A Popov-theory-based approach to digital H^∞ control with measurement feedback for Pritchard–Salamon systems

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In this paper, we deal with the digital output-measurement-feedback H^∞ control problem for Pritchard–Salamon infinite-dimensional systems with unbounded input and output operators. A discrete Popov-theory-based solution is given in terms of the solvability of Kalman–Szegő–Popov–Yakubovitch systems associated with the equivalent discrete-time time-invariant system obtained by lifting the T -periodic continuous-time system.

1. Introduction

In [25, 26], Pritchard & Salamon introduced a class of infinite-dimensional systems that allow certain unboundedness in control and observation. Specific examples of such systems are linear systems described by PDEs with point and/or boundary control and observation [9, 12], and retarded systems with delays in control and observation [25]. This class has become very popular since it has been proved to be rich enough to permit one to obtain the solution to the linear quadratic control problem (see Pritchard & Salamon [26] and van Keulen [29]) as well as the extension of the classical finite-dimensional results on H^∞ optimal control from [15] (see van Keulen [29]). For an overview of the basic results concerning perturbation theory, exponential stability, and transfer functions of Pritchard–Salamon systems, the reader is referred to the recently announced paper of Curtain *et al.* [13].

The H^∞ control problem with state and output measurement feedback for *infinite-dimensional* systems has received much attention in recent years. A state-space solution to the problem has been given by van Keulen in [29] where the main two-Riccati equation result of Doyle *et al.* from [15] has been extended to the Pritchard–Salamon class of systems.

Among the various approaches to the H^∞ control problem, the Popov-theory-based solution, originally developed for discrete-time finite-dimensional systems by Ionescu & Weiss [20], has been fully generalized to infinite-dimensional systems: the continuous-time bounded-input bounded-output case by Weiss [32], and the time-dependent discrete-time case by Dragan, Halanay, & Ionescu [17].

The digital linear quadratic control problem for Pritchard–Salamon systems has been addressed in Barb & DeKoning [3] where a digital state-feedback result was given. The case of digital output-measurement feedback was treated in Barb &

DeKoning [4]. In [5], a Popov-theory-based solution to the digital state-feedback H^∞ control of Pritchard–Salamon systems was given in terms of solvability of certain Kalman–Szegő–Popov–Yakubovitch (KSPY) systems. This paper is a natural continuation of the work in [5]. We are concerned with the (γ -suboptimal) digital output-measurement-feedback H^∞ control problem for Pritchard–Salamon systems. Our goal is to design a discrete-time controller interconnected with the continuous-time plant via a synchronized sampler and zeroth-order holder such that the effect of the disturbance input on the controlled output is lower than a prespecified bound imposed by continuous-time design considerations. The piecewise constant control is generated from the discretized measurement output $\{y_2(kT)\}_{k \in \mathbb{N}}$ (when $y_2(\cdot)$ can be sampled; we shall see that further on in this paper). The difficulty of taking into account the intersample behaviour is overcome by lifting the overall time-dependent system to a discrete-time one. This is done in the spirit of the original work of Yamamoto [33], making use of the framework developed by Bamieh *et al.* [1, 2]. A similar lifting technique was applied by Tadmor [28] who gave a solution to the digital H^∞ control problem for systems with finite-dimensional state space in terms of three Riccati equations, two algebraic equations, and a differential one. The Popov-theory approach we take in this paper has the advantage of allowing the main control result to be stated in a framework of Hilbert-space operators, while permitting the set of necessary and sufficient conditions to be expressed in terms of the solvability of a certain KSPY system (equivalent to a Riccati equation). This was not possible in [1], since the spaces of the lifted disturbance input and the lifted controlled output were infinite-dimensional. Within our approach, we remain in the classical framework of the solution based on the Riccati equation. There are two reasons for this. The first is that our original state space is *infinite-dimensional* and there is no struggle as in [1] to reduce everything to finite-dimensional spaces—the implementation issue is not one of our objectives at this stage. The second reason is that we have available a general Riccati theory for discrete-time systems on real separable Hilbert spaces that we can apply to our digital control problem.

The structure of this paper is the following. In Section 2, we formulate the problem. Some basic Pritchard–Salamon results are also given here. In Section 3, we show how to reduce the control problem to an equivalent discrete-time one. General discrete Popov-theory results are collected in Section 4, and the solution to the γ -attenuation problem for discrete-time systems is outlined in Section 5 in the case of output measurement feedback. The shortcomings of the method are discussed in Section 6. Conclusions are reached and directions of future research are indicated in Section 7.

2. Problem formulation

2.1. Pritchard–Salamon systems. Basic properties

Capital script letters will stand for real, separable Hilbert spaces. If $A : \mathcal{X} \rightarrow \mathcal{Y}$, then $\mathcal{D}(A)$ will denote the domain of A . We write $\mathfrak{L}(\mathcal{X}, \mathcal{Y})$ for the space of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$. By $L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{X})$ we denote the space of locally (Bochner) square-integrable functions from $\mathbb{R}_+ = [0, \infty)$ to $\mathcal{X}$, and $\ell_2(\mathbb{N}, \mathcal{X})$ will stand for the

Hilbert space of square-summable sequences in $\mathcal{X}$ (i.e. maps from $\mathbb{N} = \{0, 1, \dots\}$ to $\mathcal{X}$). We shall say that A generates an *exponentially stable* semigroup of bounded linear operators $\{S(t)\}_{t \in \mathbb{R}_+}$ on $\mathcal{X}$ (a C^0 semigroup) if there exists $\mu \geq 1$ and $\alpha < 0$ such that

$$\|S(t)\| < \mu e^{\alpha t} \quad \text{for } t \geq 0.$$

An operator $A \in \mathfrak{L}(\mathcal{X})$ is called *power-stable* if there exist $\mu \geq 1$ and $\rho \in (0, 1)$ such that

$$\|A^k\| < \mu \rho^k \quad \text{for } k \in \mathbb{N}.$$

Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|_{\mathcal{X}}$, and let $\mathcal{W}$ be a linear subspace of $\mathcal{X}$. Assume that another norm, $\|\cdot\|_{\mathcal{W}}$, is defined on $\mathcal{W}$ such that $\mathcal{W}$ is itself a Banach space. Consider the linear operator $I: \mathcal{W} \rightarrow \mathcal{X}$ defined by $Iw = w$. We call this a *continuous embedding* [14] if $I \in \mathfrak{L}(\mathcal{W}, \mathcal{X})$. If $\mathcal{W}$ is dense in $\mathcal{X}$ with respect to the norm in $\mathcal{X}$ (i.e. its closure is equal to $\mathcal{X}$), we shall call I a *dense injection*. In the case when I is also continuous, we shall use the notation $\mathcal{W} \hookrightarrow \mathcal{X}$.

If $\mathcal{X}$ is a real Hilbert space with the norm given by the inner product, we shall say that $\mathcal{X}$ is *separable* if $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ contains a dense subset which is countable.

Let us consider three real separable Hilbert spaces $\mathcal{W}$, $\mathcal{X}$, and $\mathcal{V}$ with continuous dense injections $\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$. We consider a C_0 semigroup $\{S(t)\}_{t \in \mathbb{R}_+}$ on $\mathcal{V}$, and we shall assume that its restriction on $\mathcal{W}$ and $\mathcal{X}$ are also C_0 semigroups. $A^{\mathcal{W}}$, $A^{\mathcal{X}}$, and $A^{\mathcal{V}}$ will denote the infinitesimal generators of $\{S^{\mathcal{W}}(t)\}_{t \in \mathbb{R}_+}$, $\{S^{\mathcal{X}}(t)\}_{t \in \mathbb{R}_+}$, and $\{S^{\mathcal{V}}(t)\}_{t \in \mathbb{R}_+}$ on $\mathcal{W}$, $\mathcal{X}$, and $\mathcal{V}$, where $A^{\mathcal{V}} = A$, and where the superscripts $\mathcal{W}$ and $\mathcal{X}$ indicate the corresponding restricted operators. The following definition is reported from [26, 13, 29].

DEFINITION 1 Let $\mathcal{U}$ and $\mathcal{Y}$ be the input and output spaces.

1. An operator $B \in \mathfrak{L}(\mathcal{U}, \mathcal{V})$ is called an *admissible input operator* for $S(\cdot)$ (see Weiss [30]) if there exist $\iota > 0$ and $\alpha > 0$ such that

$$\int_0^{\iota} S^{\mathcal{V}}(t - \tau) B u(\tau) d\tau \in \mathcal{W} \quad (2.1)$$

and

$$\left\| \int_0^{\iota} S^{\mathcal{V}}(t - \tau) B u(\tau) d\tau \right\|_{\mathcal{W}} \leq \alpha \|u(\cdot)\|_{L_2(0, \iota; \mathcal{U})} \quad (2.2)$$

for all $u(\cdot) \in L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{U})$, where

$$\|u(\cdot)\|_{L_2(0, \iota; \mathcal{U})} \triangleq \|u(\cdot) \upharpoonright [0, \iota]\|_{L_2(0, \iota; \mathcal{U})}.$$

2. An operator $C \in \mathfrak{L}(\mathcal{W}, \mathcal{Y})$ is called an *admissible output operator* for $S(\cdot)$ (see Weiss [31]) if there exist $\iota > 0$ and $\beta > 0$ such that

$$\|CS^{\mathcal{W}}(\cdot)x\|_{L_2(0, \iota; \mathcal{Y})} \leq \beta \|x\|_{\mathcal{V}} \quad (2.3)$$

for all $x \in \mathcal{W}$.

3. Let $B \in \mathfrak{L}(\mathcal{U}, \mathcal{V})$ and $C \in \mathfrak{L}(\mathcal{W}, \mathcal{Y})$ be admissible input and output operators for $S(\bullet)$. The system $\Sigma(S(\bullet), B, C, D)$ given by

$$\left. \begin{aligned} x(t) &= S^\mathcal{V}(t)x_0 + \int_0^t S^\mathcal{V}(t-\tau)Bu(\tau) d\tau \\ y(t) &= Cx(t) + Du(t) \end{aligned} \right\} \quad (t \geq 0), \quad (2.4a, b)$$

where $x_0 \in \mathcal{V}$ and $u(\bullet) \in L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{U})$, is called a *Pritchard–Salamon system*; this is well defined because, if (2.2) (resp. (2.3)) holds for some $t > 0$ and for some $\alpha > 0$ (resp. $\beta > 0$), then it holds for any $t > 0$ and some α (resp. β) depending on t (see [13]). $\square$

Note that, if $S(\bullet)$ is exponentially stable on $\mathcal{W}$, then α and β can be chosen not to depend on t , and (2.2) and (2.3) hold then for $t = \infty$.

If $x_0 \in \mathcal{W}$, then $x(\bullet)$ defined by (2.4a) is a continuous function on $\mathbb{R}_+$ with respect to the topology on $\mathcal{W}$, and the output

$$t \mapsto y(t) = CS^\mathcal{W}(t)x_0 + C \int_0^t S^\mathcal{V}(t-\tau)Bu(\tau) d\tau \quad (2.5)$$

is also a continuous function on $\mathbb{R}_+$ with values in $\mathcal{Y}$. If $x_0 \in \mathcal{V} \setminus \mathcal{W}$, then (2.5) does not make sense directly. From (2.3), the admissibility of the output operator implies that the linear map

$$\mathcal{W} \rightarrow L_2(\mathbb{R}_+, \mathcal{Y}) : x \mapsto CS^\mathcal{W}(\bullet)x$$

has a unique bounded extension

$$\mathcal{V} \rightarrow L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{Y}) : x \mapsto \overline{CS^\mathcal{W}(\bullet)}x$$

in the sense that, for every $x \in \mathcal{V}$, we shall use the expression $\overline{CS^\mathcal{W}(\bullet)}x$ to denote the function in $L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{Y})$ obtained by continuous extension to $\mathcal{V}$ of the operator that maps $x \in \mathcal{W}$ into $CS^\mathcal{W}(\bullet)x$ in $L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{Y})$. Thus, the output $y(\bullet) \in L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{Y})$ in (2.5) is interpreted as

$$t \mapsto y(t) = \overline{CS^\mathcal{W}(t)}x_0 + C \int_0^t S^\mathcal{V}(t-\tau)Bu(\tau) d\tau + Du(t)$$

and is a well defined $\mathcal{Y}$ -valued locally L_2 function.

Following [13], we assume (without loss of generality) that the Pritchard–Salamon system is smooth, i.e.

$$\mathfrak{D}(A^\mathcal{V}) \hookrightarrow \mathcal{W}. \quad (2.6)$$

Notice that assumption is not restrictive in the sense that (2.6) is satisfied by all known examples of Pritchard–Salamon systems if $\mathcal{W}$ and $\mathcal{V}$ are chosen appropriately. We quote from [13] that (2.6) is implied by $\mathfrak{D}(A^\mathcal{V}) \subset \mathcal{W}$, the latter inclusion also ensuring that the resolvent sets of $A^\mathcal{W}$ and $A^\mathcal{V}$ are the same. In contrast, the growth bounds $\omega_\mathcal{W}$ and $\omega_\mathcal{V}$ may not be the same (see Example 2.1 in [13]), and hence exponential stability on $\mathcal{V}$ (resp. $\mathcal{W}$) does not imply exponential stability on $\mathcal{W}$ (resp. $\mathcal{V}$). However, using the concept of *admissible stabilizability*, a satisfactory stability theory was developed (see [13, 29] for details).

DEFINITION 2 Let $\Sigma(S(\cdot), B, C, D)$ be a Pritchard-Salamon system.

1. The pair $(S(\cdot), B)$ is called *admissibly (boundedly) stabilizable* if there exists an admissible output operator $F \in \mathfrak{L}(\mathcal{W}, \mathcal{U})$ (an operator $F \in \mathfrak{L}(\mathcal{V}, \mathcal{U})$) such that the perturbed C^0 semigroup $S_F^R(\cdot)$, generated by $A^\nu + BF$, is exponentially stable on $\mathcal{W}$ and $\mathcal{V}$.

2. The pair $(C, S(\cdot))$ is called *admissibly (boundedly) detectable* if there exists an admissible input operator $H \in \mathfrak{L}(\mathcal{Y}, \mathcal{V})$ (an input operator $H \in \mathfrak{L}(\mathcal{Y}, \mathcal{W})$) such that the perturbed C^0 semigroup $S_H^L(\cdot)$, generated by $A^\nu + HC$, is exponentially stable on $\mathcal{W}$ and $\mathcal{V}$. $\square$

The following result [29] gives the relationship that exists between the two concepts of stabilizability quoted above.

PROPOSITION 1 (1) The pair $(S(\cdot), B)$ is admissibly stabilizable if and only if is boundedly stabilizable. (2) The pair $(C, S(\cdot))$ is admissibly detectable if and only if is boundedly detectable. $\square$

We shall focus now on the notion of *input-output stability*.

DEFINITION 3 Let $\Sigma(S(\cdot), B, C, D)$ be a Pritchard-Salamon system, and let

$$G : \mathfrak{D}(G) \subseteq L_2^{\text{loc}}(\mathbb{R}_+; \mathcal{U}) \rightarrow L_2^{\text{loc}}(\mathbb{R}_+; \mathcal{Y})$$

defined by

$$(Gu)(t) \triangleq C \int_0^t S(t-\tau)Bu(\tau) d\tau + Du(t). \quad (2.7)$$

We call $\Sigma(S(\cdot), B, C, D)$ *input-output stable* if $\mathfrak{D}(G) = L_2^{\text{loc}}(\mathbb{R}_+; \mathcal{U})$. $\square$

The next result gives the relationship between input-output stability and exponential stability (see [29]).

PROPOSITION 2 Let $\Sigma(S(\cdot), B, C, D)$ be a Pritchard-Salamon system, and suppose that $(S(\cdot), B)$ is admissibly stabilizable and $(C, S(\cdot))$ is admissibly detectable, respectively. Then the system is input-output stable if and only if is exponentially stable on $\mathcal{W}$ and $\mathcal{V}$ (i.e. $\omega_{\mathcal{W}} < 0$ and $\omega_{\mathcal{V}} < 0$). $\square$

2.2. The digital output-measurement-feedback H^∞ control problem for Pritchard-Salamon systems

Let us consider three Hilbert spaces with continuous dense injections $\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$, and assume that there exist two other real separable Hilbert spaces $\mathcal{U}_1$ and $\mathcal{Y}_1$, and let $B_1 \in \mathfrak{L}(\mathcal{U}_1, \mathcal{V})$, $B_2 \in \mathfrak{L}(\mathcal{U}_2, \mathcal{V})$, $C_1 \in \mathfrak{L}(\mathcal{W}, \mathcal{Y}_1)$, $C_2 \in \mathfrak{L}(\mathcal{W}, \mathcal{Y}_2)$, and $D_{12} \in \mathfrak{L}(\mathcal{U}_2, \mathcal{Y}_1)$. We consider the Pritchard-Salamon system

$$\Sigma\left(S(\cdot), [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix}\right)$$

satisfying

$$\left. \begin{aligned} x(t) &= S^{\mathcal{W}}(t)x_0 + \int_0^t S^{\mathcal{V}}(t-\tau)[B_1u_1(\tau) + B_2u_2(\tau)]d\tau \\ y_1(t) &= C_1x(t) + D_{11}u_1(t) + D_{12}u_2(t) \\ y_2(t) &= C_2x(t) \end{aligned} \right\} \quad (t \in \mathbb{R}_+). \quad (2.8)$$

We shall assume that the system is smooth (i.e. (2.6) is fulfilled). In our set-up, $x(t) \in \mathcal{V}$ is the state, $u_1(t) \in \mathcal{U}_1$ is the disturbance input, $u_2(t) \in \mathcal{U}_2$ is the control input, $y_1(t) \in \mathcal{Y}_1$ the controlled output, and $y_2(t) \in \mathcal{Y}_2$ is the measurement output at time t . The following assumptions are made on the Pritchard–Salamon system

1. $\mathcal{U}_2 = \mathbb{R}^m$ (we have m actuators).
2. $\mathcal{Y}_2 = \mathbb{R}^p$ (we have p sensors).
3. $x_0 \in \mathcal{W}$ (the output measurement is a continuous $\mathbb{R}^p$ -valued function with respect to the topology on $\mathcal{W}$).
4. $(S(\cdot), B_2)$ is admissibly stabilizable.
5. $(C_2, S(\cdot))$ is admissibly detectable.
6. $D_{11} = 0$.
7. $D_{12}^*D_{12}$ is coercive.

REMARK 1

- Assumptions 1–2 are based on engineering reality, in which the number of sensors and actuators must obviously be finite.
- Assumption 3 guarantees that the output measurement can be sampled.
- Assumptions 4 and 5 provide a sufficient condition for the hybrid stability of the digital control system (see Proposition 3).
- Assumption 6 is made for simplicity. For details of how this assumption can be removed, the reader is referred to [29: § 5.4].
- Assumption 7 guarantees the well-posedness of the Pritchard–Salamon system. $\square$

If $x_0 = 0$, then we can express (2.8) as

$$\begin{bmatrix} y_1(\cdot) \\ y_2(\cdot) \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} u_1(\cdot) \\ u_2(\cdot) \end{bmatrix}, \quad (2.9)$$

where

$$\begin{aligned} G_{11} &: L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{U}_1) \rightarrow L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{Y}_1), \\ (G_{11}u_1)(t) &= C_1 \int_0^t S^{\mathcal{V}}(t-\tau)B_1u_1(\tau)dt; \end{aligned}$$

$$G_{12} : L_2^{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m) \rightarrow L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{Y}_1),$$

$$(G_{12}u_2)(t) = C_1 \int_0^t S^\nu(t-\tau) B_2 u_2(\tau) d\tau + D_{12} u_2(t);$$

$$G_{21} : L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{U}_1) \rightarrow L_2^{\text{loc}}(\mathbb{R}_+, \mathbb{R}^p),$$

$$(G_{21}u_1)(t) = C_2 \int_0^t S^\nu(t-\tau) B_1 u_1(\tau) d\tau;$$

$$G_{22} : L_2^{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m) \rightarrow L_2^{\text{loc}}(\mathbb{R}_+, \mathbb{R}^p),$$

$$(G_{22}u_2)(t) = C_2 \int_0^t S^\nu(t-\tau) B_2 u_2(\tau) d\tau.$$

Let $\mathcal{K}$ be another real separable Hilbert space, and consider a controller $\Sigma(K, L, M, N)$ of the form

$$\xi_{k+1} = K\xi_k + L\eta_k, \quad \zeta_k = M\xi_k + N\eta_k, \quad (2.10)$$

where $K \in \mathfrak{L}(\mathcal{K})$, $L \in \mathfrak{L}(\mathbb{R}^p, \mathcal{K})$, $M \in \mathfrak{L}(\mathcal{K}, \mathbb{R}^m)$, $N \in \mathbb{R}^{p \times m}$, and the initial state of the controller, ξ_0 , is given. If $\xi_0 = 0$, then (2.10) can be expressed in an input-output fashion as

$$\zeta_k = (G_K \eta)_k,$$

where

$$(G_K \eta)_k \triangleq \sum_{i=0}^{k-1} M K^i L \eta_{k-i-1} + N \eta_k.$$

We want to stabilize the system (in a certain sense that will be defined below), and ensure that the influence of the disturbance input u_1 on the controlled output y_1 is smaller (in a certain sense) than a prespecified bound γ , by *digital output-measurement feedback*. Let us explain further what we mean by digital output-measurement feedback. Define first the sample and zeroth-order hold operators with period T :

$$S_T : \mathfrak{C}(\mathbb{R}_+, \mathcal{X}) \rightarrow \mathcal{X}^{\mathbb{N}}, \quad \tilde{\zeta} = S_T \zeta = (\zeta_0, \dots, \zeta_k = \zeta(kT), \dots),$$

$$H_T : \mathcal{X}^{\mathbb{N}} \rightarrow \mathfrak{C}(\mathbb{R}_+, \mathcal{X}), \quad \zeta(t) = (H_T \zeta)_k = \zeta_k \quad \text{for } t \in [kT, (k+1)T),$$

where by $\mathfrak{C}(\mathbb{R}_+, \mathcal{X})$ we have denoted the space of $\mathcal{X}$ -valued piecewise continuous functions that are bounded on compact subsets of $\mathbb{R}_+$, and where $\mathcal{X}^{\mathbb{N}}$ is the space of $\mathcal{X}$ -valued sequences defined on the set of nonnegative integers. By their definitions, the hold operator is synchronized in time with the sampling operator, and they are well defined when $\mathcal{X}$ is replaced by any other space. Since the output measurement is a continuous $\mathbb{R}^p$ -valued function (we have assumed that $x_0 \in \mathcal{W}$, so this condition is fulfilled) we can define its time-discretized version as

$$\tilde{y}_2 = S_T y_2.$$

We want to make sense of the feedback connection

$$\left. \begin{aligned} y_{2,k} &= \eta_k \\ u_2(t) &= \zeta_k \quad \text{for } kT \leq t < (k+1)T \end{aligned} \right\} \quad (k \in \mathbb{N}),$$

as depicted in Fig. 1.

Fig. 1. The closed-loop configuration of digital output-measurement feedback.

Our goal is to design a controller $\Sigma(K, L, M, N)$ with the following properties.

1 (stability). The closed-loop system $\Sigma(G_T, G_K)$, obtained by the feedback interconnection of the hybrid generalized plant G_T and the controller G_K is *hybrid-stable*. We shall explain what we mean by *hybrid stability* of the structure $\Sigma(G_T, G_K)$ in Section 3.1.

2 (γ contraction). The input-output operator

$$T_1 : L_2(\mathbb{R}_+, \mathcal{U}_1) \rightarrow L_2(\mathbb{R}_+, \mathcal{Y}_1) : u_1 \mapsto y_1$$

expressed as a linear fractional transformation of the hybrid generalized plant

$$G_T = \begin{bmatrix} G_{11} & G_{12}H_T \\ S_T G_{21} & S_T G_{22}H_T \end{bmatrix}$$

and the controller G_K , i.e.

$$T_1 = F(G_T, G_K) \triangleq G_{11} + G_{12}H_T G_K (I - S_T G_{22}H_T G_K)^{-1} S_T G_{21},$$

is well defined and is a γ contraction of spaces, i.e.

$$\|T_1 u_1(\bullet)\| < \gamma.$$

REMARK 2 Due to the sampler, T_1 is not time-invariant (see [28]), and we cannot associate a transfer function with the linear system mapping u_1 to y_1 . $\square$

2.3. The discrete-time H^∞ control problem

We conclude this section by formulating the H^∞ control problem in discrete time. Our basic model is the infinite-dimensional discrete-time system

$$\Sigma\left(A, [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix}\right)$$

representing the set of equations

$$\left. \begin{aligned} sx &= Ax + B_1 u_1 + B_2 u_2 \\ y_1 &= C_1 x + D_{11} u_1 + D_{12} u_2 \\ y_2 &= C_2 x \end{aligned} \right\}, \quad (2.11)$$

where s denotes the forward shift operator $(x_k)_{k \in \mathbb{N}} \mapsto (x_{k+1})_{k \in \mathbb{N}}$, and where $Ax \triangleq (Ax_k)_{k \in \mathbb{N}}$ etc. $A \in \mathfrak{L}(\mathcal{X})$ is assumed to define a power-stable evolution, $B_1 \in \mathfrak{L}(\mathcal{U}_1, \mathcal{X})$, $B_2 \in \mathfrak{L}(\mathcal{U}_2, \mathcal{X})$, $C_1 \in \mathfrak{L}(\mathcal{X}, \mathcal{Y}_1)$, $C_2 \in \mathfrak{L}(\mathcal{X}, \mathcal{Y}_2)$, $D_{11} \in \mathfrak{L}(\mathcal{U}_1, \mathcal{Y}_1)$, and $D_{12} \in \mathfrak{L}(\mathcal{U}_2, \mathcal{Y}_1)$. We consider $\mathcal{U}_1, \mathcal{U}_2, \mathcal{Y}_1, \mathcal{Y}_2$ to be the real separable Hilbert spaces of the disturbance input ($u_{1,k} \in \mathcal{U}_1$), control input ($u_{2,k} \in \mathcal{U}_2$), output to be controlled ($y_{1,k} \in \mathcal{Y}_1$), and measured output ($y_{2,k} \in \mathcal{Y}_2$), respectively. Here $\mathcal{X}$ is a real separable Hilbert state space, with $x \in \mathcal{X}$. Consider for the system (2.11) the controller

$$sx_c = A_c x_c + B_c y_2, \quad u_2 = C_c x_c + D_c y_2, \quad (2.12)$$

where $A_c x_c \triangleq (A_c x_{ck})_{k \in \mathbb{N}}$ etc., and where x_{ck} belongs to the real separable Hilbert controller state-space $\mathcal{X}_c$; here A_c, B_c, C_c , and D_c are operators bounded on appropriate subspaces.

The resultant closed-loop system is given by

$$sx_R = A_R x_R + B_R u_1, \quad y_1 = C_R x_R + D_R u_1, \quad (2.11)$$

where again the convention $A_R x_R \triangleq (A_R x_{Rk})_{k \in \mathbb{N}}$ etc. is used, and where

$$x_R = \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad A_R = \begin{bmatrix} A + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix}, \quad B_R = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

$$C_R = \begin{bmatrix} C_1 + D_{12} D_c C_2 \\ D_{12} C_c \end{bmatrix}, \quad D_R = D_{11}.$$

The augmented state space $\mathcal{X}_R = \mathcal{X} \oplus \mathcal{X}_c$ is a real separable Hilbert space under the inner product

$$\langle x_{ck}, x_{ck} \rangle_{\mathcal{X}_R} = \langle x_k, x_k \rangle_{\mathcal{X}} + \langle x_{ck}, x_{ck} \rangle_{\mathcal{X}_c}.$$

The system (2.11) can be written in an input-output fashion as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $T_{ij} \in \mathfrak{L}(\ell_2(\mathbb{N}, \mathcal{U}_i), \ell_2(\mathbb{N}, \mathcal{Y}_j))$ for $i, j = 1, 2$. If T_c is the input-output operator associated with the controller, then the closed-loop input-output operator is expressed as the linear fractional transformation of the system (2.11) and the controller (2.12):

$$T_1 = F\left(\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, T_c\right) = T_{11} + T_{12} T_c (I - T_{22} T_c)^{-1} T_{21}.$$

The γ -suboptimal H^∞ control problem (or γ -disturbance-attenuation problem) is to find a controller (2.12) for the system (2.11) such that

1. The closed-loop system (2.13) is power-stable (i.e. A_R generates a power stable evolution on $\mathcal{X}_R$). We shall refer to this as the *stability* requirement.
2. The closed-loop input-output operator is a γ contraction:

$$\|T_1\| < \gamma.$$

We shall refer to this as the γ -attenuation requirement.

3. The equivalent discrete-time H^∞ optimal-control problem

In this section, we shall reduce the digital control problem to an equivalent discrete-time one. The basic idea is to lift a continuous-time periodic signal to a shift-invariant discrete-time one. In this way, we describe the action from the lifted input to the lifted output of a periodically time-dependent system by an operator we shall call the *lifted* input–output operator. Since state-space formulae are important for our developments, we shall derive the expression of lifted A , B , C , and D operators of a given system. Our basic references are Bamieh & Pearson [1], Bamieh *et al.* [2], and Yamamoto [33], papers in which the reader can find a much more detailed treatment of this problem. We shall also investigate stability properties of the digital control system. Since the resulting closed loop is inherently hybrid in its nature, an appropriate stability theory should be developed. However, our work is greatly simplified by the concept of hybrid stability introduced by Chen & Francis [10], which is adequate for our purposes. We generalize the main results on hybrid stability from [10] to the Pritchard–Salamon class of hybrid systems.

3.1. Hybrid stability for Pritchard–Salamon systems

The concept of hybrid stability was initially introduced for the finite-dimensional case in [10]. In [11] it has been shown that exponential stability does not imply L_2 -input–output stability unless a low-pass filter is used prior to the sampler. However, it has been shown that, under some pathology-excluding conditions imposed on the sampling step, a certain concept called hybrid stability is equivalent to exponential stability in finite dimensions. In this section, we shall extend the concept of hybrid stability to the Pritchard–Salamon class of systems and establish the relationship between this type of stability and input–output stability as was defined in subsection 2.1.

Consider a Pritchard–Salamon system $\Sigma(S(\bullet), B, C, D)$ with respect to $\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$. Assume that $x_0 \in \mathcal{W}$. Suppose that the control is piecewise constant with $u(t) = u_k$ for $kT \leq t < (k+1)T$, and that $(x_k)_{k \in \mathbb{N}}$ is the sequence of state discrete values. Then the state equation becomes

$$x(t) = S^{\mathcal{W}}(t - kT)x_k + \int_{kT}^t S^{\mathcal{V}}(t - \tau)Bu_k d\tau.$$

Write $t - \tau = \eta$. Then, at $t = (k+1)T$, we obtain

$$x_{k+1} = S^{\mathcal{W}}(T)x_k + \int_0^T S^{\mathcal{V}}(\eta)Bu_k d\eta \quad (3.1)$$

and also

$$y_k = Cx_k + Du_k. \quad (3.2)$$

DEFINITION 4 Consider $\Sigma(S^{\mathcal{V}}(\bullet), B, C, D)$ with respect to $\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$. We shall call (Φ, Γ, C, D) a *time-discretization* of $\Sigma(S^{\mathcal{V}}(\bullet), B, C, D)$, the latter system satisfying the difference state equation (3.1) and the discrete output equation (3.2), if Φ and Γ (depending on the sampling step T) are given by

$$\Phi \triangleq S^{\mathcal{V}}(T), \quad \Gamma \triangleq \int_0^T S^{\mathcal{V}}(\eta)B d\eta. \quad \square \quad (3.3)$$

Following [10], let us introduce two compatible exogenous signals $v \in \ell_2(\mathbb{N}, \mathbb{R}^p)$ and $w \in \ell_2(\mathbb{N}, \mathbb{R}^m)$ and add them to the closed-loop system. Clearly, we define the concept of hybrid stability on the basis of the signals

$$u_1, y_1, \tilde{u}_2, w, v, \tilde{y}_2$$

and the configuration depicted in Fig. 2.

DEFINITION 5 The digital control system is called *hybrid-stable* if the mapping

$$(u_1, v, w) \mapsto (y_1, \tilde{u}_2, \tilde{y}_2)$$

is well defined and bounded from

$$L_2(\mathbb{R}_+, \mathcal{U}_1) \oplus \ell_2(\mathbb{N}, \mathbb{R}^p) \oplus \ell_2(\mathbb{N}, \mathbb{R}^m)$$

to

$$L_2(\mathbb{R}_+, \mathcal{Y}_1) \oplus \ell_2(\mathbb{N}, \mathbb{R}^m) \oplus \ell_2(\mathbb{N}, \mathbb{R}^p). \quad \square$$

REMARK 3 Notice that the equation relating (u_1, v, w) to $(y_1, \tilde{u}_2, \tilde{y}_2)$ is

$$\begin{bmatrix} I & -G_{12}H_T & 0 \\ 0 & I & G_K \\ 0 & S_T G_{22}H_T & I \end{bmatrix} \begin{bmatrix} y_1 \\ \tilde{u}_2 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & 0 \\ 0 & I & 0 \\ S_T G_{21} & 0 & I \end{bmatrix} \begin{bmatrix} u_1 \\ v \\ w \end{bmatrix}.$$

Since G_{21} is strictly proper ($D_{21} = 0$), $S_T G_{21}$ is well defined and bounded, and so is

$$\begin{bmatrix} G_{11} & 0 & 0 \\ 0 & I & 0 \\ S_T G_{21} & 0 & I \end{bmatrix}. \text{ Hence, if the exponential stability assumption on } S(\bullet) \text{ holds,}$$

then a simple sufficient condition for hybrid stability is that the operator

$$\begin{bmatrix} I & -G_{12}H_T & 0 \\ 0 & I & G_K \\ 0 & S_T G_{22}H_T & I \end{bmatrix} \text{ defined on } L_2(\mathbb{R}_+, \mathcal{U}_1) \oplus \ell_2(\mathbb{N}, \mathbb{R}^p) \oplus \ell_2(\mathbb{N}, \mathbb{R}^m) \text{ with values in}$$

$L_2(\mathbb{R}_+, \mathcal{Y}_1) \oplus \ell_2(\mathbb{N}, \mathbb{R}^m) \oplus \ell_2(\mathbb{N}, \mathbb{R}^p)$ is boundedly invertible for a given sampling step $T > 0$. $\square$

Fig. 2. The hybrid stability configuration.

DEFINITION 6 Let $\Sigma(S(\cdot), B, C, D)$ be a Pritchard–Salamon system with respect to $\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$. We shall say that it is hybrid-stabilizable if there exists a controller such that the digital control system is hybrid-stable. $\square$

We want to characterize first the stabilizability (in a certain sense). As was argued in Section 1, a pair $(S(\cdot), B)$ that is stabilizable on $\mathcal{W}$ (resp. $\mathcal{V}$) is not necessarily stabilizable on $\mathcal{V}$ (resp. $\mathcal{W}$). The next example shows that this inconvenient property of $(S(\cdot), B)$ is not alleviated by sampling, i.e. the power stabilizability of the time-discretized pair, (Φ, Γ) , on $\mathcal{W}$ (resp. $\mathcal{V}$) does not necessarily imply power stabilizability on $\mathcal{V}$ (resp. $\mathcal{W}$). Define first for every sampling step the *discrete growth constant* of $S(T)$ as

$$\beta = e^{\alpha T} / T, \quad (3.4)$$

where $S(T)$ is either $S^{\mathcal{W}}(T)$ or $S^{\mathcal{V}}(T)$, and let $\beta_{\mathcal{W}}$ and $\beta_{\mathcal{V}}$ be their corresponding discrete growth constants. The following examples, which represent the time-discretization of the translation semigroups (see [13: Ex. 2.1]), show that $\beta_{\mathcal{V}}$ and $\beta_{\mathcal{W}}$ may in general be different (this is actually a consequence of the fact that $\alpha_{\mathcal{W}}$ and $\alpha_{\mathcal{V}}$ might be different), and both situations $\beta_{\mathcal{V}} > \beta_{\mathcal{W}}$ and $\beta_{\mathcal{V}} < \beta_{\mathcal{W}}$ might appear. Let

$$\mathcal{W} = L_2(\mathbb{R}_+, \mathbb{R}), \quad \mathcal{V} = \{f \in L_2^{\text{loc}}(\mathbb{R}_+, \mathbb{R}) : e^{-\cdot} f \in \mathcal{W}\}, \quad (3.5a, b)$$

with $\|f\|_{\mathcal{V}} = \|e^{-\cdot} f(\cdot)\|_{\mathcal{W}}$. Consider the translation semigroup $S(\cdot)$ defined by

$$[S(t)f](x) = f(x+t),$$

which is a C^0 semigroup on $\mathcal{W} \hookrightarrow \mathcal{V}$. Let $T > 0$ be arbitrary chosen. It is straightforward to prove that $\|S(t)\|_{\mathcal{L}(\mathcal{W})} = 1$ and $\|S(t)\|_{\mathcal{L}(\mathcal{V})} = e^{2t}$ (see again [13: Ex. 2.1]) which implies $\alpha_{\mathcal{W}} = 0$ and $\alpha_{\mathcal{V}} = 2$ and hence $\beta_{\mathcal{W}} = 1/T$ and $\beta_{\mathcal{V}} = e^{2T}/T$. Hence, $\beta_{\mathcal{V}} > \beta_{\mathcal{W}}$. Let $\mathcal{W}$ and $\mathcal{V}$ be given by (3.5) and consider the semigroup $S(\cdot)$ defined by

$$[S(t)f](x) = \begin{cases} 0 & \text{if } 0 \leq x < t, \\ f(x-t) & \text{if } x \geq t. \end{cases}$$

Following [13], we have that $S(\cdot)$ is strongly continuous on $\mathcal{W} \hookrightarrow \mathcal{V}$ with $\|S(t)\|_{\mathcal{L}(\mathcal{W})} = 1$ and $\|S(t)\|_{\mathcal{L}(\mathcal{V})} = 1/e^{2t}$, implying that $\alpha_{\mathcal{W}} = 0$ and $\alpha_{\mathcal{V}} = -2$. Then, by (3.4) we get $\beta_{\mathcal{W}} = 1/T$ and $\beta_{\mathcal{V}} = 1/T e^{2T}$. Hence, $\beta_{\mathcal{V}} < \beta_{\mathcal{W}}$. We give now the discrete version of Definition 2.

DEFINITION 7 A pair (Φ, Γ) is called *power-stabilizable with respect to* $\mathcal{W} \hookrightarrow \mathcal{V}$ if there exists $F_1 \in \mathcal{L}(\mathcal{W}, \mathcal{U})$ such that $\Phi + \Gamma F_1$ is power-stable on $\mathcal{W}$ and $\mathcal{V}$. $\square$

DEFINITION 8 A pair (C, Φ) is called *power-detectable with respect to* $\mathcal{W} \hookrightarrow \mathcal{V}$ if there exists $H_1 \in \mathcal{L}(\mathcal{Y}, \mathcal{V})$ and $H_2 \in \mathcal{L}(\mathcal{Y}, \mathcal{W})$ such that $\Phi + CH_1$ and $\Phi + CH_2$ are both power-stable on $\mathcal{W}$ and $\mathcal{V}$. $\square$

The following central result is reported from [6].

LEMMA 1 Let $\mathcal{X}$ denote either $\mathcal{W}$ or $\mathcal{V}$, and let $\Sigma(S^{\mathcal{V}}(\cdot), B, C, D)$ be a smooth Pritchard–Salamon system with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$. Then the following statements hold.

1. If $S^{\mathcal{V}}(\cdot)$ is exponentially stable on $\mathcal{V}$ and $\mathcal{W}$, then $\Phi = S^{\mathcal{V}}(T)$ is power-stable on $\mathcal{V}$ and $\mathcal{W}$.

2. $\Gamma \in \mathfrak{L}(\mathcal{U}, \mathcal{W})$.

3. Assume that $(S^\nu(\cdot), B)$ is admissibly stabilizable. Then there exists a sufficiently small sampling period T_0 such that, for every $T < T_0$, the time-discretized pair (Φ, Γ) associated with $(S^\nu(\cdot), B)$ is power-stabilizable on $\mathcal{V}$ and $\mathcal{W}$.

4. Let $(S^\nu(\cdot), B)$ be admissibly stabilizable and let (Φ, Γ) be the time-discretized pair associated with $(S^\nu(\cdot), B)$. If an admissible feedback operator makes the perturbed C_0 semigroup $S_F^R(\cdot)$, generated by $A^\nu + BF$, to be stable both on $\mathcal{W}$ and $\mathcal{V}$, it ensures also the power stability of the perturbed time-discretized pair $\Phi + \Gamma F$ on both spaces. $\square$

COROLLARY 1 Let $\Sigma(S(\cdot), B, C, D)$ be a Pritchard-Salamon system such that $(C, S(\cdot))$ is admissibly detectable. Then there exists a sufficiently small sampling period $T_0 > 0$ such that, for every sampling step T smaller than T_0 , the pair (C, Φ) is power-detectable with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$. $\square$

The following result is the generalization to Pritchard-Salamon systems of [10: Thm 1]. Since the extension is rather trivial, we shall only sketch the proof.

PROPOSITION 3 The Pritchard-Salamon system

$$\Sigma\left(S(\cdot), [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} 0 & D_{12} \\ 0 & 0 \end{bmatrix}\right)$$

is hybrid-stabilizable if (Φ, Γ_2, C_2) , is power-stabilizable and power-detectable with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$.

Proof. Let $\Sigma(K, L, M, N)$ be a stabilizing compensator for $S_T G_{22} H_T$, and let $u_1 = 0$. Let $\tilde{y}_2 = S_T y_2$, and let $\tilde{u}_2$ be the compensator output. The operator

$$(w, v) \mapsto (\tilde{y}_2, \tilde{u}_2)$$

is bounded from $\ell_2(\mathbb{N}, \mathbb{R}^m) \oplus \ell_2(\mathbb{N}, \mathbb{R}^p)$ to $\ell_2(\mathbb{N}, \mathbb{R}^p) \oplus \ell_2(\mathbb{N}, \mathbb{R}^m)$. The power stabilizability and power detectability of (Φ, Γ_2, C_2) implies that the operator

$$(w, v) \mapsto \tilde{x}$$

is bounded from $\ell_2(\mathbb{N}, \mathbb{R}^m) \oplus \ell_2(\mathbb{N}, \mathbb{R}^p)$ to $\ell_2(\mathbb{N}, \mathcal{W})$ and to $\ell_2(\mathbb{N}, \mathcal{V})$ respectively. Let $\mathcal{X}$ denote either $\mathcal{W}$ or $\mathcal{V}$. Since

$$x(t) = S^\nu(t - kT)x_k + \int_{kT}^t S^\nu(t - \tau)B_2 d\tau u_{2,k},$$

and if $\omega_1 = \max_{t \in [0, T]} \|S^\nu(t)\|_{\mathcal{W}}$ and $\omega_2 = \int_0^T \|S^\nu(\tau)B_2\|_{\mathbb{R}^m} d\tau$, then it follows immediately that

$$\|x(t)\|_{\mathcal{X}} \leq \omega_1 \|\tilde{y}_{2,k}\|_{\mathbb{R}^p} + \omega_2 \|\tilde{u}_{2,k}\|_{\mathbb{R}^m} \quad \text{for } kT \leq t < (k+1)T,$$

which implies that $(v, w) \mapsto x$ is a bounded operator from $\ell_2(\mathbb{N}, \mathbb{R}^p) \oplus \ell_2(\mathbb{N}, \mathbb{R}^m)$ to $L_2(\mathbb{R}_+, \mathcal{X})$. It is sufficient to show that the operator

$$u_1 \mapsto (u_2, \tilde{y}_2, y_1)$$

is bounded from $L_2(\mathbb{R}_+, \mathcal{U}_1)$ to $\ell_2(\mathbb{N}, \mathcal{U}_2 \oplus \mathbb{R}^p) \oplus L_2(\mathbb{R}_+, \mathcal{Y}_1)$. Notice that the state

equation can be re-written at the sampling instants as

$$x_{k+1} = \hat{\Phi}x_k + \hat{\Gamma}_1 \hat{u}_{1,k} + \Gamma_2 u_{2,k}, \quad (3.6)$$

where $\hat{u}_{1,k}$ stands for $u_1(kT + \cdot) \upharpoonright (0, T)$, with

$$\left. \begin{aligned} \hat{\Phi} : \mathcal{W} &\rightarrow \mathcal{W}, & \hat{\Phi} &\triangleq S^\nu(T) \upharpoonright \mathcal{W} \\ \hat{\Gamma}_1 : L_2(0, T; \mathcal{U}_1) &\rightarrow \mathcal{W}, & \hat{\Gamma}_1 \hat{u}_{1,k} &\triangleq \int_0^T S^\nu(T - \tau) B_1 u_{1,k}(\tau) d\tau \\ \Gamma_2 : \mathbb{R}^m &\rightarrow \mathcal{W}, & \Gamma_2 u_{2,k} &\triangleq \int_0^T S^\nu(T - \tau) B_2 d\tau u_{2,k} \end{aligned} \right\}.$$

One can check immediately that $u_1 \mapsto \hat{u}_1$ is bounded from $L_2(\mathbb{R}_+, \mathcal{U}_1)$ to $\ell_2(\mathbb{N}; L_2(0, T; \mathcal{U}_1))$. Indeed, writing the condition for B_1 to be admissible on the time interval $\mathbb{R}_+$, we obtain that $\hat{\Gamma}_1 \in \mathfrak{L}(L_2(\mathbb{R}_+, \mathcal{U}_1), \mathcal{W})$. From the power stability of the solution to the state equation under the control $\tilde{u}_2 = G_K \tilde{y}_2$, we get that $\hat{u}_1 \mapsto (\tilde{y}_2, \tilde{u}_2)$ is bounded from $\ell_2(\mathbb{N}; L_2(0, T; \mathcal{U}_1))$ to $\ell_2(\mathbb{N}; \mathbb{R}^p \oplus \mathbb{R}^m)$. Finally, the boundedness of $u_1 \mapsto y_1$ follows from standard results. $\square$

COROLLARY 2 Let

$$\Sigma \left(S(\cdot), [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} 0 & D_{12} \\ 0 & 0 \end{bmatrix} \right)$$

be a smooth exponentially stabilizable Pritchard–Salamon system. Then there exists a sufficiently small sampling step T_0 such that, for every $T < T_0$, the Pritchard–Salamon system is hybrid-stabilizable. $\square$

3.2. Lifting continuous-time systems

In this section, we give the main results on lifting a continuous-time *periodic* system to a discrete-time time-invariant one. The idea is to rearrange the original periodic system in such a way that its periodicity is reflected by shift invariance in the new set-up. We need first to define the lifting operator. Let $\mathcal{Z}$ be a Banach space, and let $L_2(\mathbb{R}_+, \mathcal{Z})$ be the space of square integrable $\mathcal{Z}$ -valued functions and $\ell_2(\mathbb{N}; \mathcal{Z})$ the space of square summable $\mathcal{Z}$ -valued sequences. Notice that $L_2(\mathbb{R}_+, \mathcal{Z})$ and $\ell_2(\mathbb{N}; \mathcal{Z})$ are Hilbert spaces with respect to the norms induced by the inner products. Let $T > 0$ be a fixed positive constant. Let

$$\Omega : L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{Z}) \rightarrow S(\mathbb{N}, L_2(0, T; \mathcal{Z})), \quad \Omega \zeta = \hat{\zeta}, \quad (3.7)$$

where $S(\mathbb{N}, \mathcal{Z})$ is the space of $\mathcal{Z}$ -valued sequences $\hat{\zeta} = (\zeta_k)_{k \in \mathbb{N}}$, and where

$$\hat{\zeta}_k = \hat{\zeta}_k(\cdot) = \zeta(kT + \cdot) \in L_2(0, T; \mathcal{Z}).$$

It is easy to show (see [1, 2, 33] for details of the proof) that Ω is a linear bijective isometry from $L_2^{\text{loc}}(\mathbb{R}_+, \mathcal{Z})$ to $S(\mathbb{N}, L_2(0, T; \mathcal{Z}))$. Notice also that, if $\zeta(\cdot)$ is a piecewise constant function, say

$$\zeta(t) = \zeta_k \quad \text{for } t \in [kT, (k+1)T),$$

then $\tilde{\zeta} = (\zeta_0, \dots, \zeta_k, \dots)$ stands for the sample sequence of $\zeta(\cdot)$, and furthermore $\hat{\zeta} = \tilde{\zeta}$.

Let us focus on state-space formulae for lifted systems. Consider the continuous-time system $\Sigma(S(\bullet), B, C, D)$ on a real separable Hilbert space $\mathcal{X}$ given by

$$x'(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t), \quad (3.8a, b)$$

where $S(\bullet)$ is a strongly continuous semigroup generated by A . Let $\mathcal{U}$ and $\mathcal{Y}$ be the real separable Hilbert spaces of the inputs and outputs respectively. The state equation can be reformulated as

$$x(kT + t) = S(t)x(kT) + \int_{kT}^{kT+t} S(kT + t - \eta)Bu(\eta) d\eta$$

for $kT \leq t < (k+1)T$. For $\eta = kT + \tau$, equation (3.8a) becomes

$$x(kT + t) = S(t)x(kT) + \int_0^t S(t - \tau)Bu(kT + \tau) d\tau.$$

At the sampling instants, the state equation (3.8a) becomes

$$x_{k+1} = S(T)x_k + \int_0^T S(T - \tau)B\hat{u}_k(\tau) d\tau.$$

Let

$$\begin{aligned} \hat{A} : \mathcal{X} &\rightarrow \mathcal{X}, & \hat{A} &= S(T), \\ \hat{B} : L_2(0, T; \mathcal{U}) &\rightarrow \mathcal{X}, & \hat{B}\hat{u}_k &= \int_0^T S(T - \tau)B\hat{u}_k(\tau) d\tau, \end{aligned} \quad (3.9)$$

where $\hat{u}_k \triangleq u(kT + \bullet)|_{(0, T)}$. From (3.8b), we finally obtain

$$\hat{y}_k = \hat{C}x_k + \hat{D}\hat{u}_k = CS(\bullet)x_k + C \int_0^\bullet S(\bullet - \tau)B\hat{u}_k(\tau) d\tau + D\hat{u}_k.$$

Define now

$$\begin{aligned} \hat{C} : \mathcal{X} &\rightarrow L_2(0, T; \mathcal{Y}), & \hat{C}x_k &= CS(\bullet)x_k, \\ \hat{D} : L_2(0, T; \mathcal{U}) &\rightarrow L_2(0, T; \mathcal{Y}), & \hat{D}\hat{u}_k &= C \int_0^\bullet S(\bullet - \tau)B\hat{u}_k(\tau) d\tau + D\hat{u}_k. \end{aligned} \quad (3.10)$$

Then the original system admits at the sampling instants the following representation

$$sx = \hat{A}x + \hat{B}\hat{u}, \quad \hat{y} = \hat{C}x + \hat{D}\hat{u},$$

where s is the forward shift operator, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ are defined via (3.9)–(3.10), and where $x \in \mathcal{X}^{\mathbb{N}}$, $\hat{u} \in S(\mathbb{N}, L_2(0, T; \mathcal{U}))$, and $\hat{y} \in S(\mathbb{N}, L_2(0, T; \mathcal{Y}))$ respectively; here again the convention $\hat{A}x \triangleq (\hat{A}x_k)_{k \in \mathbb{N}}$ etc. is used.

REMARK 4 1. Notice that the lifting operator (3.7) is defined on locally square-integrable $\mathcal{Z}$ -valued functions and its values are sequences of square-integrable $\mathcal{Z}$ -valued functions on $[0, T]$. However, if we restrict to square-integrable $\mathcal{Z}$ -valued functions then the lifted signal $\hat{\zeta}$ is a square-summable sequence of square-integrable $\mathcal{Z}$ -valued functions on $[0, T]$.

2. Assume that, in (3.8a), the operator A is the infinitesimal generator of a stable C_0 semigroup on $\mathcal{X}$. Then, for any $u \in L_2(\mathbb{R}_+, \mathcal{U})$, the state is also square-integrable, i.e.

$x \in L_2(\mathbb{R}_+, \mathcal{X})$, which implies $y \in L_2(\mathbb{R}_+, \mathcal{Y})$. Let $\hat{u}$ and $\hat{y}$ be the lifted image of u and y . It is routine to show that $\hat{u} \in \ell_2(\mathbb{N}, L_2(0, T; \mathcal{U}))$ and $\hat{y} \in \ell_2(\mathbb{N}, L_2(0, T; \mathcal{Y}))$.

3. If we assume stability, then the input–output operator (2.7) of the linear system (3.8a) and (3.8b) is a well defined and bounded map from $L_2(\mathbb{R}_+, \mathcal{U})$ to $L_2(\mathbb{R}_+, \mathcal{Y})$. It is well known (see [2]) that the lifted image of G is then

$$\hat{G} = \Omega G \Omega^{-1} \in \mathfrak{L}(\ell_2(\mathbb{N}; L_2(0, T; \mathcal{U})), \ell_2(\mathbb{N}; L_2(0, T; \mathcal{Y}))),$$

and furthermore $\|\hat{G}\| = \|G\|$. $\square$

3.3. Lifting the hybrid Pritchard–Salamon system

The theory developed in subsection 3.2 is now applied to lifting the hybrid Pritchard–Salamon system. Consider the smooth Pritchard–Salamon system

$$\Sigma \left(S(\cdot), [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix} \right)$$

with respect to $\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$ and satisfying (2.8). Assume that there exist two other real separable Hilbert spaces $\mathcal{U}_1$ and $\mathcal{Y}_1$ such that $B_1 \in \mathfrak{L}(\mathcal{U}_1, \mathcal{V})$, $B_2 \in \mathfrak{L}(\mathbb{R}^m, \mathcal{V})$, $C_1 \in \mathfrak{L}(\mathcal{V}, \mathcal{Y}_1)$, $D_{12} \in \mathfrak{L}(\mathbb{R}^m, \mathcal{Y}_1)$. As in [3, 4] we shall restrict to the case when the system is smooth. Let us apply the lifting technique developed in subsection 3.2 under the assumption that u_2 is obtained in a piecewise fashion from the discrete values $x(kT)$ of the state, say $u_2(kT + \tau) = u_{2,k}$ ($0 \leq \tau < T$). The state equation written at the sampling instants becomes

$$x_{k+1} = \hat{\Phi}x_k + \hat{\Gamma}_1 \hat{u}_{1,k} + \hat{\Gamma}_2 u_{2,k} \quad (3.11)$$

with

$$\hat{\Phi} : \mathcal{V} \rightarrow \mathcal{V}, \quad \hat{\Phi} \triangleq S^V(T),$$

$$\hat{\Gamma}_1 : L_2(0, T; \mathcal{U}_1) \rightarrow \mathcal{W}, \quad \hat{\Gamma}_1 \hat{u}_{1,k} \triangleq \int_0^T S^V(T - \tau) B_1 \hat{u}_{1,k}(\tau) d\tau,$$

$$\hat{\Gamma}_2 : \mathbb{R}^m \rightarrow \mathcal{W}, \quad \hat{\Gamma}_2 u_{2,k} \triangleq \int_0^T S^V(T - \tau) B_2 u_{2,k} d\tau.$$

The equation for the controlled output is now

$$\hat{y}_{1,k} = \hat{C}_1 x_k + \hat{D}_{11} \hat{u}_{1,k} + \hat{D}_{12} u_{2,k}$$

with

$$\hat{C}_1 : \mathcal{W} \rightarrow L_2(0, T; \mathcal{Y}_1), \quad \hat{C}_1 \triangleq \overline{C_1 S^W(\cdot)}, \quad (3.12a)$$

$$\hat{D}_{11} : L_2(0, T; \mathcal{U}_1) \rightarrow L_2(0, T; \mathcal{Y}), \quad \hat{D}_{11} \hat{u}_{1,k} \triangleq C_1 \int_0^\bullet S^V(\bullet - \tau) B_1 \hat{u}_{1,k}(\tau) d\tau, \quad (3.12b)$$

$$\hat{D}_{12} : \mathbb{R}^m \rightarrow L_2(0, T; \mathcal{Y}), \quad \hat{D}_{12} u_{2,k} \triangleq C_1 \int_0^\bullet S^V(\bullet - \tau) B_2 u_{2,k} d\tau + D_{12} u_{2,k}. \quad (3.12c)$$

With this, we have obtained the following discrete-time representation for the original

system:

$$x_{k+1} = \hat{\Phi}x_k + \hat{\Gamma}_1\hat{u}_{1,k} + \hat{\Gamma}_2u_{2,k}, \quad \hat{y}_{1,k} = \hat{C}_1x_k + \hat{D}_{11}\hat{u}_{1,k} + \hat{D}_{12}u_{2,k}, \quad (3.13a, b)$$

$$y_{2,k} = C_2x_k, \quad (3.13c)$$

where $\hat{u}_{1,k}$, $u_{2,k}$, $\hat{y}_{1,k}$, and $\hat{y}_{2,k}$ are respective k th elements of the sequences

$$\hat{u}_1 \in L_2(0, T; \mathcal{U}_1)^{\mathbb{N}}, \quad \tilde{u}_2 \in (\mathbb{R}^m)^{\mathbb{N}}, \quad \hat{y}_1 \in L_2(\mathbb{R}_+, \mathcal{Y}_1)^{\mathbb{N}}, \quad \tilde{y}_2 \in (\mathbb{R}^p)^{\mathbb{N}}$$

REMARK 5 We have assumed u_2 to be constant on $[kT, (k+1)T)$, for $k \in \mathbb{N}$, so the lifted image of $u_2(\cdot)$ is the same as its sample sequence $\tilde{u}_2$, and hence $\hat{u}_2 = \tilde{u}_2 \in (\mathbb{R}^m)^{\mathbb{N}}$. It follows that the expression of $\hat{\Gamma}_2$ simplifies to $\Gamma_2 = \int_0^T S^\gamma(\tau)B_2 d\tau$. $\square$

COROLLARY 3 The operators in (3.13a, b) satisfy the boundedness conditions

$$\hat{\Phi} \in \mathfrak{L}(\mathcal{W}),$$

$$\hat{\Gamma} \in \mathfrak{L}(L_2(0, T; \mathcal{U}_1), \mathcal{W}), \quad \Gamma_2 \in \mathfrak{L}(\mathbb{R}^m, \mathcal{W}),$$

$$\hat{C}_1 \in \mathfrak{L}(\mathcal{W}, L_2(0, T; \mathcal{Y}_1)),$$

$$\hat{D}_{11} \in \mathfrak{L}(L_2(0, T; \mathcal{U}_1), L_2(0, T; \mathcal{Y}_1)), \quad \hat{D}_{12} \in \mathfrak{L}(\mathbb{R}^m, L_2(0, T; \mathcal{Y}_1)).$$

Proof. The result for $\hat{\Phi}$ is immediate from the definition. By Proposition 3, $\hat{\Gamma}_1 \in \mathfrak{L}(L_2(0, T; \mathcal{U}_1), \mathcal{W})$, and the result $\Gamma_2 \in \mathfrak{L}(\mathbb{R}^m, \mathcal{W})$ is obtained by applying Lemma 1 to the pair $(S(\cdot), B_2)$. This yields the results for $\hat{C}_1$, $\hat{D}_{11}$, and $\hat{D}_{12}$ immediately from their definitions. $\square$

REMARK 6 After lifting, the compensator's input and output spaces remain unaltered, i.e. $\mathbb{R}^p$ and $\mathbb{R}^m$. The lifting operator affects only the spaces where the new disturbance input $\hat{u}_1$ and the new controlled output $\hat{y}_1$ belong. They are now $L_2(0, T; \mathcal{U}_1)$ and $L_2(0, T; \mathcal{Y}_1)$ respectively. Any finite-dimensionality of $\mathcal{U}_1$ and/or $\mathcal{Y}_1$ is not preserved in the new problem, since the new spaces are infinite-dimensional. $\square$

The digital control system, being hybrid in its nature, becomes discrete-time after lifting. The resultant closed-loop system is then

$$x_{R,k+1} = A_R x_{R,k} + B_R u_{R,k}, \quad y_{R,k} = C_R x_{R,k} + D_R u_{R,k},$$

where

$$x_R = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad A_R = \begin{bmatrix} \hat{\Phi} + \Gamma_2 N C_2 & \Gamma_2 M \\ L C_2 & K \end{bmatrix}, \quad B_R = \begin{bmatrix} \hat{\Gamma}_1 \\ 0 \end{bmatrix},$$

$$C_R = [\hat{C}_1 + \hat{D}_{12} N C_2, \hat{D}_{12} M], \quad D_R = \hat{D}_{11}.$$

It is rather straightforward that, if $\mathcal{X}_R = \mathcal{W} \oplus \mathcal{K}$ stands for the augmented state space for the resultant closed-loop system, then the following boundedness conditions are

satisfied:

$$\begin{aligned} A_R &\in \mathfrak{L}(\mathcal{X}_R), & B_R &\in \mathfrak{L}(L_2(0, T; \mathcal{U}_1), \mathcal{X}_R), \\ C_R &\in \mathfrak{L}(\mathcal{X}_R, L_2(0, T; \mathcal{Y}_1)), & D_R &\in \mathfrak{L}(L_2(0, T; \mathcal{U}_1), L_2(0, T; \mathcal{Y}_1)). \end{aligned}$$

The following proposition establishes the equivalence between the hybrid stability of Pritchard–Salamon systems and the power stability of its lifted counterpart.

PROPOSITION 4 $A_R = \begin{bmatrix} \hat{\Phi} + \Gamma_2 N C_2 & \Gamma_2 M \\ L C_2 & K \end{bmatrix}$ is power-stable on $\mathcal{X}_R = \mathcal{W} \oplus \mathcal{K}$ if and only if the Pritchard–Salamon system $\Sigma\left(S(\cdot), [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix}\right)$ is hybrid-stable.

Proof. (‘if’) Suppose that A_R is power-stable on $\mathcal{X}_R$. Then G_K is a stabilizing compensator for the lifted hybrid generalized plant

$$\hat{G}_T \triangleq \begin{bmatrix} \Omega G_{11} \Omega^{-1} & \Omega G_{12} H_T \Omega^{-1} \\ \Omega S_T G_{21} \Omega^{-1} & \Omega S_T G_{22} H_T \Omega^{-1} \end{bmatrix} = \begin{bmatrix} \Omega G_{11} \Omega^{-1} & \Omega G_{12} H_T \\ S_T G_{21} \Omega^{-1} & S_T G_{22} H_T \end{bmatrix}$$

(see [2] for details about lifting $S_T G$, $G H_T$, and $S_T G H_T$ operators). It follows that G_K stabilizes $S_T G_{22} H_T$. Then, by Proposition 3, we get that the Pritchard–Salamon system $\Sigma\left(S(\cdot), [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} 0 & D_{12} \\ 0 & 0 \end{bmatrix}\right)$ is hybrid-stable.

(‘only if’) Let the Pritchard–Salamon system $\Sigma\left(S(\cdot), [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} 0 & D_{12} \\ 0 & 0 \end{bmatrix}\right)$ be hybrid-stable. It follows that the map $(u_1, v, w) \mapsto (y_1, \tilde{u}_2, \tilde{y}_2)$ is well defined and bounded from

$$L_2(\mathbb{R}_+, \mathcal{U}_1) \oplus \ell_2(\mathbb{N}, \mathbb{R}^p) \oplus \ell_2(\mathbb{N}, \mathbb{R}^m)$$

to

$$L_2(\mathbb{R}_+, \mathcal{Y}_1) \oplus \ell_2(\mathbb{N}, \mathbb{R}^m) \oplus \ell_2(\mathbb{N}, \mathbb{R}^p),$$

which implies that $(\hat{u}_1, v, w) \mapsto (\hat{y}_1, \tilde{u}_2, \hat{y}_2)$ is well defined and bounded from

$$\ell_2(\mathbb{N}, L_2(0, T; \mathcal{U}_1)) \oplus \ell_2(\mathbb{N}, \mathbb{R}^p) \oplus \ell_2(\mathbb{N}, \mathbb{R}^m)$$

to

$$\ell_2(\mathbb{N}, L_2(0, T; \mathcal{Y}_1)) \oplus \ell_2(\mathbb{N}, \mathbb{R}^m) \oplus \ell_2(\mathbb{N}, \mathbb{R}^p).$$

This follows from the fact that the lifting operator is an isometry of spaces. We conclude that the discrete-time system (3.13a), closed under output feedback via the controller (2.10), is input–output-stable. But we know that, for a system that is power-stabilizable and power-detectable, input–output stability and power stability are equivalent in the sense that the transfer function is holomorphic and bounded outside the closed unit disc. The reader is referred to [23]. We conclude that the closed-loop system is internally stable, i.e. A_R is power-stable on $\mathcal{X}_R$. $\square$

REMARK 7 Notice that power stability of the closed-loop system on $\mathcal{X}_R = \mathcal{W} \oplus \mathcal{K}$ does not necessarily imply power stability of the closed-loop system on $\mathcal{V} \oplus \mathcal{K}$. It is

an open question whether simultaneous stability of the closed-loop system on both spaces $\mathcal{X}_R$ and $\mathcal{V} \oplus \mathcal{K}$ is equivalent to hybrid stability. $\square$

We return now to the γ -attenuation condition we have imposed for our digital control system. Let us notice first that the following holds.

PROPOSITION 5 Let

$$T_1 : L_2(\mathbb{R}_+, \mathcal{U}_1) \rightarrow L_2(\mathbb{R}_+, \mathcal{Y}_1)$$

be the input-output operator mapping u_1 to y_1 , and let

$$\hat{T}_1 : \ell_2(\mathbb{N}, L_2(0, T; \mathcal{U}_1)) \rightarrow \ell_2(\mathbb{N}, L_2(0, T; \mathcal{Y}_1))$$

be the input-output operator mapping $\hat{u}_1$ to $\hat{y}_1$. Then

$$\|T_1\|_{L_2(\mathbb{R}_+, \mathcal{Y}_1)} = \|\hat{T}_1\|_{\ell_2(\mathbb{N}, L_2(0, T; \mathcal{Y}_1))}.$$

Proof. Notice that the input-output operator mapping u_1 to y_1 can be expressed as a linear fractional transformation of the hybrid generalized plant, and the controller

$$T_1 = F\left(\begin{bmatrix} G_{11} & G_{12}H_T \\ S_T G_{21} & S_T G_{22}H_T \end{bmatrix}, G_K\right)$$

and the input-output operator mapping $\hat{u}_1$ to $\hat{y}_1$ can be expressed as a linear fractional transformation of the lifted hybrid generalized plant and the controller

$$\hat{T}_1 = F\left(\begin{bmatrix} \Omega G_{11} \Omega^{-1} & \Omega G_{12}H_T \\ S_T G_{21} \Omega^{-1} & S_T G_{22}H_T \end{bmatrix}, G_K\right).$$

Exploiting the fact that the lifting operator preserves the norms, we get

$$\begin{aligned} \|\hat{T}_1\| &= \|\Omega(G_{11} + G_{12}H_T K(I - S_T G_{22}H_T K)^{-1} S_T G_{21})\Omega^{-1}\| \\ &= \|G_{11} + G_{12}H_T K(I - S_T G_{22}H_T K)^{-1} S_T G_{21}\| = \|T_1\|. \quad \square \end{aligned}$$

We can conclude this section with the following result.

THEOREM 1 Let $\Sigma\left(S(\cdot), [B_1, B_2], \begin{bmatrix} C_1 \\ C \end{bmatrix}, \begin{bmatrix} 0 & D_{12} \\ 0 & 0 \end{bmatrix}\right)$ be a smooth Pritchard-Salamon system with respect to $\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$ satisfying

$$x(t) = S(t)^{\mathcal{W}} x_0 + \int_0^t S^{\mathcal{V}}(t - \tau)[B_1 u_1(\tau) + B_2 u_2(\tau)] d\tau, \quad x_0 \in \mathcal{W},$$

$$y_1(t) = C_1 x(t) + D_{12} u_2(t), \quad y_2(t) = C_2 x(t).$$

Then a discrete-time controller $\Sigma(K, L, M, N)$ satisfying

$$\xi_{k+1} = K\xi_k + L\eta_k, \quad \zeta_k = M\xi_k + N\eta_k,$$

solves the digital output-measurement-feedback H^∞ control problem if it is a solution to the discrete-time output-measurement-feedback H^∞ control problem for the lifted

discrete-time system

$$\Sigma\left(\hat{\Phi}, [\hat{\Gamma}_1, \Gamma_2], \begin{bmatrix} \hat{C}_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} \\ 0 & 0 \end{bmatrix}\right) \quad (3.14)$$

satisfying

$$\begin{aligned} x_{k+1} &= \hat{\Phi}x_k + \hat{\Gamma}_1\hat{u}_{1,k} + \Gamma_2u_{2,k}, & x_0 &\in \mathcal{W}, \\ \hat{y}_{1,k} &= \hat{C}_1x_k + \hat{D}_{11}\hat{u}_{1,k} + \hat{D}_{12}u_{2,k}, & y_{2,k} &= C_2x_k, \end{aligned}$$

where

$$\begin{aligned} \hat{\Phi} &\triangleq S^\nu(T) \upharpoonright \mathcal{W}, \\ \hat{\Gamma}_1\hat{u}_{1,k} &\triangleq \int_0^T S^\nu(T-\tau)B_1\hat{u}_{1,k}(\tau) d\tau, & \Gamma_2 &\triangleq \int_0^T S^\nu(T-\tau)B_2 d\tau, \\ \hat{C}_1 &\triangleq C_1S^\nu(\cdot), \\ \hat{D}_{11}\hat{u}_{1,k} &\triangleq C_1 \int_0^\bullet S^\nu(\cdot-\tau)B_1\hat{u}_{1,k}(\tau) d\tau, & \hat{D}_{12} &\triangleq C_1 \int_0^\bullet S^\nu(\cdot-\tau)B_2 d\tau + D_{12}. \end{aligned}$$

Proof. Consider the disturbance-attenuation problem for the lifted system (3.14). Since $\Sigma(K, L, M, N)$ is its solution, we obtain the following results.

1. $\Sigma(K, L, M, N)$ provides power stabilizability and power detectability for $(\hat{\Phi}, \Gamma_2, C_2)$ with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$. By Proposition 3, $\Sigma(K, L, M, N)$ then provides hybrid stability for the original Pritchard–Salamon system.
2. The input–output operator from $\hat{u}_1$ to $\hat{y}_1$ is a γ -contracting map from $\ell_2(\mathbb{N}, L_2(0, T; \mathcal{U}_1))$ to $\ell_2(\mathbb{N}, L_2(0, T; \mathcal{Y}_1))$. It follows by Proposition 5 that the input–output operator mapping u_1 to y_1 is also a γ -contracting map from $L_2(\mathbb{R}_+, \mathcal{U}_1)$ to $L_2(\mathbb{R}_+, \mathcal{Y}_1)$.

We conclude that $\Sigma(K, L, M, N)$ is then also a solution to the digital output-measurement-feedback control problem. $\square$

Theorem 1 is the cornerstone result which facilitates our application of the discrete Popov-theory-based solution of the discrete-time H^∞ control problem, to solve the digital H^∞ control problem for Pritchard–Salamon systems.

4. General discrete Popov theory

Originally developed by Ionescu & Weiss [20] for finite-dimensional discrete-time systems, the main results on discrete Popov theory have been completely generalized to time-dependent discrete-time systems on Hilbert space in [16]. Among the various results of Popov theory, the one giving the link between a quadratic cost functional and the solution to the so-called Kalman–Szegő–Popov–Yakubovitch (KSPY) system, strongly related to Riccati-equation theory, is (in our opinion) the most relevant. The central result proved in [19, 16] replaced the Popov positivity condition with a more general one, expressed in terms of the invertibility of a certain operator.

An immediate application, probably the most important one, was to express the solution to the discrete-time H^∞ (γ -suboptimal) control problem. This section is devoted to collecting the main results of discrete Popov theory developed in [20, 16], which provide a basis for deriving the solution to the output-measurement-feedback discrete-time H^∞ (γ -suboptimal) control problem.

Our formal model is described by the difference state equation

$$x_{k+1} = Ax_k + Bu_k \quad (k \in \mathbb{N}), \quad x_0 = \xi, \quad (4.1)$$

where $A \in \mathfrak{L}(\mathcal{X})$ and $B \in \mathfrak{L}(\mathcal{U}, \mathcal{X})$. We shall assume that A is power-stable on $\mathcal{X}$ and consequently, for every given $(\xi, u) \in \mathcal{X} \times \ell_2(\mathbb{N}, \mathcal{U})$, that equation (4.1) has a unique solution $x^{\xi, u} \in \ell_2(\mathbb{N}, \mathcal{X})$ given by

$$x^{\xi, u} = \Phi x + \tilde{L}u, \quad (4.2)$$

where $\Phi: \mathcal{X} \rightarrow \ell_2(\mathbb{N}, \mathcal{X})$ and $\tilde{L}: \ell_2(\mathbb{N}, \mathcal{U}) \rightarrow \ell_2(\mathbb{N}, \mathcal{X})$ are defined as

$$(\Phi x)_k = A^k x, \quad (\tilde{L}u)_k = \sum_{i=0}^{k-1} A^{k-i-1} Bu^i, \quad (\tilde{L}u)_0 = 0,$$

for $k \in \mathbb{N}$. Let

$$\Pi = \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix},$$

with $Q = Q^* \in \mathfrak{L}(\mathcal{X})$, $L \in \mathfrak{L}(\mathcal{U}, \mathcal{X})$, and $R = R^* \in \mathfrak{L}(\mathcal{U})$, and call (A, B, Π) a *Popov triple* on $(\mathcal{X}, \mathcal{U})$ (in this definition, A is not necessarily power-stable). Associate with (A, B, Π) two objects, as follows.

1. The KSPY system

$$\Sigma_{\text{KSPY}}: \begin{cases} R + B^*XB = G, \\ L + A^*XB = H, \\ Q + A^*XA - X = HG^{-1}H^*, \end{cases} \quad (4.3)$$

is considered as a set of simultaneous equations in $X = X^*$, G , and H . We shall write

$$F_{G,H} \triangleq -G^{-1}H^*. \quad (4.4)$$

2. Given an operator V on $\mathcal{Z}$, we denote by $\underline{V}$ the operator on $\mathcal{Z}^{\mathbb{N}}$ defined by

$$\underline{V}(z_k)_{k \in \mathbb{N}} = (Vz_k)_{k \in \mathbb{N}}.$$

Then the self-adjoint operator $\tilde{R}: \ell_2(\mathbb{N}; \mathcal{U}) \rightarrow \ell_2(\mathbb{N}; \mathcal{U})$ is defined by

$$\tilde{R} = \underline{R} + \underline{L}^* \tilde{L} + \tilde{L}^* \underline{L} + \tilde{L}^* Q \tilde{L}. \quad (4.5)$$

It is easy to check that (4.3) is equivalent to the discrete-time algebraic Riccati equation (DTARE)

$$A^*XA - X - (A^*XB + L)(R + B^*XB)^{-1}(L^* + B^*XA) + Q = 0. \quad (4.6)$$

A triplet (X, G, H) is called a *stabilizing solution* to Σ_{KSPY} (4.3) if G^{-1} is bounded and $A + BF_{G,H}$ defines a power-stable evolution on $\mathcal{X}$ for $F_{G,H}$ defined via (4.4). Clearly, in this case, X is also a stabilizing solution to (4.6) and, as is well known, is unique.

The main result of discrete Popov theory can be stated as follows.

THEOREM 2 Let (A, B, Π) be a Popov triplet, with A power-stable. Then the following statements are equivalent.

1. $\tilde{R}^{-1}$ is well defined and bounded.
2. The KSPY system (4.3) (or equivalent DTARE (4.6)) has a stabilizing solution.

Proof. See [16]. $\square$

For more comments on the general framework of Theorem 2, the reader is referred to [16, 17].

We have assumed that A is power-stable on $\mathcal{X}$. Here we treat the situation when A does not necessarily satisfy this assumption. Let (A, B, Π) be a Popov triple on $(\mathcal{X}, \mathcal{U})$ with A not necessarily defining a power-stable evolution on $\mathcal{X}$, and suppose that (A, B) is power-stabilizable. Let $F \in \mathcal{Q}(\mathcal{X}, \mathcal{U})$. We shall call (A_F, B_F, Π_F) the F -equivalent of (A, B, Π) if

$$A_F = A + BF, \quad B_F = B, \quad \Pi_F = \begin{bmatrix} Q + LF + F^*L^* + F^*RF & L + F^*R \\ L^* + RF & R \end{bmatrix}. \quad (4.7)$$

PROPOSITION 6 Let (A, B, Π) be a discrete Popov triple, and $F \in \mathcal{Q}(\mathcal{X}, \mathcal{U})$ an arbitrary feedback operator. The stabilizing solution (X, G, H) to the KSPY system associated with Π exists if and only if the stabilizing solution to the KSPY system associated with Π_F exists. Moreover, the solution to the latter is $(X, G, H + F^*G)$.

Proof. See [17]. $\square$

Notice that X is also a solution to the discrete-time Riccati equation (4.6) for (A_F, B_F, Π_F) . Hence, in order to remove the power-stability assumption made on A we proceed as follows.

Step 1. Choose F such that $A_F = A + BF$ is power-stable, and consider the F -equivalent Popov triple (A_F, B_F, Π_F) .

Step 2. Prove via Theorem 2 the existence of the solution to the KSPY system associated with (A_F, B_F, Π_F) .

We conclude with two necessary results for the developments of the next section.

PROPOSITION 7 Assume that (X, G, H) is a stabilizing solution to (4.3), where A is assumed to be power-stable. Then $\tilde{R}$ given by (4.5) admits the factorization

$$\tilde{R} = S^* \underline{G} S, \quad (4.8)$$

where $S: \ell_2(\mathbb{N}, \mathcal{U}) \rightarrow \ell_2(\mathbb{N}, \mathcal{U})$ is the input-output operator associated with the system

$$sx = Ax + Bu, \quad x_0 = 0, \quad v = -Fx + u.$$

Proof. See [16].

(A more general result is proved in [20]). The idea is to show that the associated quadratic forms on $\ell_2(\mathbb{N}; \mathcal{U})$ are the same for the r.h.s. and for the l.h.s. in (4.8). $\square$

An immediate consequence is the following.

COROLLARY 4 Assume that the KSPY system associated with (A, B, Π) has a stabilizing solution (X, G, H) , and let (A_F, B_F, Π_F) be its F -equivalent, where $F = F_{G,H}$ is given by (4.4). Then (4.8) becomes

$$\underline{G}_F = \underline{G} = \tilde{R}_F,$$

where G_F and $\tilde{R}_F$ are associated with (A_F, B_F, Π_F) . $\square$

5. Disturbance-attenuation problem for discrete-time systems

The γ -disturbance-attenuation problem for discrete-time systems has recently received a lot of attention. Starting with the leading work of Stoorvogel [27], Iglesias [19], and Ionescu & Weiss [21], where a two-Riccati-equation set of formulae has been written for the solution to the finite-dimensional H^∞ control problem, a complete generalization of the latter results, based on the extension of the Popov function theory developed in [20], has been obtained for time-dependent systems on Hilbert spaces by Halanay & Ionescu [17] and by Halanay *et al.* [16], respectively. However, the theory in [17, 16] cannot be applied to solve the digital H^∞ (γ -suboptimal) control problem for the Pritchard–Salamon class of infinite-dimensional systems. The motivation of this is the strong-epicity assumption made on the D_{21} term ($D_{21}D_{21}^* \geq \nu I$). In our case, the γ -attenuation problem we have obtained for the lifted hybrid Pritchard–Salamon system was essentially *singular*, having the D_{21} term equal to zero (recall that this fact was a direct consequence of the fact that the sampler operator is not well defined over L_2 spaces).

The purpose of this section is to give necessary and sufficient conditions for the existence of the solution to the γ -disturbance attenuation problem under the mildest possible assumptions on the initial discrete-time data. We also set the basis for writing the solution to the digital H^∞ (γ -suboptimal) control problem for the Pritchard–Salamon class of systems in terms of two Riccati-equation formulae for the H^∞ (γ -suboptimal) control problem associated with the discrete-time system obtained by lifting the hybrid Pritchard–Salamon system.

REMARK 8 (1) If a state-space feedback law is applied, i.e. if u_2 is replaced by $u_2 + F_2 x$, then the system (2.11) is updated by keeping all the coefficients the same except A and C_1 , which become $A + B_2 F_2$ and $C_1 + D_{12} F_2$, respectively. (2) If a state-space injection is applied, then the system (2.11) is updated by keeping all the coefficients the same except A which becomes $A + K_2 C_2$. $\square$

Before we state and prove the necessary solvability conditions, we introduce the following definition.

DEFINITION 9 We say that the infinite-dimensional discrete-time system (2.11) is:

(1) *right-coercive* if (i) there exists F_2 such that the infinite-dimensional discrete-time system (2.11) updated with $A \leftarrow A + B_2 F_2$ and $C_1 \leftarrow C_1 + D_{12} F_2$ is power-stable and (ii) there exists $\delta_{12} > 0$ such that

$$\langle u_1, T_{12} u_1 \rangle_{\mathcal{Y}_2} \geq \delta_{12} \|u_1\|_2^2 \quad \text{for } u_1 \in \ell_2(\mathbb{N}, \mathcal{U}_1); \quad (5.1)$$

(2) *left-coercive* if (i) there exists K_2 such that the infinite-dimensional discrete-time system (2.11) updated with $A \leftarrow A + K_2 C_2$ is power-stable and (ii) there exists $\delta_{21} > 0$ such that

$$\langle u_2, T_{21} u_2 \rangle_{\mathcal{Y}_1} \geq \delta_{21} \|u_2\|_2^2 \quad \text{for } u_2 \in \ell_2(\mathbb{N}, \mathcal{U}_2), \quad (5.2)$$

where $T_{12} : \ell_2(\mathbb{N}, \mathcal{U}_1) \rightarrow \ell_2(\mathbb{N}, \mathcal{Y}_2)$ and $T_{21} : \ell_2(\mathbb{N}, \mathcal{U}_2) \rightarrow \ell_2(\mathbb{N}, \mathcal{Y}_1)$ are the input-output operators of the linear systems mapping u_1 to y_2 and u_2 to y_1 , respectively.

5.1 Necessary solvability conditions

In this section, we give the set of necessary conditions for the solvability of the γ -attenuation problem in terms of associated KSPY systems. The minimum set of assumptions we make on the initial data is:

- (1) (A, B_2) is power-stabilizable;
- (2) (C_2, A) is power-detectable;
- (3) The infinite-dimensional discrete-time system (2.11) is right-coercive and left-coercive.

Let us define the Popov triplets

$$\begin{aligned} (A, B_2, \Pi_2) &= \left(A, B_2, \begin{bmatrix} Q_2 & L_2 \\ L_2^* & R_{22} \end{bmatrix} \right) = \left(A, B_2, \begin{bmatrix} C_1^* C_1 & C_1^* D_{12} \\ D_{12}^* C_1 & D_{12}^* D_{12} \end{bmatrix} \right), \\ (A, B, \Pi) &= \left(A, B, \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix} \right), \end{aligned}$$

where

$$\begin{aligned} B &= [B_1, B_2], \quad Q = Q_2, \quad L = C_1^* [D_{11}, D_{12}], \\ R &= \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} [D_{11}, D_{12}] - \begin{bmatrix} \gamma^2 I_{\mathcal{U}_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^* & R_{22} \end{bmatrix}; \end{aligned}$$

and

$$\begin{aligned} (A^*, C_2^*, \hat{\Pi}_2^*) &= \left(A^*, C_2^*, \begin{bmatrix} \hat{Q}_2^* & \hat{L}_2^* \\ \hat{L}_2^* & \hat{R}_{22}^* \end{bmatrix} \right) = \left(A^*, C_2^*, \begin{bmatrix} B_1 B_1^* & 0 \\ 0 & 0 \end{bmatrix} \right), \\ (A^*, C^*, \hat{\Pi}^*) &= \left(A^*, C^*, \begin{bmatrix} \hat{Q}^* & \hat{L}^* \\ \hat{L}^* & \hat{R}^* \end{bmatrix} \right), \end{aligned}$$

where

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad \hat{Q} = \hat{Q}_2, \quad \hat{L} = B_1[D_{11}^*, 0],$$

$$\hat{R} = \begin{bmatrix} D_{11} \\ 0 \end{bmatrix} [D_{11}^*, 0] - \begin{bmatrix} \gamma^2 I_{y_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{12}^* & \hat{R}_{22} \end{bmatrix} = \begin{bmatrix} \hat{R}_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

Introduce also $J = \begin{bmatrix} -I_{u_1} & 0 \\ 0 & I_{u_2} \end{bmatrix}$ and $\hat{J} = \begin{bmatrix} -I_{y_1} & 0 \\ 0 & I_{y_2} \end{bmatrix}$.

The main result giving a necessary set of conditions for the solvability of the γ -attenuation problem is the next theorem, which depends on the following two lemmas for its proof.

LEMMA 2 Consider the Popov triplets

$$(A, B, \Pi_2) = \left(A, B_2, \begin{bmatrix} Q & L_2 \\ L_2^* & R_{22} \end{bmatrix} \right),$$

$$(A, B, \Pi) = \left(A, B, \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix} \right) = \left(A[B_1, B_2], \begin{bmatrix} Q & L_1 & L_2 \\ L_1^* & R_{11} & R_{12} \\ L_2^* & R_{12}^* & R_{22} \end{bmatrix} \right),$$

with A power-stable, and let

$$\tilde{R} : \ell_2(\mathbb{N}, \mathcal{U}_1 \oplus \mathcal{U}_2) \rightarrow \ell_2(\mathbb{N}, \mathcal{U}_1 \oplus \mathcal{U}_2),$$

$$\tilde{R} = \underline{R} + \underline{L}^* \tilde{L} + \tilde{L}^* L + \tilde{L}^* Q = \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{12}^* & \tilde{R}_{22} \end{bmatrix},$$

$$\tilde{P} : \ell_2(\mathbb{N}, \mathcal{U}_1 \oplus \mathcal{U}_2) \rightarrow \mathcal{X}, \quad \tilde{P} = \Phi^*(Q\tilde{L} + L) = [\tilde{P}_1, P_2].$$

Assume that there exists $\delta > 0$ such that

$$\tilde{R}_{22} \geq \delta I_{\mathcal{U}_2}, \quad \tilde{R}_{11} - \tilde{R}_{12} \tilde{R}_{22}^{-1} \tilde{R}_{12}^* \leq -\delta I_{\mathcal{U}_1}.$$

Then the KSPY systems

$$\left\{ \begin{array}{l} \tilde{V}_2^* \tilde{V}_2 = R_{22} + B_2^* X_2 B_2 \\ \tilde{W}_2^* \tilde{V}_2 = L_2 + A^* X_2 B_2 \\ \tilde{W}_2^* \tilde{W}_2 = Q_2 + A^* X_2 A - X_2 \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l} V^* J V = R + B^* X B \\ W^* J V = L + A^* X B \\ W^* J W = Q + A^* X A - X \end{array} \right\} \quad (5.3)$$

have stabilizing solutions.

Proof. See [16]. $\square$

LEMMA 3 Let $T_{ij} : \ell_2(\mathbb{N}, \mathcal{U}_i) \rightarrow \ell_2(\mathbb{N}, \mathcal{Y}_j)$ be the linear bounded operator mapping u_i to y_i . Then the operator $\tilde{R}$ admits a representation of the form

$$\tilde{R} = \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{12}^* & \tilde{R}_{22} \end{bmatrix} = \begin{bmatrix} T_{11}^* \\ T_{12}^* \end{bmatrix} [T_{11}, T_{12}] - \begin{bmatrix} \gamma^2 I_{y_1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. See [16]. $\square$

THEOREM 3 If there exists a γ -attenuator (2.12) for (2.11), then the KSPY systems

$$\left\{ \begin{array}{l} \tilde{V}_2^* \tilde{V}_2 = R_{22} + B_2^* X_2 B_2 \\ \tilde{W}_2^* \tilde{V}_2 = L_2 + A^* X_2 B_2 \\ \tilde{W}_2^* \tilde{W}_2 = Q_2 + A^* X_2 A - X_2 \end{array} \right\}, \quad \left\{ \begin{array}{l} \hat{V}_2^* \hat{V}_2 = C_2 Y_2 C_2^* \\ \hat{W}_2^* \hat{V}_2 = A Y_2 C_2^* \\ \hat{W}_2^* \hat{W}_2 = \hat{Q}_2 + A Y_2 A^* - Y_2 \end{array} \right\},$$

$$\left\{ \begin{array}{l} V^* J V = R + B^* X B \\ W^* J V = L + A^* X B \\ W^* J W = Q + A^* X A - X \end{array} \right\}, \quad \left\{ \begin{array}{l} \hat{V}^* \hat{J} \hat{V} = \hat{R} + C Y C^* \\ \hat{W}^* \hat{J} \hat{V} = \hat{L} + A Y C^* \\ \hat{W}^* \hat{J} \hat{W} = \hat{Q} + A Y A^* - Y \end{array} \right\}$$

have stabilizing solutions whose respective first components satisfy

$$X \geq X_2 \geq 0, \quad Y \geq Y_2 \geq 0,$$

$$\text{with } V = \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} \text{ and } \hat{V} = \begin{bmatrix} \hat{V}_{11} & \hat{V}_{12} \\ 0 & \hat{V}_{22} \end{bmatrix}.$$

Proof. Notice first that the problem of γ -disturbance attenuation is compatible with the discrete Popov function theory, i.e. Theorem 2 is applicable to finding the solution. As has been shown in [16], the first statement of Theorem 2 allows a signature for $\tilde{R}$. Let $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ and suppose that $\tilde{R}$ has the form $\begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{12}^* & \tilde{R}_{22} \end{bmatrix}$. Then $\tilde{R}^{-1}$ is well defined and bounded if

$$\tilde{R}_{22} \gg 0, \quad -\tilde{R}_{11} + \tilde{R}_{12} \tilde{R}_{22}^{-1} \tilde{R}_{12}^* \gg 0$$

where ' $\gg 0$ ' means that the operator is coercive. This situation corresponds to the typical game situation

$$\max_{u_1 \in \mathcal{U}_1} \min_{u_2 \in \mathcal{U}_2} J\left(\xi, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)$$

encountered in [7]. We have assumed that the infinite dimensional system is right-coercive and left-coercive. This implies that T_{12} and T_{21} are coercive operators. But the operator R_{22} written for (A, B, Π) is exactly $T_{12}^* T_{12}$, and its counterpart for $(A^*, C^*, \hat{\Pi}^*)$ is $T_{21}^* T_{21}$, respectively, which is obtained by applying Lemma 3. Now apply Lemma 2 to obtain the existence of the solutions to the KSPY systems (5.3) associated with the Popov triplet (A, B, Π) and do the same for their counterparts associated with $(A^*, C^*, \hat{\Pi}^*)$, respectively, and the proof is complete. $\square$

5.2. Sufficient solvability conditions

In this section, we give a set of sufficient conditions for the existence of a solution to the γ -attenuation problem. Consider the equivalent system

$$\begin{aligned} sX &= A_0 X + B_{01} \tilde{u}_1 + B_{02} u_2, \\ \tilde{y}_1 &= C_{01} X + D_{011} \tilde{u}_1 + D_{012} u_2, \quad y_2 = C_{02} X, \end{aligned}$$

where

$$\begin{aligned} A_o &= A + B_1 F_1, & B_{o1} &= \gamma B_1 V_{11}^{-1}, & B_{o2} &= B_2, \\ C_{o1} &= -V_{22} F_2, & C_{o2} &= C_2, & D_{o11} &= \gamma V_{21} V_{11}^{-1}, & D_{o12} &= V_{22}, \\ F_1 &= -V_{11}^{-1} W_1, & F_2 &= V_{22}^{-1} V_{21} V_{11}^{-1} W_1 - V_{22}^{-1} W_2, \end{aligned}$$

where $W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ is partitioned conformally with $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$, and consider also the Popov triplet

$$(A_o^*, C_o^*, \Pi^*) = \left(A_o^*, C_o^*, \begin{bmatrix} Q_o^* & L_o^* \\ L_o^* & R_o^* \end{bmatrix} \right),$$

where

$$\begin{aligned} C_o &= \begin{bmatrix} C_{1o} \\ C_{2o} \end{bmatrix}, & Q_o &= B_{o1} B_{o1}^*, & L_o &= B_{o1} [D_{o11}^*, 0], \\ R_o &= \begin{bmatrix} D_{o11} \\ 0 \end{bmatrix} [D_{o11}^*, 0] - \begin{bmatrix} \gamma^2 I_{y_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_{o11} & R_{o12} \\ R_{o12}^* & R_{o22} \end{bmatrix} = \begin{bmatrix} R_{o11} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Introduce also $\tilde{J} = \begin{bmatrix} -I_{y_1} & 0 \\ 0 & I_{y_2} \end{bmatrix}$.

THEOREM 4 If there exists a γ -attenuator (2.12) for the system (2.11), then the KSPY system

$$\begin{aligned} V_o \tilde{J} V_o^* &= R_o + C_o Y_o C_o^*, \\ W_o \tilde{J} V_o^* &= L_o + A_o Y_o C_o, \\ W_o \tilde{J} W_o^* &= Q_o + A_o Y_o A_o^* - Y_o \end{aligned}$$

has a stabilizing solution corresponding to the triple (Y_o, V_o, W_o) with $Y_o \geq 0$ and $V_o = \begin{bmatrix} V_{o11} & V_{o12} \\ 0 & V_{o22} \end{bmatrix}$.

Proof. See [16]. $\square$

The following theorem is our main result on sufficient conditions for the existence of a solution to γ -attenuation problem.

THEOREM 5 Let Assumptions 1, 2, and 3 hold, and assume that the KSPY systems

$$\left\{ \begin{array}{l} V^* J V = R + B^* X B \\ W^* J V = L + A^* X B \\ W^* J W = Q + Q^* X A - X \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l} V_o \tilde{J} V_o^* = R_o + C_o Y_o C_o^* \\ W_o \tilde{J} V_o^* = L_o + A_o Y_o C_o \\ W_o \tilde{J} W_o^* = Q_o + A_o Y_o A_o^* - Y_o \end{array} \right\}$$

have stabilizing solutions. Then the γ -disturbance-attenuation problem admits a

solution, and a compensator which simultaneously achieves stability and γ -attenuation is given by (2.10) for

$$A_c = A_o - B_{o2}D_{o12}^{-1}[C_{o1} + (I + K_{o1})V_{o12}V_{o22}^{-1}C_{o2}], \quad (5.4a)$$

$$B_c = K_{o2} + (B_{o2}D_{o12}^{-1} + K_{o1})V_{o12}V_{o22}^{-1}, \quad (5.4b)$$

$$C_c = D_{o12}^{-1}(C_{o1} - V_{o12}V_{o22}^{-1}C_{o2}), \quad D_c = -D_{o12}V_{o12}V_{o22}^{-1}. \quad (5.4c, d)$$

Proof. See [17]. $\square$

5.3. The output-measurement-feedback solution to the digital control problem

The Popov-theory-based output-measurement-feedback solution to the digital H^∞ control problem can be now written in an explicit way. Assume that (A^V, B_2) is exponentially stabilizable. It follows by Lemma 1 that the discretized pair (Φ, Γ_2) is power-stabilizable for a sufficiently small sampling step. Hence, by Corollary 3, the original Pritchard–Salamon system is hybrid-stabilizable. Consider

$$\Sigma\left(\hat{\Phi}, [\hat{\Gamma}_1(\cdot), \Gamma_2], \begin{bmatrix} \hat{C}_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} \hat{D}_{11}(\cdot) & \hat{D}_{12} \\ 0 & 0 \end{bmatrix}\right), \quad (5.5)$$

which is the system obtained by lifting the hybrid Pritchard–Salamon system

$$\Sigma\left(S(\cdot), [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix}\right). \quad (5.6)$$

If $\hat{\Phi}$ is not power-stable, then choose an arbitrary power-stabilizing feedback, and obtain the F -equivalent of the lifted system which is power stable. According to Theorem 1, finding a output-measurement-feedback solution to the digital H^∞ control problem is equivalent to finding an output-measurement-feedback solution to the discrete-time H^∞ control problem for the power-stable lifted system. Check if (5.3) is right-coercive and left-coercive. If this is the case, then the KSPY systems, written for the lifted system as

$$\left\{ \begin{array}{l} \tilde{V}_2^* \tilde{V}_2 = R_{22} + \hat{\Gamma}_2^* X_2 \hat{\Gamma}_2 \\ \tilde{W}_2^* \tilde{V}_2 = L_2 + \hat{\Phi}^* X_2 \hat{\Gamma}_2 \\ \tilde{W}_2^* \tilde{W}_2 = Q_2 + \hat{\Phi}^* X_2 \hat{\Phi} - X_2 \end{array} \right\}, \quad \left\{ \begin{array}{l} \hat{V}_2^* \hat{V}_2 = C_2 Y_2 C_2^* \\ \hat{W}_2^* \hat{V}_2 = \hat{\Phi}^* Y_2 C_2^* \\ \hat{W}_2^* \hat{W}_2 = \hat{Q}_2 + \hat{\Phi}^* Y_2 \hat{\Phi}^* - Y_2 \end{array} \right\}, \quad (5.7a, b)$$

$$\left\{ \begin{array}{l} V^* J V = R + \hat{\Gamma}^* X \hat{\Gamma} \\ W^* J V = L + \hat{\Phi}^* X \hat{\Gamma} \\ W^* J W = Q + \hat{\Phi}^* X \hat{\Phi} - X \end{array} \right\}, \quad \left\{ \begin{array}{l} \hat{V}^* \hat{J} \hat{V} = \hat{R} + \hat{C} Y \hat{C}^* \\ \hat{W}^* \hat{J} \hat{V} = \hat{L} + \hat{\Phi} Y \hat{C}^* \\ \hat{W}^* \hat{J} \hat{W} = \hat{Q} + \hat{\Phi} Y \hat{\Phi}^* - Y \end{array} \right\}, \quad (5.7c, d)$$

have stabilizing solutions such that

$$X \geq X_2 \geq 0, \quad Y \geq Y_2 \geq 0,$$

where

$$\hat{\Gamma} = [\hat{\Gamma}_1, \hat{\Gamma}_2], \quad \hat{C} = \begin{bmatrix} \hat{C}_1 \\ C_2 \end{bmatrix},$$

with $V = \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix}$ and $\hat{V} = \begin{bmatrix} \hat{V}_{11} & \hat{V}_{12} \\ 0 & \hat{V}_{22} \end{bmatrix}$ partitioned conformally with $J = \begin{bmatrix} -I_{u_1} & 0 \\ 0 & I_{u_2} \end{bmatrix}$ and $\hat{J} = \begin{bmatrix} -I_{y_1} & 0 \\ 0 & I_{y_2} \end{bmatrix}$. Then the stabilizing and γ -contracting compensator we are looking for is given by (5.4) updated with data provided by (5.5) respectively.

6. Shortcomings of the method

In this section we focus on the drawbacks of the Popov-theory-based solution to the H^∞ control problem as developed in this paper. Recall that, in Section 5.1, the minimum set of assumptions made on the discrete-time system was: (1) (A, B_2) is power-stabilizable; (2) (C_2, A) is power-detectable; and (3) the infinite-dimensional discrete-time system (2.11) is right-coercive and left-coercive. We would like to give a set of assumptions that, when made on the original data, i.e. on the Pritchard-Salamon system

$$\Sigma\left(S(\cdot), [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix}\right),$$

would guarantee that (1), (2), and (3) hold for the lifted system

$$\Sigma\left(\hat{\Phi}, [\hat{\Gamma}_1, \Gamma_2], \begin{bmatrix} \hat{C}_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} \\ 0 & 0 \end{bmatrix}\right).$$

Notice that admissible or bounded stabilizability and admissible or bounded detectability of $(S(\cdot), B_2, C_2)$ and a sufficiently small chosen sampling step, $T > 0$, ensures that $(\hat{\Phi}, \Gamma_2, C_2)$ is power-stabilizable and power-detectable with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$. This is a natural constraint which is always imposed in digital control. Let us focus now on the third item regarding left and right coercivity of the discrete-time system

$$\Sigma\left(A, [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix}\right).$$

PROPOSITION 8 Assume that (1) $D_{12}^* D_{12}$ is coercive; (2) $(\Pi_{12} C_1, A - B_2 D_{12}^\dagger C_1)$ is power-detectable, where $D_{12}^\dagger = (D_{12}^* D_{12})^{-1} D_{12}^*$ and $\Pi_{12} = I - D_{12} D_{12}^\dagger$. Then

$$\Sigma\left(A, [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix}\right)$$

is right-coercive.

Proof. The assumptions made in Proposition 8 are sufficient conditions for the existence of a positive semidefinite stabilizing solution to the DTARE with

$$Q = C_1^* C_1, \quad L = C_1^* \hat{D}_{12}, \quad R = D_{12}^* D_{12}.$$

Let $F_2 = -(R + B_2^* X B_2)^{-1} (B_2^* X A + L^*)$ be the corresponding stabilizing feedback.

Then the DTARE can be rewritten as

$$\tilde{A}^* X \tilde{A} - X + \tilde{Q} = 0, \quad (6.1)$$

where $\tilde{A} = A + B_2 F_2$ and $\tilde{Q} = Q + F_2 L + L^* F_2^* R F_2$. A simple computation performed on (6.1) gives

$$R + B^* X B = \tilde{T}_{12}^* \tilde{T}_{12}, \quad (6.2)$$

where $\tilde{T}_{12} = \tilde{C}_1 (sI - \tilde{A})^{-1} B_2 + D_{12}$ and $\tilde{C}_1 = C_1 + D_{12} F_2$. Since $R = D_{12}^* D_{12} \gg 0$ and $X \geq 0$, it follows from (6.2) that $\tilde{T}_{12}^* \tilde{T}_{12} \gg 0$, which completes the proof. $\square$

Let us see now what assumptions one should make on the original Pritchard-Salamon system such that (1) and (2) from Proposition 8 hold for the lifted system

$$\Sigma \left(\hat{\Phi}, [\hat{\Gamma}_1, \hat{\Gamma}_2], \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \end{bmatrix}, \begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} \\ 0 & 0 \end{bmatrix} \right).$$

Recall that

$$\begin{aligned} \hat{\Phi} &\triangleq S^V(T), \\ \hat{\Gamma}_1 \hat{u}_{1,k} &\triangleq \int_0^T S^V(T - \tau) B_1 \hat{u}_{1,k}(\tau) d\tau, \quad \Gamma_2 \triangleq \int_0^T S^V(T - \tau) B_2 d\tau, \\ \hat{C}_1 &\triangleq C_1 S^V(\cdot), \\ (\hat{D}_{11} \hat{u}_{1,k})(t) &\triangleq C_1 \int_0^t S^V(t - \tau) B_1 \hat{u}_{1,k}(\tau) d\tau, \\ (\hat{D}_{12} \hat{u}_{2,k})(t) &\triangleq \left(C_1 \int_0^t S^V(t - \tau) B_2 d\tau + D_{12} \right) \hat{u}_{2,k}. \end{aligned}$$

From the expression of $\hat{D}_{12}$, one can easily notice that, if $D_{12}^* D_{12} \gg 0$ for the Pritchard-Salamon system, then $\hat{D}_{12}^* \hat{D}_{12} \gg 0$ for its lifted counterpart. Some straightforward computation shows that, if $(C_1, S(\cdot))$ is considered admissibly or boundedly detectable, then $(\hat{\Pi}_{12} \hat{C}_1, \hat{\Phi} - \Gamma_2 \hat{D}_{12}^{\dagger} \hat{C}_1)$ is power-detectable, where $\hat{D}_{12}^{\dagger} = (\hat{D}_{12}^* \hat{D}_{12})^{-1} \hat{D}_{12}^*$ and $\hat{\Pi}_{12} = I - \hat{D}_{12} \hat{D}_{12}^{\dagger}$ is power-detectable. Unfortunately Proposition 8 does not apply for the dual problem, since $D_{21} = 0$ and hence $D_{21} D_{21}^*$ is no longer coercive. The best we can prove is the following.

PROPOSITION 9 Let $\Sigma \left(S(\cdot), [B_1, B_2], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix} \right)$ be such that $(S(\cdot), B_1)$ is exactly controllable on $[0, T]$ and $C_2^* C_2 \gg 0$. Then the lifted system

$$\Sigma \left(\hat{\Phi}, [\hat{\Gamma}_1, \hat{\Gamma}_2], \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \end{bmatrix}, \begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} \\ 0 & 0 \end{bmatrix} \right)$$

is *left-coercive*.

Proof. Since $(S(\cdot), B_1)$ is exactly controllable on $[0, T]$, it follows that the

controllability gramian

$$P_T \triangleq \int_0^T S(\tau) B_1 B_1^* S^*(\tau) d\tau \quad (6.3)$$

is coercive [14: Thm 4.1.7]. But $P_T = \hat{\Gamma}_1 \hat{\Gamma}_1^*$, and hence $\hat{\Gamma}_1 \hat{\Gamma}_1^* \gg 0$. Let us prove that this implies left coercivity of the lifted system. Indeed, let K_2 be an output-stabilizing injection. Then the updated system is now power-stable, and a realization for $T_{21}^* = \underline{\Gamma}_1 (I - s\hat{\Phi}^*)^{-1} s \underline{C}_2^*$ is

$$x = \Phi^* s x + \underline{C}_2^* \mu \quad \eta = \underline{\Gamma}_1^* s x \quad (6.4a, b)$$

with $\mu \in \ell_2(\mathbb{N}; \mathbb{R}^p)$, $x \in \ell_2(\mathbb{N}; \mathcal{V})$, and $\eta \in \ell_2(\mathbb{N}; L_2(0, T; \mathcal{U}_1))$. Exploiting the coercivity of the controllability gramian (6.3) as well as the fact that $\underline{C}_2^* \underline{C}_2 \gg 0$, we get

$$\|\Gamma_1^* s x\|_2 \geq \delta_1 \|s x\|_2, \quad \|\underline{C}_2^* \mu\|_2 \geq \delta_2 \|\mu\|_2 \quad (6.5)$$

for some $\delta_1 > 0$ and $\delta_2 > 0$. Hence from (6.4a) we obtain, by exploiting (6.5),

$$\begin{aligned} \delta_2^2 \|\mu\|_2^2 &\leq \|\underline{C}_2^* \mu\|_2^2 = \|x - \Phi^* s x\|_2^2 \leq 2\|x\|_2^2 + 2\|\hat{\Phi}^* s x\|_2^2 \\ &\leq 2(1 + \|\hat{\Phi}\|^2) \|x\|_2^2 \leq \frac{2(1 + \|\hat{\Phi}\|^2)}{\delta_1^2} \|\underline{\Gamma}_1^* s x\|_2^2, \end{aligned}$$

from which we obtain $\delta_3 \|\mu\|_2^2 \leq \|\eta\|_2^2$ for $\delta_3 = \delta_1 \delta_2 / \sqrt{2}(1 + \|\hat{\Phi}\|^2)^{1/2}$. Thus $\|T_{21}^* \mu\|_2 \geq \delta_3 \|\mu\|_2$, which shows that $T_{12}^* T_{12} \gg 0$. $\square$

REMARK 9 1. In general, it is hard to prove that a system is exactly controllable; indeed, large classes of partial differential and delay systems are not exactly controllable but only approximately controllable. Thus, checking the exact controllability of $(S(\cdot), B_1)$ is not a sensible way of checking whether $T_{21}^* T_{21}$ is coercive. In particular, when $\mathcal{U}_1$ has finite rank, then $(S(\cdot), B_1)$ will not be exactly controllable.

2. Testing for the coercivity of $\Gamma_1 \Gamma_1^*$ entails an investigation of the exact controllability of $(S(\cdot), B_1)$; this problem does not arise when using the coercivity of $C_2^* C_2$ as a test of that of $T_{21}^* T_{21}$. By an adequate choice of the measurement sensors, right coercivity can be achieved for large classes of systems.

3. If we restrict to the finite-dimensional case, then the exact controllability of $(S(\cdot), B_1)$ collapses to a trivial coercivity of $\Gamma_1 \Gamma_1^*$ —which i.e. the coercivity of P_T is generic—which (by a suitable choice of the design configuration) can also be achieved.

7. Conclusions and future research directions

In this paper, a discrete Popov-function-based solution to the digital H^∞ γ -suboptimal control of a Pritchard-Salamon system has been given. The set of necessary and sufficient conditions for the solvability of the problem has been expressed in terms of bounded invertibility of a certain Toeplitz-type operator

associated with the system, as well as in terms of the solvability of two Kalman–Szegő–Popov–Yakubovitch systems. In our opinion, a natural continuation of this research would cover

- obtaining a finite-dimensional controller from the digital infinite-dimensional one obtained by the discrete Popov-function approach
- a case study on a PDE model which matches the framework of the Pritchard–Salamon class
- the relationship between the discrete-time controller obtained by the discrete Popov-theory approach and the one obtained by discretizing the continuous-time controller derived by van Keulen in [29]
- finding a reliable way of expressing the coercivity of $T_{21}^* T_{21}$ in terms of the original data of the Pritchard–Salamon system.

REFERENCES

1. BAMIEH, B. A., & PEARSON JR, J. B., 1992. A general framework for linear periodic systems with applications to H^∞ sampled-data control. *IEEE Trans AC* **37**, 418–35.
2. BAMIEH, B., PEARSON JR, J. B., FRANCIS, B. A., & TANNENBAUM, A., 1991. A lifting technique for linear periodic systems with applications to sampled-data control. *Systems & Control Letters* **17**, 79–88.
3. BARB, F. D., DE KONING, W. L., & WEISS, M., 1993. The digital linear quadratic control problem for Pritchard–Salamon systems. *Proceedings of the 2nd European Control Conference, ECC'93*, Groningen, The Netherlands, 1993. Pp. 1371–6.
4. BARB, F. D., & DE KONING, W. L., 1993. Digital control of Pritchard–Salamon systems. Report 93–33 of the Department of Technical Mathematics and Informatics, Delft University of Technology (submitted).
5. BARB, F. D., IONESCU, V., & DE KONING, W. L., 1993. Digital state-feedback H^∞ control of Pritchard–Salamon systems. A Popov function approach. Report 93–96 of the Department of Technical Mathematics and Informatics, Delft University of Technology. Submitted to *SIAM J. Control & Optimiz.*
6. BARB, F. D., IONESCU, V., & DE KONING, W. L., 1995. A Popov-theory-based survey on digital control of infinite-dimensional systems with unboundedness. To appear in *IMA J. Math. Control & Info.*
7. BASAR, T., & BERNHARD, P., 1991. H^∞ optimal control and related min-max design problems; A dynamic game approach. Birkhäuser, Basel-Boston.
8. BASAR, T., & OLSDER, G. J., 1982. *Dynamic noncooperative game theory*. Academic Press.
9. BONTSEMA, J., & CURTAIN, R., 1988. Perturbation properties of a class of infinite-dimensional systems with unbounded control and observation. *IMA J. Math. Control & Info.* **5**, 333–52.
10. CHEN, T., & FRANCIS, B. A., 1991. H^2 -optimal sampled-data control. *IEEE Trans AC* **36**, 387–97.
11. CHEN, T., & FRANCIS, B. A., 1991. Input–output stability of sampled-data systems. *IEEE Trans AC* **36**.
12. CURTAIN, R., 1992. A synthesis of time-domain and frequency-domain methods for the control of infinite-dimensional systems: A system theoretic approach. Report W-8912 of the Mathematics Institute, University of Groningen.
13. CURTAIN, R., LOGEMANN, H., TOWNLEY, S., & ZWART, H., 1992. Well-posedness, stabilizability and admissibility for Pritchard–Salamon systems. Report No. 260-I of the Institute of Dynamical Systems, University of Bremen.
14. CURTAIN, R., & ZWART, H., 1993. Introduction to infinite dimensional linear systems theory. Manuscript.

15. DOYLE, J. C., GLOVER, K., FRANCIS, B. A., & KHARGONEKAR, P., 1989. State-space solutions to standard H^2 and H^∞ control problems. *IEEE Trans AC* **34**, 831–47.
16. HALANAY, A., & IONESCU, V., 1993. Generalized discrete-time Popov–Yakubovitch theory. *Systems & Control Letts* **20**, 1–6.
17. HALANAY, A., & IONESCU, V., 1994. Time-varying discrete linear systems. *Operator Theory Advances and Applications* **68**. Birkhäuser. Pp. 71–78.
18. HALANAY, A., IONESCU, V., & DRAGAN, V., 1995. Infinite horizon disturbance attenuation for discrete-time systems: a Popov–Yakubovitch approach. To appear in *J. Operator Theory*.
19. IGLESIAS, P., 1991. State-space approach to discrete-time H^∞ control. *Int. J. Control* **54**, 1031–73.
20. IONESCU, V., & WEISS, M., 1993. Continuous and discrete-time Riccati theory: a Popov function approach. *Linear Algebra and its Applications* **193**. Pp. 173–203.
21. IONESCU, V., & WEISS, M., 1993. Two-Riccati formulae for discrete-time H^∞ control problem. *Int. J. of Control* **57**, 141–95.
22. KALMAN, R., HO, Y. C., & NARENDRA, K., 1963. Controllability of linear dynamical systems. In *Contributions to Differential Equations*, vol. 1. New York: Interscience.
23. LOGEMMAN, H., 1992. Stability and stabilizability of linear infinite-dimensional discrete-time systems. *IMA J. Math. Control & Info.* **9**, 255–63.
24. POPOV, V. M., 1973. Hyperstability of control systems. Bucuresti: Editura Academiei Române. Berlin: Springer.
25. PRITCHARD, A. J., & SALAMON, D., 1985. The linear quadratic control problem for retarded systems with delays in control and observation. *IMA J. Math. Control & Info.* **2**, 335–62.
26. PRITCHARD, A. J., & SALAMON, D., 1985. The linear quadratic optimal control problem for infinite-dimensional systems with unbounded input and output operators. *SIAM J. Control & Optimiz.* **25**, 121–44.
27. STOOBVOGEL, A., 1992. The H^∞ control problem: A state-space approach. Prentice Hall.
28. TADMOR, G., 1992. H^∞ optimal sampled-data control in continuous time systems. *Int. J. Control* **56**, 99–141.
29. VAN KEULEN, B., 1993. H^∞ -control for infinite-dimensional systems: a state space approach. Ph.D. Thesis, University of Groningen.
30. WEISS, G., 1989. Admissibility of unbounded control operators. *SIAM J. Control & Optimiz.* **27**, 527–45.
31. WEISS, G., 1989. Admissible observation operators for linear semigroups. *Israel J. Math.* **65**, 17–43.
32. WEISS, M., 1993. Necessary and Sufficient conditions for the existence of stabilizing solutions of the Riccati equation in Hilbert space: a Popov function approach. Report W-9311, of the Mathematics Institute, University of Groningen.
33. YAMAMOTO, Y., New approach to sampled-data control systems. A function space method. *Proceedings of the 29th IEEE Conference on Decision & Control*, Honolulu, Hawaii. Pp. 1882–7.