HETEROGENEOUS DOMAIN DECOMPOSITIONS FOR THE EFFICIENT SIMULATION OF WAVE PROPAGATION IN COMPLEX DOMAINS

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Abstract. This paper advances the idea of a heterogeneous domain decomposition for Computational Aeroacoustics (CAA). Direct simulations of aeroacoustic problems are accelerated by sub-dividing the computational domain into smaller domains. In each of these sub-domains the equations, the discretization, the mesh and the time step may be different and are adapted to the local behavior of the solution. High order methods such as ADER-Finite Volume and ADER-Discontinuous Galerkin methods are used to reduce the total number of elements and to ensure good wave propagation properties. Here, we integrate a high order Finite Difference method into the coupling framework. In the examples section, convergence rates for the coupling procedure in 3D show, that high order of accuracy is maintained globally also for partitioned domains. A numerical example that involves multiple domains underlines the flexibility of the approach. Another example shows that the proposed domain decomposition also holds for the coupling of the Navier-Stokes equations with the Linearized Euler Equations.

1 INTRODUCTION

In the field of Computational Aeroacoustics (CAA), the requirements regarding the accuracy of the methods are very high and in general, quite large domains have to be computed. Therefore, an accurate and feasible direct simulation that considers both the generation of sound and its propagation into the far-field is hard to realize with one numerical method in a single computational domain. However, a direct approach contains automatically the interaction of the acoustic perturbations with the flow-field, a property which lacks the popular acoustic analogy models.

The method that is proposed here is basically a direct simulation, but it simplifies the problem that has to be solved for individual regions in the computational domain.
The idea is to use a non-overlapping domain decomposition method where the equations, methods, grids and time steps are adapted to meet the local requirements. Following the tradition of earlier domain decomposition approaches e.g. for jet noise by Freund et al.\textsuperscript{1} or the Chimera methods introduced by Steger et al.,\textsuperscript{2} a more universal method has been developed and proposed by Utzmann et al.\textsuperscript{3} High-order methods such as the ADER-Finite Volume (ADER-FV) and ADER-Discontinuous Galerkin (ADER-DG) schemes are used on both structured and unstructured grids, ensuring excellent wave propagation capabilities. In the domains, the Euler Equations and the Linearized Euler Equations (LEE) are solved.

In this paper, we add high order Finite Difference (FD) methods to the coupling framework. Furthermore, a numerical example illustrates that the domain decomposition method also holds for the coupling of the Navier-Stokes equations with the Linearized Euler Equations. In the examples section, convergence studies show the capability of the domain decomposition to maintain globally high order of accuracy for several 3-dimensional test cases. Finally, an example that involves multiple domains underlines the versatility of the approach.

2 NUMERICAL METHODS

The general approach in our method is to use high order schemes everywhere. The idea of ADER schemes by Toro, Titarev et al.\textsuperscript{4,5,6} has been adopted and optimized for linear and nonlinear equations on structured grids in the framework of Finite Volume schemes by Schwartzkopff et al.\textsuperscript{7,8,9,10} This idea has been extended to the Discontinuous Galerkin methods by Dumbser et al.\textsuperscript{11,12} which is favorable for the use on unstructured grids. A recent addition to our proposed methods are arbitrary high order DRP (Dispersion Relation Preserving) schemes that are also based on a Taylor expansion in time and not on the usual Runge-Kutta procedure (Lörcher\textsuperscript{13}).

2.1 High order ADER-FV schemes for structured grids

The ADER-FV method may be formulated with arbitrary accuracy in space and time. For the calculation of the time-integrated fluxes at the cell boundaries, so-called Generalized Riemann Problems with piecewise polynomial initial data on both sides of the cell boundaries are solved. Therefore, the solution is expanded in a Taylor series in time at the interface and then time-derivatives are replaced by space-derivatives making successive use of the governing equations. This is the so-called Cauchy-Kovalevskaja procedure. The method was first developed in this Finite Volume context by Toro, Titarev et al.\textsuperscript{4,5,6} The main advantage of the ADER-FV approach is that only one polynomial reconstruction of the local values at the interface has to be performed. Beside the functional values, all values of the derivatives are taken into account. By this approach fast schemes can be constructed for all situations, for which the reconstruction is simple and fast. The best candidate is a Cartesian grid or at least a structured grid. The reconstruction becomes
cumbersome on unstructured grids.

2.2 High order ADER-DG schemes for unstructured grids

The very compact ADER-DG formulation does not need a reconstruction and thus provides the possibility to achieve arbitrary high order of accuracy in space and time even on unstructured grids, which should be useful for accurate noise propagation in the time domain around complex obstacles or in complex geometries. Within the DG framework, arbitrary high order discretization of curved wall boundaries on unstructured grids is also introduced using superparametric elements. Our implementation of the ADER-DG scheme is very general concerning the mesh, the equations to solve and concerning the order of accuracy in space and time. The implemented scheme can solve general linear hyperbolic systems with variable coefficients and source terms on 2D and 3D unstructured domains and the user may chose any order of accuracy in space and time he needs. Arbitrary spatial accuracy is possible due to the availability of sets of hierarchical orthogonal basis functions for any desired polynomial degree. For linear systems, it is sufficient that the user provides the system matrices A, B and C together with their eigenvalues and left and right eigenvectors. General nonlinear hyperbolic conservation laws can be solved on unstructured meshes at theoretically arbitrary order of accuracy in space and time as well. In this case, the arbitrary accuracy is available only for those nonlinear systems for which the Cauchy-Kovalevskaja procedure and an appropriate numerical flux is implemented, see the algorithms presented in Dumbser et al. This has been done e.g. for the Euler and the magneto-hydrodynamic equations.

2.3 High order DRP schemes for structured grids

In aeroacoustic applications, the DRP scheme by Tam is widely used in order to solve the linear hyperbolic system for wave propagation. In the DRP scheme, the first order spatial derivatives are approximated by one-dimensional finite difference formulas. The time approximation is performed by Runge-Kutta schemes. Both approximations are optimized to capture wave propagation in an optimal way. In our modification (see Lorcher), the basic idea of how to get arbitrary order of consistency in time is to use a Taylor expansion in time. For linear systems with constant coefficients, the PDE also holds, if the state vector is replaced by any of its time derivatives. Consequently, we can get successively an approximation of any time derivative. This provides all necessary time derivatives in order to evaluate the Taylor expansion up to a desired order in time. This scheme can be also applied to curvilinear structured grids. A related scheme was proposed by Hu et al.
3 DOMAIN DECOMPOSITION

In order to reduce the effort of a direct simulation, our main objective is to introduce a domain decomposition and to apply the most efficient solver on every subdomain. The strategy has been discussed in detail by Utzmann et al.\(^3\) Unstructured grids are used only in the direct vicinity of complex bodies. Then, as quickly as possible, we switch to structured, Cartesian meshes. The grids may have a jump in size, hence the mesh can be adapted to the local flow phenomena in the subdomain. Explicit time integration schemes are used in the domains, therefore the time steps are chosen as large as possible for each domain. Thus, also the time step can have a jump at the domain interface. Finally, we allow the underlying equations to change. In the core regions near sound generating obstacles and where viscous effects play a role, the Navier-Stokes equations are solved. Then, we switch over to the nonlinear and the linearized equations, e.g. the Linearized Euler Equations. In the following, the integration of Finite Difference schemes into the framework of the coupling method is described. We focus on the coupling of structured FD grids with unstructured DG grids, other configurations such as FD-FV can be implemented in a straightforward manner.

3.1 Coupling of different equations

The coupling of nonlinear and linear Euler equations has been discussed by Utzmann et al.\(^3\) Reflections at the coupling boundaries can be kept very small by prescribing rather the continuity of the state variables at the coupling boundary than the continuity of the flux. The coupling of the Navier-Stokes equations with the nonlinear or linearized Euler equations follows this principle and allows a discontinuous interface flux while preserving the continuity of the state. The reflections in the numerical example (see section 4.3) turn out to be quite small again.

3.2 Coupling of FD grids with DG grids

The coupling of arbitrary FV- and DG- grids (Utzmann et al.\(^3\)) has been extended for FD grids. Two or more different domains \(\Omega_i\) are coupled at their common boundary \(\partial \Omega = \Gamma\) over the data in the ghostelements. Depending on the discretization method, this element can be a ghostcell (Finite Volume and Discontinuous Galerkin methods) or a ghostpoint (Finite Difference methods). The data are exchanged by interpolating the values from the neighbor-grid onto either the Gauss integration points of the ghost cells (FV and DG methods; with a subsequent integration in order to obtain the mean-values) or onto the ghostpoints (FD methods) themselves. For a Finite Difference scheme like the one proposed by Lörcher\(^{13}\) (see also section 2.3), there are several rows of ghostpoints (their number depends on the chosen order of accuracy). For the DG scheme, only one row of ghostcells is needed (Fig. 1 and 2). These ghostelements are then used by the numerical methods to update the inner elements in each domain. In the following, the basic coupling mechanism between the FD- and DG discretizations shall be described. The
interpolation procedure is illustrated for 2D elements while the actual implementation is also 3-dimensional.

1. **Interpolation from the FD domain onto the DG ghostcells:** As the domains are aligned along their common boundary, the DG ghostcells lie completely within the FD domain (Fig. 1). In order to obtain the values for the elements, a 3-dimensional Lagrange interpolation is performed: The state vector $\hat{U}$ is interpolated from the regular FD mesh onto the position $X_{iGP}$ of each Gauss point in the element’s volume:

$$\hat{U}_{iGP} = \mathcal{L}^{\Omega_{FD}}(X_{iGP}).$$

(1)

Then, the Gauss point values $\hat{U}_{iGP}$ are integrated and projected onto the cell’s degrees of freedom (DOF) $\hat{U}_{DG}$:

$$\hat{U}_{DG} = \sum_{iGP=1}^{nGP} \sum_{iDOF=1}^{nDOF} \omega_{iGP} \cdot \phi_{iGP,iDOF} \cdot \hat{U}_{iGP} \cdot M_{iDOF,iDOF},$$

(2)

with the integration weights $\omega_{iGP}$, the mass matrix $M_{iDOF,iDOF}$ and the value of the base function $\phi_{iGP,iDOF}$ at the Gauss point position. The interpolation stencils are taken from the FD grid and their order can be arbitrary. However, uneven numbers of interpolation points in each direction provide more symmetrical stencils and are preferred. Figure 3 depicts a close-up of one of the ghostcells: In this case, fifth order stencils are used for the interpolation.

2. **Interpolation from the DG domain onto the FD ghostpoints:** A unique neighbor element from the DG domain can be assigned to every ghostpoint of the FD coupling boundary (Fig. 2). As the DG cell’s degrees of freedom contain the complete polynomial information, there is no need for an interpolation stencil and...
the value of the state vector $\vec{U}_{FD}$ at the position of each FD ghostpoint can be obtained easily:

$$\vec{U}_{FD} = \sum_{iDOF=1}^{nDOF} \phi_{iDOF}(X_{FD}) \cdot \hat{U}_{iDOF},$$

with the degrees of freedom $\hat{U}_{iDOF}$ in the DG neighbor cell and the value $\phi_{iDOF}(X_{FD})$ of the base function at the position $X_{FD}$ of the ghostpoint. Spatial derivatives of the state vector (that may be needed by a Cauchy-Kovalevskaja procedure, see "3.3. Coupling of Time Steps" in this section) are obtained in the same way by using the derivatives of the base function.

### 3.3 Coupling of time steps

The idea of multi size meshes with different time steps has been examined by Tam et al.\textsuperscript{16} for DRP schemes on particularly designed grids. This basic idea has been extended for FV- and DG-schemes on arbitrary meshes and for arbitrary jumps in the time steps. In this regard, the so-called Cauchy-Kovalevskaja (CK) procedure takes a key position (Utzmann et al.\textsuperscript{3}). It answers the question how to treat the ghostelements of the domain with the smaller $\Delta t$, which have to be provided with an updated value before the domain can proceed with its time stepping. By replacing time derivatives with spatial derivatives, a Taylor series in time can be used to calculate the state in a ghostelement at an arbitrary time. This procedure can be also applied for the considered coupling between FD and DG domains and is an ingredient of the numerical methods in section 2.

The local CFL number of the domains is adjusted such, that the time step of one domain is a multiple $n$ of the neighboring domain with the smaller $\Delta t$. After the domain with the small time step has been updated $n$ times, all domains have the same time level again. If a Runge-Kutta time integration scheme is chosen for the FD domain,
Figure 4: Coupling different time steps: 1. Data exchange at the common time level: The states are interpolated onto the ghostpoints of the FD domain and onto the ghostcells of the DG domain. For the FD ghostpoints, also the spatial derivatives are interpolated. 2. Time stepping of the FD domain. This can be understood as either an intermediate Runge-Kutta time step or a real time step. At the same time, its ghostpoints are lifted to the new time level by the CK procedure. 3. The FD domain makes another time step. Again, its ghostpoints are pushed to the new time level, based on the original data. 4. The DG domain performs one single time step and meets the FD domain at a common time level again. Then, a new data exchange takes place.

Also intermediate time steps are needed. They can be also provided by the CK procedure. Thus, if the FD domain is the one with the smaller $\Delta t$ (e.g. in a DNS calculation like in the examples section, 4.3), the overall number of time steps for the FD domain will be $2 \cdot n + 1$ (Runge-Kutta $\mathcal{O}(4)$). In case the high order DRP scheme proposed by Lörcher is used for the propagation of sound waves into the far-field, it is most likely that the time step will be bigger than the ones in the inner domains. However, besides the usual boundary conditions, the scheme also needs time derivatives for their ghostelements, which can be again provided by the CK procedure. Figure 4 shows a whole cycle (e.g. for a $\mathcal{O}(4)$ Runge-Kutta DRP scheme coupling with a DG domain and a time step ratio of $n = 1$) between two data exchanges, including the updating of the ghostpoints via the CK procedure.
4 NUMERICAL RESULTS

4.1 Gauss point coupling: Numerical convergence

Utzmann et al.\textsuperscript{3} showed for several examples in 2D, that globally high order of accuracy can be maintained for the fully conservative coupling procedure. Now, the following test examines the behavior towards convergence for the generalized Gauss point coupling procedure in 3D.

The Linearized Euler Equations are solved with the background flow $\rho_0 = 1.0$, $u_0 = 1.0$, $v_0 = 0.0$, $w_0 = 0.0$ and $p_0 = 1.0/\gamma$ in the two domains $\Omega_1 = [-75.0 : -25.0] \times [-25.0 : 25.0]$ and $\Omega_2 = [-25.0 : 25.0] \times [-25.0 : 25.0] \times [-25.0 : 25.0]$. A Gaussian pulse in density is initialized at the center of both domains at $t = 0.0$ (Fig. 5):

$$
\rho'(t = 0.0) = e^{-\frac{1}{2} \left(\frac{(x-x_c)^2 + (y-y_c)^2 + (z-z_c)^2}{\sigma^2}\right)},
$$

with the pulse center coordinates $(x_c, y_c, z_c)_{\Omega_1} = (-50.0, 0.0, 0.0)$ and $(x_c, y_c, z_c)_{\Omega_2} = (0.0, 0.0, 0.0)$ and the pulse halfwidth $\sigma = 5.0$. The other perturbation variables $u'$, $v'$, $w'$ and $p'$ are set to zero. When the calculation stops at $t = 50.0$, the Gaussian pulse from domain $\Omega_2$ has left the calculation area through outflow boundary conditions. Meanwhile, the pulse from domain $\Omega_1$ has traveled into domain $\Omega_2$ and has taken the position of the original pulse that was initialized in $\Omega_2$ (Fig. 6). The comparison with the initial condition in $\Omega_2$ allows now the determination of error norms. In order to determine the order of convergence, both grids $\Omega_1$ and $\Omega_2$ are refined while keeping the ratio of grid cells $N_G$ per space direction. The meshes never conform at the boundary.

Three different grid configurations are tested. The first calculation is performed with two unstructured grids, using a 3rd order ADER-DG scheme. The time step ratio is 1:1, as the difference in grid spacing is a little bit smaller than 1:2. Table 1 shows good 3rd order convergence rates with slight fluctuations due to the irregularity of the grids and the initially very coarse meshes. In the second calculation, a structured 3rd order
ADER-FV domain $\Omega_1$ is coupled with an unstructured 3rd order ADER-DG domain $\Omega_2$, as shown in Fig. 5. Due to the more restrictive CFL limitation of the DG method, the time step ratio this time is $\Delta t_{\Omega_1} : \Delta t_{\Omega_2} = 1 : 27$. Nevertheless, a 3rd order convergence rate is maintained globally (Table 2). Finally, Table 3 shows 4th convergence rates for the coupling of two structured 4th order ADER-FV domains. The time step ratio here is again 1:1. Two additional refinement stages have been computed, as the memory and CPU time requirements are rather low for the ADER-FV method.

Note, that the unstructured grids are completely irregular. Therefore, a mesh refinement that is done by prescribing the spacing on the surface of the domain implicates variations in the single element sizes. Considering this, the order of accuracy was calculated using the maximum inner sphere diameter $d_{is}$ of all elements in the unstructured domain.

<table>
<thead>
<tr>
<th>$N_G(\Omega_1/\Omega_2)$</th>
<th>$d_{is}$</th>
<th>$L_\infty$</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$O_{L_\infty}$</th>
<th>$O_{L_1}$</th>
<th>$O_{L_2}$</th>
</tr>
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<tr>
<td>4/7</td>
<td>4.802</td>
<td>1.27490E-01</td>
<td>2.88274E+02</td>
<td>2.58795E+00</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
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<td>2.43708E-02</td>
<td>4.08444E+01</td>
<td>4.31417E-01</td>
<td>2.65</td>
<td>3.12</td>
<td>2.86</td>
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<td>9.71506E-03</td>
<td>1.32672E+01</td>
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<td>2.30</td>
<td>2.82</td>
<td>2.61</td>
</tr>
<tr>
<td>16/28</td>
<td>1.378</td>
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<td>4.84344E+00</td>
<td>5.54580E-02</td>
<td>3.28</td>
<td>4.51</td>
<td>4.51</td>
</tr>
</tbody>
</table>

Table 1: Convergence rates for the coupling of two ADER-DG $O_3$ domains.

<table>
<thead>
<tr>
<th>$N_G(\Omega_1/\Omega_2)$</th>
<th>$d_{is}$</th>
<th>$L_\infty$</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$O_{L_\infty}$</th>
<th>$O_{L_1}$</th>
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<td>10/7</td>
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<td>5.43559E+02</td>
<td>5.55733E+00</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>20/14</td>
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<td>6.79695E-02</td>
<td>1.26996E+02</td>
<td>1.49701E+00</td>
<td>2.20</td>
<td>2.33</td>
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<tr>
<td>30/21</td>
<td>1.724</td>
<td>2.54219E-02</td>
<td>4.53704E-01</td>
<td>5.61591E-01</td>
<td>2.46</td>
<td>2.58</td>
<td>2.46</td>
</tr>
<tr>
<td>40/28</td>
<td>1.378</td>
<td>1.19687E-02</td>
<td>2.10111E+01</td>
<td>2.62947E-01</td>
<td>3.37</td>
<td>3.45</td>
<td>3.40</td>
</tr>
</tbody>
</table>

Table 2: Convergence rates for the coupling of an ADER-FV $O_3$ domain with an ADER-DG $O_3$ domain.

<table>
<thead>
<tr>
<th>$N_G(\Omega_1/\Omega_2)$</th>
<th>$L_\infty$</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$O_{L_\infty}$</th>
<th>$O_{L_1}$</th>
<th>$O_{L_2}$</th>
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<tr>
<td>15/10</td>
<td>1.37486E-01</td>
<td>4.55379E-03</td>
<td>1.51073E-02</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>30/20</td>
<td>3.99579E-02</td>
<td>8.01283E-04</td>
<td>3.33272E-03</td>
<td>1.78</td>
<td>2.51</td>
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<td>45/30</td>
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<td>1.85627E-04</td>
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<td>3.06</td>
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<td>60/40</td>
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<td>6.18226E-05</td>
<td>2.70144E-04</td>
<td>3.70</td>
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<td>75/50</td>
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<td>2.57103E-05</td>
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<td>3.92</td>
<td>3.93</td>
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<tr>
<td>90/60</td>
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<td>1.25456E-05</td>
<td>5.50166E-05</td>
<td>3.77</td>
<td>3.94</td>
<td>3.94</td>
</tr>
</tbody>
</table>

Table 3: Convergence rates for the coupling of two ADER-FV $O_4$ domains.
4.2 Multi-domain gaussian pulse

This example has been designed in order to test a great variety of possible domain constellations and thus to test the effortless coupling of multiple domains. A 2-dimensional Gaussian pulse in density is transported through a partitioned calculation area with constant advection speed. Figure 7 shows the configuration of each individual partition: There are 6 different domains on which either the linearized or non-linear Euler Equations (EE) are solved with a Finite Volume or Discontinuous Galerkin discretization. The grids are both structured and unstructured and have individual mesh sizes $\Delta h$ (see Table 4 for the domain parameters). The so-called ”multi-domain ghostcells” described before occur in this example each time when two domains couple both with another domain’s side (e.g. at the common interface between domains 4, 5 and 6). The Gaussian pulse starts at $t = 0$ in domain 1 and crosses the other domains on its way to domain 6, where it arrives at $t = 100$. (Fig. 8-10). During the calculation, the pulse is located in up to four different domains at the same time (Fig. 11). The comparison between the multi-domain result with a solution that was obtained on an unpartitioned domain (parameters of domain 1) shows a good accordance in both x- and y-direction (Fig. 12 and 13). Note, that the

<table>
<thead>
<tr>
<th>#</th>
<th>Mesh</th>
<th>Equations</th>
<th>Method</th>
<th>Order</th>
<th>Extents</th>
<th>$\Delta h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>struct.</td>
<td>lin. EE</td>
<td>fastADER -FV</td>
<td>$\mathcal{O}4$</td>
<td>$[-75 : -25] \times [-25 : 25]$</td>
<td>1.250</td>
</tr>
<tr>
<td>2</td>
<td>struct.</td>
<td>non-lin. EE</td>
<td>ADER-FV</td>
<td>$\mathcal{O}2$</td>
<td>$[-25 : 0] \times [5.625:25]$</td>
<td>0.676</td>
</tr>
<tr>
<td>3</td>
<td>unstruct.</td>
<td>lin. EE</td>
<td>ADER-DG</td>
<td>$\mathcal{O}5$</td>
<td>$[25 : 0] \times [-25:5.625]$</td>
<td>2.770</td>
</tr>
<tr>
<td>4</td>
<td>unstruct.</td>
<td>non-lin. EE</td>
<td>ADER-DG</td>
<td>$\mathcal{O}3$</td>
<td>$[0 : 25] \times [-5 : 25]$</td>
<td>1.000</td>
</tr>
<tr>
<td>5</td>
<td>struct.</td>
<td>lin. EE</td>
<td>fastADER -FV</td>
<td>$\mathcal{O}9$</td>
<td>$[0 : 25] \times [-25 : -5]$</td>
<td>2.083</td>
</tr>
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</table>

Table 4: Domain parameters.
gaps between the grids are due to the visualization: The primal mesh is shown for the unstructured domains, the dual mesh is plotted for the Cartesian grids.

Figure 8: Contour plot of density $\rho$ at $t = 0.0$: Initial Condition.

Figure 9: Contour plot of density $\rho$ at $t = 50.0$

Figure 10: Contour plot of density $\rho$ at $t = 100.0$

Figure 11: Contour plot of density $\rho$ at $t = 50.0$, 3D view

Figure 12: Comparison of the multi-domain solution with a single-domain solution at $t = 100.0$: Cut in x-direction at $y = 0.0$.

Figure 13: Comparison of the multi-domain solution with a single-domain solution at $t = 100.0$: Cut in y-direction at $x = 50.0$. 
4.3 Coupling of an unstructured acoustic solver with a DNS code

As an example for the integration of Finite Difference schemes into the framework of the proposed domain decomposition, an acoustic code using the ADER-DG method is coupled with a DNS (Direct Numerical Simulation) code (see also Babucke et al.\textsuperscript{17}). While the previous examples were computed with one single code which couples several domains in its internal structure, the DNS solver and the acoustic solver are stand-alone codes this time.

The 3-dimensional acoustic code uses the ADER-DG method on unstructured tetrahedrons. On the DNS side, an FD solver for the three-dimensional unsteady compressible Navier-Stokes equations is used. Spatial discretization in streamwise x- and normal y-direction is done by 6th order compact finite differences. The flow is assumed to be periodic in spanwise direction, therefore a spectral ansatz is applied for the z-direction. Time integration is done using the standard 4th order Runge-Kutta scheme (see e.g. Kloker et al.\textsuperscript{18}). See Babucke et al.\textsuperscript{19,20} about more details regarding the discretization schemes and the DNS code.

The separate codes have to exchange their coupling data – the set of primitive variables $(\rho', u', v', w', p')$ near the boundary – at each subcycle. This communication is based on

<table>
<thead>
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<th>Case</th>
<th>DNS grid (x y z)</th>
<th>DG elements</th>
<th>DG order</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>401 x 201 x 5</td>
<td>368800</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>401 x 201 x 5</td>
<td>368800</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>401 x 201 x 5</td>
<td>92400</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>401 x 201 x 5</td>
<td>92400</td>
<td>5</td>
</tr>
<tr>
<td>E</td>
<td>401 x 603 x 5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Domain configurations for the computed cases.
the Transmission Control Protocol\textsuperscript{21} consisting of a server and a client. The server opens a port and waits for the client to connect. For the client, one has to specify IP-address and port of the server. The communication itself is done by simply writing on or reading from the socket.

The setup for the example is as follows: In the domains, a uniform mean flow is chosen with $Ma_\infty = 0.5$ in streamwise direction, resulting in an ambient pressure of $p_0 = \frac{1}{\gamma}Ma_\infty^2 = 2.85714$. The temperature is $T_0 = 280K$, containing a temperature disturbance of $0.001 \cdot T_0$ with a radius of 1.0 located at $x = 39.0$ and $y = 0.0$ in the DNS domain. This leads to a circular pressure pulse being emitted in all directions and a temperature spot which is convected with the streamwise mean flow. The Reynolds number $Re = 500$ and the Prandtl number $Pr = 0.71$, which describe the viscous terms, are of course only needed for the DNS domain. The DNS grid has a range of $-15.0 \leq y \leq 15.0$ in normal and $0.0 \leq x \leq 115.0$ in streamwise direction. In order to circumvent disturbances convected with the meanflow reaching the outflow, the grid is stretched in $x$ direction from $x = 60.0$ onwards. The spanwise extent has been chosen with respect to acceptable grid properties of the DG code and is $0.0 \leq z \leq 0.5$ for the fine and $0.0 \leq z \leq 1.0$ for the coarse grid, respectively. The DG domains are located above and below the DNS grid with the same extent in streamwise and spanwise direction and ranging to $y = \pm 25.0$. The wave propagation is evaluated by comparing discretization orders on different grids, which are illustrated in Fig. 14 and 15. Table 5 gives an overview of the cases which have been computed to evaluate the coupling scheme. Case E is a large DNS which has been performed to compare the results. The domain has three times the size of the original DNS ranging up to $y = \pm 45.0$. This gives enough space so the wave does not reach the boundary in the considered time. Hence, we can exclude errors due to boundary conditions and the DNS result from case E may serve as reference data.

For case C, the wave crossing the coupling interface is shown in Fig. 16, using the polynomial ansatz of the DG elements in order to visualize the smooth solution. Figure 17 illustrates the grid configuration showing only the integral mean values in the elements. Despite the different equations and discretizations used in both codes, the wave crosses the coupling plane with almost no reflections.

The pressure distribution along a constant $x$-position is shown in Fig. 18 and 20. The detailed views in Fig. 19 and 21 illustrate the differences between various DG orders and the DNS solution. At time $t = 7.8539$, when the pulse passes the coupling interface, a small amplitude error can be observed. This error may correspond to small reflections at the interface. Later in time, the amplitudes are identical for both DNS and DG scheme, as one can see in Fig. 21. The phase error compared to the DNS solution decreases with higher order for the DG domain. It must be taken into account that the Navier-Stokes equations contain viscous terms, while in the outer domain, the Linearized Euler Equations are solved. This might explain the better accordance of the amplitudes at the later time step (Fig. 21).
Figure 16: Acoustic wave crossing the coupling plane at time \( t = 9.4247 \) for 4th-order DG scheme on the coarse grid.

Figure 17: DNS and DG grids with integral mean values of the DG elements at time \( t = 9.4247 \) for 4th-order DG scheme on the coarse grid.

Figure 18: Pressure distribution at position \( x = 52.03 \) and time \( t = 7.8539 \).

Figure 19: Detailed view of pressure distribution at position \( x = 52.03 \) and time \( t = 7.8539 \).

Figure 20: Pressure distribution at position \( x = 52.03 \) and time \( t = 9.4247 \).

Figure 21: Detailed view of pressure distribution at position \( x = 52.03 \) and time \( t = 9.4247 \).
5 CONCLUSION

The proposed domain decomposition approach for the direct simulation of aeroacoustic problems allows the versatile coupling of different numerical methods, grids, equations and time steps. 3D examples show, that the method is capable of retaining globally the order of accuracy that is used by the high order methods in the domains. As the coupling scheme is very general, the integration of Finite Difference schemes into the decomposition framework is straightforward.

Due to the flexibility of the approach, an independent DNS code based on a Finite Difference scheme could be coupled with an acoustic solver. Here, the domain decomposition method holds for the coupling of the Navier-Stokes equations with the Linearized Euler Equations. The ADER-DG method was used for the acoustic domain in order to test the coupling of two stand-alone codes. However, while the unstructured DG method is highly suitable in the vicinity of complex geometries, a much cheaper structured code (e.g. high order DRP) could be used for far field computations, based on the proposed coupling mechanism.

The coupling scheme is currently prepared for parallel computing. All subdomains can be calculated independently with data exchange at the interfaces of adjacent subdomains in each time step. The subdomains can then be divided into MPI partitions.

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