we receive the $F(t, T)$ (the futures price at time $t$ for which we sold our contract) and also we owe the money amount $S(t) \exp(r(T - t))$. The total value of our position at time $T$ is $F(t, T) - S(t) \exp(r(T - t))$. From the no arbitrage principle it follows that the values of the positions at both time $t$ and time $T$ must be equal so we have $F(t, T) - S(t) \exp(r(T - t)) = 0$, or $F(t, T) = S(t) \exp(r(T - t))$.

Continuing with this formula gives us:

$$dF(t, T) = dS(t) \exp(r(T - t)) - rS(t) \exp(r(T - t))dt,$$

$$\frac{dF(t, T)}{F(t, T)} = \frac{dS(t) \exp(r(T - t))}{F(t, T)} - rdt,$$

and finally:

$$\frac{dF(t, T)}{F(t, T)} = \frac{dS(t)}{S(t)} - rdt.$$
We then remember the dynamics of the stock price: 
\[
\frac{dS(t)}{S(t)} = rdt + \sigma dW(t),
\]
so:
\[
\frac{dF(t, T)}{F(t, T)} = \frac{dS(t)}{S(t)} - rdt = rdt + \sigma dW(t) - rdt = \sigma dW(t)
\]
or we can rewrite this as
\[
dF(t, T) = \sigma F(t, T) dW(t). \tag{3}
\]
We see that the expected change in the return of the futures price is zero:
\[
E\left[\frac{dF(t, T)}{F(t, T)}\right] = E[\sigma dW(t)] = \sigma \cdot 0
\]
Continuing with equation (3) and integrating it in the same way as with the stock price we get
\[
F(t_1, T) = F(t, T) \exp\left(-\frac{1}{2} \sigma^2 (t_1 - t) + \sigma W(t_1 - t)\right), \quad t < t_1,
\]
or equivalently
\[
F(t, T) = F(0, T) \exp\left(-\frac{1}{2} \sigma^2 t + \sigma W(t)\right).
\]
Notice that this is a lognormally distributed variable.
Next, we calculate the expectation of the futures price for \( t < t_1 \):
\[
E[F(t_1, T) | F(t)] = E\left[ F(t, T) \exp\left(-\frac{1}{2} \sigma^2 (t_1 - t) + \sigma W(t_1 - t)\right) | F(t) \right]
= F(t, T) \exp\left(-\frac{1}{2} \sigma^2 (t_1 - t)\right) E[\exp(\sigma W(t_1 - t)) | F(t)]
= F(t, T) \exp\left(-\frac{1}{2} \sigma^2 (t_1 - t)\right) \exp\left(\frac{1}{2} \sigma^2 (t_1 - t)\right).
\]
We see
\[
E(F(t_1, T) | F(t)) = F(t, T),
\]
for \( t_1 > t \). When deriving this, in the last equality we used the known fact that the expectation of a lognormal variable \( \exp(x + yZ) \) with \( Z \) normal with mean \( x \) and standard deviation \( y \) is equal to \( \exp(x + \frac{1}{2} y^2) \). In our case the expectation of \( \exp(\sigma W(t_1 - t)) \) is equal to \( \exp(\frac{1}{2} \sigma^2 (t_1 - t)) \). The futures price is a martingale under the risk-neutral measure.

### 3.5 A short note on the relation between forward and future prices

A forward contract is an agreement between two parties to buy or sell an asset at a specified future time at a price agreed upon today. The main difference with a futures contract is that the forward is settled one time, namely at the end of the contract, whereas a futures contract is settled daily (and the difference is added to the margin account). Furthermore, a futures contract is standardized, meaning that the terms and conditions are set by one party and the other party has to ‘take it or leave it’. A forward price is martingale under the so-called forward measure[2], whereas the futures price is martingale under the risk-neutral measure. The equation derived in the previous section, equation (2) for the futures price also holds for the forward price. Under deterministic interest rates and in absence of credit risk (the risk that a borrower will default by failing to make payments to which he/she is obligated), forward and futures prices with the same maturity are equal [2].
3.6 Simulating the futures price

Suppose we know the futures prices of some commodity or stock over a given time period. Using the known prices we want to make a simulation of how the futures price will behave in the future. To do this we need to have a formula for the $\sigma$ so that we can use the equation for the futures price (3) derived in the previous section.

Assume we have $F_0, \ldots, F_n$ future prices at times $i = 1, \ldots, n$ in chronological order. We can calculate the returns

$$R_i = \frac{F_i - F_{i-1}}{F_{i-1}}, \quad i = 1, \ldots, n$$

We take timestep $\Delta t$ fixed for all $i$. The $R_1, \ldots, R_n$ are then normally distributed with mean $\mu$ and variance $\sigma^2 dt$.

We remember the future prices are lognormally distributed with $r = 0$. To use the formula for the returns or the formula for future price dynamics we need an estimator for the $\sigma$. To estimate this we will need $n$ data points and we will then use the maximum likelihood estimator. The maximum likelihood estimation for a sample from any density goes as follows:

Suppose we have a sample $X_1, \ldots, X_n$ from a density $p_\theta$ with $\theta = (\theta_1 \ldots \theta_k)$, so $k$ is the number of variables our density depends on. We define the maximizer $\hat{\theta}$ as the point that maximizes the likelihood function

$$L_X(\theta) = \prod_{i=1}^{n} p_\theta(X_i),$$

or that maximizes the log likelihood function

$$\log L_X(\theta) = \log(\prod_{i=1}^{n} p_\theta(X_i)) = \sum_{i=1}^{n} \log(p_\theta(X_i)).$$

If the $p_\theta$ is partially differentiable with respect to the vector $\theta$ the estimator also solves

$$\frac{\partial \log L_X(\theta)}{\partial \theta_j}(\theta) = 0, \quad j = 1, \ldots, k.$$

Let’s look at a variable $X$ which can be transformed to $Z \sim N(\mu, \sigma^2)$ by the transformation $f(X) = Z$ for some function $f$. We have that $\theta = [\mu, \sigma^2]$. The distribution function of $X$ then equals

$$P(X \leq x) = \Phi_{0,1} \left( \frac{f(x) - \mu}{\sigma} \right).$$

The density $f_X$ is given by

$$f_X(x) = \frac{d}{dx} \Phi \left( \frac{f(x) - \mu}{\sigma} \right)$$

$$= \frac{1}{\sigma} \Phi \left( \frac{f(x) - \mu}{\sigma} \right) \frac{df}{dx}$$

$$= \frac{1}{\sigma \sqrt{2\pi}} e^{-f(x)-\mu^2/(2\sigma^2)} \frac{df}{dx},$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution.
Suppose we have \( X_1, \ldots, X_n \) as a sample from the same distribution as \( X \). The log likelihood is then:

\[
\log L_X(\mu, \sigma) = \sum_{i=1}^{n} \log \left( \frac{1}{\sigma \sqrt{2\pi}} e^{-(f(X_i) - \mu)^2/(2\sigma^2)} \right) + \sum_{i=1}^{n} \log \frac{df}{dx}(X_i))
\]

\[
= n \log \sqrt{2\pi} - n \log \sigma - \frac{1}{2 \sigma^2} \sum_{i=1}^{n} (f(X_i) - \mu)^2 + \sum_{i=1}^{n} \log \frac{df}{dx}(X_i))
\]

Differentiating this with respect to \( \mu \) and \( \sigma^2 \) gives

\[
\frac{d \log L_X}{d \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (f(X_i) - \mu),
\]

\[
\frac{d \log L_X}{d \sigma^2} = -\frac{1}{2 \sigma^2} n + \frac{1}{2 \sigma^4} \sum_{i=1}^{n} (f(X_i) - \mu)^2.
\]

Setting equal to zero (we want to obtain the parameters that maximize the log likelihood function, so we set the derivatives equal to zero) we get:

\[
\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} f(X_i); \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (f(X_i) - \hat{\mu})^2.
\]

For the returns we use:

\[
\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} R_i; \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (R_i - \hat{\mu})^2.
\]

For the future price dynamics:

\[
\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} (\log(X_i)); \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (\log(X_i) - \hat{\mu})^2.
\]

The estimator for the volatility however is biased. The unbiased version is:

\[
\sigma^2 = \frac{1}{n-1} \sum_{i=1}^{n} (\log(X_i) - \hat{\mu})^2.
\]

Figure 2: Simulation of 100 future price paths with \( \sigma = 0.5 \)
4 Pricing options on futures

4.1 Black 76 model

The Black 76 model was developed to price a European option based on the futures contract price. The assumptions are:

- The interest rates are constant.
- The underlying futures price is lognormally distributed.
- It is possible to borrow and lend cash at a known constant risk-free interest rate.
- The transactions (selling and buying) do not incur any fees or costs
- It is possible to buy and sell any amount of futures.

Suppose we have a futures contract with maturity $T_2$, and an option with maturity $T_1$ and strike $K$, so that $T_2 > T_1$. For simplicity denote $F(t) = F(t, T_2)$ since the future maturity $T_2$ is fixed.

For the dynamics of the future price we have:

$$dF(t) = \sigma F(t) dW(t).$$

For the price of a call option we write $C(t, F, r, T_1, T_2, K) = C$, since $r, T_1, T_2,$ and $K$ are fixed. Using Ito’s formula, we get:

$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial F} dF + \frac{1}{2} \frac{\partial^2 C}{\partial F^2} (dF)^2$$

$$= \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial F} (\sigma F dW) + \frac{1}{2} \frac{\partial^2 C}{\partial F^2} (\sigma^2 F^2 dt).$$

The last equality follows from the fact that $dW^2$ behaves like $dt$ as $dt$ goes to 0.

We can eliminate the random factor

$$\sigma \frac{\partial C}{\partial F} F dW$$

from the equation if we short an amount $\frac{\partial C}{\partial F}$ in futures contracts at time $t$.

Denote the initial value of the futures contract by $v(t)$. Note that this is not the futures price at time $t$, it is the initial value of the contract. It follows that $v(t) = 0$, because it costs nothing to enter a futures contract with futures price $F(t)$. However at time $t + dt$, a futures contract generates a cashflow $F(t + dt) - F(t)$, so

$$dv(t) \equiv dF(t) = \sigma F(t) dW(t).$$

As our portfolio we therefore choose

$$\Pi = C - v(t) \frac{\partial C}{\partial F}.$$
i.e. a long position in a call on a futures contract, and a short in \( \frac{\partial C}{\partial F} \) futures contracts which have value \( v(t) \).

We then get:

\[
d\Pi = d\left(C - v(t)\frac{\partial C}{\partial F}\right)
= \frac{\partial C}{\partial t} + \sigma \frac{\partial C}{\partial F} F dW + \frac{1}{2} \frac{\partial^2 C}{\partial F^2} \sigma^2 F^2 dt - \sigma F dW \frac{\partial C}{\partial F}
= \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial F^2} \sigma^2 F^2 dt
\]

The second equality follows from Ito’s formula (see (4)). Now, remember the no arbitrage principle: a riskless portfolio has to grow at the same rate as money on the bank, so \( d\Pi = r\Pi dt \), or in this case

\[
d\Pi = r \left[ C - v(t)\frac{\partial C}{\partial F}\right] dt.
\]

Therefore

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial F^2} \sigma^2 F^2 dt = r \left( C - v(t)\frac{\partial C}{\partial F}\right) dt.
\]

However, since \( v(t) = 0 \) we get the Black 76 equation:

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 C}{\partial F^2} = rC,
\]

or rewriting gives us

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 C}{\partial F^2} - rC = 0. \tag{5}
\]

Notice that the term \( rF \frac{\partial C}{\partial F} \) is missing here, while present in the Black-Scholes equation. In Black’s words: The term is missing because the value of a futures contract is zero, while the value of the security is positive. Futures contracts do not cost anything to enter into, hence the term does not appear in the derivation, unlike a share under Black-Scholes which does cost something to buy. Futures are not worthless though. Once a futures contract is made, the value of that contract will change as the market moves.

The solution of (5) is:

\[
C(t, F, r, T, K) = Fe^{-r(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),
\]

where

\[
d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{F}{K} \right) + \frac{1}{2} \sigma^2 T \right],
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{F}{K} \right) - \frac{1}{2} \sigma^2 T \right],
\]

and \( \Phi \) is the normal cumulative distribution function and \( F = F(t) \).

The Greeks \( \Delta, \Gamma \) and \( \nu \) for a Call are:

\[
\Delta = \frac{\partial C}{\partial F} = e^{-rt} \Phi(d_1),
\]

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\[ \Gamma = \frac{\partial^2 C}{\partial F^2} = e^{-rt} \frac{\Phi(d_1)}{F(t)\sigma \sqrt{T}}. \]
\[ v = \frac{\partial C}{\partial \sigma} = F(t)e^{-rt}\Phi(d_1)\sqrt{T}. \]

To get the price of a put option one can use put-call parity; i.e.
\[ C(t) - P(t) = e^{-r(T-t)}(F(t) - K), \]
where \( C(t) \) and \( P(t) \) are short for the call and put prices at time \( t \).

### 4.2 Asian options on baskets of futures

A basket is a portfolio consisting of different assets. Several baskets are listed on an exchange, others are designed by banks in order to specify certain demands of customers. Trading is done over-the-counter, that is directly between the institution and its clients. Investors are often interested in derivative products on a basket rather than in the basket itself, since it is cheaper to use a basket option for portfolio insurance than to use the corresponding portfolio of separate plain vanilla options for each stock involved. Due to the correlation between the futures, a basket is less volatile than a portfolio of separate futures. A basket can help reduce portfolio volatility and add diversification. A basket of commodity futures is the weighted sum or average of different commodity futures.

To determine a fair price for these options, the distribution of the basket at expiry is required. We assume the Futures prices of the commodities all follow a geometric Brownian distribution:
\[ \frac{dF_i(t)}{F_i(t)} = \sigma_idW_i(t) \quad i = 1, \ldots, p, \]
where \( F_i(t) \) represents the futures price at time \( t \) of commodity \( i \), and \( \sigma_i \) is the constant volatility and \( p \) is the number of futures on commodities in the basket. Furthermore we have that the futures prices are correlated, meaning here
\[ \text{Cov}(W_i(t), W_j(t)) = \rho(i, j)t. \]

We then define:
\[ A_1(T) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} F_i(t_j) \]
with \( T \) the expiration time of the option, \( p \) the number of futures in our basket and \( n \) the number averaging dates: \( (t_1, \ldots, t_n) \). The option payoff of the option is then:
\[ V(T) = (A(T) - K)^+ \]
with \( K \) the strike. By the martingale theory it follows:
\[ V(0) = e^{-rT} E((A(T) - K)^+) \]

In this thesis the focus will be on pricing Asian (call) options on baskets (for a put option the methods are easily modified using the put-call parity). Now, the main problem we will face when pricing the Asian option on a basket of futures, is that in order to use the Black equation our underlying has to be lognormal. However, when pricing an Asian option on basket of futures, our underlying is a basket, so a sum, of lognormal variables (the futures), and the
sum of lognormal variables is not lognormal. We don’t have a closed form expression for that distribution, therefore we need to find an approximation of the distribution of the average, which sometimes leads to closed-form expressions for the price approximation. The problem also arises when pricing an Asian option on one future, since the Asian option payoff is already a sum of lognormal variables.

4.3 Cholesky decomposition

As mentioned our future prices are dependent in the Brownian motions, since we have:

$$\text{Cov}(W_i(t), W_j(t)) = \rho(i, j)t.$$  

When we have multiple future prices we can use the Cholesky decomposition in order to change the multi-dimensional future processes from dependent to independent.

Consider a vector-valued process $X(t)$, so that

$$dX(t) = \hat{\mu}X(t)dt + \hat{\sigma}X(t)d\tilde{W}(t),$$

where $\tilde{W}(t)$ is a vector of independent Brownian motions, $\hat{\mu}$ is the drift and $\hat{\sigma}$ the volatility. However, in the case of stock or future prices the Brownian motions are correlated. We therefore need the Cholesky theorem \cite{cholesky}, which tells us how correlations can be imposed on independent processes: Every symmetric positive definite matrix, $C$, has a unique factorization, called the Cholesky decomposition:

$$C = LL^T,$$

where $L$ is a lower triangular matrix with positive diagonal entries, and $L^T$ therefore an upper triangular matrix.

Suppose that we have correlated Brownian motions, as is the case with stock prices and future prices. Using Cholesky we can then express the system in independent Brownian motions.

If we have $W(t) = [W_1(t), W_2(t)]^T$ with a correlation $\rho$, in the portfolio we have two correlated stocks $S_1(t)$ and $S_2(t)$, with

$$dS_1 = \mu_1 S_1(t)dt + \sigma_1(t)S_1(t)dW_1(t),$$
$$dS_2 = \mu_2 S_2(t)dt + \sigma_2(t)S_2(t)dW_2(t).$$

For a (2x2)-correlation matrix $C$ we can find the Cholesky decomposition:

$$C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{bmatrix}.$$

To express our portfolio in terms of independant Brownian motions we can then say:

$$\begin{bmatrix} dS_1(t) \\ dS_2(t) \end{bmatrix} = \begin{bmatrix} \mu_1 S_1(t) \\ \mu_2 S_2(t) \end{bmatrix} dt + \begin{bmatrix} \sigma_1 S_1(t) & \sigma_2 S_2(t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} d\tilde{W}_1(t) \\ d\tilde{W}_2(t) \end{bmatrix},$$

where we basically have defined dependant Brownian motions $W_1$ and $W_2$ in terms of independant Brownian motions $\tilde{W}_1$ and $\tilde{W}_2$ by using $dW(t) = L \cdot d\tilde{W}(t)$:

$$dW_1(t) = d\tilde{W}_1(t),$$
$$dW_2(t) = d\tilde{W}_2(t).$$
\[ dW_2(t) = \rho d\tilde{W}_1(t) + \sqrt{1 - \rho^2} d\tilde{W}_2(t). \]

In case of the future price we have the following decomposition for two futures:

\[
\begin{bmatrix}
    dF_1(t) \\
    dF_2(t)
\end{bmatrix} =
\begin{bmatrix}
    \sigma_1 F_1(t) & \sigma_2 F_2(t)
\end{bmatrix} \begin{bmatrix}
    1 & 0 \\
    \rho & \sqrt{1 - \rho^2}
\end{bmatrix} \begin{bmatrix}
    d\tilde{W}_1(t) \\
    d\tilde{W}_2(t)
\end{bmatrix}.
\]
5 Monte Carlo for Asian options on baskets

Using Monte Carlo simulation to calculate the price of an option is a useful technique when the option price is dependent on the path of the underlying asset price. Monte Carlo valuation relies on risk neutral valuation. The price of the option is its discounted expected value. The Monte Carlo technique simulates a large number of samples of the underlying price path (in this case a future price path), between some starting time and the maturity of the option. The associated exercise value of the option for each path is determined and these payoffs are then averaged and discounted to today. This result is the value of the option. In the case of an Asian option on a basket, for every future in the basket samples will be generated and these samples will be used to determine the arithmetic mean over any averaging period. Here, correlation between asset returns can be incorporated using for example Cholesky decomposition (see Section 5.3).

The main concern however, when using Monte Carlo is that accurate estimates can be very time consuming to obtain, and since in the stock market there is a need for fast pricing methods (because the stock market is such a fast-moving place), it will be better to use fast approximating methods (see Chapter 6).

5.1 The general Monte Carlo method

The Monte Carlo simulation approach is a discrete numerical approximation to the true analytic solution, in this case where the underlying prices follow a geometric Brownian motion. In principle this approach converges to the true solution. An acceptable level of approximation is usually reached after a large number of simulations. Variance reduction techniques can be implemented to reduce the number of simulations. The discrete model for change in futures price is

\[ F(t_{j+1}) = F(t_j) \exp\left(-\frac{1}{2}\sigma^2 \Delta t + \sigma (W(t_{j+1}) - W(t_j))\right) \]

where \( F(t_j) \) is the futures price at time \( t_j = j \Delta t \) for \( j = 0, 1, ..., n \) and \( \Delta t = T/n \), with \( T \) the maturity of the future, \( \sigma \) a measure of the futures annual price volatility or tendency to fluctuate and \( n \) the number of dates on which we want to average the futures price. We can then simulate a sample path of futures prices \( \{F_0, F_1, F_2, ..., F_n\} \). If we perform this simulation \( M \) times we get \( M \times n \) futures prices. Then, for every simulation \( k = 1, ..., M \) we can calculate our option payoff function. Taking the mean (average) and standard deviation of the payoffs (a vector of size \( M \)) we get an approximation for the option price.

In the case of an Asian option on a basket of futures, we simulate not one futures path but let’s say \( p \) futures paths if our basket contains \( p \) futures. For each of these futures we calculate the price at \( j = 0, ..., n \), where \( n \) is the number of averaging dates. Then we value the following payoff:

\[ V(T) = (A(T) - K)^+, \]

with

\[ A(T) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} F_i(t_j), \]

and \( p \) the number of futures in our basket and \( n \) the number averaging dates: \( (t_1, ..., t_n) \). The price of the option at time zero is (by the martingale theory):

\[ V(0) = \exp(-rT)E(A(T) - K)^+. \]
5.2 Monte Carlo for an Asian on one future

There are two different methods of simulating the futures price:

- If the solution of the differential equation is known, we can use the solution. In the case of the futures with a constant volatility we have an explicit expression;

\[ F(t) = F(0) \exp\left(-\frac{1}{2} \sigma^2 t + \sigma W(t)\right), \]

or

\[ F(t_{i+1}) = F(t_i) \exp\left(-\frac{1}{2} \sigma^2 \Delta t + \sigma(W(t_{i+1}) - W(t_i))\right). \]

- Based on \( dF(t) = \sigma F(t) dW(t) \), we can simulate the price path using:

\[ F(t_{i+1}) = F(t_i) + dF(t) = F(t_i) + \sigma F(t_i)(W(t_{i+1}) - W(t_i)), \]

or

\[ F(t_{i+1}) = F(t_i) + \sigma F(t_i) Z_{i+1} \sqrt{\Delta t}, \]

where \( Z_{i+1} \) is a normally distributed random variable, and \( \Delta t \) the time step. Notice that when we want to use this method we need to use a sufficiently small \( \Delta t \), otherwise this approximation may generate a big error.

5.3 Two methods of simulating correlated futures

We have used two methods for simulating the correlated future prices, with correlation \( \rho_{1,2} \). The methods are explained for two futures however they can easily be generalised to more than two futures.

- First we can use the Cholesky decomposition (see Section 4.3). We define our correlated Brownian motions in terms of uncorrelated ones. For two futures this results in:

\[
F_1(t_{i+1}) = F_1(t_i) + dF_1(t_{i+1}) \\
= F_1(t_i) + \sigma_1 F_1(t_i)(W_1(t_{i+1}) - W_1(t_i)), \\
F_2(t_{i+1}) = F_2(t_i) + dF_2(t_{i+1}) \\
= F_2(t_i) + \sigma_2 F_2(t_i)(\rho_{1,2}(W_1(t_{i+1}) - W_1(t_i)) + \sqrt{1 - \rho_{1,2}^2}(W_2(t_{i+1}) - W_2(t_i))).
\]

Note, we use \((W_1(t_{i+1}) - W_1(t_i))\) since we only need to ‘add’ a random part from time step \( i \) to \( i + 1 \). Or, in the same way but using the known solution to the equation of the future dynamics

\[
F_1(t_{i+1}) = F_1(t_i) \exp\left(-\frac{1}{2} \sigma_1^2 \Delta t + \sigma_1(W_1(t_{i+1}) - W_1(t_i))\right), \\
F_2(t_{i+1}) = F_2(t_i) \exp\left(-\frac{1}{2} \sigma_2^2 \Delta t + \sigma_2(\rho_{1,2}(W_1(t_{i+1}) - W_1(t_i)) + \sqrt{1 - \rho_{1,2}^2}(W_2(t_{i+1}) - W_2(t_i)))\right).
\]

- The other way is to generate correlated normally distributed random variables \( Z \) (moving random variables) with mean 0 and correlation \( \rho_{1,2} \). We then have \( W_j = \sqrt{\Delta t} Z_j \), and we use:

\[
F_1(t_{i+1}) = F_1(t_i) + dF_1(t_{i+1}) = F_1(t_i) + \sigma_1 F_1(t_i)(W_1(t_{i+1}) - W_1(t_i)),
\]

\[
F_2(t_{i+1}) = F_2(t_i) + dF_2(t_{i+1}) = F_2(t_i) + \sigma_2 F_2(t_i)(\rho_{1,2}(W_1(t_{i+1}) - W_1(t_i)) + \sqrt{1 - \rho_{1,2}^2}(W_2(t_{i+1}) - W_2(t_i)))).
\]
\[ F_2(t_{i+1}) = F_2(t_i) + dF_2(t_{i+1}) = F_2(t_i) + \sigma_2 F_2(t_i)(W_2(t_{i+1}) - W_2(t_i)). \]

Using the known solution

\[ F_1(t_{i+1}) = F_1(t_i) \exp \left( -\frac{1}{2} \sigma_1^2 dt + \sigma_1 (W_1(t_{i+1}) - W_1(t_i)) \right), \]

\[ F_2(t_{i+1}) = F_2(t_i) \exp \left( -\frac{1}{2} \sigma_2^2 dt + \sigma_2 (W_2(t_{i+1}) - W_2(t_i)) \right). \]

The results are approximately the same (the 95 percent confidence intervals obtained with Matlab coincide greatly). The difference comes from the fact that the random variables used are different and the number of paths is finite.

5.4 Variance reduction in Monte Carlo

Sometimes a large number of simulations is needed in order to obtain an estimate with an acceptable standard error and this can be time-consuming. There are however other ways to reduce the standard error. Note that to reduce the standard error we have to reduce the variance since

\[ SE = \sqrt{\frac{\text{Var}}{M}}, \]

with \( M \) the number of simulations.

One way to reduce the variance is by means of antithetic variates. For every sample path obtained we take its antithetic path. So suppose we have generated a path using standard normal random variables \( \{z_1, \ldots, z_n\} \), we will now make another path using \( \{-z_1, \ldots, -z_n\} = \{\tilde{z}_1, \ldots, \tilde{z}_n\} \).

Suppose that we want to estimate some expectation \( E[X] \). We have computed \( X_1 \) using the usual paths and \( X_2 \) using the antithetic paths. We then compute the antithetic estimator:

\[ X_A = \frac{X_1 + X_2}{2}. \]

It follows that

\[ \text{Var}(X_A) = \frac{\text{Var}(X_1) + \text{Var}(X_2) + 2\text{Cov}(X_1, X_2)}{4}. \]

We can see that the variance of the new estimator will be less than that of the old one, when \( \text{Cov}(X_1, X_2) \) is negative.

Another way to reduce the variance is by means of control variates. This method is based on using a similar yet simpler problem to improve the solution. Again, suppose we want to estimate \( E[X] \) and there exists a random variable \( Y \) with a known mean \( \mu = E[Y] \). We define

\[ Z = X + \theta(Y - \mu) \]

to be our control variate for estimating \( E[X] \). Notice that for every \( \theta \), \( W \) remains an unbiased estimator of \( E[X] \), since \( E[W] = E[X] + \theta E[Y - \mu] = E[X] \). Furthermore we have

\[ \text{Var}(Z) = \text{Var}(X) + \theta^2 \text{Var}(Y) + 2\theta \text{Cov}(X, Y). \]

So, we require

\[ \theta^2 \text{Var}(Y) + 2\theta \text{Cov}(X, Y) < 0. \]

Now, when we want to minimize the variance of \( W \) we find that the \( \theta \) that does that is equal to

\[ \theta_{\text{min}} = -\frac{\text{Cov}(X, Y)}{\text{Var}(Y)}. \]
The effectiveness of the approach also depends on which $Y$ to choose. An example may be when pricing arithmetic average-rate options to choose $Y$ to be the otherwise identical geometric average-rate option’s price. This is exactly what we will do to reduce the variance for an Asian option on a basket of futures. We choose the geometric average as a control variate. This means that for the simulated paths we calculate the geometric average. Using the Black 76 formula we calculate the exact value of the price with a geometric average (the known mean). Note that this is possible to do using the Black formula because a geometric average of lognormals is again lognormal. We then calculate the new variable using these values. This has then a smaller variance than without control variates.
6 Analytical models for the pricing of fixed strike Asian options on Baskets

In this chapter we will look at four different methods for pricing a fixed strike Asian option on a basket, namely the Sophis method, the Gentle method, the Levy method and the Curran method. The main assumption in the Sophis, Levy and Gentle methods will be that the underlying arithmetic average is a lognormally distributed variable. The Curran method uses conditioning on the geometric average to determine the price of the Asian option.

6.1 Sophis model

Remember, we have for \( A(T) \), the averaged sum of the future prices (averaged from the starting time until the maturity of the option) summed over the basket:

\[
A(T) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} F_i(t_j),
\]

with \( 0 = t_0 < t_1 < ... < t_j < ... < t_n = T \).

The first, and probably the most basic approximation that we can use is to assume that \( A(T) \) is lognormal:

\[
A(T) = M \exp(-\frac{1}{2} \sigma^2 T + \sigma W(T))
\]

Note, the whole distribution only depends on the two unknowns \( M \) and \( \sigma \). So, we need to find estimators for these parameters. In short, for \( M \) we will use the expectation of \( A(T) \) and for \( \sigma \) we will use variance of the geometric average.

Now we have two formulas for the \( A(T) \):

- The Sophis approximation reads \( A_1(T) = M \exp(-\frac{1}{2} \sigma^2 T + \sigma W(T)) \).
- The true value is

\[
A_2(T) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} F_i(t_j).
\]

To determine \( M \) we use \( E[A_1(T)] = E[A_2(T)] \). We have

\[
E[A_1(T)] = E[M \exp(-\frac{1}{2} \sigma^2 T + \sigma W(T))] = M \exp(-\frac{1}{2} \sigma^2) E[\exp(\sigma W(T))] = M \exp(-\frac{1}{2} \sigma^2) \exp(\frac{1}{2} \sigma^2) = M,
\]

and

\[
E[A_2(T)] = E[\frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} F_i(t_j)] = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} E[F_i(t_j)] = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} F_i(0) = \frac{p}{i=1} F_i(0) = A(0).
\]
So, \( M = A(0) \). To determine the volatility \( \sigma \) we first define a geometric average (which is lognormal if the underlying future price is because it is a product and therefore a logical approximation for the arithmetic average):

\[
G(T) = \prod_{j=1}^{n} \prod_{i=1}^{p} \left( \frac{1}{n \alpha_i} F_i(t_j) \right)^{\alpha_i},
\]

with

\[
\alpha_i = \frac{1}{n} \frac{F_i(0)}{\sum_{i=1}^{p} F_i(0)}.
\]

We can rewrite this as

\[
G(T) = \left( \frac{1}{n} \frac{F_i(0)}{\alpha_i} \right)^{\sum_{i=1}^{p} \sum_{j=1}^{n} \alpha_i} \exp \left( \sum_{j=1}^{n} \sum_{i=1}^{p} \left( -\frac{1}{2} \alpha_i \sigma_i^2 t_j + \alpha_i \sigma_i W_i(t_j) \right) \right).
\]

Since we have \( \sum_{i=1}^{p} \sum_{j=1}^{n} \alpha_i = 1 \) we find

\[
G(T) = \frac{1}{n} \frac{F_i(0)}{\alpha_i} \exp \left( \sum_{j=1}^{n} \sum_{i=1}^{p} \left( -\frac{1}{2} \alpha_i \sigma_i^2 t_j + \alpha_i \sigma_i W_i(t_j) \right) \right)
= \sum_{i=1}^{p} (F_i(0)) \exp \left( \sum_{j=1}^{n} \sum_{i=1}^{p} \left( -\frac{1}{2} \alpha_i \sigma_i^2 t_j + \alpha_i \sigma_i W_i(t_j) \right) \right),
\]

and finally

\[
G(T) = A(0) \exp \left( \sum_{j=1}^{n} \sum_{i=1}^{p} \left( -\frac{1}{2} \alpha_i \sigma_i^2 t_j + \alpha_i \sigma_i W_i(t_j) \right) \right).
\]

We can calculate the variance of the geometric average, and use it for the variance of the arithmetic average. We do this because the geometric average is also a lognormal process, therefore it seems natural to approximate the volatility of the lognormal process \( A(T) \) by the variance of \( G(T) \). Define \( X(T) := \log(G(T)) \); and compute

\[
\text{Var}(X) = \text{Var}[\log(G(T))] = E[\log(G(T))^2] - (E[\log(G(T)])^2).
\]

We have

\[
X(T) = \log(A(0)) + \sum_{j=1}^{n} \sum_{i=1}^{p} \left( -\frac{1}{2} \alpha_i \sigma_i t_j \right) + \sum_{j=1}^{n} \sum_{i=1}^{p} \alpha_i \sigma_i W_i(t_j).
\]
In order to calculate the variance of \( X(T) \) we need the first and the second moments:

\[
E[X(T)] = \log(A(0)) + \sum_{j=1}^{n} \sum_{i=1}^{p} \left( -\frac{1}{2} \alpha_i \sigma_i t_j \right) := x_0, \\
E[X^2(T)] = E \left( \left( \log(A(0)) + \sum_{j=1}^{n} \sum_{i=1}^{p} \left( -\frac{1}{2} \alpha_i \sigma_i t_j \right) \right)^2 + \left( \sum_{j=1}^{n} \sum_{i=1}^{p} \alpha_i \sigma_i W_i(t_j) \right)^2 \right) \\
= x_0^2 + 2x_0 \cdot 0 + E \left( \sum_{j=1}^{n} \sum_{i=1}^{p} \alpha_i \sigma_i W_i(t_j) \right)^2. 
\]

Only the remaining expectation is to be calculated, i.e.

\[
E \left[ \left( \sum_{i=1}^{p} \alpha_i \sigma_i (W_i(t_1) + W_i(t_2) + \ldots + W_i(t_n)) \right)^2 \right] = \sum_{i_1,i_2=1}^{p} \alpha_{i_1} \alpha_{i_2} \sigma_{i_1} \sigma_{i_2} \cdot E \left[ (W_i(t_1) + W_i(t_2) + \ldots + W_i(t_n)) (W_{i_2}(t_1) + W_{i_2}(t_2) + \ldots + W_{i_2}(t_n)) \right].
\]

The following expression will be useful in the derivation

\[
E[W_i(t_1) \cdot W_i(t_2)] = \min(t_1, t_2),
\]

and if we assume \( t_1 < t_2 \) then:

\[
E[W_{i_1}(t_1) \cdot W_{i_2}(t_2)] = E[W_{i_1}^t(t_1) \cdot W_{i_2}(t_1)] = \rho_{i_1,i_2} t_1.
\]

Thus for the expectation we get the following expression

\[
E \left( \left( \sum_{j=1}^{n} \sum_{i=1}^{p} \alpha_i \sigma_i W_i(t_j) \right)^2 \right) = \sum_{i_1,i_2=1}^{p} \alpha_{i_1} \alpha_{i_2} \sigma_{i_1} \sigma_{i_2} \cdot \sum_{j_1,j_2=1}^{n} \rho_{i_1,i_2} \min(t_{j_1}, t_{j_2}).
\]

Thus we get

\[
\text{Var}(X) = x_0^2 + \sum_{i_1,i_2=1}^{p} \alpha_{i_1} \alpha_{i_2} \sigma_{i_1} \sigma_{i_2} \cdot \sum_{j_1,j_2=1}^{n} \rho_{i_1,i_2} \min(t_{j_1}, t_{j_2}) - x_0^2,
\]

and the variance of the geometric average is then given by

\[
\sigma^2 = \sum_{j_1,j_2=1}^{n} \sum_{i_1,i_2=1}^{p} \alpha_{i_1} \alpha_{i_2} \sigma_{i_1} \sigma_{i_2} \rho(i_1,i_2) \min(t_{j_1}, t_{j_2}). \tag{13}
\]

Next, we can apply Black’s formula for a call (or a put) using the obtained values for \( M \) and \( \sigma \):

\[
C(t, F, r, T, K) = Me^{-r(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \tag{14}
\]

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where
\[
d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \log \left( \frac{F}{K} \right) + \frac{1}{2} \sigma^2 T \right],
\]
\[
d_2 = d_1 - \sigma \sqrt{T} = \frac{1}{\sigma \sqrt{T}} \left[ \log \left( \frac{F}{K} \right) - \frac{1}{2} \sigma^2 T \right],
\]
and \( \Phi \) denotes the cumulative normal distribution.

### 6.2 Gentle model

The Gentle model approximates the arithmetic average in the basket payoff by a geometric average (see equation (10)). We use the geometric average to determine a value for both \( M \) (the expectation of the geometric average) and \( \sigma \). This is unlike Sophis, which only uses the geometric average to determine \( \sigma \). Since a geometric average of lognormal random variables is again log-normally distributed, this allows us to use the Black 76 valuation formula for pricing the approximating payoff. We approximate \( A(T) \) with
\[
G(T) = \prod_{j=1}^{n} \prod_{i=1}^{p} \left( \frac{1}{n \alpha_i} F_i(t_j) \right)^{\alpha_i},
\]
with again
\[
\alpha_i = \frac{1}{n} F_i(0) \sum_{i=1}^{p} F_i(0).
\]
Since \( G(T) \) is lognormally distributed (a product of lognormal variables is again lognormal) we have
\[
G(T) = M \exp \left( -\frac{1}{2} \sigma^2 T + \sigma W(T) \right).
\]

We can therefore apply Black's formula. However, we will need to correct for the difference between the expectation of \( A(T) \) and \( G(T) \). We wish to achieve \( E(A(T) - K) = E(G(T) - \tilde{K}) \), so it follows that we should take \( \tilde{K} = E(G(T)) + K - E(A(T)) \).

When using Black's formula we need a formula for both \( M \) and \( \sigma \). For \( \sigma \) we use the formula obtained in the previous section, see equation (13)
\[
\sigma = \sqrt{\frac{1}{T} \sum_{j=1}^{n} \sum_{i=1}^{p} \alpha_i \alpha_j \sigma_i \sigma_j \rho(i_1, i_2) \min(t_{j_1}, t_{j_2})}.
\]

We know that \( M \) is equal to the expectation of the lognormal formula for \( G(T) \). We can now calculate the expectation of the geometric average formula for \( G(T) \) and set it equal to \( M \):
\[
M = E \left[ \prod_{j=1}^{n} \prod_{i=1}^{p} \left( \frac{1}{n \alpha_i} F_i(t_j) \right)^{\alpha_i} \right] = A(0) \exp \left( \sum_{j=1}^{n} \sum_{i=1}^{p} \left( -\frac{1}{2} \alpha_i \sigma_i^2 t_j \right) + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{p} \alpha_i \alpha_j \sigma_i \sigma_j \rho(i_1, i_2) \min(t_{j_1}, t_{j_2}) \right),
\]
see Appendix 1 for more details on the calculation. The Black 76 formula can be applied now, with the modified strike value \( \tilde{K} \). So, the option price using Gentle is
\[
V(0) = \exp(-rT) E(G(T) - \tilde{K})
\]
6.3 Levy model

The basic idea in the Levy model is to approximate the distribution of the basket by a lognormal distribution such that the first two moments of the lognormal approximation (see (7)) and of the original distribution of the weighted sum of the stock prices (see (6)) coincide. So, we have again

- The approximation:
  \[ A_1(T) = M \exp(-\frac{1}{2}\sigma^2 T + \sigma W(T)). \]

- The true value:
  \[ A_2(T) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} F_i(t_j). \]

We prefer to have the following equalities:

- Equality in the first moments requires \( E[A_1(T)] = E[A_2(T)]. \) It is known that \( E[A_1(T)] = M \) (see (8)) and \( E[A_2(T)] = A(0) \) (see (9)). So we get \( M = A(0). \)

- Equality in the second moments \( E[A_1(T)^2] = E[A_2(T)^2] \) gives us
  \[
  E[(A_1(T))^2] = E[M^2 \exp(-\sigma^2 T + 2\sigma W(T))] \\
  = M^2 \exp(-\sigma^2 T) E[\exp(2\sigma W(T))] \\
  = M^2 \exp(-\sigma^2 T) \exp(2\sigma^2 T) \\
  = M^2 \exp(\sigma^2 T),
  \]
  and thus,
  \[
  E[(A_2(T))^2] = \frac{1}{n^2} \sum_{j_1,j_2=1}^{n} \sum_{i_1,i_2=1}^{p} F_{i_1}(0) F_{i_2}(0) \exp(\sigma_{i_1}\sigma_{i_2}\rho(i_1,i_2) \min(t_{j_1},t_{j_2})),
  \]

see the Appendix 1 for details on this calculation. We then set

\[
E[(A_1(T))^2] = E[(A_2(T))^2] \quad \Leftrightarrow \quad M^2 \exp(\sigma^2 T) = E[(A_2(T))^2] \\
\exp(\sigma^2 T) = E[(A_2(T))^2] \quad \Leftrightarrow \quad M^2 \\
\sigma^2 T = \log \left( \frac{E[(A_2(T))^2]}{M^2} \right). 
\]

We can now use the Black 76 formula with: \( M = A(0) \) and,

\[
\sigma^2 T = \log \left( \frac{E[(A_2(T))^2]}{M^2} \right) = \log(E[A_2(T)^2]) - 2 \log(M). 
\]
6.4 Curran model

The Curran model computes the expected option payoff by conditioning on the geometric average. Remember that the price of the option at time zero can be expressed as

\[ V(0) = \exp(-rT)E[(A(T) - K)^+]. \]

Conditioning on \( G(T) \) gives us

\[ V(0) = \exp(-rT)E[(A(T) - K)^+|G(T)]], \tag{15} \]

with \( G(T) \) the geometric average. We will now proceed to decompose equation (15):

\[ V(0) = \exp(-rT)(C_1 + C_2), \]

where \( g \) is the density function of \( G \).

We first will approximate the exact part \( C_2 \). Note that we can rewrite it as

\[ C_2 = E[(A(T) - K)^+|G(T) \geq K]. \]

We need the Jensen inequality which states that for a convex function \( \phi \), and weights \( a_i \) we have

\[ \phi \left( \frac{\sum a_i x_i}{\sum a_j} \right) \leq \sum a_i \phi(x_i) \sum a_j \]

Furthermore, we have

\[ G(T) = \prod_{j=1}^{n} \prod_{i=1}^{p} \left( \frac{1}{n\alpha_i} F_i(t_j) \right)^{\alpha_i} = \exp \left( \sum_{j=1}^{n} \sum_{i=1}^{p} \alpha_i \log \left( \frac{1}{n\alpha_i} F_i(t_j) \right) \right), \]

and

\[ A(T) = \sum_{j=1}^{n} \sum_{i=1}^{p} \frac{1}{n} F_i(t_j) = \sum_{j=1}^{n} \sum_{i=1}^{p} \alpha_i \exp \left( \log \left( \frac{1}{n\alpha_i} F_i(t_j) \right) \right). \]

From the Jensen inequality we have \( G(T) \leq A(T) \), since the exponent is a convex function. Using the property \( A(T) \geq G(T) \geq K \), so \( (A(T) - K)^+ = A(T) - K \) we have

\[ E[(A(T) - K)^+|G(T) \geq K] = E[A(T) - K|G(T) \geq K] \]

\[ = E[A(T)|G(T) \geq K] - E[K|G(T) \geq K] \]

\[ = E[A(T)|G(T) \geq K] - K \Pr(G(T) \geq K). \]

Now, define \( X = \log(G(T)) \), note that \( X \) is then normally distributed with

\[ \mu_X = E[\log(G(T))] = \log(A(0)) + \sum_{j=1}^{n} \sum_{i=1}^{p} \left( -\frac{1}{2} \alpha_i \sigma_i t_j \right); \]

\[ \sigma_X^2 = \frac{1}{T} \log(\text{Var}(G(T))) = \frac{1}{T} \sum_{j_1,j_2=1}^{n} \sum_{i_1,i_2=1}^{p} \alpha_{i_1} \alpha_{i_2} \sigma_{i_1} \sigma_{i_2} \rho(i_1,i_2) \min(t_{j_1},t_{j_2}). \]
Next, we also define $X_i = \log(F_i(t))$, which is also normally distributed, with mean $\mu_i = \log(F_i(0)) - \frac{1}{2} \sigma_i^2 t$ and variance $\sigma_i^2$. Then we determine the covariance between $X_i$ and $X$ (this correlation is different for each $i$ and $j$),

$$\text{Cov}(t_j) := \text{Cov}[(\log(G(T)), \log(F_i(T))]$$

$$= E[\log(G(T)) \log(F_i(T))] - E[\log(G(T))]E[\log(F_i(T))]$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{p} \alpha_{ij} \sigma_i \sigma_i \rho_{i,i} \min(t_j, t_{ji}).$$

Note that $X$ and $X_i$ have a bivariate normal distribution for all $i$, because they are both linear combinations of normal random variables. We know that if two random variables $X_1$ and $X_2$ have a bivariate normal distribution, the conditional distribution of $X_1$ given $X_2$ is also normally distributed[8]:

$$X_i | X_2 \sim \mathcal{N}\left(\mu_i + \frac{\sigma_i}{\sigma_2} \rho(X_2 - \mu_2), (1 - \rho^2)\sigma_i^2\right),$$

where $\rho$ is the correlation between $X_1$ and $X_2$, $\mu_i$ the mean of $X_i$ and $\sigma_i^2$ the variance of $X_i$. Furthermore we have that $\rho \sigma_1 \sigma_2 = \text{Cov}(X_1, X_2)$. These two properties we will use to determine the distribution of $X_i$ given $X$, in other words

$$X_i | X \sim \mathcal{N}\left(\mu_i + \frac{\text{Cov}(t)}{\sigma_X T} (\log(G(T)) - \mu_X), \sigma_i^2 t - \frac{\text{Cov}^2(t)}{\sigma_X^2}\right). \quad (16)$$

As a result, the conditional distribution of $F_i(t)$ is lognormally distributed.

We know that the expectation of a lognormal random variable with parameters $\beta$ and $\chi^2$ equals $\exp(\beta + \chi^2/2)$. Furthermore, $G(T)$ is also lognormally distributed, so we know the density $g(G)$. We find

$$E[F_i(t_k)]|G(T) > K]$$

$$= \int_K \exp\left(\log(F_i(0)) - \frac{1}{2} \sigma_i^2 t_j + \frac{\text{Cov}(t_j)}{\sigma_X^2 T} (\log(G(T)) - \mu_X) + \frac{1}{2} \sigma_i^2 t_j - \frac{\sigma_i^2 (t_j)^2}{2\sigma_X^2 T}\right) g(G) dG$$

$$= \int_K F_i(0) \exp\left(\frac{\text{Cov}(t_j)}{\sigma_X^2 T} (\log(G(T)) - \mu_X - \frac{1}{2} \text{Cov}(t_j))\right) \frac{\exp\left(-\frac{\log(G(T) - \mu_X)^2}{2\sigma_X^2 T}\right)}{G\sigma_X \sqrt{2\pi T}} dG$$

$$= \int_K F_i(0) \frac{1}{G\sigma_X \sqrt{2\pi T}} \exp\left(-\frac{(\text{Cov}(t_j) - \log(K + \mu_X))^2}{2\sigma_X^2 T}\right) dG$$

$$= F_i(0) \Phi\left(\frac{(\text{Cov}(t_j) - \log(K + \mu_X))}{\sigma_X \sqrt{2T}}\right),$$

where the last equality follows from a change of variables and then using the definition of the cdf of the normal distribution: $\Phi(x) = \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$. 

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So, for the exact part we find

$$E[(A(T) - K)^+ | G(T) \geq K] = E[A(T)|G(T) \geq K] - KP(G(T) \geq K)$$

$$= \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} F_i(0) \Phi \left( \frac{\text{Cov}(t_j) - \text{log}(K) + \mu_X}{\sigma_X \sqrt{T}} \right) - K \Phi \left( \frac{\mu_X - \text{log}(K)}{\sigma_X \sqrt{T}} \right).$$

Now that we have an expression for the exact part we will look at the approximated part:

$$C_1 = \int_{0}^{K} E[(A(T) - K)^+ | G(T)] f(G) dG = E[(A(T) - K)^+ | G(T) \leq K] f(G) dG.$$

We will approximate function $H_G = (A(T) - G(G)$ by a lognormal distribution, so we assume

$$H_G = M_H \exp(-\frac{1}{2} \sigma_H^2 T + \sigma_H W(T)).$$

Next, we will use the moment matching technique from Levy (see section 6.3). We have

$$M_H = E[H_G] = E[A(T) - G(G) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} E[F_i(t_j)] G - G,$$

$$\sigma_H^2 T = \text{log}(E[H_G^2]) - 2 \text{log}(M_H).$$

We have also


$$= \frac{1}{n^2} \sum_{j_1, j_2 = 1}^{n} \sum_{i_1, i_2 = 1}^{p} E[F_{i_1}(t_{j_1}) F_{i_2}(t_{j_2})|G] - 2EG[A(T)|G] + G^2.$$

Remember now, we know that the distribution of $\text{log}(F_i)$ given $\text{log}(G)$ is normally distributed (see equation (16)) and we can easily derive the expectation of $F_i|G$ (which is then lognormally distributed):

$$E[F_i(t_j)|G] = F_i(0) \exp \left( \frac{\text{Cov}(t_j)}{\sigma_X^2 T} \left( \text{log}(G) - \mu_X - \frac{1}{2} \text{Cov}(t_j) \right) \right).$$

Furthermore we have that the expectation of $F_{i_1}(t_{j_1}) F_{i_2}(t_{j_2})$ given $G(T)$ is again lognormally distributed, with mean

$$-\frac{1}{2} \sigma_{i_1}^2 t_{j_1} - \frac{1}{2} \sigma_{i_2}^2 t_{j_2} + \frac{\text{Cov}_{i_1 i_2}(t_{j_1}, t_{j_2})}{\sigma_X^2 T} (\text{log}(G) - \mu_X),$$

and variance

$$\sigma_{i_1}^2 t_{j_1} + \sigma_{i_2}^2 t_{j_2} + 2 \rho_{i_1 i_2} \sigma_{i_1} \sigma_{i_2} \text{min}(t_{j_1}, t_{j_2}) - \frac{\text{Cov}_{i_1 i_2}(t_{j_1}, t_{j_2})}{\sigma_X^2 T}.$$

This then results in:

$$E[F_{i_1}(t_{j_1}) F_{i_2}(t_{j_2})|G]$$

$$= F_{i_1}(0) F_{i_2}(0) \exp \left( \frac{\text{Cov}_{i_1 i_2}(t_{j_1}, t_{j_2})}{\sigma_X^2 T} \left( \text{log}(G) - \mu_X - \frac{1}{2} \text{Cov}_{i_1 i_2}(t_{j_1}, t_{j_2}) \right) + \rho_{i_1 i_2} \sigma_{i_1} \sigma_{i_2} \text{min}(t_{j_1}, t_{j_2}) \right),$$

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where
\[ \text{Cov}_{i_1 i_2}(t_{j_1}, t_{j_2}) := \text{Cov}[\log(G), \log(F_{i_1}(t_{j_1}) F_{i_2}(t_{j_2}))] = \text{Cov}_{i_1}(t_{j_1}) + \text{Cov}_{i_2}(t_{j_2}). \]

We assume that the mean and variance are equal to their values for \( G = K \). This makes sense, since the majority of the contribution to the value of the option when \( G < K \) will come from instances when the geometric mean is close to the strike.

We can apply Black’s formula
\[
BS = \mathbb{E}[(A(T) - K)^+ | G] = M_H \Phi(d_1(G)) - (K - G) \Phi(d_2(G)),
\]
where
\[
d_1 = \frac{\log(M_H) - \log(K - G)}{\sigma_H \sqrt{T}} + \frac{1}{2} \sigma_H \sqrt{T},
\]
\[
d_2 = d_1 - \sigma_H \sqrt{T}.
\]

We can determine \( C_1 \) by integrating this expression times \( g(G) \) from 0 to \( K \) over \( G \). Of course
\[
g(G) = \frac{\exp(-\frac{(\log(G) - \mu_X)^2}{2\sigma_X^2 T})}{G\sigma_X \sqrt{2\pi T}},
\]
which is the density function of a lognormal variable. We will use a numerical quadrature method to compute the integral. Any quadrature method relies on evaluating the integrand on a finite set of points, then processing these evaluations somehow to produce an approximation to the value of the integral. The specific method of quadrature used is the adaptive Simpson’s method, which uses an estimate of the error we get from calculating a definite integral using Simpson’s rule. If the error exceeds some level of tolerance, the interval is subdivided in two and the adaptive Simpson’s method is applied to each subinterval in a recursive manner.

6.5 Summary of the methods

Four methods for pricing a fixed strike Asian option on a basket have been discussed. In the Sophis method we assume our \( A(T) \) is lognormal, meaning we have \( A(T) = M \exp(-\frac{1}{2}\sigma^2 T + \sigma W(T)) \). To calculate \( M \) we set the expectation of the lognormal approximation equal to that of the expectation of the real \( A(T) \). Next, to determine our \( \sigma \), we define a geometric average
\[
G(T) = \prod_{j=1}^{n} \left( \prod_{i=1}^{p} \left( \frac{1}{\alpha_i} F_i(t_j) \right)^{\alpha_i} \right), \tag{17}
\]
with
\[
\alpha_i = \frac{1}{\sum_{i=1}^{p} \alpha_i} F_i(0). \tag{18}
\]
Then we calculate the volatility of the geometric average and use that for the volatility of \( A(T) \), because the geometric average is lognormally distributed. Therefore it seems natural to approximate the volatility of \( A(T) \) (which we assume is lognormal) by that of a truly lognormal process.

The Gentle method approximates the arithmetic average \( A(T) \) completely by a geometric average \( G(T) \), see (17) and (18). Since a product of lognormal variables is again lognormal, \( G(T) \) is lognormal and this approximation allows us to use the Black 76 valuation formula. Setting the volatility \( \sigma \) equal to the volatility of \( G(T) \) and \( M \) equal to the expectation of \( G(T) \), we apply the
Black 76 model to determine an option price, with a modified strike \( \tilde{K} = E(G(T)) + K - E(A(T)) \), meaning the option price is \( V(0) = \exp(-rT)E(G(T) - \tilde{K}) \).

In the Levy method we approximate the arithmetic average \( A(T) \) (which is a sum of lognormal variables, and so not lognormal) by a lognormal distribution

\[
A(T) = M \exp(-\frac{1}{2} \sigma^2 T + \sigma W(T)),
\]

using moment matching to determine values for \( M \) and \( \sigma \).

And finally, in the Curran method conditioning on the geometric average is used to calculate the price. The price then consists of two parts: an exact part and an approximated part.

The results are given for a call option, however the put price is easily obtained using put-call parity:

\[
C(t) - P(t) = e^{-r(T-t)}(F(t) - K).
\]

Due to this relation the conclusions about which method works best under what circumstances hold for both a call as well a a put option.
7 Results for pricing a fixed strike Asian option on a basket

7.1 Asian option on one future

We assume the following parameters $F(0) = 100$, $K = 100$ (at the money strike), $\sigma = 0.4$, $T = 1$, $n = 10$ (number of fixing dates), and thus $\Delta t = 0.1$, all unless otherwise mentioned. Furthermore, we use a 95 percent confidence interval for the Monte Carlo method (using $10^4$ paths) and we implemented a control variate (the geometric average) to reduce variance. The results are given for a call option.

In table 1 we varied the volatility. Starting with a low volatility (and thus a low variation of the futures price over time) we increased it until a volatility of 1 (when the value of the future fluctuates greatly).

$$\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Volatility} & \text{Sophis} & \text{Gentle} & \text{Levy} & \text{Curran} & \text{Monte Carlo} \\
\hline
0.2 & 4.9476 & 4.9312 & 4.9546 & 4.9459 & 4.9452 \pm 0.0013 \\
0.4 & 9.8761 & 9.7466 & 9.9328 & 9.8627 & 9.8655 \pm 0.0057 \\
0.6 & 14.7669 & 14.3348 & 14.9573 & 14.7222 & 14.7219 \pm 0.0156 \\
0.8 & 19.6015 & 18.5934 & 20.0496 & 19.4976 & 19.4762 \pm 0.0289 \\
1 & 24.3623 & 22.4331 & 25.2280 & 24.1640 & 24.1248 \pm 0.0523 \\
\hline
\end{array}$$

Table 1: Different volatilities for an Asian option on one future

From table 1 we can conclude that Curran works best for all volatilities. The Sophis method is second best with an accuracy of 2 significant digits, while Gentle underestimates the option value up to an error of 1.7, and Levy overestimates the value with a maximum of 1.1.

In table 2 we vary the number of fixing dates for the Asian option. For one fixing date the Sophis, Levy and Gentle methods are equivalent to the regular Black 76 model, since the underlying $A(T)$ is lognormally distributed. Furthermore, the geometric average is equivalent to the arithmetic average, and the $\sigma$ calculated using $G(T)$ is the same as the our defined $\sigma$. We see that the values for one fixing date of Sophis, Levy and Gentle coincide.

$$\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Number of fixing dates} & \text{Sophis} & \text{Gentle} & \text{Levy} & \text{Curran} & \text{Monte Carlo} \\
\hline
1 & 15.8519 & 15.8519 & 15.8519 & - & 15.8519 \pm 0.0000 \\
10 & 9.8761 & 9.7466 & 9.9328 & 9.8627 & 9.8579 \pm 0.0057 \\
20 & 9.5352 & 9.4092 & 9.5941 & 9.5213 & 9.5201 \pm 0.0055 \\
50 & 9.3300 & 9.2064 & 9.3902 & 9.3158 & 9.3166 \pm 0.0054 \\
100 & 9.2614 & 9.1387 & 9.3221 & 9.2472 & 9.2593 \pm 0.0052 \\
200 & 9.2271 & 9.1048 & 9.2881 & 9.2129 & 9.2142 \pm 0.0053 \\
\hline
\end{array}$$

Table 2: Different numbers of fixing dates for an Asian option on one future

We can conclude from table 2 that the number of fixing dates does not have an effect on the accuracy of all methods.

In table 3 we vary the strike price. A strike less than 100 is the in-the-money strike, a strike of 100 is at-the-money, and a strike price higher than 100 is out-of-the-money. It is clear that for a call option the price will be lower for a higher strike.
7.1.1 Conclusion

The Sophis method works well for all volatilities, though the error gets larger for higher volatility (a maximum error of 0.20). Furthermore we see that the standard deviation becomes bigger for bigger volatilities. This is of course obvious since for a higher volatility the future price fluctuates more. We see that the standard deviation of the option price gets smaller as the number of averaging dates increases. This is due to the fact that the underlying has less fluctuations when more fixing dates are used, this results in a lower price (the option has less risk). For a high strike (out-of-the-money strike) the error of the Sophis method increases until approximately 0.02. Overall Sophis has a maximum error of 0.25.

Gentle underestimates the Monte Carlo value for all volatilities and the higher the volatility the larger the underestimation, so for high volatilities Gentle does not work well. No big difference in the underestimation of Gentle is observed for different numbers of fixing dates. The higher the strike, the larger the underestimation of Gentle of the Monte Carlo value. Apparently, the approximation of the arithmetic average by the geometric average works worse for high strikes and high volatility. Gentle has a maximum error of 1.8, and this occurs for the high volatilities.

Levy also gets less precise for higher volatility, however the maximum error has a maximum of 1.1. Again no notable effect of the fixing dates on the Levy value is seen, the value remains reasonably accurate. An out-of-the-money strike causes the Levy value to diverge.

Curran works well under all circumstances, and always produces an error less than 0.1.

For one option we can conclude that Curran is the best method (an accuracy of 3 significant digits) and Sophis is second best (an accuracy of 2 significant digits).

7.1.2 Figures for one option

In this section the results in the tables for one option are plotted.

<table>
<thead>
<tr>
<th>Strike</th>
<th>Sophis</th>
<th>Gentle</th>
<th>Levy</th>
<th>Curran</th>
<th>Monte Carlo</th>
</tr>
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<tbody>
<tr>
<td>20</td>
<td>80.0000</td>
<td>80.0000</td>
<td>80.0000</td>
<td>80.0000</td>
<td>79.9973 ± 0.0079</td>
</tr>
<tr>
<td>80</td>
<td>22.2228</td>
<td>22.1367</td>
<td>22.2567</td>
<td>22.1179</td>
<td>22.1176 ± 0.0066</td>
</tr>
<tr>
<td>100</td>
<td>9.8761</td>
<td>9.7466</td>
<td>9.9328</td>
<td>9.8627</td>
<td>9.8670 ± 0.0058</td>
</tr>
<tr>
<td>120</td>
<td>3.6460</td>
<td>3.5473</td>
<td>3.6934</td>
<td>3.7445</td>
<td>3.7257 ± 0.0052</td>
</tr>
<tr>
<td>200</td>
<td>0.0272</td>
<td>0.0246</td>
<td>0.0289</td>
<td>0.0433</td>
<td>0.0426 ± 0.0022</td>
</tr>
</tbody>
</table>

Table 3: Different strikes for an Asian option on one future
Figure 3: Approximation vs Monte Carlo for different numbers of fixing dates on one future

Figure 4: Approximation vs Monte Carlo for different strike prices on one future
Figure 5: Approximation vs Monte Carlo for high strikes on one future

Figure 6: Approximation vs Monte Carlo for different volatilities on one future
Figure 7: Approximation vs Monte Carlo for high volatilities on one future
7.2 Asian option on two futures

We assume the following parameters $F_1(0) = 100$, $F_2(0) = 80$, $K = 180$ (in the money strike), $\sigma_1 = \sigma_2 = 0.4$, $T = 2$, $\rho_{1,2} = 0.5$, $n = 10$ and so $\Delta t = 0.1$, all unless otherwise mentioned. Furthermore, we use a 95 percent confidence interval for the Monte Carlo method (using $10^4$ paths) and we implemented a control variate (the geometric average) to reduce variance.

In table 4 we assume for the volatility that $\sigma_1 = \sigma_2$. We again vary the volatility from low to high.

<table>
<thead>
<tr>
<th>Volatility</th>
<th>Sophis</th>
<th>Gentle</th>
<th>Levy</th>
<th>Curran</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>10.9260</td>
<td>10.8129</td>
<td>10.9593</td>
<td>10.9293</td>
<td>10.9250 ± 0.0072</td>
</tr>
<tr>
<td>0.4</td>
<td>21.7888</td>
<td>20.9008</td>
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<td>21.7763 ± 0.0323</td>
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<td>0.6</td>
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<td>33.4375</td>
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<td>32.5906 ± 0.0871</td>
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<td>0.8</td>
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<td>36.4739</td>
<td>45.2576</td>
<td>43.1955</td>
<td>43.1372 ± 0.1744</td>
</tr>
<tr>
<td>1</td>
<td>53.3900</td>
<td>41.1637</td>
<td>57.6733</td>
<td>53.5512</td>
<td>53.7754 ± 0.3403</td>
</tr>
</tbody>
</table>

Table 4: Different volatilities for an Asian option on two futures

In table 5 we take a look at the preciseness of the option price for different numbers of fixing dates. A high number of fixing dates means that the arithmetic average is more averaged out.

<table>
<thead>
<tr>
<th>Fixing dates</th>
<th>Sophis</th>
<th>Gentle</th>
<th>Levy</th>
<th>Curran</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>34.9009</td>
<td>33.5490</td>
<td>35.1268</td>
<td>35.0834</td>
<td>35.1319 ± 0.0655</td>
</tr>
<tr>
<td>10</td>
<td>21.7888</td>
<td>20.9008</td>
<td>22.0567</td>
<td>21.8122</td>
<td>21.8357 ± 0.0334</td>
</tr>
<tr>
<td>50</td>
<td>20.5866</td>
<td>19.7750</td>
<td>20.8591</td>
<td>20.6014</td>
<td>20.6124 ± 0.0301</td>
</tr>
<tr>
<td>100</td>
<td>20.4357</td>
<td>19.6338</td>
<td>20.7089</td>
<td>20.4496</td>
<td>20.4479 ± 0.0299</td>
</tr>
<tr>
<td>200</td>
<td>20.3602</td>
<td>19.5632</td>
<td>20.6337</td>
<td>20.3736</td>
<td>20.3609 ± 0.0295</td>
</tr>
</tbody>
</table>

Table 5: Different numbers of fixing dates for an Asian option on two futures

In table 6 we again look at the accuracy of the method for in-the-money strikes, at-the-money strikes and out-of-the-money strikes.

<table>
<thead>
<tr>
<th>Strike</th>
<th>Sophis</th>
<th>Gentle</th>
<th>Levy</th>
<th>Curran</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>130.0000</td>
<td>130.0000</td>
<td>130.0000</td>
<td>130.0000</td>
<td>130.00044 ± 0.0377</td>
</tr>
<tr>
<td>100</td>
<td>80.4143</td>
<td>80.2885</td>
<td>80.4460</td>
<td>80.3120</td>
<td>80.3121 ± 0.0368</td>
</tr>
<tr>
<td>150</td>
<td>38.4093</td>
<td>37.6720</td>
<td>38.6141</td>
<td>38.1978</td>
<td>38.1781 ± 0.0347</td>
</tr>
<tr>
<td>250</td>
<td>4.5910</td>
<td>4.1026</td>
<td>4.7687</td>
<td>4.9618</td>
<td>4.9758 ± 0.0264</td>
</tr>
<tr>
<td>300</td>
<td>1.3568</td>
<td>1.1466</td>
<td>1.4430</td>
<td>1.6382</td>
<td>1.6376 ± 0.0199</td>
</tr>
</tbody>
</table>

Table 6: Different strikes for an Asian option on two futures

In table 7 we take a look at the accuracy of the methods for different correlation. A high negative correlation between two futures means that if one future goes up, the other future will move down. A zero correlation means that there is no relation between the two futures, whereas a high positive correlation means that if one future moves up the other one will also go up.
In table 7 we see that the Curran method again has the best performance, however, for a negative correlation, the Levy method also gives accurate values (it always has an accuracy of two significant digits or more).

7.2.1 Conclusion

The Sophis method keeps performing very accurate for all volatilities, although for higher volatility a higher deviation is observed (maximum error of 0.5). We do not see any notable effect of the number of fixing dates on the accuracy of the Sophis option price. We see that the option price for just one fixing date is much higher, since a European option’s final price has more volatility than the average price of an Asian (and thus more potential for a profit). For a higher strike (out-of-the-money strikes) the Sophis method is less exact (an error of maximum 0.3 is observed). The Sophis price deviates more as correlation decreases and a maximum error of about 1.6 is observed.

Gentle deviates more as volatility rises. A high volatility (0.6 or more) results in a large deviation from the Monte Carlo value (maximum difference of 12), therefore we can say that the Gentle method does not work well for high volatilities. We see that the Gentle value undervalues the Monte Carlo value. The fixing dates have no effect on the preciseness of Gentle (an underestimation remains). For different strikes we again see an underestimation, and this error deviates more as the strike rises, with the maximum error about 1. Apparently, approximating the arithmetic average by the geometric average, tends to be an underestimation. A negative correlation results in a bigger deviation, and this deviation is bigger than for Sophis.

Levy tends to overestimate the Monte Carlo value. A higher volatility causes more deviation in the Levy method (an error of 4). We have no noticeable effect regarding the number of fixing dates on the Levy value. It remains accurate. Levy remains reasonably accurate for all strikes, however it again deviates more for out-of-the-money strikes. Note, this deviation is smaller than the deviation observed with Sophis and Gentle. Note, that compared to Gentle and Sophis, Levy gives the most accurate value for all correlations (a maximum error of 0.6), however it still slightly overestimates the price for all correlations, this overestimation is bigger for lower correlations.

Curran again remains accurate under all circumstances (overall maximum error of about 0.3). A slight deviation can be observed when dealing with a high number of fixing dates. Note also, that Curran remains accurate for all correlations. One may be surprised that, for all correlations Curran remains precise, while Levy overestimates the price, since Curran is the sum of an exact part and an approximated part, and where the exact part is based on the Levy approach. The reason for this is that the Levy approximation is used for computing the part of Curran

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Sophis</th>
<th>Gentle</th>
<th>Levy</th>
<th>Curran</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.8</td>
<td>8.3969</td>
<td>7.7424</td>
<td>10.6928</td>
<td>9.8773</td>
<td>10.0985 ± 0.0902</td>
</tr>
<tr>
<td>-0.4</td>
<td>13.9807</td>
<td>13.0487</td>
<td>14.9527</td>
<td>14.6398</td>
<td>14.7060 ± 0.0730</td>
</tr>
<tr>
<td>0</td>
<td>17.8859</td>
<td>16.8979</td>
<td>18.3696</td>
<td>18.1604</td>
<td>18.1886 ± 0.0554</td>
</tr>
<tr>
<td>0.4</td>
<td>21.0677</td>
<td>20.1476</td>
<td>21.3559</td>
<td>21.1267</td>
<td>21.1435 ± 0.0371</td>
</tr>
<tr>
<td>0.8</td>
<td>23.8178</td>
<td>23.0565</td>
<td>24.0757</td>
<td>23.7701</td>
<td>23.7704 ± 0.0230</td>
</tr>
<tr>
<td>1</td>
<td>25.0763</td>
<td>24.4229</td>
<td>25.3638</td>
<td>25.0087</td>
<td>25.0077 ± 0.0228</td>
</tr>
</tbody>
</table>

Table 7: Different correlation for an Asian option on two futures
that deals with an out-of-the-money strike, and we see that Levy underestimates the price for these strikes, therefore an overestimation plus an underestimation results in a reasonably precise solution.

Note the following: with Levy we approximate $A(T)$ by a lognormal distribution, this overestimates the Monte Carlo value for a high volatility, in Gentle we approximate the $A(T)$ by a geometric average, this causes an underestimation for high volatility. Sophis combines both Levy and Gentle, so an overestimate and an underestimate, and therefore Sophis appears to work well for all volatilities.

We can conclude that for two futures Curran again is the most accurate method. However, for a negative correlation and extremely out-of-the-money strikes Levy works very well and is therefore second best, and for high volatility Sophis is second best.

### 7.2.2 Figures for two options

In this section the results in the tables for two options are plotted.

![Figure 8: Approximation vs Monte Carlo for different numbers of fixing dates on two futures](image)
Figure 9: Approximation vs Monte Carlo for different strike prices on two futures

Figure 10: Approximation vs Monte Carlo for out-of-the-money strikes on two futures
Figure 11: Approximation vs Monte Carlo for different volatilities on two futures

Figure 12: Approximation vs Monte Carlo for high volatilities on two futures
7.3 Asian option on ten futures

We use the following parameters: $F_i(0) = 10$ for all $i$, $\sigma_i = 0.4$ for all $i$, the correlation between future one and two is equal to 0.5, between two and three equal to 0, between three and four equal to 0.5 etc., $T$ is 2, the number of fixing dates is 10 and the strike price is 100, all unless otherwise stated. For the Monte Carlo method we used a 95 percent confidence interval and $10^4$ number of paths.

In table 8 we take a look at the effect of changing the volatility on the accuracy of the methods.

<table>
<thead>
<tr>
<th>Volatility</th>
<th>Monte Carlo</th>
<th>Sophis</th>
<th>Gentle</th>
<th>Levy</th>
<th>Curran</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>2.6784 ± 0.0815</td>
<td>2.7111</td>
<td>2.6583</td>
<td>2.7320</td>
<td>2.7303</td>
</tr>
<tr>
<td>0.4</td>
<td>5.5205 ± 0.1766</td>
<td>5.4191</td>
<td>5.0087</td>
<td>5.5907</td>
<td>5.5707</td>
</tr>
<tr>
<td>0.6</td>
<td>8.6010 ± 0.2938</td>
<td>8.6208</td>
<td>6.8020</td>
<td>8.7249</td>
<td>8.6281</td>
</tr>
<tr>
<td>0.8</td>
<td>11.8013 ± 0.4331</td>
<td>11.8132</td>
<td>7.8911</td>
<td>12.3328</td>
<td>12.0186</td>
</tr>
</tbody>
</table>

Table 8: Different volatilities for an Asian option on ten futures

In table 9 different numbers of fixing dates are used.

<table>
<thead>
<tr>
<th>Number of fixing dates</th>
<th>Monte Carlo</th>
<th>Sophis</th>
<th>Gentle</th>
<th>Levy</th>
<th>Curran</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.1025 ± 0.3025</td>
<td>8.7229</td>
<td>7.6138</td>
<td>9.2157</td>
<td>9.1760</td>
</tr>
<tr>
<td>50</td>
<td>5.3231 ± 0.1665</td>
<td>5.1184</td>
<td>4.7563</td>
<td>5.2748</td>
<td>5.2550</td>
</tr>
<tr>
<td>100</td>
<td>5.4254 ± 0.1705</td>
<td>5.0807</td>
<td>4.7245</td>
<td>5.2352</td>
<td>5.2155</td>
</tr>
</tbody>
</table>

Table 9: Different numbers of fixing dates for an Asian option on ten futures

Again, in table 10 we look at different correlations. From the results with two futures we know that Sophis, Levy and Gentle had the largest error for a highly negative correlation. Therefore, we decided to look at a highly positive correlation (0.8), a highly negative (-0.8) and a correlation of 0, since these are the most interesting values.

<table>
<thead>
<tr>
<th>Number of fixing dates</th>
<th>Monte Carlo</th>
<th>Sophis</th>
<th>Gentle</th>
<th>Levy</th>
<th>Curran</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.8</td>
<td>2.5608 ± 0.0812</td>
<td>1.9801</td>
<td>1.8155</td>
<td>2.5862</td>
<td>2.5296</td>
</tr>
<tr>
<td>0</td>
<td>4.6279 ± 0.1445</td>
<td>4.4258</td>
<td>4.0780</td>
<td>4.6064</td>
<td>4.5923</td>
</tr>
<tr>
<td>0.8</td>
<td>6.1222 ± 0.1960</td>
<td>5.9354</td>
<td>5.4960</td>
<td>6.1373</td>
<td>6.1093</td>
</tr>
</tbody>
</table>

Table 10: Different correlation for an Asian option on ten futures

Overall, the methods keep performing the same way as for two futures. Curran is still the most accurate method overall. Levy works good for all correlations, while Gentle and Sophis deviate from the Monte Carlo value for a negative correlation. Gentle again underestimates the Monte Carlo value, especially for high volatility. The Sophis method performs good for all volatilities. When comparing the relative maximum errors of the methods with ten futures compared to two and one futures we get the following:
We see that the relative error for two futures is overall bigger than for one future. For ten futures the relative error is slightly smaller than for two futures (except for the Gentle method, where the error is bigger). The difference is however not big enough to derive any conclusions about whether the method converges or not for more futures. Please note, the method we would especially expect to converge for more futures would be the Levy method, since it is the one where a sum is approximated by a lognormal (and from the central limit theorem the sum might converge to a lognormal). We can therefore also see that Levy performs better for 10 futures and a high volatility than it did for two futures and a high volatility. The absolute error for Gentle and Levy is smaller for ten futures than for two futures. Overall, it is possible to conclude that Sophis, Levy and Curran keep performing good for ten futures, while Gentle again has a high deviation for a high correlation (as was the case with one and two futures). Curran is again the best method, and the Sophis method is second best for volatility, while Levy performs good for a negative correlation, and thus is second best for that.

<table>
<thead>
<tr>
<th>Number of futures</th>
<th>Sophis</th>
<th>Gentle</th>
<th>Levy</th>
<th>Curran</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>0.37</td>
<td>0.09</td>
<td>0.04</td>
<td>0.00</td>
</tr>
<tr>
<td>Two</td>
<td>0.19</td>
<td>0.31</td>
<td>0.07</td>
<td>0.03</td>
</tr>
<tr>
<td>Ten</td>
<td>0.09</td>
<td>0.38</td>
<td>0.04</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Table 11: Relative error
8 Mathematical explanations of the numerical observations

From the results we have seen, we can conclude that approximating the arithmetic average by a geometric average tends to underestimate the option value. Furthermore, approximating the \( A(T) \) by a lognormal distribution can lead to overestimating the option value. Both a high volatility and a negative correlation resulted in bigger differences (usually underestimations) between the option price from the Monte Carlo value. However, if we combine both the approximation of \( A(T) \) by a lognormal and the approximation by a geometric average (so an overestimate and an underestimate, as in Sophis), the results remain accurate also for high volatilities.

8.1 Approximating \( A(T) \) by a lognormal distribution

Let’s first take a look at the case where an approximate distribution follows directly from the model. This is the case for a very small volatility. First, we take a simplified version of the previously known \( A(T) \) (just for the simplicity of the derivations), namely just a sum of the basket futures (not averaged over the fixing dates)

\[
L(T) = \sum_{i=1}^{p} F_i(t) = \sum_{i=1}^{p} F_i(0) \exp \left( -\frac{1}{2} \sigma_i^2 t + \sigma_i W_i(t) \right),
\]

The mean of \( \left( -\frac{1}{2} \sigma_i^2 t + \sigma_i W_i(t) \right) \) is equal to \( \tilde{\mu}_i = -\frac{1}{2} \sigma_i^2 t \) and the variance is \( \sigma_i^2 t \).

We first look more closely at

\[
F_i(t) = F_i(0) \exp \left( -\frac{1}{2} \sigma_i^2 t + \sigma_i W_i(t) \right).
\]

Denote \( X_i = \sigma_i W_i(t) \). We have \( W_i(t) \sim \mathcal{N}(0, t) \) and thus \( X_i \sim \mathcal{N}(0, \sigma_i^2 t) \). Define \( g(X_i) = \exp(X_i) \). We can then use the Delta method, so that we have

\[
\exp(X_i) - \exp(0) \sim \mathcal{N}(0, \sigma_i^2 t \cdot g'(0)).
\]

We know that \( g'(0) = \exp(0) = 1 \), and thus

\[
\exp(X_i) - 1 \sim \mathcal{N}(0, \sigma_i^2 t).
\]

We can rewrite this using the fact that \( X_i \sim \mathcal{N}(0, \sigma_i^2 t) \):

\[
\exp(X_i) \sim 1 + \mathcal{N}(0, \sigma_i^2 t) \approx 1 + X_i,
\]

and so we have

\[
F_i(t) = F_i(0) \exp \left( -\frac{1}{2} \sigma_i^2 t + \sigma_i W_i(t) \right)
= F_i(0) \exp \left( -\frac{1}{2} \sigma_i^2 t \right) \exp(X_i)
\approx F_i(0) \exp \left( -\frac{1}{2} \sigma_i^2 t \right) (1 + X_i)
\]

This results in

\[
L(T) \approx \sum_{i=1}^{p} F_i(0) \exp \left( -\frac{1}{2} \sigma_i^2 t \right) (1 + X_i),
\]
which is a sum of normal variables \((X_i)\), and thus also approximately normally distributed. Then, if the variance of the sum is also small enough, \(L(T)\) is approximately lognormal. We can conclude, the lognormal approximation should work well for very small volatilities.

Now, let’s look more closely at which impact changing the volatility, correlation, and number of fixing dates have on the distribution of \(A(T)\) for the Asian option on a basket. Changes in one or more parameters have an immediate effect on the mean and the variance of the price of the entire basket. Remember that we have for

\[
A(T) = \frac{1}{n} \sum_{i=1}^{p} \sum_{j=1}^{n} F_i(t_j)
\]

the following

\[
E[A(T)] = \sum_{i=1}^{p} F_i(0),
\]

and

\[
\text{Var}[A(T)] = E[A(T)^2] - E[A(T)]^2
\]

\[
= \frac{1}{n^2} \sum_{i_1,i_2=1}^{p} \sum_{j_1,j_2=1}^{n} F_{i_1} F_{i_2} \left[ \exp(\sigma_{i_1} \sigma_{i_2} \rho(i_1, i_2) \min(t_{j_1}, t_{j_2})) - 1 \right].
\]

Changing \(F_i(0)\) has an effect on both the mean and the variance. Raising it will result in an increase of both the mean and the variance, so the resulting distribution is more spread out over the positive real line. Conversely, lowering \(F_i(0)\) results in a more peaked probability density function. Changing either the volatility or the correlation only has effect on the variance. Higher values cause the probability density to be more flattened, whereas lower values lead up to a distribution whose mass is highly concentrated on a relatively small range. However, in both cases the mean price is the same.

Note also that if two stocks are negatively correlated then when one future price rises the other will fall and the sum of both will lie closely around some mean value.

When estimating the arithmetic average of the Asian option on a basket we could assume this was lognormally distributed with mean \(\mu = \log(M) - \frac{1}{2} \sigma^2 t\) and volatility \(\tau = \sigma \sqrt{t}\), so

\[
A(T) = M \exp \left( -\frac{1}{2} \sigma^2 t + \sigma W(T) \right)
\]

We can compare this approximated distribution with the real values of \(A(T)\). To do this we make a histogram of the real values of \(A(T)\) (using \(10^5\) samples), and fit a lognormal probability distribution. Next, we test the null hypothesis that the \(A(T)\) is drawn from a lognormal distribution using a Kolmogorov-Smirnov test. If the test tells us to reject the null hypothesis we know that the lognormal distribution is unsuitable to approximate the actual distribution of the basket. We again assume we have two futures with \(F_1(0) = 100, F_2(0) = 80, K = 180\) (in the money strike), \(\sigma_1 = \sigma_2 = 0.4, T = 2, \rho_{1,2} = 0.5\) and \(n = 10\), unless otherwise mentioned.

First we see what happens to the distribution if we use a large and a small volatility. The Kolmogorov-Smirnov test tells us to accept the null hypothesis for small volatilities, however for volatilities above 0.7 it rejects the null hypothesis. The histograms still look reasonable, although the histogram for the high volatility fits the lognormal worse.
Now, we experiment with a positive and negative correlation. From previous results we know that especially a highly negative correlation resulted in a large deviation of the approximating method from the Monte Carlo value. We want to verify that indeed the lognormal distribution is a bad approximation for the arithmetic average for a negative correlation. When testing, the Kolmogorov-Smirnov test accepts the null hypothesis for positive correlations, however for highly negative it rejects it. Again, we see that the histogram for a negative correlation looks worse, however still acceptable.
Thus, we can conclude that the lognormal distribution is an inaccurate approximation for baskets containing stocks which have a negative correlation structure. This is possibly explained by the fact that the distribution of the basket is highly concentrated around one specific value (if we have a negative correlation) as mentioned above. The lognormal distribution however spreads over the complete positive real line. Therefore whereas the basket stays concentrated around a specific value, the lognormal distribution will place mass there where the basket does not attain values. Thus, the lognormal does not fit closely. For the lognormal distribution spreads its mass over the complete positive real line, a significant amount of mass is put inside ranges where the basket price will not take its value and the lognormal density will not closely fit the observations. Also, a high volatility causes the distribution of the basket to have a relatively large right tail due to a relatively large number of high prices (since the volatility is higher the price can attain higher values), however the lognormal distribution does not match this. Therefore, for high volatility the obtained option price using the approximation will overestimate the true value (as is the case in the Levy model).
8.2 Approximating the arithmetic average $A(T)$ by a geometric average $G(T)$

We have

$$A(T) = \frac{1}{n} \sum_{i=1}^{p} \sum_{j=1}^{n} F_i(t_j),$$

and

$$G(T) = \prod_{j=1}^{n} \prod_{i=1}^{p} \left( \frac{1}{n\alpha_i} F_i(t_j) \right)^{\alpha_i},$$

with

$$\alpha_i = \frac{1}{n} \frac{F_i(0)}{\sum_{i=1}^{p} F_i(0)}.$$

First note that $G(T) \leq A(T)$ (see plot 17). This can be verified mathematically. For simplicity, we take just one future, so

$$A(T) = \frac{1}{n} \sum_{j=1}^{n} F(t_j)$$

and

$$G(T) = \prod_{j=1}^{n} \left( \frac{1}{n\alpha_i} F_i(t_j) \right)^{\alpha_i},$$

where $\alpha_i = \frac{1}{n}.$

We determine $\log(A(T))$ and $\log(G(T))$

$$\log(A(T)) = \log \left( \frac{1}{n} \sum_{j=1}^{n} F(t_j) \right),$$

and

$$\log(G(T)) = \sum_{j=1}^{n} \log(F(t_j)^{\frac{1}{n}}) = \frac{1}{n} \sum_{j=1}^{n} \log(F(t_j)).$$

Now, recall Jensen’s inequality which states the following: For a real concave function $\phi$ and the numbers $x_1, x_2, ..., x_n$ in its domain the following holds

$$\phi \left( \frac{\sum (x_i)}{n} \right) \geq \frac{\sum \phi(x_i)}{n}.$$

The function $\log(x)$ is concave, therefore

$$\log \left( \frac{1}{n} \sum_{j=1}^{n} F(t_j) \right) \geq \frac{1}{n} \sum_{j=1}^{n} \log(F(t_j)).$$

In conclusion we have $G(T) \leq A(T)$. Therefore, also $E(G(T)) \leq E(A(T))$, and we would expect that when approximating the arithmetic average by the geometric average, the option price would underestimate the true value. This is exactly the case in the Gentle approximation.
Furthermore, the geometric average follows a lognormal distribution, therefore, this approximation will work roughly the same way as the lognormal approximation.

8.3 A closer look at the correlation

Let’s assume we have

\[ Y_1 = a_1 \exp\left(-\frac{1}{2}\sigma_1^2 t + \sigma_1 W_1(t)\right), \]
\[ Y_2 = a_2 \exp\left(-\frac{1}{2}\sigma_2^2 t + \sigma_2 W_2(t)\right), \]

where \( W_1 \) and \( W_2 \) are Brownian motions, with correlation \( \rho \) and with the following property

\[ W_i(t) = \sqrt{t}X_i, \text{ where } X_i \text{ is a standard normal random variable.} \]

Now, we want to calculate the correlation between \( Y_2 \) and \( Y_2 \).

\[
E[Y_1 Y_2] = E[a_1 a_2 \exp\left(-\frac{1}{2}\sigma_1^2 t - \frac{1}{2}\sigma_2^2 t + \sigma_1 W_1(t) + \sigma_2 W_2(t)\right)].
\]

Using the properties in Appendix 1 we arrive at

\[
E[Y_1 Y_2] = a_1 a_2 \exp\left(-\frac{1}{2}\sigma_1^2 t - \frac{1}{2}\sigma_2^2 t\right) \exp\left(\frac{1}{2}t(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)\right).
\]

So,

\[
\text{Cov}(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2]
\]

\[
= a_1 a_2 \exp\left(-\frac{1}{2}\sigma_1^2 t - \frac{1}{2}\sigma_2^2 t\right) \exp\left(\frac{1}{2}t(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)\right) - 1.
\]

Furthermore, we know

\[
\text{Var}(Y_i) = E[Y_i^2] - E[Y_i]^2 = a_i^2 \exp(\sigma_i^2 t) - a_i^2
\]
Therefore, for the correlation between $Y_1$ and $Y_2$ we arrive at

$$\text{Cor}(Y_1, Y_2) = \frac{\text{Cov}(Y_1, Y_2)}{\sqrt{\text{Var}(Y_1)}\sqrt{\text{Var}(Y_2)}}$$

$$= \frac{a_1a_2(\exp(-\frac{1}{2}\sigma_1^2 t - \frac{1}{2}\sigma_2^2 t) \exp(\frac{1}{2}t(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)) - 1)}{\sqrt{a_1^2 \exp(\sigma_1^2 t)} - a_1^2 \sqrt{\exp(\sigma_2^2 t)} - a_2^2}$$

$$= \frac{\exp(\rho\sigma_1\sigma_2 t) - 1}{\sqrt{\exp(\sigma_1^2 t)} - 1 \sqrt{\exp(\sigma_2^2 t)} - 1}.$$ 

We see that the correlation between $Y_1$ and $Y_2$ depends on the correlation between $W_1$ and $W_2$. If $\rho$ is negative we see that $\text{Cor}(Y_1, Y_2)$ will also be negative, however, note that $\text{Cor}(Y_1, Y_2)$ will be larger in value than $\rho$. 
9 Pricing an Average Rate/Average Strike Asian option

Until now we have focused on the fixed strike Asian option (the underlying price is averaged), however, there are many more types of Asian options. We discuss now a new type of Asian options, the so-called Asian option on average underlying price (the spot price) with average strike (an average strike/average rate option: an AS/AR option), so that the payoff of this option depends on the difference between two arithmetic averages of the future prices. We assume this option has an observation period that determines the strike prior to the payout observation period. The buyer of an AS/AR option doesn’t know what the option’s strike will be until the final observation of the strike averaging period has passed. Once the rates have been collected to determine the strike, the option buyer has unlimited protection should the average spot price rise above the average strike. However, if the option expires out-of-the-money, the holders maximum loss is limited to the premium paid. Compared to a vanilla call, AS/AR option prices are lower due to the averaging factors. Companies most frequently use the cashsettled AS/AR options to hedge economic risk, namely projected profits, budgets and other accounting-related exposures[5].

First we assume we have one future, and thus the AS/AR option has payoff

\[ V(T) = \left( \frac{1}{n_s} \sum_{i=s}^{n} F(t_i) - \frac{1}{n_k} \sum_{j=1}^{k} F(t_j) \right)^+, \]

where for the average strike we take the average over \( n_k = k \) time steps \( t_1, ..., t_k \), and for our average price we take the average over \( n_s = n - s + 1 \) time steps \( t_s, ..., t_n = T \).

Note, we can rewrite the payoff as

\[ V(T) = \left( \frac{1}{n_k} \sum_{j=1}^{k} F(t_j) \right) \cdot \left( \frac{1}{n_s} \sum_{i=s}^{n} F(t_i) - \frac{1}{n_k} \sum_{j=1}^{k} F(t_j) \right)^+, \]

or at time zero

\[ V(0) = e^{-rT} E \left[ \left( \frac{1}{n_k} \sum_{j=1}^{k} F(t_j) \right) \cdot \left( \frac{1}{n_s} \sum_{i=s}^{n} F(t_i) - \frac{1}{n_k} \sum_{j=1}^{k} F(t_j) \right)^+ \right] \]

\[ \approx e^{-rT} E \left[ \left( \frac{1}{n_k} \sum_{j=1}^{k} F(t_j) \right) \cdot \left( \frac{1}{n_s} \sum_{i=s}^{n} F(t_i) - 1 \right)^+ \right] E \left[ \frac{1}{n_k} \sum_{j=1}^{k} F(t_j) \right]. \]

Please note, we have used here \( E[AB] = E[A]E[B] \), this of course is only the case when \( A \) and \( B \) are independent. In our case \( A \) and \( B \) are not independant, so equation (19) is an approximation.

9.1 Levy for the AS/AR option

We know from previous results that Levy works well for all correlations and is easiest to implement, therefore we would like to know the Levy price for the AS/AR option.

For the first part of the price note that the average strike and average price are not lognormally distributed, therefore using Levy’s moment matching we approximate both by a lognormal
distribution: \( M \exp(-\frac{1}{2} \sigma^2 T + \sigma W(T)) \). So, we approximate again

\[
A_s(T) = \frac{1}{n_s} \sum_{i=s}^{n} F(t_i) \quad \text{and} \quad A_k(T) = \frac{1}{n_k} \sum_{j=1}^{k} F(t_j),
\]

by lognormal variables. Levy then gives us the following parameters:

\[
M_s = F(0) \quad \text{and} \quad M_k = F(0),
\]

and

\[
\sigma^2_s T = \log(E[A_s^2(T)]) - 2 \log(M) \quad \text{and} \quad \sigma^2_k T = \log(E[A_k^2(T)]) - 2 \log(M),
\]

where

\[
E(A_s^2(T)) = \frac{1}{n_s^2} \sum_{j_1=s}^{n_s} \sum_{j_2=s}^{n_s} F(0)^2 \exp(\sigma^2 \min(t_{j_1}, t_{j_2}))
\]

\[
E(A_k^2(T)) = \frac{1}{n_k^2} \sum_{j_1=1}^{n_k} \sum_{j_2=1}^{n_k} F(0)^2 \exp(\sigma^2 \min(t_{j_1}, t_{j_2})).
\]

What we have now is the following: we approximated \( A_s \) and \( A_k \) by a lognormal distribution, so we have

\[
A_s = M_s \exp(-\frac{1}{2} \sigma^2_s T + \sigma_s W(T)) \quad \text{and} \quad A_k = M_k \exp(-\frac{1}{2} \sigma^2_k T + \sigma_k W(T)).
\]

Next, in order to obtain the lognormal approximation for \( \frac{A_s}{A_k} \), we divide the obtained lognormal variables and we get again a lognormal distribution (since the product of lognormal variables is again lognormally distributed) which satisfies the following formula:

\[
\frac{A_s}{A_k} \approx \frac{M_s}{M_k} \exp(-\frac{1}{2} \sigma^2_s T + \sigma_s W(T)) + \frac{1}{2} \sigma^2_k T - \sigma_k W(T)
\]

\[
= 1 \exp(-\frac{1}{2} (\sigma^2_s - \sigma^2_k) T + (\sigma_s - \sigma_k) W(T)).
\]

Thus, we can apply the Black 76 formula (see (14)) with

\[
M = 1, \quad \sigma^2 T = \sigma^2_s T - \sigma^2_k T.
\]

We multiply the price obtained with the Black 76 formula by the expectation of \( \frac{1}{n_k} \sum_{j=1}^{k} F(t_j) \), which is

\[
E \left[ \frac{1}{n_k} \sum_{j=1}^{k} F(t_j) \right] = F(0),
\]

to obtain the final price for the AS/AR option on one future.

Note that for more than one future (the general case with \( p \) futures in our basket) we have the following parameters:

\[
M = 1, \quad \sigma^2 T = \sigma^2_s T - \sigma^2_k T,
\]

(20) \quad (21)
Each by a lognormal age strike and average price are not lognormal, therefore using Sophis we want to approximate 

\[ p \]

Suppose we have

\[ \text{look at the performance of the Sophis method for the AS/AR option.} \]

Sophis is simple to implement and works well for all volatilities. Therefore it is interesting to

9.2 Sophis for the AS/AR option

This is our final option price approximated using Levy.

Finally we multiply the obtained Black’s price by the expectation of \( \frac{1}{n_k} \sum_{j=1}^{k} \sum_{i=1}^{p} F_i(t_j) \) which equals

\[ \sum_{i=1}^{p} F_i(0). \]  \hspace{1cm} (22)

This is our final option price approximated using Levy.

9.2 Sophis for the AS/AR option

Sophis is simple to implement and works well for all volatilities. Therefore it is interesting to look at the performance of the Sophis method for the AS/AR option.

Suppose we have \( p \) futures in our basket. For the first part of the price note that the average strike and average price are not lognormal, therefore using Sophis we want to approximate each by a lognormal \( M \exp(-\frac{1}{2} \sigma^2 T + \sigma W(T)) \) to obtain a value for \( M \). This is the same as in Levy, thus we have \( M = 1 \), see (20).

Next, to obtain a value for \( \sigma \) we approximate both \( A_s(T) \) and \( A_k(T) \) by a geometric averages \( G_s(T) \) and \( G_k(T) \). Then we will calculate the variance of \( \frac{G_s(T)}{G_k(T)} \) and use it as an approximation for the variance of \( \frac{A_s(T)}{A_k(T)} \).

First note that we can write

\[
\frac{G_s}{G_k} = \prod_{i=1}^{p} \prod_{j=s}^{p} \left( \frac{1}{n_{i,j}} \frac{F_i(t_j)}{F_i(t_j)} \right)^{\alpha_{i,s}} \\
= \sum_{i=1}^{p} \sum_{j=s}^{p} \frac{F_i(0)}{n_{i,j}} \exp \left( -\frac{1}{2} \alpha_{i,s} \sigma^2 t_j + \alpha_{i,s} \sigma W_i(t_j) \right) \]

and use it as an approximation

\[
\sum_{i=1}^{p} \sum_{j=1}^{p} \frac{F_i(0)}{n_{i,j}} \exp \left( -\frac{1}{2} \alpha_{i,k} \sigma^2 t_j + \sigma W_i(t_j) \right) \]

where

\[
\alpha_{i,s} = \frac{1}{n_{i,s}} F_i(0), \hspace{1cm} \alpha_{i,k} = \frac{1}{n_{i,k}} F_i(0).
\]
In the second equality we have used equation (12). We wish to calculate the variance of the logarithm of (24) (see Section 6.1 for more details).

We have

\[ E \left[ \log \left( \frac{G_s}{G_k} \right) \right] = \sum_{i=1}^{p} \sum_{j=s}^{n} \left( -\frac{1}{2} \alpha_i \sigma_i^2 t_j + \alpha_i \sigma_i W_i(t_j) \right) - \sum_{i=1}^{p} \sum_{j=1}^{k} \left( -\frac{1}{2} \alpha_i \sigma_i^2 t_j + \alpha_i \sigma_i W_i(t_j) \right) \]

\[ = \sum_{i=1}^{p} \sum_{j=s}^{n} \left( -\frac{1}{2} \sigma_i^2 t_j \right) - \sum_{i=1}^{p} \sum_{j=1}^{k} \left( -\frac{1}{2} \sigma_i^2 t_j \right) := x_0, \]

\[ E \left[ \log \left( \frac{G_s^2}{G_k^2} \right) \right] = x_0^2 + E \left[ \left( \sum_{i=1}^{p} \sum_{j=s}^{n} \alpha_i \sigma_i W_i(t_j) - \sum_{i=1}^{p} \sum_{j=1}^{k} \alpha_i \sigma_i W_i(t_j) \right)^2 \right] \]

\[ := x_0^2 + y_0^2. \]

Remember that

\[ \text{Var} \left( \log \left( \frac{G_s}{G_k} \right) \right) = E \left[ \log \left( \frac{G_s^2}{G_k^2} \right) \right] - E \left[ \log \left( \frac{G_s}{G_k} \right) \right]^2 = y_0. \]

We calculate

\[ y_0^2 = E \left[ \left( \sum_{i=1}^{p} \sum_{j=s}^{n} \alpha_i \sigma_i W_i(t_j) \right)^2 \right] - 2E \left[ \sum_{i=1}^{p} \sum_{j=s}^{n} \alpha_i \sigma_i W_i(t_j) \sum_{i=1}^{k} \alpha_i \sigma_i W_i(t_j) \right] \]

\[ = \sum_{j_1, j_2=s, i_1, i_2=1}^{p} \alpha_{i_1} \alpha_{i_2} \sigma_{i_1} \sigma_{i_2} \rho(i_1, i_2) \min(t_{j_1}, t_{j_2}) \]

\[ + \sum_{j_1, j_2=1, i_1, i_2=1}^{k} \alpha_{i_1} \alpha_{i_2} \sigma_{i_1} \sigma_{i_2} \rho(i_1, i_2) \min(t_{j_1}, t_{j_2}) \]

\[ - 2 \sum_{j_1=1}^{p} \sum_{j_2=s}^{n} \sum_{i_1, i_2=1}^{p} \alpha_{i_1} \alpha_{i_2} \sigma_{i_1} \sigma_{i_2} \rho(i_1, i_2) \min(t_{j_1}, t_{j_2}). \]

For more details on this calculation we refer to Section 6. So for the variance we get (25). Now, we also know \( M = 1 \), so we can use the Black 76 formula to calculate the Black price, and again multiply this by equation (22) to get our final option price obtained with Sophis.
9.3 Results

We assume the following for all futures: we calculate the price over \( n = 100 \) time steps with \( s = 95 \) and \( k = 5, F_1(0) = 100, F_2(0) = 80, \sigma = 0.4 \) and \( \rho_{1,2} = 0.5 \) and \( T = 1 \) for one future and \( T = 2 \) for two futures. Furthermore we do \( 10^5 \) Monte Carlo paths using antithetic variables to reduce the variance. Note that the value of \( k \) gives the date until which the strike is averaged. Thus for \( k = 5 \) the strike is averaged over the dates \( t_1, \ldots, t_k \). The value of \( s \) gives the date on which the averaging of the spot price starts. Thus a value of \( s = 95 \) means that the spot price is averaged on dates \( t_{95}, \ldots, t_{100} \).

In table 12 and 13 we see the results for the Levy and Sophis method for different volatilities. Again we see the standard error of the Monte Carlo value increases as the volatility of the future increases.

<table>
<thead>
<tr>
<th>Volatility</th>
<th>Monte Carlo</th>
<th>Levy</th>
<th>Sophis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>7.6624 ± 0.0348</td>
<td>7.7371</td>
<td>7.8989</td>
</tr>
<tr>
<td>0.4</td>
<td>15.2446 ± 0.0839</td>
<td>15.4010</td>
<td>15.7206</td>
</tr>
<tr>
<td>0.6</td>
<td>22.8402 ± 0.1510</td>
<td>22.9231</td>
<td>23.3901</td>
</tr>
<tr>
<td>0.8</td>
<td>30.0633 ± 0.2401</td>
<td>30.2350</td>
<td>30.8368</td>
</tr>
<tr>
<td>1</td>
<td>36.6096 ± 0.3516</td>
<td>37.2764</td>
<td>37.9966</td>
</tr>
</tbody>
</table>

Table 12: Different volatilities for an AS/AR option on one future

<table>
<thead>
<tr>
<th>Volatility</th>
<th>Monte Carlo</th>
<th>Levy</th>
<th>Sophis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>16.9463 ± 0.0805</td>
<td>17.1061</td>
<td>17.4348</td>
</tr>
<tr>
<td>0.4</td>
<td>33.8759 ± 0.2031</td>
<td>34.1329</td>
<td>34.6136</td>
</tr>
<tr>
<td>0.6</td>
<td>50.5652 ± 0.1536</td>
<td>50.9829</td>
<td>51.2917</td>
</tr>
<tr>
<td>0.8</td>
<td>66.5856 ± 0.2612</td>
<td>67.5173</td>
<td>67.2458</td>
</tr>
<tr>
<td>1</td>
<td>82.4061 ± 0.4400</td>
<td>83.5339</td>
<td>82.2834</td>
</tr>
</tbody>
</table>

Table 13: Different volatilities for an AS/AR option on two futures

In table 14 and 16 we look at different averaging dates for the AS/AR option. A high value of \( k \) means that the strike remains undetermined for a longer time. Only at time \( k \) the strike is completely known. A high value of \( s \) means that the calculation of the spot price starts earlier.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( s )</th>
<th>Monte Carlo</th>
<th>Levy</th>
<th>Sophis</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>95</td>
<td>15.2446 ± 0.0839</td>
<td>15.4010</td>
<td>15.7206</td>
</tr>
<tr>
<td>20</td>
<td>80</td>
<td>13.4810 ± 0.0728</td>
<td>14.1387</td>
<td>15.3414</td>
</tr>
<tr>
<td>5</td>
<td>80</td>
<td>14.5453 ± 0.0780</td>
<td>14.5719</td>
<td>14.9080</td>
</tr>
</tbody>
</table>

Table 14: Different averaging dates for an AS/AR option on one future
Table 15: Different averaging dates for an AS/AR option on two futures

Lastly, it is of course again interesting to look at the performance of the Levy and Gentle method for negative and positive correlations. Our guess would be that especially the Sophis method will deviate from the Monte Carlo value for a negative correlation, since this was also the case with the fixed strike option.

Table 16: Different correlations for an AS/AR option on two futures

Overall, the Levy method works best for the AS/AR option. The absolute error with Levy for negative correlations is less than the error using Sophis. For a high volatility though, the Sophis method is slightly more exact. Note, when we take a large number of $n_k$, both Sophis and Levy diverge from the Monte Carlo value (the Levy error is less than the Sophis error though). This may be because when $n_k$ is large, it takes a longer time before the strike is known, and this results in a less exact approximation.
10 Conclusion

In the first part of this thesis stocks, options and futures were explained. The market was assumed to be arbitrage-free and complete. Under these assumptions a model for the stock price was developed. This model assumed that stock prices were distributed lognormally. Using the lognormal model for the stock price we derived a model for the futures prices. These also follow a lognormal distribution, however the drift is zero.

We looked at the Black 76 model which is universally used to price an option on a future. However, this model assumes the underlying to be lognormally distributed. As we saw, with an Asian option on a basket the underlying is the arithmetic average: a sum of lognormal variables, and thus does not follow a lognormal distribution. We can of course price this Asian option on a basket using the Monte Carlo method. Sadly, this method can consume some time in order to get a precise option price. Approximating methods were developed in order to price the option in a fast way. We have seen four different methods for valuing an Asian option on a basket of futures: the Sophis method, the Gentle method, the Levy method and the Curran method. The Sophis, Levy and Gentle methods all approximate the sum of lognormal variables by a lognormal variable, and are easy to implement. Curran divides the option price into two parts by conditioning. One part can be evaluated exactly (and this part makes up the largest part of the option price) and the other part can be evaluated using Levy approximation and numerical integration. Curran is hardest to implement.

The Sophis method keeps performing very accurately for all volatilities, strikes and number of fixing dates. However for a negative correlation Sophis deviates from the Monte Carlo value. The Gentle method overall underestimates the option price, since the geometric average is less than the arithmetic average, and for high volatilities an extremely high underestimation is observed. For all strikes and number of fixing dates Gentle works acceptably. For negative correlation again a too high deviation is observed, even bigger than the deviation for Sophis. Levy sometimes overestimates the real value, and this overestimation is biggest for a high volatility. Compared to Sophis and Gentle, Levy gives the best price for negative correlations and high (out-of-the-money) strikes. Thus we can say that for a high volatility the Sophis method is preferred, whereas for negative correlations and high strikes the Levy method works best. Curran works well under all circumstances, however it is also the hardest to implement.

In short, we can say that the Curran method is the most precise method for under all circumstances. However, a hybrid of the Sophis method (to get exact values for high volatility) and the Levy method (to get exact values for all correlations and strikes) will work about as accurate as the Curran method.

We also prices an average rate/average strike option. This option is harder to price since it also has an average strike, instead of just a constant strike. Using the Sophis and Levy method, two methods were developed for pricing this option. The conclusion was that the Levy method performed best for pricing this option. Apparently the Geometric average is not a good approximation for the volatility in this case. Furthermore, it was worth noting that increasing the number of dates on which the strike is averaged resulted in a bigger absolute error. This may be due to the fact that the strike price remains unknown for a longer time.

In this thesis we only looked at pricing the Asian option with fixed strike and the option with variable strike. However, in the derivatives market there are many more options. One
An idea for further research would be to take a look at those options, and try to price them using the methods developed in this thesis. Furthermore, we saw that the developed models for the AS/AR option did not work well for when the strike price remained unknown for a longer time (a high value of $k$). An idea would be to look into this problem and try to come up with a better method. It would also be interesting to look at other models for the future price dynamics and compare what models represent the real futures prices (as observed in the market) best. We used a specific geometric average when developing the Sophis and Gentle method. It might be an idea to look at other geometric averages and see how this influences the accuracy of both Sophis and Gentle. For example, the geometric average

$$G(T) = \left( \prod_{j=1}^{n} \prod_{i=1}^{p} \frac{pF_i(t_j)}{p} \right)^{\frac{1}{pn}},$$

can be used. Furthermore, Gentle usually underestimated the option price, despite that we tried to compensate for the fact that the geometric average is lower than the arithmetic average by using a modified strike. It might be an idea to look into this problem and maybe use another modified strike or another way of compensating for the undervaluation. For traders it is important to be able to manage the risk of their portfolios. This is done by combining stocks and options, and is called hedging. One way of hedging for an option $V$ is by buying an amount $\frac{\partial V}{\partial S}$ of stocks, this is called delta-hedging. It might be useful to look into the process of delta-hedging for the Asian option on a basket of futures.
References


[5] RBC Capital Markets; *Hedge Currency Payables with an Average Strike/Average Rate Call Option*


[10] Martin Krekel, Johan de Kock, Ralf Korn, Tin-Kwai Man; *An Analysis of Pricing Methods for Baskets Options*; Wilmott magazine


11 Appendix 1: Deriving the expectations of arithmetic and geometric averages

When calculating the expectation of the averages we will need the following three items:

- To calculate the expectation of two different Brownian motions, but at the same time $t_1$:
  \[ E[\exp(\sigma_1 W^1(t_1) + \sigma_2 W^2(t_1))], \]
  we first rewrite
  \[ W^2(t_1) = W^1(t_1) \rho + \sqrt{1 - \rho^2} \tilde{W}(t_1), \]
  where $\tilde{W}(t_1)$ is independent of $W(t_1)$ so that we can write our expectation as
  \[ E[\exp(\sigma_1 W^1(t_1) + \sigma_2 W^2(t_1)) \cdot \exp(\sigma_2 \sqrt{1 - \rho^2} \tilde{W}(t_1))] = \exp(\frac{1}{2} (\sigma_1^2 + 2 \rho \sigma_1 \sigma_2 + \sigma_2^2) t_1). \]

- The expectation of two Brownian motions at different times $t_1$ and $t_2$:
  \[ E[\exp(\sigma_1 W^1(t_1)) \cdot \exp(\sigma_2 W^2(t_2))], \]
  is equal to:
  \[ E[\exp(\sigma_1 W^1(t_1) + \sigma_2 W^2(t_1)) \cdot \exp(\sigma_2 (W_2(t_2) - W^2(t_1))]]. \]
  We know the difference between the Brownian motions is independent, so we arrive at
  \[ \exp\left(\frac{1}{2} (\sigma_1^2 + 2 \rho \sigma_1 \sigma_2 + \sigma_2^2) t_1\right) \cdot \exp\left(\frac{1}{2} \sigma_2^2 (t_2 - t_1)\right). \]

- The last expectation needed is for the same Brownian motions. This one is easiest, since we know that the sum of two Brownian motions $W_1(t_1) \sim N(0, t_1)$ and $W_1(t_2) \sim N(0, t_2)$ is normally distributed with mean 0 and variance $t_1 + t_2$, so this expectation is equal to:
  \[ \exp\left(\frac{1}{2} \sigma_1^2 (t_1 + t_2)\right). \]
12 Appendix 2: Matlab codes

12.1 Monte Carlo for an Asian on one future

%% Problem and method parameters
F0 = 100;
K = 100;
sigma = .4;
r = 0;
T = 1;
N = 1;
Dt = T/N;
M = 100000;

%% MONTE CARLO METHODS
Dt2=0.0001; N2=T/Dt2;
F=ones(M,N2);
Fa=ones(M,N2);
F(:,1)=F0*F(:,1);
Fa(:,1)=F0*Fa(:,1);
for i=1:N2
    z=randn(M,1);
    F(:,i+1)=F(:,i)+sigma*F(:,i).*z*sqrt(Dt2);
end
indexen=round(t/Dt2);
indexen=indexen(2:(length(indexen)));
F=F(1:M,indexen);
AT=mean(F,2);
V=exp(-r*T)*max((AT-K),0);
aM2 = mean(V)
bM2 = std(V);
stdev=1.96*bM2/sqrt(M)
conf2 = [aM2 - 1.96*bM2/sqrt(M), aM2 + 1.96*bM2/sqrt(M)];

%% second method
F=zeros(M,N);
Fa=zeros(M,N);
for i=1:M
    z=randn(1,N);
    F(i,:) = F0*cumprod(exp((-0.5*sigma^2)*Dt+sigma*sqrt(Dt)*z),2);
end
AT = mean(F,2);
V=exp(-r*T)*max((AT-K),0);
aM2 = mean(V)
bM2 = std(V);
stdev=1.96*bM2/sqrt(M)
conf2 = [aM2 - 1.96*bM2/sqrt(M), aM2 + 1.96*bM2/sqrt(M)];
12.2 Two methods of simulating correlated futures

\begin{verbatim}
F1_0=100;
F2_0=80;
sigma1=.4;
sigma2=sigma1;
T=2;
rho12=1;
K=180;
N=10;%aantal tijdsstapjes waarover we gem nemen
Dt=T/N;
M=10^5;%aantal simulaties
r=0;

%first method (cholesky)
F1=ones(M,N);%matrices with M rows, N columns
F2=ones(M,N);
F1a=ones(M,N);%matrices with M rows, N columns
F2a=ones(M,N);

for i=1:M
  w1=randn(1,N)*sqrt(Dt);
  w2=randn(1,N)*sqrt(Dt);
  F1(i,:)=F1_0*cumprod(exp((-0.5*sigma1^2)*Dt+sigma1*w1),2);
  F2(i,:)=F2_0*cumprod(exp((-0.5*sigma2^2)*Dt+sigma2*(rho12*w1+sqrt(1-rho12^2)*w2)),2);
end
Ftot1=F1+F2;
AT=mean(Ftot1,2);
V=exp(-r*T)*max((AT-K),0);
deviatie=1.96*std(V)/sqrt(M)
conf=[waarde-1.96*std(V)/sqrt(M) waarde+1.96*std(V)/sqrt(M)];

%second method
F1=zeros(M,N);
F2=zeros(M,N);
for i=1:M
  z=mvnrnd([0,0],[1,rho12;rho12,1],N);
  F1(i,:)=F1_0*cumprod(exp((-0.5*sigma1^2)*Dt+sigma1*sqrt(Dt)*z(:,1)'),2);
  F2(i,:)=F2_0*cumprod(exp((-0.5*sigma2^2)*Dt+sigma2*sqrt(Dt)*z(:,2)'),2);
end
Ftot2=F1+F2;
AT2=mean(Ftot2,2);
V2=exp(-r*T)*max((AT2-K),0);
waarde2=mean(V2)
deviatie2=1.96*std(V2)/sqrt(M)
conf=[waarde2-1.96*std(V2)/sqrt(M) waarde2+1.96*std(V2)/sqrt(M)]
\end{verbatim}
12.3 Control variates in Monte Carlo

% Calculate the control variate exact value using Gentle
%(however without modifying the strike!)

aantalfutures = length(F0);
Dt = T/N;
alpha = (1/N*F0)/(sum(F0));
t = [0: Dt: T];
som = 0;

for j1=2:N+1
    for j2=2:N+1
        for i1=1:aantalfutures
            for i2=1:aantalfutures
                som = som + alpha(i1)*alpha(i2)*sigma(i1)*sigma(i2)*rho(i1,i2)*min(t(j1),t(j2));
            end
        end
    end
end

A0 = sum(F0);
sigma_asian = sqrt(som/T);
som2 = 0;
for j1=2:N+1
    for i1=1:aantalfutures
        som2 = som2 + (-1/2*alpha(i1)*(sigma(i1)^2)*t(j1));
    end
end

Mg = A0*exp(som2+0.5*sigma_asian^2*T);

d1 = (log(Mg)-log(K))/(sigma_asian*sqrt(T))+0.5*sigma_asian*sqrt(T);
d2 = d1 - sigma_asian*sqrt(T);

Nd1 = 0.5*(1+erf(d1/sqrt(2))); Nd2 = 0.5*(1+erf(d2/sqrt(2)));
gen = Mg*Nd1*exp(-r*T) - K*exp(-r*T)*(Nd2);

% Calculate the geometric average
prod = 1;
% control version:
Fprod1 = (((1/(N*alpha(1)))*F1).^alpha(1)).*((1/(N*alpha(2)))*F2).^alpha(2));

goave = geoave.*Fprod1(:,j);

75
end

paygeo=exp(-r*T)*max(geoave-K,0);
cova=cov(V2,paygeo);

% determine best theta
theta=-cova(1,2)/cova(2,2);

% the new variable
Z=V2+theta*(paygeo-gen);
Zmean=mean(Z)
stdevZ=1.96*std(Z)/sqrt(M)
12.4 Sophis

function sop=SOP_sophis(N, T, sigma, F0, K, r, rho)

aantalfutures=length(F0);
Dt = T/N;
alpha=(1/N*F0)/(sum(F0));

% a row vector with the weights for the different futures
t=[Dt:Dt:T];

som=0;
for j1=1:N
    for j2=1:N
        som=som+min(t(j1),t(j2));
    end
end
alphaSig = alpha .* sigma;
som = som*alphaSig * rho * alphaSig’;
sigma_asian=sqrt(som/T);

M=sum(F0);
d1 = (log(M)-log(K))/(sigma_asian*sqrt(T))+0.5*sigma_asian*sqrt(T);
d2 = d1 - sigma_asian*sqrt(T);
Nd1 = 0.5*(1+erf(d1/sqrt(2))); Nd2 = 0.5*(1+erf(d2/sqrt(2)));
sop = M*Nd1*exp(-r*T) - K*exp(-r*T)*(Nd2)
12.5 Gentle

function gen=G_gentle(N, T, sigma, F0, K, r, rho)
aantalfutures=length(F0);
Dt = T/N;
alpha=(1/N*F0)/(sum(F0));

% a row vector with the weights for the different futures

for j1=2:N+1
    for j2=2:N+1
        som=som+min(t(j1),t(j2));
    end
end

alphaSig = alpha .* sigma;
som = som*alphaSig * rho * alphaSig';
sigma_asian=sqrt(som/T);

A0=sum(F0);
M=0;
for j1=2:N+1
    for i1=1:aantalfutures
        M=M+(-0.5*alpha(i1)*sigma(i1)^2*t(j1));
    end
end
M=A0*exp(M+0.5*sigma_asian^2*T);
Knieuw=K-A0+M;
d1 = (log(M)-log(Knieuw))/(sigma_asian*sqrt(T))+0.5*sigma_asian*sqrt(T);
d2 = d1 - sigma_asian*sqrt(T);
Nd1 = 0.5*(1+erf(d1/sqrt(2))); Nd2 = 0.5*(1+erf(d2/sqrt(2)));

gen = M*Nd1*exp(-r*T) - Knieuw*exp(-r*T)*(Nd2)
12.6 Levy

function lev=L_levy(N, T, sigma, F0, K, r, rho)
aantalfutures=length(F0);
Dt = T/N;
t=[0:Dt:T];
som=0;

for j1=2:N+1
    for j2=2:N+1
        for i1=1:aantalfutures
            for i2=1:aantalfutures
                som=som+F0(i1)*F0(i2)*exp(sigma(i1)*sigma(i2)*rho(i1,i2)*min(t(j1),t(j2)));
            end
        end
    end
end

som=som/N^2;
M=sum(F0);
sigma_asian=sqrt(1/T*(log(som)-2*log(M)));
tau = T;
d1 = (log(M/K) + (0.5*sigma_asian^2)*(tau))/(sigma_asian*sqrt(tau));
d2 = d1 - sigma_asian*sqrt(tau);
Nd1 = 0.5*(1+erf(d1/sqrt(2))); Nd2 = 0.5*(1+erf(d2/sqrt(2)));

lev = M*Nd1*exp(-r*tau) - K*exp(-r*tau)*(Nd2)
12.7 Curran

Curran function file

```matlab
function cur=C_curran(N, T, sigma, F0, K, r, rho)
Dt=T/N;

%the exact part
aantalfutures=length(F0);
alpha=(1/N*F0)/(sum(F0));
t=[Dt:Dt:T];

mx=log(sum(F0))-0.5*sum(alpha.*sigma.^2)*sum(t);
sigG=0;
for j1=1:N
    for j2=1:N
        sigG=sigG+min(t(j1),t(j2));
    end
end
alphaSig = alpha .* sigma;
sigG = sigG*alphaSig * rho * alphaSig';
sigG=sqrt(sigG/T);
cov=zeros(aantalfutures,N);
for j=1:N
    for j2=1:N
        cov(:,j) = cov(:,j)+min(t(j),t(j2));
    end
end
for j=1:N
    cov(:,j)=cov(:,j).*(sigma' .* (rho * alphaSig'));
end

exact = sum(F0'.*sum(normcdf((cov-(log(K)-mx))/(sigG*sqrt(T))),2))/(N)
- K * normcdf((mx-log(K)) / (sigG*sqrt(T)));

% calculate the "approximated" part
sigG2=(sigG^2)*T;
%create function integrand
integrand = @(g) int(g,aantalfutures,F0,sigma,T,rho,mx,sigG2,cov,K,N)
approx = quad(integrand,0,K); % numerical quadrature
% price
cur = exact + approx

Integrand calculation

function result = int(g,p,F,sig,T,rho,mx,sigG2T,cov,K,N)
% evaluate the integrand

% evaluates E[ A(T)-g | G(T)=g ]
```
n = length(g);
eA = zeros(1,n);
for j = 1 : n
    for j2=1:N
        for i=1:p
            eA(j) = eA(j)+F(i)*exp((cov(i,j2)/sigG2T)*(log(g(j))-mx-.5*cov(i,j2)));
        end
    end
end
mH=eA/N-g;

% evaluates E[ A^2(T) | G(T)=g ]

Dt=T/N;
t=[Dt:Dt:T]; %the time intervals
x = zeros(1,n);
for j = 1 : n
    for i1=1:p
        for i2=1:p
            for j1=1:N
                for j2=1:N
                    cov12=cov(i1,j1)+cov(i2,j2);
                    %eFF is formula for E(Fi1Fi2|G)
                    eFF=F(i1)*F(i2)*exp((cov12/sigG2T)*(log(g(j))-mx-.5*cov12)
                        +rho(i1,i2)*sig(i1)*sig(i2)*min(t(j1),t(j2)));
                    x(j)=x(j)+eFF;
                end
            end
        end
    end
end
eA2 = x; %sum over j1 j2 i1 and i2 of eFF (needed for computing E[H_G^2]
ehg2=eA2/(N^2)-2*g.*(mH+g)+g.^2; %E[H_G^2]
sigH2T = log(ehg2) - 2*log(mH);

d1 = (log(mH)-log(K-g))./sqrt(sigH2T) + .5*sqrt(sigH2T);
d2 = d1 - sqrt(sigH2T);
condExp = mH.*normcdf(real(d1)) - (K-g).*normcdf(real(d2));
density = exp(-((log(g)-mx).^2/(2*sigG2T)))/(g*sqrt(sigG2T)*sqrt(2*pi));
result = condExp.*density;
12.8 Monte Carlo for AS/AR option

%variable strike CALL: mac(pricesum-strike, 0)
%one future
%strike: som van i=1...k (begin) F1
%pricesum: som van i=s...n (eind) n=N F2

%defining the variables:
F1_0 = 100;
F2_0=80;
sigma1 = 1;
sigma2=sigma1;
r = 0;
T = 2;
N = 100;
rho12=.5;
M=10^-4;
Dt=T/N;

aantalfutures=length(F0);
alpha=(1/N*F0)/(sum(F0));
t=[Dt:Dt:T];

F1=zeros(M,N);
F2=zeros(M,N);
F1a=zeros(M,N);
F2a=zeros(M,N);
for i=1:M
    z=mvnrnd([0,0],[1,rho12;rho12,1],N);
    F1(i,:) = F1_0*cumprod(exp((-0.5*sigma1^2)*Dt+sigma1*sqrt(Dt)*z(:,1)'),2);
    F2(i,:) = F2_0*cumprod(exp((-0.5*sigma2^2)*Dt+sigma2*sqrt(Dt)*z(:,2)'),2);
    F1a(i,:) = F1_0*cumprod(exp((-0.5*sigma1^2)*Dt+sigma1*sqrt(Dt)*-z(:,1)'),2);
    F2a(i,:) = F2_0*cumprod(exp((-0.5*sigma2^2)*Dt+sigma2*sqrt(Dt)*-z(:,2)'),2);
end

%we make two vectors Fk and Fs, Fk is for the strike, Fs for the pricesum
k=5; %we can change k and s so as to change the averaging periods
s=95;
F1k=F1(:,1:k);
F2k=F2(:,1:k);
F1s=F1(:,s:N);
F2s=F2(:,s:N);
AT1=mean(F1k+F2k,2); %strike
AT2=mean(F1s+F2s,2); %pricesum
V=exp(-r*T)*max(AT2-AT1,0);

F1ak=F1a(:,1:k);
F2ak=F2a(:,1:k);
F1as = F1a(:,s:N);
F2as = F2a(:,s:N);
AT1a = mean(F1ak + F2ak, 2); % strike
AT2a = mean(F1as + F2as, 2); % prices
Va = exp(-r*T)*max(AT2a - AT1a, 0);

aM2 = mean(.5*(V+Va))
bM2 = std(.5*(V+Va));
stdev = 1.96*bM2/sqrt(M)
conf2 = [aM2 - 1.96*bM2/sqrt(M), aM2 + 1.96*bM2/sqrt(M)];
12.9 Levy for AS/AR option

function lev2=L_levy2(N, T, sigma, F0, K, r, rho,k,s) %take K=1 always
aantalfutures=length(F0);
Dt = T/N;
t=[Dt:Dt:T];

som2=0;
for j1=s:N
    for j2=s:N
        for i1=1:aantalfutures
            for i2=1:aantalfutures
                som2=som2+F0(i1)*F0(i2)*exp(sigma(i1)*sigma(i2)*rho(i1,i2)*min(t(j1),t(j2)));
            end
        end
    end
end

som3=0;
for j1=1:k
    for j2=1:k
        for i1=1:aantalfutures
            for i2=1:aantalfutures
                som3=som3+F0(i1)*F0(i2)*exp(sigma(i1)*sigma(i2)*rho(i1,i2)*min(t(j1),t(j2)));
            end
        end
    end
end

som2=som2/((N-s+1)^2);
som3=som3/(k^2); %strike
M=1;
sig2=log(som2)-2*log(M);
sig3=log(som3)-2*log(M); %strike
sigma_asian=sqrt(1/T*(sig2-sig3));
tau = T;
d1 = (log(M/K) + (0.5*sigma_asian^2)*(tau))/(sigma_asian*sqrt(tau));
d2 = d1 - sigma_asian*sqrt(tau);
Nd1 = 0.5*(1-erf(d1/sqrt(2))); Nd2 = 0.5*(1+erf(d2/sqrt(2)));
lev = M*Nd1*exp(-r*tau) - K*exp(-r*tau)*(Nd2);
%FINAL VALUE
lev2=lev*(F0(1)+F0(2))
function sop2=S_sop2(N, T, sigma, F0, K, r, rho,k,s) %take K=1 always
alpha1=((1/k)*F0)/sum(F0));
alpha2=((1/(N-s+1))*F0)/sum(F0));
som1=0; %strike
for j1=1:k
    for j2=1:k
        for i1=1:aantalfutures
            for i2=1:aantalfutures
                som1=som1+alpha1(i1)*alpha1(i2)*sigma(i1)*sigma(i2)*rho(i1,i2)*min(t(j1),t(j2));
            end
        end
    end
end
som2=0; %price
for j1=s:N
    for j2=s:N
        for i1=1:aantalfutures
            for i2=1:aantalfutures
                som2=som2+alpha2(i1)*alpha2(i2)*sigma(i1)*sigma(i2)*rho(i1,i2)*min(t(j1),t(j2));
            end
        end
    end
end
som3=0; %strike price
for j1=s:N
    for j2=1:K
        for i1=1:aantalfutures
            for i2=1:aantalfutures
                som3=som3+alpha2(i1)*alpha1(i2)*sigma(i1)*rho(i1,i2)*sigma(i2)*min(t(j1),t(j2));
            end
        end
    end
end
sigma_asian=sqrt(1/T*(som2+som1-2*som3));
M=1;
tau = T;
d1 = (log(M/K) + (0.5*sigma_asian^2)*tau)/(sigma_asian*sqrt(tau));
d2 = d1 - sigma_asian*sqrt(tau);
Nd1 = 0.5*(1+erf(d1/sqrt(2))); Nd2 = 0.5*(1+erf(d2/sqrt(2)));
sop = M*Nd1*exp(-r*tau) - K*exp(-r*tau)*(Nd2);
%FINAL VALUE
sop2=sop*(sum(F0))