MAGNITUDE CONTROL OF COMMUTATOR ERRORS

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Abstract. Non-uniform filtering of the Navier-Stokes equations expresses itself, next to the turbulent stresses, in additional closure terms known as commutator errors. These terms require explicit subgrid modeling if the non-uniformity of the filter is sufficiently pronounced. We derive expressions for the magnitude of the mean flux, the turbulent stress flux and the commutator error for individual Fourier modes. This gives rise to conditions for the spatial variations in the filter-width and the filter-skewness subject to which the magnitude of the commutator errors can be controlled. These conditions are translated into smoothness requirements of the computational grid, that involve ratios of first-, second- and third order derivatives of the grid mapping.

1 Introduction

The desire to extend large-eddy simulation to flows in complex domains generally implies that one is confronted with strongly varying turbulence intensities within the flow-domain. In certain regions a nearly laminar flow may exist while a lively, fine-scale turbulent flow may be present simultaneously in other regions. This emphasizes the need to incorporate anisotropic, heterogeneous small scales into the large-eddy approach, to consistently address turbulent flows in complex situations \cite{1, 2}.

The usual formulation of the filtering approach to large-eddy simulation is based on convolution filters \cite{3}. This formulation assumes that the width of the filter is constant. However, the efficient extension of large-eddy simulation to turbulent flows in complex geometries and to cases with strong spatial variation of turbulence intensities, calls for the introduction of non-uniform filter-widths \cite{4, 5, 6}. To be specific, turbulent boundary layers, wake-vortex flows, multi-phase flows and strongly localized combustion phenomena are four examples of turbulent flow whose efficient modeling naturally summons spatially varying filter-widths.

The use of spatially non-uniform filters complicates the subgrid closure problem in large-eddy simulation. While the application of convolution filters gives rise to the tur-
bulent stress tensor \[1\], non-uniform filtering leads to the appearance of additional commutator errors \[7\]. These terms arise because non-uniform filtering does not commute with spatial differentiation. That is, \( \overline{\partial_x u} \neq \partial_x \overline{u} \) where \( \partial_x u \) denotes differentiation of the solution \( u \) with respect to \( x \) and the overline indicates the filter operation.

One may distinguish two different approaches to deal with commutator errors: (i) the variations in the filter-properties are kept sufficiently small so that the dynamic contributions of the commutator errors may be neglected, relative to the turbulent stress contributions, or, (ii) the spatial variations in the turbulence properties are so strong, e.g., in turbulent boundary layers \[8\], that an efficient capturing of the flow requires significant filter-non-uniformities and hence the explicit modeling of the commutator errors.

It is the purpose of this paper to establish conditions under which commutator errors can be expected to be negligible. The magnitude of commutator errors that may arise in large-eddy simulation of incompressible turbulent flow is known to be of the same order in the filter-width, or larger, as that of the turbulent stress fluxes, for general high-order spatial filters \[2\]. Consequently, one cannot reduce the size of the commutator errors independently of the turbulent stress terms by any judicious construction of the filter operators, contrary to claims in \[9, 10\]. Independent control over the commutator errors compared to the turbulent stress fluxes can, instead, be obtained by appropriately restricting the spatial variations of the filter-width and filter-skewness. Based on a Fourier-mode analysis, criteria for the level of non-uniformity of the filter are obtained, such that commutator errors are expected to be dynamically negligible and explicit modeling of the commutator error does not appear to be required. Moreover, these criteria can be translated into requirements for the grid-mapping. If it is possible to adhere to these requirements, a solution-dependent grid generation method for large-eddy simulation may be obtained such that only the closure of the turbulent stress fluxes is required to good approximation.

The organization of this paper is as follows. In section 2 we review the non-uniform high-order filtering of the Navier-Stokes equations governing incompressible flow. Section 3 is devoted to the determination of the magnitude of the commutator errors relative to the turbulent stress fluxes and the mean convective fluxes. This is specified for individual Fourier modes. Conditions are derived under which commutator errors and/or turbulent stress fluxes are negligible relative to the mean convective fluxes. In section 4 requirements are derived for the grid-mapping that yield negligible commutator errors. Concluding remarks are collected in section 5.

2 Non-uniform high-order filtering of the Navier-Stokes equations

In this section a general class of non-uniform filters with compact support is introduced and applied to the equations governing incompressible flow. The application of a non-uniform filter generates turbulent stress fluxes as well as commutator errors \[11\]. Both groups of closure terms will be written as the commutator bracket of the filter operator and either the product operator, or the derivative operator. Consequently, the basic subgrid
modeling problem in large-eddy simulation shares several formal algebraic properties with the Poisson-bracket in classical mechanics [12, 13]. Moreover, we review the dependence of the order of magnitude of the commutator error and the turbulent stress flux on the filter-width and filter-skewness to emphasize that high-order filtering does not offer an independent control over commutator errors relative to the turbulent stress fluxes [2].

The filtering approach adopted here is based on a general compact-support filter, whose application in one spatial dimension is denoted by $\mathcal{L}$:

$$\overline{u}(x,t) = \mathcal{L}(u)(x,t) = \int_{x-\Delta(x,t)}^{x+\Delta(x,t)} \frac{H(x,\xi,t)}{\Delta(x,t)} u(\xi,t) \, d\xi$$

where $H(x,\xi,t)$ is the ‘characteristic’ filter function and $\Delta_{\pm} \geq 0$ denote the upper – and lower bounding functions which define the filter-width $\Delta = \Delta_+ + \Delta_-$. The filter $\mathcal{L}$ is assumed to be normalized, i.e., $\mathcal{L}(1) = 1$. This class of filters can readily be extended to product-filters in three spatial dimensions by defining the composition $\mathcal{L} = \mathcal{L}_1 \circ \mathcal{L}_2 \circ \mathcal{L}_3$ where $\mathcal{L}_j$ with $j = 1, 2, 3$, represents filtering in the $x_j$-direction only, as in (1). In complex flows, spatial and temporal variations in turbulence intensity pose different requirements on the local length-scale with which the flow should be represented to maintain an acceptable level of accuracy. Such situations may be addressed by allowing a non-uniform filter-width, as suggested in (1).

Incompressible flow is governed by conservation of mass and momentum. These can be expressed in the continuity equation and Navier-Stokes equations as

$$\partial_j u_j = 0$$

$$\partial_t u_i + \partial_j(u_j u_i) + \partial_i p - \frac{1}{Re} \partial_{jj} u_i = 0 \quad ; \quad i = 1,2,3$$

where $u_j$ is the component of the velocity $u$ in the $x_j$-direction, $t$ denotes time and $\partial_t$, $\partial_j$ are the partial derivative operators with respect to $t$ and $x_j$ respectively. Moreover, $p$ is the pressure and $Re = (u_r \lambda_r) / \nu_r$ denotes the Reynolds number in terms of reference velocity $u_r$, length-scale $\lambda_r$ and kinematic viscosity $\nu_r$ [14]. Throughout, the summation convention is adopted, implying summation over repeated indices.

If one applies the filter $\mathcal{L}$ to the incompressible flow equations, commutator errors may arise, e.g., if $\overline{\partial_x f} - \partial_x \overline{f} = L(\partial_x f) - \partial_x (L(f)) = [L, \partial_x](f) \neq 0$ [15]. Here, the commutator error is written in terms of the commutator bracket $[L, \partial_x]$ of $L$ and the derivative operator $\partial_x$. One may show that $[L, \partial_j](f) = 0$ for $j = 1,2,3$, if and only if the filter $L$ is a convolution filter, which, by definition is spatially uniform. For the non-uniformly filtered continuity equation we may formally write

$$\partial_j \overline{\pi}_j = -[L, \partial_j](u_j) \quad (4)$$

Hence, the divergence of the non-uniformly filtered velocity differs from zero, i.e., $\overline{\pi}_j$ is not solenoidal, and the corresponding continuity equation is no longer in local conservation form. The term on the right-hand side corresponds to apparent local creation
and annihilation of ‘resolved’ mass as a consequence of variations in $\Delta_x$ and $H$. Further developments and parameterization are needed before this effect of non-uniform filtering can be consistently integrated into the large-eddy formulation. As an example, the form of (4) may motivate similarity modeling of the right-hand side: $[L, \partial_j](u_j) \rightarrow [L, \partial_j](\overline{u}_j)$. This yields specific contributions to the Poisson equation for the pressure and is a subject of current research.

Filtering the Navier-Stokes equations yields the following system:

$$
\partial_t \overline{u}_i + \partial_j(\overline{u}_j \overline{u}_i) + \partial_i p - \frac{1}{Re} \partial_{jj} \overline{u}_i = - \left\{ [L, \partial_t](u_i) + \partial_j([L, \Pi](u_i, u_j)) + [L, \partial_j](\Pi(u_i, u_j)) + [L, \partial_i](p) - \frac{1}{Re} [L, \partial_{jj}](u_i) \right\}
$$

(5)

We observe that commutator brackets emerge involving the filter $L$ and the product operator $\Pi(f, g) = fg$, as well as commutator brackets of $L$ and first–or second order partial differentiation. Filtering a linear term such as $\partial_t u_i$ gives rise to a ‘mean-flow’ term $\partial_t \overline{u}_i$ and a corresponding commutator error $[L, \partial_t](u_i)$. Filtering the nonlinear convective terms leads to two different types of closure terms. First, as in the case of convolution filtering, the divergence of the turbulent stress tensor $\tau_{ij} = \overline{u}_i \overline{u}_j - \overline{u}_i \overline{u}_j = \Pi(u_i, u_j)$ arises. The divergence of $\tau_{ij}$ will be called the turbulent stress flux. Second, an associated commutator error $[L, \partial_j](\Pi(u_i, u_j))$ emerges from filtering the convective fluxes. The local conservation form of the Navier-Stokes equations is no longer maintained as a result of the non-uniform filtering, similar to what was observed in (4) for the continuity equation.

We next investigate the magnitude of the commutator errors and turbulent stress fluxes arising from the application of high-order filters, closely following [2]. General $N$-th order filters are introduced by requiring that $L(x^k) = x^k$ for $k = 0, 1, \ldots, N-1$ [15]. Previously [9, 10] it was claimed that an increase in the order of these filters reduces the magnitude of the commutator errors. This is correct but incomplete as the same filters yield an equally strong reduction in the order of magnitude of the turbulent stress fluxes [2, 5]. Specifically, we consider compact support filters in one spatial dimension which do not explicitly depend on time $t$, i.e., $[L, \partial_t](u) = 0$. These filters may be written as:

$$
\overline{u}(x, t) = \int_{I_x} dy ~ H\left(x, x + \Delta(x)y\right) u\left(x + \Delta(x)y, t\right)
$$

(6)

in the integration-variable $y = (\xi - x)/\Delta(x)$. The support $I_x$ is given by

$$
I_x = \left\{ y \in \mathbb{R} ~ \left| \frac{\sigma(x) - 1}{2} \leq y \leq \frac{\sigma(x) + 1}{2} \right. \right\} ; \quad \sigma(x) = \frac{\Delta_+(x) - \Delta_-(x)}{\Delta_+(x) + \Delta_-(x)}
$$

(7)

in the ‘normalized skewness’ of the filter, $\sigma$, which satisfies $|\sigma| \leq 1$. The filtering of $x^k$ may now be expressed as

$$
\overline{x}^k = \int_{I_x} dy ~ H\left(x, x + \Delta(x)y\right) \left(x + \Delta(x)y\right)^k = \sum_{m=0}^{k} \binom{k}{m} x^{k-m} \Delta_m(x) m! M_m(x)
$$

(8)
in terms of the ‘moments’

\[ M_m(x) = \frac{1}{m!} \int_{\mathcal{I}_x} dy \, H(x, x + \Delta(x)y) \, y^m \] (9)

By requiring the characteristic filter function \( H \) to be such that \( M_m(x) = \delta_{m0} \) for \( m = 0, 1, \ldots, N - 1 \) an \( N \)-th order filter is obtained [15, 16]. Application of an \( N \)-th order filter to a smooth signal \( u \) yields:

\[
\bar{u} - u = \sum_{m=N}^{\infty} \left( \Delta^m(x) M_m(x) \right) u^{(m)} = \Delta^N(x) M_N(x) u^{(N)} + \ldots \] (10)

where \( u^{(m)} \) denotes the \( m \)-th spatial derivative of \( u \). We observe that the effect of the filter, expressed as the difference between \( u \) and \( \bar{u} \), scales with the \( N \)-th power of the filter-width \( \Delta \). In addition, the moment \( M_N \) determines the magnitude of the filter’s effect. We next investigate the effect of these filters on the commutator error and turbulent stress flux.

To quantify the various subgrid contributions in more detail we consider the following decomposition of a typical nonlinearity:

\[
\partial_x(u^2) = \partial_x(\bar{u}^2) + \partial_x(\bar{u}^2 - \bar{u}^2) + \left\{ \partial_x(u^2) - \partial_x(\bar{u}^2) \right\} = \partial_x(u^2) + [L, \partial_x](\Pi(u)) + [L, \partial_x](\Pi(u)) (11)
\]

in which we distinguish a mean flux contribution \( \partial_x(\bar{u}^2) \) next to the turbulent stress flux \( \partial_x([L, \Pi](u)) \) and the commutator error \( [L, \partial_x](\Pi(u)) \). Analogous to (10) we may find expressions for \( \partial_x(u^2) \) and \( \bar{u}^2 \) and hence also for \( \partial_x(\bar{u}^2) \). Based on this, after some calculation the commutator error \( [L, \partial_x](\Pi(u)) \) can be written as

\[
[L, \partial_x](u^2) = - \sum_{m=N}^{\infty} \left( \Delta^m M_m \right)'(u^{(m)}) \]

\[
= - \sum_{m=N}^{\infty} \left( m \Delta^m - 1 \Delta' M_m + \Delta^m M'_m \right) (u^{(m)}) \] (12)

where the prime indicates differentiation with respect to \( x \). Combination of \( \bar{u}^2 \) and \( \bar{u}^2 \) allows the turbulent stress tensor to be expressed as

\[
[L, \Pi](u) = \sum_{m=N}^{\infty} \left( \Delta^m M_m \right) g_m(x) \] (13)

\[
g_m(x) = \left[ (u^2)^{(m)} - 2uu^{(m)} \right] - u^{(m)}(x) \sum_{k=N}^{\infty} \left( \Delta^k M_k \right) u^{(k)}(x) \] (14)
and, correspondingly, we find for the turbulent stress flux

\[
\partial_x([L,\Pi](u)) = \sum_{m=N}^{\infty} \left(\Delta^m M_m\right) \frac{d}{dx} g_m(x) + \left(\Delta^m M_m\right) g'_m(x) \tag{15}
\]

Expressions (12) and (15) form a basis for discussing the magnitude of the commutator error and turbulent stress flux for general \(N\)-th order filters.

To estimate the magnitude of the commutator error and the turbulent stress flux, we describe the non-uniform filter-width in (1) by \(\Delta_{\pm}(x) = \kappa f_{\pm}(x)\) with constant \(\kappa\) such that \(0 \leq \kappa \ll 1\). We assume that \(f_{\pm}\) are positive, bounded functions, with bounded derivatives. The actual magnitude of the various contributions in (12) and (15) depends strongly on the specific non-uniformity of the upper and lower bounding functions \(\Delta_{\pm}\) and the specific filter that was adopted in a given application. However, the typical dominant scaling with \(\kappa\) can be inferred quite generally. We may summarize our findings as follows [2]:

- Turning to (12) both contributions under the summation are of \(m\)-th order in \(\kappa\); the first term because \(M_m\) is of \(O(\kappa^0)\) and \(\Delta^{m-1}\Delta' \sim O(\kappa^m)\), the second because \(M'_m\) is of \(O(\kappa^0)\). Therefore, \([L,\partial_x](u^2) \sim O(\kappa^N)\).

- Likewise, if the order of the filter \(N \geq 2\), the turbulent stress tensor in (13) scales with \(\kappa^N\) since \(g_m\) is of \(O(\kappa^0)\) for \(m \geq 2\). If \(N = 1\) we observed \(g_1 \sim \Delta M_1\) and so \([L,\Pi](u) \sim \kappa^2\). Consequently, the turbulent stress flux in (15) is of order \(\kappa^N\) with a characteristic contribution \(\sim \Delta'\Delta^{N-1}\) as \(N \geq 2\). If \(N = 1\) the turbulent stress flux typically scales with \(\kappa^2\) with a characteristic term \(\sim \Delta'\Delta\).

Hence, the two subgrid contributions to the total flux in (11) are formally of equal order of magnitude if \(N \geq 2\). If \(N = 1\) the commutator error scales with terms of \(O(\kappa)\) while the turbulent stress flux scales with terms which are formally of \(O(\kappa^2)\). In this case, which corresponds, e.g., to the application of a skewed top-hat or Gaussian filter, the formal order of magnitude of the commutator error is even larger than that of the turbulent stress fluxes.

The detailed evaluation of the turbulent stress flux and the commutator error indicates an alternative route toward (some) independent control over the ratio of these contributions. It is well known that commutator errors are zero if and only if the filter is a strict convolution filter. Therefore, it will be intuitively clear that if the spatial filter is ‘close’ to this case, the dynamic implications of the commutator errors are likely to be small. Specifically, this implies that variations in \(\Delta\) and in \(\sigma\) as well as the deviation of \(H(x-\xi)\) from a function \(H(x-\xi)\) should be kept sufficiently small. For general \(N\)-th order filters, a separate control over the commutator error can be obtained only by restraining these non-uniformities. In such cases one could argue that modeling of the commutator errors may not be required. Conversely, for sufficiently large variations of these filter properties,
the fluxes associated with the commutator errors may become significant and require explicit treatment. An a priori analysis along these lines has been described in [5], using direct numerical simulation data of a turbulent mixing flow. This numerical analysis illustrates the estimates above and indicates that for significant filter non-uniformities the commutator errors can no longer be neglected.

In principle, all commutator errors in (4) and (5) require explicit parameterization in much the same way as the turbulent stress fluxes do. However, in practice one would like to address this subgrid closure only for those contributions that are actually dynamically relevant. In the next section we turn to a computation of the mean flux, the turbulent stress flux and the commutator error for individual Fourier modes. From this we will extract criteria for variations in $\Delta$ and $\sigma$ that yield control over the relative magnitudes of these fluxes.

3 Magnitude of commutator errors for single Fourier modes

In this section we characterize restrictions on the non-uniformities of the filter that, when imposed, yield commutator errors that are much smaller than the turbulent stress fluxes. First, we derive expressions for the magnitude of the mean flux, the turbulent stress flux and the commutator error, associated with a general filter. We restrict to one spatial dimension and consider velocity fields consisting of a single Fourier mode, i.e., $u = \sin(kx)$. Subsequently, the relative magnitudes are quantified in detail for a top-hat filter. This will result in expressions, in terms of the wavenumber $k$, the filter-width $\Delta$, the filter-skewness $\sigma$ and their derivatives, which are required to be sufficiently small to render commutator errors much smaller than turbulent stress fluxes.

A precise assessment of the relative magnitude of the different fluxes can be obtained for single Fourier modes, by explicit computation. This requires a number of steps that will be specified in this section. First, we introduce the general decomposition of the filtered convective term

$$\overline{\partial_x(u^2)} = \partial_x(\overline{u^2}) + \partial_x([L, \Pi](u)) + [L, \partial_x](u^2) = \mathcal{M} + \mathcal{T} + \mathcal{C}$$

in which $\mathcal{M}$, $\mathcal{T}$ and $\mathcal{C}$ denote the mean flux, the turbulent stress flux and the commutator error respectively. To quantify the relative dynamic importance of these fluxes we consider

$$\delta^2 = \frac{\|\mathcal{T}\|^2}{\|\mathcal{M}\|^2} ; \quad \varepsilon^2 = \frac{\|\mathcal{C}\|^2}{\|\mathcal{T}\|^2}$$

where we adopt the norm

$$\|f\|^2 = a_0^2 + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

for functions $f$ with a Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} \left( a_k \cos(k(x + \phi)) + b_k \sin(k(x + \phi)) \right)$$
where \( \phi \) is a constant phase. This norm is closely related to the \( L_2 \)-norm \( \| f \|_2 \) of \( f \), i.e.,
\[
\| f \|_2^2 = \| f \|_2^2 + a_0^2/2,
\]
via Parseval’s theorem.

The different fluxes \( \mathcal{C}, T \) and \( M \) can readily be derived for \( u = \sin(kx) \). We find the commutator error as
\[
\mathcal{C} = [L, \partial_x](\sin^2(kx)) = \partial_x(\sin^2(kx)) - \partial_x(\sin^2(kx)) = k \sin(2kx) + \frac{1}{2} \partial_x(\cos(2kx)) \tag{20}
\]
Likewise, we obtain for the turbulent stress flux:
\[
T = \partial_x([L, \Pi](\sin(kx))) = \partial_x\left(\sin^2(kx) - \sin(kx)^2\right) = \partial_x\left(\frac{1}{2}(1 - \cos(2kx)) - \sin(kx)^2\right) \tag{21}
\]
and the mean flux is given by:
\[
M = \partial_x(\sin(kx)^2) \tag{22}
\]
The next task is to rewrite and simplify these fluxes. Basic in each of these expressions is the filtering of \( \sin(kx) \) or \( \cos(kx) \) that are derived first.

The non-uniform filtering of \( \sin(mkx) \) may be obtained from (6) as follows:
\[
\overline{\sin(mkx)} = \int_{(\sigma-1)/2}^{(\sigma+1)/2} H(x, x + \Delta(x)y) \sin(mk(x + \Delta(x)y)) \, dy = F(mk\Delta, \Delta, \sigma, x) \sin(mkx) + G(mk\Delta, \Delta, \sigma, x) \cos(mkx) = A(mk\Delta, \Delta, \sigma, x) \sin\left(mk(x + \phi(mk\Delta, \Delta, \sigma, x))\right) \tag{23}
\]
where \( m \) is a constant and we introduced the amplitude \( A \) and phase \( \phi \):
\[
A = \left(F^2 + G^2\right)^{1/2} \quad \tan(\phi) = \frac{G}{F} \tag{24}
\]
in terms of the filter structure-functions
\[
F(mk\Delta, \Delta, \sigma, x) = \int_{(\sigma-1)/2}^{(\sigma+1)/2} H(x, x + \Delta(x)y) \cos(mk\Delta(x)y) \, dy \tag{25}
\]
\[
G(mk\Delta, \Delta, \sigma, x) = \int_{(\sigma-1)/2}^{(\sigma+1)/2} H(x, x + \Delta(x)y) \sin(mk\Delta(x)y) \, dy \tag{26}
\]
For notational convenience we introduce \( A_m = A(mk\Delta, \Delta, \sigma, x) \) and \( \phi_m = \phi(mk\Delta, \Delta, \sigma, x) \) which allows to write
\[
\overline{\sin(mkx)} = A_m \sin(mk(x + \phi_m)) \tag{27}
\]
Equivalently, we may obtain

\[
\cos mkx = \mathcal{F}(mk\Delta, \Delta, \sigma, x) \sin(mkx) - \mathcal{G}(mk\Delta, \Delta, \sigma, x) \cos(mkx) = A_m \cos(mk(x+\phi_m)) \tag{28}
\]

These expressions will be used to specify the fluxes \(\mathcal{M}, \mathcal{T}\) and \(\mathcal{C}\) in further detail.

For the mean flux \(\mathcal{M}\) we obtain

\[
\mathcal{M} = \partial_x (\sin(kx)^2) = \partial_x \left( \left( A_1 \sin(k(x+\phi_1)) \right)^2 \right) = \partial_x \left( \frac{1}{2} A_1^2 (1 - \cos(2k(x+\phi_1))) \right)
\]

\[
= \left( A_1 A'_1 \right) - \left( A_1 A'_1 \right) \cos(2k(x+\phi_1)) + kA_1^2(1 + \phi'_1) \sin(2k(x+\phi_1))
\]

\[
= a_1 + a_2 \cos(2k(x+\phi_1)) + a_3 \sin(2k(x+\phi_1)) \tag{29}
\]

where we put

\[
a_1 = A_1 A'_1 ; \quad a_2 = -A_1 A'_1 ; \quad a_3 = kA_1^2(1 + \phi'_1) \tag{30}
\]

The corresponding calculation for the turbulent stress flux is somewhat more involved. First, we consider the turbulent stress tensor for which we find

\[
\tau = [L, \Pi](\sin(kx)) = \frac{1}{2} (1 - \cos(2kx)) - \sin(kx)^2
\]

\[
= \frac{1}{2} (1 - A_2 \cos(2k(x+\phi_2))) - \left( A_1 \sin(k(x+\phi_1)) \right)^2
\]

\[
= \frac{1}{2} (1 - A_1^2) - \frac{1}{2} \left[ A_2 \cos(2k(x+\phi_2)) - A_1^2 \cos(2k(x+\phi_1)) \right] \tag{31}
\]

This may be simplified by using the following identity

\[
a \cos(k(x+\alpha)) + b \cos(k(x+\beta)) = A \cos(k(x+\gamma)) \tag{32}
\]

with

\[
A^2 = \{a \cos(k\alpha) + b \cos(k\beta)\}^2 + \{a \sin(k\alpha) + b \sin(k\beta)\}^2 \tag{33}
\]

and

\[
\tan(k\gamma) = \frac{a \sin(k\alpha) + b \sin(k\beta)}{a \cos(k\alpha) + b \cos(k\beta)} \tag{34}
\]

Correspondingly, we obtain

\[
\tau = \frac{1}{2} (1 - A_1^2) - \frac{1}{2} B \cos(2k(x+\psi)) \tag{35}
\]

where

\[
B^2 = \{A_2 \cos(2k\phi_2) - A_1^2 \cos(2k\phi_1)\}^2 + \{A_2 \sin(2k\phi_2) - A_1^2 \sin(2k\phi_1)\}^2 \tag{36}
\]
and
\[ \tan(2k\psi) = \frac{A_2 \sin(2k\phi_2) - A_1^2 \sin(2k\phi_1)}{A_2 \cos(2k\phi_2) - A_1^2 \cos(2k\phi_1)} \]  
(37)

Thus, we may write the desired turbulent stress flux in the form
\[ T = \partial_x \tau = b_1 + b_2 \cos(2k(x + \psi)) + b_3 \sin(2k(x + \psi)) \]  
(38)
in which
\[ b_1 = -A_1 A' ; \quad b_2 = -\frac{1}{2} B' ; \quad b_3 = kB(1 + \psi') \]  
(39)

Finally, we turn our attention to the commutator error \( C \). By definition we may write:
\[ C = \partial_x \left( \frac{\sin^2(kx)}{2} \right) - \partial_x \left( \frac{\sin^2(kx)}{2} \right) = k \sin(2kx) + \frac{1}{2} \partial_x (\cos(2kx)) \]
\[ = kA_2 \sin(2k(x + \phi_2)) + \frac{1}{2} \partial_x (A_2 \cos(2k(x + \phi_2))) \]
\[ = \frac{1}{2} A'_2 \cos(2k(x + \phi_2)) - kA_2 \phi'_2 \sin(2k(x + \phi_2)) = c \cos(2k(x + \alpha)) \]  
(40)
in which we put
\[ c^2 = \left\{ \frac{1}{2} A'_2 \cos(2k\phi_2) - kA_2 \phi'_2 \sin(2k\phi_2) \right\}^2 + \left\{ \frac{1}{2} A'_2 \sin(2k\phi_2) + kA_2 \phi'_2 \cos(2k\phi_2) \right\}^2 \]
\[ = \left( \frac{1}{2} A'_2 \right)^2 + \left( kA_2 \phi'_2 \right)^2 \]  
(41)

and
\[ \tan(2k\alpha) = \frac{\frac{1}{2} A'_2 \sin(2k\phi_2) + kA_2 \phi'_2 \cos(2k\phi_2)}{\frac{1}{2} A'_2 \cos(2k\phi_2) - kA_2 \phi'_2 \sin(2k\phi_2)} \]  
(42)

After these calculations we have obtained the mean flux, the turbulent stress flux and the commutator error associated with \( u = \sin(kx) \). The amplitudes and phases depend in a complicated manner on the filter-width, the wavenumber, the filter-skewness and the filter-kernel. In general, the expressions for the relative magnitudes \( \delta^2 = \|T\|^2/\|M\|^2 \) and \( \varepsilon^2 = \|C\|^2/\|T\|^2 \) are quite complex. Further progress can be made for specific filters. Therefore, we will next specify the magnitude of the individual terms for the popular top-hat filter.

The top-hat filtering of Fourier modes is most directly expressed in the structure functions as introduced in (25) and (26). For the top-hat filter the characteristic filter-kernel \( H = 1 \) and we obtain
\[ F = \int_{(\sigma-1)/2}^{(\sigma+1)/2} \cos(k\Delta y) \, dy = \left( \frac{\sin(k\Delta/2)}{k\Delta/2} \right) \cos(k\Delta\sigma/2) \]  
(43)
and
\[ G = \int_{(\sigma-1)/2}^{(\sigma+1)/2} \sin(k\Delta y) \, dy = \left( \frac{\sin(k\Delta/2)}{k\Delta/2} \right) \sin(k\Delta\sigma/2) \]  
(44)
which yields

\[ A_i^2 = \left( \frac{\sin(k\Delta/2)}{k\Delta/2} \right)^2 ; \quad \tan(k\phi_1) = \frac{G}{F} = \tan(k\Delta\sigma/2) \] (45)

Hence, we infer that the phase-shift is directly related to the skewness of the filter: \( \phi_1 = \Delta\sigma/2 \). Applying the top-hat filter to a mode with wavenumber \( 2k \) we notice that \( \phi_2 = \phi_1 \). These expressions associated with the top-hat filter will next be used to obtain the Fourier-coefficients in \( \mathcal{M}, \mathcal{T} \) and \( \mathcal{C} \).

For the mean flux in (29) we may write

\[ a_1 = A_1A_1' = \left( \frac{1}{2}A_1^2 \right)' = \left( \frac{1}{2}\left( \frac{\sin(k\Delta/2)}{k\Delta/2} \right)^2 \right)' = \left( \frac{(k\Delta)^2}{12} \right) \left( \frac{\Delta'}{\Delta} \right) D(k\Delta) \] (46)

where

\[ D(z) = \frac{12}{z^4} \{ z \sin(z) - 2 + 2 \cos(z) \} = -\{ 1 - \frac{1}{15}z^2 + \frac{1}{560}z^4 - \ldots \} \] (47)

This also implies

\[ a_2 = -A_1A_1' = -\left( \frac{(k\Delta)^2}{12} \right) \left( \frac{\Delta'}{\Delta} \right) D(k\Delta) \] (48)

\[ a_3 = kA_1^2(1 + \phi_1') = k\left( \frac{\sin(k\Delta/2)}{k\Delta/2} \right)^2 (1 + \frac{1}{2}(\Delta\sigma)') \] (49)

For the turbulent stress flux we may proceed similarly. The turbulent stress is given by (35). If the top-hat filter is used we obtain specifically

\[ \frac{1}{2}(1 - A_1^2) = \frac{1}{2(k\Delta)^2} \left( (k\Delta)^2 - 2 + 2 \cos(k\Delta) \right) ; \quad B^2 = \{ A_2 - A_1^2 \}^2 \] (50)

Here use was made of the property that \( \phi_2 = \phi_1 \). The corresponding phase-shift \( \psi = \phi_1 = \Delta\sigma/2 \). For the turbulent stress flux \( \mathcal{T} \) given by (38) we infer \( b_1 = a_2 = -A_1A_1' \). Moreover, for wavenumbers \( |k\Delta| < \pi \) we find

\[ B = \frac{4\sin^2(k\Delta/2)}{(k\Delta)^2} - \frac{\sin(k\Delta)}{k\Delta} = \frac{1}{12}(k\Delta)^2 \{ 1 - \frac{1}{15}(k\Delta)^2 + \frac{1}{560}(k\Delta)^4 - \ldots \} \] (51)

and therefore

\[ B' = \left( \frac{\Delta'}{\Delta} \right) \left( \frac{(k\Delta)^2}{6} \right) \{ \frac{6(3k\Delta \sin(k\Delta) - 4 + \cos(k\Delta)(4 - (k\Delta)^2))}{(k\Delta)^4} \} \]

\[ = \left( \frac{\Delta'}{\Delta} \right) \left( \frac{(k\Delta)^2}{6} \right) \{ 1 - \frac{2}{15}(k\Delta)^2 + \frac{3}{560}(k\Delta)^4 - \ldots \} \] (52)

This directly specifies \( b_2 = B'/2 \) and \( b_3 = kB(1 + (\Delta\sigma/2)') \) that appear in (38).
Finally, we quantify the commutator error $C$ associated with top-hat filtering. The amplitude factor in (40) is given by:

$$c^2 = \left( \frac{1}{2} A_2' \right)^2 + \left( k A_2 \phi_2' \right)^2$$  \hspace{1cm} (53)

where

$$\left( \frac{1}{2} A_2' \right)^2 = \frac{1}{4} \left( \frac{\Delta'}{\Delta} \right)^2 \left( \frac{k \Delta \cos(\Delta) - \sin(\Delta)}{\Delta} \right)^2$$  \hspace{1cm} (54)

and

$$\left( k A_2 \phi_2' \right)^2 = \frac{1}{4} (k \Delta)^2 \left( \frac{\sin(\Delta)}{k \Delta} \right)^2 \left( \frac{(\Delta \phi_1')}{\Delta} \right)^2$$  \hspace{1cm} (55)

This computation fully quantifies the different contributions to the flux if the top-hat filter is adopted. We now proceed with the derivation of the relative magnitudes that may be used to assess the dynamic importance of the fluxes $\mathcal{M}$, $\mathcal{T}$ and $\mathcal{C}$.

The ratio between the commutator error and the turbulent fluxes may be obtained after some rewriting. It is given by

$$\varepsilon^2 = \frac{\|C\|^2}{\|T\|^2} = \frac{4 \mathcal{E}_1(k \Delta) \xi^2 + 36 \mathcal{E}_2(k \Delta) \eta^2}{1 + \xi^2(1 + \mathcal{E}_3(k \Delta)) + \frac{1}{2} (k \Delta)^2 \eta^2 (2 + \frac{1}{2} (k \Delta)^2 \eta)}$$  \hspace{1cm} (56)

In these expressions we introduced the characteristic variables

$$\xi = \frac{\Delta'}{k \Delta} ; \quad \eta = \frac{(\Delta \phi_1')}{(k \Delta)^2}$$  \hspace{1cm} (57)

to measure the influence of variations in the filter-width and variations in the skewness of the filter. We used the short-hand notations

$$\mathcal{E}_1(z) = \left( \frac{3z(z \cos(z) - \sin(z))}{12(z \sin(z) - 2 + 2 \cos(z))} \right)^2 = 1 - \frac{1}{15} z^2 + \frac{1}{4200} z^4 + \ldots$$  \hspace{1cm} (58)

$$\mathcal{E}_2(z) = \left( \frac{z^2}{12} \right)^2 \left( \frac{z \sin(z)}{z \sin(z) - 2 + 2 \cos(z)} \right)^2 = 1 - \frac{1}{5} z^2 + \frac{41}{4200} z^4 + \ldots$$  \hspace{1cm} (59)

and

$$\mathcal{E}_3(z) = \frac{1}{4} \left( \frac{3z \sin(z) - 4 + \cos(z)(4 - z^2)}{z \sin(z) - 2 + 2 \cos(z)} \right)^2 = 1 - \frac{2}{15} z^2 + \frac{17}{6300} z^4 + \ldots$$  \hspace{1cm} (60)

The specific way of expressing $\varepsilon$ in (56) was selected for convenience, as each of the functions $\mathcal{E}_j$ is approximately equal to 1 if $|z| \ll 1$.

The ratio between the turbulent stress flux and the mean flux may be obtained in the following form:

$$\delta^2 = \frac{\|T\|^2}{\|\mathcal{M}\|^2} = \left( \frac{(k \Delta)^2}{12} \right)^2 \mathcal{H}_1(k \Delta) \mathcal{H}_2(k \Delta, \xi, \eta)$$  \hspace{1cm} (61)
in which we put
\[ H_1(z) = \left( \frac{12}{z^2} \right)^2 \left( \frac{2 - 2 \cos z - z \sin z}{2 - 2 \cos z} \right)^2 = 1 + \frac{1}{30} z^2 + \frac{3}{3800} z^4 + \ldots \] (62)
and
\[ H_2(z, \xi, \eta) = 1 + \frac{1}{2} z^2 \eta \left( 2 + \frac{1}{2} z^2 \eta \right) + \xi^2 \left( 1 + \mathcal{E}_3(z) \right) \]
\[ 1 + \frac{1}{2} z^2 \eta \left( 2 + \frac{1}{2} z^2 \eta \right) + \xi^2 \left( \frac{z}{12} H_1(z) \right) \] (63)

The expressions (56) and (61) summarize the main result of this section. These quantify the relative importance of the mean flux, the turbulent stress flux and the commutator error. We observe that for the top-hat filter the central parameters are \( k \Delta, \xi \) and \( \eta \). If we want the computational modeling to adhere to small commutator errors then it is required that \( \varepsilon \ll 1 \). If we also require the turbulent stress flux to be small compared to the mean flux, i.e., for the simulation to be ‘close’ to a fully resolved simulation of the turbulent flow, then we should have \( \delta \ll 1 \). Correspondingly, (56) and (61) quantify precisely to what extent variations in the non-uniformity of the filter should be kept small.

The interpretation of (56) and (61) may be facilitated by restricting for convenience to the case in which \( |k \Delta| \ll 1 \) while \( \xi \) and \( \eta \) remain bounded. We then obtain
\[ \varepsilon \approx \frac{4 \xi^2 + 36 \eta^2}{1 + 2 \xi^2} ; \quad \delta \approx \left( \frac{(k \Delta)^2}{12} \right)^2 \{1 + 2 \xi^2 \} \] (64)
Correspondingly, if we require \( \varepsilon \) and \( \delta \) to be smaller than some pre-set value, this implies that only a certain range of small values for \( \xi, \eta \) and \( k \Delta \) are allowed. Thus, the relative variation of \( \Delta' \) compared to \( k \Delta \) and the relative variation of \( (\Delta \sigma)' \) compared to \( (k \Delta)^2 \) should be kept sufficiently small to be able to restrict the large-eddy closure to the turbulent stress fluxes. In cases with constant filter-width and constant filter-skewness we notice that \( \xi = \eta = 0 \), i.e., no commutator errors arise and the ratio between the turbulent stress flux and the mean flux scales with \( (k \Delta)^2 \). Similarly, if the filter-width and filter-skewness are kept sufficiently small, in such a way that the variables \( \xi \) and \( \eta \) are both sufficiently small, then the commutator errors may be expected to be under control. In such cases modeling of the commutator error appears not required. We notice from (56) that nonzero values of \( \eta \) yield a much stronger effect than variations in \( \xi \), consistent with the findings in [5]. This implies that variations in the filter-skewness require a more restrictive control compared to variations in the filter-width. A more complete analysis of (56) and (61), also if \( k \Delta \) is not very small, will be considered in the future and published elsewhere.

The restrictions in the non-uniformity of the filter-width and the filter-skewness of a top-hat filter can be directly related to the computational grid that is adopted in a large-eddy simulation. In fact, the conditions \( |\xi| \ll 1 \) and \( |\eta| \ll 1 \) can be expressed in properties of the grid-mapping. This will be considered in the next section.
4 Gridding requirements for negligible commutator errors

In the previous section we showed that the commutator error becomes negligible under appropriate smoothness requirements on the filter. In this section we relate these smoothness requirements to smoothness requirements for the grid-mapping. This leads to characteristic combinations of first-, second- and third order derivatives of this mapping which should be kept sufficiently small to warrant commutator errors to be of negligible dynamic relevance. These conditions on the grid-mapping may be adopted for generating and adapting computational grids for large-eddy simulation in which only the closure of the turbulent stress tensor is required. We restrict ourselves to the top-hat filter in one spatial dimension.

In practical large-eddy formulations a top-hat filter is commonly defined with reference to the computational grid. For this purpose we consider a flow-domain of size $\ell$ and describe the grid-points as $x_j = \ell f(j/N)$ for $j = 0, \ldots, N$ in which $N$ denotes the number of intervals and the grid-mapping $f$ is strictly increasing: $f'(s) > 0$ for all $s \in [0, 1]$. We consider $f$ to be bounded between 0 and 1. In the discrete formulation the filter-width at $x = x_j$ may be defined as

$$\Delta(x_j) = x_{j+n} - x_{j-n} = \ell \left( f\left(\frac{j+n}{N}\right) - f\left(\frac{j-n}{N}\right) \right)$$

(65)

where $n \geq 1$ is a constant that characterizes the number of grid-intervals covering the filter-width. The skewness $\sigma$ may likewise be defined as

$$\sigma(x_j) = \frac{\Delta_+(x_j) - \Delta_-(x_j)}{\Delta_+(x_j) + \Delta_-(x_j)} = \frac{f\left(\frac{j+n}{N}\right) - 2f\left(\frac{j}{N}\right) + f\left(\frac{j-n}{N}\right)}{f\left(\frac{j+n}{N}\right) - f\left(\frac{j-n}{N}\right)}$$

(66)

The numerical derivative of the filter-width in $x_j = f(s_j)$ may be written as

$$\Delta'(x_j) \approx \frac{\Delta(x_{j+1}) - \Delta(x_{j-1})}{x_{j+1} - x_{j-1}} = \frac{\ell}{x_{j+1} - x_{j-1}} \left( f\left(\frac{j+1+n}{N}\right) - f\left(\frac{j+1-n}{N}\right) \right) - \left( f\left(\frac{j-1+n}{N}\right) - f\left(\frac{j-1-n}{N}\right) \right)$$

(67)

and likewise the derivative of the skewness may be approximated as

$$\sigma'(x_j) \approx \frac{\sigma(x_{j+1}) - \sigma(x_{j-1})}{x_{j+1} - x_{j-1}} = \frac{1}{x_{j+1} - x_{j-1}} \left( \left[ f\left(\frac{j+1+n}{N}\right) - 2f\left(\frac{j+1}{N}\right) + f\left(\frac{j+1-n}{N}\right) \right] \right.$$

$$\left. - \left[ f\left(\frac{j-1+n}{N}\right) - 2f\left(\frac{j-1}{N}\right) + f\left(\frac{j-1-n}{N}\right) \right] \right)$$

(68)
These expressions allow to determine the values of the central variables $\xi$ and $\eta$ defined in (57) for a particular grid-mapping $f$. For this purpose we restrict to the situation in which the filter-width is considerably smaller than the flow-domain, i.e., $n \ll N$.

The magnitude of the commutator error can be quantified for single Fourier modes in terms of the variations of the filter-width and skewness. For the central variable $\xi$ we find:

$$
\xi_j = \frac{\Delta_j'}{k\Delta_j} = \frac{f(i+1+n/N) - f(i+1-n/N) - f(i-1+n/N) + f(i-1-n/N)}{k(x_{j+1} - x_{j-1})(f(2+n/N) - f(2-n/N))}
$$

(69)

We use the assumption that $n \ll N$ to approximate the grid-mapping in the vicinity of $j/N$ with a Taylor expansion. After some calculation we find, e.g.,

$$
f\left(\frac{j + 1 + n}{N}\right) - f\left(\frac{j + 1 - n}{N}\right) = \left(\frac{2n}{N}\right)f'\left(\frac{j}{N}\right) + \left(\frac{2n}{N^2}\right)f''\left(\frac{j}{N}\right) + \ldots
$$

(70)

and similar expressions for the other terms in (69). Combining these expansions, we find to leading order

$$
\xi_j = \alpha_j \frac{f''(j/N)}{f'(j/N)} + \ldots \quad ; \quad \alpha_j = \frac{2}{Nk(x_{j+1} - x_{j-1})}
$$

(71)

For a uniform grid we find $\alpha_j = (2/N)/(2k\ell/N) = 1/(k\ell)$. Thus, the characteristic variable $\xi$ is primarily a measure for the ratio of $f''$ and $f'$, normalized by a dimensionless parameter $\alpha_j$ which is associated with a fraction of $1/(k\ell)$ according to the local non-uniformity of the grid. To keep commutator errors sufficiently small, one may require that the maximum of $\xi_j$ over all grid-points is small enough.

A similar computation can be adopted to find $\eta$. We notice that

$$
\Delta_j\sigma_j = \ell \left( f\left(\frac{j + n}{N}\right) - 2f\left(\frac{j}{N}\right) + f\left(\frac{j - n}{N}\right) \right)
$$

(72)

which implies

$$
(\Delta_j\sigma_j)' \approx \frac{\ell}{x_{j+1} - x_{j-1}} \left( f\left(\frac{j + 1 + n}{N}\right) - 2f\left(\frac{j + 1}{N}\right) + f\left(\frac{j + 1 - n}{N}\right) - \frac{\ell}{x_{j+1} - x_{j-1}} \left( f\left(\frac{j - 1 + n}{N}\right) - 2f\left(\frac{j - 1}{N}\right) + f\left(\frac{j - 1 - n}{N}\right) \right) \right)
$$

(73)

Using a third order Taylor expansion we obtain

$$
f\left(\frac{j + 1 + n}{N}\right) - 2f\left(\frac{j + 1}{N}\right) + f\left(\frac{j + 1 - n}{N}\right) = f''\left(\frac{j}{N}\right)\left(\frac{n^2}{N^2}\right) + f''\left(\frac{j}{N}\right)\left(\frac{n^2}{N^3}\right) + \ldots
$$

(74)

and a similar expression for the contribution at $j - 1$. Therefore, to leading order we find

$$
(\Delta_j\sigma_j)' = \frac{\ell}{x_{j+1} - x_{j-1}} \left( f''\left(\frac{j}{N}\right)\left(\frac{n^2}{N^2}\right) + f''\left(\frac{j}{N}\right)\left(\frac{n^2}{N^3}\right) \right)
$$
\[- \left[ f''\left( \frac{j}{N} \right) \left( \frac{n^2}{N^2} \right) - f''\left( \frac{j}{N} \right) \left( \frac{n^2}{N^2} \right) \right] + \ldots \]

\[= \frac{\ell}{x_{j+1} - x_{j-1}} \left( f''\left( \frac{j}{N} \right) \left( \frac{2n^2}{N^2} \right) \right) + \ldots \]  

Likewise, we obtain the leading order expression

\[(k \Delta_j)^2 = \left( k \ell \left( f\left( \frac{j+n}{N} \right) - f\left( \frac{j-n}{N} \right) \right) \right)^2 = \left( \frac{2nk \ell}{N} \right)^2 (f'(\frac{j}{N}))^2 + \ldots \]  

Hence, we may finally evaluate the parameter \( \eta_j \):

\[\eta_j = \frac{(\Delta_j \sigma_j)'^2}{(k \Delta_j)^2} = \beta_j \left( \frac{f''(\frac{j}{N})}{(f'(\frac{j}{N}))^2} \right) \quad \beta_j = \frac{\alpha_j}{4k \ell} \]  

For a uniform grid we find \( \beta_j \rightarrow 1/(4k^2 \ell^2) \). We observe that the contribution of the filter-skewness is related to the ratio between the third derivative and the square of the first derivative of the grid-mapping \( f \), normalized by a factor \( \beta_j \). For a uniform grid this normalization factor is equal to \( \beta = \alpha^2 / 4 \). To keep these contributions to the commutator error small, the maximal values of \( \eta_j \) need to be kept small, which poses restrictions on the accessible grid-mappings.

The conditions for \( \xi \) and \( \eta \) in (71) and (77) express the characteristic variables in derivatives of the grid-mapping. The conditions that both \( \xi \) and \( \eta \) should remain sufficiently small, i.e., commutator errors remain sufficiently small, can hence directly be verified in terms of the grid. Conversely, one may impose such requirements and derive computational grids that are specific to large-eddy simulation such that commutator errors do not require explicit modeling. This may also be used in an adaptive gridding strategy. In such cases the local complexity of the solution is first translated into an effective wave-number \( \langle k \rangle \). In turn this provides guidance for the locations of the grid points that yield sufficient smoothness of the filter to neglect commutator errors. The level to which \( \xi \) and \( \eta \) should be restricted in actual simulations is a matter that needs further attention and will be studied in the future, in combination with actual large-eddy simulations.

5 Concluding remarks

In this paper the commutator errors associated with non-uniform filtering in large-eddy simulation were studied. For a general class of non-uniform filter operators the filtered, incompressible Navier-Stokes equations were derived and all closure terms were identified. Besides the turbulent stress contributions, commutator errors were shown to arise. The order of magnitude of the commutator errors and the turbulent stress fluxes was derived for arbitrary high-order filters, with bounded moments. The main result of this analysis is that both closure contributions scale with the same order of the filter-width, or that commutator errors are even larger, when the spatial non-uniformity is non-zero.
This implies that while an increase in the order of the spatial filter allows control over the magnitude of the commutator error, the flux due to the turbulent stress is affected simultaneously in the same order of magnitude. Hence, an independent control over the commutator errors cannot be obtained through the application of a general high-order filter.

A more detailed analysis of the commutator errors and turbulent stress fluxes for single Fourier-modes shows that the commutator errors may be reduced in size by restricting the variations in the filter-width and normalized filter-skewness. This suggests employing gradually varying filter properties in complex geometries, from the point of view of avoiding explicit modeling of the commutator errors. In view of maintaining appropriate efficiency in large-eddy simulations of turbulent flows in/around complex geometries it may be required to allow for sharp variations in $\Delta$ and $\sigma$. In such cases the dynamic importance of the commutator errors summons an explicit parameterization of the commutator errors.

The relative magnitude of the mean flux, the turbulent stress flux and the commutator error was expressed as function of $k\Delta$, $\xi = \Delta'/(k\Delta)$ and $\eta = (\Delta\sigma')/(k\Delta)^2$ for top-hat filtering. If $k\Delta$ is sufficiently small then the turbulent stress flux is considerably smaller than the mean flux. In addition, if both $\xi$ and $\eta$ are sufficiently small then the commutator error is much smaller than the turbulent stress flux. These conditions were formulated in terms of first-, second- and third order derivatives of the grid-mapping. This provides criteria for testing whether computational grids are sufficiently smooth to be able to ignore the modeling of the commutator error and the turbulent stress flux. In addition, it can be used to arrive at dynamic grid adaptation criteria that are consistent with neglecting the commutator error and possibly also the turbulent stress flux.

REFERENCES


