THE DESCRIPTION OF PATCHY ATMOSPHERIC TURBULENCE, BASED ON A NON-GAUSSIAN SIMULATION TECHNIQUE

by

G. A. J. van de Moesdijk

DELFT - THE NETHERLANDS

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1. Summary

This report deals with non-Gaussian aspects of atmospheric turbulence. A variety of complex problems posed by atmospheric turbulence is due to its essentially non-Gaussian structure. For instance in predicting the probability of exceeding loads or responses of a given magnitude, the theoretical curves computed with the classical Gaussian assumption seriously underestimate the empirical exceedance curves derived from actual turbulence measurements.

Another shortcoming of the Gaussian assumption appears when using aircraft simulators in order to study the effects of turbulence on aircraft handling qualities, performance of pilots under stress etc. In this context the non-Gaussian properties are often referred to as the "patchiness" of atmospheric turbulence.

Although the phenomenon itself has been described recently in the literature, there seems to be as yet not a clear mathematical description of patchy characteristics.

A model describing patchiness as sensed by the pilot is developed in this report, the degree of patchiness being defined in mathematical terms. An existing technique of simulating patchy turbulence, intended to be used in piloted flight simulation is extended, allowing patchiness to vary significantly. The model of patchiness is combined with the turbulence simulation technique, yielding analytical expressions for the patchiness parameters as defined in this report.

A complete description of the model of patchiness is presented as well as a complete description of the statistical properties, including patchiness of each of the simulated turbulence components.
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3. - List of symbols

A \quad \text{filter constant}

a(t) \quad \text{stochastic process, output of filter } F_a

B \quad \text{filter constant}

b(t) \quad \text{stochastic process, output of filter } F_b

C_{xx,uu \text{ etc.} (\ )} \quad \text{auto-covariance function, the indices indicating the variable concerned}

C_{xy}(\ ) \quad \text{cross-covariance function, the indices indicating the variables concerned}

c(t) \quad \text{stochastic process, output of filter } F_c

D \quad \text{filter constant}

E \quad \text{filter constant}

E[\ ] \quad \text{expected value}

erf(\ ) \quad \text{error function}

F(\ ) \quad \text{cumulative probability function}

H_{a,b,c}(\ ) \quad \text{filter transfer function, the index indicating the output variable concerned}

h(\ ) \quad \text{impulse response function}

K_{a,b,c} \quad \text{constant of input white noise power spectrum, the index indicating the variable concerned}

K \quad \text{Kurtosis of the probability distribution}

K_0(\ ) \quad \text{Modified Bessel function of the second kind of the order zero}

L_{u_g,v_g,w_g} \quad \text{scale length of turbulence, the index indicating the turbulence component concerned}

M_n \quad \text{moment of the probability distribution function of the order } n

m_n \quad \text{central moment of the probability distribution function of the order } n
\( P_T \)  
Patchiness parameter, function of the timeconstant \( \tau \)

\( p( ) \)  
probability density distribution function

\( Q \)  
ratio between the standard deviations of \( u(t) \) and \( c(t) \)

\( R \)  
ratio between the cutoff frequencies of filter \( F_a \) and filter \( F_b \)

\( R_{aa,bb \ , \ etc.}( ) \)  
auto-correlation function, the indices indicating the variable concerned

\( S \)  
set of possible outcomes of an experiment

\( S \)  
skewness of the probability distribution

\( T \)  
specific instant in time

\( t \)  
time

\( U( ) \)  
unit step function

\( u(t) \)  
stochastic process, product of \( a(t) \) and \( b(t) \)

\( u_g \)  
longitudinal component of the turbulence velocity

\( V \)  
velocity of the aircraft

\( v_g \)  
lateral component of the turbulence velocity

\( w(t) \)  
stochastic process, output of the system representing \( u_g, v_g \) or \( w_g \) respectively

\( w_g \)  
vertical component of the turbulence velocity

\( x(t) \)  
stochastic process

\( y(t) \)  
stochastic process, square of the process \( x(t) \)

\( z(t) \)  
stochastic process, output of the memory filter

\( \beta \)  
ratio between the filterconstants \( \sqrt{A} \) and \( \sqrt{B} \) or \( \sqrt{A} \) and \( \sqrt{D} \)

\( \epsilon \)  
shift in time

\( \mu_{x,y \ \text{etc.}} \)  
mean value of a stochastic variable, the index indicating the variable concerned
\( \zeta_i \) specific outcome of an experiment, i.e. a realisation of a stochastic process

\( \sigma^2_{a,b,c,u_g,etc.} \) variance of a stochastic variable, the index indicating the variable concerned

\( \phi_{u_g u_g, v_g v_g, w_g w_g, etc}() \) auto-power spectral density function of a stochastic variable, the indices indicating the variable concerned

\( \tau \) dummy variable of time

\( \Omega \) spatial frequency (rad/m)

\( \omega \) circular frequency (rad/sec)
4. - Introduction

Simulation of aircraft handling qualities in several flight regimes has become of increasing importance during the past decades. Questions concerning performance, stability and control are attacked by simulation well before the design stage. The validity of a piloted simulation experiment highly depends upon the quality of simulator hardware as well as on the degree of sophistication of the mathematical models used to represent the aircraft and its environment. In particular the representation of realistic external disturbances often is of prime importance. In this context aircraft responses to atmospheric turbulence, wind and windshear often define the critical tasks in the evaluation of aircraft system performance. It is therefore recognized that the accuracy of introducing atmospheric turbulence may well determine to a great extent the outcome of a simulation experiment. Current methods of producing simulated turbulence are usually based on the power spectrum and on the assumption of a Gaussian distribution. If actual atmospheric turbulence were Gaussian, the spectral description should provide an essentially complete statistical description of the process, which is very attractive from a mathematical point of view. The problem is, however, that actual turbulence is essentially non-Gaussian, a fact confirmed by recent extensive turbulence measurement programs, see e.g. Refs. 1 and 2. This fact causes the complex difficulties in assessing the effects of turbulence on aeronautical systems. Therefore the problem of correctly modelling the non-Gaussian properties of atmospheric turbulence is of great interest not only in simulation but also in other research areas, in particular those related to aircraft airworthiness requirements, e.g. safety aspects of control systems, studies of automatic landing, probability of exceeding limit design loads, fatigue criteria etc.

The emphasis is laid in this report on modelling the non-Gaussian aspects of atmospheric turbulence from a pilot's point of view. In this context a model of patchiness is proposed in which patchy characteristics are described and defined as sensed by a pilot. An existing technique
of simulating patchy turbulence, Refs. 3 and 4, is extended and further mathematically elaborated upon. Possible questions concerning a physical interpretation of the patchiness parameters to be defined will not be gone into, since no attempt is made to explain the patchiness of atmospheric turbulence from a physical point of view.

4.1. - Statement of the problem
The term "patchiness" of "patchy" atmospheric turbulence, sometimes also referred to as "intermittency" indicates properties of the turbulence phenomenon, which can briefly be described as follows. Atmospheric turbulence, a stochastic process by its nature, seems to exhibit localised regions of relatively higher energy concentrations. Apparently homogeneous turbulence is punctuated by relatively large bumps occurring to the pilot virtually without warning. This property represents an element of surprise to the pilot, since it may cause sudden large deviations in aircraft response (normal and lateral accelerations, bank angle etc.). Currently existing methods for turbulence simulation, based on the assumption of a Gaussian distribution seem to fail to reproduce this property. Pilots often criticize turbulence response in ground-based simulators for feeling too "regular" and therefore as being not quite realistic. Motivated by the desire to obtain a more realistic simulation of atmospheric turbulence, attempts have been made to include this property of patchiness in the simulation, see Refs. 3 and 4. However, as stated earlier in this Section, patchiness has a much wider scope and many researchers have tried to penetrate into the nature of patchiness. A brief review of work done in this area is therefore included in the next Section.

4.2. - Review of patchiness
Taking the effects of patchiness into account, requires a turbulence model, in which patchiness is defined by quantitative parameters, appearing in the mathematical turbulence model. Since the classical turbulence models are based purely on the power spectrum, it is necessary to go beyond the power spectrum description and investigate more closely the distribution of turbulence velocities. Much experimental and theore-
tical work has been done in this field e.g. by Dutton, Thompson and
Deaven, Refs. 1 and 5, and Gould and MacPherson, Ref. 2.
From the experimental data, it appears that atmospheric turbulence
cannot properly be considered as a purely Gaussian process. The
distribution of the turbulence velocities contains consistent departures
from a Gaussian behaviour, see Section 4.3. Dutton and Lane, Ref. 6,
used these departures from Gaussian behaviour in their description of
patchiness. In the simulation field, Reeves, Refs. 3 and 4; Jones and
Tomlinson, Refs. 23 and 26, mechanized turbulence simulations having
to some extent patchy characteristics, by generating certain departures
of the distribution of atmospheric turbulence from the Gaussian distri-
bution function.
As will be described in this report; it may well be that patchiness
is only partially described by departures from normality, such as can
be characterized by the fourth order central moment, see Section 5.5.
As a basis for further elaborations, it seems worthwhile to briefly
repeat some basic concepts of stochastic processes such as can be
found in the literature, Section 5. Next atmospheric turbulence is
described on a statistical basis and specific assumptions, not generally
made in the theory of stochastic processes are included, Section 6.
Section 7 contains a description of a proposed model of patchiness, using
the concept of a simple "memory-filter" as a means to assess the patchy
turbulence structure.
Finally the real-time simulation of patchy turbulence velocities is
treated in Section 8 and the patchy characteristics of the simulated
turbulence velocities are described in Section 9, according to the
proposed model of patchiness.
5. - Basic concepts of stochastic processes

Atmospheric turbulence by its nature is a stochastic process. Some well-known definitions and assumptions of stochastic processes more or less applicable to atmospheric turbulence are recapitulated here. The extensive use made of these concepts, when deriving the results presented in this report, demands for clarity in notation and mathematical form so as to avoid confusion, since some of the definitions may well be presented in the literature in a slightly different way.

After the introduction of stationarity and ergodicity, two techniques to provide information about stochastic processes are described. The first method to consider stochastic processes is based on probability concepts. Here attention is focussed only on the frequency of occurrence of a random variable of various magnitudes. Spatial or temporal sequence properties of the random variable are disregarded in this method. The second method is based on spectral analysis, in which the distribution of energy over the range of frequencies or wavelengths of interest is considered. Finally a fundamental theorem in linear system theory is mentioned.

5.1. - General

A stochastic process can be viewed as a function of two variables \( \xi \) and \( t \). The domain of \( \xi \) is the set \( S \), specified by the outcomes \( \xi \) of an experiment. The domain of \( t \) is the entire time axis. For a specific outcome \( \xi_i \) of the experiment, the expression \( x(t, \xi_i) \) signifies a single time function, called a realisation, see Fig. 1. All time functions specified by the outcomes \( \xi \) form the ensemble of the stochastic process. The dependence of a stochastic process on \( \xi \) is usually neglected. The notation \( x(t) \) is often employed to represent both a complete family of time functions or a single time function. The specific interpretation of \( x(t) \) in this report will be made clear from the context.

5.2. - Stationarity

Atmospheric turbulence is commonly considered as a stationary process. A process is stationary by definition if its statistics are not affected
by a shift of the time origin. This implies that the ensemble \( x(t) \) and the ensemble \( x(t + \varepsilon) \) of a stationary process have identical statistics for any \( \varepsilon \), e.g. a probability density function of the first ensemble equals the probability density function of the second ensemble, thus:

\[
    f(x,t) = f(x,t + \varepsilon) \tag{5.2.1.}
\]

Since this is true for any \( \varepsilon \) in a stationary situation, the probability density function \( f(x,t) \) is independent of \( t \), hence:

\[
    f(x,t) = f(x) \tag{5.2.2.}
\]

Similar conclusions can be drawn concerning the mean value \( \mu \), and the autocorrelation function \( R(t_1, t_2) \) etc., see Section 5.6.

5.3. - Ergodicity

A stationary stochastic process consists of a family - or ensemble - of functions, viz. the individual realisations of the process, as indicated in 5.1. From this ensemble of functions, in general only one function or realization is available from measurements of limited duration. Ergodicity deals with the problem of determining the statistics of the entire ensemble from this single realization.

The process \( x(t) \) is by definition ergodic with respect to certain statistical properties if these statistical properties can be determined from the single realisation \( x(t) \) of the entire ensemble. If the statistical parameters are expressed as time averages, ergodicity can be formulated as follows:

\( x(t) \) is ergodic with respect to a certain statistical property, if the time average, expressing that property equals the corresponding ensemble average (i.e. the expected value).

Since in practice not all, but only some statistical parameters of a stochastic process may be of interest, ergodicity may be defined with
respect to these parameters only.

5.4. - Probability distribution functions

A cumulative probability distribution function expresses the probability that a random variable \( x(t) \) is smaller or equal than a specified value \( x_1 \):

\[
\text{Prob}(x(t) \leq x_1) = F(x_1)
\]  

(5.4.1.)

Some important properties of the cumulative distribution function \( F(x) \) are:

\[
\begin{align*}
\text{a} & \quad F(-\infty) = 0; \quad F(\infty) = 1 \\
\text{b} & \quad F(x) \text{ is a non-decreasing function of } x: \\
& \quad F(x_1) \leq F(x_2) \quad \text{for } x_1 < x_2
\end{align*}
\]  

(5.4.2.)

(5.4.3.)

The derivative of the distribution function \( F(x) \) is called the probability density distribution function or probability density distribution, \( p(x) \):

\[
p(x) = \frac{dF(x)}{dx}
\]  

(5.4.4.)

The probability density distribution is used throughout this report. In various studies, concerning atmospheric turbulence and its influence on aircraft motions, e.g. Refs. 7 and 8, turbulence is considered Gaussian i.e. the probability density distributions of the turbulence velocities are supposed to be Gaussian:

\[
p(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp \left[ -\frac{(x - \mu)^2}{2\sigma_x^2} \right]
\]  

(5.4.5.)

where:

- \( x \) = gust velocity
- \( \sigma_x \) = standard deviation of the gust velocity
- \( p(x) \) = probability density distribution
- \( \mu \) = mean value of the random variable \( x \)
Experimental data on atmospheric turbulence, Refs. 1 and 2 indicate turbulence to be significantly non-Gaussian and emphasis is put in this report on some non-Gaussian aspects and their influence on the way a pilot senses the atmospheric turbulence. Departures from the Gaussian or normal behaviour can - as far as the probability of occurrence is concerned - be characterized by the so-called central moments, discussed in the next Section.

5.5. - Central moments

A more complete specification of the statistics of a random variable, such as a gust velocity, is possible by its moments $M_n$ defined by, see Ref. 9:

$$M_n = E(x^n) = \int_{-\infty}^{+\infty} x^n p(x) \, dx \quad n = 1, 2, 3 \ldots \quad (5.5.1.)$$

It can be seen from eq. 5.5.1. that the first order moment is the mean value of the random variable $x$.

$$M_1 = E(x) = \mu \quad (5.5.2.)$$

If the constants $M_n$ are finite and known up to any order $n$, they specify uniquely the probability density function $p(x)$, see Ref. 9.

The constants $m_n$ defined by:

$$m_n = E((x - \mu)^n) = \int_{-\infty}^{+\infty} (x - \mu)^n p(x) \, dx \quad (5.5.3.)$$

are called the central moments.

Hence:

$$m_1 = 0; \quad m_2 = \sigma_x^2 \quad (5.5.4.)$$

In the case of a Gaussian or normally distributed random variable, the central moments are, see Ref. 9:
\[ m_n = E((x - \mu)^n) = 1.3.5. \ldots (n - 1) \sigma^n_x \]

for \( n = \text{even} \)

\[ m_n = 0 \]

for \( n = \text{odd} \) \hspace{1cm} (5.5.5.)

As can be seen from eq. (5.5.5.), the Gaussian distributed variable is completely determined by its first and second order central moments as far as the probability distribution is concerned.

If the distribution of a random variable departs from the Gaussian distribution, as is the case with atmospheric turbulence velocities, the higher order central moments may be used in addition to characterize the deviations from normality. Therefore two parameters are defined.

The first one is called "skewness":

\[ S = \frac{(m_3)^2}{(m_2)^3} \] \hspace{1cm} (5.5.6.)

and the second and more important one is called "kurtosis":

\[ K = \frac{m_4}{(m_2)^2} \] \hspace{1cm} (5.5.7.)

If the skewness differs from zero, the density distribution function is not symmetric about the mean value.

However, knowledge of the skewness gives almost no clue as to the shape of the distribution, i.e. a density distribution can have a skewness, \( S = 0 \), while the density distribution is far from symmetric, see Ref. 10.

Atmospheric turbulence velocities will be assumed to possess symmetrical distributions.

Lateron in this report a particular class of probability density distribution functions departing from the Gaussian distribution function will be defined. It will be shown that the kurtosis can be used to characterize the deviations from normality for this particular class of distribution functions. Kurtosis values higher than 3, the value in the Gaussian case, will lead to a more "peaky" density distribution function, see Fig. 2.
Those aspects of turbulence known as "patchiness" are reflected in a more "peaky" measured probability density distribution and kurtosis values higher than 3, see e.g. Ref. 1. Therefore the kurtosis may reflect at least partially, patchy characteristics of atmospheric turbulence. Central moments higher than the fourth order moment, derived experimentally, are more and more sensitive to sampling fluctuations. As a consequence the values computed from observations are subject to large margins of error. Therefore in practice they are rarely used, although in principle they could specify the shape of an unknown distribution function to a larger extent.

Hitherto spatial or temporal sequence properties of a stochastic variable have been disregarded. These properties will be the subject of the next Section.

5.6. - Covariance and power spectral density

A second way to look at stochastic processes is spectral analysis. Here sequence properties in time are considered. In general two realizations, \( x_1(t) \) and \( x_2(t) \) are considered from the entire stochastic process \( x(t) \). The basic function to be considered here is the correlation function:

\[
R_{x_1x_2}(t_1,t_2) = E\{x_1(t_1) \cdot x_2(t_2)\} \quad (5.6.1)
\]

In general the correlation function is a function of \( t_1 \) and \( t_2 \). Under the assumption of stationarity, see Section 5.2., the correlation function is only dependent on \( t_2 - t_1 = \tau \). Hence:

\[
R_{x_1x_2}(\tau) = E\{x_1(t + \tau) \cdot x_2(t)\} \quad (5.6.2)
\]

If \( x_2(t) \) is replaced by \( x_1(t) \), the function \( R_{x_1x_1}(\tau) \) is called auto-correlation function, while \( R_{x_1x_2}(\tau) \) is called the cross-correlation function. If the expected value \( E\{x_1(t + \tau) \cdot x_2(t)\} \) is taken with respect to the mean value, the function is called the covariance function of \( x_1(t) \) and \( x_2(t) \): \( C_{x_1x_2}(\tau) \). The auto-covariance function is
thus defined as:

\[ C_{x_1x_1}(\tau) = E \left[ (x_1(t + \tau) - \mu_1) (x_1(t) - \mu_1) \right] \quad (5.6.3.) \]

and the cross-covariance function as:

\[ C_{x_1x_2}(\tau) = E \left[ (x_1(t + \tau) - \mu_1) (x_2(t) - \mu_2) \right] \quad (5.6.4.) \]

The covariance-function expresses the average relation between two random variables at two instants in time separated by \( \tau \). From eqs. (5.6.2.), (5.6.3.) and (5.6.4.), it appears that the correlation-function and covariance-function are related by:

\[ C_{x_1x_1}(\tau) = R_{x_1x_1}(\tau) - \mu_1^2 \quad (5.6.5.) \]

and:

\[ C_{x_1x_2}(\tau) = \kappa_{x_1x_2}(\tau) - \mu_1 \mu_2 \quad (5.6.6.) \]

In ergodic stochastic processes the definitions given above can be written as, see Ref. 9:

\[ C_{x_1x_1}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} \left[ (x_1(t + \tau) - \mu_1) (x_1(t) - \mu_1) \right] dt \quad (5.6.7.) \]

and:

\[ C_{x_1x_2}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} \left[ (x_1(t + \tau) - \mu_1) (x_2(t) - \mu_2) \right] dt \quad (5.6.8.) \]

Some important properties of the above functions are the following, see Ref. 9:

- If \( x_1(t) \) is real, then \( C_{x_1x_1}(\tau) \) is real and even:

\[ C_{x_1x_1}(\tau) = C_{x_1x_1}(-\tau) \quad (5.6.9.) \]
and \( C_{x_1 x_1}(\tau) \) has a maximum at the origin:

\[
|C_{x_1 x_1}(\tau)| \leq C_{x_1 x_1}(0) \tag{5.6.10.}
\]

The maximum is equal to the variance \( \sigma^2_x \):

\[
C_{x_1 x_1}(0) = \mathbb{E}\left[ \left( X_1(t) - \mu_1 \right)^2 \right] = \sigma^2_x \tag{5.6.11.}
\]

The covariance-function is the key to another basic feature in stochastic processes, the power spectral density function or power spectrum.

The power spectral density function \( \Phi_{x_1 x_2}(\omega) \) of a process \( x(t) \) is defined as the Fourier-transform of the covariance function.

There are two power spectral density functions to be defined. The auto-

power spectral density function:

\[
\Phi_{x_1 x_1}(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} C_{x_1 x_1}(\tau) e^{-j\omega \tau} d\tau \tag{5.6.12.}
\]

and similarly the cross-power spectral density function:

\[
\Phi_{x_1 x_2}(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} C_{x_1 x_2}(\tau) e^{-j\omega \tau} d\tau \tag{5.6.13.}
\]

From the Fourier inversion formula follows \( C_{x x}(\tau) \) expressed in terms of \( \Phi_{x x}(\omega) \):

\[
C_{x_1 x_1}(\tau) = \frac{1}{2} \int_{-\infty}^{+\infty} \Phi_{x_1 x_1}(\omega) e^{j\omega \tau} d\omega \tag{5.6.14.}
\]

and similarly:

\[
C_{x_1 x_2}(\tau) = \frac{1}{2} \int_{-\infty}^{+\infty} \Phi_{x_1 x_2}(\omega) e^{j\omega \tau} d\omega \tag{5.6.15.}
\]

Since \( C_{x_1 x_1}(\tau) \) and \( \Phi_{x_1 x_1}(\omega) \) are real functions of \( \tau \) and \( \omega \) respectively, the following relations hold, see e.g. Ref. 11:
\[
\Phi_{x_1x_1}(\omega) = \frac{2}{\pi} \int_0^\infty C_{x_1x_1}(\tau) \cos \omega \tau \, d\tau \tag{5.6.16.}
\]

and:
\[
C_{x_1x_1}(\tau) = \int_0^\infty \Phi_{x_1x_1}(\omega) \cos \omega \tau \, d\tau \tag{5.6.17.}
\]

with \( \tau = 0 \), eq. (5.6.14.), reduces to:
\[
C_{x_1x_1}(0) = \int_0^\infty \Phi_{x_1x_1}(\omega) \, d\omega = \sigma_x^2 \tag{5.6.18.}
\]

The total area enclosed by the function \( \Phi_{x_1x_1}(\omega) \) is non-negative and equals the average power of the process \( x(t) \).

Thus the variance of a random variable can be calculated in principle in two ways: by means of a calculation of the second order moment, or by taking the integral value over a measured power spectral density.

Finally a basic theorem in linear system theory has to be mentioned. Consider a linear system having a transfer function \( H(j\omega) \). If the stationary stochastic input signal, \( x(t) \), has an auto-power spectral density \( \Phi_{xx} \), the output signal, \( y(t) \), has an auto-power spectral density \( \Phi_{yy}(\omega) \) given by, Ref. 9:
\[
\Phi_{yy}(\omega) = |H(j\omega)|^2 \Phi_{xx}(\omega) \tag{5.6.19.}
\]

The cross-power spectral density \( \Phi_{yx}(\omega) \) between input and output of a linear system is expressed by:
\[
\Phi_{yx}(\omega) = H(j\omega) \Phi_{xx}(\omega) \tag{5.6.20.}
\]

where \( \Phi_{yx}(\omega) \) is the complex cross-power spectral density function between the output and input signals. It should be remarked that the sequence \( y-x \) is important here, since:
\[ \Phi_{xy}(\omega) = H^*(j\omega) \Phi_{xx}(\omega) \]  \hspace{1cm} (5.6.21.)

where \( H^*(j\omega) \) is the complex conjugate of \( H(j\omega) \), see Ref. 9.
6. - Atmospheric Turbulence

In the description of atmospheric turbulence, the tools of probability theory and spectral analysis, discussed in the preceding Section are useful in more than one respect. Knowledge of the probability density distribution and power spectral density, obtained from analyzing experimental data, may be used for turbulence description and modelling. Simultaneously this knowledge has an important influence on theoretical studies concerning the physical structure of the turbulent motion, which is not fully understood at the moment.

In this Section first of all some particular assumptions often made about atmospheric turbulence are dealt with in addition to the general assumptions of ergodicity and stationarity of the preceding Section. The scope and validity of the assumptions made are verified in Section 6.3. Next, the classical spectral description of atmospheric turbulence is presented and finally patchy characteristics are described as an introduction to a proposed model for patchy turbulence.

6.1. - Homogeneity and Isotropy

The statistics of atmospheric turbulence are in general dependent on time as well as on the spatial coordinates. The assumption of stationarity, see Section 5.2., implies that the joint distribution function characterizing the process is invariant with respect to a time-shift. Homogeneity and isotropy refer to the spatial properties. Homogeneity implies that the statistics are invariant under spatial translation of the frame of reference. Isotropy implies that the statistics of the turbulence velocity components at a particular point, defined in relation to a particular set of axes are unaltered if the axes of the reference-frame are rotated or reflected in a plane through the origin, see Ref. 12. As an immediate result of isotropy, cross-correlations between the turbulence velocities in different directions are zero, see Ref. 12. Furthermore due to isotropy the variances of the turbulence velocities in different directions are equal.
\[
\overline{u^2} = \overline{v^2} = \overline{w^2} = \sigma^2
\]  
(6.1.1.)

6.2. - Taylor's hypothesis

Taylor's hypothesis is the well-known hypothesis concerning the "frozen atmosphere". Most aircraft fly at speeds high as compared to the turbulence velocities. The velocities at a certain point in space are therefore assumed not to change with time during the short interval in which the aircraft is affected by the velocities at that point. This hypothesis offers the possibility to relate spatial covariance functions and spatial frequencies to covariance functions and frequencies in the time domain. With respect to the velocity of the aircraft the following frequently used relations are valid under Taylor's hypothesis:

\[
\Delta x = V \tau
\]  
(6.2.1.)

where \( \Delta x = \) distance between two points in space

\( V = \) velocity of the aircraft

\( \tau = \) time lag, time taken by the aircraft to cover the distance \( \Delta x \)

In the frequency domain the spatial frequency \( \Omega \) may be replaced by the frequency \( \omega \):

\[
\omega = \Omega V
\]  
(6.2.2.)

where \( \omega = \) rad/sec

\( \Omega = \) rad/m

6.3. - Limits of validity of the assumptions

It may be obvious that the assumption of stationarity of atmospheric turbulence can be satisfied only in areas of constant meteorological conditions. When turbulence is stationary and homogeneous it is also ergodic. However the fact that turbulence is patchy in nature reflects
a lack of homogeneity. At high altitudes turbulence appears to occur in large "patches", which may be effectively homogeneous within a "patch" but showing differences between the "patches", see also Ref. 13. The condition of isotropy may be fulfilled to a reasonable extent at high altitudes. At low altitudes turbulence turns out to be anisotropic due to the proximity of the earth's boundary layer and to varying terrain roughness conditions. An isotropic turbulence model is therefore not ideal to represent atmospheric turbulence for the take-off and landing conditions. In Ref. 14 a turbulence model is presented for the approach and landing phases, taking some aspects of anisotropy into account. In VTOL and STOL flight conditions Taylor's hypothesis seems to be less applicable as the aircraft velocity may be of the same order of magnitude as the local windspeed.

6.4. - Spectral equations
In early turbulence research attention generally was focussed on the problem of determining the energy density spectra and covariance-functions of atmospheric turbulence, e.g. see Refs. 12, 15 and 16. Confirmed by theoretical considerations, Ref. 16, and more recent turbulence measurement programs, Refs. 2 and 17, the spectral equations due to von Kármán seemed to fit the measured spectra well, although the spectra due to Dryden are frequently used because of their rational form.

The spectral functions and their corresponding spatial correlation functions due to von Kármán are, see Refs. 13 and 16:

\[ \Phi_{u_u}(\Omega) = \frac{2}{\pi} \sigma_{u_u}^2 L_{u_u} \frac{1}{\left[1 + (1.339 L_{u_u} \Omega)^2\right]^{5/6}} \] (6.4.1.)

Auto-covariance function:
\[ C_{u_g u_g}(\Delta x) = \sigma_{u_g}^2 \frac{2^{2/3}}{\Gamma(1/3)} \left( \frac{\Delta x}{1.339 \, L_{u_g}} \right)^{1/3} K_{1/3} \left( \frac{\Delta x}{1.339 \, L_{u_g}} \right) \]  \hspace{1cm} (6.4.2.)

where \( \Gamma(\ ) = \text{gamma function} \), see Ref. 18

\[ K_i(\ ) = \text{Modified Bessel function of the second kind of the order } i, \text{ see Ref. 18} \]

b Auto-power spectrum, vertical turbulence velocity, \( w_g \):

\[ \phi_{w_g w_g}(\Omega) = \frac{1}{\pi} \sigma_{w_g}^2 \frac{1}{L_{w_g}} \frac{1 + 8/3(1.339 \, L_{w_g} \, \Omega)^2}{\left[ 1 + (1.339 \, L_{w_g} \, \Omega)^2 \right]^{11/6}} \]  \hspace{1cm} (6.4.3.)

Auto-covariance function:

\[ C_{w_g w_g}(\Delta x) = \sigma_{w_g}^2 \frac{2^{2/3}}{\Gamma(1/3)} \left( \frac{\Delta x}{1.339 \, L_{w_g}} \right)^{1/3} \left[ K_{1/3} \left( \frac{\Delta x}{1.339 \, L_{w_g}} \right) \right. \]

\[ - \frac{1}{2} \frac{\Delta x}{1.339 \, L_{w_g}} \cdot K_{2/3} \left( \frac{\Delta x}{1.339 \, L_{w_g}} \right) \]  \hspace{1cm} (6.4.4.)

c The auto-power spectral density and auto-covariance functions of the lateral turbulence velocity, \( v_g \), are identical to corresponding functions of the vertical velocity.

Employing Taylor's hypothesis, see Section 6.2., the spectral functions and their corresponding correlations in the time domain become:

aa Auto-power spectrum, longitudinal turbulence velocity, \( u_g \):

\[ \phi_{u_g u_g}(\omega) = \frac{2}{\pi} \sigma_{u_g}^2 \frac{L_{u_g}}{v} \frac{1}{\left[ 1 + (1.339 \, L_{u_g} \, \omega)^2 \right]^{5/6}} \]  \hspace{1cm} (6.4.5.)

Auto-covariance function:

\[ C_{u_g u_g}(\tau) = \sigma_{u_g}^2 \frac{2^{2/3}}{\Gamma(1/3)} \left( \frac{v|\tau|}{1.339 \, L_{u_g}} \right)^{1/3} K_{1/3} \left( \frac{v|\tau|}{1.339 \, L_{u_g}} \right) \]  \hspace{1cm} (6.4.6.)
Auto-power spectrum, vertical turbulence velocity, \( w_g \):

\[
\phi_{w_g w_g}(\omega) = \frac{1}{\pi} \sigma_{w_g}^2 \frac{L_{w_g}}{V} \left( \frac{1 + 8/3 \left( \frac{L_{w_g}}{V} \omega \right)^2}{\left[ 1 + \left( \frac{L_{w_g}}{V} \omega \right)^2 \right]^{11/6}} \right)
\]

(6.4.7)

Auto-covariance function:

\[
C_{w_g w_g}(\tau) = \sigma_{w_g}^2 \frac{V |\tau|}{\Gamma(1/3)} \left( \frac{1.339}{L_{w_g}} \right)^{1/3} \cdot \left[ \frac{V |\tau|}{\Gamma(1/3)} - \frac{1}{2} \frac{V |\tau|}{L_{w_g}} K_{2/3} \left( \frac{V |\tau|}{1.339 L_{w_g}} \right) \right]
\]

(6.4.8)

Unfortunately the spectral expressions due to von Karman are not very convenient in real-time simulation, because they cannot be generated exactly from white noise by linear filtering. More convenient, although experimentally and theoretically less supported are the spectra due to Dryden, see Refs. 15 and 19.

Under Taylor's hypothesis the spectra can be expressed alternatively:

aa Auto-power spectral density, longitudinal turbulence velocity, \( u_g \):

\[
\phi_{u_g u_g}(\omega) = \frac{2}{\pi} \sigma_{u_g}^2 \frac{L_{u_g}}{V} \frac{1}{1 + \frac{L_{u_g}^2}{V^2} \omega^2}
\]

(6.4.9)

Auto-covariance function:

\[
C_{u_g u_g}(\tau) = \sigma_{u_g}^2 \exp \left[ -|\tau| \frac{V}{L_{u_g}} \right]
\]

(6.4.10)

bb Auto-power spectral density, vertical turbulence velocity, \( w_g \):
\[ \Phi_{w_g w_g}(\omega) = \frac{1}{\pi} \sigma_{w_g}^2 \frac{2 L_{w_g}}{V} \frac{1 + 3 \frac{L_{w_g}}{V^2} \omega^2}{\left[ 1 + \frac{L_{w_g}}{V^2} \omega^2 \right]^2} \] \hspace{1cm} (6.4.11.)

Auto-covariance function:

\[ C_{w_g w_g}(\tau) = \sigma_{w_g}^2 \exp\left[-|\tau| \frac{V}{L_{w_g}}\right] \left(1 - \frac{1}{2} \frac{V}{L_{w_g}} |\tau| \right) \] \hspace{1cm} (6.4.12.)

The spectrum and covariance-functions of the lateral turbulence velocity, \( v_g \), are identical again to the corresponding functions of the vertical velocity.

The differences between the von Kármán spectral form and the Dryden form manifest themselves at large values of the frequency \( \omega \) in terms of the asymptotic behaviour of the corresponding power spectral density functions, see Figs. 3a and 3b. On a logarithmic scale the von Kármán spectral equations show a slope of \(-5/3\) as a function of increasing \( \omega \), whereas the Dryden spectral functions show a slope of \(-2\). If the turbulence signals are to be obtained by passing white noise signals through linear filters, the Dryden spectral model can be considered as a first approximation of the von Kármán spectral model. The errors resulting from the use of linear filters can be reduced by adding lead-lag pairs (poles and zero's) to the filter transfer-functions. A complete description of such a filtering technique is presented in Ref. 14. In view of the complexity of the system to be described in Sections 8 and 9 this refined filtering technique has not been considered in this report.

In isotropic turbulence cross-covariances between the turbulence velocities of two directions are zero. Due to local anisotropy, cross-covariances may be expected to exist at low altitudes. Unfortunately
these cross-covariances are extremely hard to model, since the cross-
covariances appear to be highly dependent on characteristics of the
earth's surface and local meteorological conditions. A cross-spectrum
between $u_g$ and $w_g$, based on experimental data is presented in Ref. 13.
Although cross-spectra are not taken into account in the following
Sections; the simulation technique to be described in this report does
not basically exclude incorporation of non-zero cross-spectra, see also
Ref. 4.

6.5. - Patchy characteristics of atmospheric turbulence

Atmospheric turbulence in contrast with a Gaussian process possesses
a certain element of surprise to the pilot. Areas in the atmosphere
characterized by relative intense turbulence are often followed by more
quiet areas. This phenomenon is known as patchiness or intermittency.
It is sometimes postulated that this phenomenon is (at least partially)
reflected by the measured departures from the Gaussian or normal velo-
city distribution, see Refs. 1 and 20. This would mean that kurtosis
values, see Section 5.5., larger than 3 (Gaussian) reflect the patchiness
of atmospheric turbulence. The larger the deviations are, the more
"intense" the patchiness is, according to this point of view. Kurtosis
values characterizing the deviations of approximately 5.5. have been
measured in flight, see Ref. 1 and Fig. 4 taken from Ref. 1. A typical
measured probability distribution of the turbulence velocity, $w_g$, is
compared to the Gaussian distribution in Fig. 5.

In most simulator experiments the real-time simulation of turbulence
velocities is based on the spectral equations only, e.g. see Refs.
21 and 22. The synthetic turbulence velocities are Gaussian distributed,
neglecting the patchiness mentioned above. Pilots often complain that
the turbulence thus simulated is too "regular" and lacks the element
of surprise. Since the above described features are recognized as being
essential to the accuracy of the conclusions drawn from simulator
studies, a number of attempts have recently been made to simulate
patchy characteristics, see Refs. 3, 4 and 23.

In the next Section, patchiness is considered in mathematical terms
and parameters are defined to characterize the degree of patchiness. Furthermore a statistical technique is offered which transcends those now available to assess to a certain extent significant aspects of atmospheric turbulence.
7. - Description of a model of patchiness

Hitherto the structure of atmospheric turbulence has been described mathematically in terms of the power spectral density functions, see Section 6.4., and the probability density function of the turbulence velocities. The departures of the turbulence velocities from Gaussian or normal behaviour can be characterized by the fourth order central moment or kurtosis value. The question arises whether the fourth order central moment is a sufficient parameter to describe the patchy characteristics.

Returning to the pilot's point of view, another question to be answered is, how patchiness as sensed by the pilot might be expressed more explicitly in mathematical terms.

As stated in the Introduction, patchiness can be considered as caused by local regions of relatively higher energy concentrations, appearing in a measured turbulence record. Therefore, it is assumed in this report that the pilot bases his opinion concerning the degree of patchiness on his memory of the variations of the instantaneous energy during the immediately preceding, relatively short period of time. This point of view leads to the concept of a "memory" filter to be described in the next Section.

7.1. - The memory filter concept

In further elaborating the above verbal description of the manner in which the pilot assesses patchiness, a plausible representation of the instantaneous energy seems to be the square of the turbulence velocity. If a turbulence velocity is represented by the random variable $w(t)$, the square of the turbulence velocity is $w^2(t)$. In Fig. 6 two possible examples of $w(t)$ and the associated $w^2(t)$ are given.

The pilot's memory of the variations of the instantaneous energy during an arbitrary period of time may be expressed mathematically as follows. The representation of the instantaneous energy, $w^2(t)$ is multiplied by a weighting function or "window", see Fig. 7, representing the pilot's
memory. The product is integrated with respect to time. The resulting signal is \( z(t) \):

\[
z(t) = \int_0^\infty w^2(t - \nu) h(\nu) \, d\nu \tag{7.1.1}
\]

which in fact is a convolution in the time domain.

Let \( h(\nu) \) have the form shown in Fig. 7:

\[
h(\nu) = \begin{cases} 
\frac{1}{\tau} \exp\left[-\frac{\nu}{\tau}\right] & \nu \geq 0 \\
0 & \nu < 0
\end{cases} \tag{7.1.2}
\]

This function \( h(\nu) \) suggests the pilot's memory to decay exponentially backwards in time. Although other forms of the weighting function might be conceived, it will become clear that the exact shape of \( h(\nu) \) is not of great importance to the final results of the arguments given in Section 7.2.

In the frequency domain the exponential weighting function is represented by a simple linear filter, characterized by the transfer function, see Fig. 7:

\[
H(j\omega) = \frac{1}{1 + j\omega\tau} \tag{7.1.3}
\]

From Fig. 7, it may be seen that due to this "memory" filter the individual bumps expressed by the instantaneous intensity, \( w^2(t) \), will be "forgotten" some time after their occurrence. The time constant \( \tau \) of the filter may be considered as a measure of the "length" of the pilot's memory.

If the turbulence is not patchy but more or less continuous such as normally distributed turbulence, the output of the memory filter \( z(t) \) will be relatively constant. If, however, turbulence is patchy the out-
put of the memory filter can be expected to react on the individual patches of turbulence. As a consequence the variance \( \sigma_z^2 \) of the fluctuations of \( z(t) \) about its mean value will be larger than in case of non-patchy turbulence.

If the pilot indeed assesses the patchiness of the turbulence on the basis of his memory of the variations of the instantaneous intensity over some preceding time interval then it is the variance \( \sigma_z^2 \) of the "memory" filter output \( z(t) \) - as just described - that can serve as the parameter to express his assessment of the patchiness.

To determine the characteristics of \( \sigma_z^2 \), the power spectral density of the squared patchy turbulence velocity, \( w^2(t) \) will be studied in the following.

Consider the power density of the squared signal, \( w^2(t) \).

From the basic properties of the power spectral density function, see Section 5.6., some interesting properties with respect to the power spectral density of the squared velocity may be concluded. The power spectral density of the turbulence velocity, \( w(t) \), represents, speaking very loosely, a decomposition of the velocity record into a sum or integral of harmonic functions. An important property is, according to eq. (5.6.18.):

\[
\sigma_w^2 = \int_{0}^{\infty} \phi_{ww}(\omega) \, d\omega
\]

(7.1.4.)

Thus, the integrand \( \phi_{ww}(\omega) \) \( d\omega \) indicates how much of the variance \( \sigma_w^2 \) is contributed by harmonic functions, having frequencies in the interval \( (\omega_1, \omega_1 + d\omega) \).

It can be shown, see Ref. 1 and Appendix 4, that:

\[
(K - 1) \, \sigma_w^4 = \int_{0}^{\infty} \phi_{w^2w^2}(\omega) \, d\omega
\]

(7.1.5.)

where \( K \) is the kurtosis value of \( w(t) \).

The term \( (K - 1) \, \sigma_w^4 \) is, according to eq. (7.1.4.), the variance of the
stochastic process \( w^2(t) \). Thus the variance of \( w^2(t) \) is equal to the fourth order central moment of \( w(t) \) minus the mean value of \( w^2(t) \), which is not equal to zero, see Appendices 2 and 4. Furthermore, the quantity \( \Phi_{w^2w^2}(\omega) \right) d\omega \) indicates how much of the kurtosis value or - by means of eq. (5.5.7.) - how much of the fourth order central moment is contributed by harmonic functions having frequencies in the interval \( (\omega_1, \omega_1 + d\omega) \). Although patchiness may be expressed to a certain extent by the fourth order central moment or kurtosis, the power spectral density of the squared velocity provides additional information about patchiness.

Evidently, by giving the kurtosis value and the variance of \( w(t) \), the shape of the power spectral density of \( w^2(t) \) is not determined.

Different shapes of the second order power spectral density - \( \Phi_{w^2w^2}(\omega) \) - may exist, leading to different forms of patchy characteristics as sensed by the pilot through his "memory" filter, although the kurtosis value and the variance of \( w(t) \) may remain constant. The additional information about patchiness, expressed through a certain shape of the second order power spectral density is studied further by employing the memory filter concept.

From eq. (5.6.19.) and eq. (7.1.4.), it follows that under the condition of stationarity the variance of the output of the memory filter, \( \sigma_z^2 \) can be calculated from:

\[
\sigma_z^2 = \int_0^\infty \Phi_{zz}(\omega) \, d\omega = \int_0^\infty \Phi_{w^2w^2}(\omega) \cdot |H(j\omega)|^2 \cdot d\omega \quad (7.1.6.)
\]

The memory filter serves as a low-pass filter of the \( w^2(t) \) signal.

If the time-constant \( \tau \) in the memory filter, see eq. (7.1.3.) tends to zero, \( H(j\omega) \) approaches unity, the value of \( \sigma_z^2 \) tends to the integral value of the power spectral density of \( w^2(t) \), which is \( (K-1) \sigma_w^4 \).

If \( \tau \) is taken to infinity, the output variance \( \sigma_z^2 \) of \( z(t) \) tends to zero, since the process is assumed to be stationary.

In the next Section a patchiness parameter, \( P_\tau \), will be defined expressing both the influence of the kurtosis \( K \) and the patchy characteristics contained in the shape of the power spectral density of \( w^2(t) \).
7.2. - The patchiness parameter $P_T$

As mentioned before in Section 6.5., patchiness appears to increase if the kurtosis value $K$ increases from the value 3 in the Gaussian case. Consequently, taking into account pilot's comments, patchiness is supposed to be minimal in the case of a purely Gaussian or normal turbulence velocity distribution. Furthermore, if a turbulence process is purely Gaussian, the auto-correlation function of the squared turbulence velocity $w^2(t)$ and consequently the shape of the power spectral density of $w^2(t)$ is completely determined by the auto-correlation function of $w(t)$, see Ref. 9.

It seems therefore reasonable to use the behaviour of $\sigma_z^2(\tau)$ in the Gaussian case as a base-line or reference level in defining a patchiness parameter. A dimensionless figure for the patchiness is thus proposed as the ratio of $\sigma_z^2(\tau)$ non-normal and $\sigma_z^2(\tau)$ normal:

$$P_T = \frac{\sigma_z^2(\tau) \text{ non-normal}}{\sigma_z^2(\tau) \text{ normal}} \tag{7.2.1.}$$

The behaviour of $P_T$ is illustrated in Figs. 22 and 23 as obtained from simulated patchy turbulence, see Section 8 and 9.

The choice of the shape of the memory filter in Section 7.1. is somewhat arbitrary. However, if $\tau$ increases - the break-off frequency of the filter decreases - then the ratio $P_T$ will be relatively independent of the shape of the memory filter, provided $\tau$ is chosen sufficiently large.

It might therefore be argued that a sensible choice for the time-constant would be $\tau = \infty$, yielding:

$$P_\infty = \frac{\sigma_z^2(\tau = \infty) \text{ non-normal}}{\sigma_z^2(\tau = \infty) \text{ normal}} \tag{7.2.2.}$$

as the final measure of patchiness. In accepting the choice of $\tau = \infty$, the influence of the shape of the weighting function $h(v)$ on the patchiness parameter $P$ is in fact entirely eliminated.
If the time-constant $\tau$ tends to infinity, the value of $P$ tends in the limit to the ratio of the power spectral densities at $\omega = 0$, see Appendix 5:

$$P_\infty = \frac{\Phi_{w^2w^2}(\omega = 0)_{\text{non-normal}}}{\Phi_{w^2w^2}(\omega = 0)_{\text{normal}}} \quad (7.2.3.)$$

In view of the use of $\sigma_z^2(\tau)_{\text{normal}}$ as a reference in the patchiness parameter $P_\tau$, the behaviour of $\sigma_z^2(\tau)_{\text{normal}}$ will be the subject of the next Section. Both a Dryden spectral density of the turbulence velocity $w(t)$ and a von Kármán spectral density of $w(t)$ are considered.

7.3. - The variance of the memory filter output, $\sigma_z^2(\tau)$, Gaussian turbulence

The variance of the memory filter output, $\sigma_z^2(\tau)$, can be calculated employing eq. (7.1.6.). However, the exact shape of $\Phi_{w^2w^2}(\omega)$ is needed to perform the calculation analytically. If the turbulence is purely Gaussian, the degree of patchiness is minimal according to the description presented in Section 6.5. In this case the auto-correlation function of $w^2(t)$ is completely determined by the auto-correlation function of $w(t)$. Since the turbulence velocity $w(t)$ is assumed to be a stationary Gaussian process, $w^2(t)$ will also be stationary, see Ref. 9.

The auto-correlation function of $w^2(t)$, $R_{w^2w^2}(\tau)$ will read, see Ref. 9:

$$R_{w^2w^2}(\tau) = E\{w^2(t + \tau) \cdot w^2(t)\} \quad (7.3.1.)$$

The expected value is further elaborated to:

$$E\{w^2(t + \tau) \cdot w^2(t)\} = E\{w^2(t + \tau)\} \cdot E\{w^2(t)\} + 2E\{w(t + \tau) \cdot w(t)\} \quad (7.3.2.)$$

as derived in Ref. 9.

The mean value $u_{w^2}$ of the process $w^2(t)$ is:
\[ E\{w^2(t)\} = R_{w,w}(0) = \sigma_w^2 \]  
\hspace{10cm} (7.3.3.)

Employing the definition of the correlation function eq. (5.6.2.) and eq. (7.3.3.), eq. (7.3.1.) can be written as:

\[ R_{w^2,w^2}(\tau) = R_{w,w}^2(0) + 2R_{w,w}(\tau) \]  
\hspace{10cm} (7.3.4.)

The mean value of the process \( w^2(t) \) is not equal to zero. Therefore the auto-covariance function of \( w^2(t) \), \( C_{w^2,w^2}(\tau) \), is different from the auto-correlation function.

Employing eq. (5.6.5.), the auto-covariance function \( C_{w^2,w^2}(\tau) \) is:

\[ C_{w^2,w^2}(\tau) = R_{w^2,w^2}(\tau) - \mu_{w^2}^2 \]  
\hspace{10cm} (7.3.5.)

Substituting eq. (7.3.4.) in eq. (7.3.5.) and employing eq. (7.3.3.) yields:

\[ C_{w^2,w^2}(\tau) = 2R_{w,w}^2(\tau) \]  
\hspace{10cm} (7.3.6.)

The Fourier transform of eq. (7.3.6.) according to eq. (5.6.16.) yields the power spectral density of \( w^2(t) \):

\[ \Phi_{w^2,w^2}(\omega)_{\text{normal}} = \frac{2}{\pi} \int_0^\infty 2R_{w,w}(\tau) \cos \omega \tau \, d\tau \]  
\hspace{10cm} (7.3.7.)

Applying the frequency convolution theorem, see Ref. 11, the same result can be obtained directly from the power spectral density function of \( w(t) \), \( \Phi_{w,w}(\omega) \):

\[ \Phi_{w^2,w^2}(\omega)_{\text{normal}} = \int_{-\infty}^{+\infty} \Phi_{w,w}(\lambda) \Phi_{w,w}(\omega - \lambda) \, d\lambda \]  
\hspace{10cm} (7.3.8.)

Substituting the Dryden equations for the auto-covariance functions, the integral of eq. (7.3.7.) can be calculated analytically. The result
is, see Appendix 1:

a. Horizontal turbulence velocity, \( u_g \):

\[
\phi_{u_g} \frac{u_g^2}{L_{u_g}} \text{normal} = \frac{8}{\pi} \sigma_{u_g}^4 \frac{v/L_{u_g}}{\frac{v^2}{L_{u_g}} + \omega^2}
\]

(7.3.9.)

b. Vertical turbulence velocity, \( w_g \):

\[
\phi_{w_g} \frac{w_g^2}{L_{w_g}} \text{normal} = \frac{4}{\pi} \sigma_{w_g}^4 \frac{v}{L_{w_g}} \left[ \frac{20 v^4}{L_{w_g}^4} + 13 \frac{v^2 \omega^2}{L_{w_g}^2} + 3 \omega^4 \right]
\]

(7.3.10.)

c. Lateral turbulence velocity, \( v_g \).

The spectrum of the squared lateral turbulence velocity is identical to the spectrum of the vertical turbulence velocity.

The power spectral density functions, expressed by eqs. (7.3.9.) and (7.3.10.) are illustrated in Figs. 8 and 9 respectively.

By integrating the power spectral density of the squared normal turbulence velocity, the value of \( (K - 1) \sigma_w^4 \) is obtained. Thus:

\[
(K - 1) \sigma_w^4 = \int_0^\infty \phi_{w} w^2(\omega) \text{normal} \, d\omega = 2 \sigma_w^4
\]

(7.3.11.)

where \( w \) replaces the turbulence velocities \( u_g, w_g \) or \( v_g \).

Employing eq. (7.1.6.) the following expressions for the variance of the output of the memory filter are obtained.

a. Horizontal turbulence velocity, \( u_g \):
\begin{equation}
\sigma^2_{zu_g} \text{ normal} = 2\sigma^4_{u_g} \frac{1}{1 + 2 \frac{V}{L_{u_g}} \tau}
\tag{7.3.12.}
\end{equation}

\text{b. Vertical turbulence velocity, } \omega_g:
\begin{equation}
\sigma^2_{\omega_g} (\tau) \text{ normal} = \sigma^4_{\omega_g} \frac{\frac{V^2}{2} \tau^2 + 6 \frac{V^2}{2} \tau + 2}{(2 \frac{V}{L_{\omega_g}} \tau + 1)}
\tag{7.3.13.}
\end{equation}

\text{c. Lateral turbulence velocity, } \epsilon_g.

The expression for } \sigma^2_{\omega_g} \text{ of the output of the memory filter for the }
\text{lateral turbulence velocity, } \epsilon_g \text{ is identical to the expression }
\sigma^2_{\omega_g} (\tau) \text{ of the vertical turbulence velocity, replacing the index }
\omega_g \text{ by } \epsilon_g.

For many applications the spectral equations after von Kármán are
considered to be more accurate, see Section 6.4. Therefore, the behaviour
of the power spectral density of the squared turbulence velocities,
assuming the turbulence to be Gaussian is also calculated. Due to the
somewhat inconvenient expressions for the auto-covariance functions and
the auto-power spectral densities of } u_g(t), \omega_g(t) \text{ or } \epsilon_g(t), \text{ the }
calculations have been performed by means of numerical integration
of eq. (7.3.8.). The differences between the behaviour of the power
spectral density of the squared turbulence velocities; assuming a
Dryden or a von Kármán spectral density are presented in Fig. 8 for the
horizontal turbulence velocity. The differences in behaviour of the power
spectral density of the squared turbulence velocities in the case of
vertical or lateral turbulence are illustrated in Fig. 9.

In addition the behaviour of } \sigma^2_z (\tau) \text{ normal, assuming a von Kármán spectral}
density, is calculated by means of numerical integration of eq. (7.1.6.).
The result is presented in Fig. 10 for the horizontal turbulence velocity
comparing the behaviour of } \sigma^2_{zu_g} (\tau) \text{ normal for a Dryden and a von Kármán}
spectral model. The differences in behaviour of } \sigma^2_z (\tau) \text{ normal for the
vertical or lateral turbulence velocity are illustrated in Fig. 11.

Non-Gaussian turbulence may have different power spectral density functions of the squared velocity. To reach definite conclusions about the behaviour of these power spectral densities, first of all a complete description of the statistics of patchy turbulence velocities as generated by a non-Gaussian simulation technique is presented in Section 8. The calculations concerning the behaviour of $\sigma_z^2(\tau)_{\text{non-normal}}$ and the behaviour of $P_\tau$ are performed in Section 9.
8. - Simulation of patchy turbulence velocities

In Ref. 3 a method is described to generate time-histories of simulated turbulence velocities, possessing prescribed Dryden power spectra as well as having a certain non-normal distribution function. The method consists of a multiplication of two independent Gaussian processes $a(t)$ and $b(t)$, see Fig. 13a.

$$u(t) = a(t) \cdot b(t) \quad (8.1.)$$

The probability density function of the output signal, $u(t)$, of the system is a modified Bessel function of the second kind of the order zero, given in integral representation by, see Ref. 3:

$$K_0(x) = \int_{-\infty}^{+\infty} \exp[-x \cosh(y)] \, dy \quad \text{arg} \, |x| < \frac{\pi}{2} \quad (8.2.)$$

and

$$p(u) = \frac{1}{\pi \sigma_u} K_0 \left( \frac{|u|}{\sigma_u} \right) \quad (8.3.)$$

This probability density function $p(u)$ is compared to the normal distribution function in Fig. 12. The deviation from normality may be characterized by the fourth order central moment, which is obtained by integrating eq. (8.3.) according to eq. (5.5.3.), for $n = 4$. The result is:

$$m_4 = 9 \sigma_u^4 \quad (8.4.)$$

This calculated fourth order central moment, $9 \sigma_u^4$, is much larger than the values of $m_4$ found from measurements of actual atmospheric turbulence, see Section 6.5. and Fig. 4. In Ref. 4 the method of Ref. 3 is extended by adding a third independent Gaussian process to the process $u(t)$ just described, to obtain lower and more realistic fourth order central moments. Thus in addition to linear filtering, the system generating artificial turbulence velocities consists of a multiplication and an
addition of independent Gaussian processes. The system is illustrated by the block-diagram of Fig. 13b. This method of generating patchy turbulence velocities is extended further in this report. The respective power spectral density functions of the compounding normal processes are allowed to vary in such a way as to yield different patchy characteristics of the resulting process. The power spectral density of the resulting process \( w(t) \) however, can remain unaltered. A complete description of the system and its statistical properties is presented in the next Section.

8.1. - Probability distributions and moments

Consider the block diagram of Fig. 13b. The input signals to the linear filters \( F_a \), \( F_b \) and \( F_c \) are independent Gaussian white noise processes. Since a Gaussian process remains Gaussian when passed through a linear filter, see Ref. 9, the output signals \( a(t) \), \( b(t) \) and \( c(t) \) are also Gaussian processes.

Their probability density distributions are respectively:

\[
p_a(x) = \frac{1}{\sigma_a \sqrt{2\pi}} \exp \left[ -\frac{x^2}{2\sigma_a^2} \right]
\]

(8.1.1.)

\[
p_b(x) = \frac{1}{\sigma_b \sqrt{2\pi}} \exp \left[ -\frac{x^2}{2\sigma_b^2} \right]
\]

(8.1.2.)

\[
p_c(x) = \frac{1}{\sigma_c \sqrt{2\pi}} \exp \left[ -\frac{x^2}{2\sigma_c^2} \right]
\]

(8.1.3.)

The process \( u(t) \) is the product of \( a(t) \) and \( b(t) \). The probability density function of \( u(t) \) reads, see Ref. 3:

\[
p_u(x) = \frac{1}{\pi \sigma_a \sigma_b} \int_{-\infty}^{\infty} \exp \left[ -\frac{|x|}{\sigma_a \sigma_b \cosh(y)} \right] dy
\]

(8.1.4.)
where:
\[ \sigma_u = \sigma_a \sigma_b. \] (8.1.5.)

An independent Gaussian process \( c(t) \) is added to the process \( u(t) \).

The probability density function of the result \( w(t) = u(t) + c(t) \) is, see Ref. 4:
\[
p_w(z) = \frac{(1+Q^2)^{\frac{1}{2}}}{\pi \sigma_w} \int_0^\infty \left[ -\frac{2}{1+2\zeta^2 Q^2} \right]^{\frac{1}{2}} \exp \left[ -\zeta^2 - \frac{1}{2} \left( \frac{z}{\sigma_w} \right)^2 \frac{(1+Q^2)}{(1+2\zeta^2 Q^2)} \right] d\zeta \] (8.1.6.)

where:
\[
\sigma_w^2 = \sigma_u^2 + \sigma_c^2
\] (8.1.7.)

The probability density function \( p_w(z) \) actually defines a whole class of probability density functions, depending on the value of \( Q \), where \( Q \) is defined as, see Ref. 4:
\[
Q = \frac{\sigma_a \sigma_b}{\sigma_c} = \frac{\sigma_u}{\sigma_c}
\] (8.1.8.)

which is the ratio of the standard deviations of the processes \( u(t) \) and \( c(t) \). If \( Q \) is set equal to zero then:
\[
p_w(z) = \frac{1}{\sigma_w \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{z}{\sigma_w} \right)^2 \right] \] (8.1.9.)

which is equal to the Gaussian distribution function. If \( Q \) is taken to infinity, \( p_w(z) \) reduces to:
\[
p_w(z) = \frac{1}{\pi \sigma_w} \int_0^\infty \frac{1}{\zeta} \exp \left[ -\zeta^2 - \frac{1}{2} \left( \frac{z}{\sigma_w} \right)^2 \frac{1}{2\zeta^2} \right] d\zeta
\]

substituting \( \zeta = \frac{1}{\sigma_a \sqrt{2}} \) and \( \sigma_w = \sigma_a \sigma_b \) yields:
\[ p_w(z) = \frac{1}{\pi \sigma_a \sigma_b} \int_0^\infty \exp \left[ -\frac{1}{2} \left( \frac{\lambda}{\sigma_a} \right)^2 - \frac{1}{2} \left( \frac{z}{\lambda \sigma_b} \right)^2 \right] \frac{d\lambda}{\lambda} \quad (8.1.10.) \]

Performing the substitution:

\[ \lambda^2 = \frac{\sigma_a}{\sigma_b} |z| e^{-y} \quad (8.1.11.) \]

the result is:

\[ p_w(z) = \frac{1}{\pi \sigma_a \sigma_b} \int_0^\infty \exp \left[ -\frac{|z|}{\sigma_a \sigma_b} \cosh(y) \right] dy \quad (8.1.12.) \]

which equals the probability density function of \( u(t) \), eq. (8.1.4.) or eq. (8.2.), the integral representation of the modified Bessel function of the order zero, see Refs. 3 and 24:

\[ p_w(z) = \frac{1}{\pi \sigma_a \sigma_b} K_0 \left( \frac{|z|}{\sigma_a \sigma_b} \right) \quad (8.1.13.) \]

The integral representation of the class of probability density functions, eq. (8.1.6.), has been evaluated numerically. In Fig. 14 the result is given for various values of \( Q \). The upper and lower limits are included, viz. the "Bessel" distribution, \( Q = \infty \), and the Gaussian distribution, \( Q = 0 \), respectively.

The cumulative probability function \( F_w(z) \) of \( w(t) \) can be obtained from eq. (8.1.6.) by integration. The result is, see Ref. 4:

\[ F_w \left( \frac{z}{\sigma_w} \right) = 0.5 + \frac{1}{\sqrt{\pi}} \int_0^\infty \exp(-\zeta^2) \frac{\text{erf} \left[ z \left( \frac{1 + Q^2}{2} \right)^{\frac{1}{2}} \zeta \right]}{\sigma_w \left( 1 + 2 \zeta^2 Q^2 \right)^{\frac{1}{2}}} \, d\zeta \quad (8.1.14.) \]

where \( \text{erf}(x) \) denotes the error function defined as:

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2} \, dv \]
Eq. (8.1.14.) has also been evaluated numerically. Results for various values of $Q$ are presented in Fig. 15.

The central moments of the class of probability density distributions can be obtained from eq. (8.1.6.) by integration with respect to eq. (5.5.3.). The result is, see also Ref. 4:

$$m_{2n-1} = 0$$

$$m_{2n} = \frac{\sigma_w^{2n}}{(1+Q^2)^n} \left[ 1 - \sum_{i=1}^{n} \binom{n}{i} (1.3 \ldots (2i-1)) Q^{2i} \right]$$  \hspace{1cm} (8.1.15.)

where $\binom{n}{i}$ is the binomial coefficient, see Ref. 24.

In particular the fourth order central moment $m_4$ is:

$$m_4 = \frac{9Q^4 + 6Q^2 + 3}{(1 + Q^2)^2} \cdot \sigma_w^4$$  \hspace{1cm} (8.1.16.)

where for all $Q$:

$$m_2 = \sigma_w^2$$  \hspace{1cm} (8.1.17.)

Thus a class of non-normal processes has been defined, having related distribution functions and a fixed relationship between the second and higher order moments, depending on a parameter $Q$.

This parameter $Q$ ranges from zero to infinity. The Gaussian is the limiting case for $Q = 0$ in this class. The fourth order central moment is presented as a function of $Q$ in Fig. 16.

In the analysis of data on aircraft instrument landing approaches Burgerhout, Ref. 25, derived a class of non-normal probability density functions quite similar to the class of probability density functions described above. The class of probability density functions derived by Burgerhout has the same range with respect to the fourth order moment. For equal values of the fourth order moments both distributions show
almost the same behaviour apart from the region $|x/\sigma| < 1$, where the influence of the discontinuity at $x/\sigma$ equals zero, in the distributions of Burgerhout becomes significant. It should be remarked that the parameter $\alpha$, defined in the distributions of Burgerhout is not identical to the parameter $Q$ defined in this report.

8.2. - Power spectral density of the system output, $w(t)$

The shapes of the power spectral densities of the stochastic processes $a(t); b(t)$ and c(t), viz. Fig. 13b - are completely determined by the characteristics of the filters $F_a; F_b$ and $F_c$. They are given by the following equations:

$$
\phi_{aa}(\omega) = |H_a(j\omega)|^2 \phi_{oo}(\omega) \tag{8.2.1.}
$$

$$
\phi_{bb}(\omega) = |H_b(j\omega)|^2 \phi_{oo}(\omega) \tag{8.2.1.}
$$

$$
\phi_{cc}(\omega) = |H_c(j\omega)|^2 \phi_{oo}(\omega) \tag{8.2.1.}
$$

where:

$H_a(j\omega)$ = transfer function of filter $F_a$

$H_b(j\omega)$ = transfer function of filter $F_b$

$H_c(j\omega)$ = transfer function of filter $F_c$

$\phi_{aa}(\omega)$ = output auto-power spectrum of $a(t)$

$\phi_{bb}(\omega)$ = output auto-power spectrum of $b(t)$

$\phi_{cc}(\omega)$ = output auto-power spectrum of $c(t)$

$\phi_{oo}(\omega)$ are input auto-power spectra of independent Gaussian white noise processes.

They are constant, $K_a^2; K_b^2$ and $K_c^2$ respectively.

Multiplication of $a(t)$ and $b(t)$ produces a process $u(t)$ having a auto-power spectrum obtained by a convolution of the composing auto-power spectra, see Refs. 3 and 11:
\begin{equation}
\phi_{uu}(\omega) = \frac{1}{2} \int_{-\infty}^{+\infty} \phi_{aa}(\omega - \lambda) \phi_{bb}(\lambda) \, d\lambda \tag{8.2.2.}
\end{equation}

or in short-hand notation:

\begin{equation}
\phi_{uu}(\omega) = \frac{1}{2} \phi_{aa}(\omega) \ast \phi_{bb}(\omega) \tag{8.2.3.}
\end{equation}

The factor \( \frac{1}{2} \) in this expression is due to the definition of the Fourier transform in Section 5. Other possible definitions used in the literature lead to different factors.

Employing eqs. (8.2.1.) and (8.2.2.), appropriate filter transfer functions can be chosen to produce a desired auto-power spectrum \( \phi_{uu}(\omega) \). Since a Gaussian process remains Gaussian when passed through a linear filter, any linear filtering performed upon the Gaussian white noise processes will therefore shape only their power spectral densities and not their distribution functions.

The process \( c(t) \) is added to the process \( u(t) \), see the block diagram of Fig. 13b. Since the processes \( u(t) \) and \( c(t) \) are independent and therefore uncorrelated, the auto-power spectral density of the sum, \( w(t) \), is simply the sum of the compounding auto-power spectral densities, see Ref. 9:

\begin{equation}
\phi_{ww}(\omega) = \phi_{uu}(\omega) + \phi_{cc}(\omega) \tag{8.2.4.}
\end{equation}

No change in the shape of the auto-power spectrum of the process \( w(t) \) does therefore occur, if the two compounding spectra of \( u(t) \) and \( c(t) \) are identical. The spectra of the basic processes \( a(t) \); \( b(t) \) and \( c(t) \) are thus required to satisfy the condition:

\begin{equation}
\frac{\phi_{cc}(\omega)}{\sigma_c^2} = \frac{\phi_{uu}(\omega)}{\sigma_u^2} = \frac{1}{2\sigma_a^2 \sigma_b^2} \int_{-\infty}^{+\infty} \phi_{aa}(\omega - \lambda) \phi_{bb}(\lambda) \, d\lambda \tag{8.2.5.}
\end{equation}

The Fourier transform of eq. (8.2.3.) yields, if the processes \( a(t) \) and \( b(t) \) are uncorrelated:
\[ C_{uu}(\tau) = C_{aa}(\tau) \cdot C_{bb}(\tau) \quad (8.2.6.) \]

where:

- \( C_{aa}(\tau) \) = auto-covariance function of \( a(t) \)
- \( C_{bb}(\tau) \) = auto-covariance function of \( b(t) \)
- \( C_{uu}(\tau) \) = auto-covariance function of \( u(t) \)

Furthermore, Fourier transforming eq. (8.2.4.) yields:

\[ C_{ww}(\tau) = C_{uu}(\tau) + C_{cc}(\tau) \quad (8.2.7.) \]

The spectral equations along with their corresponding auto-correlation functions due to Dryden are used in the remainder of this report, since they are easily obtained by linear filtering. In the following Sections the filter transfer functions of \( a(t) \), \( b(t) \) and \( c(t) \) for the turbulence velocities in three directions will be derived.

8.3. - The horizontal turbulence velocity: \( u_g \)

The spectral equation for the horizontal turbulence velocity due to Dryden is given by, see Section 6.4.:

\[ \Phi_{u_g u_g}(\omega) = \frac{2}{\pi \sigma_{u_g}} \frac{2 L_{u_g}}{V} \frac{1}{1 + \omega^2 \frac{L_{u_g}}{V^2}} \quad (8.3.1.) \]

According to eq. (8.2.5.), \( \Phi_{uu}(\omega) \) and \( \Phi_{cc}(\omega) \) are chosen as to satisfy the condition:

\[ \Phi_{uu}(\omega)/\sigma_u^2 = \Phi_{cc}(\omega)/\sigma_c^2 = \Phi_{u_g u_g}/\sigma_{u_g}^2 \quad (8.3.2.) \]

Thus:
\[ \phi_{uu}(\omega) = \frac{2}{\pi} \sigma_u^2 \frac{L_{ug}}{V} \frac{1}{1 + \omega^2 \frac{L_{ug}}{V^2}} \] 

(8.3.3.)

Furthermore, see eq. (8.2.6.):

\[ C_{uu}(\tau) = C_{aa}(\tau) \cdot C_{bb}(\tau) \] 

(8.3.4.)

The Fourier transform of eq. (8.3.3.) yields a function of an exponential form:

\[ C_{uu}(\tau) = \sigma_u^2 e^{-|\tau| \frac{V}{L_{ug}}} \] 

(8.3.5.)

Therefore, a sensible choice for \( C_{aa}(\tau) \) and \( C_{bb}(\tau) \) may be:

\[ C_{aa}(\tau) = \frac{\pi A}{2\sqrt{B}} \exp\left(-|\tau|/\sqrt{B}\right) \] 

(8.3.6.)

\[ C_{bb}(\tau) = \frac{\pi D}{2\sqrt{E}} \exp\left(-|\tau|/\sqrt{E}\right) \] 

(8.3.7.)

since the power spectral density \( \phi_{aa}(\omega) \) will then take the general form:

\[ \phi_{aa}(\omega) = \frac{A}{1 + B\omega^2} \] 

(8.3.8.)

Substituting eqs. (8.3.6.) and (8.3.7.) in eq. (8.3.4.):

\[ C_{uu}(\tau) = \frac{\pi^2 AD}{4\sqrt{BE}} \exp\left(-|\tau|\left(\frac{1}{\sqrt{B}} + \frac{1}{\sqrt{E}}\right)\right) \] 

(8.3.9.)

The following conditions have to be fulfilled, to satisfy eq. (8.3.5.):

\[ \frac{1}{\sqrt{B}} + \frac{1}{\sqrt{E}} = \frac{V}{L_{ug}} \] 

and

\[ \frac{\pi^2 AD}{4\sqrt{BE}} = \sigma_u^2 \]

(8.3.10.)
From the conditions expressed by eq. (8.3.10.) it will be seen, that the constants A, B, D and E are not uniquely determined. In particular the constants B and E specifying the cutoff frequencies of the compounding filters \( F_a \) and \( F_b \), see Fig. 13b are important. The ratio:

\[
R = \frac{\sqrt{B}}{\sqrt{E}}
\]  

(8.3.11.)

...can still freely be chosen affecting neither the power spectral densities \( \Phi_{uu}(\omega) \) and \( \Phi_{ww}(\omega) \), nor the probability density distribution \( p_\omega(z) \). However, it may be obvious that variation of the ratio R, specifying the cutoff frequencies, does affect the signals \( u(t) \) and \( w(t) \). The influence of R will be considered more explicitly in the next Section.

Substituting eq. (8.3.11.) in the first condition expressed by eq. (8.3.10.) yields:

\[
\sqrt{B} = (R + 1) \frac{L_{u_b} g}{V}
\]

\[
\sqrt{E} = \left( \frac{R + 1}{R} \right) \frac{L_{u_b} g}{V}
\]  

(8.3.12.)

The ratio of the constants A and D can also freely be chosen; it affects the ratio of the standard deviations \( \sigma_a \) and \( \sigma_b \) of the signals \( a(t) \) and \( b(t) \). In the mathematical elaborations leading to the results of the statistical properties of the final signal \( w(t) \), i.e. \( u_b \), only the product \( \sigma_a \sigma_b \) appears. Therefore variation of the ratio between the standard deviations \( \sigma_a \) and \( \sigma_b \) has no effect on the statistical properties of \( w(t) \), considered in this report. Let the ratio of A and D be expressed by:

\[
\beta^2 = \frac{A}{D}
\]  

(8.3.13.)

The substitution of eqs. (8.3.12.) and (8.3.13.) in the second condition expressed by eq. (8.3.10.) yields:
\[
A = \frac{2}{\pi} \beta \sigma_u \frac{R + 1}{\sqrt{R}} \frac{L_u}{V} \\
D = \frac{2}{\pi} \beta \sigma_u \frac{R + 1}{\sqrt{R}} \frac{L_u}{V} 
\]

(8.3.14.)

Since \( \beta \) can be freely chosen, let \( \beta \) take the value:

\[
\beta = \sqrt{R} 
\]

(8.3.15.)

In this way the ratio of \( A \) and \( D \) is used to keep \( \sigma_a \) and \( \sigma_b \) constant, i.e. independent of the value of \( R \).

From the eqs. (8.3.6.) and (8.3.7.) the variances \( \sigma_a^2 \) and \( \sigma_b^2 \) now reduce to:

\[
\sigma_a^2 = \frac{\pi A}{2 \sqrt{B}} = \sigma_u 
\]

(8.3.16.)

\[
\sigma_b^2 = \frac{\pi D}{2 \sqrt{E}} = \sigma_u 
\]

Substituting eq. (8.3.16.) in eqs. (8.3.6.) and (8.3.7.) yields:

\[
C_{aa}(\tau) = \sigma_u \exp \left[ -|\tau| \frac{V}{L_u} \frac{1}{R + 1} \right] 
\]

(8.3.17.)

\[
C_{bb}(\tau) = \sigma_u \exp \left[ -|\tau| \frac{V}{L_u} \frac{R}{R + 1} \right] 
\]

(8.3.18.)

Furthermore, from eq. (8.3.2.) it follows by Fourier transforming \( \Phi_{cc}(\omega) \):

\[
C_{cc}(\tau) = \sigma_c^2 \exp \left[ -|\tau| \frac{V}{L_u} \right] 
\]

(8.3.19.)

The Fourier transform of eqs. (8.3.17.), (8.3.18.) and (8.3.19.) results in the following auto-power spectral densities of the signals \( a(t) \), \( b(t) \) and \( c(t) \):
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\[ \phi_{aa}(\omega) = \frac{2}{\pi} \sigma_u \frac{L_u g}{V} \frac{(R + 1)}{\left( 1 + \frac{L_u g}{V} \frac{R + 1}{R} \omega \right)^2} \quad (8.3.20.) \]

\[ \phi_{bb}(\omega) = \frac{2}{\pi} \sigma_u \frac{L_u g}{V} \frac{R + 1}{R} \frac{1}{\left( 1 + \frac{L_u g}{V} \frac{R + 1}{R} \omega \right)^2} \quad (8.3.21.) \]

\[ \phi_{cc}(\omega) = \frac{2}{\pi} \sigma_c \frac{L_u g}{V} \frac{1}{\left( 1 + \frac{L_u g}{V} \omega \right)^2} \quad (8.3.22.) \]

The transfer functions of the filters \( F_a, F_b \) and \( F_c \) are obtained from eq. (8.2.1.), substituting eqs. (8.3.20.); (8.3.21.) and (8.3.22.) respectively. The result is:

\[ H_a(j\omega) = \frac{1}{K_a} \sqrt{\frac{2}{\pi} \sigma_u \frac{L_u g}{V}} \frac{(R + 1)}{\left( 1 + \frac{L_u g}{V} \frac{R + 1}{R} j\omega \right)} \quad (8.3.23.) \]

\[ H_b(j\omega) = \frac{1}{K_b} \sqrt{\frac{2}{\pi} \sigma_u \frac{L_u g}{V}} \frac{R + 1}{R} \frac{1}{\left( 1 + \frac{L_u g}{V} \frac{R + 1}{R} j\omega \right)} \quad (8.3.24.) \]

\[ H_c(j\omega) = \frac{1}{K_c} \sigma_c \sqrt{\frac{2}{\pi} \frac{L_u g}{V}} \frac{1}{\left( 1 + \frac{L_u g}{V} j\omega \right)} \quad (8.3.25.) \]

8.4. - The vertical turbulence velocity: \( w_g \)

The spectral equation for the vertical turbulence velocity, due to Dryden is, see Section 6.4.:

\[ \phi_{w_g w_g}(\omega) = \frac{1}{\pi} \sigma_{w_g}^2 \frac{L_u g}{V} \frac{2 \omega^2 \frac{L_u g}{V}^2}{\left( 1 + \omega^2 \frac{L_u g}{V}^2 \right)^2} \quad (8.4.1.) \]
Similarly to the derivation of the longitudinal turbulence velocity filters:

\[
\Phi_{uu}(\omega) = \frac{1}{\pi} \sigma_u^2 \frac{L_\omega g}{V} \frac{1 + 3\omega^2 \frac{L_\omega g}{V^2}}{\left(1 + \omega^2 \frac{L_\omega g}{V^2}\right)^2}
\]

(8.4.2.)

According to eq. (8.2.6.):

\[
C_{uu}(\tau) = C_{aa}(\tau) \cdot C_{bb}(\tau)
\]

(8.4.3.)

Fourier transforming eq. (8.4.2.) yields:

\[
C_{uu}(\tau) = \sigma_u^2 \exp \left[-\frac{|\tau|}{L_\omega g} \cdot \frac{V}{2L_\omega g}\right]
\]

(8.4.4.)

Let \( C_{aa}(\tau) \) and \( C_{bb}(\tau) \) take the form:

\[
C_{aa}(\tau) = \frac{\pi A}{2\sqrt{B}} \exp \left[-\frac{|\tau|}{\sqrt{B}}\right]
\]

(8.4.5.)

\[
C_{bb}(\tau) = \frac{\pi C}{2\sqrt{D}} \exp \left[-\frac{|\tau|}{\sqrt{D}}\right] \cdot \left[1 - \frac{|\tau|}{2} \left(\frac{1}{\sqrt{B}} + \frac{1}{\sqrt{D}}\right)\right]
\]

(8.4.6.)

Substituting eqs. (8.4.5.) and (8.4.6.) in eq. (8.4.3.):

\[
C_{uu}(\tau) = \frac{\pi^2 AC}{4\sqrt{BD}} \exp \left[-\frac{|\tau|}{\sqrt{B}} \left(\frac{1}{\sqrt{B}} + \frac{1}{\sqrt{D}}\right)\right] \cdot \left[1 - \frac{|\tau|}{2} \left(\frac{1}{\sqrt{B}} + \frac{1}{\sqrt{D}}\right)\right]
\]

(8.4.7.)

The following conditions have to be fulfilled to satisfy eq. (8.4.4.):

\[
\frac{\pi^2 AC}{4\sqrt{BD}} = \sigma_u^2
\]

\[
\frac{1}{\sqrt{B}} + \frac{1}{\sqrt{D}} = \frac{V}{L_\omega g}
\]

(8.4.8.)

Substitution of the ratio's:
\[ R = \frac{\sqrt{D}}{\sqrt{B}} \]  

(8.4.9.)

and

\[ \beta^2 = \frac{C}{A} = R \]  

(8.4.10.)

in the conditions expressed by eq. (8.4.8.) yields:

\[ \sqrt{B} = (R + 1) \frac{L_w g}{V} \]  

(8.4.11.)

\[ \sqrt{D} = (R + 1) \frac{L_w g}{V} \]  

(8.4.12.)

\[ A = \frac{2}{\pi} \sigma_u \frac{R + 1}{R} \frac{L_w g}{V} \]  

(8.4.13.)

\[ C = \frac{2}{\pi} \sigma_u (R + 1) \frac{L_w g}{V} \]  

(8.4.14.)

Substituting eqs. (8.4.11.) until (8.4.14.) in eqs. (8.4.5.) and (8.4.6.) results in:

\[ C_{aa}(\tau) = \sigma_u \exp \left[ - |\tau| \frac{V}{L_w g} \frac{R}{R + 1} \right] \]  

(8.4.15.)

\[ C_{bb}(\tau) = \sigma_u \exp \left[ - |\tau| \frac{V}{L_w g} \frac{1}{R + 1} \right] \cdot \left[ 1 - \frac{|\tau|}{2} \frac{V}{L_w g} \right] \]  

(8.4.16.)

Furthermore similar arguments as in the case of the longitudinal turbulence velocity lead to the expression for \( C_{cc}(\tau) \) for the vertical turbulence velocity:

\[ C_{cc}(\tau) = \sigma_c^2 \exp \left[ - |\tau| \frac{V}{L_w g} \right] \cdot \left[ 1 - \frac{|\tau|}{2} \frac{V}{L_w g} \right] \]  

(8.4.17.)

The Fourier transform of eqs. (8.4.15.); (8.4.16.) and (8.4.17.) yields the auto-power spectral densities of the signals \( a(t) \), \( b(t) \) and \( c(t) \) for the vertical turbulence velocity:
\[ \Phi_{aa}(\omega) = \frac{1}{2} \sigma_u \frac{L_w g}{V} \frac{R + 1}{R} \frac{1}{1 + \frac{L_w g}{V^2} \left(\frac{R + 1}{R}\right)^2 \omega^2} \quad (8.4.18.) \]

\[ \Phi_{bb}(\omega) = \frac{1}{\pi} \sigma_u \frac{L_w g}{V} (R + 1) \frac{(1 - R) + (3 + R)(R + 1)^2}{\left[1 + \frac{L_w g}{V^2} \left(\frac{R + 1}{R}\right)^2 \omega^2\right]^2} \quad (8.4.19.) \]

\[ \Phi_{cc}(\omega) = \frac{1}{\pi} \sigma_c \frac{L_w g}{V} \frac{1}{1 + \frac{L_w g}{V^2} \omega^2} \quad (8.4.20.) \]

The following transfer functions of the filters \( F_a, F_b \) and \( F_c \) are obtained from eq. (8.2.1.), substituting eqs. (8.4.18.) until (8.4.20.), respectively:

\[ H_a(j\omega) = \frac{1}{K_a} \sqrt{\frac{2}{\pi}} \frac{L_w g}{V} \left(\frac{R + 1}{R}\right) \frac{1}{1 + \frac{R + 1}{R} \frac{L_w g}{V} (j\omega)} \quad (8.4.21.) \]

\[ H_b(j\omega) = \frac{1}{K_b} \sqrt{\frac{1}{\pi}} \sigma_u \frac{L_w g}{V} (R + 1) \sqrt{1 - R} + (R + 1) \frac{\sqrt{3} + R}{\sqrt{1 + (R + 1) \frac{L_w g}{V} (j\omega)}} \quad (8.4.22.) \]

\[ H_c(j\omega) = \frac{\sigma_c}{K_c} \sqrt{\frac{1}{\pi V}} \frac{1 + \sqrt{3}}{\pi V} \frac{L_w g}{V} j\omega \frac{1 + \frac{L_w g}{V} (j\omega)}{\left[1 + \frac{L_w g}{V} (j\omega)^2\right]^2} \quad (8.4.23.) \]

In the expression for the power spectral density of the process \( b(t) \), see eq. (8.4.19.) the factor \( 1 - R \) appears. If \( R \) is taken \( R > 1 \), the spectrum would have a negative area for low values of the frequency \( \omega \). Therefore the value of \( R \) has to be restricted within the range \( 0 < R < 1 \).
8.5. The lateral turbulence velocity: \( v_g \)

The spectral equation and auto-correlation function for the lateral turbulence velocity are identical to the spectral equation and auto-correlation function for the vertical turbulence velocity, they are obtained by exchanging \( w_g \) by \( v_g \), see Section 6.4. Therefore the derivation of the auto-power spectra of \( a(t) \), \( b(t) \) and \( c(t) \) as well as the transfer functions for the filters \( F_a \), \( F_b \) and \( F_c \) are identical, provided the index \( w \) is replaced by \( v \).

Summarizing the results of Section 8.4., for the lateral turbulence velocity:

\[
\phi_{aa}(\omega) = \frac{2}{\pi} \sigma_u \frac{L_v}{V} \frac{R+1}{R} \frac{1}{1 + \frac{L_v g}{V^2} \left( \frac{R}{R} + 1 \right)^2 \omega^2} \quad (8.5.1.)
\]

\[
\phi_{bb}(\omega) = \frac{1}{\pi} \sigma_u \frac{L_v g}{V} (R+1) \frac{\omega^2}{\left[ 1 + \frac{L_v g}{V^2} \left( \frac{R}{R} + 1 \right)^2 \omega^2 \right]^2} \quad (8.5.2.)
\]

\[
\phi_{cc}(\omega) = \frac{1}{\pi} \sigma_c^2 \frac{L_v g}{V} \frac{1 + 3 \frac{L_v g}{V^2} \omega^2}{\left( 1 + \frac{L_v g}{V^2} \omega^2 \right)^2} \quad (8.5.3.)
\]

The filter transfer functions are:

\[
H_a(j\omega) = \frac{1}{K_a} \sqrt{\frac{2}{\pi} \sigma_u \frac{L_v g}{V} \frac{R+1}{R} \frac{1}{1 + \frac{R+1}{R} \frac{L_v g}{V(j\omega)}}} \quad (8.5.4.)
\]

\[
H_b(j\omega) = \frac{1}{K_b} \sqrt{\frac{1}{\pi} \sigma_u \frac{L_v g}{V} (R+1) \frac{\sqrt{1 - R + \sqrt{3 + R} (R+1) \frac{L_v g}{V(j\omega)}}}{\left[ 1 + (R+1) \frac{L_v g}{V(j\omega)} \right]^2}} \quad (8.5.5.)
\]
\[ H_c(j\omega) = \frac{\sigma_c}{K_c} \sqrt{\frac{L_{Vg}}{\pi V}} \frac{1 + \sqrt{3} \frac{L_{Vg}}{V} (j\omega)}{\left[1 + \frac{L_{Vg}}{V} (j\omega)\right]^2} \] (8.5.6.)
9. - Statistical properties of the squared turbulence velocity

In the model of patchiness the square of the turbulence velocity is assumed to be a representation of the instantaneous energy of the turbulence on which the pilot bases his opinion about patchiness, see Section 7.

The statistical properties in terms of the probability density function, mean value and variance of \( w^2(t) \) will be derived in this Section. The power spectral density of \( w^2(t) \) to be used in the evaluation of the model of patchiness will be derived, allowing to find closed analytical expressions for the patchiness parameter \( P_T \).

9.1. - Probability density, mean and variance of \( w^2(t) \)

Consider the stochastic process:

\[
y(t) = w^2(t)
\]  \hspace{1cm} (9.1.1.)

where \( w(t) \) is stationary. It can be shown, see Ref. 9, that \( y(t) \) is also stationary. The probability density function of \( y(t) \) reads, see Ref. 9:

\[
p_y(y) = \frac{1}{2\sqrt{y}} \left[ p_w(\sqrt{y}) + p_w(-\sqrt{y}) \right] U(y)
\]  \hspace{1cm} (9.1.2.)

where \( U(y) \) is the unit-step function, defined as:

\[
U(y) = 1 \quad y \geq 0
\]  \hspace{1cm} (9.1.3.)

\[
U(y) = 0 \quad y < 0
\]

The non-normal distribution function of the turbulence signal \( w(t) \) is, see eq. (8.1.6.):
Substituting eq. (9.1.4.) in eq. (9.1.2.) yields:

\[ p_{w2}(y) = \frac{1}{\sqrt{\pi y}} \left( \frac{1+Q^2}{\sigma_w^2} \right)^\frac{1}{2} \int_0^\infty \left[ \frac{2}{1+2\zeta^2Q^2} \right] \frac{1}{1+2\zeta^2Q^2} \exp \left[ -\zeta^2 - \frac{y}{\sigma_w^2} \frac{1+Q^2}{1+2\zeta^2Q^2} \right] d\zeta \]

for \( y \geq 0 \) and \( p_{w2}(y) = 0 \) for \( y < 0 \) \hspace{1cm} (9.1.5.)

The mean value of the squared turbulence signal \( w^2(t) \) is, see Appendix 2:

\[ m_{1(w2)} = \mu_{w2} = \sigma_w^2 \] \hspace{1cm} (9.1.6.)

The second order moment of the squared turbulence signal \( w^2(t) = y(t) \) is, see Appendix 2:

\[ M_2(y) = \frac{\sigma_w^4}{(1+Q^2)^2} (9Q^4 + 6Q^2 + 3) \] \hspace{1cm} (9.1.7.)

According to eq. (8.1.16.):

\[ M_2(y) = m_4(w) \] \hspace{1cm} (9.1.8.)

Thus the second order moment of \( w^2(t) \) equals the fourth order central moment of \( w(t) \).

The variance of \( w^2(t) = y(t) \) then is:

\[ \sigma_{w2}^2 = M_2(y) - \mu_{w2}^2 = \frac{2\sigma_w^4}{(1+Q^2)^2} (4Q^4 + 2Q^2 + 1) \] \hspace{1cm} (9.1.9.)

Again a complete class of non-normal distribution functions, depending on the value of \( Q \) of the squared turbulence velocity \( w^2(t) \) has been defined. Eq. (9.1.5.) has been evaluated numerically, the results are presented in Fig. 17 for various values of \( Q \).
9.2. - Covariance and power spectral density of $w^2(t)$

Consider again the block diagram of Fig. 13b. The turbulence signal $w(t)$ is built up from the signals $a(t)$, $b(t)$ and $c(t)$ according to:

$$w(t) = a(t) \cdot b(t) + c(t) \quad (9.2.1.)$$

The squared turbulence signal $w^2(t)$ can be obtained from eq. (9.2.1.):

$$w^2(t) = a^2(t) \cdot b^2(t) + c^2(t) + 2a(t) \cdot b(t) \cdot c(t) \quad (9.2.2.)$$

According to the definition of the auto-correlation function, eq. (5.6.2.), the auto-correlation function of $w^2(t)$ is:

$$R_{w^2w^2}(\tau) = E[w^2(t) \cdot w^2(t + \tau)] \quad (9.2.3.)$$

Taking the expected value of eq. (9.2.2.), according to eq. (9.2.3.) yields, see Appendix 3:

$$R_{w^2w^2}(\tau) = E[w^2(t) \cdot w^2(t + \tau)] =$$

$$= E[a^2(t) \cdot a^2(t + \tau), E[b^2(t) \cdot b^2(t + \tau)] + E[c^2(t) \cdot c^2(t + \tau)]$$

$$+ 4E[a(t) \cdot a(t + \tau)] \cdot E[b(t) \cdot b(t + \tau)] \cdot E[c(t) \cdot c(t + \tau)]$$

$$+ 2E[c^2(t)] \cdot E[a^2(t + \tau)] \cdot E[b^2(t + \tau)] \quad (9.2.4.)$$

In terms of auto-correlation functions eq. (9.2.4.) can be rewritten as:

$$R_{w^2w^2}(\tau) = R_{a^2a^2}(\tau), R_{b^2b^2}(\tau) + R_{c^2c^2}(\tau) + 4R_{aa}(\tau) \cdot R_{bb}(\tau) \cdot R_{cc}(\tau)$$

$$+ 2R_{aa}(0) \cdot R_{bb}(0) \cdot R_{cc}(0) \quad (9.2.5.)$$
Bearing in mind that \( a(t) \) and \( b(t) \) as well as \( c(t) \) are independent Gaussian processes, it can be shown, see Ref. 9, that:

\[
R_{a^2a^2}(\tau) = R_{aa}(\tau) + 2R_{aa}(\tau)
\]

\[
R_{b^2b^2}(\tau) = R_{bb}(\tau) + 2R_{bb}(\tau)
\]

\[
R_{c^2c^2}(\tau) = R_{cc}(\tau) + 2R_{cc}(\tau)
\] (9.2.6.)

Substituting eqs. (9.2.6.) into eq. (9.2.5.) yields:

\[
R_{w^2w^2}(\tau) = R_{aa}(\tau)R_{bb}(\tau) + 2R_{aa}(\tau)R_{bb}(\tau) + 2R_{aa}(\tau)R_{cc}(\tau)
\]

\[
+ R_{cc}(\tau) + 2R_{cc}(\tau) + 4R_{aa}(\tau)R_{bb}(\tau)R_{cc}(\tau)
\]

\[
+ 4R_{aa}(\tau)R_{bb}(\tau) + 2R_{aa}(\tau)R_{bb}(\tau)R_{cc}(\tau)
\] (9.2.7.)

According to eq. (5.1.16.) the auto-power spectral density function of the squared turbulence velocity \( w^2(t) \) is obtained as the Fourier transform of the auto-covariance function.

The auto-covariance function of the squared turbulence signal is related to the auto-correlation function by:

\[
C_{w^2w^2}(\tau) = R_{w^2w^2}(\tau) - w^2_{1}(w^2)
\] (9.2.8.)

According to eq. (9.1.6.):

\[
\mu_{(w^2)}^2 = \sigma_{w}^2 = (\sigma_a^2 + \sigma_b^2 + \sigma_c^2)^2
\] (9.2.9.)

Thus substituting eq. (9.2.9.) into eq. (9.2.7.) and bearing in mind that generally:

\[
R_{xx}(\tau) = \sigma_x^2
\] (9.2.10.)

and

\[
C_{xx}(\tau) = R_{xx}(\tau)
\]
where \( x(t) \) is a stochastic process having zero mean value yields:

\[
C_{ww}(\tau) = 2\sigma_a^4C_{bb}^2(\tau) + 2\sigma_b^4C_{aa}^2(\tau) + 2C_{cc}^2(\tau)
+ 4C_{aa}(\tau)C_{bb}(\tau)C_{cc}(\tau) + 4C_{aa}(\tau)C_{bb}(\tau)
\]  
(9.2.11.)

The Fourier transforms of eq. (9.2.11.), see eq. (5.6.16.), is:

\[
\Phi_{ww}(\omega) = \frac{2}{\pi} \int_0^\infty C_{ww}(\tau) \cos \omega \tau \, d\tau
\]  
(9.2.12.)

The auto-covariance functions and auto-power spectral density functions will be calculated for the horizontal, vertical and lateral squared turbulence velocities. The Dryden spectral equations are chosen for the turbulence velocities \( u_g, w_g \) and \( v_g \).

9.2.1. - The squared longitudinal turbulence velocity: \( u_g^2(t) \)

The covariance functions of the signals \( a(t), b(t) \) and \( c(t) \) in the case of the horizontal turbulence velocity are, see eqs. (8.3.17.) until (8.3.19.):

\[
C_{aa}(\tau) = \sigma_u^2 \exp \left[ -\frac{|\tau|}{L_{ug}} \frac{1}{R + 1} \right]
\]

\[
C_{bb}(\tau) = \sigma_u^2 \exp \left[ -\frac{|\tau|}{L_{ug}} \frac{R}{R + 1} \right]
\]  
(9.2.1.1.)

\[
C_{cc}(\tau) = \sigma_c^2 \exp \left[ -\frac{|\tau|}{L_{uc}} \frac{\bar{v}}{L_{ug}} \right]
\]

where:

\[ \sigma_u^2 = \sigma_a^2 = \sigma_b^2 \]

The variances \( \sigma_u^2 \) and \( \sigma_c^2 \) can be expressed in terms of \( \sigma_{ug}^2 \).

According to eq. (8.1.7.) and eq. (8.1.8.):
\[ a_{ug}^2 = a_w^2 = a_u^2 + a_c^2 \]  

(9.2.1.2.)

and

\[ a_u^2 = Q^2 a_c^2 \]  

(9.2.1.3.)

Substituting eq. (9.2.1.3.) into eq. (9.2.1.2.) yields:

\[ a_u^2 = \frac{Q^2 a_{ug}^2}{1 + Q^2} \]  

(9.2.1.4.)

\[ a_c^2 = \frac{a_{ug}^2}{1 + Q^2} \]

Substituting eqs. (9.2.1.1.) and (9.2.1.4.) into eq. (9.2.11.) yields:

\[ C_{ug 2ug 2}(\tau) = \frac{a_{ug}^4}{(1 + Q^2)^2} \left[ (4Q^4 + 4Q^2 + 2) \exp(-2 |\tau| \frac{V}{L_{ug}}) \right. \]

\[ + 2Q^4 \{ \exp(-2 |\tau| \frac{V}{L_{ug}} \frac{1}{R + 1}) + \exp(-2 |\tau| \frac{V}{L_{ug}} \frac{R}{R + 1}) \} \]  

(9.2.1.5.)

The Fourier transform of eq. (9.2.1.5.) yields:

\[ \Phi_{ug 2ug 2}(\omega) = \frac{8}{\pi} \frac{a_{ug}^4}{L_{ug} \frac{V}{(1 + Q^2)^2}} \left[ \frac{1}{(2Q^4 + 2Q^2 + 1)} \frac{1}{4 + \frac{L_{ug}^2}{V^2} \omega^2} \right. \]

\[ + \frac{Q^4}{4R^2 + (R+1)^2} \frac{L_{ug}^2}{V^2} \omega^2 \]

\[ + \frac{Q^4}{4 + (R+1)^2} \frac{L_{ug}^2}{V^2} \omega^2 \]  

(9.2.1.6.)
If \( Q \) is set to zero, the power spectral density of \( \Phi_{u_g^2 u_g^2}(\omega) \)_{normal}, eq. (7.3.9.) is obtained:

\[
\Phi_{u_g^2 u_g^2}(\omega) = \frac{8}{\pi} \sigma_{u_g} \frac{4 L_{u_g}}{V} \frac{1}{4 + \frac{L_{u_g}}{V^2} \omega^2} \]  
(9.2.1.7.)

The power spectral density function \( \Phi_{u_g^2 u_g^2}(\omega) \) has been illustrated in Fig. 18 showing the influences of both \( Q \) and \( R \).

9.2.2. – The squared vertical and lateral turbulence velocities: \( w_g^2(t) \) and \( v_g^2(t) \)

The auto-covariance functions of the signals \( a(t), b(t) \) and \( c(t) \) in the case of the vertical turbulence velocities are, see eqs. (8.4.15.) until (8.4.17.):

\[
C_{aa}(\tau) = \sigma_u \exp \left[-|\tau| \frac{R}{R + 1} \frac{V}{L_{w_g}} \right] \\
C_{bb}(\tau) = \sigma_u \exp \left[-|\tau| \frac{V}{L_{w_g}} \frac{1}{R + 1} \right] \cdot \left[1 - \frac{|\tau|}{2 \frac{V}{L_{w_g}}} \right] \]  
(9.2.2.1.)

\[
C_{cc}(\tau) = \sigma_c^2 \exp \left[-|\tau| \frac{V}{L_{w_g}} \right] \\
\]

Substituting eqs. (9.2.2.1.) and (9.2.1.4.) in eq. (9.2.11.) yields:

\[
C_{w_g^2 w_g^2}(\tau) = \frac{\sigma_{w_g^2}}{(1+Q^2)^2} \left[(4Q^4 + 4Q^2 + 2) \exp (-2 |\tau| \frac{V}{L_{w_g}}) \cdot (1 - |\tau| \frac{V}{2L_{w_g}})^2 + 2Q^4 \exp (-2 |\tau| \frac{V}{L_{w_g}} \frac{1}{R + 1}) \right. \\
\left. \cdot (1 - |\tau| \frac{V}{2L_{w_g}})^2 + \exp (-2 |\tau| \frac{V}{L_{w_g}} \frac{R}{R + 1}) \right] \]  
(9.2.2.2.)

The Fourier transform of eq. (9.2.2.2.) is performed according to eq. (9.2.12.). The result is the auto-power spectral density of the squared turbulence velocity \( w_g^2(t) \):
\[
\Phi_{\omega g^2 \omega_g^2}(\omega) = \frac{4}{\pi} \frac{\sigma_{\omega g}^4}{(1 + Q^2)^2} \frac{V}{L_{\omega g}} \left(2Q^4 + 2Q^2 + 1\right) \left(\frac{20}{4} \frac{V^4}{L_{\omega g}} + \frac{13}{2} \frac{V^2}{L_{\omega g}} \omega^2 + 3\omega^4 \right)
\]

\[
+ Q^4 \left\{ \frac{2(R+1)}{4 \frac{V^2}{L_{\omega g}} + (R+1)^2 \omega^2} - \frac{4}{(4 \frac{V^2}{L_{\omega g}} + (R+1)^2 \omega^2)^2} \right\} \]

\[
+ \frac{4}{4 \frac{V^2}{L_{\omega g}} + (R+1)^2 \omega^2} \left(\frac{V^2}{L_{\omega g}} \omega^2 \right)^2 \left(\frac{V^2}{L_{\omega g}} \omega^2 \right)^3 \left(\frac{2R(R+1)}{4 \frac{V^2}{L_{\omega g}} + (R+1)^2 \omega^2} \right) \right\} \]  

(9.2.2.3.)

If Q is set equal to zero, the power spectral density of \(\Phi_{\omega g^2 \omega_g^2}(\omega)\) normal, eq. (7.3.10.), is obtained:

\[
\Phi_{\omega g^2 \omega_g^2}(\omega) \text{ normal} = \frac{4}{\pi} \frac{\sigma_{\omega g}^4}{L_{\omega g}} \frac{V}{4} \left(\frac{20}{4} \frac{V^4}{L_{\omega g}} + \frac{13}{2} \frac{V^2}{L_{\omega g}} \omega^2 + 3\omega^4 \right)
\]

(9.2.2.4.)

The power spectral density \(\Phi_{\omega g^2 \omega_g^2}(\omega)\) is illustrated in Fig. 19, showing the influences of both Q and R.

The squared lateral turbulence velocity power spectral density is identical to the power spectral density of the squared vertical turbulence
velocity, except for the exchange of the index $w$ by $v$.

9.3. - Patchiness in the frequency domain

In Section 7 a model of patchiness has been described, based on a memory filter as a means to assess the patchy characteristics. The input signal to the "memory filter" is the square of turbulence velocity, $w^2(t)$. Based on a possible simulation of such patchy turbulence velocities, described in Section 8, the statistics of the squared turbulence velocities $w^2(t)$ are derived in Sections 9.1. and 9.2. In particular the derivation of the power spectral density of the squared turbulence velocity is important. Knowledge of the shape of this second order power spectral density provides an easy means to derive the behaviour of the variance of the output of the memory filter, which is assumed in Section 7 to be a descriptive function of the patchy characteristics of atmospheric turbulence.

Consider the memory filter with input signal $w^2(t)$ and output $z(t)$, see Fig. 7. The memory filter is a simple first order filter with transfer function, see eq. (7.1.3):

$$H(j\omega) = \frac{1}{1 + j\omega \tau} \quad (9.3.1.)$$

According to eq. (5.6.19.), the output power spectral density $\Phi_{zz}(\omega)$ is given by:

$$\Phi_{zz}(\omega) = \frac{1}{1 + \omega^2 \tau^2} \cdot \Phi_{w^2w^2}(\omega) \quad (9.3.2.)$$

The variance $\sigma_z^2$ of the output $z(t)$, the descriptive function of patchiness can be obtained by integrating eq. (9.3.2.). Thus:

$$\sigma_z^2(t) = \int_0^\infty \frac{1}{1 + \omega^2 \tau^2} \Phi_{w^2w^2}(\omega) \, d\omega \quad (9.3.3.)$$

Employing the appropriate expressions of $\Phi_{w^2w^2}(\omega)$, eqs. (9.2.1.6.) and
(9.2.2.4.) for the longitudinal and vertical/lateral turbulence velocities respectively, the following results are obtained, see Appendix 4:

**a. Longitudinal turbulence velocity, \( u_g \):**

\[
\sigma_{zu_g}^2(\tau)_{\text{non-normal}} = \frac{2\sigma_{u_g}^4}{(1+Q^2)^2} \left[ \frac{2Q^4+2Q^2+1}{1+2\tau} \frac{V}{L_{u_g}} + \frac{Q^4(R+1)}{R+1+2RT} \frac{V}{L_{u_g}} + \frac{Q^4(R+1)}{R+1+2\tau} \frac{V}{L_{u_g}} \right]
\]

(9.3.4.)

**b. Vertical turbulence velocity, \( w_g \):**

\[
\sigma_{z\omega_g}^2(\tau)_{\text{non-normal}} = \frac{\sigma_{\omega_g}^4}{(1+Q^2)^2} \left[ \frac{(2Q^4+2Q^2+1)(5\frac{V^2}{L_{\omega_g}^2} \tau^2 + 6\frac{V}{L_{\omega_g}} \tau + 2)}{(2\frac{V}{L_{\omega_g}} \tau + 1)^3} \left( \frac{2}{L_{\omega_g}} \tau + 1 \right) \left( \frac{2}{L_{\omega_g}} \tau + 1 \right)^2 + \frac{V^2}{L_{\omega_g}} \tau^2(R+1)^2 \right]
\]

\[
+ \frac{2}{R+1+2\frac{V}{L_{\omega_g}} \tau R} \right]
\]

(9.3.5.)

**c. Lateral turbulence velocity, \( v_g \)**

The expression for \( \sigma_{zv_g}^2(\tau)_{\text{non-normal}} \) is identical to the expression (9.3.5.) for the vertical turbulence velocity, except for replacing the index \( w_g \) by \( v_g \).

From the above stated expressions, it follows that if \( \tau \) equals zero the value of \( \sigma_z^2 \) equals the value of \( m_4 - \sigma_w^4 \). According to eq. (7.1.5.), this is equal to \( (K-1) \sigma_w^4 \).

Thus:

\[
(K-1) \sigma_w^4 = \sigma_w^4 \frac{8Q^4 + 4Q^2 + 2}{(1 + Q^2)^2}
\]

(9.3.6.)
where $k$ is the kurtosis value of $w(t)$.

If $\tau$ tends to infinity the value of $\sigma_z^2$ tends to zero.

The behaviour of $\sigma_z^2(\tau)_{\text{non-normal}}$ as a function of $\tau$, the "length" of the pilot's memory, is illustrated in Fig. 20 for the longitudinal turbulence velocity, for various values of $Q$ and $R$. The behaviour of $\sigma_z^2(\tau)_{\text{non-normal}}$ for the vertical or lateral turbulence velocity is illustrated in Fig. 21. As can be seen from Figs. 20 and 21 the parameters $Q$ and $R$ can be considered as the variables quantitatively determining the patchy characteristics. The parameter $Q$ singles out a particular probability density distribution from the entire set of possible distribution functions, defined by eq. (8.1.6.). The parameter $Q$ does not, however, appear in the auto-power spectral density function of the turbulence velocities $w(t)$.

The parameter $R$ appears neither in the expression for the probability density function nor in the power spectral density of $w(t)$.

The influence of $R$ can only be seen in the power spectral density of the squared turbulence velocity $w^2(t)$, see Figs. 18 and 19. Variation of $R$ alters the shape of this power spectral density, while the value of the fourth order central moment, the integral value of this power spectral density may remain constant, governed by the specific value of $Q$. As can be seen from Figs. 18 and 19, the mean contributions to the fourth order central moment shift to the lower frequency range, when the value of $R$ decreases from 1. It can be seen that due to this shift of the instantaneous energy of the turbulence to lower frequencies, a pilot may be expected to experience a different sensation of patchiness. The "local regions" of relatively higher energy used in the description of patchiness, see Section 3.1., differ apparently as the value of $R$ differs from 1. The different sensation of patchiness may be expressed by the value of the variance $\sigma_z^2$ of the output of the memory filter at a certain value of $\tau$.

As the value of $R$ decreases from 1, the variance $\sigma_z^2(\tau)$ shows a slower
decay with $\tau$, see Figs. 20 and 21. As $Q$ diminishes to zero, leading to Gaussian turbulence, the parameter $R$ becomes less important, because a relatively large change in the value of $R$ then has hardly any effect on the behaviour of $\sigma_z^2(\tau)$.

If $Q$ is large, indicating a large deviation from the Gaussian behaviour of the turbulence velocity, a relatively small change in the value of $R$ produces a remarkable difference in the behaviour of $\sigma_z^2(\tau)$.

9.4. Calculation of the patchiness parameter $P_T$

According to Section 7.2., eq. (7.2.1.) the patchiness parameter $P_T$ is defined as:

$$P_T = \frac{\sigma_z^2(\tau)_{\text{non-normal}}}{\sigma_z^2(\tau)_{\text{normal}}} \quad (9.4.1.)$$

Substituting the expressions for $\sigma_z^2(\tau)_{\text{non-normal}}$ and $\sigma_z^2(\tau)_{\text{normal}}$ for the turbulence velocities in three directions, eqs. (9.3.4.), (9.3.5.) and (7.3.12.), (7.3.13.) yields the following:

a. Longitudinal turbulence velocity, $u_g$

$$P_{T_{u_g}} = \frac{2Q^2+Q^2+1}{(1+Q^2)^2} + \frac{\frac{L_{u_g}}{V}}{(1+Q^2)^2} \left[ \frac{\frac{L_{u_g}}{V}}{(R+1)} + 2\tau \right] + \frac{\frac{L_{u_g}}{V}}{(1+Q^2)^2} \left[ \frac{\frac{L_{u_g}}{V}}{(R+1)} + 2\tau \right]$$

$$P_{T_{u_g}} = \frac{2Q^2+Q^2+1}{(1+Q^2)^2} + \frac{\frac{L_{u_g}}{V}}{(1+Q^2)^2} \left[ \frac{\frac{L_{u_g}}{V}}{(R+1)} + 2\tau \right] + \frac{\frac{L_{u_g}}{V}}{(1+Q^2)^2} \left[ \frac{\frac{L_{u_g}}{V}}{(R+1)} + 2\tau \right]$$

(9.4.2.)

The patchiness parameter $P_{T_{u_g}}$ is illustrated in Fig. 22 for different values of $Q$ and $R$.

b. Vertical turbulence velocity, $w_g$
\[ P_{\tau w_g} = \frac{2Q^4 + 2Q^2 + 1}{(1 + Q^2)^2} + \frac{Q^4(R+1)}{(1 + Q^2)^2} \left( 2 \frac{V}{L_{w_g}} \right)^3 (2 \frac{V}{L_{w_g}} \tau + 1) \left( 5 \frac{V^2}{2} \tau^2 + 6 \frac{V}{L_{w_g}} \tau + 2 \right) \]

\[ + \frac{2Q^4(R+1)}{(1 + Q^2)^2} \left( 2 \frac{V}{L_{w_g}} \tau + 1 \right)^3 \left( 2 \frac{V}{L_{w_g}} \tau + 1 \right)^2 \]

(9.4.3.)

The patchiness parameter \( P_{\tau w_g} \) is illustrated in Fig. 23 for different values of \( Q \) and \( R \).

c. Lateral turbulence velocity, \( v_g \)

The expression for \( P_{\tau v_g} \) is identical to the expression (9.4.3.) except for the replacement of the index \( w_g \) by \( v_g \).

In Section 7.2, the final measure of patchiness has been defined as \( P_{\tau = \infty} \). The limit:

\[ P_\infty = \lim_{\tau \to \infty} P_\tau \]

is calculated in Appendix 5. The result is as follows:

a. Longitudinal turbulence velocity, \( u_g \)

\[ P_{u_g} = 1 + \frac{1}{6} \left( \frac{(R + 1)^2}{R} + 1 \right) \cdot [K - 3] \]  

(9.4.4.)

where \( K \) is the kurtosis value of \( u_g(t) \).
b. **Vertical and lateral turbulence velocities, \( v_g \) and \( v_g \)**

\[
P_{\infty_v g,v_g} = 1 + \frac{1}{30} \left[ R^3 - R^2 + 3R + 18 + \frac{8}{K} \right] [K - 3]
\tag{9.4.5.}
\]

It can be seen from the eqs. (9.4.4.) and (9.4.5.) that \( P_{\infty} \) is a linear function of the kurtosis, \( K \). The behaviour of \( P_{\infty} \) is plotted as a function of the kurtosis value \( K \) at various values of \( R \) in Fig. 24 for the longitudinal turbulence velocity and in Fig. 25 for the vertical or lateral turbulence velocity. From these Figures it can be seen that a pilot can be expected to sense the same patchiness, as expressed by \( P_{\infty} \), at different fourth order moments and correspondingly different values of \( R \). Furthermore, the influence of a particular value of \( R \) becomes more significant with increasing value of the kurtosis value \( K \), while at \( K = 3 \), Gaussian turbulence, the influence of \( R \) has vanished.
10. - Conclusions

The available literature produces increasingly convincing evidence that atmospheric turbulence inherently is non-Gaussian in its structure. Therefore, atmospheric turbulence cannot sufficiently be described by its probability density functions and power spectral density functions alone. The patchiness of atmospheric turbulence caused by departures from Gaussian behaviour has been investigated in this report by studying higher order moments of the probability density functions, simultaneously with second order power spectral densities of the turbulence. A class of non-Gaussian processes has been described, which can approximate or predict the measured non-Gaussian probability densities as well as the well-known measured power spectral densities.

Statistical properties are defined in a proposed model of patchiness, which are contained neither in the non-Gaussian probability densities, nor in the power spectral densities due to von Kármán and Dryden. An important aspect of the proposed model is its usefulness in the generation of time-histories of simulated turbulence velocities having specified non-Gaussian characteristics, along with the conventional power spectral densities. Furthermore, the proposed model of patchiness suggests a method to better define and investigate the patchy characteristics of actual atmospheric turbulence. Currently research is in progress to derive numerical values for the patchiness parameters defined in this report, from measurements of actual atmospheric turbulence.
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Fig. 3a. Comparison between the power spectral densities due to Von Karman and due to Dryden; longitudinal turbulence velocities.
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\[ \frac{4}{\pi} \exp \left( -\frac{\gamma^2}{\pi} \right) \quad (\gamma > 0) \]

Window, representing pilot's memory

\[ 10 \text{ sec} \]

a) Time series of the signals \( w(t) \) and \( w^2(t) \).

\[ w^2(t) \rightarrow H(j\omega) = \frac{1}{1 + j\omega T} \]

Output variance \( \sigma_z^2 \)

b) Linear filter, representing pilot's memory

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Normal distribution Lower limit
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Appendix I

The auto-power spectral density of the squared Gaussian turbulence velocities

**Longitudinal turbulence velocities**

The auto-correlation function due to Dryden is, eq. (6.4.10.):

\[ C_{u_g u_g}(\tau) = \sigma_{u_g}^2 \exp \left[ -|\tau| \frac{V}{L_{u_g}} \right] \]  \hspace{1cm} (A-1.1.)

According to Ref. 9 the auto-correlation function of a squared Gaussian process can be expressed in terms of the auto-correlation function of the Gaussian process \( u_g(t) \), with zero mean value:

\[ R_{u_g^2 u_g^2}(\tau) = R_{u_g u_g}(0) + 2R_{u_g u_g}(\tau) \]  \hspace{1cm} (A-1.2.)

Since the Gaussian process \( u_g(t) \) has zero mean value, the auto-correlation function and auto-covariance function are equal. Substituting eq. (A-1.1.) in eq. (A-1.2.) yields:

\[ R_{u_g^2 u_g^2}(\tau) = \sigma_{u_g}^4 + 2\sigma_{u_g}^4 \exp \left[ -2|\tau| \frac{V}{L_{u_g}} \right] \]  \hspace{1cm} (A-1.3.)

The factor \( \sigma_{u_g}^4 \) represents the squared mean value of the process \( u_g^2(t) \), which is not equal to zero. According to eq. (5.6.5.), the auto-covariance function of the process \( u_g^2(t) \) will be:

\[ C_{u_g^2 u_g^2}(\tau) = 2\sigma_{u_g}^4 \exp \left[ -2|\tau| \frac{V}{L_{u_g}} \right] \]  \hspace{1cm} (A-1.4.)

By Fouriertransforming eq. (A-1.4.), the auto-power spectral density of the squared Gaussian turbulence velocity \( u_g^2(t) \) is obtained:

\[ \Phi_{u_g^2 u_g^2}(\omega) = \frac{8}{\pi} \sigma_{u_g}^4 \frac{V}{L_{u_g}} \frac{1}{\sigma_{u_g}^4 \left( \frac{V^2}{L_{u_g}} \right)^2 + \omega^2} \]  \hspace{1cm} (A-1.5.)
Vertical turbulence velocities

The auto-correlation function due to Dryden is, eq. (6.4.12.):

\[ C_{\omega g \omega g}(\tau) = \sigma_{\omega g}^2 \exp \left[ - |\tau| \frac{V}{L_{\omega g}} \right] \left( 1 - \frac{1}{2} |\tau| \frac{V}{L_{\omega g}} \right) \]  
(A-1.6.)

Substituting eq. (A-1.6.) in eq. (A-1.2.) yields:

\[ R_{\omega g \omega g} = \sigma_{\omega g}^4 + 2\sigma_{\omega g}^4 \exp \left[ -2 |\tau| \frac{V}{L_{\omega g}} \right] \left( 1 - \frac{1}{2} |\tau| \frac{V}{L_{\omega g}} \right)^2 \]  
(A-1.7.)

The auto-covariance function is:

\[ C_{\omega g \omega g}(\tau) = 2\sigma_{\omega g}^4 \exp \left[ -2 |\tau| \frac{V}{L_{\omega g}} \right] \left( 1 - |\tau| \frac{V}{L_{\omega g}} + \frac{\tau^2}{4} \frac{V^2}{L_{\omega g}^2} \right) \]  
(A-1.8.)

The Fourier transform of \( C_{\omega g \omega g}(\tau) \) is:

\[ \phi_{\omega g \omega g}(\omega) = \frac{4}{\pi} \sigma_{\omega g}^4 \frac{V}{L_{\omega g}} \left[ \frac{20 \frac{V^4}{L_{\omega g}^4} + 13 \frac{V^2}{L_{\omega g}^2} \omega^2 + 3\omega^4}{(4 \frac{V^2}{L_{\omega g}^2} + \omega^2)^3} \right] \]  
(A-1.9.)
Appendix 2

Calculation of the mean value and variance of the squared simulated turbulence velocity $w^2(t)$

The probability density function of the stochastic process $w^2(t)$ is expressed by eq. (9.1.5.):

$$p_{w^2}(y) = \frac{1}{\sqrt{2\pi}} \left[ \frac{1+Q^2}{\pi \sigma_w^2} \right]^{\frac{1}{2}} \int_0^\infty \left[ \frac{1}{1+2\zeta^2 Q^2} \right]^{\frac{1}{2}} \exp \left( -\frac{1}{2} \frac{y}{\sigma_w^2} + \frac{1+Q^2}{1+2\zeta^2 Q^2} \right) d\zeta$$

for $y \geq 0$  \hspace{1cm} (A-2.1.)

and $p_{w^2}(y) = 0$ for $y < 0$

The mean value of the stochastic process $w^2(t)$ can be obtained by calculating the integral:

$$\mu_{w^2} = \int_{-\infty}^{+\infty} y \ p_{w^2}(y) \ dy$$ \hspace{1cm} (A-2.2.)

where $y = w^2$.

Substituting eq. (A-2.1.) into eq. (A-2.2.) yields:

$$\mu_{w^2} = \int_{0}^{\infty} \int_{0}^{\infty} \left[ \frac{y(1+Q^2)}{\pi \sigma_w^2} \right]^{\frac{1}{2}} \left[ \frac{1}{1+2\zeta^2 Q^2} \right]^{\frac{1}{2}} \exp \left( -\frac{1}{2} \frac{y}{\sigma_w^2} + \frac{1+Q^2}{1+2\zeta^2 Q^2} \right) d\zeta \ dy$$

\hspace{1cm} (A-2.3.)

Interchanging the order of integration in eq. (A-2.3.) and integrating with respect to $y$ results:

$$\mu_{w^2} = \frac{2\sigma_w^2}{\sqrt{\pi}(1+Q^2)} \int_0^\infty \left( 1+2\zeta^2 Q^2 \right) e^{-\zeta^2} d\zeta$$ \hspace{1cm} (A-2.4.)

Integrate eq. (A-2.4.) to obtain the final result:

$$\mu_{w^2} = \sigma_w^2$$ \hspace{1cm} (A-2.5.)
The variance of the stochastic process $w^2(t)$ can be obtained by first calculating the second moment $M_2$ of $w^2(t)$. According to eq. (5.5.1) the second non-central moment of $w^2(t)$ is:

$$M_2 = \int_{-\infty}^{+\infty} y^2 p_{w^2}(y) \, dy \quad (A-2.6.)$$

Substituting eq. (A-2.1) yields:

$$M_2 = \int_0^\infty \int_0^\infty y \left( \frac{y(1+Q^2)}{\pi \sigma_w^2} \right)^{\frac{1}{2}} \left[ \frac{2}{1+2\zeta^2 Q^2} \right]^{\frac{1}{2}} \exp \left[ -\zeta^2 - \frac{v}{\sigma_w^2} \frac{1+Q^2}{1+2\zeta^2 Q^2} \right] d\zeta \, dy \quad (A-2.7.)$$

Interchanging the order of integration in eq. (A-2.7) and integrating with respect to $y$ results:

$$M_2 = \frac{6 \sigma_w^4}{\sqrt{\pi} (1+Q^2)^2} \int_0^\infty (1+2\zeta^2 Q^2) \exp \left[ -\zeta^2 \right] d\zeta \quad (A-2.8.)$$

Integrate eq. (A-2.8) with respect to $\zeta$:

$$M_2 = \frac{\sigma_w^4}{(1+Q^2)^2} (9Q^4 + 6Q^2 + 3) \quad (A-2.9.)$$

According to eq. (8.1.16) the second order moment of $w^2(t)$ is equal to the fourth order central moment $w(t)$.

The variance of the process $w^2(t)$, which is equal to the second order central moment of $w^2(t)$, see eq. (5.5.4.), is:

$$\sigma_{w^2}^2 = M_2(\omega^2) - \mu_{\omega^2}^2 \quad (A-2.10.)$$

Therefore combining eqs. (A-2.9) and (A-2.8) yields:

$$\sigma_{w^2}^2 = \frac{2 \sigma_w^4}{(1+Q^2)^2} (4Q^4 + 2Q^2 + 1) \quad (A-2.11.)$$
Appendix 3

Calculation of the covariance-function of the squared non-Gaussian process $C_{w^2w^2}(\tau)$

The correlation-function of a stochastic process $R_{yy}(\tau)$ is defined, according to eq. (5.6.2.), as:

$$R_{yy}(\tau) = E\{y(t) \cdot y(t + \tau)\} \quad (A-3.1.)$$

Consider the stochastic process $y(t) = w^2(t)$. According to eq. (9.2.2.):

$$y(t) = w^2(t) = a^2(t) \cdot b^2(t) + c^2(t) + 2a(t) \cdot b(t) \cdot c(t) \quad (A-3.2.)$$

where $a(t)$, $b(t)$ and $c(t)$ are uncorrelated Gaussian stochastic processes. The correlation-function $R_{yy}(\tau)$ is calculated by taking the expected value of eq. (A-3.2.), according to eq. (A-3.1.):

$$R_{yy}(\tau) = E\left\{a^2(t) \cdot b^2(t) + c^2(t) + 2a(t) \cdot b(t) \cdot c(t) \cdot a^2(t+\tau) \cdot b^2(t+\tau) + c^2(t+\tau) + 2a(t+\tau) \cdot b(t+\tau) \cdot c(t+\tau)\right\} \quad (A-3.3.)$$

The expected value of the product of random variables, expressed by eq. (A-3.3.) can be written as a sum of products of random variables. Bearing in mind that the expected value of a sum of random variables equals the sum of the expected values, see Ref. 9, the following sum of expected values is obtained:

$$R_{yy}(\tau) = E\{w^2(t) \cdot w^2(t+\tau)\} = E\{a^2(t) \cdot b^2(t) \cdot a^2(t+\tau) \cdot b^2(t+\tau)\} +$$

$$+ E\{a^2(t) \cdot b^2(t) \cdot c^2(t+\tau)\} +$$

$$+ 2E\{a^2(t) \cdot b^2(t) \cdot a(t+\tau) \cdot b(t+\tau) \cdot c(t+\tau)\} +$$

$$+ E\{c^2(t) \cdot a^2(t+\tau) \cdot b^2(t+\tau)\} +$$
\[ + E\{c^2(t) c^2(t+\tau)\} + \]
\[ + 2E\{a(t+\tau) b(t+\tau) c(t+\tau) a^2(t)\} + \]
\[ + 2E\{a(t) b(t) c(t) a^2(t+\tau) b^2(t+\tau)\} + \]
\[ + 2E\{a(t) b(t) c(t) c^2(t+\tau)\} + \]
\[ + 4E\{a(t) b(t) c(t) a(t+\tau) b(t+\tau) c(t+\tau)\} \]

(A-3.3.)

If \( a(t), b(t) \) and \( c(t) \) are uncorrelated stochastic processes, \( a^2(t), b^2(t) \) and \( c^2(t) \) are also uncorrelated.

According to Ref. 9 the expected value of the product of uncorrelated stochastic processes equals the product of the individual expected values. Thus:

\[ E\{x(t) \cdot z(t)\} = E\{x(t)\} \cdot E\{z(t)\} \]

(A-3.4.)

if \( x(t) \) and \( z(t) \) are uncorrelated

Due to eq. (A-3.4.), the expression (A-3.3.) can be reduced, by carefully distinguishing between the random variables, which are uncorrelated and those which are not uncorrelated. Furthermore the terms evaluated from eq. (A-3.3.), which contain \( E\{a(t)\} \) or \( E\{b(t)\} \) or \( E\{c(t)\} \) will vanish, since the random variables \( a(t), b(t) \) and \( c(t) \) have zero mean values.

The final result is:

\[ R_{yy}(\tau) = R_{x\omega^2}(\tau) = E\{a^2(t) a^2(t+\tau)\} \cdot E\{b^2(t) b^2(t+\tau)\} \cdot E\{c^2(t) \cdot c^2(t+\tau)\} + 4E\{a(t) a(t+\tau)\} E\{b(t) b(t+\tau)\} E\{c(t) \cdot c(t+\tau)\} \]

(A-3.5.)
Using the definition of the auto-correlation function, eq. (A-3.1.):

\[
R_w^2(\tau) = R_{a2a2}(\tau) R_{b2b2}(\tau) + R_{c2c2}(\tau) + 4R_{aa}(\tau) R_{bb}(\tau) R_{cc}(\tau) \\
+ 2R_{aa}(0) R_{bb}(0) R_{cc}(0)
\]  

(A-3.6.)
Appendix 4

Calculation of the variance of the memory filter output $\sigma_{z(\tau)}^2$ non-normal
The auto-power spectral density function of the output of the memory
filter is, see eq. (9.3.2.):

$$\phi_{zz}(\omega) = \frac{1}{1 + \frac{2}{\omega^2 \tau^2}} \cdot \phi_{\omega^2 \omega^2}(\omega) \quad (A-4.1.)$$

In the case of the horizontal turbulence velocity, the auto-power spectral
density reads:

$$\phi_{zz}(\omega) = \frac{1}{1 + \frac{2}{\omega^2 \tau^2}} \cdot \phi_{u^2 u^2}(\omega) \quad (A-4.2.)$$

Substituting eq. (9.2.1.6.) into eq. (A-4.2.) yields:

$$\phi_{zz}(\omega) = \frac{8}{\pi} \sigma_u^4 \frac{L_u}{\mathcal{E}} \frac{V}{(1+Q^2)^2} \left[ \frac{2Q^4 + 2Q^2 + 1}{4 + \frac{L_u}{\mathcal{E}} \frac{V}{2} \omega^2} \frac{1}{1 + \frac{2}{\omega^2 \tau^2}} \right]$$

$$+ Q^4 \left[ \frac{\mathcal{R}(R+1)}{4R^2 + (R+1)^2} \frac{L_u}{\mathcal{E}} \frac{2}{V^2} \omega^2 \frac{1}{1 + \frac{2}{\omega^2 \tau^2}} \right]$$

$$+ Q^4 \left[ \frac{R+1}{4 + (R+1)^2} \frac{L_u}{\mathcal{E}} \frac{2}{V^2} \omega^2 \frac{1}{1 + \frac{2}{\omega^2 \tau^2}} \right] \quad (A-4.3.)$$

The variance $\sigma_{z(\tau)}^2$ non-normal can be obtained by integrating eq. (A-4.3.),
according to eq. (7.1.4.):

$$\sigma_{z(\tau)}^2 \text{non-normal} = \int_0^\infty \phi_{zz}(\omega) \, d\omega \quad (A-4.4.)$$

The integral of eq. (A-4.4.) can be computed as a sum of three integrals,
each of which can be further simplified by means of partial fraction
expansion.
The result is:
\begin{equation}
\sigma_{zg}(\tau)_{\text{non-normal}} = \frac{2\sigma_{ug}^4}{(1+Q^2)^2} \left[ \frac{2Q^4+2Q^2+1}{1+2\tau} \frac{V}{L_{ug}} + \frac{Q^4(R+1)}{R+1+2R\tau} \frac{V}{L_{ug}} + \frac{Q^4(R+1)}{R+1+2\tau} \frac{V}{L_{ug}} \right]
\end{equation}

(A-4.5.)

Equating \( \tau \) to zero, it can be seen from eq. (A-4.2.) that:

\[ \Phi_{zz}(\omega)_{\tau=0} = \Phi_{ug}^2 u_g^2(\omega) \]  

(A-4.6.)

The value of \( \sigma_{zg}^2(0) \) corresponding to the integral value of \( \Phi_{ug}^2 u_g^2(\omega) \) is obtained from eq. (A-4.5.):

\[ \sigma_{zg}^2(\omega) = \frac{2\sigma_{ug}^4}{(1+Q^2)^2} \left[ 4Q^4 + 2Q^2 + 1 \right] \]  

(A-4.7.)

The fourth order central moment of the non-Gaussian turbulence velocities is, according to eq. (8.1.16.):

\[ m_4 = \frac{9Q^4 + 6Q^4 + 3}{(1 + Q^2)^2} \sigma_w^4 \]  

(A-4.8.)

Eqs. (A-4.7.) and (A-4.8.) can be combined to yield the following relation:

\[ \sigma_{zg}^2(0) = m_4(u_g)^{-\sigma_w^4} = \int_0^\infty \Phi_{ug}^2 u_g^2(\omega) \, d\omega \]  

(A-4.9.)

or, because of eq. (5.5.7.):

\[ (K_{ug} - 1) \sigma_g^4 = \int_0^\infty \Phi_{ug}^2 u_g^2(\omega) \, d\omega \]  

(A-4.10.)
where $K_u$ is the kurtosis value of the longitudinal turbulence velocity.

Similarly the variance of the output of the memory filter in the case of the vertical or lateral turbulence velocity is calculated. Substituting eq. (9.2.2.3.) into eq. (A-4.1.) yields:

$$
\phi_{zz}(\omega) = \frac{4}{\pi} \frac{\sigma_{wg}^4}{(1+Q^2)^2} \frac{V}{L_{\omega g}} \left( 2Q^4 + 2Q^2 + 1 \right) \frac{20 \frac{V^4}{L_{\omega g}^2} + 13 \frac{V^2}{L_{\omega g}} \omega^2 + 3\omega^4}{4 \left( 4 \frac{V^2}{L_{\omega g}} + \omega^2 \right)^3}.
$$

$$
\cdot \frac{1}{1+\omega^2} + Q^4 \left\{ \frac{2R(R+1)}{4 \frac{V^2}{L_{\omega g}} + (R+1)^2 \omega^2} \frac{1}{1+\omega^2} - \frac{4 \frac{V^2}{L_{\omega g}} (R+1) - (R+1)^4 \omega^2}{4 \left[ 4 \frac{V^2}{L_{\omega g}} + (R+1)^2 \omega^2 \right]^2} \frac{1}{1+\omega^2} + \frac{4 \frac{V^4}{L_{\omega g}} (R+1)^3 - 3(R+1)^5 \frac{V^2}{L_{\omega g}} \omega^2}{4 \left[ 4 \frac{V^2}{L_{\omega g}} + (R+1)^2 \omega^2 \right]^3} \frac{1}{1+\omega^2} \right\}.
$$

(A-4.11.)

The variance $\sigma_{z_{wg}}^2(\tau)$ non-normal is obtained according to:
\[ \sigma_{z_{w_{g}}}^{2}(\tau)_{\text{non-normal}} = \int_{0}^{\infty} \phi_{z_{z}}(\omega) \, d\omega \quad (A-4.12.) \]

The integral of (A-4.12.) can be written as the sum of seven integrals. Each of these can be simplified by the technique of partial fraction expansion. The result is a sum of integrals of which the general formula of each can be written as, see Ref. 24:

\[ \int_{0}^{\infty} \frac{x^{n-1}}{(p + qx)^{n+1}} = \frac{1}{\nu_{p}} \frac{\Gamma(\frac{1}{\nu})}{\Gamma(1 + n)} (p_{q}^{n+1}) \quad (A-4.13.) \]

After some mathematical elaboration the following result is obtained:

\[ \sigma_{z_{w_{g}}}^{2}(\tau) = \frac{\sigma_{w_{g}}^{4}}{(1 + Q^{2})^{2}} \left[ \frac{5}{2} \frac{V^{2}}{L_{w_{g}}} \tau^{2} + 6 \frac{V}{L_{w_{g}}} \tau + 2 \frac{V^{2}}{L_{w_{g}}} \tau^{4} + 2Q^{4}(R+1) \left\{ \frac{2}{2} \frac{V}{L_{w_{g}}} \tau^{R+1} - \frac{2}{2} \frac{V}{L_{w_{g}}} \tau^{R+1} \right\} \right] \quad (A-4.14.) \]

If \( \tau \) is set equal to zero, the value of \( \sigma_{z_{w_{g}}}^{2}(o) \) is obtained:

\[ \sigma_{z_{w_{g}}}^{2}(o) = \frac{2\sigma_{w_{g}}^{4}}{(1 + Q^{2})^{2}} \left[ 4Q^{4} + 2Q^{2} + 1 \right] \quad (A-4.15.) \]

Thus similar to eq. (A-4.9.):

\[ \sigma_{z_{w_{g}}}^{2}(o) = m_{a}(w_{g}) - \sigma_{w_{g}}^{4} = \int_{0}^{\infty} \phi_{w_{g}}(\omega) \, d\omega \quad (A-4.16.) \]
Appendix 5

Calculation of the patchiness parameter $P_\infty$

The patchiness parameter $P_\tau$ is defined as, see eq. (7.2.1.):

$$P_\tau = \frac{\sigma_z^2(\tau)_{\text{non-normal}}}{\sigma_z^2(\tau)_{\text{normal}}} \quad (A-5.1.)$$

According to eq. (9.4.1.), the patchiness parameter $P_{\tau u_g}$ for the longitudinal turbulence velocities is:

$$P_{\tau u_g} = \frac{2Q^4 + 2Q^2 + 1}{(1 + Q^2)^2} + \frac{Q^4 (R+1) \left( \frac{U_g}{V} + 2\tau \right)}{(1 + Q^2)^2 \left[ (R+1) \frac{U_g}{V} + 2\tau \right]} + \frac{Q^4 (R+1) \left( \frac{U_g}{V} + 2\tau \right)}{(1 + Q^2)^2 \left[ (R+1) \frac{U_g}{V} + 2\tau \right]} \quad (A-5.2.)$$

If $\tau$ is taken to infinity the limit of $P_{\tau u_g}$ is obtained:

$$\lim_{\tau \to \infty} P_{\tau u_g} = \frac{Q^4 \left[ 2 + \frac{(R+1)^2}{R} \right] + 2Q^2 + 1}{(1 + Q^2)^2} \quad (A-5.3.)$$

From eq. (9.2.1.6.) the value of $\Phi_{u_g 2u_g 2}(0)$ is derived as:

$$\Phi_{u_g 2u_g 2}(0)_{\text{non-normal}} = \frac{2}{\pi} \sigma_{u_g} L_u \frac{U_g}{V} \frac{Q^4 \left[ 2 + \frac{(R+1)^2}{R} \right] + 2Q^2 + 1}{(1 + Q^2)^2} \quad (A-5.4.)$$

If $Q$ is set equal to zero in eq. (A-5.4.) the value of $\Phi_{u_g 2u_g 2}(0)_{\text{normal}}$ is obtained:

$$\Phi_{u_g 2u_g 2}(0)_{\text{normal}} = \frac{2}{\pi} \sigma_{u_g} L_u \frac{U_g}{V} \quad (A-5.5.)$$

Therefore, combining eqs. (A-5.3.) until (A-5.5.), the following relation holds:
$$P_{\infty u_g} = \lim_{\tau \to \infty} \frac{\phi_{u_g^2 u_g^2(0)}_{\text{non-normal}}}{\phi_{u_g^2 u_g^2(0)}_{\text{normal}}}$$  \hspace{1cm} (A-5.6.)

The kurtosis of the non-normal process proved to be, see eq. (8.1.16.):

$$K = \frac{m_4}{\sigma_w^4} = \frac{9Q^4 + 6Q^2 + 3}{(1 + Q^2)^2}$$  \hspace{1cm} (A-5.7.)

Using eq. (A-5.7.), the patchiness parameter $P_{\infty u_g}$ can be expressed as a function of the kurtosis:

$$P_{\infty u_g} = 1 + \frac{1}{6} \left[ \frac{(R + 1)^2}{R} + 1 \right] \cdot [K - 3]$$  \hspace{1cm} (A.5.8.)

Similarly, the patchiness parameter $P_{\infty w_g}$ for the vertical turbulence velocity can be calculated. To this end the parameter $\tau$ of $P_{\tau w_g}$ in eq. (9.4.2.) is taken to infinity.

The result is:

$$P_{\infty w_g} = \lim_{\tau \to \infty} P_{\tau w_g} = \frac{2Q^4 + 2Q^2 + 1}{(1 + Q^2)^2} + \frac{8Q^4(R + 1)}{5(1 + Q^2)^2} - \frac{4Q^4(R + 1)^2}{5(1 + Q^2)^2}$$

$$+ \frac{Q^4(R + 1)^3}{5(1 + Q^2)^2} + \frac{8Q^4(R + 1)}{5R(1 + Q^2)^2}$$  \hspace{1cm} (A-5.9.)

Employing eq. (A-5.7.) the patchiness parameter $P_{\infty w_g}$ expressed as a function of the kurtosis is:

$$P_{\infty w_g} = 1 + \frac{1}{30} \left( R^3 - R^2 + 3R + 18 + \frac{8}{R} \right) \cdot [K - 3]$$  \hspace{1cm} (A-5.10.)