Spectral Analysis of Individual Realization LDA Data

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Abstract: The estimation of the autocorrelation function (acf) or the spectral density function (sdf) from LDA data poses unique data-processing problems. The random sampling times in LDA preclude the use of the spectral methods for equi-spaced samples. As a consequence, special data-processing algorithms are used to process the LDA data. However, the random sampling causes an additional statistical variability of the spectral estimates that obscures the behaviour of the sdf in the high frequency range. The maximum frequency at which reliable estimates can be made is usually less than the mean data rate. For LDA measurements in gas flows the mean data rate is often small compared to the highest frequencies of the velocity fluctuations. As a consequence, the small scales of the turbulent fluctuations cannot be studied from the estimated sdf's with the presently available data-processing methods.

It is the objective of the present study to modify an existing data-processing method such that information on the spectral density can be revealed at much higher frequencies. The modification consists of two elements. First, a locally scaled autocorrelation function is computed. This modification of the conventional slotting technique results in a much lower statistical variance at small lag times. Next, the locally scaled acf is cosine-transformed using a lag window whose width is varied with frequency.

The modified estimator is applied to two types of simulated data to illustrate its performance. It is shown that the modified slotting technique in conjunction with a variable window forms a powerful spectral estimator for low data density flows.

Keyword(s): Spectral analysis, Random sampling, Laser Doppler anemometry

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\begin{itemize}
  \item \(a\) parameter in Pao's spectral density function
  \item \(a_1, a_2\) parameters in \(AR(2)\) model
  \item \(B_E\) energy of spectral window
  \item \(d\) data window
  \item \(d\) diameter of measuring volume
  \item \(f_s\) frequency
  \item \(g\) polynomial function
  \item \(H\) number of cross products
  \item \(H\) filter characteristic
  \item \(H_b\) number of cross products in case of velocity bias
  \item \(J\) local variance
  \item \(j_m\) integer
  \item \(L_x, L_y, L_z\) Taylor length scales for \(x, y\) and \(z\)-direction
  \item \(l\) length of measuring volume
  \item \(M\) number of slots
  \item \(m\) number of blocks
  \item \(N\) number of samples
  \item \(n(t)\) uncorrelated noise
  \item \(p\) probability density function
  \item \(R\) autocovariance function
  \item \(\hat{R}\) autocovariance calculated from slotting technique
  \item \(R_m\) volume-averaged autocovariance function
  \item \(R_T\) circular autocovariance function
  \item \(S\) power spectral density function
  \item \(\hat{S}\) Fourier transform of \(\hat{R}\)
  \item \(S_T\) periodogram
  \item \(S_{TW}\) smoothed periodogram
  \item \(S_X\) spectral density of sample-and-hold signal, Eq (2.38)
  \item \(S_{1B}\) spectral density calculated with 'short block' method, Eq (2.19)
  \item \(S_3\) spectral density computed from Eq (2.46)
  \item \(T\) measuring time
  \item \(T_B\) block time
  \item \(T_u\) integral time scale
  \item \(t\) time
  \item \(U_t\) pseudo-random velocity
  \item \(u_m\) measured velocity
  \item \(u(t)\) turbulent velocity component, random process
  \item \(V_t\) time average of absolute velocity
  \item \(w\) lag window
  \item \(\bar{x}\) location within measuring volume
\end{itemize}

Greek

\begin{itemize}
  \item \(\alpha_1, \alpha_2\) parameters in \(AR(2)\) process
  \item \(\Delta \rho\) magnitude of discontinuity
  \item \(\Delta t\) time step
  \item \(\Delta \tau\) slot width
\end{itemize}
<table>
<thead>
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<td>distance between particles</td>
</tr>
<tr>
<td>$\Delta\omega$</td>
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<td>$\omega$</td>
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<td>parameter in Pao's spectral density function</td>
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<td>$\omega_0$</td>
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Chapter 1

Introduction

The power spectral density function of a random signal is of importance in many disciplines. In fluid mechanics the random process to be studied is often the fluctuating component of the turbulent velocity. The power spectrum describes the distribution of the kinetic energy over different frequency ranges, and it can be used to determine the scales of the turbulent motion. When the random signal is sampled at equidistant intervals, as is usually the case, the spectral analysis can be carried out conveniently with the Blackmann-Tuckey correlation method [Blackman and Tuckey 1959] or with the fast Fourier transform e.g. Bendat and Piersol [1966]. These methods are well developed and computationally very efficient.

Despite these advantages, the periodic sampling gives rise to a number of problems such as the aliasing effect. To avoid aliasing the sampling rate should be at least twice the highest frequency present in the signal. This is widely known as the Nyquist criterion. Shapiro and Silverman [1960] have shown theoretically that alias-free estimates of the spectral density can be made with certain types of random sampling. In particular, they have shown that when the time intervals between successive samples, $\Delta t$, are Poisson distributed, alias-free estimates of the power spectrum can be obtained. In that case the Nyquist criterion does not apply anymore and, as a consequence, spectral estimates can be obtained at any frequency from data sampled at arbitrary low frequencies, in theory at least. However, the power spectrum cannot be obtained using the methods for equi-spaced samples. So, different methods had to be developed to perform the spectral analysis. As indicated by Gaster and Roberts [1975] the method of generating spectral estimates as given by Silverman and Shapiro was of little practical use, because the estimates were found to be very sensitive to small errors in the estimated correlation coefficients. More practical methods for spectral estimation were given by e.g. Scott[1974], Mayo [1974] and Smith and Meadows[1974]. All these contributions are concerned with a correlation method, where first a discrete version of the correlation function is obtained which is subsequently Fourier cosine transformed to yield the power spectrum.

Gaster and Roberts [1975] introduced a correlation method which is a generalization of the above mentioned technique. In another paper Gaster and Roberts [1977] proposed a direct Fourier transform of the randomly sampled signal. Both papers form a good theoretical framework for practical methods. An important conclusion of their work is that although the random sampling eliminates the aliasing effect, it also results in an increased variability of the spectral estimates. In addition, the algorithms used to form spectral estimates from randomly sampled data are computationally inefficient.

Nowadays LDA is frequently used to measure the velocity fluctuations in turbulent flows. The measuring technique is ideally suited for turbulence measurements, because of its non-intrusive character and its ability to measure instantaneous flow reversals. In addition the technique combines good spatial and temporal resolution. The LDA measures the velocity of small particles that are carried along with the fluid. Since the particles are randomly distributed
in space, the sampling times are random as well. It is known from both theory and experiment that the time between successive samples can be approximated closely by a Poisson process. The LDA thus provides a record of randomly sampled velocities. Spectral estimates can be made by applying the earlier mentioned methods to the LDA data. The random sampling causes an extra variability of the spectral estimates that obscures the behaviour of the spectral density function in the high frequency range. The maximum frequency at which reliable estimates can be made is usually less than the average data rate, which is generally small compared to the highest frequencies of the velocity fluctuations. As a consequence, the small scales of the turbulent fluctuations cannot be studied from the estimated spectral density functions with the presently available methods. It is the objective of the present study to modify the existing methods such that information on the spectral density can be revealed at much higher frequencies. To illustrate the performance of the modified estimator, it is applied to two types of simulated data.
Chapter 2

Spectral Estimators

2.1 Autocorrelation and Spectral Density Function

The autocovariance function of the stationary random process \( u(t) \) is defined as

\[
R(\tau) = \overline{u(t)u(t + \tau)}
\]

where \( \tau \) is the lag time and the overline denotes the expectation operator. When \( R(\tau) \) is scaled with \( R(0) \), the autocorrelation function is obtained

\[
\rho(\tau) = \frac{R(\tau)}{R(0)}
\]

where \( R(0) \) is the variance of \( u(t) \) given by \( R(0) = \overline{u^2} \), so that \( R(0) \) is proportional to the power of \( u(t) \). The autocorrelation function measures the similarity between a realization of \( u(t) \) and the same realization shifted over time \( \tau \). The autocorrelation has a number of properties that can be formulated as

1. \( \rho(0) = 1 \)
2. \( \rho(-\tau) = \rho(\tau) \)
3. \( \rho(\tau) \leq \rho(0) \)
4. \( \frac{d\rho}{d\tau}|_{\tau=0} = 0 \)
5. \( \lim_{\tau \to \infty} \rho(\tau) = 0 \)

The latter property is valid only when \( u(t) \) contains no periodicities. In fluid mechanics \( \rho(\tau) \) is used to determine a number of time scales of the turbulent flow. The first time scale is defined as

\[
T_u = \int_0^\infty \rho(\tau)d\tau
\]

and is therefore called the integral scale. \( T_u \) is the largest time scale that characterises \( u(t) \). The second time scale is the Taylor time scale which follows from the curvature of \( \rho(\tau) \) at zero lag time.

\[
\lambda_u^2 = -2/\left. \frac{d^2\rho}{d\tau^2} \right|_{\tau=0}
\]

The power spectral density function is defined as the Fourier transform of \( R(\tau) \), which reduces to a cosine transform because \( R(\tau) \) is a real and even function

\[
S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau = \frac{1}{\pi} \int_0^{\infty} R(\tau)\cos\omega\tau d\tau
\]
$S(\omega)d\omega$ can be interpreted as the contribution to the power of the components with frequencies between $\omega$ and $\omega + d\omega$. In practice a record of $u(t)$ can be measured only on a finite length segment with length $T$. From such a record the following estimate of $R(\tau)$ can be made

$$R_T(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} u(t)u(t + \tau)dt$$

(2.6)

The periodogram of the process $u(t)$ is defined as

$$S_T(\omega) = \frac{1}{T} \left| \int_{-T/2}^{T/2} u(t)e^{-i\omega t}dt \right|^2$$

(2.7)

and $S_T(\omega)$ can be expressed in terms of $R_T(\tau)$ as follows

$$S_T(\omega) = \int_{-\tau}^{\tau} R_T(\tau)e^{-i\omega \tau}d\tau$$

(2.8)

It can be shown [Papoulis 1991] that $S_T(\omega)$ is an unbiased estimation of $S(\omega)$ and its variance is given by

$$\text{var}\{S_T(\omega)\} = \begin{cases} 2S^2(0) & \text{if } \omega = 0 \\ S^2(\omega) & \text{if } \omega \gg 1/T \end{cases}$$

(2.9)

According to Eq (2.9) the variance becomes unacceptably large for high frequencies. To reduce the variability a window function is introduced. This means that in Eq (2.8) $R_T(\tau)$ is replaced by $R_T(\tau)w(\tau)$. The lag window smoothes the spectral estimates as can be seen from

$$S_{TW}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_T(\omega - \mu)W(\mu)d\mu$$

(2.10)

Here $W(\omega)$ is the spectral window, defined as the Fourier transform of the lag window $w(\tau)$. For $\omega \gg 1/T$ the variance of the spectral estimates becomes

$$\text{var}\{S_{TW}(\omega)\} = \frac{B_E}{T} S^2(\omega)$$

(2.11)

The quantity $B_E$ is a measure for the energy of the spectral window, defined as

$$B_E = \frac{1}{2\pi} \int_{-\infty}^{\infty} W^2(\omega)d\omega$$

(2.12)

so that $B_E = \frac{3}{4}\tau_m$ for the Tuckey-Hanning window given by $w(\tau) = 1/2 + 1/2\cos(\pi\tau/\tau_m)$ with $w(\tau) = 0$ for $\tau \geq \tau_m$.

### 2.2 Spectral Estimators for Randomly Sampled Data

The preceding analysis assumes that a continuous record of $u(t)$ is available. However, in case of LDA measurements the velocity measurements are taken at discrete, random times. Consider a record of $N$ samples $u(t_i)$ of the stationary process $u(t)$ that are measured on the interval $[0, T]$. The time intervals, $\Delta t = t_{i+1} - t_i$ between successive samples are Poisson distributed

$$p(\Delta t) = \nu e^{-\nu \Delta t}$$

(2.13)

where $\nu$ is the mean data rate given by $\nu = N/T$. For simplicity it is assumed that $u(t)$ has zero mean or that the mean has been subtracted. The objective is to estimate the spectral density function $S(\omega)$ from the $N$ samples $u(t_i)$ $i = 1, 2, \ldots, N$. Two basic methods are given in literature which are known as the direct method [Gaster and Roberts 1977] and the correlation method [Gaster and Roberts 1975], [Scott 1974] and [Mayo 1974]. These methods are described in sections 2.2.1 through 2.2.3. This is followed by a discussion of the sample-and-hold method, which is used as a representative of the much wider class of reconstruction techniques. Finally a new spectral estimator is introduced in section 2.2.5.
2.2.1 The Direct Method

The direct method [Gaster and Roberts 1977] is based on a discretised form of Eq (2.7).

$$S_1'(\omega) = \frac{1}{2\pi T} \left| \sum_{i=1}^{N} u(t_i) d(t_i) e^{-i\omega t_i} \Delta t \right|^2$$  \hspace{1cm} (2.14)

where \( d(t) \) is a data window. Eq (2.14) can be rearranged as

$$S_1'(\omega) = \frac{1}{2\pi T \nu^2} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} u(t_i) u(t_j) d(t_i) d(t_j) e^{-i\omega (t_i-t_j)} + \sum_{i=1}^{N} u^2(t_i) d^2(t_i) \right\}$$  \hspace{1cm} (2.15)

where the interval times \( \Delta t_i \) are replaced by their average value \( 1/\nu \). The second term on the right-hand side causes a false displacement of the estimates and a proper spectral estimator is therefore given by

$$S_1(\omega) = \frac{1}{2\pi T \nu^2} \left\{ \sum_{i=1}^{N} u(t_i) d(t_i) e^{-i\omega t_i} \right\}^2 - \sum_{i=1}^{N} u^2(t_i) d^2(t_i)$$  \hspace{1cm} (2.16)

Gaster and Roberts [1975] derived expressions for the bias and variance of \( S_1(\omega) \). They showed that \( S_1(\omega) \) is an asymptotically unbiased estimate of \( S(\omega) \), and that its variance is given by

$$\text{var}\{S_1(\omega)\} = \alpha + \frac{\beta}{\nu} + \frac{\gamma}{\nu^2}$$  \hspace{1cm} (2.17)

The values of \( \alpha \), \( \beta \) and \( \gamma \) depend on the measuring time \( T \), the window \( d(t) \) and the shape of the true spectral density function \( S(\omega) \). For high frequencies the values of \( \alpha \), \( \beta \) and \( \gamma \) are given by

$$\alpha = S^2(\omega), \hspace{1cm} \beta = \frac{S(\omega) R(0)}{\pi} \hspace{1cm} \text{and} \hspace{1cm} \frac{R^2(0)}{4\pi^2} |T \to 0 \leq \gamma \leq \frac{3R^2(0)}{4\pi^2} |T \to \infty$$  \hspace{1cm} (2.18)

The value of \( \gamma \) varies between \( 3R^2(0)/(4\pi^2) \) for infinitely long measuring times and \( R^2(0)/(4\pi^2) \) for short measuring times. The value for \( \alpha \) is also found in the expression for the variance in case of a continuous record, Eq (2.11), and in case of periodically sampled data (e.g. Priestley [1981]). Therefore, it can be concluded that the terms containing \( \beta \) and \( \gamma \) represent the extra variability of the spectral estimates due to the random sampling. Note that \( S_1 \) in its basic form is an inconsistent estimator, because the variance does not become zero for infinitely long measuring times. To reduce the variability the so called short block method is used. This means that the measuring time is divided into \( m \) blocks of duration \( T_B (= T/m) \) and Eq (2.16) is applied to each data segment. The resulting spectral estimates are averaged according to

$$S_{1B}(\omega) = \frac{1}{m} \sum_{j=1}^{m} S_{1j}(\omega)$$  \hspace{1cm} (2.19)

This estimator is again asymptotically unbiased, but with smaller variance

$$\text{var}\{S_{1B}(\omega)\} \approx \frac{1}{m} \text{var}\{S_1(\omega)\}$$  \hspace{1cm} (2.20)

The smaller block length, however, causes a decrease of the resolution of the spectral estimates, since the width of the spectral window is inversely proportional to the block length. For very high frequencies, i.e. small values of \( S(\omega) \), the variability of \( S_{1B}(\omega) \) is dominated by

$$\text{var}\{S_{1B}(\omega)\} = \frac{\gamma}{nm^2}$$  \hspace{1cm} (2.21)
When LDA measurements are performed in gas flows it is seldom possible to obtain data rates that are sufficient high to neglect this term. In that case the range of the estimated power spectrum is small. This can be seen from an example. Consider a process \( u(t) \) with \( \bar{u}^2 = 1.0 \, \text{m}^2/\text{s}^2 \) and \( T_u = 50 \, \text{ms} \) that is sampled at a mean rate of 1 kHz. The spectral density for which \( \text{var}\{S_{1B}(\omega)\} \) exceeds \( \delta^2 S^2(\omega) \) follows from Eq (2.20) as

\[
S_\delta = \frac{R(0) \, \frac{1}{2\pi\nu}}{\delta \sqrt{m} - 1}
\]  

(2.22)

Assuming, not quite arbitrarily, \( m = 200 \) and \( \delta = 1/3 \) yields \( S_\delta = 4.29 \times 10^{-5} \, \text{m}^2/\text{s} \). Since \( S(0) = T_u/\pi = 1.59 \times 10^{-2} \, \text{m}^2/\text{s} \) the depth of the estimated spectrum is only 2.5 decades, before the variability of the estimates exceeds the mean value.

As can be seen from Eq (2.17) the terms describing the extra variability through the random sampling are proportional to the variance \( R(0) \). With this in mind Gaster and Roberts [1980] suggested to filter out parts of the spectral density to decrease the variance of the remaining signal. If the analysis is carried out for this new signal then the variability of the spectral estimates is expected to be smaller. For the zero frequency component the method can be easily implemented if for each block the local mean is subtracted from the data instead of the mean calculated from all \( N \) samples. This effectively removes all energy for frequencies smaller than \( 1/T_B \). The remaining signal thus has a somewhat smaller variance, which will reduce the variance of the spectral estimates. This procedure can also be followed to remove the contribution from other frequencies. However, as indicated by Gaster and Roberts [1980], the spectral components to be removed must be estimated accurately, otherwise the filtering procedure could actually increase the variability. In addition, there may be a sharp increase of the bias error in the regions where the spectral components are removed. Therefore, this method of reducing the variability of the spectral estimates is not without drawbacks.

### 2.2.2 The Correlation Method

Each cross product \( u(t_i)u(t_j) \) can be regarded as a realization of the covariance function at lag time \( \tau = t_i - t_j \). According to Gaster and Roberts [1975] a spectral estimate is formed by the cosine transform of the cross products

\[
S_2(\omega) = \frac{1}{\pi\nu^2T} \sum_{i=1}^{N-1} \sum_{j=i+1}^{J_m} u(t_i)u(t_j)w(t_j - t_i) \cos(\omega(t_j - t_i))
\]

(2.23)

where \( T \) is the measuring time and \( w \) is a lag window to improve the spectral estimate. For each \( i, j_m \) is the largest integer that satisfies \( t_{jm} - t_i < \tau_m \) where \( \tau_m \) is a chosen maximum lag time. Note that \( 1/(\nu^2 T) \) is the mean lag time between the covariance estimates.

Roberts and Gaster [1975] demonstrated that for large measuring times, \( S_2(\omega) \) is an unbiased estimator of the spectral density function \( S(\omega) \). The variance of \( S_2(\omega) \) is given by

\[
\text{var}\{S_2(\omega)\} = \frac{c \tau_m}{T} \left\{ S(\omega) + \frac{R(0)}{2\pi\nu} \right\}^2
\]

(2.24)

where the constant \( c \) depends on the lag window, e.g. \( c = 3/4 \) for the Tuckey-Hanning window. This expression has the same characteristics as that for the direct method (Eq (2.17)). Again, the random sampling results in an increase of the variance of the spectral estimates. The term \( R(0)/(2\pi\nu) \) will dominate at high frequencies and obscure the behaviour of \( S(\omega) \). It is interesting to compare the variance of the spectral estimator \( S_2(\omega) \) with that of the short block direct method \( S_{1B}(\omega) \). Using Eq (2.17) and Eq (2.24), the ratio of the two variances at high frequencies reduces to

\[
\mu = \frac{\text{Var}\{S_{1B}(\omega)\}}{\text{Var}\{S_2(\omega)\}} \approx \frac{T}{mc\tau_m}
\]

(2.25)
It is reasonable to assume that the number of blocks \( m \) equals \( 2T/\tau_m \), so that with \( c = 3/4 \) (Hanning window) \( \mu \approx 8/3 \). This means that the variance of \( S_2(\omega) \) is slightly smaller than that of \( S_{1B}(\omega) \). However, an important disadvantage of the spectral estimator \( S_2(\omega) \) is the larger computational effort. To compute one spectral estimate from \( S_2, Tm\nu^2 \) cosines have to be evaluated. The \( S_{1B} \) estimator requires only \( 2mT_B\nu \) function evaluations. From this it is clear that the short-block direct method is much faster than the correlation method \( S_2 \), since

\[
\frac{2mT_B\nu}{T\tau_m\nu^2} = \frac{2}{\tau_m\nu} \ll 1
\]  

(2.26)

### 2.2.3 The Slotting Technique

The spectral estimator \( S_2(\omega) \) requires a large computational effort because each cross-product is first multiplied by a cosine and then summed according to Eq (2.23). As indicated by Gaster and Roberts [1975] the computation can be speeded up by first calculating a discretized autocovariance function from all the cross products, followed by a cosine transform of this function. The method used to estimate the discretized autocovariance function from the random samples is known as the slotting technique ([Mayo 1974], [Scott 1974]) and it is given by the following algorithm

\[
\hat{R}(k\Delta\tau) = \frac{\text{sum}\{u_i u_j\}(k\Delta\tau)}{H(k\Delta\tau)} \quad \text{with} \quad k = 0, 1, \ldots, M - 1
\]  

(2.27)

where \( \text{sum}\{u_i u_j\}(k\Delta\tau) \) denotes the sum of all cross-products \( u(t_i)u(t_j) \) having a lag time in the interval \( (k - \frac{1}{2})\Delta\tau < t_i - t_j < (k + \frac{1}{2})\Delta\tau \), \( H(k\Delta\tau) \) is the number of cross products within a slot and \( \Delta\tau \) is the, usually constant, slot width. The integer \( M \) follows from \( M = \tau_m/\Delta\tau \) where \( \tau_m \) is a chosen maximum lag time. In case of Poisson sampling the expected number of cross products per slot is given by

\[
\bar{H}(k\Delta\tau) = \nu N\Delta\tau\{1 - \frac{k\Delta\tau}{T}\}
\]  

(2.28)

This shows that the Poisson sampling process provides estimates at all lag times, even at the smallest lag times. For a fixed slot width the number of cross products per slot can be increased by increasing the data rate or by increasing the number of samples. The discretization of the lag-time axis results in a bias error due to the smearing of the time information. Instead of \( R(\tau = k\Delta\tau) \) the slotting technique estimates

\[
\overline{R}(k\Delta\tau) = \frac{1}{\Delta\tau} \int_{-\infty}^{\infty} R(\tau)h(k\Delta\tau - \mu)d\mu
\]  

(2.29)

where it is assumed that the slot width \( \Delta\tau \) is small compared to the measuring time \( T \), so that the lag times of the different cross-products are uniformly distributed over the slot, i.e.

\[
h(\mu) = \begin{cases} 
1 & \text{if } |\mu| \leq \Delta\tau/2 \\
0 & \text{if } |\mu| > \Delta\tau/2 
\end{cases}
\]  

(2.30)

It is seen from Eq (2.29) that the non-zero width causes a bias when the autocovariance function is non linear. This bias error may be especially large for the region near \( \tau = 0 \), because there the curvature is large. Taking Fourier transforms of both sides in Eq (2.29), yields ([Scott 1974])

\[
\hat{S}(\omega) = \sum_{n=-\infty}^{\infty} S(\omega - \frac{2\pi n}{\Delta\tau})\Psi(\omega - \frac{2\pi n}{\Delta\tau}) \quad \text{with} \quad \Psi(\omega - \frac{2\pi n}{\Delta\tau}) = \frac{2\sin(\omega - \frac{2\pi n}{\Delta\tau})}{\omega - \frac{2\pi n}{\Delta\tau}}
\]  

(2.31)
This illustrates that the discretization introduces aliasing, and that the high-frequency components are attenuated due to the smearing of the time information. The variance of the slotted correlation is given by the following expression, which is also due to Scott [1974]

$$\text{var}\{\hat{R}(k\Delta\tau)\} = \frac{R^2(0) + R^2(\tau)}{H(k\Delta\tau)} + \frac{4}{T} \int_{0}^{\infty} R^2(\mu) d\mu \quad (2.32)$$

The first term on the right-hand side can be considered the variability caused by the finite number of autocovariance estimates per slot. For a given $\Delta\tau$, $H(k\Delta\tau)$ can be increased by increasing the data rate $\nu$ or by increasing the number of samples $N$, see Eq (2.28). However, when the number of samples $N(=\nu T)$ is fixed (e.g. because of limited storage capacity) then an increase of the data rate results in a decrease of the measuring time, which in turn will increase the magnitude of the second term. So, to decrease the variance of the $\hat{R}(k\Delta\tau)$ estimates, both an increase of the data rate and longer measuring times are necessary. Once the discretized autocovariance function is available, a spectral estimate can be formed from

$$S_{2S}(\omega) = \frac{\Delta\tau}{\pi} \left\{ \frac{1}{2} \hat{R}(0) + \sum_{k=1}^{M-1} \hat{R}(k\Delta\tau) u(k\Delta\tau) \cos(k\omega\Delta\tau) \right\} \quad (2.33)$$

where $u(t)$ again is the lag-window function. The first slot ($k = 0$) is treated as a separate case, because it has width $\Delta\tau/2$. Due to the discretization of the lag-time axis and the smearing of the time information, the slot width must be chosen small enough so that the aliasing error and the attenuation error are negligible. In the absence of a mathematical expression for the variance of $S_{2S}$, Gaster and Roberts [1975] suggested to use Eq (2.24) that was derived for $S_2$. Although this approach seems prompted by convenience, in practice it yields satisfactory results.

### 2.2.4 The Sample-and-Hold Method

The essence of all reconstruction methods is to replace the discrete, randomly sampled time series by a continuous signal that is subsequently sampled at regular time intervals. The advantage of the reconstruction is that the spectral analysis of equi-spaced samples can be carried out efficiently with the FFT algorithm. In addition, the complication due to a correlation between the sampling process and the process being sampled, known in LDA as the velocity bias, may be eliminated by the reconstruction method. On the other hand, it is widely known that reconstruction methods require high sampling rates. The exact meaning of the adjective high will be given during the course of this section. For the moment, it suffices to remark that for LDA measurements in gas flows it is seldom possible to achieve sufficiently high sampling rates. Despite its limited importance to the spectral analysis in gas flows, the use of reconstruction techniques has become widespread, and this is the reason for its inclusion in the present overview of spectral estimators. The following discussion of reconstruction methods is limited to the most commonly used technique, namely sample-and-hold. Therefore, the discussion is incomplete and it merely serves to underline the fact that reconstructions are generally not well suited for the spectral analysis of LDA data acquired at a relatively low sampling rate.

In the sample-and-hold method a continuous signal, say $x(t)$, is obtained from the randomly sampled series $u(t_n)$ by holding the last measured value until the next measurement is obtained. This can be expressed in mathematical terms by

$$x(t) = u(t_n) \quad t_n \leq t < t_{n+1} \quad (2.34)$$

In variations on this method linear and higher-order polynomials are fitted to the measured values. A more advanced reconstruction that applies the Kalman filter ([Priestley 1981]) is described by Van Maanen and Tulleken [1994].
Spectral Estimators

It is easy to show (e.g. Boyer and Searby [1986]) that, in absence of velocity bias, the mean, variance and higher-order moments of the sample-and-hold signal are unbiased, i.e.

$$\bar{x}(t) = \bar{u}(t) = \bar{u} \quad \text{and} \quad \left( \bar{x}(t) - \bar{x} \right)^2 = \left( \bar{u}(t) - \bar{u} \right)^2 \quad j = 2, 3, \ldots \quad (2.35)$$

Boyer and Searby also showed that the spectral densities of \( x(t) \) and \( u(t) \) are related as

$$S_x(\omega) = \frac{(\pi \nu)^{-1}}{1 + (\frac{\omega}{\nu})^2} \left\{ \pi^2 - 2 \int_0^\infty \frac{S(\omega) \, d\omega}{1 + (\frac{\omega}{\nu})^2} \right\} + \frac{S(\omega)}{1 + (\frac{\omega}{\nu})^2} \quad (2.36)$$

where \( S_x(\omega) \) and \( S(\omega) \) are the spectral densities of \( x(t) \) and \( u(t) \), respectively. The first term on the right-hand side of Eq (2.36) originates from the Poisson distributed sampling intervals \( \Delta t = t_{n+1} - t_n \). It has a maximum at zero frequency and decays as \( \omega^{-2} \) for \( \omega \gg \nu \). It is important to realize that the roll-off as \( \omega^{-2} \) is an artifact of the reconstruction and it does not represent the behaviour of the spectral density function \( S(\omega) \) in the high-frequency range.

The second term represents the spectral density \( S(\omega) \) that is low-pass filtered by

$$H(\omega) = \frac{1}{1 + (\frac{\omega}{\nu})^2} \quad (2.37)$$

This filter causes a significant distortion even for frequencies well below the mean sampling rate. However, since the time constant of the filter depends only on the mean data rate, which can be considered a known quantity, the effect of the filter can be eliminated by multiplication of \( S_x(\omega) \) by \( H^{-1}(\omega) \). This yields the sample-and-hold spectral estimator \( S_{SH}(\omega) \)

$$S_{SH}(\omega) \overset{\text{def}}{=} S_x(\omega) \left\{ 1 + (\frac{\omega}{\nu})^2 \right\} = \frac{1}{\pi \nu} \left\{ \pi^2 - 2 \int_0^\infty \frac{S(\omega) \, d\omega}{1 + (\frac{\omega}{\nu})^2} \right\} + S(\omega) \quad (2.38)$$

It is seen that \( S_{SH}(\omega) \) is equal to \( S(\omega) \) only for infinite mean sampling rates. The first term between the brackets acts as a white-noise level and, as a result, it is impossible to fully restore \( S(\omega) \) from \( S_{SH}(\omega) \) if \( S(\omega) \) contains power above the mean sampling rate \( \nu \). A precise criterion for the maximum frequency for which \( S_{SH}(\omega) \) is accurately cannot be given, because it depends on the exact shape of the spectral density \( S(\omega) \) itself. However, it will be shown in section 5.3 that the maximum frequency is considerably less than the often cited value \( \omega_{\text{max}} = \nu \).

2.2.5 A New Spectral Estimator

As will be shown in section 5.3, the spectral estimators \( S_1, S_2, S_{2S} \) and \( S_{SH} \) are not able to reveal the structure of the spectral density function in the high-frequency range. This is caused by the random sampling, which gives rise to an increased variability that becomes significant at high frequencies. In this section a new estimator is proposed that overcomes some of the variability problems at high frequencies. This spectral estimator is a modified version of \( S_{2S}(\omega) \), the estimator that is based on the slotting technique. The modification consists of two elements. First, a discretized autocorrelation function is calculated from the discretized autocovariance \( \hat{R}(k\Delta \tau) \) using a local normalisation (see Tummers et al. [1995]). Secondly, the width of the lag window is varied with frequency as proposed by Gaster and Roberts [1975]. It is important to note that each of these modifications alone is insufficient to significantly improve the spectral estimate at high frequencies. The combination of the two, however, yields the desired result.

Local Normalisation of the Autocovariance

The spectral estimator \( S_{2S}(\omega) \) applies a cosine transform of a discretized version of the autocovariance function. Instead of \( \hat{R}(k\Delta \tau) \) it is also possible to transform the discrete autocorrelation \( \hat{\rho}(k\Delta \tau) \), which is defined as

$$\hat{\rho}(k\Delta \tau) = \frac{\hat{R}(k\Delta \tau)}{u^2} \quad \text{with} \quad k = 0, 1, \ldots, M - 1 \quad (2.39)$$
where $\hat{R}(k\Delta \tau)$ is given by Eq (2.27) and $\bar{u}^2 = R(0)$ is the variance of the process $u(t)$, which is usually estimated from

$$\bar{u}^2 = \frac{1}{N} \sum_{i=1}^{N} u(t_i)^2$$  \hspace{1cm} (2.40)

However, since $u(t)$ is a stationary process $\sqrt{u(t)^2 u(t+\tau)^2}$ is equivalent to $\bar{u}^2$. Therefore, the discrete autocovariance $\hat{R}(k\Delta \tau)$ can be scaled alternatively with a discretized version of $\sqrt{u(t)^2 u(t+\tau)^2}$, i.e.

$$J(k\Delta \tau) = \frac{\sqrt{\text{sum}\{u_i^2\}(k\Delta \tau) \text{sum}\{u_j^2\}(k\Delta \tau)}}{H(k\Delta \tau)} \hspace{1cm} (2.41)$$

where $H(k\Delta \tau)$ is defined as in Eq (2.27), and $\text{sum}\{u_i^2\}$ and $\text{sum}\{u_j^2\}$ are the sums of the squares of the samples $u(t_i)$ and $u(t_j)$, respectively, for which the cross product $u(t_i)u(t_j)$ has a lag time in the interval $(k-\frac{1}{2})\Delta \tau < t_i - t_j < (k+\frac{1}{2})\Delta \tau$. This means that for each time interval a separate estimate of the variance is made, involving only those samples that also contribute to the sum of the cross-products in that slot. The slotted autocorrelation function then follows from

$$\hat{\rho}(k\Delta \tau) = \frac{\hat{R}(k\Delta \tau)}{J(k\Delta \tau)} = \frac{\text{sum}\{u_iu_j\}(k\Delta \tau)}{\sqrt{\text{sum}\{u_i^2\}(k\Delta \tau) \text{sum}\{u_j^2\}(k\Delta \tau)}}$$  \hspace{1cm} (2.42)

The advantage of the local normalisation is that at very small lag times, i.e. when the autocorrelation is near 1, the use of Eq (2.41) to scale the $\hat{R}(k\Delta \tau)$ values leads to a significant reduction of the variance of the correlation values as compared to the conventional scaling of $\hat{R}(k\Delta \tau)$ with $\bar{u}^2$ according to Eq (2.40). This will be shown in section 5.2.

**Variation of the Width of the Lag Window**

The selection of the width of the lag window is a trade-off between the smoothness and the resolvability of the spectral estimates. A decrease of the width of the lag window decreases the variance of the spectral estimates (see Eq (2.24)), but the bias will be increased as can be seen from (Priestley 1981)

$$\text{bias}\{\hat{S}(\omega)\} \sim \frac{d^2 S}{d\omega^2} B_w^2$$  \hspace{1cm} (2.43)

where $B_w$ is the width of the spectral window. Suppose that the window width is chosen such that the narrowest peak in the spectral density function can be resolved. If this width is used at all other frequencies, then there is more than enough resolution everywhere. However, in regions where the spectral density function is relatively flat, i.e. in regions with small curvature, the excess resolution can be used to increase the width of the window without creating a large bias error. This increase of the window width has the beneficial effect of reducing the variance of the spectral estimates, see e.g. Priestley [1981].

In the present investigation a Tuckey-Hanning window was used

$$w(\tau) = 1/2 + 1/2 \cos \left( \frac{\pi \tau}{\tau_m} \right) \quad \text{with} \quad \tau < \tau_m$$  \hspace{1cm} (2.44)

The width of this window can be conveniently varied with frequency by varying the maximum time lag as

$$\frac{\tau_m(\omega)}{\tau_{m_0}} = \kappa \frac{\omega_0}{\omega} \quad \text{with} \quad \omega_0 = \frac{2\pi}{\tau_{m_0}}$$  \hspace{1cm} (2.45)
where \( \omega_0 \) is the lowest frequency and \( \kappa \) is approximately equal to the ratio of the centre frequency \( \omega \) and the width of the spectral window. As a result of the variation of \( \tau_m \) with the frequency, the resolvability will be relatively low at high frequencies. However, when the spectral density function does not have sharp peaks in the high-frequency range, the variation of the window width works satisfactory.

Combining the locally normalized autocorrelation \( \tilde{\rho}(k \Delta \tau) \) with the variable window width, yields the following spectral estimator

\[
S_3(\omega) = \frac{R(0) \Delta \tau}{\pi} \left\{ \frac{1}{2} + \sum_{k=1}^{M-1} \tilde{\rho}(k \Delta \tau) w(k \Delta \tau) \cos(k \omega \Delta \tau) \right\}
\]

(2.46)

where \( \tilde{\rho}(k \Delta \tau) \) is given by Eq (2.42) and \( \tau_m \) is varied with frequency as in Eq (2.45). The factor \( R(0) \) is introduced for compatibility with the spectral estimator \( S_{2S} \), and it can be determined from Eq (2.40).
Chapter 3

Lifting the Model Restrictions

The preceding spectral analysis implicitly assumed that the LDA measuring volume was infinitely small, that the time at which particles passed through the measuring volume could be determined exactly, that the velocities were measured without error, and that the sampling process was independent of the process being sampled, i.e. there was no velocity bias. To some extent all these assumptions are violated in practice. It is the aim of this chapter to reveal some of the consequences of the model restrictions. First, the model requirements will be somewhat relaxed by the allowance of velocity errors in the form of uncorrelated noise. Secondly, the influence of the velocity bias on the act will be studied. Some consequences of ignoring the velocity bias will be revealed, and ways to correct for the velocity bias are given. This is followed by a discussion of the effects of imperfect spatial resolution.

3.1 The Influence of Uncorrelated Noise

In practice each velocity sample \( u_m \) consists of the true velocity fluctuation \( u \) and an error or white-noise component \( n \)

\[
u_m(t) = u(t) + n(t)
\]  \hspace{1cm} (3.1)

where the subscript \( m \) denotes a measured value. If it is assumed that the velocity and the noise are uncorrelated then the measured covariance will, at least in theory, not be affected by the noise

\[
\overline{u_m(t)u_m(t + \tau)} = \frac{u(t)u(t + \tau) + u(t)n(t + \tau) + n(t)u(t + \tau) + n(t)n(t + \tau)}{u(t)u(t + \tau)}
\]  \hspace{1cm} (3.2)

The measured variance, however, will be too high because it contains self products, i.e. zero lag time products, of the noise

\[
\overline{u_m(t)^2} = \frac{u(t)^2}{u(t)^2} + \frac{2u(t)n(t) + n(t)^2}{u(t)n(t) + n(t)^2}
\]  \hspace{1cm} (3.3)

When the measured variance is used to scale the autocovariance \( R(\tau) \), the resulting autocorrelation \( \rho(\tau) \) exhibits a discontinuity at zero lag time with magnitude (Absil et al. [1988])

\[
\Delta \rho = \frac{n^2}{u^2 + n^2}
\]  \hspace{1cm} (3.4)

The discrete autocorrelation \( \hat{\rho}(k\Delta \tau) \) used in the spectral estimator \( S_3 \) will also exhibit a discontinuity at zero lag time, because it utilizes self products of the noise component for the scaling of the \( \hat{R}(k\Delta \tau) \) values. Note that \( \hat{R}(k\Delta \tau) \) is not directly affected by the uncorrelated noise, because
the algorithm given by Eq (2.27) excludes all self products. Since the effect of the uncorrelated noise on $\hat{\rho}(k\Delta\tau)$ is concentrated in the first slot around $\tau = 0$, it can be practically eliminated by rescaling $\hat{R}(k\Delta\tau)$ as

$$
\tilde{\rho}_c(k\Delta\tau) = \frac{\hat{\rho}(k\Delta\tau)}{1 - \Delta\rho} \quad \text{with} \quad k = 0, 1, \ldots, M - 1
$$

(3.5)

The magnitude of the discontinuity $\Delta\rho$ can be estimated accurately if the $\hat{\rho}(k\Delta\tau)$ values reveal the behaviour of the autocorrelation function near zero lag time in detail. As pointed out in section 2.2.5, the discrete autocorrelation $\hat{\rho}(k\Delta\tau)$, as determined with the local normalisation according to Eq (2.42), has a very small statistical variance in regions where the (absolute) value of $\hat{\rho}(k\Delta\tau)$ is near 1. This property enables a reliable determination of $\Delta\rho$ from the experimental data. After rescaling according to Eq (3.5), the discrete autocorrelation $\tilde{\rho}_c(k\Delta\tau)$ can be used in spectral estimator $S_3$ instead of $\hat{\rho}(k\Delta\tau)$.

It should not be concluded from the preceding that uncorrelated noise has no influence on the discrete autocorrelation or on the spectral estimates that can be obtained from it. On the contrary, it is of great importance that the signal-to-noise ratio (SNR) defined as

$$
SNR = \frac{u^2}{n^2}
$$

(3.6)

is as high as possible. A high SNR results in a small discontinuity at $\tau = 0$, since

$$
\Delta\rho = \frac{n^2}{u^2 + n^2} = \frac{1}{1 + SNR}
$$

(3.7)

A small $\Delta\rho$ will increase the beneficial effect that the local normalisation has on the variance of the $\hat{\rho}(k\Delta\tau)$ estimates. This not only enables a more accurate determination of the magnitude of $\Delta\rho$, but it also provides more reliable estimates for small lag times, which is of particular importance for inferring the behaviour of the spectral density function in the high-frequency range. Thus, the higher the SNR, the more reliable the spectral estimates become at high frequencies.

Apart from the spike at zero lag time, the uncorrelated noise has another, more subtle, effect on the spectral estimates. In deriving Eqs (3.2) and (3.3) the ensemble averaging causes correlations, such as $\bar{u}(t)\bar{n}(t)$, to vanish. But in practice the averaging process is based on a finite number of samples. As a consequence, the cross correlations will not vanish completely, and this effectively increases the variability of $\hat{R}(k\Delta\tau)$ and $\hat{\rho}(k\Delta\tau)$, thereby increasing the variance of the spectral estimates. This again illustrates the importance of increasing the SNR.

### 3.2 The Effects of Velocity Bias on the ACF

The standard slotting technique, Eq (2.39), and its locally normalised version, Eq (2.42), can only be used to estimate the autocorrelation function from randomly sampled data if there is no correlation between the sampling process and the velocity. Since this correlation is inherent to the LDA measuring technique, it should be taken into account when processing measured data.

Buchhave et al. [1979] were the first to warn of the effects that ignoring the velocity bias has on the autocorrelation function. It remained unclear what these effects were until the simulations of Edwards and Kolodzy [1986] revealed that the autocorrelation could rise above 1 at small lag times. This physically unrealistic behaviour was also observed by Van Strien [1988] in his measurements in a stirred vessel. For the simple case of a one-dimensional flow with low turbulence intensity and a Gaussian velocity distribution, it is easy to show that the velocity bias causes $\hat{\rho}(k\Delta\tau)$ to become larger than 1 (Tummers [1992])

$$
\lim_{k\Delta\tau \to 0} \hat{\rho}(k\Delta\tau) = \frac{1 + \overline{u^2}/\overline{v^2}}{1 - \overline{u^2}/\overline{v^2}} > 1
$$

(3.8)
where \( \bar{u} \) is the mean velocity and \( \bar{w}^2 \) is the variance of the velocity fluctuations. Clearly, this abnormal behaviour of \( \hat{\rho}(k\Delta T) \) at small lag times will have adverse consequences for the spectral density \( S_{22} \). This indicates that there is a strong need to modify the existing algorithms such that the effects of velocity bias are taken into account. Buchhave et al. [1979] argued that the autocorrelation \( \hat{\rho}(k\Delta T) \) should be corrected for the velocity bias by weighting each cross-product \( u_i u_j \) with the corresponding residence times of the particles according to

\[
\hat{R}(k\Delta T) = \frac{\sum \{u_i u_j \omega_i \omega_j\}(k\Delta T)}{\sum \{\omega_i \omega_j\}(k\Delta T)} \tag{3.9}
\]

where \( \omega_i \) is the residence time of the i-th particle. The correlation \( \hat{\rho}(k\Delta T) \) then follows from \( \hat{R}(k\Delta T) \) by scaling with the corrected variance

\[
\bar{w}^2 = \sum_{i=1}^{N} u_i^2 \omega_i / \sum_{i=1}^{N} \omega_i \tag{3.10}
\]

Instead of the residence times other quantities, such as the inverse of the absolute velocity or the inter-arrival times, may be used to construct a suitable weighting factor \( \omega \).

Van Strien [1988] noted that the unrealistic behaviour of \( \hat{\rho}(k\Delta T) \) could be avoided by scaling \( \hat{R}(k\Delta T) \) with a 'local variance' similar to Eq (2.42). Even though \( \hat{\rho}(k\Delta T) \) does not exhibit the unrealistic behaviour, it is incorrect to believe that \( \hat{\rho}(k\Delta T) \) is insensitive to the effects of velocity bias. On the contrary, as shown by Van Maanen and Tummers [1996] the values of \( \hat{\rho}(k\Delta T) \) are affected by the velocity bias at all lag times, except at \( T = 0 \). Fortunately, the locally normalized autocorrelation \( \hat{\rho}(k\Delta T) \) can be modified to yield unbiased values as follows

\[
\hat{\rho}(k\Delta T) = \frac{\sum \{u_i u_j \omega_i \omega_j\}(k\Delta T)}{\sqrt{\sum \{u_i^2 \omega_i \omega_j\}(k\Delta T) \sum \{u_j^2 \omega_i \omega_j\}(k\Delta T)}} \tag{3.11}
\]

This reduces to Eq (2.42) for \( \omega = 1 \). It will be shown in section 5.2 that this algorithm infers the correct autocorrelation from a 'biased' record. It is important to realize that the velocity bias itself does not cause errors in statistical quantities. The errors arise when the LDA user applies algorithms that ignore the existence of the velocity bias. However, the velocity bias further complicates the already inefficient processing algorithms, and often additional information on the flow is needed to take the bias into account. If the available information is incomplete, it is inevitable that errors will result. On the other hand the velocity bias causes some interesting phenomena, and one of these will be briefly discussed below.

In his investigation into the effects of velocity bias on the autocorrelation function, Bell [1986] derived an expression for the expected number of cross-products per slot \( H_0 \) in the presence of velocity bias

\[
H_0(k\Delta T) = H(k\Delta T)\{1 + \rho(k\Delta T) \frac{\bar{w}^2}{\bar{u}^2}\} \quad \text{for} \quad \bar{w}^2 \ll \bar{u}^2 \tag{3.12}
\]

where \( H(k\Delta T) \) is the number of cross-products in case of Poisson sampling given by Eq (2.28). It is seen from Eq (3.12) that the velocity bias may cause large discrepancies between the predictions based on the Poisson sampling and that based on the more realistic sampling process that includes the effects of the velocity bias. Eq (3.12) indicates that the deviation from the Poisson statistics depends on the autocorrelation itself. It should be mentioned, however, that Eq (3.12) is valid only for a one-dimensional flow without flow reversal. In a general flow the deviation is a measure for the correlation function of the instantaneous volume flux through the measuring volume. This reduces to the correlation function of the velocity magnitude for a spherical measuring volume. In that case the correlation of the velocity magnitude may be determined from the particle arrival statistics alone, without measuring the particle velocities themselves.
3.3 The Influence of Spatial Averaging

Buchhave et al. [1979] derived the following interesting expression for the autocovariance function measured with an individual realization LDA.

\[ R_m(t - t') = \mu V (\bar{u}^2 + \bar{u}^2) q(\tau) + \mu^2 V^2 \int \int u(\bar{x}, t) u(\bar{x}', t') d^3\bar{x} d^3\bar{x}' \]  

(3.13)

where \( \bar{x} \) and \( \bar{x}' \) are locations in the measuring volume of particles that are measured at times \( t \) and \( t' \), respectively. The subscript \( m \) denotes a volume-averaged value, and the factor \( \mu V \) represents the fraction of time that a particle is present in the measuring volume. The first term on the right-hand side contributes only for lag times smaller than the mean transit time of the particles, i.e. \( q \) is a spike with height 1 and its width equals the mean transit time. The term arises from multiple measurements on the same particle while it transits the measuring volume. This term has little practical relevance, because most LDA signal processors prohibit multiple readings from the same particle. However, all signal processors calculate some average Doppler frequency during the passage of a particle through the measuring volume. Therefore, the shape of the true acf will be obscured for lag times smaller than the mean transit time of the particles. If the mean transit time is larger than the Kolmogorov scale, then it is inevitable that some information on the turbulence is lost.

The second term represents the volume-averaged space time covariance function. It is important to realize that, although the appearance of the double integral is the same for both the individual-realization LDA and for the continuous LDA or the hot wire, the spatial averaging for the two modes of operation are radically different. In case of a continuous LDA all parts of the measuring volume contribute to a certain extent to the instantaneous output of the signal processor. So, the instantaneous velocity measured by the continuous LDA is the result of an averaging process over the complete measuring volume. In contrast, the individual-realization LDA measures the velocity of a single particle that passes through the measuring volume in an otherwise empty volume. Therefore, the spatial resolution of an individual-realization LDA is limited only by the distance that the particle travels through the measuring volume during the time span that the processor needs to analyze the Doppler burst. The instantaneous velocity measured by an individual-realization LDA thus resembles a 'point measurement'. However, it should not be concluded from this that the individual-realization LDA has a near perfect spatial resolution, because during the process of inferring statistical quantities from the velocity samples, significant spatial averaging may take place. Whether or not the spatial averaging is relevant depends on the dimensions of the probe, the local properties of the turbulence field and on the statistical quantity that is under consideration.

There is considerable confusion among the LDA users on the nature of the volume averaging in case of an individual realization LDA. This has resulted in different approaches for evaluating the volume integral in Eq (3.13). Below, a procedure is put forward that takes the 'point measurements' into account. The analysis is carried out for the simple case of a measuring volume of zero width and length \( l \). The analysis for a more realistic shape, such as a cylinder or an ellipsoid, is straightforward but it involves a much larger computational effort while it yields hardly any additional insight into the effects of the spatial averaging.

Let \( l \) be the length of the a measuring volume that is oriented along the z-axis. It will also be assumed that the turbulence is homogeneous in \( z \)-direction. The product of two velocity samples \( u(z_1, t) u(z_2, t+\tau) \) is an estimate of the autocovariance function \( R(\tau) \) only if \( z_1 = z_2 \). This condition is not fulfilled in practice, because of the finite extent of the probe. As a consequence, the cross product \( u(z_1, t) u(z_2, t+\tau) \) should be considered an estimate of the space-time covariance \( R(\tau, z_2 - z_1) \). Recall that the slotting technique (see Eq (2.27)) involves the summing of such cross products, so that some degree of spatial averaging is incurred when the autocovariance
function is calculated from the samples. The autocovariance \( \hat{R}(\Delta \tau) \) is a measure for

\[
R_m(\tau) = \int_{-l/2}^{l/2} \int_{-l/2}^{l/2} u(z_1, t)u(z_2, t + \tau)p'(z_1)p'(z_2) dz_1 \, dz_2
\]

where \( p'(z) \) is the probability density that a particle will pass through the measuring volume between \( z \) and \( z + dz \). In practice \( p' \) is likely to resemble the Gaussian distribution of the light intensity in the probe, but for simplicity it is assumed here that \( p' \) is uniformly distributed, i.e.

\[
p'(z) = \begin{cases} 
1/l & \text{if } -l/2 \leq z \leq l/2 \\
0 & \text{otherwise}
\end{cases}
\]

Because of the homogeneity of the flow in \( z \)-direction, only the difference \( \Delta z = z_2 - z_1 \) is of relevance, so that Eq (3.14) reduces to

\[
R_m(\tau) = \int_{-l}^{l} u(z, t)u(z + dz, t + \tau)p(\Delta z) d\Delta z
\]

The probability density of \( \Delta z \) is given by (e.g. Papoulis [1991])

\[
p(\Delta z) = \begin{cases} 
1/l - |\Delta z/l| & \text{if } -l \leq \Delta z \leq l \\
0 & \text{otherwise}
\end{cases}
\]

Expanding the velocity fluctuation \( u(z + \Delta z, t + \tau) \) in a Taylor series around \( t, z \) and integrating yields

\[
R_m(\tau) = \overline{u^2} - \frac{1}{12} \left( \frac{\partial u}{\partial z} \right)^2 l^2 + O(l^4) \\
+ \frac{1}{12} \frac{\partial^3 u}{\partial t \partial z^2} \tau^2 + O(l^4) \\
- \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 \tau^2 + \frac{1}{24} \frac{\partial^4 u}{\partial t^2 \partial z^2} \tau^2 l^2 + O(l^4)
\]

The Taylor expansion of the true autocorrelation \( R(\tau) \) reads (Hinze [1975])

\[
R(\tau) = \overline{u^2} + \sum_{n=1}^{\infty} \left( \frac{-1}{2n!} \right)^n \frac{\partial^n u}{\partial t^n} \tau^{2n} = \overline{u^2} - \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 \tau^2 + O(\tau^4)
\]

A comparison between Eq (3.18) and Eq (3.19) shows that the finite size of the probe volume has resulted in unwanted constant, linear and higher-order terms that may obscure the behaviour of the true autocovariance. In particular, it is seen that the finite size of the probe causes a too low covariance at small time lags. To a first approximation the discrepancy is given by

\[
R(0) - \lim_{\tau \to 0} R_m(\tau) = \frac{1}{12} \left( \frac{\partial u}{\partial z} \right)^2 l^2 = \frac{l^2}{6L_z^2}
\]

where \( L_z \) is the Taylor length scale for the \( z \)-direction. It should not be concluded from the preceding that the influence of the spatial averaging is limited to small lag times. In fact, the expansion in Eq (3.18) readily shows that the volume-averaged covariance function \( R_m(\tau) \) will differ from \( R(\tau) \) at all lag times.

Using the same procedure as for the covariance function, an expression can be derived for the influence of volume averaging on the variance of the velocity fluctuations \( \overline{u^2} \). The outcome of the calculation is simply \( \overline{u^2}_m = \overline{u^2} \), indicating that the size of the measuring volume has no
effect on the variance in homogeneous turbulence. This is in agreement with the findings of an experimental investigation carried out by Luchik and Tiederman [1985], and it illustrates the fundamental difference between the effects of spatial averaging for an individual realization LDA and instruments such as the continuous LDA or the hot wire.

Once it is known that the variance $u^2$ is unaffected by the spatial averaging, it follows from Eq (3.18) that the imperfect resolution of the measuring volume causes a discontinuity in the autocorrelation function $\rho(\tau)$. Combining the influences of the spatial averaging and the uncorrelated noise, results in the following expression for the magnitude of the discontinuity in the autocorrelation function at zero time lag

$$\Delta \rho = \frac{n^2 + u^2 C_1}{n^2 + u^2} \quad \text{with} \quad C_1 = \frac{l^2}{6L_z^2}$$  

(3.21)

Tummers et al. [1995] give the magnitude of the discontinuity in case the measuring volume is modeled as a cylinder with diameter $d$ and length $l$.

$$\Delta \rho = \frac{n^2 + u^2 C_3}{n^2 + u^2} \quad \text{with} \quad C_3 = \frac{d^2}{8L_x^2} + \frac{d^2}{8L_y^2} + \frac{l^2}{6L_z^2}$$  

(3.22)

where $L_x$, $L_y$, and $L_z$ are the Taylor length scales for the $x$, $y$, and $z$-direction, respectively. Eqs (3.21) and (3.22) illustrate the dilemma that one faces when measuring autocorrelation functions using LDA. To avoid the occurrence of a large spike at zero lag time, the dimensions of the measuring volume should be small compared to the corresponding Taylor length scale. This often necessitates that the light-collection optics are placed in a plane perpendicular to the axis of the transmitting optics, i.e. a side-scatter configuration should be used to reduce the effective length of the measuring volume. However, in that case the intensity of the received light is small compared to intensity that can be received in forward or backward scatter. This will reduce the signal-to-noise ratio (SNR) of the Doppler signals, thereby increasing the noise component $n(t)$ of the individual samples, as suggested in section 3.1. It is seen from Eqs (3.21) and (3.22) that this will increase the magnitude of the discontinuity. Therefore, it can be concluded that the measurement of an autocorrelation function is a trade-off between the effects of the uncorrelated noise and the spatial averaging.
Chapter 4

Generation of Simulated Data

4.1 Introduction

To assess the different spectral estimators, simulated data were generated on a computer. The simulated data have known spectral density, thereby enabling a direct verification of the ability of each estimator to infer the spectral density function from the data. Two methods are given that can be used to generate synthetic data. The first method can be found in Priestley [1981] and it is used to generate second-order autoregressive (AR(2)) data. This method is fast, but only little variation of the shape of the spectral density function is possible, because its characteristics are fixed by the AR(2) model. With this method a pseudo-random signal is created at closely spaced, equidistant time instants. Then a set of Poisson distributed sampling instants \( t_i \) is generated. The pseudo-random velocity at time \( t_i \) then follows from linear interpolation. The second method is given by Shinozuka [1974] and it can be used to obtain a pseudo-random velocity signal with arbitrary spectral density function that is sampled at Poisson distributed sampling instants. This method is very time consuming, because it requires a large amount of computation.

4.2 The AR(2) Model

The discrete parameter process \( U_t \) is called second-order autoregressive, AR(2), if it satisfies the following difference equation

\[
U_t + a_1 U_{t-1} + a_2 U_{t-2} = \epsilon_t
\]  

(4.1)

where \( \epsilon_t \) is a Gaussian random process with unit variance and the coefficients \( a_1 \) and \( a_2 \) are yet to be determined. \( U_t \) is the discrete parameter form of the continuous parameter process \( U(t) \) that satisfies a second order differential equation

\[
\frac{d^2U(t)}{dt^2} + \alpha_1 \frac{dU(t)}{dt} + \alpha_2 U(t) = \epsilon(t)
\]

(4.2)

The coefficients \( \alpha_1 \) and \( \alpha_2 \) are related to \( a_1, a_2 \) and the equi-spaced time step \( \Delta t \) as

\[
\alpha_1 = \frac{-(a_1 + 2a_2)}{a_2 \Delta t}, \quad \alpha_2 = \frac{(1 + a_1 + a_2)}{a_2 \Delta t^2}
\]  

(4.3)

The sampling period \( \Delta t \) of the process \( U_t \) is chosen such that there is negligible power at frequencies higher then \( f_s = 1/\Delta t \). The autocorrelation function \( \rho(\tau) \) of the continuous parameter process \( U(t) \) is given by

\[
\rho(\tau) = \frac{c_2}{c_2 - c_1} e^{c_1 \tau} - \frac{c_1}{c_2 - c_1} e^{c_2 \tau} \quad \text{with} \quad \tau > 0
\]  

(4.4)
where $c_1$ and $c_2$ are the roots of the polynomial $g(z) = z^2 + \alpha_1 z + \alpha_2$. The spectral density function of $U(t)$ is simply the Fourier transform of $R(\tau)$

$$S(\omega) = \frac{R(0)}{\pi} \frac{\alpha_1 \alpha_2}{\alpha_1^2 \omega^2 + (\alpha_2 - \omega^2)^2}$$  \hspace{1cm} (4.5)

Beyond its first corner frequency, this spectral density function behaves as $\omega^{-2}$ until the second corner frequency is reached, after which the spectral density falls-off as $\omega^{-4}$. The coefficients $c_1$ and $c_2$ can be conveniently used two prescribe two time scales that characterize $U(t)$. These time scales are the Taylor time scale $\lambda_u$ and the integral time scale $T_u$, which are defined by Eq (2.4) and Eq (2.3), respectively. It is easy to show that the relationship between $\lambda_u$, $T_u$ and the coefficients $c_1$ and $c_2$ is given by

$$T_u = \frac{-(c_1 + c_2)}{c_1 c_2}, \hspace{1cm} \lambda_u^2 = \frac{2}{c_1 c_2}$$  \hspace{1cm} (4.6)

The procedure to generate pseudo-random velocity data is as follows. First, the Taylor time scale $\lambda_u$ and the integral time scale $T_u$ are specified. This yields the coefficients $c_1$ and $c_2$, which in turn yield the values of $\alpha_1$ and $\alpha_2$ from the polynomial $g$. For a chosen time step $\Delta t$ the values of $c_1$ and $c_2$ follow from Eq (4.3). A closely spaced primary time series is subsequently generated from the difference equation Eq (4.1) after specification of the initial conditions, e.g. $U_1 = U_2 = 0$. Then a set of $N$ Poisson distributed sampling times, $t_i$, is created using Eq (2.13) and the pseudo-random velocities $U(t_i)$ follow from linear interpolation in the primary time series.

### 4.3 Pao’s Spectral Density Function

Shinozuka [1974] describes a method that generates pseudo-random samples from

$$U(t) = \sum_{k=1}^{k_{max}} \sqrt{4S(k\Delta \omega)\Delta \omega \cos(k\Delta \omega t + \theta_k)}$$  \hspace{1cm} (4.7)

where $\Delta \omega$ is the frequency spacing and $S(\omega)$ is a prescribed spectral density function. The values of $\theta_k$ are uniformly distributed on the interval $[0, 2\pi]$. The summation in Eq (4.7) can be evaluated for any value of $t$. Therefore, pseudo-random velocity data can be created at Poisson-distributed sampling times. The advantage of this method is that it can be used for arbitrary spectral densities, but, as mentioned in the introduction, it is much slower than the previous method.

The $AR(2)$ spectral density function as given by Eq (4.5) has, when plotted on double-log axes, fixed slopes of $-2$ and $-4$. It may be argued that this does not represent a realistic behaviour of the spectral densities that are found in fluid mechanics. To meet some of this criticism, the different spectral estimators will be applied to simulated data having a more realistic spectral density at high frequencies. For this the following density is chosen

$$S(\omega) = \frac{T_u R(0) e^{-\frac{3}{4} a \left( \frac{\omega}{\omega_d} \right)^\frac{3}{2}}}{\pi + \frac{1}{a} \left( \frac{\omega}{\omega_d} \right)^\frac{3}{2}}$$  \hspace{1cm} (4.8)

The constant $\pi$ has been introduced in the denominator so that $S(0)/R(0) = T_u/\pi$ in accordance with Eq (2.5). For large frequencies $S(\omega)$ behaves as $\omega^{-5/3}$ and then decreases exponentially as suggested by Pao [1965].
4.4 Biased Time Series

To introduce velocity bias the distance between the particles, rather than the time between successive particle arrivals is assumed to be Poisson distributed. Therefore, the distances $\Delta x$ can be generated using Eq (2.13) with $\Delta t$ replaced by $\Delta x$. With the so called conveyer-belt method [Tropea 1987] the arrival times follow from

$$\Delta x_i = \int_{t_{i-1}}^{t_i} |\tilde{V}(t)| dt$$

(4.9)

where $\tilde{V}(t)$ is the instantaneous velocity vector. Here, only a one-dimensional flow is considered so that $\tilde{V}(t)$ can be replaced by the component $U(t)$. To assure the correct mean data rate $\nu (= N/T)$, the Poisson process that describes the distance between the particles must have intensity parameter $N/V_t$, where $N$ is the number of samples and $V_t$ is the time average of the absolute velocity, which reduces to the time average of $U(t)$ if there are no instantaneous flow reversals. Once the arrival times $t_i$ are determined, 'biased' pseudo-random velocities $U(t_i)$ can be generated using, for example, the method described in section 4.2.
Chapter 5

Results

5.1 Synthetic Data

Three records were created with the methods described in Chapter 4. Each record consists of $2 \times 10^5$ samples and has a duration of 200 s. Records Ia and Ib were generated with the AR(2) model, Eq (4.1). The time scales in this model were prescribed as $\lambda = 3$ ms and $T_\nu = 45$ ms. A closely-spaced primary time series was subsequently generated containing $2 \times 10^7$ data points. The samples in record Ia resulted from interpolation in the primary time series at Poisson distributed sampling times. The sampling times in record Ib were generated using Tropea’s conveyer-belt method, Eq (4.9), and as a result, record Ib is a biased time series. Both records have variance $\bar{u}^2 = 1$ m$^2$/s$^2$ and zero mean.

Record II also has a duration of 200 s. It was generated with Shinozuka's method, Eq (4.7), using Pao's spectral density function, Eq (4.8). The parameters in Eq (4.7) were set at $k_{max} = 2.5 \times 10^4$ and $\Delta \omega = 1$ rad/s. The constants in Pao's spectral density function were prescribed as $a = 0.01$ and $\omega_d = 125.0$ rad/s, while the integral time was again chosen as $T_\nu = 45$ ms. This resulted in a Taylor time scale of approximately 5.1 ms. The mean velocity and variance are identical to those of records Ia and Ib.

5.2 The Autocorrelation Function

Figure 6.1 is a detailed view of the autocorrelation in the region near zero lag time for the standard slotting technique, Eq (2.39), and the locally scaled variant, Eq (2.42). The maximum lag time is 5 ms and the slot width equals 25$\mu$s. The error bar at $\tau = 1.5$ ms has length 4$\sigma$, where $\sigma^2$ is the variance of the $\hat{\rho}(k\Delta\tau)$ estimates as calculated from Scott's formula, Eq (2.32). In case the $\hat{\rho}(k\Delta\tau)$ values have a Gaussian distribution, a 4$\sigma$-band contains approximately 97% of the estimates.

Even though there is no theoretical expression for the variability of the $\hat{\rho}(k\Delta\tau)$ values, it is clear from Figure 6.1 that the local scaling results in a dramatic decrease of the variability. The local scaling enables an accurate description of the autocorrelation function, particularly at the smallest lag times. The beneficial effect of the local scaling is most pronounced in the region where the value of the correlation coefficient is near 1. The scatter increases with decreasing correlation, and in the limiting case of zero correlation, e.g. at very large lag times, the variability of $\hat{\rho}(k\Delta\tau)$ becomes equal to that of $\hat{\rho}(k\Delta\tau)$.

Figure 6.2 shows the effect of the uncorrelated noise on the correlation function $\hat{\rho}(k\Delta\tau)$. Before processing, the Gaussian uncorrelated noise was superposed on the $2 \times 10^5$ samples of record Ia. The mean square value of the noise, $\bar{n}^2$, was varied between 0 and $\bar{u}^2/10$. Figure 6.2
Results

clearly shows that the discontinuity caused by the uncorrelated noise increases with increasing $\bar{n^2}$. The magnitude of the discontinuity $\Delta \rho$ is in excellent agreement with Eq (3.4). Also visible in Figure 6.2 is the increase of the variance of the $\hat{\rho}(k\Delta \tau)$ values with increasing $\bar{n^2}$, in particular in the region near zero lag time. As a result, the beneficial effect of the local normalisation of $\hat{R}(k\Delta \tau)$ becomes less pronounced when $\bar{n^2}$ increases. This is expected to have an adverse effect on the spectral estimates, especially in the high-frequency range.

The effects of the velocity bias on the standard autocorrelation are shown in Figure 6.3. The bias error is incurred when the data of record $I_b$ are processed as if there were no correlation between the velocity and the sampling process. It is seen that $\hat{\rho}(k\Delta \tau)$ becomes larger than 1 at the smallest time lags, but there is also a significant error at larger lag times. Further evidence of the influence of the velocity bias is given in Figure 6.4, which shows the number of cross products per slot $H_b(k\Delta \tau)$. It is seen that there is a substantial deviation from $H(k\Delta \tau)$, the number predicted for the Poisson process, see Eq (2.28). This is in agreement with the findings of Bell [1986].

The effect of the velocity bias on the locally scaled autocorrelation is given in Figure 6.5. Although the physically unrealistic $\rho > 1$ behaviour is absent, it is clear that the local scaling should not be considered a bias correction method, because there are large errors at all lag times except at zero lag. The detailed view of the autocorrelation in Figure 6.6 shows that $\hat{\rho}_b(k\Delta \tau)$ is too flat for small lags. It is obvious that $\hat{\rho}_b(k\Delta \tau)$ results in a too high Taylor time scale. It is important to note that this is not a general conclusion, because the qualitative behaviour of the bias error depends on the local turbulence intensity. This is described in detail in Van Maanen and Tummers [1996]. If $\hat{\rho}_b(k\Delta \tau)$ is computed from Eq (3.11), then the original autocorrelation function can be inferred accurately for both large and small lag times as shown in Figures 6.5 and 6.6, respectively. Again, it can be concluded that the velocity bias does not cause an error if it is taken into account during the processing of the data. However, the bias correction will in general cause a small increase of the variability of the estimates as can be seen from a comparison between Figures 6.1 and 6.6.

An interesting by-product of the local scaling is the quantity $J(k\Delta \tau)$ defined by Eq (2.41). In absence of velocity bias $J(k\Delta \tau)$ has no trend, i.e. the values for the different slots scatter around the variance $R(0)$ given by Eq (2.40). However, in case of velocity bias $J(k\Delta \tau)$ depends on the autocorrelation itself, as shown in Figure 6.7. It is obvious that this phenomenon is closely related to that described by Bell [1986]. It may be argued that if the algorithm given by Eq (3.11) is able to estimate the true autocorrelation from the the biased time series $I_b$, then it should also be possible to compute the unbiased values of $J(k\Delta \tau)$ from

$$J(k\Delta \tau) = \frac{\sqrt{\sum \{u_i^2 \omega_i \omega_j\}(k\Delta \tau) \sum \{u_j^2 \omega_i \omega_j\}(k\Delta \tau)}}{\sum \{\omega_i \omega_j\}(k\Delta \tau)} \quad \text{with } k = 0, 1, \ldots, M - 1 \quad (5.1)$$

which is the analog of Eq (2.41). Figure 6.7 shows that it is indeed possible to restore the correct statistics of $J(k\Delta \tau)$ if the weighting according to Eq (5.1) is used.

### 5.3 The Spectral Density Function

The results for the sample-and-hold estimator are shown in Figure 6.8 for record $I_a$. The reconstructed, continuous signal was sampled at a rate of 10 kHz, resulting in a Nyquist frequency of $\pi \times 10^4$ rad/s. The spectral estimator $S_x$ (see Eq (2.36)) rolls-off as $\omega^{-2}$ at high frequencies. In the same frequency range the sample-and-hold spectral density $S_{SH} = S_x H^{-1}$ exhibits a white-noise level. The spectral density for which $S_{SH}$ exceeds $\delta^2 S^2$ is $S_5 = 5.90 \times 10^{-3}$ s$^{-1}$ at
\( \omega = 500 \text{ rad/s} \) for \( \delta = 1/3 \). The maximum frequency for which the spectral density is accurately predicted is therefore well below the value that is frequently cited as the upper limit \( \omega_{\text{max}} = \nu \).

Figure 6.9 compares the performance of the spectral estimators \( S_{1B} \) and \( S_2 \). A comparison is useful only if the spectral resolution of both estimators is identical. \( S_2 \) applied the Tuckey-Hanning window with \( \tau_m = 1 \text{ s} \). The lag-time axis was divided into \( 10^5 \) intervals of \( 10 \mu s \). The estimator \( S_{1B} \) applied the box-car window, so that nearly identical resolution was achieved with 125 blocks, since \( T_B \approx 1.6\tau_m \). The results shown in Figure 6.9 demonstrate that the variability for both methods is nearly the same as indicated by Eq (2.25). The spectral density for which the variance of \( S_{1B} \) exceeds \( \delta^2 S^2 \) as determined from Eq (2.22) is \( S_\delta = 5.84 \times 10^{-5} \text{ s}^{-1} \) at \( \omega = 330 \text{ rad/s} \) for \( \delta = 1/3 \). This corresponds quite well with the value determined from the experiment, indicating that Eq (2.22) can be used in practice to predict the lowest spectral density that can be accurately estimated with \( S_{1B} \). Both estimators yield reliable results up to a frequency of approximately 450 rad/s. This corresponds to just 1/4 of the average data rate. For frequencies higher than 450 rad/s the spectral density becomes increasingly obscured by the variability of the estimates.

The spectral estimator \( S_2 \) is extremely time consuming compared to \( S_{1B} \). As indicated in Section 3.2 the computation can be speeded up by first calculating the slotted autocorrelation function followed by a Fourier cosine transform. This results in the much faster estimator \( S_{2S} \). The table below summarizes the computing times for 66 logarithmically spaced spectral estimates. Of course, the spectral resolution of the estimates is identical. The correlation methods \( S_2 \) and \( S_{2S} \) employed a maximum lag time of 1 s and the lag-time axis was divided into slots of 10\( \mu \)s.

<table>
<thead>
<tr>
<th></th>
<th>direct method</th>
<th>correlation method</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_{1B} )</td>
<td>1</td>
<td>447</td>
</tr>
<tr>
<td>( S_2 )</td>
<td>15</td>
<td>23</td>
</tr>
<tr>
<td>( S_{2S} )</td>
<td>15</td>
<td>23</td>
</tr>
</tbody>
</table>

The time unit in the above table is arbitrary, since the duration depends on the hardware used. The ratio of the different computing times is more interesting. As expected, the short-block method \( S_{1B} \) requires the smallest computational effort and \( S_2 \) is the slowest estimator. The use of \( S_{2S} \) instead of \( S_2 \) results in a considerable increase in computing speed, although \( S_{2S} \) is still slower than \( S_{1B} \). The results in Figure 6.10 demonstrate that the spectral densities calculated from \( S_2 \) and \( S_{2S} \) are practically the same. Therefore, it can be concluded that the increase of the computing speed does not result in a loss of accuracy. This effectively rules out \( S_2 \) as a practical method.

The individual influences of the local scaling and the variation of the window width on the spectral density are demonstrated in Figure 6.11. The results for the estimator \( S_{2S} \) were already shown in Figure 6.10, but they are included here as a reference. First, the standard autocorrelation \( \tilde{\rho}(k\Delta \tau) \) in Eq (2.39) was replaced by the locally scaled autocorrelation \( \tilde{\rho}(k\Delta \tau) \). Theoretically, this should lead to some improvement, but this is not noticeable from the results. Apparently, the reduction of the scatter of the autocorrelation values near \( \tau = 0 \) had negligible influence on the calculated spectral density, because the integration is performed up to \( \tau = \tau_m \). As indicated by Gaster and Roberts [1975], the use of the variable window alone alleviates some of the variability problems. The value of \( S_{\delta=1/3} \) was determined as \( 2.0 \times 10^{-6} \text{ s}^{-1} \) at \( \omega = 1600 \text{ rad/s} \).

Figure 6.12 shows the results of the \( S_{2S} \) estimator together with the results of the new spectral estimator \( S_3 \) when applied to record \( I_a \). The latter estimator combines the locally
scaled autocorrelation $\tilde{\rho}(k\Delta \tau)$ and the variable window. The width of the window is varied according to Eq (2.45) with $\kappa = 6$. Both methods used the Tuckey-Hanning window and the maximum lag time was $\tau_m = 1$ s. $S_{2S}$ gives reliable estimates up to $\omega = 820$ rad/s. The variability of $S_3$ is much lower; the value of $S_{k=1/3}$ was determined as $S_{k=1/3} = 1.8 \times 10^{-7}$ s$^{-1}$ at $\omega = 4660$ rad/s. Now, advantage is taken of the smaller variance of the correlation values near $\tau = 0$, because the upper limit in the integration decreases with frequency. The use of $S_3$ results in a gain of about 2 decades in the spectral density. The same result could have been achieved with $S_{2S}$ if $2 \times 10^9$ instead of $2 \times 10^5$ samples were processed.

The preceding results, in particular Figure 6.12, show that the performance of the spectral estimator $S_3$ is superior to that of $S_{1B}$ or $S_2$. It may be argued, however, that this conclusion is based on a comparison for one particular spectral density function, namely the AR(2) spectrum, given by Eq (4.5). It will now be demonstrated that $S_3$ can be applied successfully to other spectral densities as well.

Shown in Figure 6.13 are the results for $S_{2S}$ and $S_3$ when applied to record II. Both estimators used the Tuckey-Hanning window and $S_3$ employed $\kappa = 6$. The maximum lag time was set at 1 s and the width of the time slots was 10$\mu$s. The variance of the $S_{2S}$ estimates exceeds $(1/3)^2 S(\omega)^2$ at $\omega = 650$ rad/s. The equivalent frequency for $S_3$ is determined as $\omega = 3110$ rad/s. Through the use of $S_3$ instead of $S_{2S}$ about 1.5 decades in spectral density are gained.

In the discussion following the results shown in Figure 6.2, it was speculated that the uncorrelated noise would have adverse effects on the spectral density function. This is confirmed by the results given in Figure 6.14, which show the spectral estimates obtained from $S_3$ with $\kappa = 6$. The bullets can be seen as a reference, since they correspond to $n^2 = 0$. As expected, the results indicate that an increase of $n^2$ increases the variability of the spectral estimates. Fortunately, the increase of the variability becomes significant only in the high frequency range, $\omega > 2000$ rad/s, so that, even if a substantial amount of uncorrelated noise is added to the data, $S_3$ significantly improves the estimates as compared to $S_{1B}$ or $S_{2S}$.

It is possible to exert some control over the width of the spectral window by means of the parameter $\kappa$. A decrease of $\kappa$ leads to an increase of the spectral width, thereby resulting in a smoother spectrum. Whether or not this is accompanied by a prohibitively large increase of the bias error can be checked by comparing the results for different values of $\kappa$. Figure 6.15 shows that the shape of the spectrum does not vary with $\kappa$ in the region where the estimates are stable. This illustrates that the AR(2) spectrum is sufficiently smooth at high frequencies for the increase of the spectral width to be legitimate.
Chapter 6

Conclusions

An overview is given of existing methods for the estimation of the spectral density function from randomly sampled data. It is shown that these estimates exhibit large variability due to the random sampling, and, as a result, the existing estimators are unable to reveal the the spectral density in the high frequency range. A new spectral estimator is proposed that overcomes some of the variability problems. This estimator combines a window of variable width and a locally normalized autocorrelation \( \hat{\rho} \).

At small lag times the locally normalisation results in a strongly reduced variability of the autocorrelation estimates compared to \( \hat{\rho} \), the estimates resulting from the standard slotting technique. The characteristics of the locally-scaled autocorrelation \( \hat{\rho} \) were investigated regarding the effects of uncorrelated noise, velocity bias and spatial averaging.

- uncorrelated noise; a low signal-to-noise ratio of the Doppler signals causes an increase of the error in the individual velocities measured by the LDA. These noise errors are presumed to be uncorrelated to the velocity itself and result in a discontinuity of \( \hat{\rho} \) at zero lag time.

- velocity bias; the correlation between the instantaneous velocity and the sampling probability is inherent to the use of LDA and should therefore be taken into account. A scheme is given to restores the correct autocorrelation from a biased time series while using the local normalisation.

- spatial resolution; the effects of the spatial averaging for an individual realization LDA are fundamentally different from that of an instrument such as the hot wire. A method is proposed that can be used to investigate the effects spatial averaging for an individual realization LDA. The model indicates that the finite extent of the probe distorts the shape of the autocorrelation. In particular, the measured autocorrelation shows a discontinuity at zero lag time whose magnitude depends on the size of the probe relative to the Taylor length scale.

The new spectral estimator was applied to two types of simulated data, i.e. the AR(2) process and a Pao-like spectral density. The major findings of the simulation with regard to the spectral density estimators are listed below.

- The sample-and-hold estimator \( S_{SH} \) produces valid results only for frequencies significantly smaller than \( \nu \) rad/s. For higher frequencies the sample-and-hold estimator asymptotes to a constant level.

- The estimator \( S_2 \) and the slotted version \( S_{2S} \) give practically identical results at all frequencies, but \( S_{2S} \) is much faster then \( S_2 \). The variance of the short-block spectral estimates \( S_{1B} \) is comparable to that of \( S_{2S} \).
Conclusions

- The new spectral estimator $S_3$ proved to be superior to the existing estimators $S_{SH}$, $S_{2S}$ and $S_{1B}$. The present investigation showed that approximately two decades in spectral density were gained through the use of $S_3$. 
Bibliography


Figure 6.1: The influence of local scaling on the autocorrelation function (record 1a).

Figure 6.2: The influence of uncorrelated noise on the autocorrelation function (record 1a).

Figure 6.3: The influence of velocity bias on the standard autocorrelation function (record 1b).

Figure 6.4: The influence of velocity bias on the number of cross products (record 1b).
Figure 6.5: The influence of velocity bias on the locally scaled autocorrelation function (record Ia).

Figure 6.6: Detailed view of the locally scaled autocorrelation function near zero lag time (record Ia).

Figure 6.7: The local variance $J(k\Delta \tau)$.

Figure 6.8: The sample-and-hold spectrum $S_{SH}$ for record Ia.
Figure 6.9: Spectral estimators $S_{1B}$ (short-block method) and $S_2$ (correlation method) applied to record $Ia$.

Figure 6.10: Spectral estimators $S_2$ (correlation method) and $S_{2S}$ (slotted correlation method) applied to record $Ia$.

Figure 6.11: The individual influences of local scaling and the variable window on the spectral density (record $Ia$).

Figure 6.12: Comparison between the performance of the spectral estimators $S_{2S}$ (slotted correlation method) and the new estimator $S_3$ (record $Ia$).
Figure 6.13: The spectral estimators $S_{25}$ (slotted correlation method) and $S_3$ applied to record II.

Figure 6.14: The effect of uncorrelated noise on $S_3$ (record Ia).

Figure 6.15: The effect of $\kappa$ variation on $S_3$ for record II.