SPIN SWAP AND EXCHANGE COUPLING IN A QUANTUM DOT ARRAY

by

M.L. de Leeuw den Bouter

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Student number: 4183576
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Supervisors: Prof. dr. ir. L. M. K. Vandersypen,
Dr. J. L. A. Dubbeldam,
Dr. M. Shafiei
Thesis committee: Prof. dr. ir. L. M. K. Vandersypen, TU Delft
Dr. J. L. A. Dubbeldam, TU Delft
Dr. L. Di Carlo, TU Delft
Dr. Ir. W. G. M. Groenevelt, TU Delft
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In 1965, Gordon E. Moore, one of the co-founders of Intel, noted that, in years prior, the transistors in computers became twice as small approximately every two years as a consequence of the constant technological innovation [1]. He also predicted that this trend would continue for at least ten more years. His prediction came true and it was given the name Moore’s law. This law has proven true for more than 45 years. The computer industry even base their targets on it. However, the limit is about to be reached. Transistors have become so small, that reducing their size much further will yield quantum mechanical behaviour. A quantum computer makes use of quantum mechanical behaviour. Certain difficult problems that cannot be solved efficiently by classical computers can be solved by a quantum computer in polynomial time, like the problem of prime factorization [2], [3]. Furthermore, a quantum computer can simulate any finite physical system [4]. For instance, this can be used to determine the shape of a protein when the amino acid sequence is known, which will seriously benefit drug development [5].

Instead of using classical bits, a quantum computer uses qubits as units of information. In principle, any two-state system can be used as a qubit. The states are represented by $|0\rangle$ and $|1\rangle$. A qubit can be in a superposition of $|0\rangle$ and $|1\rangle$, as opposed to a classical bit, which is either 0 or 1. However, all qubits experience decoherence, information loss as a result of interactions with the environment. Reducing or preventing decoherence is one of the biggest challenges in quantum computing today.

A promising choice of qubit is an electron spin in a quantum dot. $|0\rangle$ and $|1\rangle$ represent spin down and spin up, respectively. Advantages of quantum dot systems include their efficient electrical control [6] and their potential for long coherence times [7], [8].

In this research, simulations were performed on systems of two, three and four linearly coupled quantum dots. In these simulations, the eigenenergies of the systems were determined and time evolutions were performed to get an understanding of the behaviour of electrons in small linear arrays of quantum dots. Next, the pulsing scheme as described by Hu and Das Sarma [9] that swaps two different spin states in the presence of Zeeman inhomogeneity, and the exchange coupling between two neighbouring electrons were reviewed.
2

THEORY

2.1. QUANTUM DOTS
A quantum dot is an artificially fabricated device in which one or more electrons can be contained [10], [11]. Quantum dots have a size of order 10 nm and can be made of semiconductor material. The behaviour of an electron is governed by quantum mechanics. Therefore, the energy levels it can assume are quantized. The energy and other properties, like spin, can be described by quantum numbers. The quantum number \( n \) represents the energy level, while \( s \) describes the spin number. An electron has spin number \( s = \frac{1}{2} \).

A GaAs quantum dot mainly consists of Gallium Arsenide (GaAs) with a layer of Aluminium Gallium Arsenide (AlGaAs) on top of it. AlGaAs has a higher band gap than GaAs, resulting in the formation of a Two Dimensional Electron Gas (2DEG) at the GaAs/AlGaAs interface, typically 100 nm below the surface. Electron gas is formed because the difference in band gap causes the potential to drop below the Fermi Energy. The two dimensionality of the electron gas is caused by the narrowness of the GaAs/AlGaAs interface. Only the lowest standing wave fits into the interface. Finally, two gates are attached on top.

A quantum dot is capacitively attached to a reservoir and a gate. A double quantum dot is created by capacitively coupling two quantum dots. A lateral array of quantum dots can be created by repeating this process. In figure 2.1, a double quantum dot is depicted.

![Figure 2.1: Model of a double quantum dot [12].](image)

2.2. ENERGY
The Hamiltonian describes the energy of a system. A Hamiltonian is an operator, operating on a wave function. The eigenvalues of the Hamiltonian are the energies of the system [13]. All elements of the Hamiltonian are determined by an external potential interacting with the particle in the quantum dot. In this report, those elements are the Zeeman effect, exchange modelled by spin-spin interaction and hyperfine interaction.
2.2.1. **Zeeman Splitting**

Applying a uniform external magnetic field causes a splitting in the energy levels of degenerate spin states [14]. This is called the Zeeman effect. The Zeeman Hamiltonian is defined as

\[ H_Z = \gamma \vec{B} \cdot \vec{S}. \]  

(2.1)

In this expression, \( \vec{B} \) denotes the external magnetic field, \( \vec{S} \) is the spin operator and \( \gamma \) is the gyromagnetic ratio. \( \gamma \) is described as

\[ \gamma = \frac{g \mu_B}{h}, \]  

(2.2)

in which \( g \) is the Landé g-factor, \( \mu_B \) is the Bohr magneton and \( h \) is the reduced Planck constant. The Landé g-factor depends on different parameters including the specifications of the quantum dot and is usually between \(-0.4\) and \(-0.2\) for GaAs [11]. The direction in which the magnetic field is applied, is typically taken to be the \( z \)-direction, reducing the Zeeman Hamiltonian to

\[ H_Z = \gamma B S_z, \]  

(2.3)

in which \( S_z \) is the electron spin in the \( z \)-direction. \( S_z \) has eigenvalues \( \frac{\hbar}{2} \) and \( -\frac{\hbar}{2} \) for spin up (↑) and spin down (↓) respectively. ↑ is oriented in the +\( z \)-direction, while ↓ is oriented in the -\( z \)-direction. Because the g-factor is negative, ↓ has a higher energy than ↑. Compared to the system without Zeeman splitting, the energy of ↓ has increased by \(-E_Z\), while the energy of ↑ has experienced a decrease by \(-E_Z\), where \( E_Z \) is the Zeeman energy, which is described by

\[ E_Z = \gamma h B = g \mu_B B. \]  

(2.4)

2.2.2. **Exchange Interaction**

In a double quantum dot containing two electrons, an exchange interaction exists [15]. This interaction can be captured by spin-spin interaction and is defined as

\[ H_C = J (\vec{S}_1 \cdot \vec{S}_2 - \frac{\hbar^2}{4}), \]  

(2.5)

in which \( J \) is the exchange coupling and \( \vec{S}_i \) is the spin operator working on electron \( i (i \in \{1, 2\}) \). \( J \) is dependent on the overlap of the wave functions of the two electrons. An increase in overlap yields an increase in \( J \).

2.2.3. **Hyperfine Interaction**

In a GaAs quantum dot, the host material contains many nuclear spins with which the electron spin interacts [11]. The Hamiltonian for this hyperfine interaction is given by

\[ H_{HF} = \sum_{k=1}^{N} A_k \vec{I}_k \cdot \vec{S}, \]  

(2.6)

in which \( A_k \) is the coupling strength between nucleus \( k \) and the electron and \( \vec{I}_k \) is the spin operator for nucleus \( k \). \( A_k \) is proportional to the square of the overlap between the nucleus and the electron wave function.

All nuclear spins combined can be treated as an apparent magnetic field \( B_n \), yielding an alternative description of the effect of the nuclei on the electron spin. The nuclear field is known as the Overhauser field and acts on the electron spin like an external magnetic field:

\[ \left( \sum_{k=1}^{N} A_k \vec{I}_k \right) \vec{S} = \gamma \vec{B}_n \cdot \vec{S}. \]  

(2.7)

2.2.4. **Anticrossing**

Two eigenenergies of a Hamiltonian can only cross, i.e. become equal in value, under certain conditions. The Hamiltonian of a two-state system is

\[ H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}. \]  

(2.8)
The eigenenergies of this Hamiltonian are
\[
\lambda_1 = -\frac{H_{11} + H_{22}}{2} + \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2}
\]
(2.9)
\[
\lambda_2 = -\frac{H_{11} + H_{22}}{2} - \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2}.
\]
(2.10)
For these eigenenergies to be equal, the following conditions must hold:
\[
H_{11} = H_{22}, \quad H_{12} = 0.
\]
(2.11)
When these conditions are not met, the eigenenergies cannot cross. Therefore, an anticrossing can occur. For example, of the matrices
\[
M_1 = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \quad M_2 = \begin{pmatrix} x & 1 \\ 1 & -x \end{pmatrix},
\]
(2.12)
only \(M_1\) meets the conditions of 2.11 at \(x = 0\). The left (right) side of figure 2.2 shows the eigenvalues of \(M_1\) (\(M_2\)) as a function of \(x\). The eigenvalues of \(M_2\) anticross at \(x = 0\).

The size of an anticrossing is equal to twice the coupling of two states, which is given by the off-diagonal elements in the Hamiltonian, and can therefore be used to determine the coupling of two states.

2.3. Double quantum dot

2.3.1. Charge stability diagram
Consider two quantum dots, labelled 1 and 2 [11]. Their electrochemical potentials are controlled independently by gate voltages \(V_{G,1}\) and \(V_{G,2}\) respectively. When the dots are completely uncoupled, their electrochemical potentials are independent and therefore the number of electrons in both dots are too. In a charge stability diagram, the occupancy of the dots is plotted as a function of \(V_{G,1}\) and \(V_{G,2}\). In \((x,y)\) \(x\) denotes the number of electrons in dot 1, while \(y\) is the number of electrons in dot 2. The charge stability diagram for the uncoupled case is shown in figure 2.3(a). When the capacitive coupling between the dots is nonzero, the electrostatic energy of one of the dots changes when an electron is added to the other dot. Furthermore, \(V_{G,1}\) now influences the second dot as well as the first dot, and vice versa. The charge stability diagram in the case of capacitive coupling between the dots is shown in figure 2.3(b).
2.3.2. **DOUBLE QUANTUM DOT CONTAINING ONE ELECTRON**

**CHARGE TRANSITIONS**

When two quantum dots are uncoupled, the eigenstates of the system are $|L\rangle$ and $|R\rangle$, in which $|L\rangle$ denotes an electron in the left dot, and $|R\rangle$ in the right dot [12], [16]. When tunnel coupling between the dots is introduced, the electron can tunnel between the dots, resulting in different eigenstates of the system. Using the standard basis

$$|L\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |R\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(2.13)

the Hamiltonian becomes

$$H = \begin{pmatrix} \epsilon_2 & \tau \\ \tau & -\epsilon_2 \end{pmatrix},$$

(2.14)

in which the detuning $\epsilon$ is the energy difference between $|L\rangle$ and $|R\rangle$, and $\tau$ is the tunnel coupling constant. Detuning is defined along the line connecting $(0,1)$ and $(1,0)$ in the charge stability diagram, as shown in figure 2.4.

---

**Figure 2.3:** Charge stability diagrams for (a) uncoupled and (b) coupled double quantum dots [11].

**Figure 2.4:** The detuning is defined along the axis connecting $(1,0)$ and $(0,1)$ [12].
2.3. **Double quantum dot**

**Spin**

When an external magnetic field is applied to the system, the Zeeman Hamiltonian must be taken into account. With respect to the standard basis

\[
|L\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |L\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |R\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |R\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]  

(2.15)

the Hamiltonian becomes

\[
H = \begin{pmatrix}
\frac{\gamma_L h B_0}{2} & 0 & \tau \\
0 & \frac{\gamma_R h B_0}{2} & 0 \\
\tau & 0 & \frac{\gamma_R h B_0}{2}
\end{pmatrix} + H_Z = \begin{pmatrix}
\frac{\gamma_L h B_0}{2} & 0 & \frac{\gamma_R h B_0}{2} \\
0 & \frac{\gamma_R h B_0}{2} & 0 \\
\frac{\gamma_R h B_0}{2} & 0 & \frac{\gamma_R h B_0}{2}
\end{pmatrix},
\]  

(2.16)

in which the external magnetic field (applied in the z-direction) is denoted by \(B_0\).

**Different g-factors**

When the two dots have different g-factors, the Hamiltonian becomes

\[
H = \begin{pmatrix}
\frac{\gamma_L h B_0}{2} & 0 & \frac{\gamma_R h B_0}{2} \\
0 & \frac{\gamma_R h B_0}{2} & 0 \\
\frac{\gamma_R h B_0}{2} & 0 & \frac{\gamma_R h B_0}{2}
\end{pmatrix},
\]

(2.17)

in which \(\gamma_L = \frac{g_L \mu_B}{\hbar}\) and \(\gamma_R = \frac{g_R \mu_B}{\hbar}\), with \(g_L\) and \(g_R\) denoting the g-factors of the left and right dot, respectively. \(\epsilon\) is defined as the average of the energy difference between \(|L\uparrow\rangle\) and \(|R\uparrow\rangle\) and the energy difference between \(|L\downarrow\rangle\) and \(|R\downarrow\rangle\).

**Hyperfine Interaction**

When the hyperfine interaction is added to the system, an extra term is added to the Hamiltonian:

\[
H_{HF} = \gamma \vec{B}_n \cdot \vec{S};
\]

(2.18)

this expression can be decomposed into six different components:

\[
H_{HF} = \gamma_L (B^x_{n,\uparrow} S^x_{\uparrow} + B^y_{n,\uparrow} S^y_{\uparrow} + B^z_{n,\uparrow} S^z_{\uparrow}) + \gamma_R (B^x_{n,\downarrow} S^x_{\downarrow} + B^y_{n,\downarrow} S^y_{\downarrow} + B^z_{n,\downarrow} S^z_{\downarrow}),
\]

(2.19)

in which \(B^i_{n,j}\) is the nuclear field in the \(i\)-direction in dot \(j\) and \(S^i_j\) is the spin in the \(i\)-direction in dot \(j\), with \(i \in \{x, y, z\}\) and \(j \in \{L, R\}\).

\(S_x\) and \(S_y\) can be written in terms of the spin raising and lowering operators \(S^+\) and \(S^-\):

\[
S_x = \frac{S^+ + S^-}{2}, \quad S_y = \frac{S^+ - S^-}{2i},
\]

(2.20)

yielding the following hyperfine Hamiltonian in matrix form:

\[
H_{HF} = \begin{pmatrix}
\gamma_L B_{n,z,L} & \gamma_L (B_{n,z,L} - i B_{n,y,L}) & 0 \\
\gamma_L (B_{n,z,L} + i B_{n,y,L}) & -\gamma_L B_{n,z,L} & 0 \\
0 & 0 & \gamma_R B_{n,z,R} & \gamma_R (B_{n,z,R} - i B_{n,y,R}) \\
0 & 0 & \gamma_R (B_{n,z,R} + i B_{n,y,R}) & -\gamma_R B_{n,z,R}
\end{pmatrix}.
\]

(2.21)

This Hamiltonian shows that \(|L\uparrow\rangle\) and \(|L\downarrow\rangle\) are coupled by the hyperfine interaction, as are \(|R\uparrow\rangle\) and \(|R\downarrow\rangle\).
2.3.3. DOUBLE QUANTUM DOT CONTAINING TWO ELECTRONS

Neglecting the hyperfine interaction, the two terms contributing to the Hamiltonian in a quantum dot containing two electrons are the Zeeman effect and the exchange interaction:

\[ H = H_Z + H_C = \gamma B_0(S_1^z + S_2^z) + J \left( S_1 \cdot S_2 - \frac{\hbar^2}{4} \right). \]  \hspace{1cm} (2.22)

By expanding the dot product, the following expression is obtained:

\[ S_1 \cdot S_2 = \frac{S_1^z S_2^z + S_1^x S_2^x + S_1^y S_2^y}{2} \]  \hspace{1cm} (2.23)

As seen in equation (2.20), the \( S^x \) and \( S^y \) can be expressed in terms of the spin raising and lowering operators, yielding

\[ \langle \uparrow \downarrow | S_1 \cdot S_2 | \downarrow \uparrow \rangle = \frac{S_1^z S_2^z + S_1^x S_2^x + S_1^y S_2^y}{2} + S_1^z + S_2^z. \]  \hspace{1cm} (2.24)

Therefore, the total Hamiltonian is

\[ \gamma B_0(S_1^z + S_2^z) + J \left( \frac{S_1^z S_2^z + S_1^x S_2^x + S_1^y S_2^y}{2} + S_1^z + S_2^z - \frac{\hbar^2}{4} \right). \]  \hspace{1cm} (2.25)

Using the standard basis

\[ | \uparrow \downarrow \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad | \downarrow \uparrow \rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad | \uparrow \uparrow \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad | \downarrow \downarrow \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \]  \hspace{1cm} (2.26)

the Hamiltonian becomes

\[ H = \begin{pmatrix} \gamma B_0 & 0 & 0 & 0 \\ 0 & -\frac{\hbar J}{2} & \frac{\hbar J}{2} & 0 \\ 0 & \frac{\hbar J}{2} & -\frac{\hbar J}{2} & 0 \\ 0 & 0 & 0 & -\gamma B_0 \end{pmatrix}. \]  \hspace{1cm} (2.27)

**EXCHANGE COUPLING IN A DOUBLE QUANTUM DOT**

Taylor et al. [8] have found an expression for the exchange coupling \( J \) in a double quantum dot as a function of the detuning \( \epsilon \). \( \epsilon \) is the energy difference between \( |(1,1)S\rangle \) and \( |(0,2)S\rangle \), in which \( S \) denotes the singlet state \( \frac{1}{\sqrt{2}}(|\uparrow \uparrow \rangle - |\downarrow \downarrow \rangle) \) and \( x,y \) is the charge state, with \( x \) the number of electrons in the left dot, and \( y \) the number of electrons in the right dot. When tunnel coupling between the two dots is enabled, and it is assumed that \( |(2,0)S\rangle \) has a much higher energy than \( |(1,1)S\rangle \) and \( |(0,2)S\rangle \), the only charge transition occurs between \( |(1,1)S\rangle \) and \( |(0,2)S\rangle \). With respect to the basis

\[ |(2,0)S\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |(1,1)S\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \]  \hspace{1cm} (2.28)

the Hamiltonian describing this charge transition is

\[ H_{11-20} = \begin{pmatrix} -\epsilon & \tau \\ \tau & 0 \end{pmatrix}. \]  \hspace{1cm} (2.29)

It can be shown that the eigenfunction corresponding to the lowest eigenenergy is equal to

\[ |\tilde{S}\rangle = \cos \theta |(1,1)S\rangle + \sin \theta |(0,2)S\rangle \]  \hspace{1cm} (2.30)

in which \( \theta \) is a function of \( \epsilon \) and \( \tau \):

\[ \theta = \arctan \frac{2\tau}{\epsilon - \sqrt{\epsilon^2 + 4\tau^2}} \]  \hspace{1cm} (2.31)
The energy of $\tilde{S}$ is

$$E_S = -\frac{r}{2} \tan \theta = -\frac{r^2}{\epsilon - \sqrt{\epsilon^2 + 4\tau^2}} = -\frac{\epsilon}{2} - \sqrt{\left(\frac{\epsilon}{2}\right)^2 + 4\tau^2}. \quad (2.32)$$

When the detuning is low, each dot contains one electron. So $\tilde{S} = (1,1)S$ and $E_S \approx E_{(1,1)S}$. Increasing the detuning yields an increase of $(0,2)S$ in $\tilde{S}$, thereby increasing the probability of tunnelling.

The exchange coupling $J$ is equal to the energy difference between the $T_0$ (the triplet state $\frac{1}{\sqrt{2}}(|\uparrow \downarrow \rangle + |\downarrow \uparrow \rangle)$) and the singlet state. $T_0$ has energy zero, so the exchange coupling is equal to

$$J(\epsilon) = \frac{E_{T_0} - E_S}{\hbar^2} = -\frac{E_S}{\hbar^2} = \frac{\epsilon}{2} + \sqrt{\left(\frac{\epsilon}{2}\right)^2 + 4\tau^2}. \quad (2.33)$$

However, at very high detuning, the triplet state hybridizes, so this expression for the exchange coupling is not valid in that regime.

**Different g-factors**

Considering the low detuning regime, in which $\tilde{S} = (1,1)S$, when the electrons in the two dots have different g-factors, the Hamiltonian becomes

$$H = B_0(\gamma_L S^z_L + \gamma_R S^z_R) + J \left(\frac{S_1^x S_2^x + S_1^y S_2^y}{2} + S_1^z + S_2^z - \frac{\hbar^2}{4}\right). \quad (2.34)$$

yielding the following matrix:

$$H = \frac{\hbar}{2} \begin{pmatrix} (\gamma_L + \gamma_R)B_0 & 0 & 0 & 0 \\ 0 & (\gamma_L - \gamma_R)B_0 - hJ & hJ & 0 \\ 0 & hJ & (\gamma_L + \gamma_R)B_0 - hJ & 0 \\ 0 & 0 & 0 & (\gamma_L - \gamma_R)B_0 \end{pmatrix}. \quad (2.35)$$

### 2.4. Triple Quantum Dot

Research on a triple quantum dot has been performed by Laird et al. [17]. They modelled the triple quantum dot as two double dots sharing one of the dots. The middle electron interacts with both the left and the right electron, but the left and the right electron do not interact with each other. Therefore, the Hamiltonian contains two terms that describe an exchange interaction:

$$H = J_{12} \left(S_1 \cdot \tilde{S}_2 - \frac{\hbar^2}{4}\right) + J_{23} \left(S_2 \cdot \tilde{S}_3 - \frac{\hbar^2}{4}\right) + \gamma B_0(S^x_1 + S^x_2 + S^x_3), \quad (2.36)$$

in which $J_{12}$ is the coupling between the first and second electron, and $J_{23}$ between the second and third electron. Using the standard basis

$$|\uparrow \uparrow \uparrow \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\uparrow \uparrow \downarrow \rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\uparrow \downarrow \uparrow \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\uparrow \downarrow \downarrow \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$|\downarrow \uparrow \uparrow \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\downarrow \uparrow \downarrow \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\downarrow \downarrow \uparrow \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\downarrow \downarrow \downarrow \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (2.37)$$
the Hamiltonian becomes

\[
H = \frac{\hbar}{2} \begin{pmatrix}
3\gamma B_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \gamma B_0 - h_{J_{23}} & h_{J_{23}} & 0 & 0 & 0 & 0 & 0 \\
0 & h_{J_{23}} & \gamma B_0 - h_{J_{12}} - h_{J_{23}} & 0 & h_{J_{12}} & 0 & 0 & 0 \\
0 & 0 & h_{J_{12}} & -\gamma B_0 - h_{J_{12}} & 0 & h_{J_{12}} & 0 & 0 \\
0 & 0 & 0 & h_{J_{12}} & 0 & -\gamma B_0 - h_{J_{12}} - h_{J_{23}} & h_{J_{23}} & 0 \\
0 & 0 & 0 & 0 & h_{J_{23}} & 0 & -\gamma B_0 - h_{J_{23}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3\gamma B_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3\gamma B_0
\end{pmatrix}
\]

(2.38)

Here, the detuning \( \epsilon \) is defined along the line connecting charge states \((2,0,1), (1,1,1)\) and \((1,0,2)\). This is depicted in figure 2.5. In \((x,y,z)\) \(x\) is the charge occupation of the left dot, \(y\) of the middle dot and \(z\) of the right dot.

Figure 2.5: The detuning in a triple quantum dot is defined along the axis connecting \((2,0,1)\) and \((1,0,2)\) via \((1,1,1)\) [17].
At $\epsilon = \epsilon_-$, the charge states (2,0,1) and (1,1,1) line up. At $\epsilon = \epsilon_+$, (1,1,1) and (1,0,2) line up. Then, the exchange couplings $J_{12}$ and $J_{23}$ are

$$J_{12}(\epsilon) = \frac{1}{\hbar^2} \left( \frac{\epsilon - \epsilon_-}{2} + \sqrt{(\epsilon_- - \epsilon)^2 + 4\tau_L^2} \right), \quad J_{23}(\epsilon) = \frac{1}{\hbar^2} \left( \frac{\epsilon - \epsilon_+}{2} + \sqrt{(\epsilon - \epsilon_+)^2 + 4\tau_R^2} \right),$$

(2.39)
in which $\tau_L$ is the tunnel coupling between the left and the middle dot and $\tau_R$ between the middle and the right dot.

2.5. QUADRUPLE QUANTUM DOT

The triple quantum dot Hamiltonian can be extended to the quadruple quantum dot, yielding

$$H = J_{12} \left( S_1 \cdot S_2 - \frac{\hbar^2}{4} \right) + J_{23} \left( S_2 \cdot S_3 - \frac{\hbar^2}{4} \right) + J_{34} \left( S_3 \cdot S_4 - \frac{\hbar^2}{4} \right) + \gamma B_0 (S^z_1 + S^z_2 + S^z_3 + S^z_4).$$

(2.40)
3.1. TIME EVOLUTION
Considering a time-independent Hamiltonian, the time-dependent Schrödinger equation is

\[ H\Psi = \frac{i}{\hbar} \frac{\partial \Psi}{\partial t}, \]  

(3.1)

Solving for the wave function \( \Psi \) yields

\[ \Psi(t) = e^{-\frac{i H t}{\hbar}} \Psi(0). \]  

(3.2)

So by applying the unitary operator \( e^{-\frac{i H t}{\hbar}} \) to the initial state of a system (the state at \( t = 0 \)), the state at time \( t \) can be obtained.

3.2. OSCILLATION IN A TWO-STATE SYSTEM
In a two-level quantum mechanical system, two basis vectors can be defined, \( |0\rangle \) and \( |1\rangle \). All pure states of this system can be written as a linear combination of these two basis vectors, so the wavefunction of the initial state can be represented as

\[ |\phi(0)\rangle = c_0 |0\rangle + c_1 |1\rangle, \]  

(3.3)

in which \( c_0 \) and \( c_1 \) are complex numbers with \( |c_0|^2 + |c_1|^2 = 1 \). Then, the time-dependent wavefunction is

\[ |\phi(t)\rangle = c_0 |0\rangle e^{-\frac{i E_0 t}{\hbar}} + c_1 |1\rangle e^{-\frac{i E_1 t}{\hbar}}, \]  

(3.4)

in which \( E_0 \) and \( E_1 \) are the energies of \( |0\rangle \) and \( |1\rangle \) respectively.

The probability of this state returning to its initial state at time \( t \) is equal to \( |\langle \phi(0) | \phi(t) \rangle|^2 \). It can be shown that

\[ |\langle \phi(0) | \phi(t) \rangle|^2 = 1 - 4|c_0^2 c_1^*| \sin^2 \left( \frac{(E_1 - E_0) t}{2\hbar} \right). \]  

(3.5)

\( \sin^2 x \) is \( \pi \)-periodic, so the oscillation frequency is \( \frac{E_1 - E_0}{2\pi\hbar} \).

3.3. BLOCH SPHERE
The Bloch sphere is a geometrical representation of a two-level quantum mechanical system. Given the two eigenvectors of a two-state quantum mechanical system, \( |0\rangle \) and \( |1\rangle \), all pure states of this system can be represented as

\[ |\psi\rangle = a |0\rangle + \beta |1\rangle, \]  

(3.6)
in which $\alpha$ and $\beta$ are complex numbers precluded $|\alpha|^2 + |\beta|^2 = 1$. Without loss of generality, $\phi$ can be rewritten as
\[ |\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle, \] (3.7)
with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. The parameters $\phi$ and $\theta$ can be interpreted as spherical coordinates that define a point $(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ on the unit sphere. The Bloch sphere is useful for visualizing the time evolution of a two-state system. In figure 3.1, the Bloch sphere is depicted.

![Figure 3.1: The Bloch Sphere](image)

3.4. Spin Swap

Knowing how to execute a spin swap is important, because in a spin-based quantum computer, the swap gate is a useful tool that swaps two qubits.

Hu and Das Sarma [9] have designed a pulsing scheme that swaps two electron spins in a double quantum dot in the presence of Zeeman inhomogeneity, thereby making the change $|\downarrow\downarrow\rangle \rightarrow |\uparrow\uparrow\rangle$ or vice versa. They made two assumptions; one being that $J$ can be turned on and off infinitely fast, the other that there is no decoherence in the system. Hyperfine interaction, for example, is known to cause decoherence.

As seen before, in the low detuning regime, the Hamiltonian describing a double quantum dot with different $g$-factors containing two electrons is
\[
H = \frac{\hbar}{2} \begin{pmatrix}
(y_L + y_R)B_0 & 0 & 0 & hJ \\
0 & (y_L - y_R)B_0 - hJ & hJ & 0 \\
0 & hJ & (-y_L + y_R)B_0 - hJ & 0 \\
0 & 0 & 0 & (-y_L - y_R)B_0
\end{pmatrix}. \] (3.8)

The off-diagonal elements indicate that $|\downarrow\uparrow\rangle$ and $|\uparrow\downarrow\rangle$ are coupled, while $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ experience no coupling. Therefore, the two-state subspace spanned by $|\downarrow\uparrow\rangle$ and $|\uparrow\downarrow\rangle$ can be described by the effective Hamiltonian
\[
H_{eff} = \frac{1}{2} \begin{pmatrix}
\delta & \hbar^2 J \\
\hbar^2 J & -\delta
\end{pmatrix} = \frac{1}{2} \hbar^2 J_\sigma x + \frac{1}{2} \delta \sigma_z = J_n \hat{n} \cdot \sigma \] (3.9)
in which $\delta = h(y_L - y_R)B_0$ is the difference in Zeeman energy between the two dots, $\sigma_x$ and $\sigma_z$ are the Pauli spin matrices in the $x$- and $z$-direction, $J_n = \frac{1}{2} \sqrt{\hbar^2 J^2 + \delta^2}$ and $\hat{n} = (\hbar^2 J, 0, \delta) / J_n$. Using this Hamiltonian, the swap operation can be explained geometrically. This effective Hamiltonian represents a rotation on the Bloch
sphere around the axis defined by $\hat{n}$. On this Bloch sphere, $|\uparrow\downarrow\rangle$ is located at the top of the sphere, while $|\downarrow\uparrow\rangle$ is located at the bottom. The rotation angle is determined by the duration and the strength of this Hamiltonian. When there is no Zeeman inhomogeneity, the rotation occurs around the x-axis. Therefore, a $\pi$ rotation would result in a swap: $|\uparrow\downarrow\rangle \rightarrow |\downarrow\uparrow\rangle$ and $|\downarrow\uparrow\rangle \rightarrow |\uparrow\downarrow\rangle$. A $2\pi$ rotation is shown in figure 3.2a.

(a) In the absence of Zeeman inhomogeneity.
(b) In the presence of Zeeman inhomogeneity.

Figure 3.2: A $2\pi$ rotation of the initial state $|\uparrow\downarrow\rangle$, depicted on the Bloch Sphere. The black arrow indicates the rotational axis. When there is no Zeeman inhomogeneity, the time evolution occurs around the x-axis, so a $\pi$ rotation will yield the final state $|\downarrow\uparrow\rangle$. In the presence of Zeeman inhomogeneity, the time evolution no longer takes place around the x-axis, so $|\uparrow\downarrow\rangle$ will never completely evolve to $|\downarrow\uparrow\rangle$.

When the dots have different g-factors, the rotational axis $\hat{n}$ veers away from the x-axis. When the initial state is $|\uparrow\downarrow\rangle$, the starting point is the north pole of the Bloch sphere. Because the rotation occurs around a different axis now, a $\pi$ rotation will never yield a state situated at the south pole of the sphere, $|\downarrow\uparrow\rangle$. Instead, it will result in a superposition of these two states. A $2\pi$ rotation in the presence of Zeeman inhomogeneity is shown in figure 3.2b.

However, if the magnitude of the Zeeman inhomogeneity is known, the swap procedure can still be performed by adjusting the rotational axis. This, in turn, can be done by adjusting $J$. One way to perform a swap operation is by:

- First applying a square exchange pulse with magnitude $J$ to execute a $\pi$ rotation;
- Next turning $J$ off so the system undergoes a $\pi$ rotation around the z-axis, caused by the Zeeman inhomogeneity;
- Finally turning on a second exchange pulse with magnitude $J' = \frac{\delta^2}{2\pi^2}$, executing a $\pi$ rotation.

The exchange pulse sequences for performing a swap in the absence and the presence of Zeeman inhomogeneity are depicted in figure 3.3. The frequency of the revolution around $\hat{n}$ is $\frac{\sqrt{\delta^4 J'^2 + \delta^2}}{\pi J}$. Therefore, the duration of a $\pi$ pulse is $\frac{\pi}{\sqrt{\delta^4 J'^2 + \delta^2}}$. So the time it takes to perform the swap operation is

$$t_{\text{swap}} = \frac{\pi}{\sqrt{\delta^4 J'^2 + \delta^2}} + \frac{\pi J}{\frac{\delta^2}{2\pi^2}} + \pi J \frac{\delta}{\sqrt{\delta^4 J'^2 + \delta^2}},$$

where $J, \delta > 0$. (3.10)

This pulse sequence will yield a swap of the two spins: $|\uparrow\downarrow\rangle \rightarrow |\downarrow\uparrow\rangle$. 

When the dots have different g-factors, the rotational axis $\hat{n}$ veers away from the x-axis. When the initial state is $|\uparrow\downarrow\rangle$, the starting point is the north pole of the Bloch sphere. Because the rotation occurs around a different axis now, a $\pi$ rotation will never yield a state situated at the south pole of the sphere, $|\downarrow\uparrow\rangle$. Instead, it will result in a superposition of these two states. A $2\pi$ rotation in the presence of Zeeman inhomogeneity is shown in figure 3.2b.

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- Next turning $J$ off so the system undergoes a $\pi$ rotation around the z-axis, caused by the Zeeman inhomogeneity;

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The exchange pulse sequences for performing a swap in the absence and the presence of Zeeman inhomogeneity are depicted in figure 3.3. The frequency of the revolution around $\hat{n}$ is $\frac{\sqrt{\delta^4 J'^2 + \delta^2}}{\pi J}$. Therefore, the duration of a $\pi$ pulse is $\frac{\pi}{\sqrt{\delta^4 J'^2 + \delta^2}}$. So the time it takes to perform the swap operation is

$$t_{\text{swap}} = \frac{\pi}{\sqrt{\delta^4 J'^2 + \delta^2}} + \frac{\pi J}{\frac{\delta^2}{2\pi^2}} + \pi J \frac{\delta}{\sqrt{\delta^4 J'^2 + \delta^2}},$$

where $J, \delta > 0$. (3.10)

This pulse sequence will yield a swap of the two spins: $|\uparrow\downarrow\rangle \rightarrow |\downarrow\uparrow\rangle$.
Figure 3.3: (a) In the absence of Zeeman inhomogeneity, a swap operation can be performed with a single exchange pulse. (b) In the presence of Zeeman inhomogeneity, a swap operation can be performed by applying two $\pi$ pulses with magnitudes $J$ and $J' = \frac{2\delta}{\pi J}$ and a $\pi$ pulse for free evolution in between [9].
4.1. DOUBLE QUANTUM DOT

4.1.1. DOUBLE QUANTUM DOT CONTAINING ONE ELECTRON
The eigenstates and corresponding eigenenergies of Hamiltonian (2.14) are shown in table 4.1.

<table>
<thead>
<tr>
<th>Eigenstate</th>
<th>Eigenenergy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\sin\theta</td>
<td>L\rangle + \cos\theta</td>
</tr>
<tr>
<td>$\cos\theta</td>
<td>L\rangle + \sin\theta</td>
</tr>
</tbody>
</table>

Table 4.1: Eigenstates and corresponding eigenenergies of a double dot containing two electrons, neglecting the different spin states, with $\theta = \arctan\frac{\epsilon}{\sqrt{\epsilon^2 + 4\tau^2}}$.

A plot of the eigenenergies as a function of the detuning $\epsilon$, the energy difference between $| (1, 0) \rangle$ and $| (0, 1) \rangle$, is shown in figure 4.1. An anticrossing occurs at $\epsilon = 0$. The size of the anticrossing is $2\tau$, which is consistent with the tunnel coupling $\tau$ that couples the two states.

Figure 4.1: Eigenenergies of a double dot system containing one electron as a function of detuning $\epsilon$. The dashed black lines represent the eigenenergies in the absence of tunnel coupling. An anticrossing occurs at $\epsilon = 0$. 

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When the Zeeman effect is taken into account, as in Hamiltonian (2.16), the eigenenergies are split by the Zeeman energy. This can be seen in table 4.2, in which the eigenstates and the corresponding eigenenergies are shown.

<table>
<thead>
<tr>
<th>Eigenstate</th>
<th>Eigenenergy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\sin \theta</td>
<td>L \uparrow\rangle + \cos \theta</td>
</tr>
<tr>
<td>$\cos \theta</td>
<td>L \uparrow\rangle + \sin \theta</td>
</tr>
<tr>
<td>$-\sin \theta</td>
<td>L \downarrow\rangle + \cos \theta</td>
</tr>
<tr>
<td>$\cos \theta</td>
<td>L \downarrow\rangle + \sin \theta</td>
</tr>
</tbody>
</table>

Table 4.2: Eigenstates and corresponding eigenenergies of a double dot containing two electrons, in the presence of an external magnetic field, with $\theta = \arctan \frac{\epsilon}{\sqrt{\epsilon^2 + 4\tau^2}}$. The Zeeman effect causes the different spin states to be separated by $E_Z$.

The Zeeman splitting can also be observed in figure 4.2. At $\epsilon = 0$, the spin up eigenstates, as well as the spin down eigenstate, experience an anticrossing with size $2\tau$, as expected.

In each subplot of figure 4.3 one of the basis components $|L \uparrow\rangle$, $|L \downarrow\rangle$, $|R \uparrow\rangle$ and $|R \downarrow\rangle$ is plotted on top of the eigenenergies. The color bar shows that the lighter the line is, the larger the basis component. This plot shows that around $\epsilon = 0$, the eigenstates are mixed states, two consisting of $|L \uparrow\rangle$ and $|R \uparrow\rangle$ and two consisting of $|L \downarrow\rangle$ and $|R \downarrow\rangle$, while when $|\epsilon| >> 0$, the eigenstates approach the basis components, due to the large energy difference between the left and the right dot.
4.1. Double quantum dot

In Hamiltonian (2.17), different g-factors are taken into account. In figure 4.4, the corresponding eigenenergies are plotted as a function of the detuning $\epsilon$. The different g-factors cause a difference in Zeeman energies in the dots. Therefore, there is a shift in the eigenenergies. Anticrossings occur when energy levels in the left and the right dot are equal. This happens when

$$\frac{\epsilon}{2} \pm \frac{\mu_B B_0 g_L}{2} = -\frac{\epsilon}{2} \pm \frac{\mu_B B_0 g_R}{2}. \quad (4.1)$$

Solving equation (4.1) for $\epsilon$ yields

$$\epsilon = \pm \frac{\mu_B B_0 (g_R - g_L)}{2}. \quad (4.2)$$

Figure 4.3: Eigenenergies of a double dot system containing one electron, as a function of detuning, in the presence of an external magnetic field. On top, the component of the basis vectors $|L \uparrow\rangle$, $|L \downarrow\rangle$, $|R \uparrow\rangle$ and $|R \downarrow\rangle$ in the corresponding eigenvectors is plotted. The lighter the line, the larger the basis component, so around $\epsilon = 0$, the eigenstates consist of either $|L \uparrow\rangle$ and $|R \uparrow\rangle$ or $|L \downarrow\rangle$ and $|R \downarrow\rangle$, while at large values of $|\epsilon|$, the eigenstates approach the basis states.
4. RESULTS: EIGENENERGIES, TIME EVOLUTION AND SPIN SWAP

The energies of $|L\uparrow\rangle$ and $|R\uparrow\rangle$ align when $
abla = \frac{\mu B_0 (g_R - g_L)}{2} = 28.94 \mu \text{eV}$. The energies of $|L\downarrow\rangle$ and $|R\downarrow\rangle$ align when $
abla = \frac{-\mu B_0 (g_L - g_R)}{2} = -28.94 \mu \text{eV}$.

Considering equal g-factors and the initial state $|L\uparrow\rangle$, when the system is pulsed into $\nabla = 0$, an oscillation between $|L\uparrow\rangle$ and $|R\uparrow\rangle$ occurs. This can be seen in the uppermost two plots in 4.5. At $\nabla \neq 0$, the oscillation no longer has a peak-to-peak amplitude of 1, as can be seen in the two plots at the bottom of figure 4.5. $|L\uparrow\rangle$ will never completely evolve to $|R\uparrow\rangle$. The oscillation frequency is

$$f = \frac{\sqrt{\epsilon^2 + 4\tau^2}}{2\pi \hbar}$$

while the oscillation peak-to-peak amplitude is

$$A_{pp} = \frac{4\tau^2}{\epsilon^2 + 4\tau^2}.$$  

When the initial state is $|L\downarrow\rangle$, an oscillation between $|L\downarrow\rangle$ and $|R\downarrow\rangle$ occurs.
Considering different g-factors and the initial state $|L\uparrow\rangle$, an oscillation between $|L\uparrow\rangle$ and $|R\uparrow\rangle$ occurs at the anticrossings. The oscillation between the spin up states occurs at $\epsilon_\uparrow = \frac{\mu B_0 (g_R - g_L)}{2}$, while the oscillation between the spin down states occurs at $\epsilon_\downarrow = \frac{\mu B_0 (g_L - g_R)}{2}$. The oscillation between $|L\uparrow\rangle$ and $|R\uparrow\rangle$ at the anticrossing $\epsilon = \epsilon_\uparrow$ can be seen in the uppermost plots in figure 4.6. The two plots at the bottom of this figure show the time evolution at a distance of 10 $\mu$eV of the anticrossing. In the case of an oscillation between the spin up states, when different g-factors are involved, the oscillation frequency and the peak-to-peak amplitude are:

\[
f = \frac{\sqrt{(\epsilon - \epsilon_\uparrow)^2 + 4\tau^2}}{2\pi \hbar}
\]
\[
A_{pp} = \frac{4\tau^2}{(\epsilon - \epsilon_\uparrow)^2 + 4\tau^2}
\]

In Hamiltonian (2.21), the hyperfine interaction is taken into account. A plot of the eigenenergies in the case of equal g-factors is shown in the left plot of figure 4.7. Zooming in on this plot reveals two new anticrossings. One is shown in the right plot of figure 4.7.

In the case of an oscillation between the spin down states, $\epsilon_\uparrow$ must be replaced by $\epsilon_\downarrow$. 

Figure 4.6: Time evolution of a system with different g-factors, initially in the state $|L\uparrow\rangle$: probability of measuring $|L\uparrow\rangle$ and $|R\uparrow\rangle$ as a function of time. In the upper two plots, the time evolution occurs at the anticrossing, $\epsilon = \epsilon_\uparrow$, causing an oscillation between $|L\uparrow\rangle$ and $|R\uparrow\rangle$. In the lower two plots, the time evolution occurs at a detuning 10 $\mu$eV from the anticrossing, so $|L\uparrow\rangle$ will never completely evolve to $|R\uparrow\rangle$. 

In Hamiltonian (2.21), the hyperfine interaction is taken into account. A plot of the eigenenergies in the case of equal g-factors is shown in the left plot of figure 4.7. Zooming in on this plot reveals two new anticrossings. One is shown in the right plot of figure 4.7.
Figure 4.7: Eigenenergies of a double dot system containing one electron as a function of detuning, in the presence of an external magnetic field. Hyperfine interactions are taken into account, causing an anticrossing to occur. In the right plot, a small part of the left plot is magnified, revealing an anticrossing.

In the following, the dependency of the size of this anticrossing on the different parameters is studied. The results show the following relation between the anticrossing size and the parameters:

\[ \text{Anticrossing size} \propto \tau \sqrt{\Delta B_{n,x}^2 + \Delta B_{n,y}^2}. \]  

(4.7)

The anticrossing size as a function of \( \Delta B_{n,x} = B_{n,x,L} - B_{n,x,R} \) for different values of \( \Delta B_{n,y} = B_{n,y,L} - B_{n,y,R} \) is shown in figure 4.8. The plot shows the dependency of the anticrossing size on \( \Delta B_{n,x} \) for different values of \( \Delta B_{n,y} \). When the anticrossing size is plotted as a function of \( \Delta B_{n,y} \) for different values of \( \Delta B_{n,x} \), the results are similar.

Figure 4.8: Size of the anticrossing induced by the hyperfine interaction as a function of \( \Delta B_{n,x} \) for different values of \( \Delta B_{n,y} \). The shapes of the curves suggest that the anticrossing size depends linearly on \( \sqrt{\Delta B_{n,x}^2 + \Delta B_{n,y}^2} \).

In figure 4.9, the anticrossing size is plotted as a function of \( \Delta B_{n,x} \) for different values of \( \tau \). The fit is
described by

\[ c\sqrt{\Delta B_{n,x}^2 + \Delta B_{n,y}^2} \]

(4.8)

in which \( c \) is the fitting constant. The fits are indistinguishable from the data. The fitting constant depends linearly on \( \tau \), so the anticrossing size depends linearly on \( \tau \sqrt{\Delta B_{n,x}^2 + \Delta B_{n,y}^2} \).

![Figure 4.9: Size of the anticrossing that is induced by the hyperfine interaction as a function of \( \Delta B_{n,x} \) for different values of \( \tau \). The curves have been fitted with \( c\sqrt{\Delta B_{n,x}^2 + \Delta B_{n,y}^2} \). The fits are equal to the curves. The fitting constant \( c \) depends linearly on \( \tau \).](image)

### 4.1.2. Double Quantum Dot Containing Two Electrons

The eigenenergies and the corresponding eigenstates of Hamiltonian (2.27) with \( J \) as in (2.33) can be found in Table 4.3. The eigenstates of a double dot containing two electrons are the three triplet states and the singlet state.

<table>
<thead>
<tr>
<th>Name</th>
<th>Eigenstate</th>
<th>Eigenenergy</th>
<th>Total spin</th>
<th>Spin number</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_+ )</td>
<td>(</td>
<td>\uparrow\uparrow\rangle )</td>
<td>( \gamma B_0 )</td>
<td>1</td>
</tr>
<tr>
<td>( T_0 )</td>
<td>( \frac{1}{\sqrt{2}} (</td>
<td>\uparrow\downarrow\rangle +</td>
<td>\downarrow\uparrow\rangle) )</td>
<td>0</td>
</tr>
<tr>
<td>( T_- )</td>
<td>(</td>
<td>\downarrow\downarrow\rangle )</td>
<td>( -\gamma B_0 )</td>
<td>1</td>
</tr>
<tr>
<td>( S )</td>
<td>( \frac{1}{\sqrt{2}} (</td>
<td>\uparrow\downarrow\rangle -</td>
<td>\downarrow\uparrow\rangle) )</td>
<td>( -\hbar^2 J )</td>
</tr>
</tbody>
</table>

Table 4.3: Eigenstates and corresponding eigenenergies of a double dot containing two electrons.

The triplet eigenenergies are constant as a function of \( \epsilon \), while the singlet eigenenergy depends on \( J \), as expected, because \( \hbar^2 J \) is the difference in energy between \( T_0 \) and \( S \). A plot of the eigenenergies as a function of the detuning is shown in Figure 4.10.
4. Results: Eigenenergies, Time Evolution and Spin Swap

Considering equal g-factors and the initial state $|\uparrow\downarrow\rangle$, the state will oscillate between $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$, due to the exchange coupling $J$, that couples $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$. The probability of measuring $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ as a function of time is shown in the two uppermost plots of figure 4.11. When the two dots have different g-factors, and the initial state is $|\uparrow\downarrow\rangle$, the oscillation will never completely reach the state $|\downarrow\uparrow\rangle$, as is shown in the bottom two plots of figure 4.11.

Suppose the initial state of the two electron spins is $|\uparrow\downarrow\rangle$ and the goal is to achieve a spin swap, but there is a Zeeman inhomogeneity. As described in chapter 3, it is not possible to do this with a constant exchange coupling, but the procedure described by Hu and Das Sarma can be applied to reach $|\downarrow\uparrow\rangle$. The probabilities of measuring $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ as a function of time during this procedure have been plotted in figure 4.12.
4.1. Double Quantum Dot

B₀ = 5 T, τ = 1 µeV, δ = 0.12 µeV, ℏ²J = 0.115 µeV

Figure 4.12: Time evolution of the initial state |↓↑⟩ during the spin swap operation as described by Hu and Das Sarma. The first π pulse, in which the exchange coupling has magnitude J, is followed by a π pulse with exchange coupling δ. The third π pulse has magnitude \( \frac{δ^2}{τ J} \). This plot shows that during this procedure, |↓↑⟩ evolves to |↑↓⟩.

In figure 4.13, the effect of the swapping procedure as described by Hu and Das Sarma is shown for different ratios of J and δ.

As mentioned in chapter 3, the duration of a π pulse is \( \frac{π ℏ}{\sqrt{ℏ^2J^2 + δ^2}} \), so the total duration of the swapping procedure is given by

\[
t_{swap} = \frac{π ℏ}{\sqrt{ℏ^2J^2 + δ^2}} + \frac{π ℏ}{|δ|} + \frac{π ℏ}{\sqrt{δ^2/τ^2 + δ^2}}, \quad J, δ > 0. \tag{4.9}
\]

Using equation (4.9), it is possible to determine upper and lower bounds for the swapping time, depending on the magnitude of the Zeeman inhomogeneity δ. In appendix B, it is shown that

\[
\frac{2π ℏ}{|δ|} < t_{swap} ≤ \frac{(\sqrt{2} + 1) π ℏ}{|δ|}. \tag{4.10}
\]
The upper bound, \( \frac{\sqrt{2} + 1}{|\delta|} \pi \hbar \), is reached when \( J = |\delta| \), while the lower bound, \( \frac{2\pi \hbar}{|\delta|} \), is approached when \( J \to 0 \) and when \( J \to \infty \). The swapping time as a function of the exchange coupling is plotted in figure 4.14.

Figure 4.14: Duration of the swapping procedure as described by Hu and Das Sarma, as a function of the exchange coupling. The swap time is bounded from below by \( \frac{2\pi \hbar}{|\delta|} \) and bounded from above by \( \frac{\sqrt{2} + 1}{|\delta|} \pi \hbar \). The upper bound is reached at \( \hbar^2 J = |\delta| \), while the lower bound is approached when \( \hbar^2 J \to 0 \) and \( \hbar^2 J \to \infty \).

In GaAs quantum dots, the decoherence time is in the order of 10 ns [19]. One of the DiVincenzo requirements of the implementation of quantum computation is that the gate operation times are much shorter than the decoherence time [20]. Previously, in measurements in GaAs quantum dots, a Zeeman inhomogeneity of approximately 0.1 \( \mu \)eV has been observed, when applying a magnetic field of 5 T. Using equation (4.10), the total required swap time is 37 ns, which is substantially larger than the coherence time of 10 ns in this material. Therefore, the Hu and Das Sarma pulsing scheme is not practically useful for obtaining a perfect spin swap in GaAs quantum dots. However, this pulsing scheme could be useful if a material or system with a longer coherence time were to be used.

4.2. TRIPLE QUANTUM DOT

Using Hamiltonian 2.36, the eigenenergies of the triple quantum dot can be calculated. In figure 4.15, the eigenenergies are shown as a function of the detuning.
4.2. TRIPLE QUANTUM DOT

The eigenstates are a quadruplet state, \( Q \), and two doublet states, \( \Delta \) and \( \Delta' \). The eigenstates and corresponding eigenenergies can be found in Laird et al.

When \( \epsilon_+ - \epsilon_- \gg \tau_L, \tau_R \), initializing the system in the state \(|\uparrow \downarrow \uparrow\rangle\) and letting it evolve freely at detuning level \( \epsilon = \epsilon_+ \) will yield oscillations not only between the initial state \(|\uparrow \downarrow \uparrow\rangle\) and \(|\uparrow \uparrow \downarrow\rangle\), but also between \(|\uparrow \downarrow \uparrow\rangle\) and \(|\uparrow \uparrow \uparrow\rangle\). However, since \( f_{23} \gg f_{12} \) at \( \epsilon = \epsilon_+ \), the latter oscillation will be very slow compared to the former. In figure 4.16, the time evolution is shown for a relatively small time scale. A few oscillations between \(|\uparrow \downarrow \uparrow\rangle\) and \(|\uparrow \uparrow \downarrow\rangle\) occur, but the oscillation between \(|\uparrow \downarrow \uparrow\rangle\) and \(|\uparrow \uparrow \uparrow\rangle\) is negligible.

In figure 4.17, the time evolution is shown for a larger time scale, in which one oscillation between \(|\uparrow \downarrow \uparrow\rangle\) and \(|\uparrow \uparrow \uparrow\rangle\) occurs. The number of oscillations between \(|\uparrow \downarrow \uparrow\rangle\) and \(|\uparrow \uparrow \uparrow\rangle\) is so large that they are indistinguishable.
If the initial state is $|\uparrow\uparrow\downarrow\rangle$ and the desired final state is $|\downarrow\uparrow\uparrow\rangle$, first the two rightmost spins must be swapped, followed by the two leftmost spins. This can be achieved by first pulsing to $\epsilon = \epsilon_+$ in order to obtain a high $J_{12}$ and then pulsing to $\epsilon = \epsilon_-$ in order to obtain a high $J_{23}$. This is shown in figure 4.18. Both pulses should yield a $\pi$ rotation. Therefore, the duration of the first pulse and the second pulse must be $\frac{\pi \hbar}{2 J_{23}}$ and $\frac{\pi \hbar}{2 J_{12}}$ respectively.

In a triple dot, Zeeman inhomogeneities can be present as well. $\delta_{LM} = (\gamma_L - \gamma_M) B_0$ and $\delta_{MR} = (\gamma_M - \gamma_R) B_0$ are the Zeeman inhomogeneities between the two leftmost dots and the two rightmost dots, respec-
4.3. QUADRUPLE QUANTUM DOT

tively. Here, a pulsing scheme is given that swaps the leftmost and the rightmost spins when $g_L \neq g_R$ and $g_M \neq g_R$. Considering the initial state $|\uparrow\uparrow\downarrow\rangle$, the desired final state is $|\downarrow\uparrow\uparrow\rangle$. First, the two rightmost spins must be swapped. As long as $J_{12} \approx 0$, the swapping of the two leftmost spins will be negligible, so during the first three pulses the condition $J_{12} \approx 0$ must hold. First, a $\pi$ pulse with magnitude $\tilde{J}_{23}$ is applied, followed by a $\pi$ pulse with magnitude $\delta_{MR}$, then a $\pi$ pulse with magnitude $\delta_{LM}$. Then, the state will have evolved to $|\uparrow\downarrow\uparrow\rangle$. During the final three pulses, $J_{23} \approx 0$ must hold, so no swapping of the rightmost spins occurs. The final three pulses are a $\pi$ pulse with magnitude $\tilde{J}_{23}$, a $\pi$ pulse with magnitude $J_{12}$ and a $\pi$ pulse with magnitude $J_{34}$, yielding the final state $|\downarrow\uparrow\uparrow\rangle$. All steps of this pulsing scheme can be found in table 4.4.

<table>
<thead>
<tr>
<th>Pulse number</th>
<th>$J_{12}$</th>
<th>$J_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$\tilde{J}_{23} \gg 0$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$\delta_{MR}/4\tilde{J}_{23} \gg 0$</td>
</tr>
<tr>
<td>4</td>
<td>$J_{12} \gg 0$</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>$\delta_{LM}/4\tilde{J}_{12} \gg 0$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.4: Pulsing scheme that lets the initial state $|\uparrow\uparrow\downarrow\rangle$ evolve to $|\downarrow\uparrow\uparrow\rangle$ when Zeeman inhomogeneities are present.

The time evolution of the initial state $|\uparrow\uparrow\downarrow\rangle$ during the pulsing scheme described above is shown in figure 4.19.

Figure 4.19: The initial state $|\uparrow\uparrow\downarrow\rangle$ will evolve to $|\downarrow\uparrow\uparrow\rangle$ when the six $\pi$ pulses described in table 4.4 are applied. The probabilities of measuring $|\uparrow\uparrow\downarrow\rangle$, $|\uparrow\downarrow\uparrow\rangle$ and $|\downarrow\uparrow\uparrow\rangle$ are plotted as a function of time during this pulsing scheme.

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In a quadruple quantum dot, the spin swap $|\uparrow\uparrow\downarrow\rangle \rightarrow |\downarrow\uparrow\uparrow\rangle$ can be obtained by three $\pi$ pulses in sequence: $J_{34} \gg J_{12}, J_{23}$ first, $J_{23} \gg J_{12}, J_{34}$ next and finally $J_{12} \gg J_{23}, J_{34}$. In figure 4.20, the probabilities of measuring $|\uparrow\downarrow\uparrow\rangle$, $|\uparrow\uparrow\downarrow\rangle$, $|\downarrow\uparrow\uparrow\rangle$ and $|\uparrow\uparrow\downarrow\rangle$ is plotted as a function of time during this pulsing scheme.
Figure 4.20: Four electrons initialized in the state $|↑↑↑↓\rangle$ can evolve to the state $|↓↑↑↑\rangle$ by applying three $\pi$ pulses in sequence: $J_{34} \gg J_{12}, J_{23}$ first, $J_{23} \gg J_{12}, J_{34}$ next and finally $J_{12} \gg J_{23}, J_{34}$. Probabilities of measuring $|↑↑↑↓\rangle$, $|↑↑↑↑\rangle$, $|↑↑↑↑\rangle$ and $|↓↑↑↑\rangle$ have been plotted as a function of time during this pulsing scheme. During the first pulse, $\hbar^2 J_{12} = 1 \mu eV$ and $\hbar^2 J_{23} = \hbar^2 J_{34} = 0$. During the second pulse, $\hbar^2 J_{23} = 1 \mu eV$ and $\hbar^2 J_{12} = \hbar^2 J_{34} = 0$. During the third pulse, $\hbar^2 J_{34} = 1 \mu eV$ and $\hbar^2 J_{12} = \hbar^2 J_{23} = 0$. 
Upon further inspection, Taylor et al.’s approach of getting an expression for the exchange coupling $J$ yields an interesting result. As seen before, neglecting $|\langle 2,0|S\rangle$, the Hamiltonian describing the charge transition between $|\langle 1,1|S\rangle$ and $|\langle 0,2|S\rangle$ in a double dot with respect to the basis $|\langle 0,2|S\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\langle 1,1|S\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

is

$$H_{11-02} = \begin{pmatrix} -\epsilon & \tau \\ \tau & 0 \end{pmatrix}.$$  

(5.1)

The eigenvalues are

$$\lambda = -\frac{\epsilon}{2} \pm \frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2}.$$  

(5.2)

The ground state has the lowest eigenenergy, $\lambda_{gs} = -\frac{\epsilon}{2} - \frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2}$. The corresponding eigenvector can be found by solving $H_{11-02} - \lambda_{gs} I = 0$, yielding

$$|\langle 1,1|S\rangle + \frac{2\tau}{\epsilon - \sqrt{\epsilon^2 + 4\tau^2}} |\langle 0,2|S\rangle.$$

(5.3)

This eigenvector can be normalized using the identities $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$ and $\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$ with $x = \frac{2\tau}{\epsilon - \sqrt{\epsilon^2 + 4\tau^2}}$. Therefore, the ground state is the new singlet state as described by Taylor et al.:

$$|\langle \tilde{S}| = \cos \theta |\langle 1,1|S\rangle + \sin \theta |\langle 0,2|S\rangle$$

(5.4)

with $\theta = \arctan \frac{2\tau}{\epsilon - \sqrt{\epsilon^2 + 4\tau^2}}$, as Taylor et al. calculated. As seen in equation (2.33), $\hbar^2 J_{11-02} = \frac{\epsilon}{2} + \frac{1}{2}\sqrt{\epsilon^2 + 4\tau^2} = -\lambda_{gs}$. Using the equality $H |\langle \tilde{S}| = \lambda_{gs} |\langle \tilde{S}|$, the following set of equations is obtained:

$$\left( \begin{pmatrix} -\epsilon \sin \theta + \tau \cos \theta \\ \tau \sin \theta \end{pmatrix} \right) = -\hbar^2 J_{11-02} \left( \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} \right).$$

(5.5)

The bottom equation yields

$$\hbar^2 J_{11-02} = -\tau \frac{\sin \theta}{\cos \theta} = -\tau \frac{|\langle 0,2|S|\tilde{S}| \rangle}{|\langle 1,1|S|\tilde{S}| \rangle}.$$ 

(5.6)

So $J_{11-02}$ is linearly dependent on the ratio of the magnitude of the components of $|\langle 0,2|S\rangle$ and $|\langle 1,1|S\rangle$ in $|\langle \tilde{S}|$. 

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The same can be applied to the system in which only $|1, 1\rangle_S$ and $|2, 0\rangle_S$ are considered. With $E_c$ the charging energy of the left quantum dot and $\epsilon$ the energy difference between $|(0, 2)\rangle_S$ and $|(1, 1)\rangle_S$, as before, $|1, 1\rangle_S$ and $|2, 0\rangle_S$ line up at $\epsilon = -E_c$. In this case,

$$h^2 J_{11-20} = \frac{-\epsilon - E_c}{2} + \frac{1}{2} \sqrt{(-\epsilon - E_c)^2 + 4\tau^2}$$  \hspace{1cm} (5.8)

and in terms of the ratio of the different components the expression is

$$h^2 J_{11-20} = -\tau \frac{|\langle 2, 0\rangle_S \rangle \langle \tilde{S}|}{\langle \langle 1, 1\rangle_S | \langle \tilde{S}|}$$  \hspace{1cm} (5.9)

Henceforth, $J_{11-02}$ and $J_{11-20}$ in terms of component ratios will be denoted as $J_{11-02}^*$ and $J_{11-20}^*$ respectively, while $J_{11-02}$ and $J_{11-20}$ will still be defined as in equation (2.33) and (5.8). When all three states are taken into consideration, the charge transition Hamiltonian with respect to the basis

$$|2, 0\rangle_S = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |1, 1\rangle_S = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |0, 2\rangle_S = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$  \hspace{1cm} (5.10)

is

$$H = \begin{pmatrix} \epsilon + E_c & \tau & 0 \\ \tau & 0 & \tau \\ 0 & \tau & -\epsilon \end{pmatrix}.$$  \hspace{1cm} (5.11)

Again $-h^2 J$ is equal to the lowest eigenvalue of this Hamiltonian. However, the expressions for the eigenvalues of this Hamiltonian are very extensive. But, when $E_c \gg \tau, J$ can be approximated by the sum of $J_{11-02}$ and $J_{11-20}$:

$$h^2 J = h^2 (J_{11-02} + J_{11-20}) = -\frac{E_c}{2} + \frac{1}{2} \sqrt{\epsilon^2 + 4\tau^2} + \frac{1}{2} \sqrt{(-\epsilon - E_c)^2 + 4\tau^2}.$$  \hspace{1cm} (5.12)

On the right side of figure 5.1, $h^2 J$, $h^2 (J_{11-02} + J_{11-20})$ have been plotted as a function of $\epsilon$. It is clear that $h^2 J$ can be approximated by $h^2 (J_{11-02} + J_{11-20})$. On the left side of figure 5.1, $h^2 J$, $h^2 J_{11-02}$ and $h^2 J_{11-20}$ have been plotted. This plot shows that $h^2 J$ approaches $h^2 J_{11-02}$ asymptotically for large $\epsilon$, and $h^2 J$ approaches $h^2 J_{11-20}$ asymptotically for large $-\epsilon$, due to $|\langle 2, 0\rangle_S |S\rangle$ being negligibly small around the charge transition of $|1, 1\rangle_S$ and $|0, 2\rangle_S$ and $|\langle 2, 0\rangle_S |S\rangle$ is negligibly small around the charge transition of $|1, 1\rangle_S$ and $|2, 0\rangle_S$. 
\[ \tau = 20 \mu eV, E_c = 500 \mu eV \]

Figure 5.1: In red, the lowest eigenvalue of the charge state Hamiltonian (5.11) is plotted as a function of the detuning \( \epsilon \) with \( E_c = 25 \tau \). The relation between the exchange coupling \( J \) and the lowest eigenvalue \( E_{\text{lowest}} \) is given by \( \hbar^2 J = -E_{\text{lowest}} \). In the left figure, \( \hbar^2 J_{11-02} \) and \( \hbar^2 J_{11-20} \) have been plotted as dashed lines, showing that \( \hbar^2 J \) approaches \( \hbar^2 J_{11-02} \) asymptotically for large \( \epsilon \), and \( \hbar^2 J \) approaches \( \hbar^2 J_{11-20} \) asymptotically for large \( -\epsilon \). In the right figure, \( \hbar^2 J_{11-02} + \hbar^2 J_{11-20} \) is plotted as a dashed black line. The two lines are indistinguishable, showing that \( \hbar^2 J \) can be approximated by \( \hbar^2 (J_{11-02} + J_{11-20}) \).

However, when the condition \( E_c \gg \tau \) does not hold, equation (5.12) does not hold either. The right plot of figure 5.2, clearly shows that \( \hbar^2 J \) and \( \hbar^2 J_{11-02} + \hbar^2 J_{11-20} \) are not equal. For \( \epsilon \gg 0 \), \( J_{11-02} \) can still be used to approximate \( \hbar^2 J \), and for \( \epsilon \ll -E_c \), the approximation \( \hbar^2 J \approx \hbar^2 J_{11-20} \) is valid.
Figure 5.2: In red, the lowest eigenvalue of the charge state Hamiltonian (5.11) is plotted as a function of the detuning \( \epsilon \) with \( E_c = 2.5 \tau \). The relation between the exchange coupling \( J \) and the lowest eigenvalue \( E_{\text{lowest}} \) is given by \( \hbar^2 J = -E_{\text{lowest}} \). In the left figure, \( \hbar^2 J_{11-02} \) and \( \hbar^2 J_{11-20} \) have been plotted as dashed lines, showing that \( \hbar^2 J \) approaches \( \hbar^2 J_{11-02} \) asymptotically for large \( \epsilon \), and \( \hbar^2 J \) approaches \( \hbar^2 J_{11-20} \) asymptotically for large \(-\epsilon\). In the right figure, \( \hbar^2 J_{11-02} + \hbar^2 J_{11-20} \) is plotted as a dashed black line. In this case, the two lines are clearly different, showing that \( \hbar^2 J \) cannot be approximated by \( \hbar^2 (J_{11-02} + J_{11-20}) \).

In the right plot of figure 5.3, \( \hbar^2 J \) and \( \hbar^2 (J_{11-02}^* + J_{11-20}^*) \) have been plotted as a function of \( \epsilon \). In the left plot, \( \hbar^2 J_{11-02} \) and \( \hbar^2 J_{11-20} \) have been plotted separately. The graphs of \( \hbar^2 J \) and \( \hbar^2 (J_{11-02} + J_{11-20}) \) are indistinguishable, showing that

\[
\hbar^2 J = \hbar^2 (J_{11-02}^* + J_{11-20}^*).
\]

Equation (5.13) was valid for all tested values of \( E_c \) and \( \tau \). In all cases, relative errors had order of magnitude \( 10^{-16} \), which indicates that the errors are purely numerical.

---

1Tested values were in the range of a few \( \mu\text{eV} \) up until 10000 \( \mu\text{eV} \).
In a double quantum dot, the relation between the exchange coupling \( J \) and the lowest eigenvalue \( E_{\text{lowest}} \) is given by \( \hbar^2 J = -E_{\text{lowest}} \).

In the left figure, \( \hbar^2 J_{11-02}^r \) and \( \hbar^2 J_{11-20}^r \) are plotted on top, showing that \( \hbar^2 J \) approaches \( \hbar^2 J_{11-02}^r \) asymptotically for large \( \epsilon \), and \( \hbar^2 J \) approaches \( \hbar^2 J_{11-20}^r \) asymptotically for large \(-\epsilon\). In the right figure, \( \hbar^2 J_{11-02}^r + \hbar^2 J_{11-20}^r \) is plotted on top. In this case, the two lines are indistinguishable. Relative errors are in the order of \( 10^{-16} \), which indicates that the errors are numerical. So \( \hbar^2 J = \hbar^2 (J_{11-02}^r + J_{11-20}^r) \).

As discussed in chapter 2, Laird et al. found expressions for the exchange coupling between the two electrons on the left (in dot 1 and 2) and the two electrons on the right (in dot 2 and 3) in a triple quantum dot by assuming that there was no interaction between non-neighbouring electrons. The expressions are

\[
J_{12}(\epsilon) = \frac{1}{\hbar^2} \left( \frac{\epsilon - \epsilon_+}{2} + \sqrt{(\epsilon - \epsilon_-)^2 + 4\tau_L^2} \right), \quad J_{23}(\epsilon) = \frac{1}{\hbar^2} \left( \frac{\epsilon - \epsilon_-}{2} + \sqrt{(\epsilon - \epsilon_+)^2 + 4\tau_R^2} \right),
\]

in which \( \epsilon_+ \) denotes the detuning value at which \((1,0,2)\) and \((1,1,1)\) line up, while \( \epsilon_- \) denotes the detuning value at which \((2,0,1)\) and \((1,1,1)\) line up.

In this three dot system, the charge state Hamiltonian with respect to the basis

\[
|2,0,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1,1,1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1,0,2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

is

\[
H = \begin{pmatrix}
\epsilon - \epsilon_- & \tau_L & 0 \\
\tau_L & 0 & \tau_R \\
0 & \tau_R & \epsilon_+ - \epsilon
\end{pmatrix},
\]

in which \( \tau_L \) and \( \tau_R \) denote the tunnel couplings between the two dots on the left and the two dots on the right, respectively. Similar to the double dot, when \( \epsilon_+ - \epsilon_- \gg \tau_L, \tau_R \), the expressions obtained by Laird et al. can be used to approximate the exchange coupling in the triple dot:

\[
\hbar^2 J = \hbar^2 (J_{12} + J_{23}).
\]
In the right plot of figure 5.4, $\hbar^2 J$ and $\hbar^2 (J_{12} + J_{23})$ have been plotted, showing that $\hbar^2 J$ can be approximated by $\hbar^2 (J_{12} + J_{23})$ when $\epsilon_+ - \epsilon_- \gg \tau_L, \tau_R$.

\[
\tau_L = 10 \mu eV, \tau_R = 20 \mu eV, \epsilon_+ = 200 \mu eV, \epsilon_- = -200 \mu eV
\]

Figure 5.4: In red, the lowest eigenvalue of the charge state Hamiltonian (5.16) has been plotted as a function of the detuning $\epsilon$ with $\epsilon_+ - \epsilon_- = 40 \tau_L = 20 \tau_R$. The relation between the total exchange coupling $J$ and the lowest eigenvalue $E_{\text{lowest}}$ is given by $\hbar^2 J = -E_{\text{lowest}}$. In the left figure, $\hbar^2 J_{12}$ and $\hbar^2 J_{23}$ have been plotted as dashed lines. In the right figure, $\hbar^2 (J_{12} + J_{23})$ is plotted as a black dashed line. In this case, the two lines are indistinguishable, showing that $\hbar^2 J$ can be described by $\hbar^2 (J_{12} + J_{23})$.

However, when the condition $\epsilon_+ - \epsilon_- \gg \tau_L, \tau_R$ is not met, equation (5.17) is not valid anymore. Defining

\[
J^*_\text{12} = -\tau_L \frac{\langle 2,0,1\vert \psi \rangle}{\langle 1,1,1\vert \psi \rangle}, \quad J^*_\text{23} = -\tau_R \frac{\langle 1,0,2\vert \psi \rangle}{\langle 1,1,1\vert \psi \rangle},
\]

in which $\psi$ as the eigenstate corresponding to the lowest eigenvalue of Hamiltonian (5.16), $J$ can be written in terms of $J^*_\text{12}$ and $J^*_\text{23}$:

\[
\hbar^2 J = \hbar^2 (J^*_\text{12} + J^*_\text{23}). \quad (5.18)
\]

In the right plot of figure 5.5, the left and right sides of equality in (5.18) of are plotted and they are indistinguishable. The quality holds for all tested values\(^2\).

\(^2\)Tested values of $\tau_L, \tau_R, \epsilon_+, \epsilon_-$ were in the range of several $\mu$eV up until 10000 $\mu$eV.
\[ \tau_L = 50 \text{ \textmu eV}, \tau_R = 100 \text{ \textmu eV}, \varepsilon_+ = 200 \text{ \textmu eV}, \varepsilon_- = -200 \text{ \textmu eV} \]

The same procedure can be repeated for the quadruple dot. A charge state is denoted as \((w,x,y,z)\), in which \(w, x, y\) and \(z\) are the number of electrons in dot 1, 2, 3 and 4, respectively. Dot 1 is the leftmost dot and dot 4 is the rightmost dot. In this system, the basis

\[
|2, 0, 1, 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
|1, 1, 1, 1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},
|1, 0, 2, 1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},
|1, 1, 0, 2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
\]

(5.19)

with the Hamiltonian

\[
H = \begin{pmatrix} -\varepsilon_{12} + E_{c,12} & \tau_L & 0 & 0 \\ \tau_L & 0 & \tau_M & 0 \\ 0 & \tau_M & -\varepsilon_{23} + E_{c,23} & 0 \\ 0 & 0 & \tau_R & -\varepsilon_{34} + E_{c,34} \end{pmatrix},
\]

(5.20)
is considered. Possible couplings between non-neighbouring electrons are not taken into account. \(\tau_L\) denotes the tunnel coupling between dot 1 and 2, \(\tau_M\) the tunnel coupling between dots 2 and 3 and \(\tau_R\) the tunnel coupling between dots 3 and 4, \(E_{c,12}\) is the charging energy of the leftmost dot, \(E_{c,23}\) is the charging energy of the third dot from the left, and \(E_{c,34}\) is the charging energy of the rightmost dot. \(\varepsilon_{12} - E_{c,12}\) is the energy difference between \(|(2,0,1,1)\rangle\) and \(|(1,1,1,1)\rangle\), \(\varepsilon_{23} - E_{c,23}\) the energy difference between \(|(1,0,2,1)\rangle\), \(|(1,1,1,1)\rangle\) and \(\varepsilon_{34} - E_{c,34}\) the energy difference between \(|(1,1,0,2)\rangle\) and \(|(1,1,1,1)\rangle\). Denoting \(\phi\) as the eigenstate corresponding to the lowest eigenvalue of Hamiltonian (5.20), \(J^*_r, J^*_s, J^*_t\) can be defined as

\[
J^*_r = -\tau_L \langle (2,0,1,1|\phi\rangle \langle (1,1,1,1)|\phi\rangle, J^*_s = -\tau_M \langle (1,0,2,1|\phi\rangle \langle (1,1,1,1)|\phi\rangle, J^*_t = -\tau_R \langle (1,1,0,2|\phi\rangle \langle (1,1,1,1)|\phi\rangle.
\]

(5.21)
As in the double and the triple dot, the equality

\[ h^2 J = h^2 (J^*_{12} + J^*_{23} + J^*_{34}), \]  

(5.22)

holds for all tested values of \( \tau_L, \tau_M, \tau_R, E_c_{12}, E_c_{23}, \) and \( E_c_{34} \), because the relative errors had order of magnitude \( 10^{-15} \), which indicates that the errors are purely numerical.

When the states \(|(0,2,1,1)\rangle, |(1,2,0,1)\rangle \) and \(|(1,1,2,0)\rangle \) are taken into account as well, \( J^*_{12}, J^*_{23}, \) and \( J^*_{34} \) have to be extended to

\[
\begin{align*}
J^*_{12} &= -\tau_L \left( \frac{|(2,0,1,1)\rangle\langle\phi| + |(0,2,1,1)\rangle\langle\phi|}{|1,1,1,1\rangle\langle\phi|} \right) \\
J^*_{23} &= -\tau_M \left( \frac{|(1,0,2,1)\rangle\langle\phi| + |(1,2,0,1)\rangle\langle\phi|}{|1,1,1,1\rangle\langle\phi|} \right) \\
J^*_{34} &= -\tau_R \left( \frac{|(1,1,0,2)\rangle\langle\phi| + |(1,1,2,0)\rangle\langle\phi|}{|1,1,1,1\rangle\langle\phi|} \right) 
\end{align*}
\]  

(5.23) \hspace{1cm} (5.24) \hspace{1cm} (5.25)

for equation (5.22) to be valid.

---

3Tested values ranged from a few µeV up until 10000 µeV for all parameters.
CONCLUSIONS

• The hyperfine interaction couples the up and down states in a double quantum dot containing one electron. The strength of the coupling depends linearly on $\tau \sqrt{\Delta B_{n,x}^2 + \Delta B_{n,y}^2}$.

• The duration of the pulsing scheme described by Hu and Das Sarma that causes the initial state $|\uparrow \downarrow\rangle$ to evolve to $|\downarrow \uparrow\rangle$ in the presence of Zeeman inhomogeneity is bounded from below by $\frac{2\hbar}{\delta}$.  

• In a double quantum dot, the magnitude of the exchange coupling can be described by the equation $\hbar^2 J = -\frac{\hbar e^2}{2} + \frac{1}{2} \sqrt{\epsilon^2 + 4\tau^2} + \frac{1}{2} \sqrt{(-\epsilon - E_c)^2 + 4\tau^2}$ only when $E_c >> \tau$.

• In a double quantum dot, the magnitude of the exchange coupling can always be described by $\hbar^2 J = -\tau \left( \frac{\langle 0.2 | \Delta S \rangle}{\langle 1.1 | \Delta S \rangle} + \frac{\langle 2.0 | \Delta S \rangle}{\langle 1.1 | \Delta S \rangle} \right)$.

• In a triple quantum dot, the magnitude of the total exchange coupling is $\hbar^2 J = -\tau_L \frac{\langle 2.0,1 | \psi \rangle}{\langle 1.1,1 | \psi \rangle} - \tau_R \frac{\langle 1.0,2 | \psi \rangle}{\langle 1.1,1 | \psi \rangle}$, in which $\psi$ is the eigenstate corresponding to the lowest eigenvalue of the charge state Hamiltonian. The exchange coupling between electrons 1 and 2, and 2 and 3 can be described by $\hbar^2 J_{12}^* = -\tau_L \frac{\langle 2.0,1 | \psi \rangle}{\langle 1.1,1 | \psi \rangle}$ and $\hbar^2 J_{23}^* = -\tau_R \frac{\langle 1.0,2 | \psi \rangle}{\langle 1.1,1 | \psi \rangle}$, respectively.

• In a quadruple quantum dot, the magnitude of the total exchange coupling is $\hbar^2 J = \hbar^2 (J_{12}^* + J_{23}^* + J_{34}^*)$, in which $J_{12}^* = -\tau_L \left( \frac{\langle 2.0,1,1 | \phi \rangle}{\langle 1.1,1,1 | \phi \rangle} + \frac{\langle 0.2,1,1 | \phi \rangle}{\langle 1.1,1,1 | \phi \rangle} \right)$, $J_{23}^* = -\tau_M \left( \frac{\langle 1.0,2,1 | \phi \rangle}{\langle 1.1,1,1 | \phi \rangle} + \frac{\langle 1.0,1,2 | \phi \rangle}{\langle 1.1,1,1 | \phi \rangle} \right)$, and $J_{34}^* = -\tau_R \left( \frac{\langle 1.1,1,0,2 | \phi \rangle}{\langle 1.1,1,1 | \phi \rangle} + \frac{\langle 1.1,2,0 | \phi \rangle}{\langle 1.1,1,1 | \phi \rangle} \right)$ denote the exchange couplings between electrons 1 and 2, 2 and 3, and 3 and 4, respectively.
7

Outlook

7.1. Fitting the exchange coupling in a double quantum dot

As found in chapter 5, the magnitude of the exchange coupling between the two electrons in a double quantum dot is a function of $\langle (2, 0) S | \tilde{S} \rangle$, $\langle (0, 2) S | \tilde{S} \rangle$ and $\langle (1, 1) S | \tilde{S} \rangle$:

$$h^2 J = h^2 (J_{11-02} + J_{11-20}) = -\tau \left( \frac{\langle (0, 2) S | \tilde{S} \rangle}{\langle (1, 1) S | \tilde{S} \rangle} + \frac{\langle (2, 0) S | \tilde{S} \rangle}{\langle (1, 1) S | \tilde{S} \rangle} \right).$$  \hspace{1cm} (7.1)

When $E_c \gg \tau$, $h^2 J_{11-02}$ reduces to $h^2 J_{11-02} = \frac{\xi}{2} + \frac{1}{2} \sqrt{\xi^2 + 4\tau^2}$ and $h^2 J_{11-20}$ reduces to $h^2 J_{11-20} = \frac{-\epsilon - E_c}{2} + \frac{1}{2} \sqrt{(\epsilon - E_c)^2 + 4\tau^2}$. However, when the condition $E_c \gg \tau$ is not met, $J_{11-02}$ and $J_{11-20}$ cannot be used to approximate $J_{11-02}$ and $J_{11-20}$.

$\langle (2, 0) S | \tilde{S} \rangle$, $\langle (0, 2) S | \tilde{S} \rangle$ and $\langle (1, 1) S | \tilde{S} \rangle$ have been plotted as a function of $\epsilon$ in figure 7.1. It can be noted that the plot of $\langle (0, 2) S | \tilde{S} \rangle$ as a function of $\epsilon$ resembles a sigmoid function.

![Figure 7.1: $\langle (2, 0) S | \tilde{S} \rangle$, $\langle (0, 2) S | \tilde{S} \rangle$ and $\langle (1, 1) S | \tilde{S} \rangle$ have been plotted as a function of the detuning $\epsilon$. It can be noted that $\langle (0, 2) S | \tilde{S} \rangle$ resembles a sigmoid function.](image)
\[ |\langle (0,2)S|\tilde{S}\rangle| \text{ has been fitted with the sigmoid function} \]
\[ f_{02}(\epsilon) = \frac{1}{2} + \frac{1}{2} \frac{\epsilon - b}{c \sqrt{1 + \left( \frac{\epsilon - b}{c} \right)^2}}, \tag{7.2} \]
as shown in the upper left plot of figure 7.2. \[ |\langle (2,0)S|\tilde{S}\rangle| \text{ has been fitted with the function} \]
\[ f_{20}(\epsilon) = \frac{1}{2} - \frac{1}{2} \frac{\epsilon + b + E_c}{c \sqrt{1 + \left( \frac{\epsilon + b + E_c}{c} \right)^2}}, \tag{7.3} \]
as shown in the upper right plot of figure 7.2. In the two lower plots of figure 7.2, the relative errors have been plotted as a function of \( \epsilon \).

\[ \tau = 100 \mu eV, E_c = 100 \mu eV \]

![Graphs showing relative errors](image)

Figure 7.2: \[ |\langle (0,2)S|\tilde{S}\rangle| \text{ and } |\langle (2,0)S|\tilde{S}\rangle| \text{ have been plotted as a function of the detuning } \epsilon \text{ in the upper left and upper right plot, respectively. } |\langle (0,2)S|\tilde{S}\rangle| \text{ has been fitted with the sigmoid function } f_{02}(\epsilon), |\langle (2,0)S|\tilde{S}\rangle| \text{ has been fitted with } f_{20}(\epsilon). \]
The relative errors have been plotted in the two lower subfigures. The left plot shows that the relative errors in the left plot are below 0.05 when \( \epsilon < -300 \mu eV \), but when \( \epsilon \) decreases further, the fit \( f_{02}(\epsilon) \) becomes less accurate. The right plot shows that \( f_{20}(\epsilon) \) approximates \[ |\langle (2,0)S|\tilde{S}\rangle| \] with a relative error of less than 0.05 when \( \epsilon < 200 \mu eV \), but when \( \epsilon \) increases further, \( f_{20}(\epsilon) \) is less accurate.

Because \( S \) is normalized, \[ |\langle (1,1)S|\tilde{S}\rangle| = \sqrt{1 - |\langle (0,2)S|\tilde{S}\rangle|^2 - |\langle (2,0)S|\tilde{S}\rangle|^2}, \] so \( \hbar^2J \) can be fitted with \( f(\epsilon) = \frac{f_{02}(\epsilon)}{\sqrt{1-f_{02}(\epsilon)^2}} + \frac{f_{20}(\epsilon)}{\sqrt{1-f_{20}(\epsilon)^2}} \). Both \( \hbar^2J \) and \( f(\epsilon) \) have been plotted in figure 7.3. \( f(\epsilon) \) approximates \( \hbar^2J \) well for \(-300 \mu eV < \epsilon < 200 \mu eV \), but the farther outside of this range, the more \( f(\epsilon) \) deviates from \( \hbar^2J \), due to propagating errors in \( f_{02}(\epsilon) \) and \( f_{20}(\epsilon) \). However, outside of this interval, \( \hbar^2J \) approaches \( \hbar^2 (J_{11-02} + J_{11-20}) \).
7.2. SUGGESTIONS FOR FURTHER RESEARCH

Further research could focus on finding an expression or a fit for $\hbar^2 J$ in a double quantum dot that is valid in all detuning regimes.

Secondly, in the derivation of the expressions that describe $\hbar^2 J$ in a triple and a quadruple quantum dot in chapter 5, the possibility of non-neighbouring electrons interacting with each other was not taken into account. It is unclear how valid this assumption is. It could be useful to find expressions describing $\hbar^2 J$ when coupling between non-neighbouring electrons is taken into account.

Furthermore, the effect of spin-orbit coupling and hyperfine interactions on the spin swapping procedure in a quantum dot array could be explored.

While the spin swap gate is useful in a spin-based quantum computer, the square-root-of-swap gate is even more important. $\sqrt{SWAP}$ is one of the building blocks of the cNOT operation, which is used for entangling operations. Further research could be dedicated to designing a pulsing scheme that carries out a $\sqrt{SWAP}$. 

Figure 7.3: $\hbar^2 J = -E_{\text{lowest}}$ and $f(\epsilon)$ have been plotted as a function of $\epsilon$. For $-300 \mu eV < \epsilon < 200 \mu eV$, $f(\epsilon)$, $\hbar^2 J$ and $f(\epsilon)$ are nearly equal. Relative errors are smaller than 0.02. However, the farther outside of this range, the more $f(\epsilon)$ deviates from $\hbar^2 J$, due to propagating errors in $f_{02}(\epsilon)$ and $f_{20}(\epsilon)$. Outside of the interval $-300 \mu eV < \epsilon < 200 \mu eV$, $\hbar^2 J$ approaches $\hbar^2 (J_{11-02} + J_{11-20})$. 

So, close to $\epsilon = 0$, $\hbar^2 J$ can be approximated by $f(\epsilon)$, and when $|\epsilon| \gg E_c$, $\hbar^2 J$ can be approximated by $\hbar^2 (J_{11-02} + J_{11-20})$. However, in the intermediate regime, $\hbar^2 J$ cannot be approximated accurately by either of them.

The coefficients $b$ and $c$ in $f_{02}(\epsilon)$ and $f_{20}(\epsilon)$ have been determined for several different values of $\tau$ and $E_c$ in the range of a few to 30000 $\mu eV$. The results can be found in appendix C. The values of $b$ and $c$ were fitted, yielding

$$b_{\text{fit}} = \left(0.02988 \left(\frac{\tau}{E_c}\right)^{-1.023} - 0.375\right) E_c$$

$$c_{\text{fit}} = 0.2653 \tau - 0.02465 E_c^{1.025} + 1.4142 \tau.$$ 

$b_{\text{fit}}$ yields values of $b$ within all 95% confidence intervals. However, $c_{\text{fit}}$ yielded values within the 95% confidence intervals of only 58 of the 63 tested combinations of $\tau$ and $E_c$. 

7.2. SUGGESTIONS FOR FURTHER RESEARCH

Further research could focus on finding an expression or a fit for $\hbar^2 J$ in a double quantum dot that is valid in all detuning regimes.
First of all, I would like to express my sincere gratitude to Lieven Vandersypen, for giving me the opportunity to do my Bachelor’s thesis at the Quantum Transport group, one of the most renowned research groups in the world. It has been a great experience to be a part of QT for the past ten months.

Next, I would like to thank Johan Dubbeldam for supervising the mathematical side of this project, solving some administrative issues for me and looking at the preliminary version of my report back in January and giving me useful feedback.

Thank you, Mohammad Shafiei, for being my supervisor, for answering all my questions and for helping me whenever I needed help.

Last but not least, a big thank you to all the students in the QT students room. It has been amazing working alongside all of you. Ferdy, thank you for giving me useful insights about spin qubits and Kian, thank you for helping me solve many a programming problem I encountered.
The standard basis is

\[
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

\[
|\uparrow\uparrow\uparrow\uparrow\rangle = 0, \quad |\uparrow\uparrow\uparrow\downarrow\rangle = 0, \quad |\uparrow\uparrow\downarrow\uparrow\rangle = 0, \quad |\uparrow\uparrow\downarrow\downarrow\rangle = 0, \quad |\uparrow\downarrow\uparrow\uparrow\rangle = 0, \quad |\uparrow\downarrow\uparrow\downarrow\rangle = 0, \quad |\uparrow\downarrow\downarrow\uparrow\rangle = 0, \quad |\uparrow\downarrow\downarrow\downarrow\rangle = 0.
\]
The Hamiltonian is

\[
H = \begin{pmatrix}
2\gamma B_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\gamma B_0 - \hbar \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\gamma B_0 - \hbar \omega & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\gamma B_0 - \hbar \omega & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2\gamma B_0 - \hbar \omega & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2\gamma B_0 - \hbar \omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2\gamma B_0 - \hbar \omega & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\gamma B_0 - \hbar \omega & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\gamma B_0 - \hbar \omega \\
\end{pmatrix}
\]
\[
\begin{array}{cccccccc}
& h_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
-2y B_0 - h_{12} & 0 & 0 & h_{23} & 0 & h_{34} & 0 & h_{45} \\
-2y B_0 - h_{12} & 0 & h_{23} & 0 & 0 & h_{34} & 0 & h_{45} \\
-2y B_0 - h_{12} & 0 & h_{23} & 0 & h_{34} & 0 & 0 & h_{45} \\
-2y B_0 - h_{12} & 0 & h_{23} & 0 & h_{34} & 0 & h_{45} & 0 \\
-4y B_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
\textbf{DERIVATION OF THE BOUNDS OF THE SWAPPING TIME}

The total swapping time is given by

\[ t_{\text{swap}} = \frac{\pi h}{\sqrt{h^4 j^2 + \delta^2}} + \frac{\pi h}{\sqrt{\frac{\delta^4}{h^4 j^2} + \delta^2}} , \quad J, \delta > 0. \] (B.1)

We claim that

\[ \frac{2\pi h}{|\delta|} < t_{\text{swap}} \leq \frac{(\sqrt{2} + 1)\pi h}{|\delta|}. \] (B.2)

\textbf{Proof.} Given a certain \( \delta \neq 0 \), the duration of the \( \pi \) rotation with the exchange coupling fixed to 0 is \( \frac{\pi h}{|\delta|} \), so only the left and right terms in equation (B.1) can influence the swapping time. Therefore, it suffices to show that

\[ \frac{1}{|\delta|} < \frac{1}{\sqrt{h^4 j^2 + \delta^2}} + \frac{1}{\sqrt{\frac{\delta^4}{h^4 j^2} + \delta^2}} \leq \sqrt{2} \] (B.3)

for all \( J > 0 \).

First, we will rewrite

\[ \frac{1}{\sqrt{h^4 j^2 + \delta^2}} + \frac{1}{\sqrt{\frac{\delta^4}{h^4 j^2} + \delta^2}} = \frac{\sqrt{h^4 j^2 + \delta^2} + \sqrt{\frac{\delta^4}{h^4 j^2} + \delta^2}}{\sqrt{h^4 j^2 + \delta^2 \sqrt{\frac{\delta^4}{h^4 j^2} + \delta^2}}} = \frac{\sqrt{h^4 j^2 + \delta^2 + \sqrt{\frac{\delta^4}{h^4 j^2} + \delta^2}}}{|\delta| \sqrt{h^4 j^2 + 2\delta^2 + \frac{\delta^4}{h^4 j^2}}} \leq \frac{\sqrt{h^4 j^2 + \delta^2}}{|\delta| \sqrt{h^4 j^2 + 2\delta^2}} \] (B.4)

in which we have made the substitutions \( x = h^4 j^2 + \delta^2 \) and \( y = \frac{\delta^4}{h^4 j^2} + \delta^2 \). \( x, y > 0 \) because \( J \neq 0 \) and \( \delta \neq 0 \).

Now we will prove that

\[ \frac{1}{\sqrt{h^4 j^2 + \delta^2}} + \frac{1}{\sqrt{\frac{\delta^4}{h^4 j^2} + \delta^2}} > 1: \]

\[ \frac{\sqrt{h^4 j^2 + \delta^2} + \sqrt{\frac{\delta^4}{h^4 j^2} + \delta^2}}{|\delta| \sqrt{h^4 j^2 + 2\delta^2 + \frac{\delta^4}{h^4 j^2}}} = \frac{\sqrt{x} + \sqrt{y}}{|\delta| \sqrt{x + y}} \geq \frac{\sqrt{x}}{|\delta| \sqrt{x + y}} = \frac{1}{|\delta|}. \]

in which (*) follows from the well known inequality \( \sqrt{a} + \sqrt{b} > \sqrt{a + b} \) for all \( a, b > 0 \).

Finally, we will prove that

\[ \frac{\sqrt{h^4 j^2 + \delta^2} + \sqrt{\frac{\delta^4}{h^4 j^2} + \delta^2}}{|\delta| \sqrt{h^4 j^2 + 2\delta^2 + \frac{\delta^4}{h^4 j^2}}} \leq \frac{\sqrt{h^4 j^2 + \delta^2}}{|\delta| \sqrt{h^4 j^2 + 2\delta^2}} \leq \frac{\sqrt{2}}{|\delta|}. \] (B.5)
(II) follows from the inequality $\sqrt{a} + \sqrt{b} \leq 2\sqrt{a + b}$ for all $a, b > 0$. Below, a proof of this statement is given:

\[
0 \leq (a - b)^2 \quad \text{ (B.6)}
\]
\[
\iff 0 \leq a^2 - 2ab + b^2 \quad \text{ (B.7)}
\]
\[
\iff 4ab \leq a^2 + b^2 + 2ab \quad \text{ (B.8)}
\]

Taking the square root on both sides yields

\[
2\sqrt{ab} \leq a + b \quad \text{ (B.9)}
\]
\[
\iff a + b + 2\sqrt{ab} \leq 2a + 2b \quad \text{ (B.10)}
\]
\[
\iff (\sqrt{a} + \sqrt{b})^2 \leq 2(a + b) \quad \text{ (B.11)}
\]

Again, we take the square root on both sides of the inequality:

\[
\sqrt{a} + \sqrt{b} \leq \sqrt{2\sqrt{a + b}}. \quad \text{ (B.12)}
\]

We have shown that both sides of equation (B.3) hold, so equation (B.1) holds too.

Furthermore,

\[
\lim_{J \to 0} t_{\text{swap}} = \lim_{J \to \infty} t_{\text{swap}} = \frac{2\pi h}{|\delta|},
\]

so the lower bound is approached when $J \to 0$ and when $J \to \infty$.

The upper bound is reached when $J = |\delta|$, because then $x = y$, and $\sqrt{x} + \sqrt{y} = 2\sqrt{x} = \sqrt{2} \sqrt{2x} = \sqrt{2} \sqrt{x + y}$. 

FITTING THE EXCHANGE COUPLING IN A DOUBLE QUANTUM DOT: RESULTS
C. Fitting the Exchange Coupling in a Double Quantum Dot: Results

<table>
<thead>
<tr>
<th>τ (eV)</th>
<th>Yes</th>
<th>1.4135</th>
<th>1.414</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000</td>
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<td>0.0000</td>
<td></td>
<td></td>
</tr>
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<td>0.9999</td>
<td>0.9999</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Results of the fitting process for the exchange coupling constant τ within bounds. The table lists the values of τ for which the fit is within the confidence intervals of the fitted parameters f and q, and also indicates whether the fit is within the parameter space as defined by the boundary conditions. Yes indicates that the fit is within both bounds, while No indicates that it is not.

The values in the table are approximate and may vary slightly due to the nature of the fitting process and the inherent uncertainties in the data.

The table also includes the parameter values for the exchange coupling constant f and q within the confidence intervals, as well as the parameter uncertainties for further analysis.

The overall fit quality is assessed by the goodness-of-fit statistics, which indicate how well the model fits the data.

The table highlights the effectiveness of the fitting process in capturing the essential features of the data within the given parameter space constraints.

The results demonstrate the robustness of the fitting method in identifying the exchange coupling constant τ, providing valuable insights into the underlying physics of the system.

The detailed parameter values and uncertainties are essential for a comprehensive understanding of the exchange coupling dynamics and its implications for quantum dot systems.
<table>
<thead>
<tr>
<th>Country</th>
<th>Death Rate / 100,000 Population</th>
<th>Sex</th>
<th>Age Group</th>
<th>Disease Category</th>
<th>Cause of Death</th>
<th>Deaths</th>
<th>Percent of Total Deaths</th>
</tr>
</thead>
</table>
| USA | 100 | Male | 0-40 | Cancers | 100 | 10 | 0.1%
| USA | 200 | Female | 41-60 | Cancers | 200 | 20 | 0.2%

**Note:** This table represents the number of deaths in the USA by sex, age group, disease category, and cause of death. The percent of total deaths is calculated by dividing the number of deaths by the total number of deaths in the respective category.


