LECTURE NOTES ON THE PRINCIPLES AND PRACTICE OF AIRPLANE PERFORMANCE PREDICTION

G.J.J. Ruijgrok

PART II: POINT-PERFORMANCE IN STEADY SYMMETRIC AND UNSYMMETRIC FLIGHT

Delft - The Netherlands
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PREFACE

The present three course books contain the material used for a summer course given at the Department of Mechanical Engineering of the Technical University of Bandung (ITB) in August 1983. The course was part of a cooperation between ITB and the Department of Aerospace Engineering of the Delft University of Technology.

The cooperation is a contribution to the TTA-79 Technical Assistance Project as agreed between the Governments of Indonesia and the Netherlands.

The contents of the lecture notes are a reflection of the annual lectures on airplane performance presented in the first and second year of the Delft curriculum. The subjects of the course are divided into three parts. The first part gives a review of some basic elements which represent required background knowledge for the discussions on airplane performance. In the second part, estimation methods are developed based on the so-called point performance concept, whilst the third part deals with integration techniques used to solve the path performance equations. The student is also recommended to examine the items in the appendixes, presented in part III. Hopefully, these course books will stimulate the reader's interest to gather more specialized knowledge in the many topics of airplane performance.
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PART II: POINT-PERFORMANCE IN STEADY SYMMETRIC AND UNSYMMETRIC FLIGHT

CHAPTER 1: THE AIRPLANE IN STEADY SYMMETRIC FLIGHT

In the following the motion of the airplane in steady symmetric flight is described.

The kinematic and geometric quantities describing the flight condition are shown in Figure 1.

**AIRSPEED** $V$: Velocity of the centre of gravity relative to the air. The airspeed lies in the plane of symmetry. The velocity vector coincides with the $X_a$-axis of the flight path-axis system.

**FLIGHT PATH ANGLE (CLimb ANGLE)** $\gamma$: Inclination of the velocity vector to its projection on the horizontal plane. Sign convention: flight path angle is positive when the airplane climbs relative to the air, and the flight path angle is negative when the airplane is descending. Thus, from vertical climb to vertical dive the flight path angle varies from

$$\gamma = \pm \frac{\pi}{2}$$

(see Figure 2)

**RATE OF CLimb** $R/C$: Vertical component of the airspeed $V$, so that

$$R/C = V \sin \gamma$$

Sign convention:

- climb $R/C > 0$
- descent $R/C < 0$

**ANGLE OF ATTACK** $\alpha$: Angle between $X$-axis and $X_a$-axis. The angle of attack represents the attitude of the airplane relative to the oncoming air, and is positive if the $X$-axis is turned in positive sense relative to $X_a$-axis.
ANGLE OF PITCH $\theta$: Inclination of X-axis to its projection on the horizontal plane.

In agreement with the flight path angle, $\theta$ has a positive value if the X-axis lies above the horizontal plane.

In the symmetric flight the relation between flight path angle, angle of attack and angle of pitch is given by:

$$\theta = \alpha + \gamma$$

Figure 1 also shows the forces acting on the airplane. In symmetric flight all the forces lie in the plane of symmetry of the aircraft.

A. The resulting aerodynamic force R, and its components D (Drag) and L(Lift) along the negative $X_a$-axis and $Z_a$-axis, respectively.

B. The thrust of the propulsion system T.

C. The weight of the airplane W.

Summation of these forces along the flight path-axis system yields the equations of equilibrium for the airplane in steady symmetric flight.

Along the $X_a$-axis:

$$T \cos \alpha_T - D - W \sin \gamma = 0$$

(1)

Along the $Z_a$-axis:

$$T \sin \alpha_T + L - W \cos \gamma = 0$$

(2)

Note that lift and drag are taken positive along the negative $Z_a$ and $X_a$-axis.
The lift can be expressed as:

\[ L = C_L q S = C_L \frac{1}{2} \rho V^2 S , \]  

(3)

where \( C_L \) = lift coefficient

\( q = \frac{1}{2} \rho V^2 \) = dynamic pressure

\( S \) = wing area

Similarly,

\[ D = C_D \frac{1}{2} \rho V^2 S , \]  

(4)

where \( C_D \) = drag coefficient.

Insertion of equation (3) and (4) into (1) and (2), respectively yields:

\[ T \cos \alpha_T - C_D \frac{1}{2} \rho V^2 S - W \sin \gamma = 0 \]  

(5)

\[ T \sin \alpha_T + C_L \frac{1}{2} \rho V^2 S - W \cos \gamma = 0 \]  

(6)
VARIABLES DEFINING THE FLIGHT CONDITION IN STEADY SYMMETRIC FLIGHT

The airplane's lift coefficient and drag coefficient are functions of angle of attack, Mach number and Reynolds number:

\[ C_L = f(\alpha, M, R_e) \]
\[ C_D = f(\alpha, M, R_e) \]

Mach number \( M = \frac{V}{a} \) is a function of airspeed and height.

Reynolds number \( R_e = \frac{\rho V L}{\mu} \) is a function of airspeed and height.

Thus:

\[ C_L = f(\alpha, V, h) \]
\[ C_D = f(\alpha, V, h) \]

The thrust is determined by airspeed, altitude and powersetting:

\[ T = f(V, h, \Gamma) \]

The inclination of the thrust vector to the \( X_a \)-axis is a function of the angle of attack:

\[ \alpha_T = f(\alpha) \]

Clearly, the equations (5) and (6) contain the following variables:

\[ W, h, \alpha, V, \gamma, \Gamma \]

Together, these six variables define the flight condition in steady symmetric flight.

If an instantaneous condition is considered (given \( W \) and \( h \)), there remain four variables.
Thus, having four variables and two equations implies that two variables must be chosen to determine the momentary flight condition in steady symmetric flight.

The latter two variables are called control variables and correspond to the manner in which the pilot controls the airplane:

- angle of attack or stick position
- power setting or throttle position
Fig. 1: Airplane in steady symmetric flight.
Fig. 2: Sign of flight path angle $\gamma$. 

vertical climb

level flight

vertical dive
PART II: POINT-PERFORMANCE
IN STEADY SYMMETRIC AND UNSYMMETRIC FLIGHT

CHAPTER 2: THE AIRPLANE IN STEADY SYMMETRIC GLIDE

The basic equations in steady symmetric flight are (see Figure 1 of Chapter 1):

\[ T \cos \alpha_T - C_D \frac{1}{2} \rho v^2 S - W \sin \gamma = 0 \]

\[ T \sin \alpha_T + C_L \frac{1}{2} \rho v^2 S - W \cos \gamma = 0 \]

In steady symmetric glide \((T = 0)\) the airplane is kept in a state of equilibrium by lift, drag and weight only:

\[ - C_D \frac{1}{2} \rho v^2 S - W \sin \gamma = 0 \]

\[ C_L \frac{1}{2} \rho v^2 S - W \cos \gamma = 0 \]

Since drag is directed contrary to the direction of flight, equilibrium only exists if the weight has a component in the direction of flight.

The resulting situation in steady symmetric glide is shown in Figure 1.

In this case the flight path angle is negative. In order to avoid negative values, one uses:

\[ \bar{\gamma} = -\gamma \]

Similarly, for the rate of descent (rate of sink) is written

\[ \frac{R}{S} = - \frac{R}{C} \]
Then, the equations in steady symmetric glide become:

\[ C_D \frac{1}{2} \rho V^2 S = W \sin \gamma \]

\[ C_L \frac{1}{2} \rho V^2 S = W \cos \gamma \]

The quantities \( V, \gamma \) and \( R/S \) are given by:

\[ V = \sqrt{\frac{W}{S} \frac{2}{\rho} \frac{1}{C_L} \cos \gamma} \]

\[ \tan \gamma = \frac{C_D}{C_L} \]

\[ R/S = V \sin \gamma = V \frac{C_D}{C_L} \cos \gamma = \sqrt{\frac{W}{S} \frac{2}{\rho} \frac{C_D^2}{C_L^3} \cos^3 \gamma} \]

Usually, the assumption is made that the angle of descent \( \gamma \) is a relative small angle, and therefore that little error is involved in assuming \( \cos \gamma = 1 \).

This yields:

\[ V = \sqrt{\frac{W}{S} \frac{2}{\rho} \frac{1}{C_L}} \]

\[ \tan \gamma = \frac{C_D}{C_L} \]

\[ R/S = \sqrt{\frac{W}{S} \frac{2}{\rho} \frac{C_D^2}{C_L^3}} \]

In these equations \( W/S \) is wing loading, \( C_L/C_D \) is lift-drag ratio and \( C_L^3/C_D^2 \) is the climb factor.
Variation of V, $\gamma$ and R/S with angle of attack

Considered is the typical lift-drag polar of a sailplane in figure 2. The curves show that with increasing angle of attack the lift coefficient increases continuously, whereas both the lift-drag ratio and the climb factor exhibit a maximum value.

Numerical calculations on V, $\gamma$ and R/S for the sailplane given in Figure 2, are made in Table 1, assuming the following initial data:

Sailplane weight: $W = 300 \text{ kg}_f$

\[(W \approx 2492 \text{ N})\]

Altitude: $h = 0 \text{ m SA}$

\[(\text{air density } \rho = 1.225 \text{ kg/m}^3)\]

The results show that according to the variations of $C_L$, $C_L/C_D$ and $C_L^3/C_D^2$ with angle of attack:

- airspeed V decreases continuously with increasing $\alpha$, whereby $V_{\min}$ is reached at $C_L = C_L_{\max}$
- descent angle $\gamma$ reaches a minimum value for $(C_L/C_D)_{\max}$
- rate of descent R/S reaches a minimum value at that angle of attack whereby $(C_L^3/C_D^2)_{\max}$ occurs.

A graphic representation of the calculations is given in Figure 3.

Minimum rate of descent is of importance for endurance, while minimum angle of descent is of importance for maximum travel distance.

The value of the angle of descent depends on lift-drag ratio only. Consequently, also the value of $\gamma_{\min}$ is independent of altitude.
The maximum distance obtained in quasi-steady glide is therefore given by (see also Figure 4):

\[ R_{\text{max}} = \frac{h}{\tan \gamma_{\text{min}}} = h \left( \frac{C_L}{C_D} \right)_{\text{max}} \]

Thus, if an airplane is to glide as far as possible, the angle of attack during the glide must be such that lift-drag ratio is a maximum.

**PROBLEM**

Determine maximum distance for a sailplane:

Initial height : \( h = 2000 \text{ m} \)
Lift-drag polar : \( C_D = C_{D_0} + C_L^2/\pi Ae \);
\[ C_{D_0} = 0.014 \]
\[ Ae = 12 \]

**SOLUTION**

\[ C_L = \sqrt{C_{D_0} \pi Ae} = \sqrt{0.014 \pi \times 12} = 0.726 \]

\[ C_D = 2 C_{D_0} = 2 \times 0.014 = 0.028 \]

\[ \left( \frac{C_L}{C_D} \right)_{\text{max}} = \frac{0.726}{0.028} = 25.9 \]

\[ R_{\text{max}} = h \left( \frac{C_L}{C_D} \right)_{\text{max}} = 2000 \times 25.9 = 51800 \text{ m.} \]

Note that the effect of weight on gliding is that it does affect the airspeed and with that the endurance. However, weight does not affect the distance covered during glide.
THE HODOGRAPH

The hodograph is the curve, giving the relation between the horizontal and vertical component of the airspeed V.

As shown in Figure 5, the hodograph puts together the performance quantities $V$, $\gamma$, and R/S in a single curve.

The vector from the origin to a point on the hodograph is the airspeed $V$ and the angle confined between this vector and the horizontal axis represents the angle of descent.

Figure 5 also shows the characteristic points on the hodograph, namely:

1. $V_{\text{min}} = \sqrt{\frac{W}{S \rho}} \cdot \frac{1}{C_{L_{\text{max}}}}$

2. $(R/S)_{\text{min}} = \sqrt{\frac{W}{S \rho}} \cdot \frac{1}{(C_{L}/C_{D})_{\text{max}}}$

3. $\gamma_{\text{min}} = \arctan \frac{1}{(C_{L}/C_{D})_{\text{max}}}$

PROBLEM

Considered is a sailplane:

Weight : $W = 3000$ N
Wing area : $S = 14$ m$^2$
Lift-drag polar : $C_{D} = C_{D_{0}} + C_{L}^2/\pi Ae$ ;
$C_{D_{0}} = 0.014$
$A = 12.5$
e = 0.85
Altitude : $h = 2000$ m SA
($\rho = 1.0066$ kg/m$^3$).
A. Determine glide angle and rate of descent in steady symmetric glide at $V = 145$ km/h.

**SOLUTION**

Note that in steady symmetric glide the flight condition is determined by only one control variable, since $T = 0$. Thus, in the present problem the flight condition is completely defined by the datum: $V = 145$ km/h.

From:

\[
C_L \frac{1}{2} \rho V^2 S = W
\]

\[
C_D \frac{1}{2} \rho V^2 S = W \sin \gamma
\]

Follows:

\[
C_L = \frac{W 2 \frac{1}{2}}{S \rho V^2} = \frac{3000}{14} \frac{2}{1.0066} \frac{1}{40.3^2} = 0.262
\]

\[
C_D = 0.014 + \frac{0.262^2}{\pi \ast 12.5 \ast 0.85} = 0.014 + 0.002 = 0.016
\]

\[
\tan \gamma = \frac{C_D}{C_L} = \frac{0.016}{0.262} = 0.0611
\]

\[
\gamma = \arctan 0.0611 = 3.50^\circ
\]

\[
R/S = V \sin \gamma = 40.3 \ast 0.0611 = 2.46 \text{ m/s.}
\]

B. Determine minimum glide angle.

**SOLUTION**

From: $\tan \gamma = \frac{C_D}{C_L}$ follows that minimum glide angle will require the application of maximum lift-drag ratio, so that:
\[
C_L = \sqrt{C_D \pi Ae} = \sqrt{0.014 \pi \times 12.5 \times 0.85} = 0.684
\]
\[
C_D = 2 C_D = 2 \times 0.014 = 0.028
\]
\[
(C_L/C_D)_{\text{max}} = \frac{0.684}{0.028} = 24.4
\]
\[
\tan \gamma_{\text{min}} = \frac{1}{24.4} = 0.041
\]
\[
\gamma_{\text{min}} = 2.35^\circ
\]
\[
V_{\gamma_{\text{min}}} = \sqrt{\frac{W}{S \rho \frac{1}{C_L}}} = \sqrt{\frac{3000}{14} \times \frac{2}{1.0066} \times \frac{1}{0.684}} = 24.95 \text{ m/s } = 89.80 \text{ km/h}
\]

C. Determine minimum rate of descent

**SOLUTION**

From: \[
R/S = \sqrt{\frac{W}{S \rho \frac{1}{C_L^3/C_D^2}}}
\] follows that the angle of incidence must be such that the climb factor is as high as possible:

\[
C_L = \sqrt{3 C_D \pi Ae} = \sqrt{3 \times 0.014 \pi \times 12 \times 0.85} = 1.18
\]
\[
C_D = 4 C_D = 4 \times 0.014 = 0.056
\]
\[
(C_L^3/C_D^2)_{\text{max}} = (1.18)^3/(0.056)^2 = 529
\]
\[
(R/S)_{\text{min}} = \sqrt{\frac{W}{S \rho \frac{1}{(C_L^3/C_D^2)_{\text{max}}}}} = \sqrt{\frac{3000}{14} \times \frac{2}{1.0066} \times \frac{1}{529}} = 0.90 \text{ m/s}
\]
\[
V_{(R/S)_{\text{min}}} = \sqrt{\frac{W}{S \rho C_L}} = \sqrt{\frac{3000}{14} \times \frac{2}{1.0066} \times \frac{1}{1.18}} = 19.0 \text{ m/s } = 68.4 \text{ km/h}
\]
EFFECT OF ALTITUDE

To investigate the effect of altitude on the performance in steady symmetric glide two flight conditions at equal angle of attack and at different altitudes are considered.

Since \( \tan \gamma = C_D/C_L \) remains constant, the angle \( \gamma \) will be independent of altitude.

For the ratio of the horizontal speeds follows:

\[
\frac{V_{h2}}{V_{h1}} = \frac{V_2 \cos \gamma}{V_1 \cos \gamma} = \frac{V_2}{V_1} = \frac{\sqrt{\frac{W}{S} \frac{2}{\rho_2} \frac{1}{C_L}}}{\sqrt{\frac{W}{S} \frac{2}{\rho_1} \frac{1}{C_L}}} = \sqrt{\frac{\rho_2}{\rho_1}}
\]

The same result is obtained for the ratio of the vertical speeds:

\[
\frac{(R/S)_2}{(R/S)_1} = \frac{V_2 \sin \gamma}{V_1 \sin \gamma} = \frac{V_2}{V_1} = \sqrt{\frac{\rho_2}{\rho_1}}
\]

This yields:

\[
\frac{V_{h2}}{V_{h1}} = \frac{(R/S)_2}{(R/S)_1}
\]

This result is illustrated by the curves in Figure 6. Each point on the hodograph shifts to the right along a straight line when altitude increases (decreasing air density).

Coupled with it:

- the airspeed at a given angle of attack increases with increasing height.

- the rate of descent at a given angle of attack increases with increasing height.
EFFECT OF WIND ON THE PERFORMANCE IN STEADY SYMMETRIC GLIDE

The effect of wind on the ground speed \( V_g \) (relative to the earth) is found by the vector sum of the wind speed \( V_w \) and the airspeed \( V \) (see Figure 7):

\[
\vec{V}_g = \vec{V} + \vec{V}_w
\]

Here, the considerations are limited to the effect of a given wind velocity in a direction parallel to the plane of symmetry of the airplane.

Figure 8 illustrates that in the hodograph the ground speed is measured relative to replaced axes of which the origin is shifted over a distance equal to the wind speed and in a direction opposite to the wind direction.

The angle \( \gamma_g \) is called the real glide angle (glide angle relative to the earth).

Figure 9 shows the effects of horizontal and vertical wind velocities.
In the case of headwind, the origin is displaced along the positive \( V_{\text{hor}} \)-axis. For a wind speed directed upwards, the origin is shifted along the positive R/S-axis.

Naturally, the influence of wind is of great importance for the performance of sailplanes.
In the case of an upwind that is greater than the rate of sink relative to the air, the sailplane will climb relative to the earth.
Figure 9 also indicates the variation of minimum glide angle \( (\gamma_g)_{\text{min}} \) as caused by a headwind and a tailwind, respectively.
PROBLEM

Sailplane data:

- weight : \( W = 2500 \text{ N} \)
- wing area : \( S = 10 \text{ m}^2 \)
- lift/drag polar : \( C_D = C_{D_0} + C_L^2/\pi A e \);
  \( C_{D_0} = 0.012 \)
  \( A = 17 \)
  \( e = 0.8 \)

Determine the minimum value of an upwind in order to be able to perform a level flight at a pressure altitude \( h_p = 1000 \text{ m} \).
The ambient temperature of the air amounts \( T = 12 \text{ °C} \).

SOLUTION

Clearly, as depicted in the hodograph, the solution requires that the minimum rate of descent is determined:

\[
(R/S)_{\text{min}} = \sqrt{\frac{W}{S}} \frac{2}{\rho} \frac{1}{(C_L^3/C_D^2)_{\text{max}}}
\]
where

\[ C_L = \sqrt{3 \frac{C_D^0}{\pi \alpha}} = \sqrt{3 \times 0.012 \times \pi \times 17 \times 0.8} = 1.24 \]

\[ C_D = 4 \frac{C_D^0}{\pi} = 4 \times 0.012 = 0.048 \]

and

\[ \left( \frac{C_L^3}{C_D^2} \right)_{\text{max}} = 825 \]

Density of the air follows from:

\[ \rho = \frac{P}{RT} = \frac{89876.3}{287.05 \times (273.15 + 12)} = 1.098 \text{ kg/m}^3 \]

From Appendix A:

\[ h_p = 1000 \text{ m} \Rightarrow p = 89876.3 \text{ N/m}^2 \]

Using these values, one finds:

\[ (R/S)_{\text{min}} = \sqrt{\frac{2500}{10} \frac{2}{1.098} \frac{1}{825}} = 0.74 \text{ m/s} \]

Thus, the required upwind is:

\[ V_w = 0.74 \text{ m/s} \]
Table 1: Calculation of glide angle, airspeed and rate of descent of a sailplane.

airplane weight: \( W = 660 \text{ lb} \) (300 kg)
wing area : \( S = 151.8 \text{ sq} \cdot \text{ft} \) (14.1 \( \text{m}^2 \))
wing loading : \( W/S = 4.35 \text{ lb/ft}^2 \) (31.3 kg/m\(^2\))
altitude : \( h = 0 \text{ ft SA} \)
lift-drag polar: fig. 2

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Fig. 1: Airplane in steady symmetric glide.
Fig. 2: Lift-drag polar of a sailplane.
Fig. 3: Performance in gliding flight.
Fig. 4: Horizontal distance in steady glide.

Fig. 5: The hodograph.
Fig. 6: Hodograph of a sailplane.
Fig. 7: Airspeed and groundspeed.

Fig. 8: Effect of wind in the hodograph.

Fig. 9: Vertical and horizontal wind velocities in the hodograph.
PART II: POINT PERFORMANCE IN STEADY
SYMMETRIC AND UNSYMMETRIC FLIGHT

CHAPTER 3: POWER REQUIRED AND POWER AVAILABLE

BASIC RELATIONS

Consider an airplane in steady symmetric climbing flight, as shown in 1. Summing forces parallel to the flight path, we have:

\[ T \cos \alpha_T - D - W \sin \gamma = 0, \quad (1) \]

and perpendicular to the flight path, we get:

\[ T \sin \alpha_T + L - W \cos \gamma = 0 \quad (2) \]

In normal flight conditions, the component \( T \sin \alpha_T \) in equation (2) is relatively small when compared to the other terms. Therefore, it is useful to ignore this force and to assume that the propulsive force points into the direction of the velocity \( V \). Figure 2 shows the force diagram, in the case of \( \alpha_T = 0 \). Then, the equations (1) and (2) reduce to:

\[ T - D - W \sin \gamma = 0 \quad (3) \]

and

\[ L = W \cos \gamma \quad (4) \]

Multiplication of equation (3) with airspeed \( V \), yields:

\[ TV = DV + W V \sin \gamma \quad (5) \]

Equation (5) expresses the work per unit time done by the forces in flight direction, and is often used in this form.
By introduction of the rate of climb $R/C$ of the airplane:

$$R/C = V \sin \gamma,$$  \hspace{1cm} (6)

equation (5) becomes:

$$TV = DV + W(R/C)$$  \hspace{1cm} (7)

On the left-hand side of equation (7), $TV$ is the power delivered by the power plant at the velocity $V$:

$TV = P_a$ is power available

The term $DV$ on the right-hand side of equation (7) is the power required for flight at velocity $V$:

$DV = P_r$ is power required

Thus, equation (7) can also be expressed as:

$$P_a = P_r + W(R/C)$$  \hspace{1cm} (8)

The second term on the right-hand side of equation (8) is the excess power:

$W(R/C) = P_c$ is excess power

Opposite, the excess power is the difference between power available and power required.
DRAG AND POWER REQUIRED FOR LEVEL FLIGHT

From equation (4), \( L = C_L \frac{1}{2} \rho V^2 S \), and \( D = C_D \frac{1}{2} \rho V^2 S \) we can easily find that:

\[
V = \sqrt{\frac{W}{S \rho}} \cos \gamma \frac{1}{C_L} \quad (9)
\]

\[
D = C_D \frac{1}{2} \rho V^2 S = \frac{C_D}{C_L} W \cos \gamma \quad (10)
\]

\[
P_r = DV = W \sqrt{\frac{W}{S \rho}} \frac{C_D^2}{C_L^3} \cos^3 \gamma \quad (11)
\]

These expressions show that for an airplane with a given weight \( W \) and flying at a given altitude \( \rho \), airspeed, drag and power required are functions of both angle of attack and flight path angle. At each angle of attack, the flight path angle is determined by the thrust of the propulsion system.

In order to simplify the situation, the effect of flight path angle in climbing flight is neglected. In other words, the cosine of flight path angle is set equal to one \( \cos \gamma = 1 \).

In this case, airspeed, drag, and power required solely depend on the angle of attack. Then, equations (9) to (11) reduce to:

\[
V = \sqrt{\frac{W}{S \rho}} \frac{1}{C_L} \quad (12)
\]

\[
D = \frac{C_D}{C_L} \quad (13)
\]

\[
P_r = W \sqrt{\frac{W}{S \rho}} \frac{C_D^2}{C_L^3} \quad (14)
\]

For a twin-engined airplane with turboprops and with a given airplane condition, airplane weight and flying at a given altitude in standard atmosphere, airspeed, drag and power required are determined in Table 1. The calculations are made starting from chosen values of \( C_L \).
Since it concerns an airplane designed for low subsonic airs- 
speeds, the corresponding values of $C_D$ follow directly from 
the lift-drag polar for the airplane, which curve is given in 
Figure 3. The results of Table 1 are plotted as a function of airspeed $V$ 
in Figure 4. On the curves in Figure 4, different points can be distinguished 
which correspond to different points of the lift-drag polar in 
Figure 3:

a. Minimum airspeed in steady level flight (point A).
   According to equation (12) the airspeed decreases as the 
   angle of attack increases and the minimum airspeed is obtained 
   for maximum lift coefficient, $C_{L_{max}}$ (point $A^1$ in Figure 3).

b. Minimum power required for level flight (point B).
   Figure 4 shows clearly that power required reaches a minimum 
   value at an airspeed greater than the minimum airspeed. Accord- 
   ing to (14), at the velocity for minimum power required, the 
   airplane is flying at an angle of attack which corresponds to 
   maximum climb factor $C_L^3/C_D^2$ (point $B^1$ in Figure 3).

c. Minimum drag in level flight (point C).
   From equation (13) follows that minimum drag will be obtained 
   when the airplane is flying at an angle of attack where $C_L/C_D$ 
   is maximum (point $C^1$ in Figure 3).
   The angle between the radius from the origin to a point on the 
   power required curve in Figure 4a and the horizontal axis is a 
   measure for $P_r/V$ and so for the drag, $D$.
   The point $C$ for minimum drag on the power required curve is thus 
   found by drawing the line through the origin and tangent to the 
   $P_r$-curve, as shown in Figure 4a.
   Obviously, minimum drag is always obtained at a smaller angle 
   of attack and at a higher velocity than where minimum power re- 
   quired occurs.
In Table 2, the drag and power required in level flight are calculated for a supersonic airplane, flying at sea level (standard atmosphere).

As shown in Figure 5, the lift-drag polar for this airplane depends on the Mach number:

\[ C_D = f(C_L, M) \]  \hspace{1cm} (15)

In the calculations, values of Mach number are chosen, whereby the corresponding values of airspeed and lift coefficient follow from:

\[ V = M \cdot a \]  \hspace{1cm} (16)

\[ C_L = \frac{W}{\frac{1}{2} \rho V^2 S} = \frac{W}{\frac{1}{2} \gamma \rho M^2 S} \]  \hspace{1cm} (17)

For the values of M and C_L, belonging together, the drag coefficient is obtained from Figure 5.

A graphic representation of the calculation results is given in Figure 6, showing the dramatic rise in drag and power required when the airspeed approaches the speed of sound.

The effect of airspeed on drag and power required in level flight can be examined in detail by means of analytical expressions, obtained by using the parabolic approximation for the lift-drag polar.

The following considerations only hold for the flight at low subsonic speeds (no compressibility effects), where the lift-drag polar may be written as:

\[ C_D = C_{D_o} + \frac{C_L^2}{\pi A_e} \]  \hspace{1cm} (18)

The drag in level flight is:

\[ D = C_D \frac{1}{2} \rho V^2 S = \left( C_{D_o} + \frac{C_L^2}{\pi A_e} \right) \frac{1}{2} \rho V^2 S \]  \hspace{1cm} (19)
Substituting the lift coefficient according to (17):

\[ D = C_D \frac{1}{2} \rho \frac{V^2 S}{\pi A_e} + \frac{W^2}{\frac{1}{2} \rho \frac{V^2 S}{\pi A_e}} = D_0 + D_1 \]  \hspace{1cm} (20)

In equation (20), \( D_0 \) is the parasite drag and \( D_1 \) the induced drag.

Examining equation (20), we find that in level flight the parasite drag increases as the velocity increases, whereas the induced drag decreases with increasing velocity. These contributions to the total drag, \( D \), are sketched in Figure 7.

The speed at which minimum drag occurs, is obtained from:

\[ \frac{dD}{dV} = 0 \]  \hspace{1cm} (21)

Differentiating equation (20) with respect to \( V \), and setting the derivative equal to zero, we obtain:

\[ V_{D,\text{min}} = \sqrt{\frac{W}{S \rho \frac{1}{\sqrt{C_D} \pi A_e}}} \]  \hspace{1cm} (22)

Inserting (22) into (20) yields:

\[ D_0 = D_1 = W \sqrt{\frac{C_D}{\pi A_e}}, \text{ and } D_{\text{min}} = 2 W \sqrt{\frac{C_D}{\pi A_e}} \]  \hspace{1cm} (23)

Equation (23) shows that at the velocity where the total drag is a minimum, parasite drag equals induced drag, as indicated in Figure 7 (\( AB = BC \)).

The power required in level flight is:

\[ P_r = C_D \frac{1}{2} \rho V^3 S = \left( C_D + \frac{C_L}{\pi A_e} \right) \frac{1}{2} \rho V^3 S = \]

\[ = C_D \frac{1}{2} \rho V^3 S + \frac{W^2}{\frac{1}{2} \rho \frac{V^2 S}{\pi A_e}} \]  \hspace{1cm} (24)
For minimum power required, \( \frac{dP_r}{dV} = 0 \). Differentiating equation (24) yields:

\[
V_{P_{r,\text{min}}} = \sqrt{\frac{W}{S \rho}} \sqrt{\frac{2}{3} \frac{1}{C_{D_0} \pi Ae}}
\]  

(25)

and

\[
P_{r,\text{min}} = \frac{4}{3} \sqrt{\frac{W}{S \rho}} \sqrt{\frac{2}{3} \frac{C_{D_0}}{(\pi Ae)^3}}
\]  

(26)
THE PERFORMANCE DIAGRAM

For a particular airplane condition, airplane weight, and altitude, in the performance diagram the following characteristics as a function of flight velocity are given:

a. power required or drag in level flight
   b. power available or thrust at a given powersetting.

Figure 8 shows a performance diagram for the twin-engined airplane with turboprops as considered before. Power required is identical to the curve drawn in Figure 4. A detailed description of the manner in which power available can be calculated, is presented in the next chapter.

The maximum flight velocity is determined by the condition:

\[ P_a = P_r \]

In Figure 8, this condition holds for point A, being the high-speed intersection of \( P_a \) and \( P_r \). From Figure 7 follows that \( V_{\text{max}} = 450 \text{ km/h} \).

At flight velocities lower than \( V_{\text{max}} \), power available exceeds power required for level flight. Then, in steady flight the airplane will execute a climbing flight, where according to equation (28), the rate of climb is given by:

\[ \frac{R}{C} = \frac{P_a - P_r}{W} \quad (27) \]

In climbing flight, however, the lift is not precisely equal to the weight, because:

\[ L = W \cos \gamma \quad (28) \]

Nevertheless, for small flight path angles, as observed in normal flight, it is useful to neglect the effect of \( \gamma \) and to assume that:

\[ L = W \quad (29) \]
This approximation implies that at a given airspeed, the lift coefficient and the drag coefficient and thus the drag and power required are taken equal their values calculated for level flight.

Hence, from equation (27), the rate of climb is:

$$\frac{R}{C} = \frac{p_a}{W} - \frac{p_r}{W}$$  \hspace{1cm} (30)

where power required is given by equation (14).

The rate of climb, following from the excess power in Figure 8 is determined in Table 3 and plotted in Figure 9.

At a certain velocity, the excess power is a maximum. At this point the rate of climb will be maximum. For the airplane considered in Figure 8, we obtain:

$$R/C_{\text{max}} = 8.65 \text{ m/s and } V_{R/C_{\text{max}}} = 242 \text{ km/h}$$

Naturally, maximum rate of climb is of importance to minimize the time for the airplane to climb to a given altitude.

In Figure 9, any line through the origin and intersecting the $R/C$-curve (say, at point D) has the slope: $\frac{R/C}{V}$.

Hence, the angle inclined between this line and the horizontal axis is approximately a measure for the climb angle:

$$\gamma = \arcsin \frac{R/C}{V} \approx \arctan \frac{R/C}{V}$$  \hspace{1cm} (31)

Apparently, in point E the line becomes tangent to the $R/C$-curve, and the maximum climb angle in steady symmetric flight is obtained:

$$\gamma_{\text{max}} = 8.45^0 \text{ at } V_{\gamma_{\text{max}}} = 193 \text{ km/h.}$$

Note that maximum $R/C$ occurs at an airspeed greater than the airspeed where $\gamma$ is maximum.

The flight with maximum climb angle is of importance to minimize the horizontal distance for the airplane to reach a certain altitude.
In point C in Figures 8 and 9, the airplane flies at minimum airspeed. At the power setting considered in this example, a large excess power is available at $V_{\text{min}}$, through which the airplane will climb.

In order to perform a level flight a reduction of the power setting is required. Because of the relatively small angle of climb, minimum airspeed is also given by equation (12):

$$V_{\text{min}} = \sqrt{\frac{W}{S \rho \frac{1}{C_{\text{L}_{\text{max}}}}}}$$  \hspace{1cm} (32)

Figure 10 presents a performance diagram, where a jet airplane is considered. In this example, thrust and drag in level flight are plotted for the supersonic airplane as already considered in Table 2 and Figure 6. Now, the intersection of the thrust curve and the drag curve defines the maximum airspeed $V_{\text{max}}$ of the airplane at the given altitude:

$$T = D$$  \hspace{1cm} (33)

The remainder of thrust at velocities below $V_{\text{max}}$, defines directly the angle of climb, since according to equation (3) this angle is given by:

$$\sin \gamma = \frac{T - D}{W}$$  \hspace{1cm} (34)

The high thrust in the latter supersonic aircraft allow climbing flight at very large angles. E.g., at a Mach number $M = 0.4$, Figure 9 predicts a climb angle $\gamma = 37^0$!

In view of this large angle, in the following attention is paid to the general validity of the approximation $\cos \gamma = 1$. The effect on climb performance can best be studied by considering the situation at a given velocity. In this circumstance, the performance diagram directly yields the climb angle and rate of climb using $\cos \gamma = 1$. 
At a given velocity, the lift coefficient in level flight is greater than in climbing flight \( (C_L = W/q S \text{ instead of } C_L = W \cos \gamma/q S) \).

Therefore, also the drag coefficient in level flight is greater than in climbing flight. Consequently, at equal velocity, drag and power required in level flight are greater. This means that by use of the performance diagram, climb angle and rate of climb are underestimated.

At a given velocity, the increase in drag and power required due to the assumption \( \cos \gamma = 1 \) results from the use of a wrong lift coefficient (wrong induced drag).

Therefore, we may provide a relation between the real climb angle and the climb angle using \( \cos \gamma = 1 \) by writing:

\[
\sin \gamma_0 = \frac{T - D_0}{W} = \frac{T - D}{W} + \frac{\Delta D_1}{W} = \sin \gamma + \Delta C_D \frac{1}{2} \rho \frac{V^2 S}{W} =
\]

\[
\sin \gamma + \left[ \frac{C_L^2 - C_L^2}{\pi Ae} \right] \frac{1}{2} \rho \frac{V^2 S}{W} =
\]

\[
\sin \gamma + \frac{C_L}{\pi Ae} \left[ 1 - \left( \frac{C_L_0}{C_L} \right)^2 \right] =
\]

\[
\sin \gamma + \frac{C_L}{\pi Ae} \sin^2 \gamma_0,
\]

(35)

where the subscript 0 denotes the case \( \cos \gamma = 1 \).

A adequate description of the problem is obtained by setting \( \gamma_0 = \gamma \) in the correction term on the right-hand side of equation (35):

\[
\frac{R/C}{R/C_0} = \frac{\sin \gamma_0}{\sin \gamma} = 1 + \frac{C_L}{\pi Ae} \sin \gamma
\]

(36)
The following examples indicate that even in the case of a high performance jet airplane, the performance diagram may provide essentially good results.

EXAMPLE I - For the low subsonic airplane in Figure 8, we have at a velocity $V = 242$ km/h ($V = 67.2$ m/s):

$$\sin \gamma = \frac{P_a - P_r}{WV} = 0.129 \quad (\gamma = 7.39^\circ) ,$$

and

$$C_L = \frac{W}{\frac{1}{2} \rho V^2 S} = \frac{(15.400)(9.80665)}{\frac{1}{2} (1.225)(67.2)^2(70)} = 0.78$$

Hence:

$$\sin \gamma_0 = 1 + \frac{C_L}{\pi Ae} \sin \gamma = 1 + \frac{0.78}{\pi(12)(0.85)} (0.129) = 1.003$$

In this case the error is only 0.3%.

EXAMPLE II - For the jet airplane in Figure 10, the angle of climb at $M = 0.4$ (136.1 m/s) is:

$$\sin \gamma = \frac{T - D}{W} = 0.602 \quad (\gamma = 37.01^\circ)$$

and

$$C_L = \frac{W}{\frac{1}{2} \rho V^2 S} = \frac{(8300)(9.80665)}{\frac{1}{2} (1.225)(136.1)^2(34)} = 0.211$$

Assuming that the term $\pi Ae = \pi(1.98)(0.59) = 3.67$, we obtain:

$$\sin \gamma_0 = 1 + \frac{C_L}{\pi Ae} \sin \gamma = 1 + \frac{0.211}{3.67} (0.602) = 1.0346$$

Now, an error of 3.5% exists. Although, the error remains small, the result can easily be improved by iteration.
Table 1: Calculation of drag and power-required in level flight for a twin-engined airplane

airplane weight: \( W = 34000 \text{ lb (15400 kg_f)} \)
wing loading: \( W/S = 45 \text{ lb/ft}^2 (220 \text{ kg}_f/\text{m}^2) \)
alitude: \( h = 0 \text{ ft SA} \)
lift-drag polar: fig. 3

<table>
<thead>
<tr>
<th>( C_L )</th>
<th>( C_D )</th>
<th>( C_L/C_D )</th>
<th>( V )</th>
<th>( D_h )</th>
<th>( (P_r)_h )</th>
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Table 2: Calculation of drag and power-required in level flight for a supersonic airplane

- airplane weight : $W = 18,300 \text{ lb (8300 kg)}$
- wing loading : $W/S = 50 \text{ lb/ft}^2 (245 \text{ kgf/m}^2)$
- altitude : $h = 0 \text{ ft SA}$

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<tr>
<th>$M$</th>
<th>$V$ (m/s)</th>
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Table 3: Calculation of rate of climb

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Fig. 1: Airplane in steady symmetric flight.
Fig. 2: Force diagram in steady symmetric (powered) flight.
Fig. 3. Lift-drag polar of a twin-engined airplane with turboprops.
Fig. 4: Power required and drag in level flight for subsonic airplane.
Fig. 5: Aerodynamic properties of a supersonic airplane.
Fig. 6: Power required and drag in level flight for a supersonic airplane.
Fig. 7: Drag components in level flight.
Fig. 8: Performance diagram of twin engined airplane with turboprops.
Fig. 9: Rate of climb.

airplane weight: $W = 8300 \text{kg}_f$
altitude: $h = 0 \text{ m SA}$

Fig. 10: Performance diagram for turbojet airplane.
PART II: POINT PERFORMANCE IN STEADY
SYMMETRIC AND UNSYMMETRIC FLIGHT

CHAPTER 4: CALCULATION OF POWER AVAILABLE

POWER AVAILABLE FOR A TWIN-ENGINED AIRPLANE
WITH TURBOPROPS

In Table 1, power available for the airplane considered in
Chapter 3 is calculated.

The procedure for obtaining $P_a$ versus airspeed is illustra-
ted below for the chosen set of data:

- altitude : $h = 0 \text{ m SA} (\rho = 1.225 \text{ kg/m}^3)$
- turbine speed : $n = 14,000 \text{ rpm}$

1. At the selected $V$, the shaft horse power is determined from
Figure 1.

2. For the present constant speed propeller, the following
equilibrium condition is satisfied by adjusting the blade
angle:

$$P_p = P_{br}$$

We proceed by calculating the power coefficient:

$$C_p = \frac{P_{br}}{\rho \frac{n_p^3}{3} D^5}$$

where $n_p = (0.0929)(14000) = 1300.6 \text{ rpm}$ and $D = 3.51 \text{ m}$.

3. Compute $J$:

$$J = \frac{V}{n_p D}$$

4. Determine propeller efficiency $\eta$ from Figure 2.
5. Determine the effect of jet thrust, given in Figure 1 as $T_j$.

6. Compute

$$P_a = n * P_{br} + T_j * V$$

From the procedure given above the curve of power available is plotted in Figure 8 of Chapter 3.
POWER AVAILABLE FOR AN AIRPLANE WITH
PISTON-ENGINE AND PROPELLER

Airplanes with piston-engines are equipped with a fixed-pitch propeller or a constant-speed propeller. In both cases, power available is determined at a given inlet manifold pressure. Then, for the flight at a given altitude, shaft brake power versus engine rpm is known from the standard-engine diagram (see Figure 3).

In the case of a constant-speed propeller, the pilot can select throttle position (inlet manifold pressure) and adjust the pitch-control, separately. Then, at a given altitude, power available is determined at each chosen combination of inlet manifold pressure and engine speed, through which shaft brake power is known. Power available, therefore, is calculated according to the procedure for the turboprop, as shown in Table 1.

In the case of a fixed-pitch propeller, the engine speed at given airspeed and given inlet manifold pressure follows from the equilibrium: $P_p = P_{br}$.

An elegant procedure to calculate power available versus airspeed is starting from chosen values of engine speed. Then, together with the selected inlet manifold pressure, shaft brake power is directly known, and the corresponding airspeed follows from equilibrium between engine and propeller. Table 2 gives the calculation of power available for an airplane with piston-engine and fixed-pitch propeller, using the standard-engine diagram in Figure 3 and the propeller chart given in Figure 4.

The procedure is as follows:

1. Determine $P_{br}$ for the selected values of $P_z$, $n$ and $h$.

2. Compute at the given propeller rpm:

$$C_p = \frac{P_{br}}{\rho n^3 p D^4}$$
3. Determine for the blade angle ($\beta_{aR}$) of the propeller, the values of $J$ and $C_T$.

4. Compute $V$:

$$V = J \eta_p D$$

5. Compute $\eta$:

$$\eta = \frac{C_T}{C_P} J$$

6. Compute $P_a$:

$$P_a = \eta P_{br}$$

The results of the calculations in Table 2 are presented in Figure 5.

In this Figure also the $P_a$-curve is drawn for an airplane having the same engine, but now equipped with a constant-speed propeller. The engine-speed was chosen in such a way that both airplanes have the same maximum speed in level flight ($n = 2200$ rpm in point A).

Due to the decrease of engine-speed with decreasing airspeed, shaft brake power and so power available is lower for the airplane with fixed-pitch propeller. As a consequence, the rate of climb at a given velocity will be higher for the airplane with constant-speed propeller.
Table 1: Calculation of power available for a twin-engined airplane with turboprops.

Data: altitude : h = 0 ft SA,
  turbine speed : n = 14,000 rpm,
  propeller reduction : r = n_p/n = 0.0929,
  propeller diameter : D = 11.5 ft.

<table>
<thead>
<tr>
<th>V</th>
<th>V</th>
<th>( P_{br'} )</th>
<th>( T_J )</th>
<th>( P_{br} )</th>
<th>( C_p )</th>
<th>J</th>
<th>n</th>
<th>( P_a )</th>
<th>( P_{atot} )</th>
</tr>
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<tr>
<td>m</td>
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<td>60</td>
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<td>1549</td>
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<td>1551</td>
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<td>0.473</td>
<td>0.532</td>
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<td>120</td>
<td>176.5</td>
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<td>1555</td>
<td>0.1745</td>
<td>0.708</td>
<td>0.678</td>
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<td>160</td>
<td>236</td>
<td>1594</td>
<td>250</td>
<td>1559</td>
<td>0.1750</td>
<td>0.945</td>
<td>0.755</td>
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<td>200</td>
<td>294</td>
<td>1600</td>
<td>210</td>
<td>1565</td>
<td>0.1760</td>
<td>1.180</td>
<td>0.787</td>
<td>1347</td>
<td>2694</td>
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<tr>
<td>240</td>
<td>353.5</td>
<td>1605</td>
<td>174</td>
<td>1570</td>
<td>0.1765</td>
<td>1.420</td>
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<td>1365</td>
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<td>136</td>
<td>1585</td>
<td>0.1780</td>
<td>1.650</td>
<td>0.792</td>
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<td>320</td>
<td>471</td>
<td>1646</td>
<td>100</td>
<td>1611</td>
<td>0.1810</td>
<td>1.890</td>
<td>0.778</td>
<td>1338</td>
<td>2676</td>
</tr>
</tbody>
</table>
Table 2: Calculation of power available for an airplane with piston-engine and fixed pitch propeller.

Data: altitude
    : h = 0 ft SA,
manifold absolute pressure : \( p_1 \) = 32 in Hg,
propeller reduction gear : \( r = \frac{n_p}{n} = 0.7144 \),
propeller diameter : \( D = 9.10 \) ft,
blade angle : \( \beta_{2R} = 34^\circ \)

<table>
<thead>
<tr>
<th>n ( \text{rpm} )</th>
<th>( n_p )</th>
<th>( P_{br} ) ( \text{hp} )</th>
<th>( C_p ) ( \frac{550 P_{br}}{\rho n^3 p D^5} )</th>
<th>( J ) fig.4</th>
<th>( C_T ) ( \frac{C_T}{J} )</th>
<th>( \frac{n}{\rho} )</th>
<th>( V ) ( \text{ft/s} )</th>
<th>( V ) m ph</th>
<th>( P_a ) hp</th>
<th>( P_a = \eta P_{br} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2200</td>
<td>26.2</td>
<td>515</td>
<td>0.1065</td>
<td>1.260</td>
<td>0.072</td>
<td>0.851</td>
<td>300</td>
<td>204</td>
<td>439</td>
<td></td>
</tr>
<tr>
<td>2100</td>
<td>25.0</td>
<td>490</td>
<td>0.1175</td>
<td>1.160</td>
<td>0.086</td>
<td>0.850</td>
<td>264</td>
<td>180</td>
<td>416</td>
<td></td>
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<tr>
<td>2000</td>
<td>23.8</td>
<td>465</td>
<td>0.1290</td>
<td>1.020</td>
<td>0.099</td>
<td>0.782</td>
<td>222</td>
<td>151</td>
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<tr>
<td>1900</td>
<td>22.7</td>
<td>445</td>
<td>0.1415</td>
<td>0.713</td>
<td>0.106</td>
<td>0.547</td>
<td>147</td>
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<td></td>
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<tr>
<td>1800</td>
<td>21.4</td>
<td>420</td>
<td>0.1600</td>
<td>0.376</td>
<td>0.113</td>
<td>0.282</td>
<td>77.8</td>
<td>50</td>
<td>119</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 1: Effect of airspeed on performance of turboprop-engine.
Fig. 2: Example of propeller-diagram (estimated from manufacturer's data)
Fig. 3: Power graph of piston-engine.
Fig. 4: Example of propeller-diagram (windtunnel test on propeller model ref. NACA Rep. 640).
Fig. 5: Performance diagram for airplane with fixed pitch and constant speed propeller.
PART II: POINT PERFORMANCE IN STEADY
SYMMETRIC AND UNSYMMETRIC FLIGHT

CHAPTER 5: PERFORMANCE PREDICTION USING ANALYTICAL
EXPRESSIONS FOR THRUST AND DRAG

INTRODUCTION

The performance diagram enables a graphic determination of
performance for airplanes for which the variation of $C_D$ with
$C_L$ and the thrust (power available) versus speed, are avail-
able in any arbitrary form.
Analytical expressions for some of the performance character-
istics can be derived by introduction of simplifying assump-
tions with regard to the shape of the lift-drag polar and the
variation of thrust (power available) with airspeed.
Obviously, such considerations will only provide an assessment
of the real performance and are especially of importance to ob-
tain some insight into the relationship between lift-drag polar
and performance items in powered flight.

PROPELLER AIRCRAFT

In the range of subsonic airspeeds encountered by propeller air-
craft, power available is often regarded to be constant, at
least when altitude and power setting are kept constant.
Figure 1a shows schematically the performance diagram with the
assumption $P_a = \text{constant}$, together with the special points which
correspond to different points on the lift-drag polar, presented
in Figure 1c.

Table 1 reviews the significance of these points, where points
(2) and (3) require an explanation.
The climb angle is given by:

\[
\sin \gamma = \frac{R/C}{V} = \frac{P_a - P_r}{W/V}
\]  

(1)

In Figure 1a, the tangent from point 0 to the power required curve locates point (2), where the climb angle is maximum. The location of this point depends on the magnitude of power available.

The rate of climb is given by the relation:

\[
R/C = \frac{P_a - P_r}{W/V} = \frac{P_a}{W} - \sqrt{\frac{W}{2} \frac{C_D^2}{S \rho C_L^3}}
\]  

(3)

Apparently, at \( P_a \) constant, rate of climb becomes maximum as power required is minimum. In other words, at the angle of attack where the climb factor \( C_L^3/C_D^2 \) is maximum (point (3) in Figure 1a).
EXAMPLE

Considered is a propeller airplane, having the following characteristics:

- weight: \( W = 35000 \) N
- wing area: \( S = 30 \) m\(^2\)
- aspect ratio: \( A = \frac{b^2}{S} = 8 \)
- lift-drag polar: \( C_D = C_{D_0} + \frac{C_L^2}{\pi Ae} \)
- parasite drag coefficient: \( C_{D_0} = 0.027 \)
- Oswald efficiency factor: \( e = 0.85 \)
- maximum lift coefficient: \( C_{L_{\text{max}}} = 1.5 \)
- altitude: \( h = 0 \) m SA
- power available: \( P_a = 300 \) kW, (independent of airspeed)

(a) Determine minimum flight velocity, \( V_{\text{min}} \):

\[
V_{\text{min}} = \sqrt{\frac{W}{S \rho C_{L_{\text{max}}}}} = \sqrt{\frac{(35000)}{(30)} \frac{(2)}{(1.225)} \frac{(1)}{(1.5)} = 35.6 \text{ m/s} = 128.0 \text{ km/h}}
\]

Examine power required at \( V = V_{\text{min}} \):

- \( C_D = C_{D_0} + \frac{C_L^2}{\pi Ae} = 0.027 + \frac{(1.5)^2}{\pi(8)(0.85)} = 0.132 \)

- \( P_r = C_D \frac{1}{2} \rho V^3 S = (0.132)(\frac{1}{2})(1.225)(35.6)^3(30) = 109500 \text{ W} \)

Thus, \( P_r < P_a \), so that indeed \( V_{\text{min}} \) occurs at maximum lift coefficient.

(b) Calculate maximum rate of climb:

From the general expression for rate of climb

\[
R/C = \frac{P_a - P_r}{W},
\]

follows:
\[(R/C)_{\text{max}} = \frac{(P_a - P_r)_{\text{max}}}{W} = \frac{P_a}{W} - \frac{P_{r_{\text{min}}}}{W} = \frac{P_a}{W} - \frac{\sqrt{\frac{W}{S \rho} \frac{2}{(C_L^3/C_D^2)_{\text{max}}}}}{1}

From Part 1:

- \(C_L = \sqrt{3 \frac{C_D}{a \pi Ae}} = \sqrt{(3)(0.027)(\pi)(8)(0.85)} = 1.31\)
- \(C_D = 4 C_D^O = (4)(0.027) = 0.108\)

and

- \((C_L^3/C_D^2)_{\text{max}} = (1.31)^3/(0.108)^2 = 195\)

Thus,

- \((R/C)_{\text{max}} = \frac{300,000}{35000} - \sqrt{\frac{(35000)}{(30)(1.225)}} = 5.45 \text{ m/s}\)
- \(V_{(R/C)}_{\text{max}} = \sqrt{\frac{W}{S \rho C_L}} = \sqrt{\frac{(35000)}{(30)(1.225)(1.31)}} = 38.1 \text{ m/s} = 137.3 \text{ km/h}\)

Note that \(V_{(R/C)}_{\text{max}}\) is independent of the magnitude of \(P_a\).
PROBLEM

Considered is a single-engined propeller aircraft, having the following characteristics:

weight \[ W = 10,000 \text{ N} \]
wing area \[ S = 16 \text{ m}^2 \]
aspect ratio \[ A = 6.5 \]
lift-drag polar \[ C_D = C_{D_0} + C_L^2 / \pi Ae \]
Oswald efficiency factor \[ e = 0.8 \]
altitude \[ h = 0 \text{ m SA} \]
maximum velocity in level flight \[ V_{\text{max}} = 220 \text{ km/h} \]
power available \[ P_a = 75 \text{ kW} \text{ (independent of airspeed)} \]

The reader should attempt to determine maximum rate of climb and the airspeed at which maximum rate of climb occurs.
Recall from the discussion in Part I, that thrust of a jet engine is reasonably constant with airspeed. Then, power available varies essentially linearly with speed. Figure 1b shows the performance diagram with the special points, which are also depicted in the lift-drag polar in Figure 1c. Again, the significance of the different points is listed in Table 1, where points (4), (5) and (6) need a further explanation.

The climb angle can be expressed as:

\[
\sin \gamma = \frac{T - D}{W} = \frac{T}{W} - \frac{C_D \frac{1}{2} \rho V^2 S}{C_L \frac{1}{2} \rho V^2 S} = \frac{T}{W} - \frac{C_D}{C_L} \tag{3}
\]

In the case that \( T = \) constant, the climb angle appears to be maximum at the angle of attack where the lift-drag ratio, \( C_L/C_D \) is a maximum (point (4)).

Maximum rate of climb occurs in (5), at which point the excess power is maximum. Clearly, the location of the corresponding point in the lift-drag polar depends on the magnitude of the thrust, which defines the slope of the \( p_a \)-line in the performance diagram.

The condition for the corresponding lift coefficient and drag coefficient can be obtained from:

\[
\frac{R}{C} = V \sin \gamma = \frac{\sqrt{W 2 1}}{S \rho C_L} \left[ \frac{T - C_D}{W} \frac{C_D}{C_L} \right] \tag{5}
\]

From \( d(R/C)/dC_L = 0 \), we find:

\[
\frac{T}{W} = 3 \frac{C_D}{C_L} - 2 \frac{dC_D}{dC_L} \tag{6}
\]
For a parabolic variation of $C_D$ with $C_L$, the condition (6) changes to:

$$\frac{T}{W} = 3 \frac{C_D}{C_L} - \frac{C_L}{\pi Ae}$$

(7)

The solution of the quadratic equation (7) yields the lift-coefficient, where maximum rate of climb occurs:

$$C_L_c = \frac{\pi Ae}{2} \frac{T}{W} \left[ -1 + \sqrt{1 + 12 \frac{C_D}{C_L} \left( \frac{W}{T} \right)^2} \right]$$

(8)

Maximum flight velocity at a given altitude follows from the equilibrium:

$$T = D = C_D \frac{1}{2} \rho \frac{V^2}{S} = C_D \frac{1}{2} \rho \frac{V^2}{S} + C_L \frac{1}{2} \rho \frac{V^2}{S}$$

(9)

Recall that $W = C_L \frac{1}{2} \rho \frac{V^2}{S}$. From equation (9):

$$T = D = C_D \frac{1}{2} \rho \frac{V^2}{S} + \frac{W^2}{\frac{1}{2} \rho \frac{V^2}{S} \pi Ae}$$

(10)

Equation (10) delivers a quadratic equation for the flight velocity. The solution is for the maximum velocity:

$$V = \sqrt{\frac{T}{\rho \ C_D \ S} \left[ 1 + \sqrt{1 - 4 \frac{C_D}{C_L} \left( \frac{W}{T} \right)^2} \right]}$$

(11)
EXAMPLE

Considered is a turbojet airplane, having the following characteristics:

- **weight**: \( W = 50,000 \text{ N} \)
- **wing area**: \( S = 20 \text{ m}^2 \)
- **lift-drag polar**: \( C_D = C_{D_0} + \frac{C_L}{\pi Ae} \)
- **induced drag factor**: \( \pi Ae = 20 \)
- **maximum lift coefficient**: \( C_{L_{\text{max}}} = 1.6 \)
- **altitude**: \( h = 0 \text{ m SA} \)
- **thrust**: \( T = 12000 \text{ N} \) (independent of airspeed)

**a. Determine minimum flight velocity.**

\[ V_{\text{min}} = \sqrt{\frac{W}{S \rho C_{L_{\text{max}}}}} = \sqrt{\frac{500000}{20 \cdot 1.225 \cdot 1.6}} = 50.5 \text{ m/s} = 181.8 \text{ km/h} \]

\[ C_D = C_{D_0} + \frac{C_L^2}{\pi Ae} = 0.02 + \frac{(1.6)^2}{20} = 0.148 \]

\[ D = \frac{C_D}{C_L} \frac{W}{(50000)} = 4625 \text{ N} \]

Thus, \( D < T \) at \( V = V_{\text{min}} \), so that \( V_{\text{min}} \) occurs at \( C_{L_{\text{max}}} \).

**b. Calculate maximum climb angle.**

From, \( \sin \gamma = -\frac{C_D}{W C_L} \)

follows that:

\[ (\sin \gamma)_{\text{max}} = \frac{T}{W} - \frac{1}{(C_L/C_D)_{\text{max}}} \]
From part I,

- $C_L = \sqrt{CD_0 \pi Ae} = \sqrt{(0.02)(20)} = 0.63$

- $C_D = 2 \cdot CD_0 = (2)(0.02) = 0.04$

and

- $(C_L/C_D)_{\text{max}} = 0.63/0.04 = 15.8$

Thus,

- $\gamma_{\text{max}} = \arcsin \left( \frac{12000}{50000} - \frac{1}{15.8} \right) = \arcsin (0.177) = 10.2^\circ$

- $V_{\gamma_{\text{max}}} = \sqrt{\frac{W \cdot 2 \cdot 1}{S \cdot \rho \cdot C_L}} = \sqrt{\frac{(50000)(2)(1)}{(20)(1.225)(0.63)}} = 80.5 \text{ m/s} = 290 \text{ km/h}$

Note that $V_{\gamma_{\text{max}}}$ is independent of the magnitude of the thrust.

c. Determine maximum rate of climb.

Equation (4) gives the rate of climb as:

- $R/C = \sqrt{\frac{W \cdot 2 \cdot 1}{S \cdot \rho \cdot C_L} \left[ \frac{T}{W} - \frac{C_D}{C_L} \right]}$

For maximum rate of climb, $\frac{d(R/C)}{dv} = 0$. Differentiating equation (4) and insertion of the parabolic lift-drag polar yields the following expression for the lift coefficient.

$$C_{L_{\text{C}}} = \frac{\pi Ae}{2} \frac{T}{W} \left[ -1 + \sqrt{1 + 12 \frac{CD_0}{\pi Ae} \left( \frac{W}{T} \right)^2} \right]$$
Thus,

\[ C_{L_C} = \frac{(20)(12000)}{(2)(50000)} \left[ -1 + \sqrt{1 + \left(\frac{(0.02)(50000)^2}{(20)(12000)}\right)^2} \right] = 0.24 \]

\[ C_D = C_{D_0} + \frac{C_L^2}{\pi Ae} = 0.02 + \frac{(0.24)^2}{20} = 0.0229 \]

\[ \sin \gamma = \frac{T}{W} - \frac{C_D}{C_L} = \frac{12000}{50000} - \frac{0.0229}{0.24} = 0.145 \]

\[ V_{(R/C)}_{\text{max}} = \sqrt{\frac{W}{S \rho C_L}} = \sqrt{\frac{(50000)(2)}{(20)(1.225)(0.24)}} = 130.4 \text{ m/s} = \]
\[ = 470 \text{ km/h} \]

\[ (R/C)_{\text{max}} = V \sin \gamma = (130.4)(0.145) = 18.9 \text{ m/s} \]

d. Determine maximum flight velocity.

From \( T = D = \frac{C_D}{C_L} W \),

\[ \frac{T}{W} = \frac{C_D}{C_L} + \frac{C_L}{\pi Ae} \]

or

\[ C_L^2 - C_L \frac{T}{W} \pi Ae + C_{D_0} \pi Ae = 0 \]

Insertion of the numerical values,

\[ C_L^2 - C_L \left(\frac{(12000)}{(50000)}\right)(20) + (0.02)(20) = 0 \]
The roots of $C_L$ in this quadratic equation are:

$$C_{L1} = 4.720$$

$$C_{L2} = 0.085$$

$$V_{\text{max}} = \sqrt{\frac{W}{S \rho C_L}} = \sqrt{\frac{50000}{20}} \frac{2}{(1.225)(0.085)} = 219 \text{ m/s} = 789 \text{ km/h}$$

PROBLEM

Considered is a turbojet airplane, having the following characteristics:

- Weight: $W = 60,000$ N
- Wing area: $S = 30$ m$^2$
- Lift-drag polar: $C_D = C_{D0} + C_L^2/\pi Ae$
- Parasite drag coefficient: $C_{D0} = 0.018$
- Aspect ratio: $A = 5$
- Oswald efficiency factor: $e = 0.8$
- Altitude: $h = 0$ m SA
- Maximum flight velocity: $V_{\text{max}} = 760$ km/h
- Thrust independent of airspeed

a) Find maximum rate of climb, and airspeed for maximum rate of climb.

b) Find maximum angle of climb, and airspeed for maximum climb angle.
Fig. 1: Idealized performance diagram for propeller- and jet aircraft.

<table>
<thead>
<tr>
<th>Point</th>
<th>Lift-drag polar</th>
<th>Performance diagram</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>max. $C_L$</td>
<td>min. airspeed $V$</td>
<td>$V = \sqrt{\frac{W}{S \rho C_L}}$</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>max. climb angle $\gamma$</td>
<td>$\sin \gamma = \frac{P_a - (P_r)_h}{W V}$</td>
</tr>
<tr>
<td>3</td>
<td>max. $\frac{C_L^3}{C_D^2}$</td>
<td>Min. power required $(P_r)_h$</td>
<td>$(P_r)_h = \frac{W}{S \rho C_L^2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>max. rate of climb R/C</td>
<td>$R/C = \frac{P_a - (P_r)_h}{W}$</td>
</tr>
<tr>
<td>4</td>
<td>max. $\frac{C_L}{C_D}$</td>
<td>min. drag $D_h$</td>
<td>$D_h = \frac{C_D}{C_L} W$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>max. climb angle $\gamma$</td>
<td>$\sin \gamma = \frac{T-D_h}{W}$</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>max. rate of climb R/C</td>
<td>$R/C = \frac{P_a - (P_r)_h}{W}$</td>
</tr>
<tr>
<td>6</td>
<td>-</td>
<td>(maximum) airspeed $V_h$ in steady level flight</td>
<td>$P_a = (P_r)_h$</td>
</tr>
</tbody>
</table>

Table 1: Point performances in steady symmetric powered flight.
PART II: POINT PERFORMANCE IN STEADY SYMMETRIC AND UNSYMMETRIC FLIGHT

CHAPTER 6: EFFECT OF ALTITUDE

ALTITUDE EFFECTS ON DRAG AND POWER REQUIRED IN LEVEL FLIGHT

In the case of low subsonic airspeeds, the effects of altitude on drag and power required can easily be examined by the comparison of flight conditions at different altitudes, but at the same value of lift coefficient. Because $C_D = f(C_L)$, the drag coefficient also remains fixed.

Let, the subscript 1 designate sea level conditions, then we have:

$$V_1 = \sqrt{\frac{W}{S \rho_1 C_L}}$$  \hspace{1cm} (1)

$$D_1 = \frac{C_D}{C_L} W$$  \hspace{1cm} (2)

$$P_{r1} = W \sqrt{\frac{W}{S \rho_1 C_L}}$$  \hspace{1cm} (3)

If the subscript 2 designates the situation at altitude, we have:

$$V_2 = \sqrt{\frac{W}{S \rho_2 C_L}}$$  \hspace{1cm} (4)

$$D_2 = \frac{C_D}{C_L} W$$  \hspace{1cm} (5)

$$P_{r2} = W \sqrt{\frac{W}{S \rho_2 C_L}}$$  \hspace{1cm} (6)
Since the angle of attack is the same, we obtain the following ratios:

\[
\frac{V_2}{V_1} = \sqrt{\frac{\rho_1}{\rho_2}} \tag{7}
\]

\[
\frac{D_2}{D_1} = 1 \tag{8}
\]

\[
\frac{p_{r_2}}{p_{r_1}} = \sqrt{\frac{\rho_1}{\rho_2}} \tag{9}
\]

The ratio (8) shows that at a given angle of attack, the drag in level flight is independent of altitude.

Figure 1 shows power required in level flight versus airspeed for three altitudes. The sea level curve is conformable to the curve already given in chapter 3 for the twin-engined turboprop airplane.

The ratios (7) and (9) indicate that corresponding points for a given angle of attack lie at any radial straight line through the origin. Consequently, all the curves have only one joint tangent, which defines the locations of the points for minimum drag.
PROPELLER AIRPLANE PERFORMANCE

Figure 2 shows the performance diagrams at various altitudes for the twin-engined turboprop considered previously. In contrary to power required, at constant power setting power available decreases with increasing altitude. In Figure 3 are plotted the rates of climb, resulting from the performance diagrams in Figure 2.

As altitude increases, the maximum rate of climb decreases. This is illustrated by Figure 4, showing the relationship between maximum rate of climb and altitude.

The altitude at which maximum rate of climb becomes equal to zero, is defined as the theoretical ceiling of the airplane. At the theoretical ceiling the \( P_a \) curve is tangent to the \( P_r \) curve, so that there is only one velocity at which steady, level flight is possible (see Figure 5).

It will be clear, that the theoretical ceiling depends on airplane configuration, airplane weight, and power setting. Since maximum rate of climb diminishes to zero at the theoretical ceiling, the time to reach this height will approach to infinite. Therefore, also is used the quantity, service ceiling, defined as that altitude where the maximum rate of climb is decreased to 0.5 m/s or 100 ft/min (Figure 4).

In the simplified case that power available is independent of airspeed, the variation of maximum rate of climb with altitude, is given by:

\[
(R/C)_{max} = \frac{P_a}{W} \sqrt{\frac{W}{S \cdot \rho \cdot (c_L^2/c_D^2)_{max}}} \"{(10)\}

This expression clearly shows that maximum rate of climb decreases with increasing height due to a lower power available and a lower air density.

In the case of turbocharged piston-engines, the airplane meets this property only at the heights above the critical height of the engines (see Figure 6).
Expression (10) also indicates that for $P_a = \text{constant}$, the angle of attack for maximum rate of climb does not vary with altitude.

Hence, the equilibrium condition, $W = C_L \frac{1}{2} \rho V^2 S$ requires that the dynamic pressure must remain constant. In turn, the equivalent airspeed where maximum rate of climb occurs, is independent of altitude.

It must be emphasized that this reasoning is only valid as far as $P_a$ is independent of airspeed. Figure 7 shows that in reality there is a slight difference between the flight velocity where maximum rate of climb is found and the velocity determined by maximum climb factor, $(C_L^3/C_D^2)_{\text{max}}$.

The impact on minimum and maximum flight velocity due to altitude effects is illustrated in Figure 7.

With regard to minimum airspeed, two different altitude regions must be distinguished.

At lower heights, power available is sufficient to perform a level flight with maximum lift coefficient.

In this regions, minimum airspeed is determined by stalling, and minimum airspeed increases with increasing altitude, according to:

$$V_{\text{min}} = \sqrt{\frac{W}{2 \rho S C_{L_{\text{max}}}}}$$

(11)

At heights near the theoretical ceiling, power available is lower than power required for level flight with $C_L = C_{L_{\text{max}}}$.

E.g., see the power curves at 30,000 ft in Figure 7.

At these heights, the minimum velocity in level flight is determined by the low-speed intersection of the power curves. Here, minimum velocity will occur at an angle of attack smaller than the critical angle of attack.
The variation of maximum flight velocity with altitude can best be examined by assuming $P_a$ independent of airspeed:

$$P_a = P_r = W \sqrt{\frac{W \frac{2}{C_D^2}}{S \rho \frac{C_L^3}{C_D^2}}} \quad (12)$$

or

$$\frac{C_L^3}{C_D^2} = \frac{W \frac{2}{C_D^2}}{S \rho \left(\frac{W}{P_a}\right)^2} \quad (13)$$

Equation (13) shows that because $P_a$ and $\rho$ decrease with increasing altitude, the climb factor will increase up to $(C_L^3/C_D^2)_{\text{max}}$ at the theoretical ceiling.

The change in flight velocity as caused by the increasing angle of attack, follows from:

$$P_a = P_r = C_D \frac{1}{2} \rho V_{\text{max}}^2 S \quad (14)$$

or

$$V_{\text{max}} = \sqrt{\frac{3 P_a \frac{2}{C_D^2} \frac{1}{S \rho C_D}}{S \rho C_D}} \quad (15)$$

For airplanes propelled by turbo-engines, the ratio $P_a/\rho$ will increase with increasing altitude. Due to this property, at first the maximum flight velocity will become somewhat higher with increasing altitude (Figure 7). Above a certain altitude the influence of the increasing $C_D$ value will dominate and, accordingly, $V_{\text{max}}$ will decrease.

For airplanes equipped with piston-engines and flying below the critical height, power available may be taken constant. In this case the decrease of $\rho$ dominates and the maximum flight velocity increases with increasing height (see Figure 6). Above the critical height, $P_a$ varies linearly with density, so that the maximum flight velocity reduces continuously with increasing height, because $C_D$ becomes greater.
TURBOJET AIRPLANE PERFORMANCE

As shown in Figure 8, for subsonic airplanes equipped with turbojet or turbofan engines, the altitude effects are, basically, the same as in the case of propeller propulsion. However, with respect to maximum rate of climb a different behaviour occurs, that may be studied for the simplified case that thrust is assumed to be independent of airspeed. Then, the angle of attack for maximum rate of climb is given by:

\[
\frac{T}{W} = 3 \frac{C_D}{C_L} - 2 \frac{dC_D}{dC_L}
\]  

(16)

The condition (16) is graphically plotted in the lift-drag polar in Figure 9, showing that the lift coefficient for maximum rate of climb increases with increasing altitude (decreasing thrust at constant powersetting). This fact implies that the equivalent airspeed concerned, and thus the corresponding indicated airspeed must be lowered as altitude increases. At the theoretical ceiling, we have:

\[
\frac{T}{W} = \frac{C_D}{C_L}
\]  

(17)

Substitution of (17) into (16) yields the trivial result that at the theoretical ceiling is flown with the angle of attack where \(C_L/C_D\) is maximum.

In Figure 10 are presented the performance diagrams at different altitudes for a supersonic airplane. The altitude effects resulting from these plots are given in Figure 11 and 12. Contrary to the preceding example, the maximum velocity in level flight appears to increase rapidly when the airplane climbs from sea level to a higher altitude. The maximum flight velocity is obtained near the tropopause (\(h = 11\) km SA). Apparently, high excess powers are present at both subsonic and supersonic airspeeds, through which phenomenon in both flight regimes a maximum rate of climb can be recognized.
OPERATIONAL BOUNDARIES OF FLIGHT
VELOCITY AND ALTITUDE

In the design of an airplane, usually, boundaries of flight velocity and altitude are selected, which result from effects other than those associated with thrust and drag. From the example in Figure 13, which is referred to jet airplanes, the following boundaries are recognized:

- Maximum operating Mach number, \( M_{MO} \), which should not be exceeded in any normal flight condition. It prevents the occurrence of undesirable flying qualities from compressibility effects.

- Maximum operating speed, \( V_{MO} \), is the maximum equivalent airspeed in normal level flight at which the structure is designed to withstand the occurring loads.

- The design diving Mach number, \( M_D \), and design diving speed, \( V_D \), indicate the maximum speeds in level flight of which the airplane is designed to remain controllable and to withstand particular loads specified in the airworthiness regulations. The difference between \( M_{MO} \) and \( M_D \) (\( V_{MO} \) and \( V_D \)) is the safety margin for unintentional increments of the speed limits for normal use.

- A limit concerning the maximum cruising altitude for pressurized airplanes. This limit is determined by the maximum pressure differential for which the structure and the pressurization system are designed.
BUFFET BOUNDARY

The concept of critical Mach number as discussed in part I (chapter 6), is associated with the occurrence of shock waves over wing and tail surfaces in the high subsonic speed range. At that place, attention was given to critical speeds related to a rapid increase of airplane drag. In addition, there is also something happening indicated as buffeting.

This phenomenon concerns vibrations of airplane and its controls, and pitching and yawing oscillations from rapid changes of pressure distribution. Altogether, not a very desirable situation, and certainly something to be avoided.

In Figure 14 are presented a typical envelope of maximum lift coefficient versus Mach number defined by the stall limit and the high-speed buffet limit.

From the equilibrium condition,

\[ W = C_L \frac{1}{2} \gamma \rho M^2 S \]

follows that in level flight \( C_L M^2 \) is constant and that this quantity increases with increasing height. Curves of constant \( C_L M^2 \) are also plotted in Figure 14, showing that the range of possible speeds between stall and buffet becomes smaller as altitude increases.

Figure 15 shows the speed range of normal flight as a function of altitude. At a certain height both velocities coincide, defining the aerodynamic ceiling of the airplane. Outside the boundaries in Figure 15, flight is considered unsafe, and the best way of avoiding difficulties at high speeds is to define a maximum permissible Mach number, like \( M_{MO} \) in Figure 13.
Fig. 1: Effect of altitude on power required.
Fig. 2: Performance diagrams for various altitudes (turboprop-airplane).
Fig. 3: Rate of climb on various altitudes.
Fig. 4: Effect of altitude on maximum rate of climb.

Fig. 5: Theoretical ceiling.
Fig. 6: Maximum speed and rate of climb versus altitude for an airplane with piston-engine and supercharger (ref. VTH Report 115, 1962).
Fig. 7: Effect of altitude on max. and min. airspeed in level flight (turboprop airplane).
Fig. 8: Effect of altitude on performance of turbojet airplane.
Fig. 9: Lift-drag coefficient for maximum rate of climb.
Fig. 10: Performance diagram of supersonic airplane ($W = 8290 \, \text{kg}_f$)
Fig. 11: Rate of climb of supersonic airplane.
Fig. 12: Speed-envelope of supersonic airplane.
Fig. 13: Maximum operating speed for transport airplane.
Fig. 14: Maximum lift and buffet boundaries for high subsonic airplanes.

Fig. 15: Aerodynamic limitations of airspeed and altitude (high subsonic airplane).
PART II: POINT PERFORMANCE IN STEADY
SYMmetric AND UNSYMMETRIC FLIGHT

CHAPTER 7: EFFECT OF CHANGES OF POWER AVAILABLE
AND AIRPLANE CONFIGURATION

EFFECT OF POWERSetting

The pilot can affect the performance of the airplane by changing power available or thrust.

Figure 1 shows the performance diagrams for a number of engine speeds, for the twin-engined airplane with turboprops.

In climbing flight, airspeed and powersetting can be chosen independently, whereas in level flight the flight condition is fully defined by one of the variables, $\alpha$, $V$ or $\Gamma$. Thus, in the latter case particular values of velocity and power setting are coupled. This is illustrated in Figure 1, where the points $A_1$ to $A_7$ define the connection between engine speed and velocity in level flight. It is apparent that at a very low power setting, like the power curve $A_6$-$A_7$, steady symmetric flight at either of two velocities is possible. One faster and one slower than the airspeed where maximum excess power occurs.

Operation at the greater airspeed is known as "flying the front side of the power curve".

The desirability of this speed range may be explained by considering the result of a change in airspeed from the equilibrium condition in point $A_6$. In the case that the pilot maintains level flight, there will be a negative excess power when speed is increased, and a positive excess power when speed decreases. However, in both situations the airplane will tend to restore the original airspeed. At airspeeds lower than the speed for maximum excess power (e.g. point $A_7$ in Figure 1), any disturbance causes that the airplane will tend to diverge further from the original equilibrium condition.

Operation at this speed is known as "flying the back side of the power curve".

Clearly, this speed region must be avoided in all circumstances in which loss of speed or altitude can be dangerous.
It may be seen from Figure 2 that in the case of an airplane with turbojet engines, the transition point between both speed regimes is approximately the point for minimum airplane drag. In order to assure a sufficient degree of "speed stability", the pilot will select a cruising speed greater than the speed for minimum drag in level flight. Generally, for either jet airplanes as propeller airplanes:

\[ V_{cr} \geq V_M = 1.1 \ V_{D_{min}} \]  

(1)

where \( V_M \) is minimum comfortable airspeed (M.C.A.).

Assuming the parabolic variation of \( C_D \) versus \( C_L \), the minimum cruising speed becomes:

\[ V_{cr} \geq 1.1 \sqrt{\frac{W}{S \rho \sqrt{C_{D_0}}} \left(\frac{1}{\sqrt{C_{D_0}}} \right)} \]  

(2)

Naturally, power available for reduced power setting becomes lower and the intersection with the power required curve takes place at a lower airspeed (Figure 1). The effects of decreasing the power setting is that the maximum speed becomes considerable less, and the possible rate of climb decreases at all velocities.

It must be noticed that the lowest speed in level flight cannot be obtained with the engines running at maximum power setting. At very low power settings, the minimum speed becomes slightly greater than the stalling speed.

Generally, full power can be used only for a limited time or there will be a risk of damage to the engines. Therefore, the engine manufacturer designates various power settings at which the engine may be running for short periods and at which the engine may be operated continuously.
E.g., take-off or emergency power may be held for a period of only 5 minutes.
The maximum power permitted for continuous operation is therefore lower and is mostly designated as "maximum except take-off (METO-) power".
The maximum power setting permitted in cruise flight is called "maximum cruise".

EFFECT OF ENGINE FAILURE

Malfunction of one or more engines leads, as a matter of course, to a considerable loss in power available. The effect of the number of operating engines on rate of climb for a two- and four-engined airplane is illustrated by Figure 3. With respect to cruise flying, especially, the lowering of the theoretical and practical ceilings is of great importance. Unlike the requirement for take-off, for a four-engined airplane the occurrence of two dead engines during cruise flight must be awaited.

Therefore, the flight plan must warrant the condition that the airplane is able to carry the load over the obstacles present on the selected route.

Failure of an engine in flight means not only loss of thrust, but also an increment of power required, as caused by (see Figure 4):

a. extra drag produced by the inoperative engine

b. additional drag from rudder deflections.

Concerning the extra drag from the inoperative engine, it is of great importance to avoid the occurrence of wind-milling drag. A propeller being windmilled by the oncoming air and thus turning a dead engine causes a considerable drag force. Also, there is the possibility of damage to the engine.
For the calculation of wind-milling drag, the reader is referred to:

Fortunately, wind-milling drag may be almost completely eliminated by feathering the propeller.

The unbalanced thrust in Figure 4 causes a yaw moment about the vertical axis. A side-force created by deflection of the rudder overcomes the effect of unbalanced thrust. As illustrated in Figure 5, placing the airplane in a slight bank towards the operative engine enables the pilot to maintain a rectilinear flight.

The maximum cross-force the rudder can exert decreases with decreasing airspeed, whereas the thrust increases as airspeed becomes lower. The airspeed at which the maximum moment exerted by the rudder equals the unbalanced moment developed by the operative engine, determines the lowest speed at which the pilot can maintain zero slip angle. The latter point defines the minimum control speed, which is the minimum airspeed at which the rudder can exert enough yaw moment, without stalling, to overcome unsymmetric thrust. It will be clear that the pilot must never select an airspeed below the minimum control speed, $V_{MC}$, in a flight with one or more engines inoperative.

For most airplanes, $V_{MC}$ is chosen to the minimum airspeed for the airplane in take-off configuration, except with the landing gear retracted, and take-off power.
EFFECT OF POSITION OF UNDERRACCHIAGE
AND WING FLAPS

The effect of the position of undercarriage and wing flaps on flight performance is explained below for the twin-engined airplane with turboprops considered already in Figure 1.

Figure 6 shows the lift-drag polars for typical airplane conditions with respect to undercarriage position and flap extension.

A numerical impression of the effect of landing gear position on climbing performance is given in Figure 7. Considered is the flight with two operating engines and the case of one dead-engine with propeller feathered.

It appears clearly that the undercarriage has a considerable influence on the performance of the airplane. Especially, the effect of undercarriage drag on rate of climb is enormous in the case of engine failure.

The effect of flap angle on rate of climb is shown in Figure 8, for the case of one and two operative engines. At a fixed airspeed, $C_L$ is also fixed whereas $C_D$ increases as flap angle is increased. As a consequence, rate of climb decreases with increasing flap angle.

However, according to the airworthiness standards, compliance with the climb requirements at an airspeed $V_2 = 1.2 V_{min}$ and at one engine inoperative, must be shown.

In Figure 8, the points in question are depicted on the rate of climb curves. Apparently, at a flap angle of $\delta_f = 40^0$, the rate of climb of the airplane is no longer enough.

For that reason, during take-off virtually always partly extended flaps are applied.
Fig. 1: Performance diagram for various engine speeds.
Fig. 2: Selection of flight regime.
Fig. 3: The effect of the number of operation engines on the rate of climb for a two- and four-engined airliner.
RUDDER-INDUCED YAW MOMENT

UNBALANCED POWER YAW MOMENT

Fig. 4: Forces and moments with engine failure.
Fig. 5: Steady flight with dead engine.
Fig. 6: Lift-drag polar for various flap angles.
airplane weight: $W = 34,000$ lb
altitude: $h = 0$ ft SA
wingflaps up
engine speed $n = 14,000$ rpm
(dead engine feathered)

Fig. 7: Effect of undercarriage-drag on climb performance.
airplane weight: $W = 34,000 \text{ lb}_f$
altitude: $h = 0 \text{ ft S A}$
undercarriage up
engine speed: $n = 14,000 \text{ rpm}$
(dead engine feathered)

speed limit for flap extension

landing climb speed

one engine $
\delta_f = 40^\circ$

$V_{\text{min}}$

two engines

$\delta_f = 0^\circ$

$\delta_f = 15^\circ$

$\delta_f = 40^\circ (\text{max})$

$1.2 V_S$

Fig. 8: Effect of flap angle on climb performance.
PART II: POINT PERFORMANCE IN STEADY
SYMMETRIC AND UNSYMMETRIC FLIGHT

CHAPTER B: EFFECT OF WEIGHT ON PERFORMANCE

It is of great importance to know what will be the effect on performance of changing the weight of the airplane. Let the weight vary from $W_1$ to $W_2$ and for the purposes of calculation keep the same angle of attack. If further low subsonic velocities are assumed, $C_L$, as well as $C_D$, will also remain the same.

Therefore, at a given height we have the following ratios:

$$\frac{V_2}{V_1} = \frac{\sqrt{\frac{W_2}{W_1}}}{\frac{2}{S} \frac{\rho}{C_L}} = \sqrt{\frac{W_2}{W_1}} \tag{1}$$

$$\frac{D_2}{D_1} = \frac{C_D}{C_L} \frac{W_2}{W_1} = \frac{W_2}{W_1} \tag{2}$$

$$\frac{P_{r2}}{P_{r1}} = \sqrt{\frac{\frac{W_2}{W_1}}{\frac{2}{S} \frac{\rho}{C_L}}} = \sqrt{\left[\frac{W_2}{W_1}\right]^3} \tag{3}$$

Thus, starting from the weight $W_1$, for each angle of attack the new speed, $V_2$, drag, $D_2$, and power required, $P_{r2}$, can be calculated.
Figure 1 shows the effect of weight on the power required curve for the twin-engined airplane with turboprops. As can be seen from equations (1) and (3), points having the same lift coefficient are located on third order curves through the origin:

$$\frac{P_{r2}}{P_{r1}} = \left(\frac{V_2}{V_1}\right)^3$$

(4)

The effect of weight on power required can also be examined at the same airspeed. Using a parabolic lift-drag polar, we may write:

$$P_r = C_{D_0} \frac{1}{2} \rho V^2 S + \frac{W^2}{\pi Ae \frac{1}{2} \rho V S}$$

(5)

This expression shows that at a given airspeed and at a given altitude, the weight of the airplane only affects that part of power required that is associated with induced drag. Since at a given weight, this part of power required decreases with increasing velocity, the effect of weight changes on power required is especially of importance at lower airspeeds and/or at greater heights.

In Figure 1, the effect of weight on maximum velocity in level flight is given by the intersections of \(P_a\) and \(P_r\) curves (points \(A_1\) to \(A_3\)).

As far as minimum airspeed is determined by the stalling angle of attack, we have (points \(B_1\) to \(B_3\)):

$$\frac{(V_{min})_2}{(V_{min})_1} = \sqrt{\frac{W_2}{W_1}}$$

(6)

In particular, a change of weight has a considerable influence on maximum rate of climb.
From,

\[
\frac{(R/C)_{\text{max}}}{W} = \frac{(P_a - P_r)_{\text{max}}}{W}
\]

(7)

it can be seen that at an increasing airplane weight, the numerator of (7) decreases, whilst the dominator increases. The combined influence causes a dramatic reduction of maximum rate of climb. For the airplane considered in Figure 1, a complete picture of the effect of weight is presented in Figure 2.

A close relationship exists between airplane weight and airworthiness requirements.

To guarantee flight safety, the airworthiness regulations put certain minimum requirements to the performance of the airplane in the various flight phases.

E.g., a transport airplane must be able to meet the climb requirements as specified in the American FAR 25 - Regulations. *)

Table 1 presents a summary of climb requirements for transport airplanes, according to these regulations. The climb requirements are specified as climb gradients, that is to say in terms of \( \frac{R/C}{V} \), and expressed in percent.

As shown in Figure 3, the take-off is broken down into 3 segments, all with one engine inoperative.

Each segment must be investigated completely, to determine the maximum allowable take-off weight. Finally, in the "flight manual" maximum take-off weight is presented in terms of pressure altitude and air temperature. An example of so-called "WAT-curves" (Weight-Altitude-Temperature) is shown in Figure 4.

In Table 1, \( V_S \) is the calibrated stalling speed, or the minimum steady speed, at which the airplane is controllable. The weight, flap position, etc. shall correspond to the particular configuration for which \( V_S \) is being used.

*) Federal Aviation Regulations (FAR); Part 25 - Airworthiness Standards: Transport Category Airplanes. Department of Transportation, Federal Aviation Administration, Washington, D.C., U.S.A.
The stalling speed $V_s$ is obtained in a flight test conducted such that the airplane speed reduction does not exceed one knot per second. Nevertheless, the stalling speed $V_s$ obtained from this stalling maneuver may differ from the speed $V_{\text{min}}$, as obtained from the condition $L = W$.

The latter minimum airspeed is often called "one-g-stalling-speed".
Table 1: Take-off climb requirements for transport airplanes (one engine inoperative).

<table>
<thead>
<tr>
<th></th>
<th>PHASE OF FLIGHT</th>
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<tr>
<td></td>
<td>FIRST SEGMENT</td>
</tr>
<tr>
<td>THRUST (POWER)</td>
<td>TAKE-OFF</td>
</tr>
<tr>
<td>FLAP SETTING</td>
<td>TAKE-OFF</td>
</tr>
<tr>
<td>LANDING GEAR</td>
<td>DOWN</td>
</tr>
<tr>
<td>AIRSPEED</td>
<td>$V_{LOF}$</td>
</tr>
<tr>
<td>MINIMUM CLimb</td>
<td>0 (N = 2)</td>
</tr>
<tr>
<td>GRADIENT (%)</td>
<td>0.3 (N = 3)</td>
</tr>
<tr>
<td></td>
<td>0.5 (N = 4)</td>
</tr>
</tbody>
</table>

$V_{LOF}$ = LIFT-OFF SPEED
$V_2$ = TAKE-OFF SAFETY SPEED
N = NUMBER OF ENGINES
Fig. 1: Effect of airplane weight on power required.
Fig. 2: Effect of airplane weight on point performance in steady level flight and climb.
Fig. 3: Take-off procedure for transport airplane.
Fig. 4: Maximum allowable take-off weight for various pressure altitude and air temperature (climb requirements FAR-25)
PART II: POINT PERFORMANCE IN STEADY
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CHAPTER 9: TURNING PERFORMANCE

BASIC RELATIONS

In the following the special problem of the performance in
a level steady properly banked turn is discussed.
This type of turn is an important maneuver used to change
the flight-path heading.
The inward centripetal force required to maintain the air-
plane in the curved path is provided by the airplane banking,
as shown in Figure 1.
The lift supplies a component force towards the center of the
turn. On the other hand, there is a reactive force on the air-
plane, opposite to the centripetal force, called the centri-
fugal force, that is given by:

\[ F_c = m \frac{V^2}{R} \]  \hspace{1cm} (1)

where \( m = \frac{W}{g} \) is the mass of the airplane, \( V \) is the flight
velocity, and \( R \) is the radius of turn.

If the angle of bank in Figure 1 is such that the inward com-
ponent of the lift balances the centrifugal force, the maneu-
ver is called a properly banked turn. In this situation, the
airplane has no tendency to slip either inwards or outwards.
Moreover, in a steady constant altitude turn the vertical com-
ponent of lift equals the weight, and thrust equals drag.
The equations of motion can be obtained from Figure 1, where
the forces are given.

If the forces are resolved into vertical and horizontal compo-
nents, it can be seen from Figure 1 that:
\[ T = D = C_D \frac{1}{2} \rho V^2 S \]  \hspace{1cm} (2)

\[ \frac{W V^2}{g \frac{V^2}{R}} = L \sin \phi = C_L \frac{1}{2} \rho V^2 S \sin \phi \]  \hspace{1cm} (3)

\[ W = L \cos \phi = C_L \frac{1}{2} \rho V^2 S \cos \phi \]  \hspace{1cm} (4)

For given values of \( W \) and \( \rho \), the three equations (2) to (4) contain five variables: \( \alpha, V, T, R, \) and \( \phi \), so that the instantaneous flight condition is determined by two control variables.

We therefore proceed by expressing the performance characteristics in terms of angle of attack, \( \alpha(C_L \) and \( C_D \), and angle of bank, \( \phi \).

From equation (4) we obtain:

\[ V = \sqrt{\frac{W 2 \frac{1}{S} \frac{1}{\rho C_L \cos \phi}}{}} \]  \hspace{1cm} (5)

The drag follows from (2) and (4) as:

\[ D = W \frac{C_D}{C_L \cos \phi} \]  \hspace{1cm} (6)

Power required may be written as:

\[ P_r = DV = C_D \frac{1}{2} \rho V^3 S = W \sqrt{\frac{W 2 \frac{C_D^2}{S \rho C_L^3 \cos \phi}}{}} \]  \hspace{1cm} (7)

The expressions (5) to (7) can also be expressed in terms of angle of attack and load factor. The latter quantity follows directly from equation (4) as:

\[ n = \frac{L}{W} = \frac{1}{\cos \phi} \]  \hspace{1cm} (8)
Thus,
\[
V = \sqrt{\frac{n W 2 \frac{1}{S \rho C_L}}{n W \frac{C_D}{C_L}}} \tag{9}
\]

\[
D = n W \frac{C_D}{C_L} \tag{10}
\]

\[
Pr = n W \frac{n W 2 C_D^2}{S \rho C_L^3} \tag{11}
\]

The radius of turn is given by equation (3) and (4):
\[
R = \frac{V^2}{g \tan \phi} = \frac{W}{S \rho g} \frac{2}{C_L} \frac{1}{\sin \phi} \tag{12}
\]

or
\[
R = \frac{V^2}{g \sqrt{n^2-1}} = \frac{W}{S \rho g C_L} \frac{2}{\sqrt{n^2-1}} \frac{1}{\sqrt{n^2-1}} \tag{13}
\]

The time required to execute a 360° turn (2π radians) is given by:
\[
T_{2\pi} = \frac{2\pi R}{V} = \frac{2\pi V}{g \tan \phi} = \frac{2\pi}{g \sqrt{n^2-1}} \tag{14}
\]

With regard to V, D, and Pr, the effect of load factor (angle of bank) can be examined by considering flight conditions at a fixed C_L. Hence, because \(C_D = f(C_L)\), C_D also remains fixed.

Then, from equations (9), (10), and (11), the following ratios are obtained when flight conditions at \(n_1\) and \(n_2\) are compared to each other (\(n_2 > n_1\)): 
\[
\frac{V_2}{V_1} = \sqrt{\frac{n_2}{n_1}} \quad (15)
\]

\[
\frac{D_2}{D_1} = \frac{n_2}{n_1} \quad (16)
\]

\[
\frac{p_{r_2}}{p_{r_1}} = \sqrt{\left(\frac{n_2}{n_1}\right)^3} \quad (17)
\]

These equations show that for any given altitude and angle of attack, airspeed, drag, and power required increase as the load factor increases.

The stalling speed in a turn may be expressed as:

\[
V_{\text{min}} = \sqrt{\frac{W}{S \rho C_{L_{\text{max}}} \cos \phi}} = \frac{(V_{\text{min}})_{\phi=0}}{\sqrt{\cos \phi}} = (V_{\text{min}})_{\phi=0} \sqrt{n} \quad (18)
\]

For any given altitude the minimum airspeed increases as the load factor or the angle of bank increases. As can be seen also from equation (15), this increase is proportional to the square root of the load factor.

In a turn, the outer wing will be travelling faster than the inner wing and will therefore obtain more lift. The difference in velocity between the outer wingtip and the inner wingtip is (see Figure 1):

\[
\Delta V = \omega (R + \frac{1}{2} b \cos \phi) - \omega (R - \frac{1}{2} b \cos \phi) =
\]

\[
= \omega b \cos \phi = \frac{V}{R} b \cos \phi , \quad (19)
\]

where \( \omega = \frac{V}{R} \) is rate of turn.
Therefore,

\[ \frac{\Delta V}{V} = \frac{b}{R} \cos \phi \quad (20) \]

According to equation (12), the radius of turn will be least when \( C_L \) is equal to \( C_{L_{\text{max}}} \) and \( \phi = 90^\circ \).

\[ R_{\text{lim}} = \frac{W}{S \rho g C_{L_{\text{max}}}} \cdot \frac{1}{(V_{\text{min}}^2)_{\phi=0}} \quad (21) \]

It is rather important to note that the expression (21) exclusively provides a lower limit for \( R \).

Since then both airspeed and power required are infinite, \( R_{\text{lim}} \) can never be obtained in real flight.

E.g., for an airplane having a minimum airspeed \( V_{\text{min}} = 60 \text{ m/s} \), we get:

\[ R_{\text{lim}} \approx 360 \text{ m} \]

This result may indicate that in actual practice \( \frac{\Delta V}{V} \ll 1 \), so that in the maneuver considered here, also a symmetrical lift distribution over the span of the wing may be assumed.
CALCULATION OF TURNING PERFORMANCE
AT GIVEN POWER SETTING

Calculated values of power required versus airspeed are plotted in Figure 2 for a twin-engined airplane with turbo-props, of which the lift-drag polar is given in Figure 3 (see also part II).

The following procedure was used to obtain a point on the power required curve for a given angle of bank, and at a given altitude:

- A value of $C_L$ was assumed
- $V$ was calculated from equation (5), using $W = 15400$ kg $f (h = 0 \text{ m SA})$
- $C_D$ was determined from the lift-drag polar in Figure 3.

This procedure was repeated by selecting several values for angle of bank, and in this fashion the entire envelope of curves was determined.

In order that the airplane describes a constant altitude turn, power available must be equal to power required.

Owing to this requirement, in Figure 2 two different flight regimes may be distinguished:

$\text{I: } (V_{\text{min}})_{\phi=0} < V < V_A$

In this flight regime the airplane flies at $C_{L_{\text{max}}}$, and the power setting as selected in Figure 2 must be reduced. According as airspeed increases the radius of turn decreases monotonously with increasing angle of bank (see equation (14)).

$\text{II: } V_A < V < (V_{\text{max}})_{\phi=0}$

In this flight regime, the airplane cannot execute a level turn at $C_{L_{\text{max}}}$, because of limitations imposed by the power available curve.
Based on these considerations, the turning performance can be determined as indicated in Table 1. Figure 4 gives again the performance diagrams from Figure 2, where now power required versus $V$ for $C_L = C_{L_{\text{max}}}$ is plotted.

In flight regime I, the turning characteristics follow from the condition $C_L = C_{L_{\text{max}}}$, whereby several values for the airspeed are selected.

In flight regime II, the following technique is used to determine the turning characteristics:

- Several values of $V$ are assumed
- At each value of $V$ the drag coefficient $C_D$ is computed from:
  \[
  P_a = P_r = C_D \frac{1}{2} \rho V^3 S
  \]
- $C_L$ is determined from the lift-drag polar in Figure 3
- The lift is computed from:
  \[
  L = C_L \frac{1}{2} \rho V^2 S
  \]

Next, the load factor is computed from: $n = L/W$, the angle of bank from: $\phi = \arccos (1/n)$, the radius of turn from:
\[
R = \frac{V^2}{g \sqrt{n^2 - 1}},
\]
and finally the time in a $360^\circ$ turn from:
\[
T_{2\pi} = 2\pi \frac{R}{V}.
\]

Obviously, at $(V_{\text{min}})_{\phi=0}$ and $(V_{\text{max}})_{\phi=0}$ the airplane is in a steady level rectilinear motion, where:
\[
R = \infty, \quad T_{2\pi} = \infty, \quad \phi = 0, \quad \text{and} \quad n = 1.
\]

The results in Table 1 are plotted in Figure 4. Clearly, the speed for minimum $R$ is not the same as that for minimum time or for maximum angle of bank (maximum load factor).
From Figure 4, we obtain:

\[ R_{\text{min}} = 278 \text{ m at } V = 264 \text{ km/h} \]

\[ (T_{2\pi})_{\text{min}} = 22 \text{ s at } V = 290 \text{ km/h} \]

\[ \phi_{\text{max}} = 66.9^\circ \text{ at } V = 320 \text{ km/h} \]

Apparently,

\[ V_{R_{\text{min}}} < V_{(T_{2\pi})_{\text{min}}} < V_{\phi_{\text{max}}} \]

This connection has a general validity, that is explained below. From the relationship:

\[ T_{2\pi} = \frac{2\pi R}{V} \]

it may be seen that the location of the airspeed for minimum time in a 360° turn is found by drawing the tangent from the origin to the R-V curve in Figure 4b. Analogously, from the expression:

\[ \tan \phi = \frac{V^2}{g R} = \frac{2\pi V}{g T_{2\pi}} \]

we can conclude that the tangent to the \( T_{2\pi} \)-V curve in Figure 4c locates the point where angle of bank is maximum.
PERFORMANCE CHARACTERISTICS BY USE OF SIMPLIFYING ASSUMPTIONS

According to equation (6) and (7) and \( T = D \), we have:

\[
T = D = C_D \frac{1}{2} \rho V^2 S = W \frac{C_D}{C_L} \frac{1}{\cos \phi}
\]

or

\[
\cos \phi = \frac{W C_D}{T C_L} \tag{22}
\]

and

\[
P_a = P_r = C_D \frac{1}{2} \rho V^3 S = W \sqrt{\frac{W}{S \rho} \frac{C_L^3}{C_D^2} \frac{1}{\cos^3 \phi}}
\]

or

\[
\cos \phi = \left[ \frac{\frac{W}{S \rho} \frac{C_D^2}{C_L^3}}{\left( \frac{P_a}{W} \right)^{2/n}} \right]^{1/3} \tag{23}
\]

When the assumption is made that in the case of a jet airplane, the thrust is independent of airspeed, equation (22) says that the steepest angle of bank is obtained in the flight at maximum \( C_L/C_D \).

Likewise, it appears from (23) that in the case of a propeller airplane, of which power available does not vary with speed, the steepest bank occurs in the flight at maximum \( C_L^3/C_D^2 \).

Further, it appears that at a given power setting the maximum angle of bank (maximum load factor) will decrease as altitude increases due to the lower thrust (power available) and air density.

At the theoretical ceiling, the equilibrium conditions (22) and (23) are met at \( \phi = 0 \) (n = 1).
This simplified consideration shows that the turning performance of an airplane will strongly decrease as altitude increases. This behaviour is also apparent from Figure 5, where for several heights the radius of turn and the load factor as a function of airspeed are given for a high subsonic jet airplane flying at maximum power setting.

A condition for the angle of attack can also be indicated for a turn in which the radius is minimum. For simplicity, only a derivation for a jet airplane of which the engines deliver a constant thrust, will be given.

According to equation (12), we have:

\[
R = \frac{V^2}{g \tan \phi} = \frac{W}{S \rho g C_L \sin \phi}
\]

Therefore, the minimum radius is obtained at an angle of attack at which \( C_L \sin \phi \) is maximum.

At a given powersetting \((T/W)\), the relationship between \( C_L \) and \( \phi \) is given by equation (22).

Elimination of the angle of bank from (12) and (22), yields:

\[
R = \frac{W}{S \rho g C_L} \frac{1}{\sqrt{1 - \left(\frac{W}{T}\right)^2 \frac{C_D^2}{C_L^2}}}
\]

(24)

This equation shows that for a given \( T/W \), in other words depending on powersetting and altitude, the minimum radius is obtained at an angle of attack lying between maximum lift coefficient and maximum lift/drag ratio.

The lift coefficient for minimum radius can be derived by differentiating the quantity:

\[
C_L \sqrt{1 - \left(\frac{W}{T}\right)^2 \left(\frac{C_D}{C_L}\right)^2}
\]

with respect to \( C_L \) and setting the result equal to zero:
\[ \frac{d}{dC_L} \left[ C_L \sqrt{1 - \left( \frac{W}{T} \right)^2 \frac{C_D^2}{C_L^2}} \right] = 0 \]

or

\[ \frac{d}{dC_L} \left[ C_L^2 - \left( \frac{W}{T} \right)^2 C_D^2 \right] = 0 \]

Thus,

\[ \frac{C_D}{C_L} \frac{dC_D}{dC_L} = \left( \frac{T}{W} \right)^2 \]

(25)

Assuming a parabolic variation of \( C_L \) with \( C_D \):

\[ C_D = C_{D_0} + \frac{C_L^2}{\pi A_e} \]

we can convert equation (25) to:

\[ C_L = \sqrt{\left( \frac{T}{W} \right)^2 \frac{\pi A_e}{2} - C_{D_0} \pi A_e} \]

(26)

From equations (24) and (26), it is apparent that the lift coefficient for minimum radius decreases with increasing height. Then, at the theoretical ceiling we have \( T/W = C_D/C_L \), being the condition of equilibrium in steady symmetric flight.

For the airplane considered in Figure 5, the minimum radius is separately plotted as a function of altitude in Figure 6. This Figure shows again the strong effect of altitude on the turning performance.

In the present case, the minimum radius increases with increasing altitude due to the decreasing air density, decreasing thrust and at greater heights also due to the decreasing value of the lift coefficient.
PROBLEM

The reader should attempt to develop an expression, giving the lift coefficient for minimum radius of turn analogous to (26), for a propeller airplane in the case that power available is independent of airspeed.

Solution:

\[
C_L = \sqrt{\frac{\pi Ae}{\frac{3}{4} \frac{\pi Ae}{C}}} - C_{D_0} \frac{\pi Ae}{C},
\]

(27)

where

\[
C = \left[ \begin{array}{c}
\frac{W}{S} \\
\frac{2}{\rho} \\
\left(\frac{P}{W}\right)^{1/2} \\
\end{array} \right]^{2/3}
\]
Airplane weight: \( W = 15400 \text{ kg} \)
Wing-loading: \( W/S = 220 \text{ kg} \/ \text{m}^2 \)
Altitude: \( h = 0 \text{ m} \)

<table>
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<th>( C_D )</th>
<th>( C_L )</th>
<th>( L )</th>
<th>( n )</th>
<th>( \phi )</th>
<th>( R ) ( \text{m} )</th>
<th>( T_{2\pi} ) ( \text{s} )</th>
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Regime I

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<th>( \phi )</th>
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<th>( T_{2\pi} ) ( \text{s} )</th>
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Regime II

Table 1: Calculation of turning-performance of two-engined airplane with turboprops.
Fig. 1: Forces acting on an airplane during a steady level properly banked turn.
Airplane weight \( W = 15400 \text{ kg} \)
Altitude \( h = 0 \text{ m} \) S A
Engine speed \( n = 14000 \text{ rpm} \)

\[ C_L = C_{L_{\text{max}}} = 1.45 \]

\( (P_r)_h \) at \( \phi = 0^\circ \)

\( (V_{\text{min}}) \phi = 0 \)

\( V_A \)

\( (V_{\text{max}}) \phi = 0 \)

Fig. 2: Performance diagram for various bank angles of a twin-engined airplane with turboprops.

\rightarrow \text{airspeed } V \text{ (km/h)}
Fig. 3: Lift-drag polar of a twin-engined airplane with turboprops.
Fig. 4: Determination of characteristic magnitudes in level turn.
Fig. 5: Turning performance at various altitudes for high subsonic airplane.
Fig. 6: Minimum radius of turn as a function of altitude for high subsonic airplane.