Computational Methodologies for the Non-linear Analysis of Concrete and Masonry Structures
Computational Methodologies for the Non-linear Analysis of Concrete and Masonry Structures

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A mio fratello Lucio
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Flavio M.B. Galanti
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Chapter 1

Introduction

The development of reliable earthquake resistant design of concrete and masonry structures presupposes the study of the potential failure mechanisms of structural components. Apart from relying on experimental analysis, the research in this field has been supported by sophisticated computer analysis, based for example on non linear finite elements. However, computer analysis is at a relatively early stage of development and is compromised by a number of problems in the underlying theory. Thus, the available methods rarely find their application directly in design and engineering.

In this thesis, some of the problems which arise in the non linear finite element analysis of concrete and masonry structures will be addressed. This introductory chapter discusses current issues in earthquake engineering, the problems in the application of computational methods in this branch of civil engineering and the aims of the thesis.

1.1 Earthquake-resistant design of structures

Earthquakes constantly demonstrate the necessity to provide adequate resistance against lateral loads in buildings and structures in seismic regions. At the beginning of the 20th century, recognition of the problem and recognition of the effect of inertial forces led engineers to take into account in their designs a set of equivalent static lateral forces proportionate to the total weight of the structure. Given the unpredictable nature of earthquakes it was soon observed that forces could be generated during an intense earthquake well in excess of the lateral strength provided; however it was also observed that in spite of this, many structures remained standing even though they had sustained some form of damage. From these observations it became apparent that rationally built structures could withstand the most severe earthquakes
via dissipation of energy and that strength was not the most important criterion on which the design had to be based. Further, it was shown in studies of elasto-plastic systems around the 1960's that certain simple relationships exist between the response of an inelastic system and the equivalent elastic system, which led to the formulation of the equivalent displacement and equal energy principles. The design codes were thus adjusted to these new developments, allowing a simple approach towards the problem of designing a structure capable of withstanding earthquakes in an inelastic manner. The latest developments were initiated towards the late 1970's, when it was realized that for the inelastic design approach to be most effective, a certain amount of attention needed to be paid to structural details. This aspect of the design became increasingly important, since it was seen that by providing rational detailing, the structure could be induced to sustain damage in a predefined manner. The various developments have paved the way towards a simple but effective design philosophy, nowadays known as capacity design, which attempts to prevent the sudden collapse of a structure during a severe earthquake by providing it with a controlled energy dissipating mechanism. In this philosophy, structural elements which are likely to fail in a brittle manner are designed with a strength well in excess of the strength of those neighbouring elements which do possess the ability to respond in a ductile manner. During an earthquake, the maximum forces which can be induced in the structure are thus determined by the various yield strengths of the ductile members. At a global level, this ensures that the structure responds in a ductile manner, without sudden failure, even during the severest of earthquakes. Most earthquake-resistant design codes have incorporated this approach, so that there is relatively little difference between codes across the world[1]. The structures for which capacity design has seen its greatest application are moment resistant frames, cantilever walls and duals systems (coupled frames and walls) made from reinforced concrete, see for example Paulay and Priestley[40]. Whilst the concepts of capacity design are well developed for these types of structures, it remains to be seen whether these same concepts could be applied to other types of structures, such as unreinforced masonry buildings and composite structures such as frames which contain unreinforced masonry walls. Limitations in the knowledge of the behaviour of these types of structures are such that certain codes, particularly those in force in countries where capacity design has been originally developed (NZ and USA), restrict or even prohibit the use of unreinforced masonry as a structural material and as an infill material. However, these provisions would be considered too restrictive in other parts of the world with a seismic risk, where unreinforced masonry structures exist and are still made, and infills are commonly used in frame buildings.
1.1.1 Unreinforced masonry

The limitations that codes impose on the use of unreinforced masonry can be attributed to its mechanical characteristics and the lack of insight in the structural behaviour of the material. Masonry is a composite material made from bricks and mortar, showing great variety in strength and stiffness depending on the properties of the individual components and the way the bricks are laid. Further, masonry behaves in a brittle manner unless it is confined or reinforced, and thus has a relatively low ductility. This last aspect is particularly important in a capacity design framework as structures are required to have a certain amount of ductility. Lack of insight in the behaviour of the material, on the other hand, is partially due to the difficulty in modelling the complex composite nature of the material, and also due to the current trend in civil engineering education, in which the subject of masonry is almost completely absent, see Lourenço[31]. These problems have been such that the progress in the research and design of masonry structures has not kept up to pace with the developments in other materials such as steel or concrete. In spite of this, masonry has such qualities as a building material that will make its use persist in the future. Furthermore, it has been observed that masonry structures which met basic anti-seismic requirements possess the ability to dissipate energy and behave adequately during an earthquake, Tomazević[54]. Taking the above into consideration, further developments in the design codes related to masonry should be encouraged, and the possibility of developing specific earthquake-resistant design rules for masonry structures needs to be investigated.

1.1.2 Reinforced concrete frame buildings with masonry infills

Often, the space in buildings is subdivided by placing masonry walls or partitions between the columns and beams of the main structure. The design of such walls is based on two different approaches. In one case, the walls are treated as non-structural elements, and provisions are taken in the design to make sure that the walls are kept separated from the main structure. In the second case, the walls are constructed by placing them directly between the columns and beams of the main structure, and the design should take into consideration the influence of the walls on the response of the entire structure to lateral loads. In most cases, the infills are placed after completion of the main structure in which case the walls are not expected to carry any vertical load. In some regions of the world a different technique is used, whereby the wall is constructed first and the reinforced concrete frame is cast around it afterwards, achieving in this manner a particularly good fit between frame and wall.

Behaviour of infilled frame buildings  The main effect of the infills in the case where they form an integral part of the structure is that of constraining the lateral
movement of structure. In other words, the structure is made to act as a braced frame. Due to the presence of the infills the structure is stiffer and stronger with respect to the bare frame. These effects are such that, during moderate earthquakes, infilled frames behave adequately and sustain relatively little damage. However, the increased stiffness means also that the structure has a shorter period of vibration and hence, due to this, may be subject to larger seismic forces, Figure 1.1. The added strength of the structure is not necessarily beneficial as this would imply a potentially greater seismic loading acting on the structure during an earthquake.

Generally, the changes in strength and stiffness with respect to the bare frame are such that it would be erroneous not to take the infills into consideration in the design process. Due to the bracing effect of the infills, the main structure does not act as a moment resisting frame, but rather as a cantilever wall with the columns acting as flanges. Irregularity in the distribution of the infills can lead to exceptional ductility demands on certain parts of the structure. For example, walls placed irregularly at each floor level lead to a misalignment of the centres of gravity and rigidity. During an earthquake this will lead to a torsional deformation of the structure. Irregularity in elevation on the other hand may lead to large deformations being imposed where the stiffness of the structure changes abruptly.

Another important aspect to consider is the local influence of the infill on the distribution of forces in the surrounding frame. The interaction forces which act between infill and frame may in some cases be so high that an unexpected failure of the frame occurs. Typically, these types of failure are shear failures of the columns or beams, occurring at locations where the infill does not cover the complete height or width of the frame to leave space for openings. Similar situations may arise even where the infill covers the entire width and height of the frame, according to the way the
infill sustains damage during an earthquake. Such situations may arise, for example, when there is a partial collapse of the infill, or when the infill cracks into several partitions which can move relative to each other. Failure of these types, whereby the main load bearing structure is damaged due to the effects of interaction between frame and infill (e.g. shear failure of the frame, failure of the beam-column joint), have to be avoided as these types of failure occur abruptly and with relatively little energy dissipation, Figure 1.2.

The response of the infill to a horizontal in-plane load occurs initially with the frame and infill acting in a composite fashion. As the load is increased, the incompatibility of the deformation frame and infill leads to a separation of the infill along the corners of the infill on the tension diagonal. This separation occurs at about 50 to 70% shear capacity for a concrete frame or at much lower levels for a steel frame, Paulay and Priestley[40]. At the ultimate load and beyond, a number of failure modes can occur, which depend on the relative stiffness and strength of the infill and frame and also on the distribution of loads in the structure at the moment of failure. The failure modes can be distinguished as to where they occur in the composite structure. In the infill, flexural cracking, diagonal tension failure (cracking) of the diagonal in compression, crushing at the corners in compression or shear sliding of the bed joints can occur. The frame on the other hand can fail in flexure, shear, yielding of the tension column or at the beam-column joint. The actual failure mechanism which occurs will often be a combination of these different modes. Tomaževič[54] describes three modes of failure which have been found to occur typically during earthquakes, see Figure 1.3. The first is a mode of failure whereby a relatively weak infill is divided into several sections, allowing the free movement of the frame. In this case plastic
Figure 1.3: Modes of failure of infills. After Tomazevic[54].

hinges form at the sections of maximum bending moment in the frame. The second type of failure is one where a shear sliding mechanism occurs along a horizontal bed joint in the wall. This failure then leads to a failure of the frame due to a short column effect. The third type of failure is initiated in the infill which forms a main diagonal crack. The infill constrains the movement of the frame in such a way that the column in tension fails in shear. More details as to the various failure modes and details as to the effect of the infill on the distribution of bending moment, shear and axial force in the frame can be found in an extensive study on the subject by Crisafulli[16].

Design and modelling of infilled frame buildings The codes related to reinforced concrete frame structures with masonry infills make essential provisions for the influence of the infill, by taking into consideration a shorter period of vibration of the structure, and certain penalties in case the infills are placed irregularly in plan and elevation. Further, a number of detailing aspects are dealt with, however no guidelines are given as to how the designer should model the behaviour of the infills.

From the previous section it was seen how, depending on the constituent materials, geometry and loading, different degrees of interaction between the reinforced concrete frame and the infill can be expected, leading to different modes of failures. Although many studies have been carried out on the subject, there is a constant need for further development of the subject. The major part of the work done up to now has concentrated on modelling the infilled reinforced concrete frame by resorting to simple models, for example by replacing the wall with diagonal struts, Stafford Smith[52], or using plasticity theory, as proposed by Liauw and Kwan[33]. The models are extremely useful in view of a practical application in the design phase of reinforced concrete buildings. However, because of the simplifications on which they are based, they are not capable to represent in a detailed manner the interactions that take place between the various structural parts and the development of damage within the structure.
1.1.3 Future development of design codes: the need for modelling

In the above discussion, certain limitations in current design codes for earthquake-resistant structures have come to light. The capacity design concept seems to be well adapted for conventional types of construction; however, when it comes to masonry structures and composite structures such as infilled reinforced concrete (r.c.) frames, some questions may arise as to the adequacy of current design codes. The issues not only concern the effective safety achieved in the design, but also whether the codes are too conservative and whether they limit unjustly the use of specific materials and construction systems.

These design issues are mainly due to the current limitations in the modelling of the structures in question. In the past, an enormous amount of effort has been put in developing simple models for basic structural elements (e.g. beam and column elements, wall elements etc.), with which the response of a structure was studied globally. These types of studies represent the majority of work done in modelling in earthquake engineering related problems. In developing these models, certain assumptions as to the non-linear behaviour of the basic elements used in the analysis are made, which may be based on a certain amount of theoretical and experimental evidence. Given these assumptions and the resolution of the structural representation, the models can give only a very approximate description of the behaviour of the real structure. Whilst this type of representation may be well suited for structures made from linear elements (e.g. frames, trusses), it is far less appropriate for structures with complex structural configurations.

In order to clarify potential issues in the design codes, numerical models need to be developed which are capable of simulating accurately and in detail the behaviour of structures. The simulations should enable visualization of structural damage and deformation, and reproduce accurately the evolution of the structural response, not only up to the ultimate strength of the structure, but also beyond, revealing its possible residual strength. The latter aspect is essential in a modern approach to earthquake-resistant design, as the capacity design philosophy is geared towards ensuring safety via "controlled failure" of structures. Another important aspect of numerical simulation is that it can complement experimental programs. The types of structures involved are such that a complete (in a statistical sense) experimental analysis of the structures cannot be achieved, simply due to the costs and size of the structures. Detailed numerical analysis could thus be used in conjunction with experimental work to calibrate the "simpler" engineering models and develop new design concepts.
1.2 Current modelling possibilities of finite element analysis

One of the most appropriate tools which can be used for the analysis of masonry and concrete structures is the finite element method. This method enables the analysis of structures using continuum and discrete elements with the possibility of modelling non-linear effects occurring at a material level.

In chapter 2 a number of preliminary analyses of a wall and of an infilled reinforced concrete frame are discussed showing the current possibilities of modelling and analysis by the use of the finite element package DIANA[20], a finite element program developed by TNO (Netherlands Organisation for Applied Research).

The analyses reveal some limitations in the current capabilities of non-linear finite element programs, in the form of incomplete and sometimes doubtful numerical results. Generally, the analyses of the types of structures described above manifest an excessive numerical sensitivity to the problem. The difficulties which are encountered derive essentially from the loss of strength which is typical of the materials being considered and can be categorised as follows: (a) structural instability and formation of mechanisms due to loss of strength of the material; (b) the rigour or lack of it in the theory underlying the material models; and (c) the accuracy and efficiency of the solution methods used in the program.

1.2.1 Current limitations of finite element analysis

The current approach to non-linear finite element analysis leads to a set of non-linear equations which is typically solved using a Newton (or Newton-Raphson) type method. This approach requires the calculation and factorization of the stiffness matrix of the discretised structure, which is assembled from the stiffnesses of the individual elements in the mesh. The method’s success is based on the correct formulation of these element stiffnesses which in turn depend on the current state of stress, loading and damage history of the material at the element sampling points (integration points).

An essential problem which arises in the analyses is related to the use of the Newton method in conjunction with materials which soften, i.e. which show strength and stiffness degradation beyond a certain loading stage. Further, some limitations exist in the constitutive modelling of masonry and concrete. The problems can be listed as follows:

a) the attainment of limit points in the structure is accompanied by a degradation of the material stiffness leading to terms in the global stiffness matrix which are very small or which are equal to zero. Beyond limit points, strength degradation is responsible for terms in the global stiffness matrix which are negative, and thus
responsible for structural instability. Terms which are zero or close to zero cause the
global stiffness matrix to become ill-conditioned. Consequently, an accurate factorization (inversion) of the stiffness matrix is difficult to achieve. The presence of negative
terms in the stiffness matrix may require particular attention in the factorization
algorithms.

b) the material is path dependent, that is, different loading paths produce differ-
ent patterns of damage in the structure. It is therefore possible to envisage situations
whereby different solution methods could lead to solutions to the same problem which
differ from each other (as will be seen in some of the examples in the following chap-
ters).

c) recent research suggests that the underlying mathematical formulation of the
problem is not complete. In most available material models softening of the material
is modelled via a negative stiffness branch in the stress-strain relationship, imitating
the force-displacement diagrams obtained from laboratory tests of material samples.
The implication of a negative stiffness is that the original partial differential equations
describing the problem change character. The problem becomes thus mathematically
ill-posed and a physically realistic solution to the problem may be absent. In finite
element computations this results in a severe dependence on the discretization.

d) another consequence of a mathematically ill-posed material model is the pos-
sibility of bifurcation, whereby multiple equilibrium states are possible. A possible
interaction with the implemented solution method is foreseeable here, with the solu-
tion jumping from one equilibrium state to another at each iteration of the solution
process. In the context of the Newton method, such an interaction is obviously detri-
mental for the entire solution process.

e) materials such as concrete and masonry exhibit an extremely complicated non-
linear behaviour. Each of the various failure modes, namely, separation (cracking),
shear failure and crushing, occurs in its own particular manner, which may or may
not be correlated with other modes. Whilst many models may exist, which reproduce
quite accurately the behaviour of a specific failure mode, it is difficult to develop a
model which takes into account all the possible failure modes. From this point of
view, the problem of infilled r.c. frames poses a particular challenge since the various
failure mechanisms, which could occur in such structures, are triggered by different
modes of failure of the material.

All of the above problems, may lead to results which are not close to the real be-
behaviour of the structure. A recent study on masonry buildings by Bartoli and Blasi[5]
evidences some of the problems and the lack of confidence in the method among en-
genies. Lack of diffusion of the method is also evident from recent conferences on
the subject of earthquake engineering, see for example [2]. Those studies in which
the method has been used for real applications, e.g. Combesecure[15] and Schmidt-
Hurtienne[48], put in evidence the necessity of further investigations, particularly into
material models.

The tendency to use simple structural representations thus persists to the prejudice of the finite element method. However, it must be said that also simpler non-linear structural analysis programs, can suffer from similar problems. The basis for these programs is the displacement method, leading to a similar approach as in the finite element method. When considering non-linear effects, a set of non-linear equations is solved using the Newton method. The same issues as in point (a) above related to the solution of these equations may then arise.

1.3 Objectives and scope of this study

The applicability of the finite element method to the study of infilled r.c. frame structures and masonry structures is undermined by a series of problems related to the solution of systems of equations and the mathematical description of the material. With reference to the solution methods, the main problems occur when reaching limit points in the global structural response and when strong or abrupt localized non-linear effects take place in the structure. At these points in the analysis, the implemented methods can fail to obtain a converged solution. Considering the various methods available for the analysis of non-linear problems, the emerging picture is that there is no general, adaptive and robust method which allows for the solution of the problem without a substantial degree of user intervention, nor is there any consensus as to which method is optimal. This deficiency in finite element non-linear analysis compromises its applicability in the detailed modelling of real structures and consequently undermines the possibility of achieving further advances in research related to structural engineering. Another aspect of importance is related to material modelling of concrete and masonry. Currently available models are specifically geared towards modelling particular modes of failure. With this premise, the aim of this thesis is

- to delineate a general method with which non-linear dynamic and static analysis of structures can be carried out. This method should overcome the problems related to the solution of a set of non linear equations arising at critical points in the analysis, preferably without the intervention of the user. Possibly, such a method should adapt itself automatically according to the degree of non-linearity which arises during an analysis. Since one of the main sources of difficulties in non-linear analysis stems from the necessity of solving a set of non-linear equations, the main focus will be towards methods in which this is avoided.

- to develop a material model for concrete and masonry which can be used in conjunction with the above method, and can be applied to real scale structures.
Such a model should portray some of the basic characteristics of cementitious materials, e.g. softening, opening and closing of cracks, development of permanent deformations and the response under biaxial and triaxial states of stress.

An important point to consider is that the interaction between solution method and material model, determines the performance of a computational analysis. Often, computations of small scale or academic problems will deliver reasonable results, whilst they may fail completely, in the case of detailed large scale problems. In this thesis, the focus lies in finding a combination of a robust solution method and a sound material model, which performs accurately and efficiently for large scale computations. In order to develop such a method and material model, a new program, LARES\(^1\), has been written based on the structure of the non-linear finite element analysis program CAPA-3D[47]. Specific developments are based directly on the theory presented in chapters 3 to 5.

### 1.4 Overview

This thesis is organized in 7 chapters. In chapter 2, results from the static non-linear analyses of typical concrete and masonry structures are discussed. These results have been obtained with DIANA, using the standard available analysis options. The presentation of the results is preceded by a brief discussion on material modelling and the types of solution methods used in finite element (f.e.) programs.

In chapter 3, after a brief overview of elastic damage models and elasto-plastic models, two new types of material models are proposed. The first concerns a combined elastic damage and plasticity model based on a fracture energy regularization approach. The second type of model concerns the application of a special regularization technique, based on a rate of damage law, in several elastic damage type models. These models will be necessary for the analyses presented in subsequent chapters.

The next two chapters investigate the possibility of using alternative types of methods for dynamic and static analysis, in an attempt to delineate a general and robust approach to non-linear analysis. Specifically, chapter 4 deals with methods for dynamic analysis. Here, the method by Newmark is reviewed in its original context, avoiding the necessity of solving for a non-linear set of equations. In an attempt to avoid this also in static analyses, a method based on a dynamic relaxation approach, is developed in chapter 5 for tracing equilibrium paths. The method is applied to the same types of structures of chapter 4.

In chapter 6, the various material models and methods which have been developed are applied to a number of case studies; e.g. the simulation of reinforced con-

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\(^1\)From the name of the ancient Roman deities protectors of the house.
crete frames with masonry infills. The final chapter contains the conclusions for this research project and a discussion of possible future developments.
Chapter 2

Analysis of concrete and masonry structures using standard finite element algorithms

2.1 Introduction

In this chapter, a series of analyses are presented to assess the capabilities of current finite element programs using the standard options they are provided with. The analyses are closely related to problems that arise in the field of earthquake engineering, and concern the simulation of a masonry panel loaded in its plane and the simulation of a reinforced concrete frame with a masonry infill. The simulations have been carried out using DIANA, under static loading conditions using the Newton method. From the analyses it appears that the extent of the results is limited by certain numerical and modelling problems.

Before discussing the numerical results, the nature of the materials which have to be modelled is looked into, as well as the typical numerical procedure which is used to solve non-linear equations.

The potential causes of numerical instability and lack of convergence are discussed.
2.2 Aspects of finite element analysis of concrete and masonry structures

2.2.1 Material modelling

Reinforced concrete

Throughout this thesis, reinforced concrete (r.c.) will be simply modelled using plane stress elements to model the concrete material, and truss elements to model the reinforcement bars. The reinforcement bars are assumed to be perfectly bonded to the concrete, so that nodes belonging to reinforcement elements are thus directly connected to the concrete elements. In reality, the bond between the reinforcement and the concrete is never perfect and a certain amount of bond-slip can occur. The steel reinforcement typically responds elastically up to the yield strength. Beyond this point the steel yields and hardens, until the ultimate tensile strength is subsequently reached. Nevertheless, in the context of this thesis, the response of the reinforcement will be approximated simply using an ideal elasto-plastic model for the steel.

The behaviour of concrete, on the other hand, is much more complicated and requires a more sophisticated model. Concrete responds in typically extremely different ways according to whether the material is subjected to tension, shear or compression. Beyond the ultimate strength the material softens, i.e. loses its strength, see Figure 2.1. This behaviour needs to be modelled as it is responsible for much of the redistribution of forces which occurs in structures (for example the transfer of tensile forces to the steel, due to cracking in the concrete). In general, the behaviour of concrete under compressive states of stress is described as frictional. Due to the softening behaviour, the damage tends to localize in so-called shear bands. On the other hand, concrete under tension simply cracks and separates. The behaviour of the two mechanisms (represented in Figure 2.2) are difficult to correlate, and the characterization of the material behaviour in the case of shear poses particular difficulties and has not been yet fully achieved. Under cyclic loading, the material exhibits both accumulation of permanent deformations and stiffness degradation.

Masonry

Masonry, like concrete, can also be characterized as a frictional material. Unlike concrete, its response is anisotropic, due to the brick-layout and the internal brick structure. Further, the response of the material is affected by the properties of the mortar and joint thickness in relation to the brick properties. Although the masonry is a composite material, modelling it as such for the purposes of a general structural analysis would be excessively demanding from a computational point of view and also in terms of the user's time. Therefore, the material will be dealt with as a
homogeneous material.

More details as to the properties of concrete and masonry are given in chapter 3. The formation of shear bands and cracks in the materials effectively constitutes the formation of a discontinuity in the material, and can be modelled either by introducing discrete elements into the finite element mesh or by implicitly simulating the discontinuity in an appropriate manner within the material model (e.g. smeared crack approach). The latter approach is the preferred one in this case, as it does not require the a-priori determination of the discontinuity. In the cases where the likely position of the discontinuity is known a-priori, use of discrete interface elements will be made (for example to model shear-slip in the interface between adjacent masonry concrete in the infilled r.c. frame examples in this chapter).
2.2.2 The Newton method and derived methods

The analysis of a structure proceeds in a stepwise fashion in which, at each specified load increment, a set of non-linear equations must be solved. In most cases this set of equations is solved using the Newton method, or some variation thereof. The basic problem is to find a solution $x$ to the problem

$$f(x) = 0$$  \hspace{1cm} (2.1)

where $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$. Given some starting point $x_c$ a model (the affine model) of the system is set up (Dennis[21]):

$$M_c(x_c + \Delta x) = f(x_c) + J(x_c) \Delta x$$  \hspace{1cm} (2.2)

where $J(x)$ is the Jacobian of $f$ at $x$. The above is simply an approximation for the real behaviour of the system at $x_c$, given a small perturbation $\delta x$:

$$f(x_c + \Delta x) = f(x_c) + \int_{x_c}^{x_c + \Delta x} J(z) \, dz$$  \hspace{1cm} (2.3)

The same approximation as in eq. 2.2 can be obtained by expanding the functional $f$ into a Taylor series about the point $x_c$ and leaving out higher order terms:

$$f(x_c + \Delta x) \approx f(x_c) + \frac{\partial f(x_c)}{\partial x} \Delta x$$  \hspace{1cm} (2.4)
A new approximation to the solution of the problem is found by solving for a value of \( \delta x \) which makes \( M_c (x_c + \Delta x) = 0 \) and adding this to \( x_c \).

Carrying out several times this operation, the Newton method is obtained, which can be summarized as follows:

\[
J \left( x^{(j)} \right) \Delta x^{(j+1)} = -f \left( x^{(j)} \right) \tag{2.5}
\]

\[
x^{(j+1)} = x^{(j)} + \Delta x^{(j+1)} \tag{2.6}
\]

where \( j \) is an iteration counter and \( x^{(0)} \) is the initial guess for the solution.

**Implementation in structural analysis problems**

At each step \( i \) of the analysis force equilibrium must be satisfied at each node of the finite element mesh, i.e. the sum of the external forces, \( S \), and internal forces, \( R \), must be equal to zero:

\[
S_i - R_i = 0 \tag{2.7}
\]

In general, \( S \) represents the sum of inertial forces and externally applied loads, \( P \) and \( R \) represent all forces inherent to the structural response. For a static geometrically linear analysis, \( S \) is equal to the externally applied load \( P \), and assuming \( R \) is only a function of nodal displacements, \( u \), the problem can be translated to that of finding a solution, \( u \), such that the residual forces in the system are equal to zero:

\[
r_i(u) = P_i - R(u) = 0 \tag{2.8}
\]

Applying the Newton method to the above problem, the following algorithm is obtained for a given step, \( i \), and iteration \( j \):

\[
\delta u_i^{(j+1)} = \left[ K_i^{(j)} \right]^{-1} r_i^{(j)} \tag{2.9}
\]

with

\[
K_i^{(j)} = \frac{\partial R(u_i^{(j)})}{\partial u} \tag{2.10}
\]

\[
r_i^{(j)} = P_i - R(u_i^{(j)}) \tag{2.11}
\]

\[
u_i^{(j+1)} = u_i^{(j)} + \delta u_i^{(j+1)} \tag{2.12}
\]

\[
u_i^{(0)} = u_{i-1} \tag{2.13}
\]

The procedure for a single iteration is summarized in the following expression:

\[
u_i^{(j+1)} = \left[ K_i^{(j)} \right]^{-1} \left[ P_i - R(u_i^{(j)}) \right] + u_i^{(j)} \tag{2.14}
\]
The procedure is initiated with \( u_i^{(0)} = u_{i-1} \), i.e. the value of the solution at the last step. Once the residual is sufficiently small, (e.g. when its norm is smaller than a prescribed value) the iterative procedure for the current step can be stopped, and the analysis can proceed with the next step. The method requires updating the stiffness matrix and calculating its inverse at every iteration. This operation, however, does not necessarily have to be made at every iteration in the procedure and may be made at the start of each step or even only once at the start of the entire analysis, giving rise to the methods otherwise known as modified Newton method and initial stiffness method, respectively.

**Displacement control**

If besides external loads displacements are prescribed at some nodes, then eq. 2.8 becomes

\[
   r_i(u) = P_i - R(u, p_i) = 0
\]  

(2.15)

where \( p_i \) is the set of prescribed displacements for step \( i \).

**Arc-length method**

In a typical analysis where instability plays a role, a load controlled or displacement controlled analysis may not be capable of tracing the complete load displacement path. In order to overcome this problem, Riks[45] developed a method, known as the arc length method, in which the magnitude of the externally applied load can be adapted according to the traced equilibrium path. The load is controlled via a multiplication factor, \( \lambda \), and an extra equation (the constraint or parameterization equation) is added to the equilibrium equations:

\[
   r_i(u) = \lambda P - R(u) = 0
\]

(2.16a)

\[
   f_i(u, \lambda) = 0
\]

(2.16b)

The parameterization equation can take several forms, some examples are the Spherical arc-length constraint,

\[
   (\Delta u_i^{(1)})^T \Delta u_i^{(j)} + [\Delta \lambda_i^{(j)}]^2 \beta = \Delta l^2
\]  

(2.17)

and the linearized constraint:

\[
   (\Delta u_i^{(1)})^T \Delta u_i^{(j)} = \Delta l^2
\]  

(2.18)

where \( \Delta u_i^{(j)} = u_i^{(j)} - u_i^{(0)} \), \( \Delta \lambda_i^{(j)} = \lambda_i^{(j)} - \lambda_i^{(0)} \) and \( \beta \) and \( \Delta l \) are two parameters. The constraint equation can be thought of as a condition that limits the total magnitude of incremental displacements of the structure during a single step.
Again, this set of equations is solved for \( u \) and \( \lambda \), using the Newton method. A disadvantage of this approach is that the Jacobian of this system loses its bandedness and symmetry. In order to avoid this, Crisfield\[17\] developed a method to solve for the parameterization equation directly. The equation of equilibrium is written as

\[
\mathbf{r}_i^{(j)} = \Delta \lambda_i^{(j)} \mathbf{P} + \lambda_{i-1} \mathbf{P} - \mathbf{R}(\mathbf{u}_i^{(j)})
\]  

(2.19)

and the Newton iteration is expressed as

\[
\delta \mathbf{u}_i^{(j+1)} = \Delta \lambda_i^{(j)} \left[ \mathbf{K}_i^{(j)} \right]^{-1} \mathbf{P} + \left[ \mathbf{K}_i^{(j)} \right]^{-1} \left[ \lambda_{i-1} \mathbf{P} - \mathbf{R}(\mathbf{u}_i^{(j)}) \right] \\
= \Delta \lambda_i^{(j)} \mathbf{a}_i^{(j+1)} + \mathbf{b}_i^{(j+1)}
\]  

(2.20)

The iterative displacement can then be thought of as consisting of two parts: one part associated with the difference between the external load at the end of the previous step and the current internal load vector and the other part associated purely with the load component being applied in this step. To derive an expression for the incremental load factor, \( \Delta \lambda_i^{(j)} \), consider eq. (2.18) for two consecutive iterations, \( i \) and \( i-1 \), and eliminate \( \Delta \mathbf{P}^2 \):

\[
\left( \Delta \mathbf{u}_i^{(1)} \right)^T \delta \mathbf{u}_i^{(j)} = 0
\]  

(2.21)

Substitution of eq. (2.20) into the above leads to

\[
\left( \Delta \mathbf{u}_i^{(1)} \right)^T \left( \Delta \lambda_i^{(j)} \mathbf{a}_i^{(j+1)} + \mathbf{b}_i^{(j+1)} \right) = 0
\]  

(2.22)

so that

\[
\Delta \lambda_i^{(j)} = -\frac{\left( \Delta \mathbf{u}_i^{(1)} \right)^T \mathbf{b}_i^{(j)}}{\left( \Delta \mathbf{u}_i^{(1)} \right)^T \mathbf{a}_i^{(j)}} \\
= -\frac{\left( \Delta \mathbf{u}_i^{(1)} \right)^T \left[ \mathbf{K}_i^{(j)} \right]^{-1} \left[ \lambda_{i-1} \mathbf{P} - \mathbf{R}(\mathbf{u}_i^{(j)}) \right]}{\left( \Delta \mathbf{u}_i^{(1)} \right)^T \left[ \mathbf{K}_i^{(j)} \right]^{-1} \hat{\mathbf{P}}}
\]  

(2.23)

For the first iteration the internal force vector \( \mathbf{R}(\mathbf{u}_i^{(0)}) \) that appears in the above equation is simply equal to the external load vector at the end of the last step (assuming convergence has been achieved). The first iterative displacement is therefore given by

\[
\delta \mathbf{u}_i^{(1)} = \Delta \mathbf{u}_i^{(1)} = \Delta \lambda_i^{(1)} \left[ \mathbf{K}_i^{(j)} \right]^{-1} \mathbf{P}
\]  

(2.24)

In the above equation, the constraint, \( \Delta \lambda_i^{(1)} \), still needs to be specified in order to evaluate the first iterative displacement. However, this can be done in an indirect way by specifying a user supplied load increment, \( \Delta \mathbf{P} \), from which \( \delta \mathbf{u}_i^{(1)} \) can be calculated directly. The iterative mechanism can be visualized by considering eq. (2.23). If the internal load vector is larger than external load vector at the end of the previous step, then the load factor \( \Delta \lambda_i \) for the current iteration will be positive. Otherwise if for
some reason (e.g. cracking, instability) the internal load vector decreases with respect to the external load in the previous step, then \( \Delta \lambda \) will be negative, indicating that load must be removed. The load factor \( \Delta \lambda \) should be seen as a factor that scales the load according to the ability of structure to carry the load.

### 2.3 Case studies using DIANA

#### 2.3.1 Analysis of masonry panels

These analyses refer to a masonry panel 2 m high by 3.6 m wide, with a thickness of 0.3 m, see Figure 2.3. The panel is fixed at the bottom and is bounded at the top by a rigid steel beam. A mesh of 8 by 14 quadrilateral 8 noded plane stress elements has been used. The vertical load is applied as a concentrated load in the middle of the rigid beam.

Two types of analyses have been executed, one using the full Newton method with arc-length (load controlled), and the other using the Newton method with constant initial stiffness and displacement control of the rigid beam. Further, two types of material models have been used. The first type is a multiple fixed crack model, Rots\cite{46}, and the second type is a rotating crack model based on the Rankine tension cut-off model, Feenstra\cite{22} (see also \cite{20}). A Mohr-Coulomb model is implemented in both cases to limit the maximum compressive stresses in the masonry, however, without softening effects. Although the simulation concerns a masonry panel, both material models are initially isotropic.

![Figure 2.3: Masonry panel geometry and loading](image)

The masonry panel has a stiffness \( E = 5000 \) MPa, a compressive strength of 4 MPa, a fracture energy (tension) \( G_f = 20 \) J/m\(^2\) and a masonry tensile strength of 0.1 MPa. For each analysis type and material mode, four cases have been considered.
using vertical loads of 0, 400, 800 and 1200 kN.

Results Figure 2.4 shows the load-displacement diagrams for the 16 different analyses. Clear differences in the peak loads obtained can be seen depending on the type of material model being used and the type of solution method used. Note that the points at which the curves end indicate either divergence in the iteration process or failure to converge within the maximum number of specified iterations. It is likely that these points coincide with a limit in the structural response, although this can be asserted with more certainty once the complete response is known.

The panels with a vertical constant load present cracks along the base and diagonally through the panel patterns, see Figure 2.5. It is interesting to note that the diagonal crack is more marked in the panel with higher vertical load (see the panel with 400 kN vertical load compared to the panel with 1200 kN vertical load). In the analyses obtained with the full Newton method with arc-length control, cracks appear localized in narrow bands. On the other hand, the analyses using the initial stiffness method show more "diffuse" patterns of cracking, i.e. the cracking is less localized. The difference evidences a typical slower rate of convergence of the initial stiffness method with respect to the full Newon method. It is interesting to note that in the analyses with the initial stiffness method, the obtained maximum loads are consistently higher than those obtained with the arc-length method. Although the initial stiffness method presents these problems, it seems to be less sensitive, which can be seen from the fact that the curves obtained from it extend further. On a final note, the maximum calculated response differs substantially depending on the material model used. In some cases, the response obtained using the rotating crack model is about 50 % higher than the response using the fixed crack model in an equivalent analysis.

2.3.2 Analysis of infilled reinforced concrete frames

The design of the frame that is analyzed here has been taken from the reports of experimental work on scaled frames at the University of Thessaloniki by Stylianidis[53] and is depicted in Figure 2.6. In these experiments a 1/3-scale model of the bottom storey of a 4-storey frame was tested. Lateral load is applied at the beam level. In the numerical analyses the entire frame has been modelled, however, non-linear material behaviour has been specified only for the bottom storey. Given this assumption, the top three storeys are modelled using standard beam elements, whereas the bottom storey is modelled using 2D quadrilateral 8 noded elements. The span of the frame is 4.77 m, and the storey height is 3.18 m. Out of plane effects are not considered. Specific details as to member dimensions, reinforcement details and material properties can be found in Galanti[24].
Figure 2.4: Load-displacement diagrams of masonry panel analyses: (a) and (c) using a fixed crack model; (b) and (d) using a rotating crack model.
Figure 2.5: Deformations and principal crack strains for 3 different analyses using a rotating crack model.
In the analyses the frame was subjected to a concentrated horizontal load applied at the first storey level. Two alternative analyses have been carried out, one for the bare frame, and one for the frame with an infill in the first storey only. In the infilled case, the connection between wall and frame is modelled using interface elements capable of representing a discrete crack. An ideal elastic perfectly plastic model with yielding being controlled by a von Mises criterion is specified for the steel of reinforcement, whereas the concrete is modelled using an ideal elasto-plastic behaviour in compression via a Mohr-Coulomb yield criterion, and a Rankine cutoff criterion for tension. The masonry infill is also modelled as a frictional material using a Mohr-Coulomb yield criterion and a tension cut-off criterion. A discrete crack initiation criterion is specified for the simulation of the fracturing behaviour of the interface elements between wall and frame.

Results The results for both analyses have been obtained using a linearized arclength method (eq. 2.18), see Figure 2.7 for the corresponding load-displacement diagrams. The graphs show the substantial effect of the masonry infill in adding strength and stiffness to the structure. Again, the ends of the curves do not represent failure of the structure but rather, failure of the solution method to continue the analysis. Figure 2.8 shows two different stages of deformation for the infilled frame.
Figure 2.7: Load displacement diagrams for bare r.c. frame and infilled r.c. frame.

The results show the formation of horizontal cracks at the base and the top of the masonry infill, separation of the masonry from the r.c. frame, and the formation of a diagonal crack in the panel.

2.4 Some remarks on the performance of the Newton method

The Newton method is essentially a procedure which comprises in a series of consistent linearizations of the non-linear problem. Clearly the speed of convergence will be dependent on the accuracy of these linearizations. In the extreme case, the convergence of the entire process may be undermined. Another important aspect is that the solution of the linear problem requires the factorization of the Jacobian. These and other aspects related to the Newton method are discussed below.

2.4.1 System condition

The factorization technique is an important aspect in numerical analysis. A low machine accuracy combined with a system which is not well-conditioned may lead to large inaccuracies of the delta-incremental displacement of eq. 2.9. Consider a small perturbation $\Delta K$ in the Newton iteration of eq. 2.9 with perturbed solution $\delta u + \eta$:

$$\delta u_i^{(j+1)} + \eta = \left[K_i^{(j)} + \Delta K\right]^{-1} r_i^{(j)} \quad (2.25)$$
Figure 2.8: Displacements and principal crack strains in infilled frame for two different stages of loading: (a) before peak load of 400 kN; (b) at the end of the run.
Leaving out the various indices, it can be shown that the error in the perturbed solution obeys (see for example Noble[36]):

\[
\frac{\|\eta\|}{\|\delta u\|} \leq M \cdot c(K) \frac{\|\Delta K\|}{\|K\|}
\]  

(2.26)

where \(M\) is some constant and \(c(K)\) is the condition number of the stiffness \(K\). The above equation shows that the error in the iterative procedure may be extremely sensitive to perturbations of the current stiffness matrix, when this matrix is ill-conditioned. This problem evidences the necessity of an accurate factorization technique when dealing with such situations.

### 2.4.2 Quadratic convergence characteristics

It can be shown that the Newton method possesses a quadratic convergence characteristic. Mathematically this means that at every iteration the new solution \(u^{(j+1)}\) obeys the following condition:

\[
\left\| \hat{u} - u^{(j+1)} \right\| \leq c \left\| \hat{u} - u^{(j)} \right\|^2
\]  

(2.27)

where \(c\) is a constant and \(\hat{u}\) is the exact solution to the problem. Although the Newton method possesses this important characteristic, it is only locally convergent. This means that if the initial guess is too far from the exact solution, it will fail to converge at all.

It is important to note here, that the quadratic convergence is achieved only if the derivative of the function being solved for, accurately represents the behaviour of the system. In non-linear structural analysis, this can be achieved via consistent tangent matrices, more on which will follow in 2.4.4.

### 2.4.3 Convergence characteristics of initial stiffness method

In the initial stiffness method the initial elastic stiffness of the system, \(K_0\), is used instead of the updated stiffness matrix. The Newton iteration as represented by eq. 2.9 becomes:

\[
\delta u_i^{(j+1)} = K_0^{-1} r_i^{(j)} = K_0^{-1} \left[ P - R \left( u_i^{(j)} \right) \right]
\]  

(2.28)

The advantage in using the same stiffness matrix at every iteration is that its factorization only has to be carried out once. Assuming that over a particular range the response is linear with stiffness \(K_s \neq K_0\), i.e. \(R(u) = R_x + K_s (u - u_x)\) where \(R_x\) and \(u_x\) are constant, then eq.2.28 can be written as:

\[
\delta u_i^{(j+1)} = K_0^{-1} \left[ P - R_x - K_s \left( u_i^{(j)} - u_x \right) \right]
\]  

(2.29a)

\[
\delta u_i^{(j+1)} = K_0^{-1} \left[ P - R_x - K_s \left( u_i^{(j-1)} + \delta u_i^{(j)} - u_x \right) \right]
\]  

(2.29b)
or
\[ \delta u_i^{(j+1)} = K_0^{-1} \left[ P - R_x - K_s \left( u_i^{(j-1)} - u_x \right) \right] - K_0^{-1} K_s \delta u_i^{(j)} \] (2.30)

From eq. 2.29a, the first term on the right hand side is equal to \( \delta u_i^{(j)} \), therefore
\[ \delta u_i^{(j+1)} = (I - K_0^{-1} K_s) \delta u_i^{(j)} = A \delta u_i^{(j)} \] (2.31)

\( K_0 \) can be expressed in terms of its eigenvalues \( \lambda_{0;1}, \lambda_{0;2}, \ldots \) and eigenvectors \( \phi_1, \phi_2, \ldots \) as
\[ K_0 = \Phi \Lambda_0 \Phi^T \], where \( \Phi = [\phi_1, \phi_2, \ldots] \) and \( \Lambda_0 \) is a matrix containing the eigenvectors \( \lambda_{0;1}, \lambda_{0;2}, \ldots \) along the diagonal. Assuming that \( K_s \) has eigenvectors coinciding with those of \( K_0 \), so that \( K_s = \Phi \Lambda_s \Phi^T \), then eq. 2.31 can be re-written in terms of generalized coordinates as
\[ \delta u_i^{(j+1)} = \Phi^T (I - K_0^{-1} K_s) \Phi \delta u_i^{(j)} \] (2.32)
\[ = (I - \Lambda_0^{-1} \Lambda_s) \delta u_i^{(j)} = A^* \delta u_i^{(j)} \] (2.33)

where \( u^* = \Phi^T u \) and
\[ A^* = \begin{bmatrix} 1 - \mu_1 \\ & 1 - \mu_2 \\ & & \ddots \\ & & & 1 - \mu_n \end{bmatrix}, \mu_k = \frac{\lambda_{s;i}}{\lambda_{0;i}} \] (2.34)

The spectral radius of \( A^* \), \( \rho(A^*) = \rho(A) = \max_k |1 - \mu_k| \). If for any of the diagonal coefficients \( 1 - \mu_k \) > 1, then the method diverges. Hence for convergence to be guaranteed \( 0 < \frac{\lambda_{s;i}}{\lambda_{0;i}} < 2 \). If this is the case, the individual eigenvalue ratios \( \mu_k \) determine the rate at which the displacements corresponding to mode 'k' converge to the exact solution. By expressing eq. 2.31 in terms of previous iterative changes as follows
\[ \delta u_i^{(j+1)} = A^* \delta u_i^{(j)} = A^* \left( A^* \delta u_i^{(j-1)} \right) \]
\[ = (A^*)^j \delta u_i^{(1)} \]
it can be seen that the rate of convergence of the method depends on the smallest absolute value of \( \mu_k \) and is equal to \( 1 - \mu_k \). If one or more of modes exist with small eigenvalues with respect to the corresponding eigenvalues of the initial stiffness matrix, \( K_0 \), then the rate of convergence will be slow. In the limit, if \( K_s \) has a zero eigenvalue then the corresponding eigenmode is never eliminated. In a non-linear problem characterized by local loss of stiffness, slow convergence is typical of the method and the errors that can be introduced by not carrying out a sufficient number of iterations are accumulated over each load increment. Finally, it should be noted that the initial stiffness method, unlike the full Newton method, is only linearly convergent. This will be true also for the modified Newton method whereby the stiffness is updated once only at the start of each step.
2.4.4 Path-dependency

In classical mathematics text books, the Newton method is developed for non-linear functions representing one-to-one mappings. The non-linear structural problems treated here deal with non-conservative systems, implying that energy is lost and damage is accumulated in locations determined by the history of the stress distribution in the structure. Because of this behaviour, it is possible that the solution to the non-linear problem is not unique.

A necessary condition for retaining the quadratic convergence characteristics of the full Newton method is to work with linear approximations which represent as precisely as possible the behaviour of the system. To do this, the stiffness matrix is assembled from the element stiffness matrices, which in turn are obtained from the slope of the stress-strain relationships of the material at integration point level. This requires that the material algorithms not only return the response of the material to a given state of strain, but also the derivative of the response (the so-called tangent stiffness) of the material. In fact, it has been shown that the optimal stiffness matrix not only depends on the derivative of the stress-strain relationship, but may also be intricately related to the algorithms which are used in the material subroutines and to the global solution method used, Simo and Taylor[50]. Stiffness matrices which are adjusted to the characteristics of the algorithms are referred to as consistent tangent stiffness matrices.

Setting up stiffness matrices can be a rather complicated task; the mathematical derivation of the tangent stiffness and its implementation may be difficult to achieve and in some cases it may not even be possible; the resulting stiffness matrices may not be symmetrical, in which case the factorization algorithms need to be adjusted to take this into account.

Given the complexity of the mathematical derivation and implementation of tangent stiffness matrices, approximations may be introduced in the computation of tangent stiffness matrices. This leads to a reduction of the effectiveness of the full Newton procedure, in the sense that the quadratic convergence can be lost, and may even compromise the convergence of the method. The risk of divergence can be even more present when considering the possibility of an ill-conditioned system and the local convergence characteristics of the method. In addition, the possibility of a non-unique solution could lead to a dependency between the response of the system and the iterative solution method.

The use of the arc-length control as represented by eq. 2.23 cannot represent by itself a solution to these various issues, as the problem of factorization of the stiffness matrix is still present.
2.5 Conclusions

In this chapter, results from a number of finite element analyses of masonry panels and infilled reinforced concrete have been presented. Although the results contain useful information as to the behaviour of a structure in terms of its response and potential failure mechanisms, several problems have emerged, related to the convergence of the solution method used in the non-linear problem. These problems undermine the successful application of finite element programs in research related work and can be summarized as follows:

- Numerical failure of the solution methods leads to a premature termination of the analysis. Of all the analyses presented here, none terminated normally. This was due to divergence or lack of convergence of the iterative procedure.

- The results are sensitive to the solution method used in the analysis. In the wall analyses, the maximum load level reached was dependent on the solution procedure used.

- The implemented material models can affect the performance of the solution method and can also have a considerable influence on the maximum calculated response. Again, this is visible from the wall analyses: the rotating crack model leads to maximum calculated responses which are consistently higher (approx. 50%) than those obtained with the multiple fixed crack model. Further, premature termination of the analyses, whether obtained with the full Newton method in combination with arc-length control or with the initial stiffness method with displacement control, occurs later.

- A considerable amount of time may be required in adjusting analysis options in order to obtain reasonable results. Although the preparation of the input may take some time, more time can be spent on interpreting causes of numerical failure and trying to adjust the analyses in order to overcome critical points. In some cases, only a trial and error based technique can be used to continue the non linear analysis. Quoting Belytschko[8]: FEM-performance of nonlinear problems with discontinuities in time and space and geometrical and/or material instabilities leads to 'computer games'. This time consuming interaction with the program makes the use of non-linear finite elements relatively inefficient and unattractive for general use at engineering practice level.

The failure of the numerical methods is essentially a consequence of the material degradation (e.g. the formation of cracks) at various locations in the structure and the apparent attainment of a limit point in the load-displacement curve. In the present analyses, the failure derives effectively from the loss of stiffness both at the local
and the structural level. In the full Newton method, failure can be caused by ill-conditioning of the stiffness matrix, possibly leading to a divergence of the method. In the initial stiffness method it is due to a lack of compatibility between the initial stiffness and the actual response of the system, whereby convergence is extremely slow. On top of this, the initial stiffness method can suffer from a greater accumulation of errors, which may lead to a difference in the development of damage in the structure (see the more localized solution for the masonry panel in Figure 2.5 using the full Newton method).

A systematic study of the various problems associated with available solution procedures and their appropriateness for non-linear structural analysis is currently not available in the literature. Although the methods are treated thoroughly in mathematics textbooks, their interrelationship with the structural problem is not, and often they are used by the structural analyst as 'black boxes'. The issue is made clear from the fact that presently a robust, automatic method for tracing equilibrium branches does not exist, Belytschko[9].

It was seen that the performance of the solution methods can be affected by the implemented material models. Before dwelling on methods for structural analysis, a number of material models for concrete and masonry will be developed in the following chapter.
Chapter 3

Material modelling of fracturing materials: concrete and masonry

An important aspect of non-linear structural finite element analysis is related to finding an appropriate mathematical description for the behaviour of the material. In this chapter, two types of material models are proposed which will be used in the analyses of later chapters. The first type is based on a combination of two classical material models which are often used for simulating cracking and crushing, namely elastic damage or rotating crack models, and plasticity models. With this combined model, crack closure as well as biaxial and triaxial states of stress can be correctly modelled. The second type of model is an enhanced elastic damage model in which a strain rate dependency has been introduced. It will be seen that this model does not lead to mesh dependency as is encountered in the first type of model. The models have been implemented in LARES for 2-dimensional analysis.

3.1 Introduction

Concrete and masonry are composite materials exhibiting intricate mechanical properties, which depend on the characteristics, the proportions and the arrangement of the constituent materials. In literature, many mathematical models for concrete and masonry can be found which attempt to reproduce the response of these materials under any given condition. The reader is referred to Chen[13] for an exhaustive selection of models. The models can be distinguished in their application in the finite element method (e.g. smeared vs. discrete cracks), in the nature of the constitutive
relationships (e.g. elastic damage models vs. plasticity based models, strain based vs. stress based formulations), and in the particular material response characteristic they focus on. Many models are commonly based on the theory of plasticity because of its ability to simulate several material states (e.g. frictional behaviour, hardening and softening, dilation). In concrete or masonry, two basic modes of failure can be distinguished, namely the separation mode (cracking) and the sliding mode. The failure of such materials is characterized by hardening and softening accompanied by pronounced dilation in the case of compression. Although elasto-plastic models can be used to simulate all these states, a major drawback lies in the approximate modelling of the pure cracking mode. Due to the accumulation of unrecoverable inelastic strains, open cracks lack the capability to close. This is acceptable if the crack strains remain relatively limited, but in the case of cyclic or dynamic simulations the accumulation of unrecoverable crack strains may not be desirable. An elastic damage model, in which the stiffness of the material decreases with increasing damage is more appropriate. In order to retain this characteristic and also that of plastic strains in the crushing mode, a model is proposed which combines elastic damage in tension with plasticity in compression. Some work along the lines of combined damage and plasticity, although different from the approach that is used here, is reported in Yazdani et al[62].

In the previous chapter it was mentioned that failure of the numerical methods used in the analysis may be caused by ill-posedness in the mathematical formulation of the problem. In order to provide insight into this issue and to experiment with possible solutions, a second type of model has been developed in which a form of rate dependency has been introduced.

Following a brief description of the observed behaviour of concrete and masonry, a series of elastic damage models (section 3.2) and elasto-plastic models (section 3.3) are reviewed. The combined model, discussed in section 3.4, is obtained by coupling the algorithms for elastic damage directly into the return mapping algorithm used in the plasticity models. The rate dependent model, which is based only on the elastic damage model, is discussed in section 3.5. In order to verify the correct response of the material models in biaxial states of stress, a verification of the shear behaviour of the models is provided in 3.6. Further, in section 3.7 some suggestions are made for modelling orthotropic materials such as masonry. The various models have been implemented in LARES and their performance is verified using a material model driver, which has been specifically developed for this purpose.

3.1.1 Observed behaviour

Concrete

In uniaxial compression, concrete exhibits a linear response up to about 30% of its peak strength, at which point the slope of the load-deformation diagram (see Fig-
Figure 3.1: Typical response of concrete in uniaxial compression.

Figure 3.1) starts to decrease. Beyond the peak strength, under continued deformation, the measured stress decreases giving rise to a so-called softening branch in the load-deformation curve. In the initial linear branch small lateral deformations can be detected, with the Poisson ratio ranging between 0.15 and 0.20. At the peak load, the lateral deformations are such that there is no volumetric change in the material, and within the softening branch the lateral deformations can reach values up to 4 to 5 times the axial deformation. The strength of concrete in compression is usually given in terms of its 28 day characteristic cylinder strength, $f_{ck}$, which has typical values between 15 MPa to 75 MPa. From this value, other material parameters can be deduced, see Table 3.1. Since deformations localize during failure in compression, the post-peak response needs to be characterized by a fracture energy in compression, $G_c$, which needs to be determined from experimental curves. Typical stress-strain curves of concrete under uniaxial compression are shown in Figure 3.2a. In uniaxial tension, concrete is characterized by a relatively brittle response and a much lower strength than in compression, see Figure 3.2b. Once the peak strength is attained following a linear branch, localized cracking of the material takes place. Again, material degradation is characterized by a fracture energy in tension, $G_t$, typically ranging between 100 and 120 J/m². Concrete also exhibits a complex response in multi-axial states of stress. The strength of concrete in these states of stress can be shown using failure contours. Typically in a biaxial compression state of stress, the measured strength can be 15 to 20% higher than the uniaxial strength, see Figure 3.3. For a broader treatment of the subject of concrete refer to van Mier[56].
Figure 3.2: Experimentally observed stress-strain curves for different concrete qualities: (a) in uniaxial compression (Wischers[59]); and (b) in uniaxial tension (Hughes and Chapman[27]).

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean compressive strength(^1)</td>
<td>( f_{cm} = f_{ck} + 4 )</td>
</tr>
<tr>
<td>Mean tensile strength(^2)</td>
<td>( f_{ctm} = 0.3(f_{ck})^{\frac{2}{3}} )</td>
</tr>
<tr>
<td>Secant modulus of elasticity(^2)</td>
<td>( E_{cm} = 9500(f_{ck} + 8)^{\frac{2}{3}} )</td>
</tr>
<tr>
<td>Poisson ratio(^2)</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Table 3.1: Mechanical properties of concrete as a function of the 28 day characteristic (5% fractile) cylinder strength, \( f_{ck} \), in MPa.
Figure 3.3: Biaxial strength envelope of concrete, after Kupfer et al.[30].

Masonry

Masonry consists of clay or concrete bricks bonded together with mortar. Typical compressive strengths can range from 5 to 30 MPa for mortar, from 20 to 100 MPa for clay bricks and from 10 to 35 MPa for concrete blocks. The compressive strength of masonry is a function of the individual strengths of the mortar and the bricks, the thickness of the joints and the Young's modulus of the mortar and the bricks. Whilst the strength of the masonry is usually lower than that of the individual brick strength, it may significantly exceed the mortar strength. A number of investigations reported by Hendry[25] show that the compressive strength of masonry can be related to that of the brick units, \( f_u \), and the mortar, \( f_m \), using a relationship of the type

\[ f_c \approx (f_u)^{\frac{2}{3}} (f_m)^{\frac{1}{3}} \]

Reducing the joint thickness has the effect of increasing the strength of the masonry. Two types of failure modes can be observed in compression, depending on the relative stiffness of the joint. A tensile splitting mode with cracks parallel to the direction of loading takes place when the joint material is deformable and relatively thick. If the joint material is stiff and the joint made thin, a shear type of failure along a diagonal line of weakness occurs. The stiffness of masonry is a function of the compressive

---

\(^1\)Based on a normal distribution of the strength, and a standard deviation \( \sigma = 2.5 \text{ MPa} \).

\(^2\)According to Eurocode 2.
strength and can be expressed as

\[ E_m = k f_c \]

where \( f_c \) is the compressive strength of masonry and \( k \) is a coefficient ranging between 500 and 1500. The compressive strength normally relates to the one measured in the direction perpendicular to the bed joints; however, the strength in the direction parallel to the joints can be substantially lower.

### 3.1.2 Verification of models

For an initial verification of the performance of the models a material model driver has been developed for LARES, allowing testing at the level of a single integration point rather than performing a complete non-linear structural analysis. In order to allow prescription of stresses rather than only strains, the driver performs a small non-linear analysis in the same way as would take place in the case of a structure. Rather than working with forces and displacements, the driver uses the six stress components and the six strain components representing the state of the integration point. Specific stresses and strain components can be prescribed and, for each step, equilibrium iterations are performed using the initial stiffness method.

### 3.2 Elastic damage models

#### 3.2.1 A 3D rotating elastic damage model

In the following, a rotating damage model is presented, similar to the type described in Crisfield[18] and Cervera et al.[12]. The model is based on the idea of an isotropic damage variable which increases according to a specific relationship with the value of the largest principal strain. The formulation presented here is geared towards the implementation of the model. A more rigorous approach, including thermodynamic considerations can be found in Yagamuchi[61].

The model includes effects of damage both due to tension and due to compression. The two damage effects are modelled as independent phenomena, which will be controlled by two single parameters \( \omega_t \in [0,1] \) and \( \omega_c \in [0,1] \), which represent the amount of damage undergone in the two different modes, respectively. These variables take values ranging from 0 (no damage) to 1, which is the point at which the material has lost all its load bearing capacity. The main characteristic of the model is a reduction of the secant stiffness of the material, which in the original derivations is introduced via a formula of the type:

\[ \sigma = (1 - \omega) E \varepsilon = (1 - \omega) \sigma_t \quad (3.1) \]
where $\omega$ is the scalar damage parameter, $E$ is a constant representing Young’s modulus and $\sigma_t$ can be considered as an effective stress, i.e. the stress over a section of intact material which effectively resists forces, see Lemaitre and Chaboche[32].

The approach used here will be slightly different but essentially leads to the same kind of model. Throughout the text, the factor $(1 - \omega)$ of eq. 3.1 will be substituted with the stiffness reduction factor, $d$, which in general will not be proportional to $\omega$, but will be a function thereof, i.e. $d = d(\omega)$. The model is based on the concept of equivalent uniaxial strains, see for example Feenstra [23]. The equivalent uniaxial strains are simply defined as the principal effective stresses divided by Young’s modulus, $E$:

$$
\varepsilon' = \begin{bmatrix} \varepsilon'_1 \\ \varepsilon'_2 \\ \varepsilon'_3 \end{bmatrix} = \frac{1}{E} \begin{bmatrix} \sigma_{1,t} \\ \sigma_{2,t} \\ \sigma_{3,t} \end{bmatrix} = \frac{\sigma_t}{E} \tag{3.2}
$$

where the principal effective stresses, $\sigma_t$, are determined from:

$$
\sigma_t = D\varepsilon
$$

in which $D$ is the elastic stiffness matrix for the principal stress case and $\varepsilon$ is the principal strain vector. Each of the uniaxial strain components represents the strain that a specimen undergoes when subjected solely to the corresponding stress component. Computation of the nominal stresses occurs always along the principal strain directions (which are identical to the principal stress and the equivalent uniaxial strain directions) and the material undergoes equal degradation in all directions. Throughout this section and in later sections use of the equivalent uniaxial strain concept will be made. The uniaxial strain can be thought of as a strain measure which accounts for the effects of lateral expansion. This can be seen from the following expressions for the strain and the uniaxial strain, respectively, in terms of the deviatoric strain and volumetric strain, in the 3-dimensional case:

$$
\varepsilon = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \varepsilon_v
$$

$$
\varepsilon' = \frac{1}{1 + \nu} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \frac{1}{3(1 - 2\nu)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \varepsilon_v
$$

The introduction of the equivalent uniaxial strain concept is necessary in that it allows each nominal stress component to be expressed as a function only of its corresponding equivalent uniaxial strain component. Consequently, the evaluation of the nominal
stress reduces to three decoupled uniaxial stress calculations even in the presence of lateral expansion effects.

In order to determine how damage evolves in the material, the largest and smallest of the uniaxial strains are determined as follows:

\[
\begin{align*}
\varepsilon_{\text{max}} &= \max (\varepsilon'_1, \varepsilon'_2, \varepsilon'_3) \\
\varepsilon_{\text{min}} &= \min (\varepsilon'_1, \varepsilon'_2, \varepsilon'_3)
\end{align*}
\] (3.3) (3.4)

Using these values, the two damage parameters, \(\omega_t\) and \(\omega_c\) are updated according to the relationships which will follow below. Introducing two stiffness reduction factors, \(d_t\) and \(d_c\) which are, respectively, functions of the two independent damage parameters, the evaluation of the effective principal stresses, \(\sigma\), proceeds using the following equation:

\[
\sigma = E \left\{ d_t \left[ \frac{\max (\varepsilon_1, 0)}{\max (\varepsilon_2, 0)} \right] + d_c \left[ \frac{\min (\varepsilon_1, 0)}{\min (\varepsilon_2, 0)} \right] \right\}
\] (3.5)

Although the damage in tension and compression is isotropic, the above expression introduces a distinction in the direction of loading (i.e. compressive strains vs. tensile strains) so that the reduced stiffness of the material may not be necessarily isotropic.

**Tension damage update**

If \(\varepsilon_{\text{max}}\) is larger than zero then there is at least one direction in which the material is in tension. A critical (variable) strain, \(\varepsilon_{t,\text{cr}}\), beyond which the material deteriorates is defined as:

\[
\varepsilon_{t,\text{cr}} = (\varepsilon_{t,u} - \varepsilon_t)\omega_t + \varepsilon_t
\] (3.6)

where \(\varepsilon_{t,u}\) is the ultimate crack strain, and \(\varepsilon_t\) is the strain corresponding to the tensile strength of the material, \(f_t\), see Figure 3.4(a). The material damage parameter is updated according to the magnitude of \(\varepsilon_{t,\text{cr}}\) with respect to \(\varepsilon_{\text{max}}\):

\[
\omega_t \left\{ \begin{array}{ll}
\varepsilon_t \text{(unchanged)} & \varepsilon_{\text{max}} \leq \varepsilon_{t,\text{cr}} \\
\frac{\varepsilon_{\text{max}} - \varepsilon_t}{\varepsilon_{t,u} - \varepsilon_t} & \varepsilon_{t,\text{cr}} < \varepsilon_{\text{max}} < \varepsilon_{t,u} \\
\frac{\varepsilon_{t,u} - \varepsilon_t}{1} & \varepsilon_{t,u} \leq \varepsilon_{\text{max}}
\end{array} \right.
\] (3.7)

Finally, the stiffness reduction factor, \(d_t\), is evaluated:

\[
d_t = \frac{\varepsilon_t}{\varepsilon_{t,\text{cr}}}(1 - \omega_t)
\] (3.8)

where \(\varepsilon_{t,\text{cr}}\) is the updated value of the critical strain calculated using equation 3.6 and the updated value of \(\omega_t\). The adjusted response of the material, which is always
along a secant loading/unloading branch as shown in Figure 3.4(a), is obtained by multiplying $d_t$ with the Young’s modulus and the associated principal strain.

Instead of a linear softening model, an exponential model can be used. In this case the softening branch of the curve is given by:

$$\sigma = f_t \exp \left( -\frac{\varepsilon - \varepsilon_t}{c} \right)$$  \hspace{1cm} (3.9)

where $c$ is a constant. Eqs. 3.6 and 3.7 need to be replaced by:

$$\varepsilon_{t,cr} = \varepsilon_t + c \ln (1 - \omega_t)$$  \hspace{1cm} (3.10)

and

$$\omega_t \left\{ \begin{array}{ll}
\omega_t \text{(unchanged)} & \varepsilon_{\text{max}} \leq \varepsilon_{t,cr} \\
1 - \exp \left( -\frac{\varepsilon - \varepsilon_t}{c} \right) & \varepsilon_{t,cr} < \varepsilon_{\text{max}} < \varepsilon_{t,u}
\end{array} \right. \hspace{1cm} (3.11)$$

### Compression damage update

Compression damage is treated in an almost identical manner. The initial elastic branch is followed by a flat branch where the strength of the material remains unaltered up to a strain $\varepsilon_{c,\text{max}}$. Similar to the tension model, the critical strain and the updated value of $\varepsilon_c$ are obtained using formulas identical to those in 3.7 and 3.8 simply replacing the subscript’s ‘t’ with ‘c’. The only change is in the stiffness reduction factor which now is given by:

$$d_c = \left\{ \begin{array}{ll}
\varepsilon_c \frac{1 - \omega_c}{\varepsilon_{c,cr}} & \varepsilon_{\text{min}} \leq \varepsilon_{c,\text{max}} \\
\varepsilon_c \frac{1 - \varepsilon_{c,\text{cr}} - \varepsilon_{c,\text{max}}}{\varepsilon_{c,u} - \varepsilon_{c,\text{max}}} & \varepsilon_{c,\text{max}} < \varepsilon_{\text{min}} < \varepsilon_{c,u} \\
0 & \varepsilon_{c,u} \leq \varepsilon_{\text{min}}
\end{array} \right. \hspace{1cm} (3.12)$$

where $\varepsilon_{c,\text{max}}$ represents the end of the flat part of the stress-strain diagram; and $\varepsilon'_{c,cr}$ refers to the updated value of $\varepsilon'_{c,cr}$.

### Fracture energy regularization

In the elastic damage model, the ultimate tensile strain $\varepsilon_{t,u}$ or the strain constant $c$ and the ultimate compression strain $\varepsilon_{c,u}$ need to be specified, according to the desired softening response of the material. It has been shown in the past that using constant values of these parameters throughout the structure to be modelled leads to results which are dependent on the resolution of the mesh. In order to circumvent this problem, which arises because the damage process tends to localize in a single row of elements, it is necessary to relate the values of the strains defining the various points of the stress-strain diagrams in tension and compression to the fracture energy
Figure 3.4: Material response: (a) in tension; (b) in compression.

of the material in these two modes. To do this, Hillerborg et al.[26] argued that the area under the stress-strain curves should be equal to the fracture energy divided by a length scale parameter, \( h \), which represents a typical length of the finite element (e.g. the square root of the average area of the elements or the average area represented by the Gauss points in an element). Such a procedure ensures that the same energy is dissipated when damage occurs within one element, independently of the size of the element. The ultimate strains in tension and compression which follow from using such an approach must be evaluated for each integration point, on the basis of the corresponding element size, \( h_e \). For linear softening the ultimate strains are given by:

\[
\varepsilon_{t,u} = \frac{2G_t}{f_t h_e}
\]

(3.13)

and

\[
\varepsilon_{c,u} = \frac{2G_c}{f_c h_e} + \varepsilon_c - \varepsilon_{c,max}
\]

(3.14)

where \( G_t \) and \( G_c \) are the fracture energies for tension and compression, respectively. In the case of exponential softening in tension, the constant \( c \) in eq. 3.9 (which plays a similar role to the ultimate strain measure of the linear softening case) is related to the fracture energy in tension by:

\[
c = \frac{G_t}{f_t h} - \frac{1}{2}\varepsilon_t
\]

(3.15)

In order to obtain a unique relationship between stress and strain, a limit on the maximum size of the elements can be introduced. For tension this limit is given by (see also Rots [46]):

\[
h_{\text{max}} = \frac{2EG_t}{f_t^2}
\]

(3.16)
If the element size is taken larger than $h_{\text{max}}$, then snap-back behaviour at the integration point level is observed (i.e. once the ultimate strength is obtained, complete damage will occur independently of the strain).

Although the fracture energy approach does lead to results which in terms of the load displacement diagrams are mesh independent (i.e. the dissipated energy remains the same regardless of the mesh which is used, as will be seen in later chapters), it does not constitute a complete approach to the problem of localization. First of all, from a physical point of view the idea of relating the parameters concerning the ultimate strains to the element size seems an artifact. Mathematically it can be shown that a softening branch modelled using a negative slope (a negative stiffness) even if related to the fracture energy as above, does not lead to a well posed problem in a standard continuum, Bazant and Cedolin[7], Sluys[51]. Further, results will be shown in the following chapters showing that the established damage patterns are still mesh dependent. In section 3.5, the elastic damage model will be enhanced to include a different type of regularization technique, which does not require the introduction of mesh related parameters (as in the case of the fracture energy approach).

2D Implementation

The concepts developed above can be easily incorporated in plane stress problems, simply by removing the third stress and strain component from the various expressions used. The effective stress is determined from the principal strains using the plane stress stiffness matrix given in eq. 3.50.

3.3 Models based on the theory of plasticity

Models based on the plasticity theory have been devised in order to model the fracturing behaviour of concrete under tension and compression. These models consist of either a unique yield surface, e.g. the enhanced five parameter model by Pearce[41], or a combination of yield surfaces, e.g. Feenstra[22], where cracking is controlled via a tension cut-off (Rankine) criterion and crushing is controlled via a specific yield surface. Some difficulties may be encountered in the implementation of these models due to the relatively large number of material parameters in the former type of model, and due to the problem of handling corners in the latter type. In the following, a review of simplified return-mapping procedures and yield surfaces is given, which will serve as a basis for the combined damage/plasticity model described subsequently.

Given a yield function $f(\sigma)$, an elastic state of stress exists if $f(\sigma) < 0$, otherwise if $f(\sigma) = 0$ plastic flow occurs. $f(\sigma) > 0$ cannot occur. Plastic flow occurs according
to an associated flow rule:

$$\Delta \varepsilon^p = \Delta \lambda \frac{\partial f}{\partial \sigma} = \Delta \lambda (A\sigma + b)$$

(3.17)

where it has been assumed that the derivative of \( f \) w.r.t. \( \sigma \) can be written as a linear function of the stress, \( \sigma \). In the above equation, \( \Delta \lambda \) is the plastic multiplier, \( A \) is a matrix constant and \( b \) is a vector constant. The state of stress at the end of a step is given by

$$\sigma_1 = D (\varepsilon_1 - \varepsilon^p_1)$$

(3.18)

where \( D \) is the elastic stiffness matrix,

$$D = \begin{bmatrix}
  x & y & y \\
  y & x & y \\
  y & y & x
\end{bmatrix}$$

(3.19)

with \( x = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \), \( y = \frac{Ey}{(1+\nu)(1-2\nu)} \) and \( z = \frac{E}{2(1+\nu)} \), and \( \varepsilon_1 \) and \( \varepsilon^p_1 \) are the total and plastic strains, respectively, at the end of a step:

$$\varepsilon_1 = \Delta \varepsilon + \varepsilon_0$$

$$\varepsilon^p_1 = \Delta \varepsilon^p + \varepsilon^p_0$$

Substituting the above expressions into eq. 3.18, the following expression for the final stress state is obtained,

$$\sigma_1 = D (\Delta \varepsilon + \varepsilon_0 - \Delta \varepsilon^p - \varepsilon^p_0) = \sigma_t - D \Delta \varepsilon^p$$

(3.20)

where \( \sigma_t \), the trial stress state, can be fully determined from known quantities at the start of the step:

$$\sigma_t = D (\Delta \varepsilon + \varepsilon_0 - \varepsilon^p_0)$$

(3.21)

Assuming the plastic strain increment in 3.17 is a function of the final stress state, and substituting this into 3.20 leads to an implicit expression in the final stress:

$$\sigma_1 = \sigma_t - \Delta \lambda D (A\sigma_1 + b)$$

(3.22)

which in rearranged format gives

$$\sigma_1 = H^{-1} (\sigma_t - \Delta \lambda Db)$$

(3.23)

where

$$H = I + \Delta \lambda DA$$

(3.24)
The next step consists in finding a value of $\Delta \lambda$ which satisfies the consistency condition $f(\sigma_1) = 0$. In eq. 3.23, the final stress is only a function of $\Delta \lambda$, which implies indirectly that the yield function can be considered as a function of $\Delta \lambda$. Using this fact, $\Delta \lambda$ can be determined numerically using any root finding method. In the model implemented in LARES, this is achieved using the bisection method, see for example Press et al.[42]. The effective plastic strain increment can then be determined from 3.17.

**Damage behaviour**

Concrete and masonry subjected to compression do not behave in an ideal elastoplastic manner and show response degradation beyond the peak load. This requires the model to exhibit hardening before reaching peak response and softening thereafter. The yield function is now a function of stresses and a damage parameter, $\kappa$, and is given by

$$f = f(\sigma, \kappa)$$

(3.25)

The material follows a strain hardening rule with the damage parameter set equal to the equivalent plastic strain,

$$\kappa = \sqrt{\frac{2}{3} \Delta \varepsilon^p \cdot \Delta \varepsilon^p}$$

(3.26)

At the end of each step, the equivalent plastic strain is updated as follows

$$\kappa_1 = \kappa_0 + \sqrt{\frac{2}{3} \Delta \varepsilon^p \cdot \Delta \varepsilon^p}$$

(3.27)

It should be noted from the above equation that the current plastic strain increment cannot be expressed as function of $\kappa$, since the implicit return mapping scheme implies that the plastic strain is expressed in terms of the final state. If the plastic strain increment is a function of stress and damage, a semi-implicit scheme has to be used instead. The plastic strain increment is then expressed in terms of the initial converged state of stress and damage:

$$\Delta \varepsilon^p = \Delta \lambda \frac{\partial f(\sigma_0, \kappa_0)}{\partial \sigma}$$

(3.28)

The final stress is given by

$$\sigma_1 = \sigma_1 - \Delta \lambda D (A \sigma_0 + b)$$

(3.29)

and the plastic multiplier is determined in the same manner as in the implicit scheme, by ensuring that $f(\sigma_1, \kappa_1) = 0$. 

45
Figure 3.5: (a) Drucker-Prager and (b) quadratic Drucker-Prager models in $\sqrt{J_2}-I_1$ space.

### 3.3.1 Drucker Prager

The implicit return mapping algorithm described above will be elaborated here for a general Drucker-Prager model. The yield function is defined as follows:

$$f(\sigma) = J_2 - (\alpha I_1 - k)^2$$  \hspace{1cm} (3.30)

where

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$  \hspace{1cm} (3.31)

$$J_2 = \frac{1}{6} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]$$  \hspace{1cm} (3.32)

or

$$J_2 = \frac{1}{6} \left[ (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 \right] + \tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2$$  \hspace{1cm} (3.33)

and $\alpha$ and $k$ are two material parameters. Substituting 3.30 into 3.17, the following expression for the plastic strain increment is obtained:

$$\Delta \varepsilon^p = \Delta \lambda \left( \frac{\partial f}{\partial J_2} \frac{\partial J_2}{\partial \sigma} + \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma} \right)$$  \hspace{1cm} (3.34)

which reduces to the form in 3.17 with:

$$A = B \quad -2\alpha C$$  \hspace{1cm} (3.35)

$$b = 2\alpha kd$$  \hspace{1cm} (3.36)
where

\[
B = \frac{1}{3} \begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

\[
d = [1, 1, 1, 0, 0, 0]^T
\]

The plastic strain increment, which is formulated in terms of the final stress, is thus given by:

\[
\Delta \varepsilon^p = \Delta \lambda \left[ (B - 2 \alpha^2 C) \sigma_1 + 2 \alpha k d \right] \tag{3.37}
\]

Following the same steps leading up to eq. 3.23 the following expression is obtained for the reduced stresses at the end of the step:

\[
\sigma_1 = H^{-1} (\sigma_t - 2 \Delta \lambda \alpha k D d) \tag{3.38}
\]

with

\[
H = I + \Delta \lambda D (B - 2 \alpha^2 C) \tag{3.39}
\]

or

\[
H = \begin{bmatrix}
X & Y & Y \\
Y & X & Y \\
Y & Y & X \\
\end{bmatrix}
\]

\[
Z
\]

\[
Z
\]

where

\[
X = 1 + x \Delta \lambda \left( \frac{2}{3} - 2 \alpha^2 \right) + y \Delta \lambda \left( \frac{1}{3} - 2 \alpha^2 \right)
\]

\[
Y = x \Delta \lambda \left( \frac{1}{3} - 2 \alpha^2 \right) + y \Delta \lambda \left( \frac{1}{3} - 4 \alpha^2 \right)
\]

\[
Z = 1 + 2 z \Delta \lambda
\]

Here, x, y and z are the elastic stiffness parameters of eq. 3.19. Given the form of \( H \) and \( d \), it can be deduced that the normal stress components of \( \sigma_1 \) in 3.38 are
independent of the shear stress components in the trial stress, $\sigma_1$. Likewise, given
the form of $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{d}$ in 3.37, it follows that plastic strains in the three normal
directions are independent of any shear stress components of $\sigma_1$. This allows the
problem to be formulated in terms of principal stresses, the directions of which are
defined by the principal directions of $\sigma_1$. Making use of this fact and if all stress and
strain vector quantities are expressed in terms of this frame of reference, then the
inverse of $\mathbf{H}$ can be implemented in the following form:

$$
\mathbf{H}^{-1} = \frac{1}{X^2 + XY - 2Y^2} \begin{bmatrix}
X + Y & -Y & -Y \\
-Y & X + Y & -Y \\
-Y & -Y & X + Y
\end{bmatrix}
$$

(3.41)

with

$$
X = 1 + \left[-2\alpha^2 (x + 2y) + \frac{2}{3} (x - y)\right] \Delta \lambda
$$

$$
Y = \left[-2\alpha^2 (x + 2y) - \frac{1}{3} (x - y)\right] \Delta \lambda
$$

3.3.2 Quadratic Drucker-Prager

In this model the yield function 3.30 is replaced by

$$
f(\sigma) = J_2 + \alpha I_1 - k^2
$$

(3.42)

The expressions for the plastic strain increment and the reduced stress (compare with
3.37 and 3.38) are now given by

$$
\Delta \varepsilon^p = \Delta \lambda (B \sigma_1 + \alpha d)
$$

(3.43)

$$
\sigma_1 = \mathbf{H}^{-1} (\sigma_1 - \Delta \lambda \alpha Dd)
$$

(3.44)

where

$$
\mathbf{H} = \mathbf{I} + \Delta \lambda \mathbf{DB}
$$

(3.45)

Here, $\mathbf{H}$, has the same form as in 3.40 but now with $X = 1 + \frac{2}{3} \Delta \lambda (x - y)$, $Y = -\frac{1}{3} \Delta \lambda (x - y)$ and $Z = 1 + 2 \Delta \lambda z$.

Damage evolution

Degradation of the material is simulated in the quadratic Drucker-Prager(QDP) model
by making the parameter $k$ in 3.42 dependent on the hardening/softening parameter
$\kappa$, the current value of which is determined from eq. 3.27. The function used for $k$ is
given by:

$$
k(\kappa) = \begin{cases}
  k_0 \left[ 1 - a_1 + \frac{a_1}{a_2} \sqrt{a_2^2 - (\kappa - a_2)^2} \right] & \kappa < a_2 \\
  k_0 \exp \left[-b (\kappa - a_2)\right] & a_2 \leq \kappa
\end{cases}
$$

(3.46)
where

$$b = \frac{k_0 h}{2G}$$  \hspace{1cm} (3.47)

$G$ is a parameter directly related to the fracture energy of the material in compression, $a_1$ is a parameter determining the stress level at which hardening initiates, and $a_2$ is a parameter representing the equivalent plastic strain at which softening is initiated. In order to avoid convergence problems in the algorithm it may be desirable to limit the amount by which $k$ can decrease, for example by setting $k = \frac{k_0}{100}$ or $k = \frac{k_0}{1000}$. Just as in the cracking models it is necessary to limit the maximum size of the elements, $h$, in order to retain a unique relationship between $k$ and the equivalent plastic strain.

Given the choice of hardening/softening, whereby only the parameter $k$ varies with damage, a value of $\alpha \neq 0$ does not allow a complete degradation of the yield surface. Because of this, there is a residual strength of the material in compression. The following expressions can be used to determine values of $k_0$ and the residual strength in uniaxial compression $f_{\text{resid}}$, from given values of $f_c$ and $\alpha$.

$$k_0 = \sqrt{\frac{1}{3} f_c^2 - \alpha f_c}$$  \hspace{1cm} (3.48)

$$f_{\text{resid}} = -3\alpha$$  \hspace{1cm} (3.49)

2D implementation

The models described so far can also be implemented in plane stress situations. For the quadratic Drucker-Prager model in principal stress/strain space, the form of the
equations remain the same but with

\[
D = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}
\]

(3.50)

\[
B = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}
\]

d = [1, 1]^T

and the plastic strain increment in the third direction given by

\[
\Delta \varepsilon_3^p = \Delta \lambda \left[ -\frac{1}{3} (\sigma_1 + \sigma_2) - \alpha \right]
\]

3.4 An approach towards modelling fracture and crushing: a combined elastic damage/plasticity model

In the introduction to this chapter, the two basic modes (mode I and mode II) of failure of concrete-like materials have been discussed. Although the models discussed in 3.2 and 3.3 are capable of capturing the essence of one of these modes of failure, neither can simulate both types of failure. In an attempt to simulate both types of failure, a model is proposed, which retains the characteristics of the damage model for the tensile behaviour, and combines it with a plasticity model for compression. The combination of the two models is obtained by inserting the elastic damage algorithm directly inside the return mapping algorithm for plastic behaviour.

3.4.1 Outline

In the combined model, the trial stress is reduced initially to take possible cracking of the material into account. If this reduced stress lies within the yield surface then it is returned as the converged value. Otherwise, plastic flow takes place. Just as in the plastic models described previously, the increment in the plastic multiplier, \( \Delta \lambda \), is obtained by trial and error using a root finding algorithm such as the bisection method. The function of \( \Delta \lambda \) for which a root is sought is a modified yield function which takes into account the reduction in stress due to cracking of the material. This value is obtained by calculating at first the reduced stress corresponding to the current plastic multiplier value as in standard plasticity. A further reduction takes place to account for possible cracking and the final value of the stress which is obtained is used to calculate the corresponding plastic strains and damage parameters. From the
values of stress and damage thus obtained, the yield function value is obtained. The algorithm is summarized in Box 3.1.

```
sub CombinedModel
  \[ \varepsilon_t \leftarrow \varepsilon_1 - \varepsilon_0^p \]
  \[ \sigma_t \leftarrow D\varepsilon_t \]
  \[ \sigma \leftarrow \sigma_t \]
  call subroutine Crack(\sigma) defined below
  if \( f(\sigma, \kappa_0) \leq 0 \) exit sub
  call subroutine to find root \( \Delta \lambda \) of function \( F \) defined below
  \[ \varepsilon_1^p \leftarrow \varepsilon_0^p + \Delta \varepsilon^p \]
end sub

function \( F(\Delta \lambda) \)
  \[ \sigma \leftarrow (I + \Delta \lambda DA)^{-1}(\sigma_t - \Delta \lambda Db) \quad (eq. 3.23) \]
  call subroutine Crack(\sigma)
  \[ \Delta \varepsilon^p \leftarrow \Delta \lambda (A\sigma + b) \quad (eq. 3.17) \]
  \[ \kappa_1 \leftarrow \kappa_0 + \sqrt{\frac{2}{3} \Delta \varepsilon^p \cdot \Delta \varepsilon^p} \quad (eq. 3.27) \]
  \[ F \leftarrow f(\sigma, \kappa_1) \]
end function

sub Crack(\sigma) \quad (reduce \sigma according to tension damage model)
  \[ \varepsilon_{max} \leftarrow \max(\sigma/E) \quad (eqs. 3.2 and 3.3) \]
  update \( \omega_t \) and \( d_t \) on the basis of \( \varepsilon_{max} \)
  \[ \sigma \leftarrow d_t (\sigma) - (-\sigma) \quad (eq. 3.5) \]
end sub
```

Box 3.1: Combined model algorithm.

The converged state of stress which is obtained using the above algorithm is consistent in the sense that plastic strains which accumulate are immediately accounted for in the algorithm for cracking. For example, if both plastic flow and cracking take place, accumulated plastic strains lead to a reduction of the damage sustained due to cracking.

To clarify the strategy further, Figure 3.7 shows how the stress is reduced for various cases. In general, three different regions of stress space can be distinguished and are identified by (1), (2) and (3) in Figure 3.7: the first contains trial stress points which, if reduced according to the cracking model, lie within the yield surface of the plasticity model. In this case, the material undergoes only damage due to tension. The second region contains all stress points which map back onto points on the yield surface which do not exceed the tensile strength of the material. The third region contains all trial stress points which lead to a combination of cracking in tension and
Material modelling of fracturing materials: concrete and masonry

Figure 3.7: In standard plasticity, the return mapping algorithm maps the trials stress \( \sigma_t \) onto \( \sigma_1 \) according to 3.38. In the proposed model, \( \sigma_1 \) must satisfy both yield and crack criteria.

plastic flow in compression, and effectively constitutes a continuous transition zone from pure cracking to pure crushing.

3.4.2 Combined quadratic Drucker-Prager and rotating crack model

This model is implemented by combining the quadratic Drucker-Prager model of section 3.3.2 with the tension damage model of section 3.2.1. The model is formulated in the principal strain space. The behaviour of the combined quadratic Drucker-Prager model is shown in Figure 3.8 for strain-controlled uniaxial compression/tension and in Figure 3.9 under different biaxial stress conditions for a single integration point with length scale \( h = 0.1 \) m, and a material with the properties listed in Table 3.2.

<table>
<thead>
<tr>
<th>( E )</th>
<th>10000 MPa</th>
<th>( f_t )</th>
<th>1 MPa</th>
<th>( k )</th>
<th>5.322 MPa</th>
<th>( G )</th>
<th>1000 J/m²</th>
<th>( \nu )</th>
<th>0.15</th>
<th>( G_t )</th>
<th>20 J/m²</th>
<th>( \alpha )</th>
<th>0.5 MPa</th>
<th>( a_1 )</th>
<th>0.35</th>
<th>( a_2 )</th>
<th>0.001</th>
</tr>
</thead>
</table>

Table 3.2: Material properties for combined quadratic Drucker-Prager and damage model
Under uniaxial compression, the material presents a response which is initially linear, followed by a hardening branch until the peak strength of 10 MPa is reached, see point A in Figure 3.8. Beyond the peak load, the material exhibits softening as represented by the exponentially decaying response. At point B the load is removed. The unloading takes place along an elastic branch BE and further, until the tensile strength of the material is reached at point C. As the strain is decreased, the material softens linearly until the stress has reduced to zero in point F. Prior to reaching point F, the strain is increased again at point D and then decreased at point E showing that unloading in tension takes place along a secant branch. At point G the strain is decreased and the response follows the horizontal branch GFE followed by the elastic branch EB. At B, the softening branch in compression is resumed.

![Figure 3.8: Cyclic uniaxial test for the combined QDP and rotating damage model.](image)

The biaxial failure contours of Figure 3.9(a) have been obtained by specifying stress paths in the material model driver described briefly in 3.1.2. Failure of the material is assumed to occur when the driver fails to converge to an equilibrium point. For example, this would correspond to trying to find an equilibrium point for a load which is higher than the stress at point A in the uniaxial compression part of Figure 3.8. A single failure contour is obtained from a number of these failure points, obtained by following different stress paths in which the shear stress is kept constant, whilst the normal stresses are incremented at a constant proportion to each
other. Further, each path is specified to begin at the same initial loading point (in the case reported in the figure, \( \sigma_{xx} = -5 \text{ MPa} \) and \( \sigma_{yy} = -5 \text{ MPa} \)). The various paths followed can be pictured as forming a rosette of straight lines in the \( \sigma_{xx} - \sigma_{yy} \) space emanating from a common origin.

Figure 3.9(b) shows all the failure points shown in Figure 3.9(a), in \( \sqrt{J_2} - I_1 \) space. The failure points form two distinct lines, one which is relatively shallow and corresponds to failure of the material due to plastic deformation and the other (steeper) line showing those points for which failure has occurred due to cracking of the material.

Figure 3.9: Failure contours for the combined QDP and rotating damage model: (a) iso-shear contours \( \tau = 0.0, 1.5, 3.0, 4.0, 4.5 \text{ MPa} \) (from outside to inside); (b) failure points in \( \sqrt{J_2} - I_1 \) space.

### 3.5 An enhanced regularization technique applied to elastic damage models

In section 3.2.1, the concept of regularization was introduced. The fracture energy approach has been applied in both the elastic damage models and the plastic softening models (consequently also in the combined model). Such an approach was necessary to ensure that during fracture the same amount of energy is dissipated independently of the finite element discretization that is used. However, this approach does not satisfy the necessary requirements for well-posedness of the problem: in the presence of softening dictated by a stress-strain law with a negative slope, the partial differential equations underlying the problem change type. The issue has been exemplified in Bazant and Belytschko[6], and Bazant and Cedolin[7], and is a consequence of an approach which assumes that the experimentally observed (global) response, in terms of
load-displacement diagrams, can be directly translated into a (local) stress-strain law. It has been shown that such a direct "engineering" approach to material modelling leads to mesh dependent results, Sluys [51]. Various "regularised" models have been developed to overcome this problem. Among the regularised models or regularization techniques are strain rate dependent models, gradient models for static problems and polar (Cosserat) continuum models. Apart from the ensuing mesh objectivity, "regularised" material models can be more appealing from a physical point of view and may also improve numerical stability of the solvers.

In order to achieve results which are not dependent on the mesh, a technique is proposed in the present section, similar to the strain rate dependent regularization technique. The regularization is achieved by introducing a time dependency in the expression for the damage, in the form of a rate of damage law. The approach is applied to three types of elastic damage type models.

In later chapters, the superiority of this approach compared to fracture energy approach will be proven, especially in terms of mesh objectivity.

### 3.5.1 Rate dependent elastic damage model in one dimension

In the standard elastic damage model (cf. eq. 3.1), the factor \(1 - \omega\) (expressed as the stiffness parameter \(d\)) depends on an expression in terms of the damage parameter \(\omega\), which was determined from the current equivalent uniaxial strain (cf. eqs. 3.8 and 3.12). The damage parameter is not updated in terms of a maximum stress or strain value, but it is assumed to change according to a rate law:

\[
\frac{d\omega}{dt} = c \left( \frac{\varepsilon}{\varepsilon_t} - 1 \right)^{n_1} (1 - \omega)^{n_2}
\]  

(3.51)

where \(c, n_1\) and \(n_2\) are constants. The above law states that damage can only occur when the strain \(\varepsilon\) reaches a limiting (tensile) strain \(\varepsilon_t\) as expressed by the term in the McCauley brackets. The damage occurs at a rate which is proportional to the strain "excess", and in proportion to the amount of material which is still intact (as represented by the term \(1 - \omega\)). Indirectly, this term ensures that the damage can never become greater than 1. The constant \(c\) has the dimension of 1/time and determines the rate at which damage occurs, i.e. the lower its value is, the more slowly damage will occur. Further, it is assumed that the damage is directly proportional to the secant stiffness of the material so that \(d = (1 - \omega)\).

To update the damage parameter at a given integration point and step, an explicit integration scheme is used:

\[
\omega_1 = \omega_0 + \dot{\omega}_0 \Delta t
\]  

(3.52)

where

\[
\dot{\omega}_0 = c \left( \frac{\varepsilon}{\varepsilon_t} - 1 \right)^{n_1} (1 - \omega_0)^{n_2}
\]  

(3.53)
The stability of the procedure can be studied for the simple case \( n_2 = 1 \) and assuming that the term in the McCauley brackets is smaller than 1. Taking \( n_2 = 1 \), recalling that \( d = 1 - \omega \) and substituting 3.52 into 3.53, one has:

\[
d_1 = d_0 - c \left( \frac{\varepsilon}{\varepsilon_t} - 1 \right)^{n_1} d_0 \Delta t = \left( 1 - c \left( \frac{\varepsilon}{\varepsilon_t} - 1 \right)^{n_1} \Delta t \right) d_0
\]

To ensure stability, it is necessary that

\[
\left| 1 - c \left( \frac{\varepsilon}{\varepsilon_t} - 1 \right)^{n_1} \Delta t \right| < 1
\]

which implies

\[
0 < \Delta t < \frac{2}{c \left( \frac{\varepsilon}{\varepsilon_t} - 1 \right)^{n_1}} \quad (3.54)
\]

Under the assumption that the term in the McCauley brackets is smaller than 1 (i.e. \( \varepsilon < 2\varepsilon_t \)), the procedure will be stable if the following condition is satisfied:

\[
\Delta t < \frac{2}{c} \quad (3.55)
\]

The response of the model for a single material point is depicted in Figures 3.10 and 3.11 for different strain rates and different combinations of the parameters \( c, n_1, n_2 \). Although expression 3.51 does not contain a strain rate term \( \dot{\varepsilon} \), the (proportional) dependency of the response on the strain rate is evident. The parameter \( n_1 \) controls the initial peak value in the response (the higher its value, the higher the peak response), whereas the parameter \( n_2 \) controls the final part of the response curve (the higher its value, the more gradual the final response). From the first two curves of Figure 3.11, it can be observed that the parameter \( c \) controls the overall decay of the response (the higher its value, the faster the decay).

From the comparison of the response under different strain rates, it can be seen that the areas under the curves tend to grow with increasing strain rate. At first sight, this would seem in contrast with experimental evidence which suggests that the fracture energy of materials such as concrete is relatively independent of the strain rate, Weerheijn[57]. However, an effective judgement of the performance of the material models can only ensue after a proper calibration of the material model parameters and a verification through a series of structural analyses.

The various parameters should be chosen on the basis of the desired structural response curve. Precise relationships between these parameters and actual experimental data are at this stage not available; however some considerations can be made, particularly on the basis of a one-dimensional analysis. Specifically, the parameter \( \varepsilon_t \)
Figure 3.10: Response of rate dependent model at the material point level for different material parameters and strain rates.
Figure 3.11: Comparison of rate dependent model at material point level for different parameters and at a constant strain rate $\dot{\varepsilon} = 20 \text{ s}^{-1}$.

is related directly to the tensile strength of the material (i.e. $f_t = E\varepsilon_{t}$) and $c$ can be related to a characteristic length scale of the material via (see also Needleman[34]):

$$l_c = \frac{c_e}{c}$$

where $c_e = \sqrt{E/\rho}$, i.e. the bar velocity or speed of longitudinal waves in the material. This length scale, however, is related only to the elastic material properties. An effective localization zone (or what could be considered a smeared crack width) needs to be defined, for example

$$w = \frac{f_t}{E} l_c$$

Some relation with the fracture energy $G_f$ of the material can be assumed, knowing that

$$G_f \propto f_t w$$

A detailed study is necessary to discern the various parameters, in the case that a specific experimental curve needs to be reproduced. A more general damage rate law may be more useful then, such as

$$\frac{d\omega}{dt} = cg (...) (1 - \omega)^n$$

where $g$ is a function which depends on the state of stress or strain of the material or any other variable describing the state of the material.
Stress-strain relationship at constant strain rate

Assuming that the strain increases with a constant rate, \( \dot{\varepsilon}_0 \), that at \( t = 0 \) the initial strain is equal to the limiting elastic strain, \( \varepsilon_t \), and that \( n_1 = n_2 = 1.0 \), the damage law in eq. 3.51 can be rewritten as

\[
\frac{d\omega}{dt} = c \frac{\dot{\varepsilon}_0}{\varepsilon_t} (1 - \omega)
\]  
(3.59)

The solution to the above ordinary differential equation with \( \omega(0) = 0 \) is

\[
\omega = 1 - \exp \left( -c \frac{\dot{\varepsilon}_0}{2\varepsilon_t} t^2 \right)
\]  
(3.60)

or substituting \( t = \frac{\varepsilon - \varepsilon_t}{\dot{\varepsilon}_0} \),

\[
\omega = 1 - \exp \left[ -\frac{c}{2\varepsilon_t \dot{\varepsilon}_0} (\varepsilon - \varepsilon_t)^2 \right]
\]  
(3.61)

Substituting the above into the basic stress-strain law, \( \sigma = (1 - \omega) E\varepsilon \), the following expression is obtained relating the stress directly in terms of the strain and applied (constant) strain rate:

\[
\sigma = \exp \left[ -\frac{c}{2\varepsilon_t \dot{\varepsilon}_0} (\varepsilon - \varepsilon_t)^2 \right] E\varepsilon
\]  
(3.62)

The above equation shows the dependency of the stress on the strain rate and the strain. Beyond the limiting elastic strain, the stress is a decaying function of the strain, the decay of which is less pronounced with increasing values of the strain rate.

3.5.2 Rate dependent elastic damage model for multi-dimensional states of stress

1.) Cut-off damage rate law

The above rate dependent elastic damage model can be extended to obtain a complete description of the material, including tension, compression, shear and biaxial or triaxial states of stress. The starting point is an elastic damage stress-strain relationship of the type given in eq. 3.5 in the principal stress space:

\[
\sigma = E \left[ (1 - \omega_t) \langle \varepsilon' \rangle - (1 - \omega_c) \langle -\varepsilon' \rangle \right]
\]  
(3.63)

where \( \varepsilon' \) is the vector of equivalent uniaxial strains (see 3.2), \( \omega_t \) is a tension damage variable and \( \omega_c \) is a compression damage variable. The evolution of the damage variable is controlled via a set of damage rate laws such as

\[
\frac{d\omega_t}{dt} = c_t \left( \frac{\varepsilon_{\text{max}}}{\varepsilon_t} - 1 \right)^{n_{t1}} (1 - \omega_t)^{n_{t2}} g_t(\varphi)
\]  
(3.64a)

\[
\frac{d\omega_c}{dt} = c_c \left( \frac{\varepsilon_{\text{min}}}{\varepsilon_c} - 1 \right)^{n_{c1}} (1 - \omega_c)^{n_{c2}} g_c(\varphi)
\]  
(3.64b)
where $\varepsilon_{\text{max}}$ and $\varepsilon_{\text{min}}$ are the maximum and minimum equivalent uniaxial strains respectively, $\varepsilon_t$ and $\varepsilon_c$ mark the limiting elastic strains in tension and compression respectively, and $g_t$ and $g_c$ are functions taking values between 0 and 1, which introduce a differentiation between tension and compression damage according to the value of the strain invariants:

$$
g_t(\varphi) = g_c(-\varphi) = \begin{cases} 
0 & \varphi \leq -\sqrt{3} \\
1 + \frac{1}{\sqrt{3}} \varphi & -\sqrt{3} < \varphi < 0 \\
1 & \varphi \geq 0
\end{cases} \quad (3.65)
$$

where $\varphi = \frac{I_1(\varepsilon)}{\sqrt{J_2(\varepsilon)}}$. In the $I_1 - \sqrt{J_2}$ space, $\varphi$, is the slope of a line connecting the strain point $\left( I_1(\varepsilon), \sqrt{J_2(\varepsilon)} \right)$ to the origin. Values of $\varphi = \pm\sqrt{3}$ correspond to uniaxial states of strain. The functions $g_t$ and $g_c$ are necessary in order to differentiate a pure tension damage mode from a pure compression mode, avoiding thus that both types of damage occur simultaneously when this is not desired. For a pure shear state ($I_1 = 0$) both types of damage can occur. In Figure 3.12, the dependency of $g_t$ on the value of the strain invariants is depicted. Failure contours, corresponding to different strain rates, are depicted by the graph in Figure 3.13(a). The shape of the contours approximates that of a square, and the dependency of the contours on the strain rate is clearly observable. For later reference the model which has been described here will be referred to as a rate dependent cut-off damage law.

2.) Quadratic Drucker-Prager damage rate law

As can be seen from the failure contour diagram of Figure 3.13(a), the rate dependent cut-off damage law does not take into consideration biaxial effects. The following alternative rate law can be used instead to model biaxial and triaxial states of stress:
Figure 3.13: Failure contour for multi-dimensional rate dependent models, obtained from the peak values of stress along different stress paths. Left: cut-off damage law; Right: QDP damage law. The specified material strengths in both cases are \( f_t = 1 \) MPa and \( f_c = 10 \) MPa. The various curves are obtained for different time increments per step or rates (thick line: \( \Delta \tau = 2 \times 10^{-4} \); thin line: \( \Delta \tau = 5 \times 10^{-5} \); dotted line \( \Delta \tau = 2 \times 10^{-5} \)).

\[
\frac{d\omega_t}{dt} = h_t(\varphi) f(\varepsilon)(1 - \omega_t)^{n_t} \tag{3.66a}
\]

\[
\frac{d\omega_c}{dt} = h_c(\varphi) f(\varepsilon)(1 - \omega_c)^{n_c} \tag{3.66b}
\]

where \( f(\varepsilon) \) is a quadratic Drucker-Prager(QDP) strain based failure function similar to the yield surface in the plastic model introduced in paragraph 3.3.2:

\[
f(\varepsilon) = \left( \frac{J_2(\varepsilon) + \alpha I_1(\varepsilon) - k^2}{k^2} \right)^{\frac{1}{2}} \tag{3.67}
\]

The values of \( \alpha \) and \( k \) in the above function can be determined from the uniaxial strengths in tension and compression, from

\[
\alpha = \frac{f_c^2 - f_t^2}{3(f_c + f_t)} \tag{3.68}
\]

\[
k = \sqrt{\frac{f_c^2}{3} - \alpha f_c} \tag{3.69}
\]

In eqs. 3.66, \( h_t \) and \( h_c \) are functions which control the tension and compression damage rates, respectively, according to the current state of strain as represented by the variable \( \varphi \). The advantage of this formulation over that of 3.64, is that the damage rates can be made identical in tension and in compression for particular values of \( \varphi \).
For example, in a pure shear strain state \((\varphi = 0)\), by making \(h_t(0) = h_c(0)\), the material undergoes a unique shear degradation whereby the tension and compression components degrade in a proportional manner, see Figure 3.14. In this case the functions \(h_t\) and \(h_c\) are given by:

\[
\begin{align*}
    h_t(\varphi) &= \begin{cases} 
    c_c & \varphi \leq -\sqrt{3} \\
    c_s + \frac{1}{\sqrt{3}}(c_s - c_c)\varphi & -\sqrt{3} < \varphi < 0 \\
    c_t & \varphi \geq \sqrt{3}
    \end{cases} \\
    &= \begin{cases} 
    c_c & \varphi \leq -\sqrt{3} \\
    c_s + \frac{1}{\sqrt{3}}(c_s - c_c)\varphi & -\sqrt{3} < \varphi < 0 \\
    c_t & \varphi \geq \sqrt{3}
    \end{cases}
\end{align*}
\]

\(h_c(\varphi) = \begin{cases} 
    c_c & \varphi \leq -\sqrt{3} \\
    c_s + \frac{1}{\sqrt{3}}(c_s - c_c)\varphi & -\sqrt{3} < \varphi < 0 \\
    c_s - \frac{1}{\sqrt{3}}c_s\varphi & 0 \leq \varphi < \sqrt{3} \\
    0 & \varphi \geq \sqrt{3}
    \end{cases}
\]

\(3.71\)  

3.) Mixed damage law

A disadvantage of the QDP damage law is that the failure surface must be specified in terms of the tensile and compressive strengths of the material. In order for the surface to fit these strengths, the resulting biaxial strength is exaggerated (see Figure 3.13). A mixed law consisting of a cut-off criterion in tension and a QDP law in
Figure 3.15: Failure contours for the mixed damage law for a material with $f_t = 1$ MPa, $\tau_{\text{max}} = 4$ MPa and $f_c = 10$ MPa. The various curves are obtained for different time increments per step or rates (thick line: $\Delta \tau = 2 \times 10^{-4}$; thin line: $\Delta \tau = 5 \times 10^{-5}$; dotted line $\Delta \tau = 2 \times 10^{-5}$).

The compression is then more appropriate:

\[
\frac{d\omega_t}{dt} = h_t(\varphi) \left( \frac{\varepsilon_{\text{max}}}{\varepsilon_t} - 1 \right)^{n_{\varepsilon_t}} (1 - \omega_t)^{n_{\omega_t}} \tag{3.72a}
\]

\[
\frac{d\omega_c}{dt} = h_c(\varphi) f(\varepsilon)^{n_{\varepsilon_t}} (1 - \omega_c)^{n_{\omega_c}} \tag{3.72b}
\]

With this formulation, the QDP failure surface can be determined from the values of the compressive strength and a fictitious shear strength of the material independently of the tensile strength. The QDP parameters now follow from:

\[
\alpha = \frac{f_c}{3} - \frac{\tau_{\text{max}}^2}{f_c}
\]

\[
k = \tau_{\text{max}}
\]

where $\tau_{\text{max}}$ is the fictitious shear strength of the material. The failure contour corresponding to this model is depicted in Figure 3.15.
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<table>
<thead>
<tr>
<th>a) rotating crack</th>
<th>b) combined QDP</th>
<th>c) rate dep. QDP</th>
<th>d) rate dep. mixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_t = 1 \text{ MPa} )</td>
<td>( f_t = 1 \text{ MPa} )</td>
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</tr>
<tr>
<td>( G_f = 20 \text{ J/m}^2 )</td>
<td>( G_f = 20 \text{ J/m}^2 )</td>
<td>( f_c = 10 \text{ MPa} )</td>
<td>( \tau_{\text{max}} = 5 \text{ MPa} )</td>
</tr>
<tr>
<td>( k = 5.774 \text{ MPa} )</td>
<td>( \alpha = 0 \text{ MPa} )</td>
<td>( f_c = 10 \text{ MPa} )</td>
<td>( \tau_{\text{max}} = 5 \text{ MPa} )</td>
</tr>
<tr>
<td>( G = 1500 \text{ J/m}^2 )</td>
<td>( h = 0.1 \text{ m} )</td>
<td>( c_t = 10000 \text{ s}^{-1} )</td>
<td>( \Delta t = 2 \times 10^{-4} \text{ s} ) (pure shear test)</td>
</tr>
<tr>
<td>( c_s = 2000 \text{ s}^{-1} )</td>
<td>( c_s = 2000 \text{ s}^{-1} )</td>
<td>( \tau_{\text{max}} = 5 \text{ MPa} )</td>
<td>( \Delta t = 2 \times 10^{-5} \text{ s} ) (Willam test)</td>
</tr>
<tr>
<td>( c_c = 1000 \text{ s}^{-1} )</td>
<td>( n_{t1} = n_{c1} = 1.0 )</td>
<td>( n_{t2} = n_{c2} = 1.0 )</td>
<td>( )</td>
</tr>
<tr>
<td>( \Delta t = 2 \times 10^{-4} \text{ s} ) (pure shear test)</td>
<td>( \Delta t = 2 \times 10^{-5} \text{ s} ) (Willam test)</td>
<td>( )</td>
<td>( )</td>
</tr>
</tbody>
</table>

\[ E = 10000 \text{ MPa}; \nu = 0.15 \]

Table 3.3: Selected material properties and models for shear tests.

### 3.6 Verification of the shear behaviour of the models

In order to study the response of rotating crack and multiple fixed crack models, Willam[58] and later Rots[46] performed a series of numerical tests subjecting a material point to a (2-dimensional) combination of shear and normal strains. In the following, the same types of tests are applied to verify the performance of the combined material models.

Two sets of results are presented for the two types of combined models under pure shear and under a continuously rotating set of strains. The first test is done by prescribing the shear strain component whilst setting the normal strains to zero. The response is shown in Figures 3.16, 3.17 and 3.18 for four different material models, namely: (a) the rotating crack model (elastic damage model with tension damage only), (b) the combined QDP and rotating crack model and the rate dependent, (c) the QDP damage and (d) the mixed damage models. The material parameters for each type of model are given in Table 3.3. In the test series, the shear strain is gradually increased from 0% to 1% in 250 increments.

The second test is performed in two phases: first, the normal strain, \( \varepsilon_{xx} \), is gradually increased from 0% to 0.01% in 100 increments; then the two normal strains, \( \varepsilon_{xx} \), \( \varepsilon_{yy} \), and the shear strain, \( \gamma_{xy} \), are increased in the proportions 0.025% : 0.0375% : 0.05% in 250 increments. In the test, the material is subjected to a state of stress in which the principal axes of stress rotate continuously during the second stage of loading. The aim of the test is to verify that the models exhibit a realistic response and that a situation of so-called stress-locking in shear does not occur. Stress-strain response diagrams are given in Figures 3.19, 3.20 and 3.21. In these figures, the

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Figure 3.16: Response in pure shear test.

Figure 3.17: Principal stress/strain response in pure shear (direction in tension).
Figure 3.18: Principal stress/strain response in pure shear (direction in compression).

The response of the rotating crack model is identical to that of the combined QDP and rotating crack model.

Figure 3.19: Willam test: shear stress-strain response.
Figure 3.20: Normal stress-strain response in $x$ direction.

Figure 3.21: Normal stress-strain response in $y$ direction.

The second series of tests shows essentially that failure of the material occurs due to tension failure independently of the material model used. This can be observed from Figure 3.20, which shows that the normal stress in the $x$ direction reaches the specified tensile strength in the first loading phase (strains up to 0.01%). Softening takes place immediately at the onset of the second loading phase. In spite of the
rotation of the principal stress directions, this softening is reflected also in the other normal direction and also in the shear stress.

It is interesting to note that the response for the rotating crack and the combined QDP model coincide, indicating that failure does not involve compression damage at all. In contrast, the pure shear strain test series does, see Figure 3.18. In the rotating crack model, the minor principal stress (which is compression) increases linearly with the strain, whereas the combined QDP model exhibits a decay after the ultimate compressive strength has been attained. The overall shear strength obtained in this model is ultimately dictated by the specified compressive strength, and is precisely half the compressive strength, Figure 3.16. This stems from the fact that the contribution to the shear resistance from the major principal direction is lost due to tension failure before degradation takes place in the other principal direction. The asynchronous degradation in tension and compression is noticeable from the "hump" in the initial response in shear. Although not shown in the graphs, a similar behaviour for which the compressive strength dictates the shear strength will also be observed for the rate dependent cut-off model.

A simple consideration on frictional materials will show that the shear strength displayed by the rate independent combined QDP and rate dependent cut-off model is relatively high. The relationship between shear strength and compressive strength for a Mohr-Coulomb material is given by:

\[ \tau_{\text{max}} = \frac{(1 - \sin \phi)}{2} f_c \]

where \( \phi \) is the internal angle of friction of the material. Taking \( \phi = 30^\circ \), which is a typical value for cementitious-like materials, one has \( \tau_{\text{max}} = \frac{1}{2} f_c \), which is a factor 2 lower than the value determined by the two models in question.

Referring to Figure 3.16, it can be seen that the rate dependent QDP and mixed models give a lower shear strength, which may be more realistic. The major difference however is to be noted in the various areas under the shear stress-strain curves. Substantially lower shear fracture energies are observed with respect to the combined QDP model.

An important feature of the mixed model is that the shear strength can be adjusted independently of the tensile strength. Often this is the case in real life, given that the tensile strength of materials cannot be easily correlated to its frictional properties. The importance of modelling correctly the shear behaviour has been shown in a number of reinforced shear panel analyses by Shing and Lofti[10]. A number of their analyses which were based on the rotating model revealed that failure occurred predominantly in flexure rather than shear. The observed spurious shear strength resulted from the fact that any shear distortion in a cracked element is inevitably associated with a compressive strain in one of the principal directions.
In later numerical examples, it will be seen that the choice of material model and parameters influences substantially the global response of a structure. In particular, the shear behaviour exhibited by a particular model will appear to have an important role in determining the actual failure mechanism of the structure.

### 3.7 Fast approach to orthotropy

Orthotropic material response can be introduced into any of the above isotropic models via a simple approach. It is assumed that the constituent material is effectively an isotropic material containing voids. Along cutting planes at different orientations, it is presumed that the void area differs, giving rise to an anisotropic behaviour on a macroscopic scale. In the finite element method, the material response in terms of stresses at a point is obtained from strains at that point. Hence this procedure can be used to obtain the response of the constituent (isotropic) material. Then by multiplying the obtained stresses by factors representing the ratio of true (material) area to nominal (material+void) area, the average stress is obtained. The procedure can be described mathematically as follows

\[
\sigma_n = a \sigma_1 a \tag{3.73}
\]

where all symbols represent tensors, \(\sigma_n\) is the nominal stress, \(\sigma_1\) is the true stress obtained from any of the models described previously, and

\[
a = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{bmatrix} \tag{3.74}
\]

Here, \(a\) is a symmetrical transformation tensor. If the axes of the local reference system are aligned with axes of material orthotropy, then the off diagonal terms \(\alpha_{jk}\) \((j \neq k)\) are zero, and the diagonal terms can be determined from

\[
\alpha_{ii} = \sqrt{\frac{A_{T,i}}{A_{n,i}}} \tag{3.75}
\]

where \(A_{T,i}\) and \(A_{n,i}\) are the true and nominal areas in the \(i^{th}\) direction. If the axes of orthotropy do not coincide with the local reference system, then the transformation tensor, \(a\), has to be related to the ratios of true and nominal areas perpendicular to the axes of orthotropy. In this system the nominal stresses can be written as

\[
\bar{\sigma}_n = r a \sigma_1 r^T = r a^T \sigma_1 r a = \bar{\sigma}_1 \bar{a}
\]

where \(r\) is a rotation transformation matrix. It follows that

\[
a = r a \bar{a} \tag{3.76}
\]
3.8 Conclusions

In this chapter, a combined damage/plasticity model for concrete and masonry has been developed. Starting from an isotropic damage model for cracking and a standard plasticity algorithm based on the quadratic Drucker-Prager model for the compressive regime, an algorithm has been obtained which ensures concurrent satisfaction of both damage and plasticity criteria. This has been made possible given the formulation of the plastic return mapping algorithm which is based on an Euler backward (implicit) approach and a single unknown. The mathematical description of the model is made relatively easy, as it is based on a principal strain formulation and because it does not require a formal thermodynamic framework. As an extension of the rotating crack model, a new model for including rate effects has also been developed.

The various models have been verified using biaxial and uniaxial loading schemes, and using the shear tests by Willam, showing a satisfactory and complete description of the material behaviour. Verification and calibration of the models will have to be carried out in the future. This remains a tricky task, since it is difficult to make a direct correlation between the model which defines the homogeneous response at a material point level, and the experimentally obtained response, which may show a strong dependency on the test setup (boundary conditions, specimen size, loading mechanism etc.). Recent studies based on new testing techniques suggest that it may be easier to carry out such a verification in the future. In this respect, tests such as the cable loaded uniaxial tensile tests of Shi and van Mier[49] are particularly interesting given the independency of the results on the experimental setup and the possibility of determining the precise loading situation at every moment in the experiment. In spite of the fact that the models developed here have not been compared directly with experiments, they reproduce approximately the response characteristics and the typical failure contour observed in experiments, as can be seen by the comparison of the stress-strain diagrams of Figures 3.8 and 3.11 with typical experimental response curves in Figure 3.2 and by the comparison of the failure contours of Figures 3.9 and 3.15 with experimentally obtained contours in Figure 3.3.

Other aspects which will require attention are the inclusion of hysteresis in the model (an important aspect here is the modelling of energy dissipation via frictional sliding along cracks, Raguennau et al[43]), and possibly the development of the tangent stiffness for the model for use in full Newton schemes. This last aspect has not come into consideration, since this thesis is primarily concerned with the use of indirect methods and which do not require the use of stiffness matrices.

In the next chapters the performance of the models developed in this chapter and their effectiveness in capturing actual failure modes will be verified in simulations of real structures.
Chapter 4

Methods for dynamic nonlinear analysis

In chapter 2, it was seen that the success of a finite element analysis is limited by problems of numerical nature. The problems, manifested by lack of convergence of the method used for solving the ensuing non-linear equations, are caused by ill-posedness in the formulation of the material models and their interaction with the solution method.

With this latter point in mind, a way to overcome part of the encountered numerical difficulties is to eliminate the process of solving for a system of equations. In a dynamic analysis, this is possible by making use of predictor-corrector methods for time integration such as the original method proposed by Newmark[35]. A direct advantage of using such an approach is that part of the problem is eliminated, enabling a direct evaluation of the performance of the material model.

4.1 Introduction

In a dynamic analysis the equations of motion arising from the finite-element discretisation need to be integrated over time. Many methods exist in order to carry out such an integration, see for example Wood[60] for an overview of methods applicable in structural dynamics, or Dahlquist[19] for a more mathematical background. Most methods can be described generally as predictor-corrector methods. In these methods, the solution at a new step is found by making a prediction based on values of the solution at previous time steps, and then correcting the prediction by carrying out a small number of iterations. The corrector phase has to be carried out regardlessly of whether the problem is linear or non-linear. Convergence is guaranteed if the time
step satisfies certain conditions (convergence criterion). A particular choice of integration parameters which implies the elimination of the corrector phase leads to what is commonly referred to as an explicit method.

Alternatively, the solution at the new step can be also found directly without going through the iterative corrector process. For a linear problem, this implies solving a set of linear equations. If the problem is non linear, then the ensuing set of equations has to be solved for using a Newton type method.

Whatever the choice of method for the corrector phase is, the time step has to satisfy stability conditions (i.e. small errors must not grow exponentially). In some cases it is possible to achieve unconditional stability by making an appropriate choice for the time integration parameters. In these cases, the use of a direct solver in the corrector phase is particularly attractive since it allows the use of a relatively large time step. It must be noted though, that if the structural problem involves material instability and is not well-posed in the formulation of the material model, then this will affect the performance of the equation solver used. In the worst case, the solver will fail to find a solution at all. Thus the necessity of a robust solver arises just as in the static case. From this point of view, the indirect approach is more desirable, as it avoids the introduction of numerical errors from the use of a solver, it does not require the evaluation of derivatives (i.e. the stiffness matrix whether updated or not), it can be used directly in the evaluation stages of a particular material model and it is easier to implement.

An attempt to overcome the problems related to the use of the direct method, by reducing the step size will increase the computational burden. In the limit, this method will deliver identical results to those obtained in an analysis carried out using the indirect method. The necessity of a direct solver thus seems diminished, particularly if the same order of step size is to be used as in the indirect method.

The use of the indirect method thus seems very attractive, since it allows carrying out an analysis without resorting to equation solving techniques. One of the most well known methods for structural analysis is the method originally developed by Newmark[35]. In the following it will be seen how this method can be implemented in the first of the two forms described above (the indirect type without equation solving). Aspects related to the convergence and stability of the method will be treated, and the method will be applied in a number of case studies relating to structures modelled using both rate independent and rate dependent models. The results show the validity of the approach, which in chapter 5 will be extended to the treatment of static problems.
4.2 Original implementation of Newmark’s method

Consider the semi-discrete equation of motion for a structure with \( n \) degrees of freedom,

\[
Ma + Cv + Ku = P
\]  

(4.1)

where \( a, v, u \) are the acceleration, velocity and displacement of each degree of freedom; \( M, C, K \) represent the mass, damping and stiffness matrices of the system; and \( P \) is some time-varying loading applied at each degree of freedom. Since eq. 4.1 is a second order differential equation, two parameters are required to initiate the motion of system, for example the initial displacements and velocities \( u_0 \) and \( v_0 \) and a single state variable is sufficient to describe the motion of the system in time.

In a numerical scheme, eq. 4.1 is integrated over finite time intervals, intervals over which the kinematics of the system is assumed to obey a particular relationship. The relations adopted in Newmark’s method are

\[
\begin{align*}
    \mathbf{v}_1 &= \mathbf{v}_0 + \Delta t \left[ (1 - \gamma) \mathbf{a}_0 + \gamma \mathbf{a}_1 \right] \quad (4.2a) \\
    \mathbf{u}_1 &= \mathbf{u}_0 + \Delta t \mathbf{v}_0 + \frac{\Delta t^2}{2} \left[ (1 - 2\beta) \mathbf{a}_0 + 2\beta \mathbf{a}_1 \right] \quad (4.2b)
\end{align*}
\]

where

\[
\mathbf{a}_j = M^{-1} \left[ \mathbf{P}_j - \mathbf{Cv}_j - \mathbf{Ku}_j \right] = M^{-1} \left[ \mathbf{P}_j - \mathbf{F} \left( \mathbf{u}_j, \mathbf{v}_j \right) \right] \quad j = 0, 1 
\]

(4.3)

and, \( \gamma \) and \( \beta \) are two integration parameters taking values between 0 and 1. The indeces 0 and 1 indicate values of the state variables at the beginning and at the end of the current step respectively. A more common notation is to replace the indeces 0 and 1 by \( n \) and \( n + 1 \), where \( n + 1 \) is the current step number, however the present notation is retained for clarity. Newmark’s method consists in the following predictor-corrector algorithm:

\[
\begin{align*}
    \tilde{\mathbf{v}}_1 &= \mathbf{v}_0 + \Delta t \left( 1 - \gamma \right) \mathbf{a}_0 \quad (4.4a) \\
    \tilde{\mathbf{u}}_1 &= \mathbf{u}_0 + \Delta t \mathbf{v}_0 + \frac{\Delta t^2}{2} \left( 1 - 2\beta \right) \mathbf{a}_0 \quad (4.4b) \\
    \mathbf{v}^{(i+1)}_1 &= \tilde{\mathbf{v}}_1 + \gamma \Delta \mathbf{t} \mathbf{a}^{(i)}_1 \quad (4.4c) \\
    \mathbf{u}^{(i+1)}_1 &= \tilde{\mathbf{u}}_1 + \beta \Delta t^2 \mathbf{a}^{(i)}_1 \quad (4.4d)
\end{align*}
\]

Alternatively, equations 4.2a and 4.2b can also be written in incremental form,

\[
\begin{align*}
    \Delta \mathbf{v} &= \Delta t \left[ (1 - \gamma) \mathbf{a}_0 + \gamma \mathbf{a}_1 \right] \quad (4.5a) \\
    \Delta \mathbf{u} &= \Delta t \mathbf{v}_0 + \frac{\Delta t^2}{2} \left[ (1 - 2\beta) \mathbf{a}_0 + 2\beta \mathbf{a}_1 \right] \quad (4.5b)
\end{align*}
\]

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and then the algorithm can be formulated as follows:

\[
\begin{align*}
\Delta v^{(1)} &= \Delta t M^{-1} [(1 - \gamma) P_0 + \gamma P_1 - F_0] \\
\Delta u^{(1)} &= \Delta t v_0 + \frac{\Delta t^2}{2} M^{-1} [(1 - 2\beta) P_0 + 2\beta P_1 - F_0] \\
\delta v^{(i+1)} &= \gamma \Delta t M^{-1} (F_1^{(i-1)} - F_1^{(i)}) \\
\delta u^{(i+1)} &= \frac{\beta}{\gamma} \Delta t \delta v^{(i+1)}
\end{align*}
\]  

(4.6a)  (4.6b)  (4.6c)  (4.6d)

where \( F_0 = F(u_0, v_0) \) and \( F_1^{(i)} = F(u_1^{(i)}, v_1^{(i)}) \). In 4.4 and 4.6 the superscript in brackets indicates the iteration number \( (i = 1, 2...) \). The iterative changes are defined in terms of the incremental changes as

\[
\begin{align*}
\delta v^{(i+1)} &= \Delta v^{(i+1)} - \Delta v^{(i)} \\
\delta u^{(i+1)} &= \Delta u^{(i+1)} - \Delta u^{(i)}
\end{align*}
\]

(4.7)

\[4.2.1 \text{ Convergence criterion}\]

The iterative process can be stopped once the difference in the internal forces between one iteration and the next are negligible. E.g. when

\[
\| F_1^{(i-1)} - F_1^{(i)} \| \leq \varepsilon_{\text{tol}}
\]

(4.8)

where \( \varepsilon_{\text{tol}} \) is some user specified tolerance. Further, one can judge the accuracy of the iterative process after say, \( k \), iterations, by calculating the norm of the residual forces, \( f \), given by:

\[
f = P_1 - M a^{(k)}_1 - F(u_1^{(k)}, v_1^{(k)})
\]

Here the accelerations at the end of the step can be found by rearranging equation 4.2a:

\[
a^{(p)}_1 = \frac{v^{(p)} - v_0}{\gamma \Delta t} - \frac{(1 - \gamma)}{\gamma} a_0
\]

(4.9)

The correct behaviour of the algorithm can be further monitored by calculating the external power at the end of each step:

\[
\hat{W}_1 = -(P_1 \cdot v_1 + F_1 \cdot \bar{v}_1)
\]

(4.10)

where \( \bar{F}_1 \) and \( \bar{v}_1 \) are the internal forces and the velocities of nodes with prescribed displacements. Alternatively, an energy monitoring approach can be implemented, Belytschko[9].
4.2.2 Convergence

Consider in both cases the linear, undamped, free vibration problem with $\mathbf{F} = \mathbf{Ku}$ and $\mathbf{P} = 0$. Now eq. 4.6d can be written as follows:

$$
\delta \mathbf{u}^{(i+1)} = \beta \Delta t^2 \mathbf{M}^{-1} \mathbf{K} [\mathbf{u}^{(i-1)} - \mathbf{u}^{(i)}] = -\beta \Delta t^2 \mathbf{M}^{-1} \mathbf{K} \delta \mathbf{u}^{(i)}
= ( -\beta \Delta t^2 \mathbf{M}^{-1} \mathbf{K})^i \Delta \mathbf{u}^{(1)}
$$  \hspace{1cm} (4.11)

The integration of the equations of motion at each step is thus defined by a sequence of the type

$$
\mathbf{y}^{(i+1)} = A \mathbf{y}^{(i)}
$$  \hspace{1cm} (4.12)

The convergence of such a sequence will be assured when all the absolute values of the eigenvalues of $A$ are smaller than 1. This condition poses a restriction on the size of the time step $\Delta t$, which can be deduced from the following eigenvalue problem:

$$
(\beta \Delta t^2 \mathbf{M}^{-1} \mathbf{K} - \lambda \mathbf{I}) \mathbf{y} = 0
$$  \hspace{1cm} (4.13)

which can also be written as

$$
(\mathbf{M}^{-1} \mathbf{K} - \omega^2 \mathbf{I}) \mathbf{y} = 0
$$  \hspace{1cm} (4.14)

where $\omega^2 = \frac{\lambda}{\beta \Delta t^2}$. The condition to be imposed is $|\lambda_j| < 1$, so that

$$
\Delta t < \frac{1}{\sqrt{\beta \omega_{\text{max}}}}
$$  \hspace{1cm} (4.15)

where $\omega_{\text{max}}$ is the highest angular frequency of the system. It should be noted that the criterion on the step size, $\Delta t$, loosens as $\beta$ is chosen smaller, see Figure 4.1. The convergence rate of the iterative process thus depends on the largest eigenvalue of $A$, which in turn depends on the highest frequency of the system and the chosen step size.

Damped case

If damping is present then eqs. 4.6c and 4.6d can be written as

$$
\delta \mathbf{v}^{(i+1)} = -\gamma \Delta t \mathbf{M}^{-1} \left( \mathbf{C} \delta \mathbf{v}^{(i)} + \mathbf{K} \delta \mathbf{u}^{(i)} \right)
$$  \hspace{1cm} (4.16)

$$
\delta \mathbf{u}^{(i+1)} = -\beta \Delta t^2 \mathbf{M}^{-1} \left( \mathbf{C} \delta \mathbf{v}^{(i)} + \mathbf{K} \delta \mathbf{u}^{(i)} \right)
$$  \hspace{1cm} (4.17)

which again can be thought of as a sequence of the type described by eq. 4.12. From the condition that the eigenvalues of $A$ must be smaller than 1, the following condition
on the time step can be deduced for each individual mode, $i$, of the system:

$$
\Delta t < \frac{\sqrt{\gamma^2 \xi_i^2 + \beta - \gamma \xi_i}}{\beta \omega_i}
$$

(4.18)

where $\xi_i$ is the damping ratio in mode $i$ (as defined in Clough and Penzien[14]).

### 4.2.3 Central difference method

By setting $\beta = 0$ and $\gamma = \frac{1}{2}$ eqns. 4.5 become:

$$
\Delta v = \frac{\Delta t}{2} (a_0 + a_1) \quad (4.19a)
$$

$$
\Delta u = \Delta tv_0 + \frac{\Delta t^2}{2} a_0 \quad (4.19b)
$$

The iterative process outlined in 4.6 thus becomes:

$$
\Delta v^{(1)} = \Delta t M^{-1} \left[ \frac{P_0 + P_1}{2} - F_0 \right] \quad (4.20a)
$$

$$
\Delta u^{(1)} = \Delta tv_0 + \frac{\Delta t^2}{2} M^{-1} [P_0 - F_0] \quad (4.20b)
$$

$$
\delta v^{(i+1)} = \frac{\Delta t}{2} M^{-1} (F_1^{(i-1)} - F_1^{(i)}) \quad (4.20c)
$$

$$
\delta u^{(i+1)} = 0 \quad (4.20d)
$$

If $M$ is diagonal and $F$ is independent of the velocity (i.e. $F = F(u)$) then the procedure can stop at the first iteration, since for $i = 1$, $F_1^{(i+1)} = F(u_1^{(i+1)}) = F(u_1^{(i)} + \delta u^{(i+1)}) = F(u_1^{(i)}) = F_1^{(i)}$ and the method is basically explicit. This method is equivalent to the central difference method.

If the system is damped, then the procedure needs to be carried out as in the general case by carrying out a certain number of corrector iterations. Further, the time step will be subject to a convergence criterion, which is given by:

$$
\Delta t < \frac{1}{2 \gamma \xi_i \omega_i} \quad (4.21)
$$

### 4.2.4 Direct approach

Alternatively to the predictor-corrector approach shown above, the solution at the end of the current step can be also obtained directly. This can be done by substituting

$$
a_1^{(p)} = \frac{\Delta u^{(p)} - \Delta tv_0 - (\frac{1}{2} - \beta) \Delta t^2 a_0}{\beta \Delta t^2} \quad (4.22)
$$
into the following functional

\[ f = P_1 - M a_1 - F(u_1, v_1) \]  \hspace{1cm} (4.23)

A solution to the problem \( f = 0 \) can be achieved using Newton's method in a manner similar to that outlined in 2.2.2.

### 4.2.5 Stability conditions for linear problems

To study the stability and accuracy of Newmark's method, rewrite the Newmark relations (4.2) for the linear case:

\[ v_1 = v_0 - \Delta t M^{-1} K [(1 - \gamma) u_0 + \gamma u_1] \]  \hspace{1cm} (4.24a)

\[ u_1 = u_0 + \Delta t v_0 - \frac{\Delta t^2}{2} M^{-1} K [(1 - 2\beta) u_0 + 2\beta u_1] \]  \hspace{1cm} (4.24b)

and consider the problem in terms of its natural modes of vibration. For each individual mode of vibration the Newmark relations can be written as:

\[ v_1^* = v_0^* - \Delta t \omega^2 [(1 - \gamma) u_0^* + \gamma u_1^*] \]  \hspace{1cm} (4.25a)

\[ u_1^* = u_0^* + \Delta t v_0^* - \frac{\Delta t^2}{2} \omega^2 [(1 - 2\beta) u_0^* + 2\beta u_1^*] \]  \hspace{1cm} (4.25b)

where \( \omega \) takes on different values for each mode. Dropping the use of the \( * \) and introducing the dimensionless variables \( \tau = \omega \Delta t \), \( y = \frac{v}{u_{ref}} \) and \( z = \frac{u}{u_{ref}} \), the above equations can be rewritten as

\[ y_1 = y_0 - \tau^2 [(1 - \gamma) z_0 + \gamma z_1] \]  \hspace{1cm} (4.26)

\[ z_1 = z_0 + y_0 - \frac{\tau^2}{2} [(1 - 2\beta) z_0 + 2\beta z_1] \]  \hspace{1cm} (4.27)

or

\[
\begin{bmatrix}
1 & \gamma \tau^2 \\
0 & 1 + \beta \tau^2 \\
\end{bmatrix}
\begin{Bmatrix}
y_1 \\
z_1 \\
\end{Bmatrix}
= 
\begin{bmatrix}
1 & -(1 - \gamma) \frac{\tau^2}{2} \\
1 & 1 - (1 - 2\beta) \frac{\Delta \tau^2}{2} \\
\end{bmatrix}
\begin{Bmatrix}
y_0 \\
z_0 \\
\end{Bmatrix}
\] \hspace{1cm} (4.28)

Solving for \( y_1 \) and \( z_1 \):

\[
\begin{Bmatrix}
y_1 \\
z_1 \\
\end{Bmatrix}
= 
\frac{1}{1 + \beta \Delta \tau^2}
\begin{bmatrix}
1 + (\beta - \gamma) \tau^2 & (\gamma - 2\beta) \tau^2 - 1 \\
1 & 1 - (1 - 2\beta) \frac{\tau^2}{2} \\
\end{bmatrix}
\begin{Bmatrix}
y_0 \\
z_0 \\
\end{Bmatrix}
\] \hspace{1cm} (4.29)

In compacted form this can be written as
\[ y_1 = Ay_0 \quad (4.30) \]

where \( A \) is called the amplification matrix. In order for the algorithm to provide a stable numerical solution of the motion, the motion must not be amplified from one step to the next and hence the spectral radius of \( A \) must be smaller than or equal to 1. To determine the spectral radius, compute the eigenvalues, \( \mu_1, \mu_2 \) of \( A \) from the characteristic equation:

\[ \mu^2 - 2 \left[ 1 - \frac{(\gamma + \frac{1}{2})}{2(1 + \beta \tau^2)} \right] \mu + \left[ 1 + \frac{(\gamma - \frac{1}{2})}{1 + \beta \tau^2} \right] = 0 \quad (4.31) \]

the roots of which are given by

\[ \mu_{1,2} = A_1 \pm \sqrt{A_1^2 - A_2} \quad (4.32) \]

where

\[
A_1 = 1 - \frac{(\gamma + \frac{1}{2})}{2(1 + \beta \tau^2)} \tau^2 \\
A_2 = 1 - \frac{(\gamma - \frac{1}{2})}{1 + \beta \tau^2} \tau^2
\]

From the condition \( |\mu_{1,2}| \leq 1 \), a set of conditions on \( \beta, \gamma \) and \( \tau \) can be determined. Table 4.1 summarizes the converge and stability requirements of the method. For \( \gamma = \frac{1}{2} \), energy is always conserved in the system. If \( \gamma > \frac{1}{2} \), the system will be artificially damped (algorithmic damping) and if \( \gamma < \frac{1}{2} \) spurious energy will be added to the system. Further, any specific choice of these paramters leads to some amount of error. Complete derivations of these stability conditions including damping effects and details as to the errors introduced by the scheme for different values of \( \beta \) and \( \gamma \) can be found in texts such as Hughes[28] or Belytschko et al[9]. In general, the errors introduced for \( \gamma = \frac{1}{2} \) are \( O(\Delta t^2) \), otherwise they are \( O(\Delta t) \).

**Remarks** Figure 4.1 shows the limits imposed by conditions for convergence and stability for \( \gamma = \frac{1}{2} \). Note the critical dependency of both types of requirement on the parameter \( \beta \). If \( \beta \) is chosen large to ensure stability, a small step size is still necessary to ensure convergence. Likewise if \( \beta \) is small, or equal to zero, as in the central difference method, then the method convergences within one or a small number of iterations, though a small step size is still necessary to ensure stability. Hence, whatever the choice of parameters, the step size can only be chosen in relation to the highest frequency in the system. As pointed out by Park[39], the maximum stability limit which can be achieved is \( \omega \Delta t \leq 2.82 \) when \( \beta = \frac{1}{8} \). Consequently, problems
modelled with fine meshes will require the use of very small step sizes in the analysis. The combination of small step size and the level of discretisation in terms of the large number of degrees of freedom increases the computational effort dramatically. However, the method presented here does not require the use of a stiffness matrix at all, bringing important programming and computational advantages. All programming related to proper stiffness matrix formulation is unnecessary, nor are factorisation routines required. The algorithm requires computation of internal forces only, and if the mass matrix is diagonal, all computations within the algorithm are reduced to vector operations.

4.2.6 Choice of time step and parameters

In order to choose a value for the time step which satisfies the conditions for convergence and stability, the maximum eigenvalue \( \omega_{\text{max}} \) needs to be calculated. Computationally this is a relatively costly task, so it is preferable to work with a simple
estimate. For example, an estimate can be based on the calculation of the traversal time of a wave through the material across the distance between adjacent nodes in the finite element mesh. Belytschko et al.[9] give the following estimate of the time step for the central difference scheme without damping:

\[ \Delta t = \min \frac{l_e}{\varepsilon c_e} \]  \hspace{0.5cm} (4.33)

where \( l_e \) is the smallest distance between any two nodes of element 'e' and \( c_e \) is the speed of sound in the element. Eq. 4.33 serves only as an indication and may vary according to the type of element and mass matrix formulation, Hughes[28]. To judge whether the time step has been chosen sufficiently small, the norm of the residuals or the rate of work of the external actions should be monitored as suggested in 4.2.1.

As a final remark, it is necessary to come to a proper selection of integration parameters. From the argument in the previous section, it would seem that by selecting \( \gamma = \frac{1}{2} \) and \( \beta = \frac{1}{3} \), the least stringent restriction on the time step can be obtained. This, however, requires a certain number of 'corrector' iterations. From a computational point of view, the explicit method is more advantageous as it requires a single iteration (in the non-viscous case). For \( \beta \neq 0 \), Acton[3] points out that the accuracy to which one iterates should not be very high, since this would only serve to improve accuracy of the numerical solution, which is simply an approximation of the exact solution. The step size should thus be chosen in terms of a specific accuracy e.g. \( \varepsilon \) and a maximum number of iterations, \( m \). Using eq. 4.11, it is required that

\[ \frac{\| \delta u^{(m)} \|}{\| \Delta u^{(1)} \|} = \left[ \beta \Delta t^2 \omega_{max}^2 \right]^{m} \leq \varepsilon \]  \hspace{0.5cm} (4.34)

or

\[ \Delta t \leq \frac{2\sqrt{\varepsilon}}{\sqrt{\beta \omega_{max}}} \]  \hspace{0.5cm} (4.35)

In Figure 4.1 the limit on the step size imposed by the above condition has been represented by a dashed line for the case \( m = 5 \) and \( \varepsilon = 10^{-2} \).

### 4.2.7 Non-linear problems

In a nonlinear problem, the tangent stiffness of the system changes with time, implying that the frequencies of vibration, \( \omega_i \), of the system can vary. If they remain bounded, that is \( 0 \leq \omega_i^2 \leq \omega_{max}^2 \), where \( \omega_{max} \) is the highest angular frequency of the initial (linear) system, then the criteria for convergence and stability are satisfied in the non-linear case as long as they are satisfied for the initial system. In other words, if the stiffness of the system remains positive definite and its eigenvalues do not increase, then the initial choice of step size is sufficient to insure convergence and stability. When the material has a softening branch the tangential stiffness may
lose its positive definiteness and thus for some values of \( \omega_i, \omega_i^2 < 0 \). The necessary condition to ensure convergence is (cf. eq. 4.15):

\[
\Delta t < \frac{1}{\sqrt{\beta |\omega_i^2|}} \quad i = 1, 2, \ldots n
\] (4.36)

where \( n \) is the number of modes of vibration of the system. The problem of stability of the method requires more attention, because the concept of stability used in 4.2.5 cannot be used here. This is due to the fact that numerical stability was achieved by dictating that the eigenvalues of the amplification matrix were less than unity. This would ensure that the numerical solution remained bounded as in the physical system. However, the physical system with a negative eigenvalue admits an unbounded growing solution. For example, the one degree of freedom system:

\[
\ddot{y} = \Omega^2 y
\] (4.37)

has a solution given by

\[
y = \frac{1}{2} \left( y_0 + \dot{y}_0 \frac{\dot{y}_0}{\Omega} \right) e^{\Omega t} + \frac{1}{2} \left( y_0 - \dot{y}_0 \frac{\dot{y}_0}{\Omega} \right) e^{-\Omega t}
\] (4.38)

the first term of which grows exponentially with time. The problem thus has to be treated by considering the growth of the relative error with respect to the exact solution. This type of study can be found in Kulkarni et al [29] for the central difference operator (and also for the 4th order Runge-Kutta operator). There, the authors concluded that central difference method has a relatively bounded error but for specific initial conditions. These conditions are represented by the case where \( y_0 + \frac{\dot{y}_0}{\Omega} = 0 \) (or nearby), when the solution is represented only by the decaying exponential.
4.3 Case studies I: rate independent models

4.3.1 Introduction: dynamic tests on concrete panels

Figure 4.2: Geometry of tested panel.

In the following, a number of dynamic analyses are presented showing the possibilities of simulation using the method outlined in this chapter. For models which are not rate dependent, it was seen in section 4.2.6 that the explicit central difference scheme has the least computational burden and so this is the method that is used here. The simulations mimic the experimental testing of a square panel of side 1.2 m and thickness 0.15 m (see Figure 4.2) in uniaxial compression or tension. The panel is constrained horizontally and vertically at the top and bottom edges and free at the sides, and is loaded by prescribing an equal inward or outward displacement of the top and bottom edges. The material model used is the combined quadratic Drucker-Prager and rotating crack model of section 3.4.2 with parameters given in table 4.2.

<table>
<thead>
<tr>
<th>$E$</th>
<th>10000 MPa</th>
<th>$\nu$</th>
<th>0.15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>1500 kg/m$^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_t$</td>
<td>1 MPa</td>
<td>$G_t$</td>
<td>20 J/m$^2$</td>
</tr>
<tr>
<td>$k$</td>
<td>4.5 MPa</td>
<td>$\alpha$</td>
<td>0.0 MPa</td>
</tr>
<tr>
<td>$G$</td>
<td>1000 J/m$^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_1$</td>
<td>1.0</td>
<td>$a_2$</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Table 4.2: Material properties for panel tests
<table>
<thead>
<tr>
<th>Mesh</th>
<th>Run Time</th>
<th>$\Delta t$</th>
<th>$u_{end}$</th>
<th>$T_{end}$</th>
<th># steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>5x5</td>
<td>22</td>
<td>$1.0 \times 10^{-5}$</td>
<td>0.4</td>
<td>0.02</td>
<td>2000</td>
</tr>
<tr>
<td>10x10</td>
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<td>$0.5 \times 10^{-5}$</td>
<td>0.4</td>
<td>0.02</td>
<td>4000</td>
</tr>
<tr>
<td>20x20</td>
<td>678</td>
<td>$0.25 \times 10^{-5}$</td>
<td>0.4</td>
<td>0.02</td>
<td>8000</td>
</tr>
</tbody>
</table>

Table 4.3: Analysis data for tension tests

4.3.2 1/4 Constrained panel tests

Given the symmetry of the problem, only the top left corner of the panel has to be simulated. The following results thus refer to a panel with half size side, constrained at the top in both directions and constrained horizontally at the right edge and vertically at the bottom edge. Two sets of results are shown: one referring to the panel in tension, and the other referring to the panel in compression.

**Tension tests**

Three different meshes have been used to model the panel, see Table 4.3. The prescribed top and bottom edge displacement is applied at a rate of 0.02 m/s for a total duration of 0.02 s. Figure 4.5 shows the patterns of cracking which appear in the 3 meshes at two different stages of the analysis. At the beginning of the analysis, cracking is initiated at the top left corner and propagates horizontally towards the right along the displaced edge. A secondary horizontal crack appears at a location below the primary crack, but does not extend across the entire width of the panel. The crack patterns are similar in all meshes, with the exception of the coarse mesh. The total vertical reaction force measured at the top edge, shown in the load-displacement diagrams of Figure 4.3, is similar for all cases. The response is linear up to the moment of fracture and then drops to zero. The maximum calculated response of about 90 kN corresponds closely with the specified material strength and the geometrical properties of the panel. A careful examination of the analysis reveals the presence of a residual vibratory motion even after the cracks have appeared. The correct behaviour of the integration scheme is shown by Figure 4.4, which shows the residual forces at each step of the analysis for the 20x20 mesh.
Figure 4.3: Vertical response at the top of the panel (half response). Tension test.

Figure 4.4: Norm of residual forces for 20x20 mesh. Tension test.
Figure 4.5: Damage distribution (cracks) in panel for tension test.
<table>
<thead>
<tr>
<th>Mesh</th>
<th>Run Time [s]</th>
<th>$\Delta t$ [s]</th>
<th>$u_{end}$ [mm]</th>
<th>$T_{end}$ [s]</th>
<th># steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>5x5</td>
<td>50</td>
<td>$1.0 \times 10^{-5}$</td>
<td>-4.0</td>
<td>0.02</td>
<td>2000</td>
</tr>
<tr>
<td>10x10</td>
<td>300</td>
<td>$0.5 \times 10^{-5}$</td>
<td>-4.0</td>
<td>0.02</td>
<td>4000</td>
</tr>
<tr>
<td>20x20</td>
<td>1412</td>
<td>$0.25 \times 10^{-5}$</td>
<td>-4.0</td>
<td>0.02</td>
<td>8000</td>
</tr>
<tr>
<td>40x40</td>
<td>10199</td>
<td>$0.125 \times 10^{-5}$</td>
<td>-4.0</td>
<td>0.02</td>
<td>16000</td>
</tr>
</tbody>
</table>

Table 4.4: Analysis data for compression tests

**Compression tests**

In this case, four different meshes have been used to model the panel, see Table 4.4. The prescribed top and bottom edge displacement is applied at a rate of 0.2 m/s for a total duration of 0.02 s. Figure 4.8 shows the patterns of plastic damage and cracks which appear in the meshes at the end of the analysis. Except for the coarser 5x5 mesh, all meshes develop a diagonal shear band which is initiated by plastic deformation at the top left corner of the panel. As the formation of the plastic shear band progresses, cracks start to develop close by. The formation of this band leads to a division of the panel into two triangular sections, the lower of which is forced to move sideways to the left. As this occurs, a vertical crack appears at the lower right corner of the panel. Further, a number of spurious cracks appear in the two finer meshes. The response of the panel is given in Figure 4.6. Although the response is generally similar, there is a slight anticipation of the moment of collapse in the two finer meshes. The maximum response is about 700 kN, which corresponds closely with the strength of the material in compression. Finally, the evolution of the residual forces for the 40x40 mesh are given in Figure 4.7.
Figure 4.6: Vertical response at the top of the panel (half response). Compression test.

Figure 4.7: Norm of residual forces for 20x20 mesh. Compression test.
Figure 4.8: Damage distribution in panel in compression test (left: equivalent plastic strains and right: cracks).
4.3.3 Full panel compression test

Here, the previous analyses on one-quarter sections of the panel are compared with compression simulations of the entire panel. Two analyses are carried out for a 20x20 mesh, one in which the top and bottom edges are displaced towards each other at the same rate, the other in which the bottom edge is kept fixed (see Table 4.5).

In the first case (both top and bottom edges displaced, Figure 4.10), the deformations localize in a diagonal cross pattern, the arms of which are similar to the diagonal shear bands which occurred in the previous one-quarter panel analyses. Similarly, a vertical crack appears in the panel as it separates into four different parts, two of which are forced to displace sideways.

In the second case (only top edge displaced, Figure 4.11), the pattern of damage is quite different. The vertical symmetry of the damage is lost, and the plastic damage concentrates along two diagonals emanating from the bottom two corners towards the centre. Again, cracks appear along the region where the plastic damage accumulates. The panel effectively separates into two regions, the lower of which acts as a wedge against the top region. This has the effect of splitting the top region as evidenced by two vertical cracks which develop from the centre of the panel. The progress of these vertical cracks which move upwards is abruptly put to a stop, as a series of horizontal cracks are created in the panel. Although the two panels develop quite different patterns of damage, their response is similar as shown by Figure 4.9.

The results demonstrate that the material model behaves in a correct manner, with the expected maximum response, and a localization of damage once the peak strength is attained. An interesting phenomenon is the occurrence of cracking in areas where localized yielding of the material takes place. This could be observed from the presence of a crack pattern which follows closely the shape of the localized plastic yielding. The results show also a certain sensitivity of the problem to the boundary conditions and also the mesh, as can be seen from the thickness of the localization band which tends to decrease with the finer meshes. Modification of the imposed prescribed displacement led to the formation of a very different damage pattern in the panel. These problems are related to the loss of well-posedness of the underlying mathematical description, as a consequence of the way softening is modelled. Such loss of well-posedness of the problem leads to a dependency of the solution on infinitesimally small variations in the boundary conditions, Bazant and Cedolin[7].
<table>
<thead>
<tr>
<th>Mesh</th>
<th>RunTime</th>
<th>$\Delta t$</th>
<th>$u_{\text{bottom}}$</th>
<th>$T_{\text{end}}$</th>
<th># steps</th>
</tr>
</thead>
<tbody>
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<td>2x2x10x10a</td>
<td>1840</td>
<td>$0.5 \times 10^{-5}$</td>
<td>-4.0</td>
<td>4.0</td>
<td>0.02</td>
</tr>
<tr>
<td>2x2x10x10b</td>
<td>1852</td>
<td>$0.5 \times 10^{-5}$</td>
<td>-8.0</td>
<td>0.0</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Table 4.5: Analysis data for full panel compression tests

![Graph](image)

Figure 4.9: Comparison of response for panel in compression with different boundary conditions.
Figure 4.10: Damage at 3 different stages of loading for full panel test with both end plates moving: equivalent plastic strains (left) and cracks (right).
Figure 4.11: Damage at 3 different stages of loading for full panel test with only top end plate moving: equivalent plastic strains (left) and cracks (right).
4.4 Case studies II: rate dependent models

4.4.1 Verification of mesh objectivity: one-dimensional bar in tension

In Chapter 3 two kinds of regularization techniques were introduced, namely the fracture energy approach, applied in the previous case studies, and the rate dependent approach via a rate of damage law. The performance of the latter approach is verified on the basis of a numerical problem, studied in the past by Sluys[51]. The problem consists in applying a sudden axial load at the free end of a bar which is fixed at one end. The bar has a length of 100 mm, a height of 10 mm and a thickness of 0.1 mm (resulting in a 1 mm² cross-sectional area). The material has a strength of 2 MPa, a density of 20000 kg/m³, a Young's modulus of 20000 MPa, and a Poisson ratio equal to zero. The applied load is 1.5 N, in a direction so as to pull the bar. The reflected wave at the fixed end is superimposed on the incoming wave leading to a doubling of the tensile stress in the bar at the fixed end. In this manner a crack is induced to form at the fixed end of the bar. Five different meshes are used to model the bar with 5, 10, 20, 40 and 80 elements respectively. An explicit time integration scheme is used with a time step of $5.0 \times 10^{-7}$ s for all meshes except that with 80 subdivisions. Out of stability considerations, a shorter time step of $2.5 \times 10^{-7}$ s has been chosen for this mesh.

For the sake of comparison, the problem is first considered using a rate independent material with a linear softening branch and a fracture energy of 2 J/m². Figures 4.12 and 4.13 show the response of the bar and the damage which occurs after the reflection of the incoming wave. The localization zone appears to be restricted to a single element, thus demonstrating the occurrence of mesh dependence as a result of using the rate independent material model. The damage in the three finer meshes localizes in zones at a small distance from the fixed edge. The post-peak response is also affected by the mesh as can be seen from Figure 4.12.

The results obtained using a rate dependent material show much less dependence on the mesh which is used. Figure 4.14 shows the response of the bar modelled using the rate dependent elastic damage model of section 3.5.1, with $c = 100000$ s⁻¹, $n₁ = 1$ and $n₂ = 1$ and alternatively $n₂ = 2$. The deformation of the bar (in terms of axial strains) after the crack has formed at the fixed end is shown in Figures 4.15 and 4.16. The results show a similar response for all meshes and a localization of the damage in a zone which is almost the same length in all cases (about 0.015 m). The actual deformations and crack patterns for both series are given in Figures 4.17(a) and 4.17(b). It can be noted that in the second series the response is less abrupt after the peak has been reached, and the strain distribution in the localization zone is more gradual.

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Figure 4.12: Reaction force at fixed end for rate independent bar at $t = 0.2 \times 10^{-3}$ s.

Figure 4.13: Deformations (x500) and damage ($\alpha_t$) of bar in axial test problem at $t = 0.2 \times 10^{-3}$ s (left end fixed). Rate independent model.
Figure 4.14: Reaction force at fixed end in one-dimensional rate dependent bar problem at $t = 0.2 \times 10^{-3}$ s: (a) case with $n_2 = 1.0$; (b) case with $n_2 = 2.0$;

Figure 4.15: Strain along bar in one-dimensional rate dependent bar problem at: (a) $t = 1.3 \times 10^{-4}$ s; and (b) $t = 2.0 \times 10^{-4}$ s. Case with $n_2 = 1.0$. 
Figure 4.16: Strain along bar in one-dimensional rate dependent bar problem at $t = 2.0 \times 10^{-4}$ s. Case with $n_2 = 2.0$.

Figure 4.17: Deformations ($\times 500$) and damage ($\alpha_t$) of bar in axial test problem at $t = 0.2 \times 10^{-3}$ s (left end fixed) for rate dependent model: (a) $n_2 = 1$, (b) $n_2 = 2$. 
Figure 4.18: Response of panel under compression modelled using rate dependent cut-off model.

4.4.2 Full panel in compression

The analyses of the panel of section 4.3.3 are proposed here with the material modelled using the rate dependent cut-off model described in section 3.5.2. The nonlinear material properties are: \( f_t = 1 \text{ MPa} \); \( c_t = 50000 \text{ s}^{-1} \); \( n_{t1} = 1 \); \( n_{t2} = 2 \); \( f_c = 10 \text{ MPa} \); \( c_c = 1000 \text{ s}^{-1} \); \( n_{c1} = 1 \); \( n_{c2} = 1 \).

Figure 4.18 shows the calculated response and Figure 4.19 shows the corresponding deformations for three different meshes: a 20 by 20 mesh, a 40 by 40 mesh and a 20 by 20 mesh with randomly offset nodes. The deformation of the panel after the peak response is concentrated in diagonals branching from the corners of the panel and in a main vertical crack in the centre of the panel. Compression damage occurs through the entire panel in the form of an X, whereas tension damage appears both in a central region as a series of vertical lines and along the vertical edges of the panel as a series of horizontal lines directed towards the centre of the panel. The general mechanism can be explained as follows. The first damage which occurs is damage in compression appearing at the four corners of the panel (here the stresses are larger due to the fixed boundary conditions in horizontal direction). This damage gradually propagates along the diagonals, inducing a sort of separation of the panel into four triangular sections. As the prescribed displacement is increased, the movement of the top and bottom triangular sections into the panel induces the remaining sections of the panel to move away from each other in a horizontal direction, hence the appearance of the central
Figure 4.19: Panel under compression at 1.25 mm displacement (rate dependent cut-off model). Left, deformations (x50); center, cracks in terms of the damage variable $\alpha_t$; right, compression damage ($\alpha_c$). Top row: 20x20 mesh; middle row, 40x40 mesh; bottom row: 20x20 random mesh.
vertical crack. The relative movement along the various section boundaries leads to friction which resists the horizontal movement of the left and right triangular sections. As a consequence, these triangular sections are led to bulge outwards rather than move outwards as rigid blocks. The ensuing curvature of the side edges eventually leads to the formation of a series of distributed horizontal cracks propagating towards the centre of the panel.

For comparison, the results of a similar series of analyses are given for the QDP rate dependent material model also described in section 3.5.2. The material parameters are identical except $n_{c_2} = 2$ and the extra shear parameter $c_s = 10000 \, \text{s}^{-1}$. Figures 4.20 and 4.21 show the results for this analysis series. As in the previous case, the results are nearly identical and independent from the mesh. In comparison, the damage pattern is different and the softening branch in the response curve is anticipated. A careful examination of the results shows that now the separation of the panel into various sections is sharper and that the distribution of damage in tension and compression coincide. The main damage is concentrated along the diagonal boundaries where sliding occurs, whilst some damage appears in a central horizontal band. The difference in the behaviour of the panel is a consequence of the parameter, $c_s$, which was introduced in order to control the shear behaviour of the model. In section 3.6 it was seen how the QDP model could be used to obtain overall lower values of the shear strength and the shear fracture energy with respect to the cut-off model.
Figure 4.21: Panel under compression at 1.5 mm displacement (rate dependent QDP model). Left, deformations (x50); center, cracks in terms of the damage variable $\alpha_t$; right, compression damage ($\alpha_c$). Top row: 20x20 mesh; middle row, 40x40 mesh; bottom row: 20x20 random mesh.
Figure 4.22: Comparison of response of panel under compression for cut-off and (alternative) QDP rate models in analysis with fixed bottom edge.

The effect is reflected in the results of the panel analysis, which seem to indicate a lower shear transfer along the diagonal bands in the panel. This is also evidenced from the fact that the lateral edges of the panel are straight and do not crack as in the previous case.

Finally, Figure 4.22 and Figure 4.23 show the results of a third analysis series whereby the bottom of the panel is held fixed and only the top end plate is displaced, comparing the cut-off rate model with the QDP rate model. The loss of symmetry in the movement of the top and bottom edges is reflected in the results which show loss of symmetry in the deformation and damage in the vertical direction. Again, due to the differences in the used material models, considerable differences in the total response can be observed.
Figure 4.23: Comparison of deformations and damage of panel for standard (top) and alternative QDP rate (bottom) models in analysis with only top end plate moving. Left, deformations (x50); center, cracks in terms of the damage variable $\alpha_d$; right, compression damage ($\alpha_c$).
4.5 Conclusions

Newmark's method is generally implemented in a form which requires the solution of a set of non-linear equations. The original method, on the other hand, corresponds to an iterative predictor-corrector method for solving dynamic problems, without the necessity of solving a set of equations. Because of certain time-step restrictions in this method and possibly long calculation times, it is almost not used in this form. However, considering the ever increasing computational power of modern day computers, the use of a small time step should not pose a severe problem. Further, a sub-class of Newmark's method, known as explicit methods ($\beta = 0$), is used commonly in car-crash analysis and of underground explosions. This option has not been used very often in the simulation of masonry and concrete structures. Further, applications of Newmark's original method to these types of non-linear problems is completely missing along with a discussion of the results which could be obtained using values of $\beta$ other than zero. In fact, it can be shown that the Newmark method applied to a non-linear dynamic problem yields practically the same solution as the implicit method in which a set of non-linear equations is used. This demonstrates that the use of an implicit method becomes superfluous in those cases which require a short time step in order to obtain a meaningful and accurate solution of the non-linear problem.

In this chapter the Newmark method has been reviewed and applied in its original form in LARES. Dynamic analyses of various problem cases have been carried out, using an explicit time integration scheme. The results show the feasibility of the use of such a method for non-linear analysis. An important aspect to consider is the simplification of the programming which this method allows: sophisticated factorisation techniques and non-linear equation solvers are not necessary, and setting up stiffness matrices is not required. All these aspects should be taken into consideration when developing a material model for the first time, as an analysis of the type carried out here can show immediately whether the material model behaves as expected in the non-linear regime.

As far as the actual analyses are concerned, an important development has been the introduction of rate dependency in the material models. Whilst the rate independent models do lead to results in which the response and the failure mechanism (distribution of damage and deformation) are physically plausible, they lead to results which are generally not independent of the mesh. This was seen in the analyses of the panels in compression and also that of the bar under tension. In the former (see for example Figure 4.5), the damage tends to localize always in a band of elements which is only 1 element thick. A similar tendency is even clearer from Figure 4.8, where the shear bands tend to become less thick with the finer meshes. The uniaxial bar problem also clearly shows the dependence of the results on the mesh. In these analyses, the mesh dependency was reflected even in the calculated response (refer to
Figure 4.12).

The use of a rate dependent model leads to a considerable improvement in the results, especially as far as mesh objectivity is concerned. In all cases which have been considered, the response and the failure mechanism were almost identical, independently of the mesh which was used. Models such as the rate dependent model used here, with effective regularization properties, thus deserve to be developed further and utilized in the non-linear analysis of structures. In the next chapter, it will be seen that the rate dependent model also leads to improved results in the static case.

In order to develop a general model which could cover different types of damage mechanisms, the models presented in chapter 3 have been obtained starting from simple models and combining these, for example, with the use of an algorithm such as that used in the combined elastic damage and quadratic Drucker-Prager model. The combination of different effects and the choice of mathematical formulation has been based only on an empirical approach. In general, a basic starting point is to develop a model which represents accurately the uniaxial response. Whilst the extension to biaxial or triaxial states of stress may be achieved relatively easily, the examples treated in this chapter have shown that small variations in the way models are formulated lead to great variations not so much in the load-displacement curve, but in the distribution of damage and in the localization of the deformation. This is particularly evident from the analysis of the panels under compression using the rate dependent models. A small addition to the model in the form of a weaker shear response altered considerably the distribution of the damage in the panel (compare Figures 4.19 and 4.23). To a lesser extent, the sensitivity of the results to the boundary conditions may also be of importance (compare the panel where both top and bottom edges were displaced with the panel where only the top edge was displaced). The various examples also show that in spite of the use of damage variables which describe supposedly independent damage mechanisms, a certain degree of interaction between the different types of damage mechanisms is unavoidable.

In large scale problems, variations between material models may lead to essentially different structural failure mechanisms, as will be seen in the analysis of some real structures in chapter 6. The issue of calibrating a model to reproduce closely the response at a structural level therefore needs to be addressed, and may require an approach whereby the actual physical phenomena related to material fracture are taken into account.
Chapter 5

A dynamic relaxation method for static nonlinear analysis

5.1 Introduction

The previous chapter showed how it is possible to obtain the dynamic response of a structure using an iterative algorithm of the predictor-corrector type consisting of simple vector operations. The solution was obtained without the need to solve a system of equations, and the validity of the results seemed to depend mostly on the material models used. Thus, it would seem natural to extend these methods to obtain the static response of a structure. This has been done in the past, using so-called dynamic relaxation (DR) methods, which obtain the static or steady state solution by performing a dynamic analysis (at constant loading) and eliminating the transient part via some form of damping. The method is similar to methods used for the iterative solution of linear system of equations such as Richardson's method or Frankel's method, also termed respectively 1st and 2nd order stationary iterative methods. All these methods including DR and methods such as the conjugate gradient can be reduced to a form given by a three-term recursion formula, Papadrakakis[38].

In the following, a dynamic relaxation method is proposed based on a dynamic solution of the problem using the method from the previous chapter. A form of artificial damping is introduced by adjusting the velocities of the system according to the magnitude of the accelerations of the system, without the necessity of calculating the optimal damping parameters.

Prior to that, the theory for iterative methods, namely on 1st order and 2nd order stationary iterative methods, is briefly reviewed. The similarity of the 2nd order stationary iterative method with dynamic algorithms (in this case the central
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difference scheme) is shown.

5.2 A brief outline of stationary iterative methods

5.2.1 1st order method

A method to determine the solution to the classical static structural problem, \( P = Ku \), is to carry out a transient analysis of the following 1st order problem:

\[
M\ddot{u} + Ku = P
\]

(5.1)

The motion of this system is such that it tends to dampen out until a displacement field is obtained that satisfies \( P = Ku \). This can be seen by recalling the type of solution to the homogeneous problem, which is of the form \( u = ce^{-\lambda t} \), where \( c \) is a vector constant and \( \lambda \) is an eigenvector of \( M^{-1}K \). In order to solve 5.1 the following explicit scheme will be used:

\[
u^{(n+1)} = u^{(n)} + \Delta tM^{-1}\left(P - Ku^{(n)}\right)
\]

(5.2)

The necessary condition to ensure stability of the solution is

\[
\Delta t < \frac{2}{\lambda_{\text{max}}}
\]

(5.3)

where \( \lambda_{\text{max}} \) is the highest eigenvalue of \( M^{-1}K \). Interestingly, by setting \( M = K \) and \( \Delta t = 1 \) then the classical Newton method for the solution of a static linear problem is obtained:

\[
u^{(n+1)} = u^{(n)} + K^{-1}\left(P - Ku^{(n)}\right)
\]

(5.4)

If the system is large and complex the inverse of \( K \) may be difficult to obtain accurately, in which case it may be necessary to carry out the calculation of 5.4 for several steps. Further, given storage and computation time limitations a number of methods have been devised to approximate \( K^{-1} \) using a preconditioner \( M^{-1} \) which will help reduce the number of steps required when using 5.2. If the system has a negative eigenvalue, then the solution will grow without bounds, as shown in section 2.4.3.

In the extreme case that computation of a stiffness matrix is not possible, then one should proceed with the calculation of 5.2 for a number of steps that is proportional to the condition number of the system,

\[
c = \frac{\lambda_{\text{min}}}{\lambda_{\text{max}}}
\]

(5.5)

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Clearly a large condition number will be detrimental for the total computation time. Axelsson[4] shows that the optimal value of $\Delta t$ leading to an optimal convergence rate is given by

$$\Delta t < \frac{2}{\lambda_{\text{max}} + \lambda_{\text{min}}}$$

(5.6)

5.2.2 2nd order method

The basic 2nd order method is similar to the first order method (eq. 5.2), but uses information from the previous two steps:

$$u^{(i+1)} = a_i u^{(i)} + (1 - a_i) u^{(i-1)} + b_i M^{-1} \left( P - Ku^{(i)} \right)$$

(5.7)

The above equation is known as the three-term recursion formula, which is closely related to many iterative methods. If constant parameters are chosen, $a_1 \equiv a$ and $b_1 \equiv b$ then the method is termed stationary. Axelsson[4] shows that the following stability condition needs to be met for convergence

$$0 < a < 2; \quad 0 < b < 2a/\lambda_{\text{max}}$$

(5.8)

and that optimal convergence rate can be achieved with

$$a = \frac{2}{1 + \sqrt{1 - \rho^2}} \quad b = \frac{2a}{\lambda_{\text{max}} + \lambda_{\text{min}}}$$

(5.9)

where $\rho = \frac{1 - \lambda_{\text{min}}/\lambda_{\text{max}}}{1 + \lambda_{\text{min}}/\lambda_{\text{max}}}$. The method originally was developed by Frankel in 1950. Compared to the 1st order method a faster convergence rate can be achieved.

Relationship with explicit iterative methods for dynamic analysis  In the following it will be shown that the central difference method ($\beta = 0$, $\gamma = \frac{1}{2}$) is identical to the 2nd order iterative method with specific parameters $a$ and $b$. From eq. 4.20 and considering a linear case:

$$v^{(i+1)} = v^{(i)} + \Delta t M^{-1} \left[ P - K \frac{u^{(i+1)} + u^{(i)}}{2} \right]$$

(5.10a)

$$u^{(i+1)} = u^{(i)} + \Delta t v^{(i)} + \frac{\Delta t^2}{2} M^{-1} \left( P - Ku^{(i)} \right)$$

(5.10b)

Rearranging 5.10b,

$$v^{(i)} = \frac{u^{(i+1)} - u^{(i)}}{\Delta t} - \frac{\Delta t}{2} M^{-1} \left( P - Ku^{(i)} \right)$$

(5.11)

substituting into 5.10a and reducing the indices by one unit, the following relationship between velocity and displacement is obtained:
\[ \mathbf{v}^{(i)} = \frac{\mathbf{u}^{(i)} - \mathbf{u}^{(i-1)}}{\Delta t} + \frac{\Delta t}{2} \mathbf{M}^{-1} \left( \mathbf{P} - \mathbf{Ku}^{(i)} \right) \]  
(5.12)

Substitution of 5.12 into 5.10b gives:

\[ \mathbf{u}^{(i+1)} = 2\mathbf{u}^{(i)} - \mathbf{u}^{(i-1)} + \Delta t^2 \mathbf{M}^{-1} \left( \mathbf{P} - \mathbf{Ku}^{(i)} \right) \]  
(5.13)

The above equation has the same format as the general 2\textsuperscript{nd} order method with constant parameters \( a = 2 \) and \( b = \Delta t^2 \) (compare with eq. 5.7).

### 5.3 Dynamic relaxation methods

Dynamic relaxation methods aim at finding the displacements of a structure under a given set of forces, by considering the problem as one of structural dynamics. Knowing that the structure will ultimately come to rest under the effects of damping forces, the static solution can be obtained by carrying out an analysis given a sufficient number of steps. An extensive review of DR can be found in Underwood[55]. In the past, the DR method has been formulated using mass matrices and damping matrices which are not necessarily related to the physical problem, for example, by choosing these matrices to be diagonal and proportional to the diagonal elements of the stiffness matrix. Using a suitable choice of mass, damping and time step, an optimal convergence to the steady state solution can be achieved. The optimal DR parameters are related to the extreme eigenvalues of the problem just as the iterative methods outlined above, and in fact it can be shown that the DR parameters are identical to those used in Frankel's method, see Papadrakakis[38]. Essentially, DR can serve to give a physical interpretation to the 2\textsuperscript{nd} order iterative method used for solving systems of equations.

#### 5.3.1 Kinetic damping

In a nonlinear problem where changes in stiffness can occur, it is necessary to adjust the DR parameters continuously. Underwood[55] and Papadrakakis[37] propose two different methods for adjusting the DR parameters automatically to account for the changes in the problem due to nonlinearities. In the following, rather than using either of these schemes for the automatic evaluation of the DR parameters, a method is proposed which does not require the continuous evaluation of the highest eigenvalue of the system. The method is similar to the method of kinetic damping described by Papadrakakis[37, 38], where an explicit scheme is used (e.g. the central difference method). At each step the kinetic energy of the system is monitored and when a peak in the kinetic energy of the system is detected, all nodal velocities are set to zero.

In the method proposed here, the choice of when to set the nodal velocities to zero is based on the norm of the accelerations rather than the magnitude of the kinetic
Figure 5.1: DR approach to finding the equilibrium point for a free vibrating single degree of freedom system. At the points A, B and C, the kinetic energy of the system has been removed.

energy. Note that the method described here is not necessarily restricted to an explicit scheme, and any predictor-corrector scheme may be used. Two acceleration norms are calculated at the start and at the end of the step, $q_0$ and $q_1$, respectively, with

$$q_x = \|a_x\|$$

and

$$a_x = M^{-1} [P - F(u_x, \nu_x)] \quad x = 0, 1$$

If the accelerations have increased (i.e. $q_1 > q_0$) the step is recalculated with the current velocities, $\nu_0$ set equal to zero. Fig. 5.1 shows schematically how the procedure works for a free vibrating single degree of freedom system. In essence, kinetic energy is removed from the system when a configuration of minimal acceleration is found. The procedure can be stopped once the calculated acceleration norm is lower than a user specified minimum. Further, some attention needs to be paid to those particular cases where the system can only move away from equilibrium. These situations are represented by those cases where for example the applied load is larger than the strength of the structure, or when the stiffness has become negative. In these cases unless special measures are taken, the velocity of the system will be set to zero at every iteration, and the solution will be 'locked'. In order to avoid this from occurring, the nodal velocities should not be reduced to zero completely, but should
be allowed to be maintained, the moment there is an indication of this type solution locking. This can be done by introducing a velocity multiplication factor given by

\[ \theta = 1 - 2^{-n} \]  

(5.16)

where \( n \) is an iteration counter which is set to zero whenever \( q_1 \leq q_0 \). Further, if displacement control is applied, care should be taken not to apply the imposed displacement too abruptly as this might lead to a concentration of damage in those elements with prescribed nodal displacements. In order to avoid this from happening it may be desirable to apply the load or displacement increment gradually using sub-increments prior to proceeding with the DR algorithm. The complete procedure is summarized by the following algorithm:

```
do i=1,m_steps
    get force increment
    get displacement increment
    apply load gradually in sub-increments:
    do j = 1, m_subincrements
        increment u and v according to 4.4 or 4.6
    end do
    DR-procedure:
    q_0 = |M^{-1} [P - F(u_0, v_0)]|
    do j = 1, max_iterations
        save initial variables: u_0, v_0
        calculate u_1 and v_1 according to 4.4 or 4.6
        q_1 = |M^{-1} [P - F(u_1, v_1)]|
        if q_1 > q_0 then
            reset initial variables
            \[ \theta = 1 - 2^{-n} \]
            \[ v_0 = \theta v_0 \]
            recalculate u_1 and v_1 according to 4.4 or 4.6
            n = n + 1
        else
            n = 0
        end if
    end do
    update material parameters
end do
```

Box 5.1: Dynamic relaxation procedure.
Figure 5.2: Geometrically non-linear problems (a) Case 1; (b) Case 2. Point masses $m_2 = m_3 = 1000$ kg placed at nodes 2 and 3.

A note on non-linear parameter update

In the DR algorithm shown, the material parameters are updated at the end of each step. If the material models are rate dependent, then the update of the material requires a fictitious time step, $\Delta \tau$, to be specified. This time step is not related to the step, $\Delta t$, specified in the DR procedure, but simply introduces a term in the analysis, describing a pseudo-loading rate $\frac{\Delta P}{\Delta \tau}$. In a static analysis, this term determines the evolution of the calculated response. Varying the number of steps in an analysis whilst keeping the loading rate unchanged, will lead to the same solution.

5.3.2 Application to simple geometrically non-linear structures

The procedure outlined above is applied to two classical problems represented by the structures in Figure 5.2. The exact force-displacement relation for the 1st case is given by

$$F_2 = -k_1 \left[ 1 - \sqrt{\frac{a^2 + b^2}{a^2 + (b - u_2)^2}} \right] (b - u_2)$$

and for the 2nd problem the displacement of the top node is given by

$$u_3 = u_2 + \frac{F_2}{k_2}$$
Figure 5.3: Load displacement diagrams (numerical vs. exact solution) and internal force history for (a) Case 1 and (b) Case 2 under load control and (c) Case 2 under displacement control.

Results of the analyses are given in Figure 5.3 where both cases have been analyzed under both load control and displacement control. The history of the internal forces for each iteration of the analysis is shown. To be noted is the sudden change in internal forces as the solution 'jumps' from the initial configuration to the final one. A final remark, is that although the method is not capable of reproducing the complete equilibrium path due to 'jumps', it is promising in that it can trace the equilibrium path beyond the critical points. Further results for real structures are presented in the following examples.
5.4 Case studies I: rate independent models

5.4.1 Panel compression test

In the following, the same panel of section 4.3.3 is simulated in compression under static loading conditions and using both the Newton method with constant initial stiffness and the dynamic relaxation method proposed here. An overview of the two analyses is given in Table 5.1. The response of the panel for both analyses and residual forces are given in Figure ???. In the first case, a pattern of damage develops which shows some similarity to the ones of the dynamic compression cases of section 4.3.3. Initially, plastic damage accumulates around an X-shaped region, Figure ???. As the imposed displacement increases, plastic damage localizes and at the same time cracks appear in the same X-shaped region. Finally, the global deformation loses its symmetry as the damage localizes more markedly along one of the diagonals. For the second case, the damage initiates in a similar manner, however the localization of plastic damage appears more suddenly and is concentrated along a horizontal band slightly above the central axis of the panel. At the same time, a central vertical crack and a horizontal crack in the bottom portion of the panel appear. The moment of collapse calculated using the dynamic relaxation method is anticipated with respect to the first analysis.

![Static compression test](image)

Static compression test: (a) load-displacement diagram; (b) residuals.

Table 5.1: Analysis data for static tests
Damage (equivalent plastic strains and cracks) in static compression test of panel using the Newton method with initial stiffness.
Damage (equivalent plastic strains and cracks) in static compression test of panel using the dynamic relaxation method.
5.5 Case studies II: rate dependent models

The previous case studies concerned the performance of the combination of the dynamic relaxation method with a rate independent model. It is interesting to verify the performance of the method when using a rate dependent model.

5.5.1 Bar problem

The problem of a bar in tension similar to that of section 4.4.1 is considered. The bar has a slight taper, the short side being at the fixed end of the bar, so that damage is initiated at a preferred "location". At every step, the displacement is applied in 200 subincrements and the analysis is continued for 1000 iterations more, using a time step $\Delta t = 5 \times 10^{-7}$ s (except for the bar modelled with 80 elements, in which the subincrements and iterations are doubled and the time step is divided by two). In the analysis, a prescribed displacement is applied at the right edge of the bar using increments of $5 \times 10^{-7}$ m. A total of 100 steps is carried out.

Figure 5.4 shows the distribution of damage in the bar for a set of analyses in which a fictitious time step is used which is 4 times larger than the DR time step, i.e. $\Delta \tau = 2 \times 10^{-6}$ s. An identical well-distributed damage pattern is obtained independently of the used mesh. The same is true also for the response of the bar, which is reported in Figure 5.5. The same analyses carried out using the Newton method with constant initial stiffness show identical results.

For comparison, the localization which occurs when using a rate independent model is shown in Figure 5.6. The localization now is limited to a single element and depends on the mesh. The deformation of the bar is also not independent of the mesh. This can be seen by closely observing the difference in the alignment of element edges from one edge to another. In the rate dependent analyses, the element edges corresponding to the same section of the bar are perfectly aligned, showing that the strain distribution is the same in all the meshes. The mesh dependency of the rate independent model is also observable from the response of the bar shown in Figure 5.7. It should be noted that for the finer meshes, the load displacement curves seem to converge to the same curve. After reaching the peak, these curves drop immediately to zero. However, this is due to snap-back, which is only possible to trace using an arc-length technique.

5.5.2 Panel compression analyses

Figures 5.8 and 5.9 show the results of the static panel compression tests using the rate dependent models proposed in section 3.5 and the dynamic relaxation method. The same material parameters as in the panels of section 4.4.2 are used, and the same DR parameters are used as in Table 5.1. Both the cut-off model and the QDP model
Figure 5.4: Tension damage ($\alpha_t$) and deformations (x500) in static bar problem at 0.05 mm displacement. Update only at every step.

Figure 5.5: Response of static bar problem for rate dependent model and for different meshes, for analysis with update only at every step.
Figure 5.6: Tension damage ($\alpha_t$) and deformations (x500) in static bar problem at 0.01 mm displacement for rate independent model, with $G_f = 2 \text{ J/m}^2$ and linear softening diagram.

Figure 5.7: Response of static bar problem for rate independent model and for different meshes.
Figure 5.8: Comparison of response of panel under compression for the cut-off standard and alternative QDP rate models.

are compared. In the same type of analysis using the rate independent model, Figure ??, the damage which occurred lacked symmetry (in the vertical direction) and the position of the compression localization zone occurred in a horizontal band at what seemed an arbitrary position. The actual position and shape of the localization may have been determined by numerical imprecision rather than a physical process. On the other hand, the use of a rate dependent model seems to lead to a lower sensitivity to "numerical perturbations" and the localization which takes place (Figure 5.9) retains also the symmetry which was observable in the dynamic simulations of the previous chapter. It is interesting to note, that just as in the dynamic panel analyses, the two models lead to different failure patterns: in the cut-off model, the compression damage localized in two horizontal bands next to the top and bottom panel edges, whereas the QDP model leads to an X-shaped failure mode.
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Figure 5.9: Comparison of deformations (x40) and damage of panel for cut-off (top) and QDP rate (bottom) models at 1.6 mm vertical displacement.

5.6 Conclusions

In this chapter, a dynamic relaxation method has been proposed and applied in the analysis of isotropic panels. The method proposed here does not make use of algorithmic damping which results from the specific dynamic integration parameters used, as this requires the estimation of the extreme eigenvalues of the system. Instead, the accelerations of the system representing indirectly the "static" residual forces are monitored, and when these reach a minimum according to a specific norm, the nodal velocities are set to zero. In the presented theory, relationships between iterative methods used in mathematics to solve linear systems and methods of structural dynamics have been treated.

In the analyses presented in this chapter, the proposed dynamic relaxation method has been applied in the analysis of a tapered bar in tension and the analysis of a square panel under compression. In the analysis of the panel under compression, in which the material was modelled using a rate independent model, it was seen that the use of the dynamic relaxation method leads to a similar response as the one obtained using the Newton method with initial stiffness. However, the damage patterns obtained using the two analysis methods differ considerably. In the analysis using the Newton method, the failure pattern has the form of an X, while in the DR analysis, the
damage simply localizes in a horizontal band. Despite the symmetry of the boundary conditions, neither analysis results in a perfectly symmetrical pattern of damage.

The introduction of rate dependency in the material model leads to a substantial improvement in the results. In the bar problem, the damage is distributed in the same manner and in the same region, and the response is the same, independently of the implemented mesh. The results were also independent of the used solution method. In contrast, the rate independent model never led to the same result with any of the meshes. Returning to the panel analyses, the symmetry in the damage patterns and deformation, which was lacking in the analyses using the rate independent models, is completely recovered.

In general, the use of the dynamic relaxation method has enabled the achievement of the solution of the static problem without resorting to the use of stiffness matrices. At the cost of a relatively slower convergence rate with respect to the initial stiffness method, this leads to a simplification of the programming.

Since the results obtained in this chapter are provisional, future research should address the issues related to convergence, accuracy and computational efficiency and make more comparisons with traditional Newton type methods. Some future developments can be envisaged in the relationship which exists between iterative methods for linear systems with the physical problem. Other ideas could be based on a combination of methods, for example, an approach in which Newton iterations are carried out as long as the system is relatively stable, followed by dynamic relaxation when the system becomes ill-conditioned.
Chapter 6

Applications

In order to demonstrate the applicability of the proposed material models of chapter 3 and the solution methods of chapter 4 and 5, two types of structural problems will be treated in this chapter. Both problems are related to the types of earthquake engineering issues which have been treated in chapter 1. The first relates to the analysis of concrete and masonry shear panels such as those that are typically tested in laboratory set-ups. The second problem concerns the study of infilled reinforced concrete frames, a problem which poses considerable difficulties due to the different interacting components.

6.1 Static analysis of shear panels

The shear panel consists of a square panel, fixed at the top and bottom edges as shown in Figure 6.1, and with thickness $t = 0.15$ m. In the analysis, the top edge undergoes a horizontal displacement applied in increments of $2.5 \times 10^{-5}$ m, while the vertical displacement is kept fixed at 0. Two cases are considered: an isotropic case and an anisotropic case. In the anisotropic case, use is made of the anisotropic material modelling developed in section 3.7. The anisotropy consists in stiffness and strength properties in the horizontal direction which have been reduced by 1/4 with respect to the vertical direction. For both cases, three different material models are used: (a) the rate independent combined damage and QDP model; (b) the rate dependent cut-off model; and (c) the rate dependent mixed model. The parameters used in the three material models are reported in Table 6.1. In the analyses using the rate dependent models, a fictitious time step $\Delta \tau = 2.5 \times 10^{-4}$ s is used. The analyses are carried out using the dynamic relaxation (DR) method (with max. 3000 iterations, $\Delta t = 0.5 \times 10^{-5}$ s) and for comparison, using the Newton method with initial stiffness (with max. 1000 iterations). The results of these analyses are discussed in section
6.1.1 for the isotropic case and section 6.1.2 for the anisotropic case.

6.1.1 Shear panels - isotropic modelling

The results of the analyses using the DR method are presented in the form of load-displacement graphs in Figure 6.2, and in terms of deformation and cracking at two different displacement levels in Figure 6.3. For all cases, the failure mode is initially characterised by horizontal flexural cracks at the top and bottom edges starting at the corners in tension. As the imposed displacement is increased, a diagonal crack is initiated simply due to high stresses along the tension diagonal. The response of the panel is approximately linear up to this stage, until at the peak of the response large localized deformations take place at the corners of the panel in compression, indicating evidently a failure mode of the material due to crushing. Concurrently, the diagonal crack, which was formed initially, opens up further and a distinct separation of the panel into two blocks can be observed. Beyond this point, the load-displacement curve changes abruptly and follows a softening branch. In spite of the similarity of the models (all are based on the rotating crack concept) and in spite of the similar strength and stiffness parameters specified, all the models lead to load-displacement curves with different peak strengths and ductility. The peak strengths are 150 kN, 140 kN and 180 kN for the three different material models, (a), (b) and (c) respectively. Whilst the panels based on models (a) and (b) present a similar energy dissipation, the introduction of a shear parameter, \( c_s \), in the mixed model (c) leads to a much more
Figure 6.2: Load-displacement graphs for shear panel analyses with different material models. The thick line refers to DR analysis, whereas the thin line refers to the analysis using the initial stiffness method. The 3 cases presented (a) to (c) refer to the different material models used.
brittle behaviour. Interestingly, model (c) presents the largest peak strength. The deformation and cracking patterns in the three cases also present subtle differences. In general, it can be concluded that the cases with higher ductility present more distributed damage. For example, this can be seen by comparing model (b) with model (c), whereby in the latter case the diagonal crack is thinner.

For comparison, the deformations and cracking patterns obtained with the Newton method with initial stiffness are presented in Figure 6.4. The response is similar to that obtained with the DR method, see Figure 6.2. The crack patterns are also similar to those in the DR analyses, however some small differences can be observed. Whilst the patterns are very similar for model (a), the analyses using the rate dependent models (b) and (c) present the largest differences. Also, what is noticeable is the slight asymmetry in the DR analyses, in particularly for model (c). On the other hand, the analyses using the Newton method are completely symmetrical.

### 6.1.2 Shear panels - anisotropic modelling

In order to verify the influence of anisotropy of the material on the global response of structures, a similar set of shear panel analyses has been carried out as in the above, this time introducing the anisotropic model developed in section 3.7. The types of analyses carried out and the basic material model parameters remain the same, however anisotropy is introduced into the material model by specifying a true area to nominal area ratio of 1/4 in the horizontal direction (see section 3.7).

The load-displacement diagrams for these analyses are presented in Figure 6.5. The initial response presents a lower stiffness, and the peak strengths of the panels are somewhat reduced, particularly for material model (c). The peak average strengths for the three models (a), (b) and (c) are 120 kN, 140 kN and 90 kN, respectively. Similarly to the isotropic panel analyses failure of the panels occurs first with flexural
Figure 6.3: Deformation (x100) and cracking patterns for shear panel analysis using the DR method, at 0.75 mm and 1.5 mm horizontal displacement. The 3 cases presented (a) to (c) refer to the different material models used.
Figure 6.4: Deformation (x100) and cracking patterns for shear panel analysis using the Newton method with initial stiffness, at 0.75 mm (left) and 1.5 mm (right) horizontal displacement.
cracking along the top and bottom edges of the panel, followed by cracking along the tension diagonal and crushing of the compression toes, see Figures 6.6 and 6.7. It is interesting to note that the formation of the diagonal crack is retarded with respect to the isotropic panel analysis. This can be seen by comparing the crack patterns in the isotropic panel analyses with those in the anisotropic analyses at the same displacement level. For models (a) and (b) in the isotropic case, diagonal cracking was already present at 0.375 mm displacement whilst in the anisotropic case this is not the case. For model (c), diagonal cracking is present at 0.75 mm displacement in the isotropic panel, whilst in the anisotropic panel the diagonal crack is only just forming. As a final remark, it should be noted that model (b) presents a series of multiple cracks in the analysis using the DR method.

![Graphs](image)

Figure 6.5: Load-displacement graphs for anisotropic shear panel analyses with different material models. The thick line refers to DR analysis, whereas the thin line refers to the analysis using the initial stiffness method.
Figure 6.6: Deformation (x100) and cracking patterns for anisotropic shear panel analysis using the DR method, at 0.375 mm (left) and 0.75 mm (right) horizontal displacement.
Figure 6.7: Deformation (x100) and cracking patterns for anisotropic shear panel analysis using the Newton method with initial stiffness, at 0.375 mm (left) and 0.75 mm (right) horizontal displacement.
6.1.3 Concluding remarks on the shear panel analyses

The results obtained for both the isotropic and the anisotropic panel analyses reproduce the basic features of the observed failure modes and response of similar panels tested in the laboratory, see for example the experiments of Rajmakers and Vermeltfoort [44]. The failure mode of such panels is characterized by flexural cracks along the fixed edges and a single diagonal shear crack. In all the analyses which were carried out, these fundamental modes of failure were reproduced independently of the material model which was used. On the other hand, although all the models used are based on the rotating crack concept, the global responses which have been obtained and the way damage actually developed in the panel was different for each model. In particular, it was seen how the rate dependent mixed model (model c) showed the least ductility and, in the anisotropic case, the lowest peak strength. The results suggest that an extremely important factor in determining the post-peak response of structures is the way the shear stresses across an open crack are transferred. As mentioned in chapter 3, important tasks in future investigations will be to study, in further detail, the fundamental failure mechanisms and to calibrate the material models using experimental evidence.

As far as the numerical solution is concerned, it has been shown that the dynamic relaxation method achieves results which are similar to those obtained with the Newton method. However, some small differences have arisen both in the response and in the distribution of the damage. This is in contrast with the results of the one-dimensional bar analyses discussed in section 5.5.1, which did not present differences depending on the used solution method. The differences which were observed in the panel analyses must be sought in the choice of analysis parameters, such as the size of displacement increments and the maximum number of iterations per step. Rules for making an appropriate choice of analysis parameters must therefore be established in a parametric study on the performance of the various solution methods and the quality of the delivered solutions.

The case studies which have been treated in this section have demonstrated the feasibility of the proposed material models and of the proposed solution methods in the non-linear analysis of masonry panels. In the next section, these models and methods will be applied to the more general case of an infilled reinforced concrete frame.
6.2 Analysis of an infilled reinforced concrete frame

The analyses discussed in this section refer to a one-storey framed masonry structure, which has been experimentally investigated in New Zealand, see Crisafulli[16]. In that investigation, the unit was constructed at a reduced scale of 3/4, representing the lower storey of a two-storey structure. In the following, two cases are discussed, one using the combined quadratic Drucker-Prager and rotating crack model and in the other, the rate dependent mixed damage model, whereby in both cases, the masonry is assumed to be isotropic. In the discussion, the possibilities and limits of the numerical tools which have been developed will be discussed. These aspects are important to review since the mechanics of infilled frames is not yet completely understood. The development of reliable computational methods must be seen as fundamental in providing answers to the issues which arise in the study of such structures.

Figure 6.8: Modelling details: mesh and reinforcement. The r.c. frame and the wall have a thickness of 0.15 m and 0.09 m, respectively.
6.2.1 Case I: Rate independent model-combined damage and plasticity

Details of the model used for the infilled reinforced concrete frame are shown in Figure 6.8, which shows the implemented mesh, and the steel reinforcement. The concrete frame and the masonry wall are modelled using a 26 by 35 mesh of quadrilateral 8-noded elements and the steel is modeled using discrete 2-noded truss elements. The combined quadratic Drucker-Prager and rotating crack model of section 3.4.2 are used to model both the concrete and masonry, whilst the steel is modelled as an ideal elasto-plastic material.

<table>
<thead>
<tr>
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<td>$\nu$ 0.15</td>
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<tr>
<td>$f_t$</td>
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<td>$G_t$ 60 J/m²</td>
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<tr>
<td>$f_c$</td>
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<td>$\rho$ 2500 kg/m³</td>
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<tr>
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</tr>
<tr>
<td>$G$</td>
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<td>$a_2$</td>
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<table>
<thead>
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<td>$f_c$</td>
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<tr>
<td>$G$</td>
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<table>
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<tr>
<td>$f_y$</td>
<td>340 MPa</td>
<td>$E$ 205000 MPa</td>
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</tbody>
</table>

Table 6.2: Material properties for infilled reinforced concrete frame.

The various model parameters are reported in Table 6.2 for each material and have been chosen on the basis of the strength parameters available in Crisafulli’s original report[16]. The possible effect due to the anisotropic nature of the masonry has not been taken into account. In the analysis, a prescribed displacement of (the nodes along) the top edges of the two columns is applied in the horizontal direction. In order to avoid potential damage accumulation in the elements connected to these nodes, the top row of elements belonging to each column is given only elastic properties. This ensures a proper transfer of forces into the structure. The frame and wall are assumed to be perfectly bonded. Several cases are presented: a static analysis of the frame without infill using the Newton method with constant initial stiffness, a dynamic explicit analysis of the infilled frame and the static analysis of the infilled frame using
Table 6.3: R.c. frame analysis summary

<table>
<thead>
<tr>
<th>Case</th>
<th>Method</th>
<th>$\Delta u_{Total}$ [m]</th>
<th>max. # iter.</th>
<th># steps</th>
<th>Run time (s)</th>
<th>$\Delta t$ (µs)</th>
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<td>2000</td>
<td>867</td>
<td>260000</td>
<td>2.5</td>
</tr>
</tbody>
</table>

the Newton and dynamic relaxation methods. A summary of the various analyses is presented in Table 6.3.

**Bare frame**

Figure 6.9 shows the deformations and crack patterns of the bare frame at 25 mm lateral displacement. Although the mesh is relatively coarse, typical flexural cracking at maximum bending moment sections can be recognized. Figure 6.10 shows the horizontal and vertical reaction forces at the column bases. The maximum lateral strength of the frame is 13 kN, which corresponds closely to the strength which would be obtained in a model of the frame using beam elements and which also corresponds to the strength predicted by Crisafulli. From the horizontal column base reactions in Figure 6.10a, it can be seen that both columns contribute almost equally towards sustaining the lateral imposed load. The presence of axial tension in the left column and the axial compression in the right column (observable from the vertical reactions at the column bases in Figure 6.10b) leads to a slight difference in the bending stiffness of the two columns. Consequently, the right column, which is stiffer than the left column, carries a slightly greater lateral force. Finally, Figure 6.11 shows the accuracy of the solution in terms of the euclidean norm of the residual forces at each step of the analysis. The residual forces are small only for the initial part of the analysis corresponding with the linear part of the response of the frame. At the onset of non-linear effects, in the form of flexural cracks, the convergence becomes relatively worse. Once the formation of cracks has stabilized, also the residual norm stabilizes.
Figure 6.9: Deformations (x10) and cracks at 25 mm displacement.

Figure 6.10: Response of bare frame: horizontal reactions (a) and vertical reactions (b) at base against applied horizontal displacement.
Figure 6.11: Residual forces in bare frame problem.

**Infilled frame**

**Dynamic analysis**  In this analysis, a dynamic simulation of the the infilled frame problem has been carried out. The frame is subjected to a prescribed displacement of the top edges of the columns, applied at a constant rate of 0.1 m/s. Figure 6.12 shows the deformation and formation of cracks at various stages of the analysis, and Figure 6.13 shows the response of the frame. During the initial loading stage, a crack appears at the bottom of the left column, and a small crack occurs in the top right edge of the beam. Further, cracks appear to develop in various regions of the wall (visible as lighter grey areas in Figure 6.12) along the main diagonal from the top left to the bottom right, along the bottom left and top right edges of the wall and along a series of diagonals at the bottom left corner of the wall. The initiating cracks along the top and bottom edges suggest a separation of the wall from the frame, however as the imposed displacement is increased, only the main diagonal crack continues to develop into a full crack. At a later stage, a series of horizontal cracks develop along the left column of the frame, suggesting the presence of a large axial tensile load in that column. It is only at a later stage that the wall separates from its support along the bottom left edge as evidenced by the presence of a large horizontal crack. Note also the presence of flexural cracking at the bottom of the right column towards the final stages of the analysis. In Figure 6.13 it can be seen that the total reaction of the infilled frame presents a somewhat irregular but cyclic response, with peaks of about 120 kN. The cyclic nature of the reaction forces is a consequence of the
Figure 6.12: Deformations (x50) and crack patterns in dynamic analysis of r.c. infilled frame.
dynamic motion of the frame in those regions in which it is free to move. Finally, it is interesting to note from the same figure, that the horizontal reaction at the top of the right column tends to vanish and that most of the imposed load is transferred via the top edge of the left column.

**Static analysis** The static analysis of the infilled frame has been carried out twice, once using the Newton method with constant initial stiffness (N-IS) and in the other case using the dynamic relaxation (DR) method proposed in chapter 5. The deformations of the frame, the emerging crack patterns and its response are shown in Figures 6.14 and 6.15 for the N-IS method and in Figures 6.16 and 6.17 for the DR method. The results, in terms of deformation, damage patterns and response, are similar for both cases. Compared to the dynamic analysis, the patterns of damage differ only slightly. As in the dynamic analysis, cracks appear first along the base of the left column, and opening cracks appear at various locations in the wall. There is an indication of frame/wall separation in the early stages of the analysis, particularly in the bottom right corner and top right wall edge, however this separation is limited in comparison to the crack which develops along the main diagonal. At increased displacement levels, a series of horizontal cracks develop in the left column indicating the presence of an axial tension in the column. Further, a horizontal crack develops at the left bottom edge of the wall.

The response is characterised by an initially linear branch, followed by a peak at 0.5 mm displacement of 100 kN for the N-IS method and 91 kN for the DR method.
Figure 6.14: Deformations (x25) and crack patterns for static analysis of infilled frame using the Newton method with constant initial stiffness.
Figure 6.15: Analysis using the Newton method with initial stiffness: (a) calculated response at prescribed displacement locations; (b) horizontal reaction forces at base; (c) vertical reaction forces. The displacement ranges shown in the top graph refer to the formation of the infill diagonal crack (A) and the formation of tension cracks in the left column (B).
Figure 6.16: Deformations (x25) and crack patterns for static analysis of infilled frame using the dynamic relaxation method.
Figure 6.17: Analysis using the DR method: (a) calculated response at prescribed displacement locations; (b) horizontal reaction forces at base; (c) vertical reaction forces at base.
This peak in the response corresponds with the formation of a main diagonal crack in the infill, see Figure 6.15(a). Beyond this initial peak, the response drops slightly to about 90 kN for the N-IS method and 85 kN for the DR method. Afterwards, the response increases at a much slower rate, and is more jagged. This section of the load-displacement diagram corresponds to the formation of tension cracks in the left column. From the vertical reaction force at the base of the left column shown in Figures 6.15(c) and 6.17(c), it can be seen that the left column is subject to a large tensile load, in agreement with the presence of horizontal cracks distributed along the length of the column. At about 5 mm displacement, the response reaches a plateau of about 130 kN for both analysis types. This point corresponds with the attainment of the tensile strength of the columns. The graphs indicate a vertical reaction force at the base of the left column of about 105 to 108 kN, which corresponds precisely with the yield strength of the longitudinal column reinforcement (2 * 157 mm² * 340 MPa = 107 kN). The high tensile force which develops as a consequence of the presence of the wall (which effectively acts as a brace) decreases drastically its capacity and stiffness in bending. This can be observed from the graphs of the horizontal reactions, which show that no horizontal load is transferred by the left column.

As in the dynamic analysis, the horizontal reaction force at the top of the right column works against the direction of displacement only in the initial stages of the analysis. This can be seen from the reaction forces in Figures 6.15(a) and 6.17(a). After the first peak in the response, i.e. after the main diagonal crack has formed in the wall, there is a decrease in the reaction force arising at the top of the right column and later an inversion of direction. This effect is less marked in the DR analysis, where a minimum reaction force is observed of -20 kN compared to -50 kN for the N-IS analysis. The inversion of the force may result from a possible extension of the top beam due to damage occurring in the structure.

**Remarks and comparison with experimental evidence** The general response of the infilled frame corresponds to that which would have been obtained with a simplified beam and column analysis, as performed with the program Raumoko[11], in which the wall is modelled using diagonal braces. Figure 6.18 compares the results of the various numerical analyses with those of the original experiment. Essentially, the response obtained in the simplified analysis is very similar to that of the finite element analyses presented here. However, in the experiment, the lateral strength obtained was less than 50% of the strength predicted by both types of analysis. The reason for this difference lies in the fact that the failure mode that occurred in the original experiment by Crisafulli was completely different from the failure mode predicted in the analyses shown here. In the numerical analyses the failure is determined by tension yielding of longitudinal reinforcement of the left column of the frame, and therefore the maximum lateral strength is essentially determined by the tensile strength of the
Figure 6.18: Comparison of the response of the infilled r.c. frame as obtained with Ruaumoko[11], the experiments by Crisafulli[16] and the present finite element analyses.

column. Failure of the infill is characterized by the formation of a diagonal crack. In the experiment, a diagonal crack did not appear in the wall, but rather a much shallower crack was observed along the bed joints of the masonry in the top section of the wall. This in turn led to a failure of the left beam column joint in the form of a crack at the bottom of the joint traversing the complete section of the column. Resistance to this mechanism was delivered by the combined effect of dowel action of the longitudinal reinforcement and friction along the opposite crack surfaces, and by the shear resistance provided by the right column in double flexure, resulting in a much lower total frame strength. A diagram representing the observed failure mode is shown in Figure 6.19.

When comparing the numerical results with the experiment, it should be mentioned that the failure mechanism which occurred in the experiment was unexpected. In fact, an experimental result similar to that predicted by the present analyses was expected given that the original design of the frame assumed a failure mechanism in which yielding of the longitudinal reinforcement of the column in tension would occur. Crisafulli mentions that the failure which occurred in the experiment has very seldomly been observed in other tests (which generally show diagonal cracking of the infill) and attributes a possible cause for this to the loading system which was used.
Another cause could lie in the quality of the materials, which could lead to the frame to be weaker in the beam/column joint zone. It could be concluded that the results of the numerical analysis are essentially correct, and that the actual failure mode could be obtained with more knowledge and data about the experiment. The type of failure mechanism which is predicted in the present numerical analyses may be of a type which prevails over other types of failure mode due to lack of calibration of the material parameters or due to the specific choice of material model. In an attempt to verify the influence of material on the failure mode of the experiment, an analysis based on a different material model, which could better describe a shear type of failure, will be presented in the next section. From this analysis, it will be seen that the problem could indeed lie with the calibration and the description of the material, as a mechanism similar to that observed in the experiment has been obtained using a different type of material model.

The numerical results related to the infilled frame, which have been obtained in the dynamic and static analyses show much similarity, Figure 6.20(a). In the static analysis, a better convergence has been obtained with the DR method compared to the N-IS method as shown in Figure 6.20(b); however this is at the cost of more iterations and computing time, see Table 6.3. Figure 6.20(c) shows the relative rate of convergence of the methods defined as the change in residual norm divided by the
Figure 6.20: (a) comparison of structural response between static N-IS and DR analyses; (b) norm of residual forces; (c) average relative rates of convergence (a value closer to 1 indicates a higher rate of convergence).

total number of iterations:

\[ C = \frac{\|r_0\| - \|r_n\|}{n \|r_0\|} = \frac{(1 - R)}{n} \quad (6.1) \]

Here, \(\|r_0\|\) is the norm of the residual forces at the beginning of the step, \(\|r_n\|\) is the norm of the residual forces at the end of \(n\) iterations, and \(R\) is the ratio \(\frac{\|r_0\|}{\|r_0\|}\). From Figure 6.20(c), \(C = 0.025\) for the N-IS method, whilst \(C = 0.000453\) for the DR method. Thus each iteration of the DR method reduces the residual norm by a factor which is about 50 times less than that obtained with the N-IS method.
Table 6.4: Case II, material properties for infilled reinforced concrete frame (concrete and masonry modelled using rate dependent mixed model).

6.2.2 Case II: Rate dependent model - mixed damage

In this section, the same infilled frame which was considered previously is simulated using the rate dependent mixed damage model for both the masonry wall and concrete frame under static conditions. The main differences with the material model used in the previous analysis are, apart from the rate dependency, the control of damage in compression via an elastic damage law and the special treatment of the shear behaviour of the material, through the use of the shear parameter $c_s$. Two analyses will be discussed, the first related to the simulation of the bare frame, and the second related to the infilled frame as summarized in Table 6.5. For the implemented material parameters, see Table 6.4. In both analyses, the prescribed horizontal displacement is incremented in steps of $2.0 \times 10^{-5}$ m. A fictitious time step, $\Delta \tau = 2.0 \times 10^{-4}$ s, has been used.

Bare frame The results for the analysis of the bare frame obtained using the initial stiffness method in terms of the response and deformation are given in Figures 6.21 and 6.22. Compared to the previous example, the maximum response is 5 kN, which is less than half than that obtained using the rate independent combined QDP and
<table>
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<td>-</td>
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</table>

Table 6.5: Rate dependent r.c. frame analysis summary

rotating crack model. The shear strength of the material is apparently so much lower in the model used in the present examples that shear slip is simulated along the reinforcement bars. This can be seen from the patterns of cracks which appear around the bars, and from the relatively little amount of bending and flexural (horizontal) cracking that can be observed in the members of the frame in comparison with the analysis of the bare frame which was discussed previously, see Figure 6.9. The low shear strength also causes a shear failure of the beam-column joint at a later stage in the analysis. Details of these phenomena are depicted in Figure 6.23a. In Figure 6.23a, it can be seen that bending leads to a single flexural crack at the base of the column. The formation of other flexural cracks is limited due to the presence of cracks along the interface between the longitudinal reinforcement and the concrete on the left side of the column. Slippage of the reinforcement is also clearly visible from the deformation of the elements around the reinforcement bar. It must be mentioned that in comparison to the bare frame analysis in the previous section, the present frame presents a much less ductile behaviour. From Figure 6.21, the response of the frame does present a constant maximum strength and drops rapidly beyond 15 mm displacement.

Infilled frame Although the rather weak shear response of the mixed rate dependent model led to an underestimation of the response of the bare frame and an exaggeration in the shear type failure modes, the infilled frame problem using the DR method reveals a failure mechanism of the frame which is similar to the one observed in the experiment. Figure 6.24 shows the deformation and cracking of the infilled frame for various levels of horizontal displacement. Initially, the infill presents cracks along the bottom left and top right corners of the wall/frame interface. Unlike the previous analyses no crack forms along the diagonal of the infill, and sliding and separation at the infill/frame boundaries can be clearly observed. At a higher horizontal displacement level, the frame-infill separation increases, in particular along the bottom edge of the panel. Due to the fact that failure in shear implies also compression failure (this is a characteristic of the model), crushing of the right bottom corner of the panel occurs. Beyond this stage, diagonal compression in the panel takes place via the inner column faces, rather than via the top and bottom edges of the panel. This observation is inferred from the fact that beyond 0.25 mm displacement, the
horizontal reaction at the base of the wall gradually decreases, and the shear at the bottom of the right column increases, see the graphs of the horizontal base reactions in Figure 6.25b. At this point, the response of the frame has reached a peak of 40 kN, see Figure 6.25a. Beyond 0.5 mm displacement, shearing of the left beam-column joint at the column interface occurs, and the total response of the structure drops considerably. A detail of the failure mechanism is shown in Figure 6.26.

Although the failure mechanism and the maximum strength of the frame correspond closely to those of the experiment, the ductility of the structure is much lower. Whilst in the experiment a deformation of 30 mm was reached, the ductility reached in the numerical analysis is only 0.5 mm. The reason for this partly lies in the fact that the longitudinal reinforcement is modelled using 2 noded truss elements, which lack the ability to transfer shear forces across the crack in the joint. The main reason, however, must be sought in the shear response of the model which is clearly too brittle. As a consequence, the friction, which developed along the surfaces of the crack in the beam/column joint of the real structure, is lacking completely in the numerical simulation. The value for the shear parameter $c_s$, in both the bare frame and infilled frame examples, clearly has to be reviewed.

Nevertheless, the numerical example of the infilled frame is interesting because the formation of cracks and shearing along the frame/wall boundaries has been achieved through the use of a model which attempts to simulate shear failure and which is embedded directly in the continuum elements. When using conventional models, the
modelling of such effects would otherwise require the use of special interface elements to be placed along the frame/wall boundaries, precluding the possibility of modelling shear failure and shear sliding along a crack that forms at an arbitrary point in the structure.

Figure 6.22: Deformations (x100) and cracks in bare frame at different displacement levels.
Figure 6.23: Details of bare frame problem at 7.5 mm displacement: (a) left column at base; (b) top left beam-column joint (deformations x100).
Figure 6.24: Deformations (x500) and crack patterns for infilled frame using DR method.
Figure 6.25: Response of infilled frame using DR method (a) applied forces; (b) horizontal reaction forces at base.

Figure 6.26: Details of beam-column joint failure in infilled frame problem at 0.5 mm displacement (deformations x100).
6.3 Conclusions

A series of analyses of shear panels and infilled reinforced concrete frames have been discussed in this chapter. The analyses have been carried out using the material models and the solution methods which have been developed in the preceding chapters of this thesis, with the intent to verify their capability in describing structural problems associated with extreme lateral loads similar to those which can occur during an earthquake.

In the various analyses of the shear panel, both in the isotropic and in the anisotropic case, mechanisms similar to those obtained in experiments have been obtained. However, the results of each analysis using different material models present differences in response and evolution of the damage (distribution and location). In spite of the small conceptual differences in the models, great variations in the calculated strength and ductility can be found. The results suggest that the shear behaviour of the models substantially affects the global response of the structure.

The case of the infilled reinforced concrete frame which has been analysed refers to that of an experimental investigation carried out by Crisafuli. The analysis has been carried out using the combined QDP and rotating crack model in one case, and in a second case using the rate dependent mixed model. Using the latter model, a failure mode similar to the one of the experiment has been achieved. Although a similar failure mode has been described and a similar strength has been obtained as in the experiment, the ductility which has been obtained is much lower and an immediate drop in the response occurs at displacement levels of less than 1 mm. Nevertheless, the example is interesting, as it has predicted the occurrence of shear sliding mechanisms along the frame/wall boundaries which in the first case were not present. These results, together with those of the shear panels, clearly indicate the importance of calibrating the material models on the basis of an appropriate series of experimental investigations focussing on the various failure modes of brittle materials.

As far as the solution methods are concerned, it has been shown that the DR method can be applied with success to the analysis of structural problems. The method produces solutions which are similar to those of the Newton method with a constant initial stiffness. Nevertheless, small differences in the solutions by using both methods and relatively large residual errors in the analyses indicate that a more in-depth investigation into the performance of these methods and into the selection of appropriate analysis parameters is required. A subject related to this investigation is also the influence of rate dependency in the material models on the solution. From the present analyses, the benefits of such models has not emerged as clearly as in chapters 4 and 5, where it was seen that rate dependency in the materials models has important regularization effects.
Chapter 7

Conclusions

This thesis deals with the problem of simulating reinforced concrete and masonry structures subjected to exceptional loads such as those arising during an earthquake. From the preliminary application of the standard finite element method to such problems, a series of numerical problems have emerged, which have lead to a detailed investigation into the actual working of the algorithms and models used in this method.

The main difficulty that was encountered in the computational analyses was the lack of convergence of the solution finding process. Lack of convergence in the structural problem can be attributed directly to difficulties in correctly formulating the material models, which are required to simulate the strongly non-linear nature of frictional materials. Nonetheless, a robust and automatic method for tracing the equilibrium path in a structural problem is currently missing. In an attempt to delineate a possible general approach to the issue of tracing equilibrium paths, this thesis has experimented with methods which do not require setting up stiffness matrices and hence do not require the solution of a set of non-linear equations. As the numerical solution depends directly on the material modelling, a number of material models have also been developed with the intent to improve numerical performance. Further, these material models have been developed in such a way so as to reproduce the typical failure modes of concrete and masonry. In order to develop and apply the presented theory, a finite element structural analysis program, LARES, has been developed using the Fortran 90 programming language.

As a first step, the Newmark method for dynamic analysis has been reviewed and implemented in its original form. Independently of the chosen time-integration parameters, this method does not require the solution of a set of non-linear equations. The only restriction in the method lies in the time step, which is subject to conditions of stability and convergence. In chapter 5, having introduced the concept of kinetic damping, a method has been obtained which is essentially a dynamic
relaxation method, and allows the solution of static problems.

The implementation of these methods for dynamic and static analysis allows a direct evaluation of the material model without undue disruption from the numerical solution finding process. In view of the continuously increasing speed of modern computers, the use of an explicit method should not be seen as a restriction but rather a way to evaluate quickly the performance of models without resorting to the implementation of updated consistent tangent stiffness matrices. Because it is not necessary to set up these matrices, an important reduction in the programming effort can be achieved. This approach allows for results which are complete (as opposed to being cut off at some stage of the analysis) and can be interpreted in terms of their ability to predict the physical reality.

Two new types of material models have been developed in this thesis. The first was based on a combined elastic damage/plastic approach in an attempt to include both fracture and crushing in a single model. Regularization in this model has been achieved via a fracture energy approach. The second type concerned elastic damage models in which regularization was achieved via a rate of damage law. Indirectly, the damage rate approach introduces a dependency on the strain rate in the model. Several analyses of the bar problem have demonstrated the regularization effect of this approach. A number of 2D problems using different meshes have confirmed this effect, showing the validity of the approach in delivering mesh insensitive results. In contrast, the fracture energy approach used in the combined model leads to mesh dependent results even though the response at a global level shows relatively little variation.

In chapter 6, the various material models and solution methods, which have been developed and implemented in LARES, have been applied in the analysis of a shear panel and an infilled reinforced concrete frame. From the comparison of the analyses which have been carried out with typical experimental results it was seen that the program is able to reproduce basic failure mechanisms which occur in real structures. Because of the lack of an experimental program focussing on the determination of basic material response parameters, a direct calibration of the material models was not possible. Consequently, differences in the models and relatively arbitrary choices of model parameters have led to important variations on the global structural level. In the shear panel analyses, such variations were reflected in differences in the global response and location of damage. The examples of the infilled r.c. frame, although extreme, presented different failure mechanisms. Referring to the panel compression analyses in chapters 4 and 5, different failure patterns have been obtained. These differences clearly arise due to the different ways of fracturing of the materials under different states of stress and the way such mechanisms are modelled. The evolution of these mechanisms in space and in time and the way they interact make the problem highly non-linear. It is of no surprise then, that the results of the present analysis show
an important sensitivity to the choice of material model and material parameters.

In future developments, investigations should focus on the calibration of the parameters in the models so as to reproduce accurately the experimentally observed behaviour. Such investigations should, in first instance, be concentrated on the individual material failure modes, and should serve to calibrate the related material model parameter. In second instance, more complex tests should be carried out whereby specimens are subjected to more complex loading configurations. Such tests should reveal the interaction between various material failure mechanisms and should serve as a basis to assess and calibrate the interaction between the various 'model failure mechanisms'.

From the theoretical point of view, an important result in this work has been the development of the rate dependent material models. The regularization properties of such models have been brought to light, however a deeper insight needs to be obtained as it has not been possible to grasp the full potential of these models in general structural analysis. The dynamic relaxation method which has been developed for the analysis of static problems, also requires further elaboration in order to improve the quality of the obtained static solutions, as it was seen that variations in the calculated response can arise according to the solution method used. Insight into these issues needs to be gained both through a more in-depth development of the background theory and through a series of "numerical experiments" which could be carried out in conjunction with the investigation into material model calibration described above.
Bibliography


Summary

Despite the widespread use of the finite element method for structural analysis, its application in the non-linear analysis of concrete and masonry structures is made difficult by a series of numerical and modelling problems. The main problem which can be encountered is the lack of convergence and instability in the algorithms used to solve non-linear system of equations. These aspects can be attributed directly to difficulties in formulating correctly the material models which are required to simulate the strongly non-linear nature of frictional materials such as masonry and concrete. The development of robust solution methods and the formulation of consistent material models are essential if computational methods are to be used successfully in solving complex engineering problems which cannot be tackled by using simple analytical models or by carrying out experiments. A typical example where the use of such investigation methods is limited due to the complexity of the interaction of structural components and to the size of the structures involved, is the study of the response of masonry structures and infilled concrete frames under earthquake loading.

The objectives of this thesis are twofold: to investigate the possibility of developing a robust solution method applicable in the non-linear dynamic and static analysis of structures and to develop consistent material models for concrete and masonry. In order to implement and test the methods and models a finite element program has been written.

In chapter 1, the problems related to the study of the earthquake survivability of masonry and concrete structures are discussed and the necessity of developing robust computational methods for this type of studies is put into perspective.

Chapter 2 gives a brief overview of material models and the types of solution methods which are typically used in finite element programs. A number of case studies of typical static non-linear analyses of concrete and masonry structures (such as masonry shear walls and infilled reinforced concrete frames) are reviewed. The case studies reveal the sensitivity of results to the type of solution method and the type of material model which are used. Often there is a point in the analysis beyond which it is not possible to obtain a solution for the next step.

Prior to a discussion of alternative solution methods, two new types of material
models are proposed in chapter 3. A brief discussion on the basic characteristics of the response of concrete and masonry is followed by the development of a combined elastic damage and plasticity model based on a fracture energy regularization approach. This model has been developed with the intent to combine fracture and crushing in a consistent manner including a realistic response upon load reversal. The second type of model concerns the application of a regularization technique in elastic damage models. The proposed regularization is based on a time-dependent evolution of the damage. In order to control the response of the model under pure shear (which in an elastic damage model is not optimal) a number of variants of this model are proposed. Finally, an approach towards anisotropic modelling of masonry is proposed. The response of each model under different loading combinations is verified using a material model driver which has been specifically developed for this purpose. On a qualitative basis it is shown that the models reproduce the typical response observed from tests on concrete and masonry specimens.

In chapters 4 and 5, alternative methods for dynamic and static analysis are proposed. The main focus is on methods avoiding the use of Newton iterations as this requires factorization of stiffness matrices. The reason for avoiding this is that in highly non-linear problems the stiffness matrix could become ill-conditioned and hence could lead to problems in the performance of the Newton method.

In chapter 4, Newmark’s method is reviewed. As opposed to current implementations, the original formulation of the method does not rely on the direct solution of a set of non-linear equations. A particular choice of parameters leads to a simply explicit scheme; this is an option that has been rarely explored in the analysis of the kind of problems which are dealt within the thesis. This method is then applied in the case study of a concrete panel under a vertical loading and that of a bar under tension using the previously developed combined damage and plasticity model and the rate dependent damage model.

In chapter 5, a method based on a dynamic relaxation approach is developed for carrying out static analyses. The method is based on the fact that a dynamic system under the action of damping forces will ultimately come to rest. By introducing an artificial form of damping, it is possible to achieve the same result, and consequently it is possible to trace the static response of a structure to a given set of load levels. The method is applied to a similar set of structures as in chapter 4.

In chapter 6, the various material models and methods which have been developed are applied in the analysis of two typical problems in earthquake engineering i.e. the simulation of shear walls and the simulation of an infilled reinforced concrete frame.

From the results of this work a number of conclusions have been drawn on the applicability of the proposed methods and models. As far as the solution of the structural problem is concerned, it has been shown that explicit methods can be successfully applied in the dynamic analysis of complex non-linear structural problems.
In spite of limitations on step size, such methods do not depend on the use of an algorithm for the direct solution of a set of non-linear equations. Hence, all the numerical problems related to the use of such methods are avoided and a direct evaluation of the performance of the material models, used in the analyses, is made possible. Further, this approach requires less effort in terms of programing.

The proposed dynamic relaxation method also performs satisfactorily and offers the same advantages as explicit methods, in terms of reduced sensitivity to non-linear behaviour and reduced programming effort. Although the method has a slower convergence than those based on the Newton method, it has been possible to achieve a solution to the problem without recurring to the use of stiffness matrices.

Both the combined elastic damage and plasticity model and the rate dependent damage model reproduce the basic response characteristics of concrete in compression and tension. Although the response obtained using the first type of model in problems of a panel under uniaxial tension and uniaxial compression is realistic, there is a clear dependency of the damage patterns on the mesh. The problem is highlighted in the analysis of a bar under uniaxial tension which shows that the rate independent model leads to distributions of damage which are clearly dependent on the resolution of the mesh. In the static analysis, the use of different solution methods in the panel compression tests also leads to different results. On the other hand, the use of the rate dependent damage model leads to a distribution of damage and strain along the bar which is practically independent of the mesh and the solution method. The same result is observed when the model is used in the analysis of the panel using different meshes.

In the practical analyses of chapter 6 it has been possible to reproduce the typical response which is observed in shear panels and infilled reinforced concrete frames. Nonetheless, these analyses have brought to light some issues which in the future will need to be addressed in more detail. One of these issues is related to the necessity of calibrating material model parameters on the basis of clear experimental evidence. Furthermore, not only the choice of parameters but also the formulation of the model can have important consequences on the final structural response. This has emerged clearly from the interaction that occurs between tension damage and compression damage and the effect of this interaction, for example, in determining the response of the structure to shear deformation.
Samenvatting

Ondanks het algemeen gebruik van de eindig elementenmethode voor de analyse van constructies, wordt zijn toepassing in de niet-lineaire analyse van beton en metselwerk constructies moeilijk gemaakt door een reeks van numerieke- en modelleringsproblemen. Het belangrijkste probleem dat kan ontstaan is het gebrek aan convergentie en stabiliteit in de algoritmen die worden gebruikt om niet-lineaire stelsel vergelijkingen op te lossen. Deze aspecten kunnen direct toegeschreven worden aan moeilijkheden in het correct formuleren van de materiaal modellen die worden vereist om de sterk niet-lineaire aard van wrijvingsmaterialen zoals metselwerk en beton te simuleren. De ontwikkeling van robuuste oplossingsmethoden en de formulering van consistente materiaal modellen is essentieel, wil men numerieke methoden met succes kunnen toepassen voor het oplossen van de complexe problemen welke niet kunnen worden aangepakt door middel van eenvoudige analytische modellen of experimenten. Een typisch voorbeeld waarin het gebruik van dergelijke onderzoeksmethoden uitgesloten wordt wegens de ingewikkeldheid van de wisselwerking van de componenten en de grootte van de constructies in kwestie, is de studie van de responsie van metselwerkconstructies en de met deze constructies opgevoerde gewapende betonraamwerken onder aardbevingsbelasting.

De doelstellingen van dit proefschrift zijn tweevoudig: het ontwikkelen van een robuuste oplossingsmethode toepasselijk in de niet-lineair dynamisch en statisch analyse van constructies en het ontwikkelen van consistent materiaalmodellen voor beton en metselwerk. Om de methoden en de modellen te implementeren en te testen is een eindig elementenprogramma geschreven.

In hoofdstuk 1 worden problemen rond de aardbevingsbestendigheid van metselwerk- en betonconstructies besproken en wordt de noodzaak voor robuuste numerieke methoden voor dit type problemen in perspectief gezet.

Hoofdstuk 2 geeft een kort overzicht van materiaalmodellen en oplossingsmethoden die typisch in huidig eindig elementenprogrammas gebruikt worden. Een aantal gevallenstudies van typisch statische niet-lineaire analyses van beton- en metselwerkconstructies worden besproken. De gevallenstudies openbaren de gevoeligheid van resultaten aan het type oplossingsmethode en het type materiaalmodel die gebruikt
worden. Vaak wordt er een punt in de analyse bereikt waarna het niet mogelijk is om een oplossing voor de volgende stap te verkrijgen.

Voorafgaand aan een bespreking van alternatieve oplossingsmethoden, worden twee nieuwe materiaalmodellen in hoofdstuk 3 voorgesteld. Na een korte discussie over de kenmerken van de responsie van beton en metselwerk wordt een gecombineerde elastisch schade en plasticiteitsmodel ontwikkeld waarin regularisatie is bereikt door middel van een op breukenergie gebaseerde benadering.

Dit model is ontwikkeld met de bedoeling om falen onder trek en onder druk op een consistente manier te combineren met inbegrip van een realistische responsie onder cyclische belasting. Het tweede type model betreft de toepassing van een regularisatie-techniek in elastische schademodellen. De voorgestelde regularisatie is gebaseerd op een tijdsafhankelijke evolutie van schade. Om de responsie van het model onder zuiver afschuivingsgeconstanteerd (wat in een elastisch schademodel niet optimaal is) te controleren zijn een aantal varianten van dit model voorgesteld. Tenslotte wordt een benadering voor de anisotrope modellering van metselwerk voorgesteld. De responsie van elk model onder verschillende belastingscombinaties wordt geverifieerd, gebruik makend van een 'material driver' die specifiek voor deze bedoeling is ontwikkeld. Op een kwalitatieve basis wordt aangetoond dat de modellen de typische responsie reproduceerende die in proeven op beton- en metselwerkmonsterwaargenomen is.

In hoofdstukken 4 en 5, zijn de alternatieve methoden voor dynamische en statische analyse voorgesteld. De aandacht gaat naar methoden die het gebruik van Newton iteraties vermijden aangezien dit factorisatie van de stijfheidsmatrices vereist. De reden hiervoor is dat in sterk niet-lineaire problemen de stijfheidsmatrix slechtsgeconditioneerd kan worden en vandaar tot problemen kan leiden in de prestatie van de methode van Newton.

In hoofdstuk 4 wordt de methode van Newmark voor dynamische analyse herzien. Tegengesteld aan huidige implementaties, is de originele formulering van de methode niet op directe oplossing van een stelsel niet-lineaire vergelijkingen gebaseerd. Een bijzondere keuze van de parameters leidt tot een expliciete methode; dit is een optie die zelden onderzocht is in de analyse van de soort problemen die worden behandeld binnen het proefschrift. Deze methode wordt dan toegepast in de analyse van een betonnen paneel onder een verticale belasting en in de analyse van een op-trek belaste staaf, gebruik makend van het eerder ontwikkelde gecombineerde schade- en plasticiteitsmodel en tijdsafhankelijke schademodel.

In hoofdstuk 5 wordt een methode ontwikkeld die op dynamische-relaxatie gebaseerd is voor het uitvoeren van statische analyses. De methode is gebaseerd op het feit dat een dynamische systeem onder de actie van de dempingsschakeren uiteindelijk tot rust zal komen. Door een kunstmatige vorm van demping te introduceren, is het mogelijk hetzelfde resultaat te bereiken en vervolgens is het mogelijk de statische responsie van een structuur op een gegeven reeks belastingen te vinden. De methode wordt
toegepast bij dezelfde soort problemen als in hoofdstuk 4.

In hoofdstuk 6 worden de diverse materiaalmodellen en de methoden toegepast in de analyse van twee typische aardbevingsproblemen, d.w.z. de simulatie van een afsluitwand en de simulatie van de met metselwerk opgevuilde gewapende betonraamwerken.

Uit de resultaten van dit werk zijn een aantal conclusies getrokken op toepasselijkheid van de voorgestelde methoden en modellen. Met betrekking tot de oplossing van het mechanisch probleem, is aangetoond dat expliciete methoden met succes in de dynamische analyse van complexe niet-lineair problemen toegepast kunnen worden. Ondanks beperkingen op stappgroottes, maken dergelijke methoden gebruik van een algoritme voor de directe oplossing van stelsels niet-lineaire vergelijkingen. Vandaar dat alle numerieke problemen met betrekking tot het gebruik van zulke methoden worden vermeden en een directe beoordeling van de prestaties van de materiaalmodellen, die in de analyses worden gebruikt, mogelijk wordt gemaakt. Verder leidt deze benadering tot een vereenvoudiging van de programma's.

De voorgestelde dynamische-relaxatiemethode is ook met success toegepast en biedt dezelfde voordelen zoals expliciete methoden, in termen van verminderde gevoeligheid voor niet-lineair gedrag en verminderde inspanning van de programmering. Hoewel de methode een langzamere convergentie heeft dan degenen die op de methode van Newton gebaseerd zijn, is het mogelijk geweest om een oplossing van het probleem te verkrijgen zonder het gebruik van de stijfheidsmatrices.

Zowel het gecombineerde elastische schade- en plasticiteitsmodel als het tijdsafhankelijke schademodel reproduceren de belangrijkste kenmerken van beton onder druk- en trekspanning. Hoewel de responsie van het eerste type model in de analyse van een paneel onder éénassige trek en éénassige druk realistisch is, is er een duidelijke afhankelijkheid van de schade patronen op de gebruikte mesh. Het probleem wordt benadrukt in de analyse van een staaf onder éénassige trekspanning hetgeen aantoont dat het tijdsafhankelijke model leidt tot distributies van schade die duidelijk van de resolutie van de mesh afhankelijk zijn. In de statische analyse leidt het gebruik van verschillende oplossingsmethoden in de paneel drukproeven ook tot verschillende resultaten. Anderzijds leidt het gebruik van het tijdsafhankelijke schademodel tot een distributie van schade en rekken langs de staaf die van het mesh en de oplossingsmethode onafhankelijk is. Hetzelfde resultaat wordt bereikt bij de analyses van een paneel dat met verschillende meshes gemodellerd is.

In de praktische analyses van hoofdstuk 6 is het mogelijk geweest om de typische responsie te reproduceren die wordt waargenomen in afsluitwanden en met metselwerk opgevuilde gewapende betonraamwerken. Niettemin, hebben deze analyses een aantal kwesties naar voren gebracht die in de toekomst in meer detail zullen moeten gehandeld worden. Eén van deze kwesties is verwant met de noodzaak om de parameters van de materiaalmodellen te kalibreren op basis van duidelijke exper-
imentele gegevens. Verder hebben niet alleen de keuze van parameters maar ook de formulering van het model belangrijke gevolgen voor de uiteindelijke responsie van de constructie. Dit is duidelijk naar voren gekomen door de wisselwerking die tussen trekschade en drukschade ontstaat en het effect van deze wisselwerking, bijvoorbeeld, op de responsie van de constructie onder afschuiving.
Curriculum Vitae

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