Memorandum M-365

The Double Helmholtz Resonator as a Possible Tool for the Determination of the Response Function of Solid Rocket Propellants in the Low Frequency Range

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Delft, The Netherlands
March 1980
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<td>$A_t$</td>
<td>Nozzle throat area</td>
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<td>$a$</td>
<td>Velocity of sound</td>
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<td>$c_p$, $c_v$</td>
<td>Specific heats at constant pressure, volume</td>
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<td>Characteristic length, $L^* = (V_1 + V_2)/A_t$</td>
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3. Introduction

During their development, many solid rocket motors exhibit oscillatory or unstable combustion. Instead of a smooth pressure history, pressure fluctuations may be observed, the combustion may cease or many very short periods of burning may be observed. This last phenomenon is known as chuffing. Older rockets which have been stored for prolonged periods may also start to exhibit combustion instability, though originally such motors have been stable.

These instabilities are undesirable for a variety of reasons. To name a few, the vibrations caused by the fluctuating thrust may damage the payload or guidance and control systems, increased heat transfer that is associated with pressure fluctuations may lead to mechanical failure, while ceasing of the combustion will result in a complete failure of the mission.

The curing of such instability problems is often quite expensive, both in terms of money and in terms of sacrificing performance. To cure the problem, one needs to have a thorough understanding of the non-stationary combustion of solid rocket propellants. To the author's knowledge there exist no theoretical tools to predict reliably the burning behavior of solid rocket propellants during non-steady conditions. As even the theoretical prediction of the steady burning rate has only met with limited success, it is not surprising that the non-steady case poses still more unsolved difficulties.

The usual approach until now has been to define certain propellant properties that might be calculated theoretically, provided that our insight in the various processes, which are associated with the combustion of solid propellants, are well-known and well-understood. These properties, however, may also be determined experimentally. One such a propellant property obviously is the steady burning rate, which is usually determined by standard techniques (1-3).

If the environment of the burning propellant is subject to limited variations of its prime variables, i.e. the pressure and the velocity of the combustion gases, the burning rate will react to these variations. In fact one deals with a kind of dynamic burning rate, but usually this is expressed as the propellant Admittance or Response function.

In those cases that one considers the influence of fluctuations of the gas-velocity parallel to the burning surface, one speaks of the
velocity-coupled response (admittance) function. If one is interested in the response of the burning propellant to fluctuations in the pressure, one speaks of the pressure-coupled response (admittance) function. As has recently been pointed out by Price \(^4\) the velocity-coupled response is closely related to erosive burning. Here, we will concern ourselves with the pressure-coupled response of a solid rocket propellant.

It is conceivable to construct a theoretical model for this pressure-coupled response function but no really successful models exist to the author's knowledge. The best-known models, belong to a class described by Brown and Muzzy \(^5\) but in many cases this model does not fully agree with experimental observations. Therefore, the propellant, pressure-coupled response function has to be determined experimentally.

If the response function is known, it is possible, provided that the computer codes are available, to predict whether pressure oscillations may be expected in solid rocket motors or whether one may expect stable, steady burning.

The (pressure-coupled) response function expresses how the propellant burning rate responds to a (small) variation in pressure. Usually, the variables are non-dimensionalized by division by their mean (time-averaged) values:

\[
R_b = \frac{r'\bar{r}}{p'\bar{p}}
\]  

(3-1)

It is convenient to express the pressure-and burning rate- variations as complex quantities. This allows to express the phase-lag between pressure and burning rate variations as the imaginary part of the response function:

\[
R_b^i = \frac{\bar{p}r}{\bar{p}} \sin \varphi
\]  

(3-2)

where \(\varphi\) stands for the phase-lag.

As has been mentioned before, the response function may be regarded as a kind of dynamic burning rate. The steady burning rate of propellants is known to depend on the propellant temperature and the pressure. Therefore, it seems reasonable to assume that the response function also depends on these variables. Moreover, there is experimental evidence \(^6,7\)
that the propellant response function also depends on the frequency of the oscillations. Therefore one should at least assume the propellant response function to depend on the propellant temperature, the (mean) pressure and the frequency of the oscillations:

\[ R_b = R_b \{ T_1, \bar{P}, F \} \]

To obtain the steady state burning rate of a solid propellant, a piece of propellant is ignited and, either the time the flame front needs to traverse some known distance is measured, or the velocity of the burning surface is measured directly by means of a doppler technique. During these measurements the mean pressure is kept constant while the initial propellant temperature is known. By repeating such measurements at various pressure levels and for different temperatures, one obtains the burning rate versus pressure and initial temperature. It should be emphasized here, that during these experiments, the mean pressure is kept constant within small margins and the obtained relations between burning rate and pressure, in no way reflect any effect of varying pressure on the burning rate. The burning rate obtained in this way therefore cannot be applied with any confidence to problems of dynamic combustion of propellant. In the case of oscillatory combustion, i.e. in those cases where the variations in the pressure are limited, one might apply the response function as a kind of dynamic burning rate, in combination with the steady-state burning rate.

As outlined before, there exist no accurate, theoretical models for the response function, and for every propellant such a (pressure-coupled) response function has to be determined experimentally.

There is a variety of tools available to determine the admittance - or response - function.

For oscillations with frequencies roughly higher than 300 Hz, the T-burner can be used \(^{(8,9)}\). The T-burner derives its name from its shape. It is essentially a closed organ pipe, with propellant at both ends. A vent, which is usually located in the center, allows for the exhaust of combustion products. Oscillations that occur, correspond to the natural frequency of the T-burner. This frequency is determined primarily by the length of the burner and the temperature of the combustion products.
From the growth of the amplitude of the oscillations the response function of the propellant may be determined. For measurements at different frequencies one has to change the length of the T-burner. It is very essential to be aware of the fact that the T-burner is a tuned device. As the composition and temperature of the combustion products is fairly well known, and hence the speed of sound, the oscillatory frequency is fully determined by the length of the T-burner. The propellant response function has to overcome losses due to friction and heat transfer, and so, if the real part of the response function is sufficiently large, oscillations with an initially growing amplitude will occur.

If the response function is not large enough no oscillations will occur; however, there may be many reasons for also being interested in these lower values of the response function. Moreover, for every frequency, at which one might want to take response function measurements, a different T-burner configuration has to be used. For the frequencies, roughly below 300 Hz, the T-burner length becomes such large, that heat transfer losses and non-uniform temperature distributions, make results unreliable. For example, if the speed of sound of the combustion products is 1200 m/s, the acoustic wave length at 300 Hz is 4 m, and as the T-burner length corresponds to half the wave length, a T-burner length of 2 m is required. Though the T-burner has some inherent disadvantages, it is presently the standard instrument for the determination of the propellant admittance or response function. To overcome some of the disadvantages of the T-burner, a device has been developed in France\(^{(10)}\) where the nozzle throat of a small rocket motor is varied periodically. In this way periodic pressure fluctuations are imposed on the burning propellant and the way in which the burning propellant responds to the fluctuations is a measure for the admittance or response function. By changing the frequency of the oscillations, measurements may be made at the required frequencies, without changing the geometry of the test device. Though the results look promising and the basic idea behind this tool seems sound, there still seem to be difficulties which have prevented this tool to become adopted as a standard test device. The impression exists that the device is less suited for the lower frequencies. Based on the same basic idea, is the rotating valve method\(^{(11)}\) as it has been developed by Brown. Again
the propellant is placed in an environment of oscillating pressures and the response of the burning propellant to these pressure oscillations is measured. One advantage of this last method is that also velocity fluctuations may be introduced, so that the velocity-coupled response may be determined too.

The rotating valve device seems able to take reliable measurements down to 100 Hz.

A quite different approach is followed by Strand(12). Again a burning propellant is placed in an environment of fluctuating pressures. However, by means of a microwave technique, the instantaneous regression of the burning surface is measured. It is interesting to note that the environment of the fluctuating pressure is also created a rotating valve.

For measurements of really low frequency oscillations, which may be of the order of a few cycles per second only, the $L^\star$ burner has been used for a long time. This is a small end-burning rocket motor with a variable volume combustion chamber and interchangeable nozzles. The choice of a particular nozzle will determine the mean pressure level, while the size of the chamber volume, in combination with the propellant will determine the oscillatory frequency. If oscillations occur in the $L^\star$-burner, these are of the bulkmode type. The oscillatory frequency depends on the phase-lag between the pressure and burning rate variations, and the residence time of the combustion products. This residence time is proportional to the characteristic length, $L^\star$, i.e. the ratio of the chamber volume to the nozzle throat area of the rocket motor. As the phase-lag is related to the imaginary part of the response function and as it is this response function that has to be measured for a given propellant, one cannot predict at what frequencies the $L^\star$-burner may generate pressure oscillations. In fact, one does not know whether oscillatory combustion will take place at all. If oscillatory combustion takes place in the $L^\star$-burner, two conditions have to be met simultaneously:

- The imaginary part of the propellant response function must be positive.
- The real part of the propellant response function should be large enough to overcome losses.
Therefore, it is difficult to measure systematically in the \( \text{L}^* \)-burner the propellant response function at low frequencies. As it turns out, that measurements of the low frequency propellant response function, with the other devices also poses difficulties, a new tool is proposed. This is the Helmholtz Resonator Burner. Contrary to the \( \text{L}^* \)-burner and like the T-burner, this is a tuned device. The HR-Burner consists of two chambers connected by a small tube, the neck. The primary chamber contains the propellant, the secondary chamber contains a supersonic nozzle. If we indicate the chamber volumes by \( V_1 \) and \( V_2 \), the length and cross-sectional area of the neck by \( L_n \) and \( O_n \), respectively, then, the frequency, \( F \), at which the HR-burner is to oscillate is approximately given by

\[
F = \frac{a}{2\pi} \sqrt{\frac{O_n}{L_n}} \left( \frac{1}{V_1} + \frac{1}{V_2} \right)
\]

where \( a \) stands for the mean speed of sound of the combustion products. With chamber dimensions of 10 cm diameter and 25 cm length, a neck length of 50 cm and a neck diameter of 1 cm, this yields, assuming a speed of sound of 1200 m/s, a basic frequency of \( \sim 75 \) Hz. The total length of such an HR-burner is 1 m. If a basic frequency of 75 Hz were to be achieved in a T-burner, a burner length of 8 m would be required. In fact some corrections have to be applied to Eq. (3-3) which will slightly change the numerical value of the basic frequency.

In the present studies the HR-burner is analyzed and it also is indicated which quantities have to be measured in order to deduct meaningful information about the propellant from test runs. First an elementary analysis of the acoustic resonator without flow is presented. The most simple analysis is by the mass-spring analogue, which also serves the purpose of giving a clear physical picture of what happens. When burning propellant is introduced to the primary chamber, and a supersonic nozzle to the secondary chamber, pressure and velocity fluctuations are imposed on a mean pressure and mean flow. Also the boundary conditions at both ends of the HR-burner change.
The elementary analysis as presented in the present studies, does not account for entrance- and exit- corrections of the connecting tube, the neck. In earlier studies \(^{(13,14)}\) the endeffects of the neck have been analyzed in detail. The corrections for these endeffects are applied directly to the analysis of the HR-burner.

The present elementary analysis, according to which it is possible to construct a tuned burner for the investigation of propellant characteristics, is the basis for the realization of such a burner.

This realization only will demonstrate the feasibility of the concept, as the proof of the pudding is the eating. A device, more or less similar to the HR-burner has been tested by Price \(^{(17)}\) at the U.S. Naval Ordnance Test Station (now NWC). Though this device, which got the nick-name "Dumb-bell Resonator", has never been developed into an operational laboratory tool, it seems to have given satisfactory results. One of the differences with the HR-burner is the location of the vent. This vent was located at the middle of the neck and connected to a pressure vessel.

It should also be emphasized here, that the HR-burner may also be suitable for high frequency measurements, and hence serve to compare the results as obtained by T-burner, Rotary Valve and Microwave technique measurements.
4. Description of a Double Helmholtz Resonator

The Helmholtz Resonator (H.R.) consists of two cavities containing gas and connected by a (small) tube, the neck.

Figure 1 is an outline of this resonator. The cavities 1 and 2, in general, will have different volumes, $V_1$ and $V_2$. The length of the neck connecting the two cavities is $L$ and its cross-sectional area is $A$. At rest the pressures, densities and temperatures in the two cavities equal. If, for some reason, the pressure in cavity 1 rises a little, the mass of gas in the connecting tube will move towards cavity 2 and compress the gas in that volume.

The gas in the tube is regarded as a piston, and if losses (mainly due to friction) are negligible, an oscillation will result at a well tuned frequency. The "piston of gas" will oscillate back and forwards in the tube and the pressures in the two volumes will fluctuate accordingly. The pressure oscillations in the two volumes will show a phase-difference of $\pi$ radians.

If one enlarges the volume(s) the frequency will decrease. It is even possible to make one of the cavities infinitely large. This is the case, for instance, if the volume 1 is filled with ambient air, and the surrounding atmosphere forms volume 2. Then still a finite frequency of the oscillations will result, as long as the remaining cavity has a finite volume.

In the next section an elementary analysis of the H.R. is presented.
5. Elementary Analysis of the Helmholtz Resonator

Let the pressure in the cavities 1 and 2 under steady conditions be \( \bar{p} \). The corresponding temperature and density are \( \bar{T} \) and \( \bar{\rho} \). Now if the "gas piston" in the tube is displaced over a distance \( x \) (positive to the right, see Fig. 1) the density in the primary cavity becomes

\[
\rho_1 = \frac{\bar{\rho}}{1 + \frac{Ox}{V_1}} \tag{5-1a}
\]

and the density in the secondary cavity becomes

\[
\rho_2 = \frac{\bar{\rho}}{1 - \frac{Ox}{V_2}} \tag{5-1b}
\]

As long as the change in volume (\( \Delta V = \pm Ox \)) is small as compared to the volumes \( V_1 \) and \( V_2 \), the expressions (5-1) may be linearized to:

\[
\rho_1 = \bar{\rho} \left(1 - \frac{Ox}{V_1}\right) \tag{5-2a}
\]

\[
\rho_2 = \bar{\rho} \left(1 + \frac{Ox}{V_2}\right) \tag{5-2b}
\]

Introducing pressure, density and temperature perturbations \( p', \rho' \) and \( T' \) one gets

\[
\frac{p'_1}{\bar{p}} = -\frac{Ox}{V_1} \tag{5-3a}
\]

\[
\frac{p'_2}{\bar{p}} = \frac{Ox}{V_2} \tag{5-3b}
\]

The corresponding pressure perturbations are given by

\[
\frac{p'}{\bar{p}} = \gamma \rho' \tag{5-4}
\]

It is assumed here that the density and pressure variations are completely reversible, i.e. isentropic, so that the Poisson relations hold.

Let us now consider the forces on and the acceleration of the "piston of gas":

At the right-hand-side there is a pressure \( \bar{p} + p'_1 \) on the surface area \( 0 \) and at the left-hand-side there is a pressure \( \bar{p} + p'_2 \).
Hence, as the total mass of the "gas piston" equals

\[ M = \rho_0 L, \quad (5-5) \]

the equation of motion for the "gas-piston" is

\[ \rho_0 L \ddot{x} = p_1' 0 - p_2' 0 \quad (5-6) \]

With the Equations (5-3) and (5-4) Equation (5-6) may be written as

\[ \ddot{x} + \frac{a^2}{L} \left( \frac{1}{V_1} + \frac{1}{V_2} \right) x = 0, \quad (5-7) \]

where use has been made of the expression for the speed of sound:

\[ a^2 = \gamma \frac{\bar{p}}{\rho} \quad (5-8) \]

A solution of Eq. (5-7) obviously is

\[ x = x_0 \sin \left( \sqrt{\frac{\bar{p}}{L} \left( \frac{1}{V_1} + \frac{1}{V_2} \right) a} t \right) \quad (5-9) \]

It has been assumed that at \( t = 0 \), there is no displacement of the "piston of gas" but only a velocity in the positive \( x \)-direction. The amplitude of the oscillation of the "gas-piston" is \( x_0 \) while the frequency of the oscillation is given by

\[ F = \frac{a}{2\pi} \sqrt{\frac{\bar{p}}{L} \left( \frac{1}{V_1} + \frac{1}{V_2} \right) a} \quad (5-10a) \]

It is easily seen that if one increases the volumes of the cavities, the frequency at which the H.R. will oscillate, decreases. If the volume of one of the cavities is fixed, while the other volume is replaced by the surrounding atmosphere (i.e. \( V_2 \to \infty \)) the frequency becomes

\[ F = \frac{a}{2\pi} \sqrt{\frac{\bar{p}}{LV_1}} \quad (5-10b) \]

Hence, by varying the volumes of the cavities, one may tune the H.R. to a required frequency.

Another way to tune the H.R. is by changing the dimensions of the neck.
By changing the ratio \( O/L \), the resonator frequency also changes. The velocity of sound, \( a \), is an important variable, it varies with the temperature of the gas, its composition, and so the frequency at which the H.R. is to oscillate, also depends on the composition and temperature of the gas.

With the Eqs. (5-3), (5-4) and (5-9), one finds:

\[
\frac{p_1'}{p} = -\frac{O}{V_1} x_0 \sin \left\{ \sqrt{\frac{O}{L} \left( \frac{1}{V_1} + \frac{1}{V_2} \right)} \right\},
\]

\[
\frac{p_2'}{p} = \frac{O}{V_2} x_0 \sin \left\{ \sqrt{\frac{O}{L} \left( \frac{1}{V_1} + \frac{1}{V_2} \right)} \right\},
\]

or

\[
\frac{p_1'}{V_1} = \frac{O}{V_1} x_0 \frac{p}{p} \sin \left\{ \sqrt{\frac{O}{L} \left( \frac{1}{V_1} + \frac{1}{V_2} \right)} \right\} + \pi, \]

\[
\frac{p_2'}{V_2} = \frac{O}{V_2} x_0 \frac{p}{p} \sin \left\{ \sqrt{\frac{O}{L} \left( \frac{1}{V_1} + \frac{1}{V_2} \right)} \right\},
\]

hence, the pressure variations in the two cavities are \( \pi \) radians out of phase. The same holds for the density variations.

We have analyzed the Helmholtz Resonator, by considering the two cavities as springs connected by a solid piston, a similar result may also be obtained by considering the equations of motion for a fluid. Such an approach is a necessity if less simple boundary conditions have to be applied.

In the absence of a mean flow the following perturbation equations for the motion of an inviscid fluid hold:

**Equation for conservation of mass**:

\[
\frac{\partial p'}{\partial t} + \rho \frac{\partial v'}{\partial x} = 0,
\]

By use of the equations (5-4) and (5-8) this equation may also be written as

\[
\frac{1}{a} \frac{\partial p'}{\partial t} + \rho \frac{\partial v'}{\partial x} = 0.
\]
Equation for conservation of momentum

\[
\frac{\partial p'}{\partial t} + \frac{1}{\rho} \frac{\partial \rho' p'}{\partial x} = 0 .
\]  

(5-15)

From these equations one obtains the wave equation:

\[
\frac{\partial^2 p'}{\partial t^2} - a^2 \frac{\partial^2 p'}{\partial x^2} = 0 .
\]  

(5-16)

A general solution of this equation is:

\[
p' = A \cos [k(t + x/a)] + B \cos [k(t - x/a)] + C \sin [k(t + x/a)]

+ D \sin [k(t - x/a)] .
\]  

(5-17)

One finds for the perturbed fluid velocity:

\[
\frac{\partial \psi'}{\partial t} - \frac{1}{\rho} \frac{\partial \rho' \psi'}{\partial x} = - \frac{k}{\rho a} \left\{ \begin{align*}
-A \sin [k(t + x/a)] & + B \sin [k(t - x/a)] \\
+ C \cos [k(t + x/a)] & - D \cos [k(t - x/a)]
\end{align*} \right\},
\]

and hence,

\[
\psi' = - \frac{1}{\rho a} \left\{ \begin{align*}
A \cos [k(t + x/a)] & - B \cos [k(t - x/a)] \\
+ C \sin [k(t + x/a)] & - D \sin [k(t - x/a)]
\end{align*} \right\} .
\]  

(5-18)

We will apply these solutions to both the cavities \( V_1 \) and \( V_2 \). For simplicity the problem is regarded as being one-dimensional. The cavities are cylindrical, with a cross-sectional area, \( A_i \), and length, \( L_i \), such that

\[
V_i = A_i \cdot L_i .
\]  

(5-19)

The total length of the cavities and their connecting channel, the neck, is (see fig. 1):

\[
L_t = L_1 + L + L_2 .
\]  

(5-20)
The boundary conditions are such that the velocity perturbation vanishes at 
\[ x = 0 \text{ and } x = L_t \quad \text{i.e.} \]
\[ x = 0 : v_1' = 0 \]
\[ x = L_t : v_2' = 0 \quad \text{(5-21)} \]

For convenience, let us introduce the following dimensionless variables:

\[ \chi = xk/a, \quad \tau = tk, \quad \lambda = Lk/a. \quad \text{(5-22)} \]

We will denote the cavities and the variables in the cavities by the indices 1 and 2. Applying the boundary conditions (5-21) yields:

\[ \chi = 0: \]
\[ v_1' = 0 = -\frac{1}{pa} [A_1 \cos \tau - B_1 \cos \tau + C_1 \sin \tau - D_1 \sin \tau] \]

and one finds,

\[ B_1 = A_1, \quad D_1 = C_1. \quad \text{(5-23)} \]

Hence, the pressure and velocity perturbation in cavity 1 may be written as:

\[ p_1' = 2 A_1 \cos \tau \cos \chi + 2 C_1 \sin \tau \cos \chi, \quad \text{(5-24)} \]
\[ v_1' = \frac{2}{pa} [A_1 \sin \tau \sin \chi - C_1 \cos \tau \sin \chi]. \quad \text{(5-25)} \]

At \( \chi = \lambda_t \):
\[ v_2' = 0 = -\frac{1}{pa} \left\{ A_2 \cos (\tau + \lambda_t) - B_2 \cos (\tau - \lambda_t) + C_2 \sin (\tau + \lambda_t) - D_2 \sin (\tau - \lambda_t) \right\}, \]

which yields
\[ B_2 = A_2 \cos 2\lambda_t + C_2 \sin 2\lambda_t, \]
\[ D_2 = -A_2 \sin 2\lambda_t + C_2 \cos 2\lambda_t, \]

and

\[ p_2' = (A_2 \cos \tau + C_2 \sin \tau) \left( \cos \chi + \cos (2\lambda_t - \chi) \right) + \]
\[ -(A_2 \sin \tau - C_2 \cos \tau) \left( \sin \chi + \sin (2\lambda_t - \chi) \right), \quad \text{(5-26)} \]
\[ v'_2 = \frac{-1}{\rho a} \left\{ (A_2 \cos \tau + C_2 \sin \tau) \left( \cos \chi - \cos(2\lambda_t - \chi) \right) + \right. \\
\left. - (A_2 \sin \tau - C_2 \cos \tau) \left( \sin \chi - \sin(2\lambda_t - \chi) \right) \right\} \]  
\hspace{1cm} (5-27)

For simplicity the flow in the cavities and the neck is assumed to be one-dimensional. This implies an instantaneous expansion and contraction of the flow near the ends of the neck. This introduces an error. To account for the effects of this error Nielsen \(^{13}\) and Ingard \(^{14}\) have determined end-corrections. These end-corrections may be incorporated in the expression for the resonance frequency.

It is also assumed that the pressure perturbation at the ends of the neck equals the corresponding pressure perturbation in the cavity adjacent to the connection of the neck. This in fact is a matching condition for the pressure.

The equation for conservation of mass (steady state) yields:

\[ (\rho + \rho'_i) \ v'_i \ o'_i = (\rho + \rho') v' o' , \]
with \( i = 1 \) or \( 2 \).

or

\[ \rho \left( 1 + \frac{1}{\gamma} \frac{p'_i}{p} \right) \ v'_i \ o'_i = \rho \left( 1 + \frac{1}{\gamma} \frac{p'}{p} \right) v' o' . \]

As \( p'_i = p' \), i.e. the pressure perturbation in the vessel \( i \) equals the pressure perturbation \( p' \) at that end of the neck that is connected to vessel \( i \), we have

\[ v'_i \ o'_1 = v' \ o \ (\chi = \lambda_1) , \]
\[ v'_2 \ o'_2 = v' \ o \ (\chi = \lambda_t - \lambda_2) . \]

Again consider the gas in the neck as an oscillating solid piston, then, the velocity of the gas in the tube is independent of the location and,

\[ (v'_i \ o'_1)_{\lambda_1} = (v'_2 \ o'_2)_{\lambda_t - \lambda_2} \]  
\hspace{1cm} (5-29)
The assumption that the displacement of the gas in the neck can be treated as if the gas in the neck behaves like a solid mass is questionable. If the neck length, \( L \), is small as compared to the wavelength, \( \lambda \), this assumption is acceptable. For neck lengths that are of the same order of magnitude as the wave length, \( \lambda \), a more refined solution, where the wave equation for the flow in the neck is solved and matched with the flow in the cavities, may be required. The present studies are primarily aimed at investigating the feasibility of the concept of the Helmholtz Resonator Burner, and it is assumed that a more refined solution for the flow through the neck, as outlined above will not be necessary for that purpose at this stage.

Inserting Eq. (5-29) into the equation (5-25) and (5-27) yields:

\[
A_1 O_1 \sin \lambda_1 = - O_2 \sin \lambda_2 \{ A_2 \cos \lambda_t + C_2 \sin \lambda_t \}, \tag{5-30}
\]

\[
C_1 O_1 \sin \lambda_1 = + O_2 \sin \lambda_2 \{ A_2 \sin \lambda_t - C_2 \cos \lambda_t \},
\]

and \((v'_1)^\lambda_1 = (v'_2)^\lambda_2\) is given by:

\[
O_1(v'_1)^\lambda_1 = (v'_2)^\lambda_2 = \frac{O_2}{\lambda_2 - \lambda_2}.
\]

\[
= - \frac{20_2}{\rho a} [ (A_2 \sin \lambda_t - C_2 \cos \lambda_t) \cos \tau + (A_2 \cos \lambda_t + C_2 \sin \lambda_t) \times \\
\times \sin \tau ] \cdot \sin \lambda_2. \tag{5-31}
\]

For the pressure perturbations at both ends of the tube one finds

\[
(p'_1)^\lambda_1 = -2 \frac{O_2 \sin \lambda_2}{O_1 \sin \lambda_1} \{ \cos \tau \cos \lambda_1 (A_2 \cos \lambda_t + C_2 \sin \lambda_t) + \\
- \sin \tau \cos \lambda_1 (A_2 \sin \lambda_t - C_2 \cos \lambda_t) \}, \tag{5-32}
\]

\[
(p'_2)^\lambda_2 = 2 \left[ (A_2 \cos \tau + C_2 \sin \tau) \cos \lambda_t \cos \lambda_2 + \\
- (A_2 \sin \tau - C_2 \cos \tau) \sin \lambda_t \cos \lambda_2 \right]. \tag{5-33}
\]
If the gas in the channel is treated as a rigid body, the equation of motion is given by Eq. (5-6):

\[ \rho \circ L \frac{d^2 x}{dt^2} = p_1' \circ - p_2' \circ \]

Now \( \frac{d^2 x}{dt^2} = \frac{dv'}{dt} = k \frac{dv'}{dt} = k \frac{O_2}{O} \left( \frac{dv'}{dt} \right)_{\lambda_{1}-\lambda_{2}} \)

Hence,

\[ \rho L k \frac{O_2}{O} \left( \frac{dv'}{dt} \right)_{\lambda_{1}-\lambda_{2}} = (p_1')_{\lambda_{1}} - (p_2')_{\lambda_{2}} \]  \( (5-34) \)

and by using Eq. (5-31) one obtains

\[ \left( \frac{dv'}{dt} \right)_{\lambda_{1}-\lambda_{2}} = \frac{2}{\rho a} \left[ (A_2 \sin \lambda_t - C_2 \cos \lambda_t) \sin \tau - (A_2 \cos \lambda_t + C_2 \sin \lambda_t) \cos \tau \right] \]

\[ \cdot \cos \tau \right) \sin \lambda_2 \]  \( (5-35) \)

Combining the equations (5-32) through (5-35) finally yields:

\[ \left( \frac{\lambda}{\lambda} - \frac{\cos \lambda_1}{O_1 \sin \lambda_1} - \frac{\cos \lambda_2}{O_2 \sin \lambda_2} \right) \left[ (A_2 \sin \lambda_t - C_2 \cos \lambda_t) \sin \tau + \right. \]

\[ - (A_2 \cos \lambda_t + C_2 \sin \lambda_t) \cos \tau \right] = 0 \]  \( (5-36) \)

with \( \lambda = \frac{KL}{a} \)  \( (5-37) \)

This equation determines the eigenvalues of the system. The equation is only satisfied if

\[ \frac{\lambda}{\lambda} - \frac{\cos \lambda_1}{O_1 \sin \lambda_1} - \frac{\cos \lambda_2}{O_2 \sin \lambda_2} = 0, \text{ or if} \]

\[ \frac{KL_1}{a_0} - \frac{\cos \lambda_1}{O_1 \sin \lambda_1} - \frac{\cos \lambda_2}{O_2 \sin \lambda_2} = 0, \]
which is equivalent to

\[
\frac{KL}{ao} \left( \cotg \frac{KL_1}{a} \right)_{O_1} - \frac{KL}{ao} \left( \cotg \frac{KL_2}{a} \right)_{O_2} = 0. \tag{5-38}
\]

This equation can only be solved numerically. For smaller values of \( \frac{KL}{a} \), the cotangents may be expanded in a series, and neglecting terms of order \( \frac{KL}{a}^3 \) and higher, we obtain:

\[
\frac{KL}{o} = \frac{1}{O_1} \left( \frac{a}{kL_1} - \frac{KL_1}{3a} \right) - \frac{1}{O_2} \left( \frac{a}{kL_2} - \frac{KL_2}{3a} \right) = 0,
\]
yielding

\[
k = a \sqrt{\frac{1}{V_1} + \frac{1}{V_2}} \frac{L/0 + L_1/(30_1) + L_2/(30_2)}{L/0 + L_1/(30_1) + L_2/(30_2)} \tag{5-39}
\]

In the case that

\[
\frac{L_1}{30_1} \ll \frac{L}{0} \quad \text{and} \quad \frac{L_2}{30_2} \ll \frac{L}{0}
\]

the same result as before (Eq. 5-10) is found:

\[
k = a \sqrt{\frac{O}{L} \left( \frac{1}{V_1} + \frac{1}{V_2} \right)} \quad \text{or}
\]

\[
F = \frac{a}{2\pi} \sqrt{\frac{O}{L} \left( \frac{1}{V_1} + \frac{1}{V_2} \right)} \tag{5-40}
\]

It is important to realize that Eq. (5-39) is a refinement as compared to the earlier solution i.e. Eq. (5-10) or Eq. (5-40).
6. The Helmholtz Resonator as a Tool for Determining the Propellant Response Function.

The idea is to modify the HR in such a way that it becomes suitable for the determination of the propellant response function. The propellant response or admittance function is usually determined in a T-burner for the higher frequency region (F \(\geq\) 300 Hz), while for the lower frequency region, \(L^x\)-burners may be used. The T-burner is a tuned device, i.e. the frequency at which any oscillation takes place, corresponds to the acoustic (organ pipe) frequency of the T-burner. Therefore, one can more or less adjust the frequency to a required value, prior to a test run. For the \(L^x\)-burner this is not the case. If oscillations take place, their frequency is solely determined by propellant characteristics. This is a serious draw-back of the \(L^x\)-burner for it is not always well possible to take response function measurements at a prescribed frequency.

T-burner data at low frequencies require such long T-burners that losses seriously degrade the experimental results, and hence make them unreliable. Therefore it is tried to design a device in which tuned low-frequency oscillations may occur, while circumventing the draw-backs of long T-burners.

The Helmholtz Resonator is modified for that reason to contain propellant at one end of the primary chamber while the secondary chamber is provided with a supersonic nozzle. Variations on this idea include propellant in both cavities, or only propellant in the secondary cavity that also contains the nozzle. In this last case there is no mean flow through the neck. Analogue to the classical HR this modified HR possesses an eigenfrequency, and oscillations take place at that particular frequency. This should allow the experimenter to obtain response function data at prescribed, low frequencies, analogous to the high frequency admittance measurements in the T-burner.

An outline of a modified HR is shown in figure 2.
7. Analysis of the Helmholtz Resonator Burner

The analysis proceeds in a similar way as the analysis in Section 5. Essential differences are the non-vanishing mean flow terms, and the different boundary conditions at the burning surface and at the nozzle.

The boundary condition at the burning propellant is given by:

\[ v' = \frac{v}{\gamma_p} p' (\gamma R_b - 1), \]  

(7-1)

where \( R_b \) is the propellant response function.

The boundary condition at the nozzle end is given by:

\[ v' = \frac{v}{\gamma_p} p' \left( \frac{\gamma - 1}{2} \right) \]  

(7-2)

where the zero-length nozzle approximation has been made.

The wave equation for a non-vanishing mean velocity is\(^{15}\):

\[ \frac{\partial^2 p'}{\partial t^2} + 2v \frac{\partial^2 p'}{\partial x \partial t} + (v^2 - a^2) \frac{\partial^2 p'}{\partial x^2} = 0, \]  

(7-3)

with the general solution

\[ p' = A e^{i[\tau(1+M) - \chi]} + B e^{i[\tau(1+M) + \chi]}, \]  

(7-4)

\[ + C e^{i[\tau(1-M) - \chi]} + D e^{i[\tau(1-M) + \chi]} \]

Again \( \tau = kt \), \( \chi = kx/a \), and \( M \) stands for the Mach number.

The wave equation may alternatively be written as

\[ p'_{\tau\tau} + 2M p'_{\tau\chi} + (M^2 - 1)p'_{\chi\chi} = 0. \]  

(7-5)
The equation for conservation of momentum (with a vanishing right-hand side) is (15):

\[ \frac{1}{\rho} \frac{\partial p'}{\partial x} = -\frac{\partial v'}{\partial t} - v \frac{\partial v'}{\partial x}, \]  

(7-6)

\[ \frac{1}{ap} \frac{\partial p'}{\partial \chi} = -\frac{\partial v'}{\partial t} - M \frac{\partial v'}{\partial \chi}. \]  

(7-7)

With the boundary conditions (7-1) and (7-2) we obtain:

\[ \chi = 0 : \quad \left[1 + (\gamma R_D - 1)M_1^2\right] \frac{\partial p'}{\partial \chi} = -M_1 (\gamma R_D - 1) \frac{\partial p'}{\partial t}, \]  

(7-8)

\[ \chi = \lambda_e : \quad \left[1 + \frac{\gamma - 1}{2} M_2^2\right] \frac{\partial p'}{\partial \chi} = -M_2 \frac{\gamma - 1}{2} \frac{\partial p'}{\partial t}. \]  

(7-9)

Now

\[ \frac{\partial p'}{\partial t} = \frac{1}{(1+M)A} e^{i(1+M)(1-\chi)} - \frac{1}{(1+M)B} e^{i(1+M)(1+\chi)} \]

\[ + \frac{1}{(1-M)C} e^{i(1-M)(1+\chi)} - \frac{1}{(1-M)D} e^{i(1-M)(1-\chi)} \]  

(7-10)

and

\[ \frac{\partial p'}{\partial \chi} = \frac{1}{-A e^{i(1+M)(1-\chi)} + B e^{i(1+M)(1+\chi)}} \]

\[ + \frac{1}{C e^{i(1-M)(1+\chi)} - D e^{i(1-M)(1-\chi)}} \]  

(7-11)

Inserting these relations, Eqs. (7-10) and (7-11), into the boundary conditions for \( \chi = 0 \) and \( \chi = \lambda_e \) yields:

\[ C_1 = A_1 \left[1 - 2 M_1 (\gamma R_D - 1)\right] \left[1 + \frac{i}{M_1} \right], \]  

(7-12)

\[ D_1 = B_1 \left[1 - 2 M_1 (\gamma R_D - 1)\right] \left[1 - \frac{i}{M_1} \right], \]  

(7-13)

or similarly
\[ C_1 = A_1 \left[ 1 - 2 M_1 \left( \gamma R_B - 1 - i\tau \right) \right], \quad (7-14) \]
\[ D_1 = B_1 \left[ 1 - 2 M_1 \left( \gamma R_B - 1 + i\tau \right) \right], \quad (7-15) \]
\[ C_2 = A_2 e^{-2i\lambda t} \left[ 1 - 2 M_2 \left( \frac{\gamma - 1}{2} - i\tau \right) \right], \quad (7-16) \]
\[ D_2 = B_2 e^{2i\lambda t} \left[ 1 - 2 M_2 \left( \frac{\gamma - 1}{2} + i\tau \right) \right], \quad (7-17) \]

where terms of order \( N^2 \) and higher have been neglected.

For the velocity at the connection between a cavity and the neck we have:

\[ \rho \dot{v}_i O_i = \rho \dot{v} O. \quad (7-18) \]

Assuming small perturbations, one finds:

\[ \frac{\rho'}{\rho} \frac{v'_i}{v_i} (1 + \frac{\rho'}{\rho}) (1 + \frac{v'_i}{v_i}) = \frac{\rho}{\rho} \frac{v}{O} (1 + \frac{\rho}{\rho}) (1 + \frac{v}{v}). \]

As, in general, \( \rho' / \rho = \frac{1}{\gamma} \frac{P'}{P} \), (isentropic fluctuations), and as the pressure and the pressure fluctuations are assumed the same at both sides of the entrance of the neck, one has:

\[ \frac{v'_i}{v'_i} = \frac{v}{v}, \quad \text{and with Eq. (7-18)} \]

we find:

\[ \frac{v'_i}{v} = \frac{O}{O_i} \quad (7-19) \]

which is the same as what had been found for the case without flow, Eq. (5-28), and where for simplicity, the gas in the neck has been treated as if it were a solid mass.

For the pressure perturbation in the primary cavity one finds:

\[ p'_1 = 2A_1 e^{i\tau} \left[ \cos(\tau M_1 - \chi) - e^{-i(\tau M_1 - \chi)} \right] M_1 (\gamma R_B - 1 - i\tau) + \]
\[ +2B_1 e^{-i\tau} \left[ \cos(\tau M_1 - \chi) - e^{i(\tau M_1 - \chi)} \right] M_1 (\gamma R_B - 1 + i\tau). \quad (7-20) \]
From the continuity equation, one finds

\[
\frac{\partial \rho'}{\partial t} + v \frac{\partial \rho'}{\partial x} = -\rho \frac{\partial v'}{\partial x}, \text{ or rewriting this equation:}
\]

\[
\frac{\partial v'}{\partial x} = -\frac{a}{\gamma \rho} \left( \frac{\partial p'}{\partial t} + M \frac{\partial p'}{\partial x} \right), \tag{7-21}
\]

which yields:

\[
v'_1 = -\frac{2a}{\gamma \rho} \left[ A_1 e^{i\tau} \left\{ -i \sin(TM_1 - \chi) - M_1 (\gamma R - 2 - i \tau) e^{-i(TM_1 - \chi)} \right\} + 
    B_1 e^{-i\tau} \left\{ -i \sin(TM_1 - \chi) + M_1 (\gamma R - 2 + i \tau) e^{i(TM_1 - \chi)} \right\} \right]. \tag{7-22}
\]

In the same way we obtain:

\[
p'_2 = 2A_2 e^{i(T - \lambda_\tau)} \left[ \cos(TM_2 + \lambda_\tau - \chi) - M_2 (\frac{\gamma - 1}{2} - i \tau) e^{-i(TM_2 + \lambda_\tau - \chi)} \right] + 
    2B_2 e^{-i(T - \lambda_\tau)} \left[ \cos(TM_2 + \lambda_\tau - \chi) - M_2 (\frac{\gamma - 1}{2} + i \tau) e^{i(TM_2 + \lambda_\tau - \chi)} \right], \tag{7-23}
\]

\[
v'_2 = -\frac{2a}{\gamma \rho} \left[ A_2 e^{i\tau} \left\{ -i \sin(TM_2 + \lambda_\tau - \chi) - M_2 (\frac{\gamma - 1}{2} - 1 - i \tau) e^{-i(TM_2 + \lambda_\tau - \chi)} \right\} + 
    B_2 e^{-i\tau} \left\{ -i \sin(TM_2 + \lambda_\tau - \chi) + M_2 (\frac{\gamma - 1}{2} - 1 + i \tau) e^{i(TM_2 + \lambda_\tau - \chi)} \right\} \right]. \tag{7-24}
\]

For the pressure and velocity perturbations at the positions \( \chi = \lambda_1 \) and \( \chi = \lambda_\tau - \lambda_2 \) one finds:

\[
(p'_1)_\lambda_1 = 2A_1 e^{i\tau} \left[ \cos \lambda_1 - M_1 \left\{ (\gamma R - 1) e^{i\lambda_1} - i \tau \cos \lambda_1 \right\} \right] + 
    2B_1 e^{-i\tau} \left[ \cos \lambda_1 - M_1 \left\{ (\gamma R - 1) e^{-i\lambda_1} + i \tau \cos \lambda_1 \right\} \right], \tag{7-25}
\]
\begin{align}
(v'_1)_{\lambda_1} &= \frac{2a}{\gamma P} \left[ A_1 e^{i\tau} \left\{-i \sin \lambda_1 + M_1 \left(\gamma R_D - 2\right) e^{-i\lambda_1} + i \sin \lambda_1 \right\} + 
\right.
-B_1 e^{-i\tau} \left\{-i \sin \lambda_1 - M_1 \left(\gamma R_D - 2\right) e^{i\lambda_1} + i \sin \lambda_1 \right\} \right], \quad (7-26) \\
(p'_2)_{\lambda_{l-1}} &= 2A_2 e^{i(\tau-\lambda_2)} \left\{ \cos \lambda_2 - M_2 \left(\gamma - \frac{1}{2}\right) e^{-i\lambda_2} - i \tau \cos \lambda_2 \right\} + 
+2B_2 e^{-i(\tau-\lambda_2)} \left\{ \cos \lambda_2 - M_2 \left(\gamma - \frac{1}{2}\right) e^{i\lambda_2} + i \tau \cos \lambda_2 \right\} , \quad (7-27) \\
(v'_2)_{\lambda_{l-1}} &= \frac{2a}{\gamma P} \left[ A_2 e^{i(\tau-\lambda_2)} \left\{ i \sin \lambda_2 + M_2 \left(\gamma - \frac{3}{2}\right) e^{-i\lambda_2} - i \sin \lambda_2 \right\} + 
\right.
-B_2 e^{-i(\tau-\lambda_2)} \left\{ i \sin \lambda_2 - M_2 \left(\gamma - \frac{3}{2}\right) e^{i\lambda_2} - i \sin \lambda_2 \right\} \right]. 

(7-28)
\end{align}

If it is again assumed that the gas in the neck may be regarded as a rigid piston, relations between the coefficients $A_1$, $A_2$, $B_1$ and $B_2$ may be obtained. If the wave equation for the neck were solved the solution would become more complex. We will not elaborate on this, as these effects are assumed to be of minor importance, and the prime purpose of this study is to investigate the validity of the concept of an HR-burner. If necessary such refinements may be incorporated in follow-on studies.

For the velocity perturbation at the connection of the neck to the primary cavity, we find:

\begin{equation}
(v'_1)_{\lambda_1} = \frac{O_1}{O} \left( v'_1 \right)_{\lambda_1}, \quad (7-29)
\end{equation}

and for the velocity perturbation at the connection of the neck to the secondary cavity, we find

\begin{equation}
(v'_1)_{\lambda_{l-1}} = \frac{O_2}{O} \left( v'_2 \right)_{\lambda_{l-1}}, \quad (7-30)
\end{equation}

Regarding the gas in the neck as a rigid piston yields

\begin{equation}
O_1 (v'_1)_{\lambda_1} = O_2 (v'_2)_{\lambda_{l-1}}. \quad (7-31)
\end{equation}
In combination with the equations (7-26) and (7-28) this yields:

\[
O_1 A_1 \left\{ -i \sin \lambda_1 + M_1 \left[ (YR_B -2)e^{-i\lambda_1} + \tau \sin \lambda_1 \right] \right\} = \\
= O_2 A_2 e^{-\lambda_2 t} \left\{ i \sin \lambda_2 + M_2 \left( \frac{\gamma - 3}{2} e^{-i\lambda_2} - \tau \sin \lambda_2 \right) \right\}, 
\]

(7-32)

\[
O_1 B_1 \left\{ -i \sin \lambda_1 - M_1 \left[ (YR_B -2)e^{-i\lambda_1} + \tau \sin \lambda_1 \right] \right\} = \\
= O_2 B_2 e^{i\lambda_2 t} \left\{ i \sin \lambda_2 - M_2 \left( \frac{\gamma - 3}{2} e^{i\lambda_2} - \tau \sin \lambda_2 \right) \right\}.
\]

(7-33)

The equation of motion for the gas piston is, Eq. (5-34),

\[
\rho L \frac{dv'}{dt} \lambda_1 - \lambda_2 = (P_1')_t \lambda_1 - (P_2')_t \lambda_2
\]

(7-34)

\[
\frac{dv'}{dt} \lambda_1 - \lambda_2 = \frac{2a}{\gamma P} \left[ \lambda_0 e^{i(\tau-\lambda_1)} \left\{ -\sin \lambda_1 + i M_2 \left( \frac{\gamma - 3}{2} e^{i\lambda_2} + \right. \right. \right. \right.
\]

\[
\left. - \tau \sin \lambda_2 + i \sin \lambda_2 \right\} + \\
\left. + B_2 e^{i(\tau-\lambda_2)} \left\{ -\sin \lambda_2 - i M_2 \left( \frac{\gamma - 3}{2} e^{-i\lambda_2} + \right. \right. \right. \right.
\]

\[
\left. - \tau \sin \lambda_2 - i \sin \lambda_2 \right\} \right].
\]

(7-35)

Combining the equations (7-35), (7.27), (7-34) and (7-35) finally yields

\[
\frac{\lambda_0}{\rho} \left[ A_2 e^{i(\tau-\lambda_1)} \left\{ -\sin \lambda_1 + i M_2 \left( \frac{\gamma - 3}{2} e^{i\lambda_2} - \tau \sin \lambda_2 + i \sin \lambda_2 \right) \right\} + \\
+ B_2 e^{i(\tau-\lambda_2)} \left\{ -\sin \lambda_2 - i M_2 \left( \frac{\gamma - 3}{2} e^{-i\lambda_2} - \tau \sin \lambda_2 - i \sin \lambda_2 \right) \right\} \right] =
\]

\[
= \lambda_0 e^{i\tau} \left[ \cos \lambda_1 - N_1 \left\{ (YR_B -1) e^{-i\lambda_1} - \tau \cos \lambda_1 \right\} + \\
+ B_1 e^{-i\tau} \left\{ \cos \lambda_1 - N_1 \left\{ (YR_B -1) e^{+i\lambda_1} + \tau \cos \lambda_1 \right\} + \\
- \lambda_2 e^{i(\tau-\lambda_1)} \left[ \cos \lambda_2 - M_2 \left( \frac{\gamma - 1}{2} e^{+i\lambda_2} - \tau \cos \lambda_2 \right) \right] + \\
- B_2 e^{i(\tau-\lambda_2)} \left[ \cos \lambda_2 - M_2 \left( \frac{\gamma - 1}{2} e^{-i\lambda_2} + \tau \cos \lambda_2 \right) \right].
\]

(7-36)
This equation shows an interesting feature. If one replaces \( k \) (the frequency) by \(-k\), \( \lambda \) is replaced by \(-\lambda\) and \( \tau \) is replaced by \(-\tau\). In that case the expressions which multiply the coefficients \( A \) and \( B \) are simply interchanged. Therefore, one has only to investigate either the \( e^{-i\tau} \) part of the equation as the other part of the equation will yield a similar result.

Moreover the Mach numbers are small for a typical propellant one may estimate:

\[
R = 400 \, \text{m}^2/(\text{s}^2\text{K}), \quad p \approx 5 \times 10^6 \, \text{Pa}, \quad T \approx 2500 \, \text{K}
\]

\[
\rho_d = 1600 \, \text{kg/m}^3 \quad \text{and} \quad v \approx 0.01 \, \text{m/s}.
\]

This yields \( \nu = 1 \, \text{m/s} \). Hence even for a cold gas \( v/a \approx 0.01 \).

For a hot gas \( M_b \approx 0.01 \).

The quantity \( \lambda \) is small too. If the chamber lengths are assumed to be of the order \( 0.1 \, \text{m} \) and even with \( k = 600 \, \text{rad/s} \) we have, (with \( a = 300 \, \text{m/s} \)) \( \lambda \approx 0.2 \). Usually the speed of sound, \( a \), will be much higher \((a = 1000 \, \text{m/s})\) and then \( \lambda \approx 0.2 \).

Therefore, we approximate \( \cos \lambda \approx 1 \) \( \sin \lambda \approx \lambda \) and \( e^{i\lambda} \approx 1 \). A refinement may be obtained by putting \( e^{i\lambda} \approx 1 + i\lambda \).

After this linearization, Eq. (7-32) becomes

\[
A_1 = \frac{O_2 a_2 e^{-i\lambda t}}{O_1} \left[ \frac{-\lambda_2 + iM_2 \left( \frac{\gamma - 3}{2} - \tau\lambda_2 \right)}{\lambda_1 + iM_1 \left\{ (\gamma R_b - 1) + \tau\lambda_1 \right\}} \right]. \tag{7-37}
\]

For the part of Eq. (7-36) which is multiplied by \( e^{i\tau} \) we obtain

\[
\frac{\lambda}{O_1} \left\{-\lambda_2 + iM_2 \left( \frac{\gamma - 3}{2} - \tau\lambda_2 + i\lambda_2 \right)\right\} =
\]

\[
\frac{1}{O_1} \left[ 1 - M_1 \left\{ (\gamma R_b - 1) - i\tau \right\} \right] \left[ \frac{-\lambda_2 + iM_2 \left( \frac{\gamma - 3}{2} - \tau\lambda_2 \right)}{\lambda_1 + iM_1 \left\{ (\gamma R_b - 2) + \tau\lambda_1 \right\}} \right] +
\]

\[
- \frac{1}{O_2} \left[ 1 - M_2 \left( \frac{\gamma - 1}{2} - i\tau \right) \right].
\]
Neglecting terms of order $M^2$ and higher:

$$\frac{\lambda}{\mathcal{O}} \left[ -\lambda_1 \lambda_2 - i\lambda_2 M_1 \left( (\gamma R_b - 2) + \tau \lambda_1 \right) + i\lambda_1 M_2 \left( \frac{\gamma - 3}{2} - \tau \lambda_2 \right) \right] =$$

$$= \frac{1}{\mathcal{O}_1} \left[ -\lambda_2 + \lambda_2 M_1 \left( (\gamma R_b - 1) - i\tau \right) + iM_2 \left( \frac{\gamma - 3}{2} - \tau \lambda_2 \right) \right] +$$

$$- \frac{1}{\mathcal{O}_2} \left[ \lambda_1 M_2 \lambda_1 \left( \frac{\gamma - 1}{2} - i\tau \right) + iM_1 \left( (\gamma R_b - 2) + \tau \lambda_1 \right) \right].$$

Now $\lambda_1 = L_1 k/a$ and $\tau = k t$.

The last equation can only be satisfied for specific values of $k$, i.e. the eigenvalues, or the frequency at which the HR will resonate. To determine these values of $k$, the expressions for $\lambda_1$ and $\tau$ are substituted in this last equation, and its terms are rearranged to the order of $k$:

$$k^4 \left\{ -\frac{i t L}{a^3} \left( M_1 + M_2 \right) \right\} + k^3 \left\{ -\frac{L}{a^3} \left( 1 + M_2 \right) \right\} +$$

$$+ k^2 \left\{ \frac{i L}{a^2} \left( \frac{M_2}{L_2} \frac{\gamma - 3}{2} - \frac{M_1}{L_1} (\gamma R_b - 2) \right) + \frac{i}{a} (M_1 + M_2) \left( \frac{1}{V_1} + \frac{1}{V_2} \right) \right\} +$$

$$+ k \left\{ \frac{1}{a V_1} [1 - M_1 (\gamma R_b - 1)] + \frac{1}{a V_2} [1 - M_2 (\gamma - 1)] \right\} +$$

$$+ i \left\{ \frac{M_1}{L_1 V_2} (\gamma R_b - 2) - \frac{M_2}{L_2 V_1} \frac{\gamma - 3}{2} \right\} = 0.$$

This is a fourth order, algebraic equation in $k$. There exist exact analytic solutions of this equation. However, by virtue of the small perturbing terms, an approximate solution, correct up to first order terms in $M_1$ and $M_2$ is more easily obtained.

The equation is written as

$$\varepsilon_1 k^4 + (-A + \varepsilon_2) k^3 + \varepsilon_3 k^2 + (B + \varepsilon_4) k + \varepsilon_5 = 0.$$
The coefficients $\varepsilon_i$ are first order terms in the (small) Mach numbers:

$$\varepsilon_1 = - \frac{i t L}{o a} (M_1 + M_2),$$

$$\varepsilon_2 = \frac{-L M_2}{o a^3},$$

$$\varepsilon_3 = \frac{L L}{a^2} \frac{M_2}{L_2} \frac{Y-3}{2} - \frac{M_1}{L_1} (\gamma R_b - 2) + \frac{i t}{a} (M_1 + M_2) \left( \frac{1}{V_1} + \frac{1}{V_2} \right),$$

$$\varepsilon_4 = \frac{-M_1}{a V_1} (\gamma R_b - 1) - \frac{M_2}{a V_2} \left[ \frac{Y-1}{2} \right],$$

$$\varepsilon_5 = \left\{ \frac{M_1}{L_1} \frac{V_2}{V_1} (\gamma R_b - 2) - \frac{M_2}{L_2} \frac{Y-3}{2} \right\},$$

while the coefficients A and B stand for:

$$A = \frac{L}{3 o a},$$

$$B = \frac{1}{a V_1} + \frac{1}{a V_2}.$$

The zeroth order equation is:

$$-A k^3 + B k = 0$$

which yields $k = 0$, \hspace{1cm} (7-38)

and $k = \pm \sqrt{B/A}$, or

$$k = \pm a \sqrt{\frac{O}{L} \left( \frac{1}{V_1} + \frac{1}{V_2} \right)} \hspace{1cm} (7-39)$$

This is the same basic frequency that had been found already in Eq. (5-10) or Eq. (5-40). We will indicate this basic-frequency by $k_0$. Moreover, as we will disregard negative frequencies, we have two solutions:
\[ k_{o1} = 0 , \quad (7-40a) \]
\[ k_{o2} = a \sqrt{\frac{0}{L} \left( \frac{1}{v_1} + \frac{1}{v_2} \right)} \quad (7-40b) \]

The frequency \( k_{o2} \) corresponds with the solution previously found for the H.R. The solution \( k_{o1} = 0 \) will be discussed later.

Now assume that the actual frequency differs from the zeroth order frequency only by a small value, of order \( \varepsilon \). We will indicate this correction by \( \phi \), and for the actual frequency we obtain:

\[ k = k_o + \phi \quad (7-41) \]

Substitution of this value for \( k \) in the fourth order equation, and neglecting small terms of second order and higher yields

If \( k_o = 0 \), \( k = -\varepsilon_5/B \quad (7-42) \)

and if \( k_o \neq 0 \) then

\[ k = k_o + (\varepsilon_1 k_o^4 + \varepsilon_2 k_o^3 + \varepsilon_3 k_o^2 + \varepsilon_4 k_o + \varepsilon_5)/2B \quad (7-43) \]

Substitution of the expressions for \( \varepsilon_1 \) through \( \varepsilon_5 \) and for \( B \) finally yields

\[ k = k_o + \frac{\Gamma \sqrt{RT}}{2L^*} \left[ i \left\{ \frac{Y-3}{2} \frac{V_1}{V_2} - (\gamma_{RB} - 2) \frac{V_2}{V_1} \right\} - \frac{k_o}{a} x \right. \]
\[ \left. \times \left\{ \frac{V_2}{V_1} (\gamma_{RB} - 1) + L_2 \frac{V_1}{V_2} \left( \frac{Y+1}{2} + 1 \right) \right\} \right] \quad (7-44) \]

for \( k_o \neq 0 \),

and

\[ k = -\frac{i \Gamma \sqrt{RT}}{L^*} \left( \gamma_{RB} - \frac{Y+1}{2} \right) \quad (7-45) \]

if \( k_o = 0 \).

Now \( \sqrt{RT} \approx 300 \) to \( 1000 \) m/s, depending mainly on the temperature and composition of the combustion products,

\[ \frac{\Gamma}{(2L^*)} \approx 0.05. \]
\[ \kappa_0 \leq 900 \text{ s}^{-1} \]
\[ a \approx 300 \text{ to } 1000 \text{ m/s} \]
\[ L_1, L_2 \approx 0,1 \]

So, \[ k = \kappa_0 + O(5) \left[ i \left\{ \frac{\gamma - 3 V_1}{2 V_2} - (\gamma R_D - 2) \frac{V_2}{V_1} \right\} \right] = O(0,3) \]

For this reason the last expression in Eq. (7-44) may be neglected in comparison with the first and second term.

The frequency, hence, may be expressed approximately as

\[ k = \kappa_0 + \frac{\Gamma a}{2 L^* \sqrt{\gamma}} \left\{ \frac{\gamma - 3 V_1}{2 V_2} - (\gamma R_D - 2) \frac{V_2}{V_1} \right\} \]

(7-46)

for the case that \( \kappa_0 \neq 0 \).

To find an expression for the pressure fluctuation at the burning surface, \( p_1' (\chi = 0) \), we first linearize Eq. (7-20):

\[ p_1' = 2A_1 e^{i \tau} [\cos \chi + \tan_1 \sin \chi - (1 - i (\tan_1 \chi)) M_1 (\gamma R_D - 1 - i \tau)] + \]
\[ + 2B_1 e^{-i \tau} [\cos \chi + \tan_1 \sin \chi - (1 + i (\tan_1 \chi)) M_1 (\gamma R_D - 1 + i \tau)] , \]

or \[ p_1' (\chi = 0) = 2A_1 e^{i \tau} (1 - M_1 (\gamma R_D - 1 - i \tau)) + \]
\[ + 2B_1 e^{-i \tau} (1 - M_1 (\gamma R_D - 1 + i \tau)) , \]

Substituting \( \tau = \kappa t \) we find

\[ p_1' (\chi = 0) = 2A_1 \exp \left[ \left\{ i \left( k_0 + \frac{\Gamma a \sqrt{\gamma}}{2 L^* \sqrt{\gamma}} \frac{V_2}{V_1} \right) - \frac{\Gamma a}{2 L^* \sqrt{\gamma}} \frac{\gamma - 3 V_1}{2 V_2} + \right. \]
\[ - (\gamma R_D - 2) \frac{V_2}{V_1} \right\} (1 + M_1) t - M_1 (\gamma R_D - 1) \right] + \]
\[ + 2B_1 \exp \left[ \left\{ - i \left( k_0 + \frac{\Gamma a \sqrt{\gamma}}{2 L^* \sqrt{\gamma}} \right) + \frac{\Gamma a}{2 L^* \sqrt{\gamma}} \left\{ \frac{\gamma - 3 V_1}{2 V_2} + \right. \]
\[ - (\gamma R_D - 2) \frac{V_2}{V_1} \right\} (1 + M_1) t - M_1 (\gamma R_D - 1) \right] . \]

(7-47)
In case that
\[
\left\{ \frac{y-3}{2} \frac{v_1}{v_2} - (y_{R_b} - 2) \frac{v_2}{v_1} \right\} \neq 0
\]
either one of the two exponentials will die out.

In a similar way one finds for the pressure fluctuation near the nozzle, \((x = \lambda t)\):
\[
p_2'(x = \lambda t) = 2a_2 e^{i\{t(M_2 + 1) - \lambda t\} - \frac{y-1}{2} M_2} +
\]
\[
- \frac{1}{2} \frac{v_2}{v_1} \frac{r_2}{r_1} \{t(M_2 + 1) - \frac{y-1}{2} M_2\}.
\]

Substituting \(\tau = kt\) and \(\lambda t = k L_t / a\) we find
\[
p_2'(x = \lambda t) = 2a_2 \exp \left[ i \left\{ \frac{k_o}{2} + \frac{\Gamma a \sqrt{\gamma}}{2L^2} \frac{v_2}{v_1} r_b \right\} \{t(M_2 + 1) - \frac{L_t}{a}\} +
\]
\[
- \frac{1}{2} \frac{v_2}{v_1} \frac{r_2}{r_1} \{t(M_2 + 1) - \frac{y-1}{2} M_2\} \right] +
\]
\[
+ 2 B_2 \exp \left[ -i \left\{ \frac{k_o}{2} + \frac{\Gamma a \sqrt{\gamma}}{2L^2} \frac{v_2}{v_1} r_b \right\} \{t(M_2 + 1) - \frac{L_t}{a}\} +
\]
\[
+ \frac{1}{2} \frac{v_2}{v_1} \frac{r_2}{r_1} \{t(M_2 + 1) - \frac{y-1}{2} M_2\} \right] \right].
\]
(7-48)

The equations (7-47) and (7-48) hold for \(k_o \neq 0\).

In the case that \(k_o = 0\), we deal with an actual frequency that is completely determined by the propellant response function. This case fully agrees with the classical \(L^*_r\) oscillation, that has been discussed extensively elsewhere.\(^16,18\).
8. Summary of Results

The analysis of the Helmholtz resonator, based on the mass-spring analogue, predicts that the Helmholtz resonator, if it oscillates, oscillates at a frequency:

\[ F = a \sqrt{\frac{0}{L \left( \frac{1}{V_1} + \frac{1}{V_2} \right)/2\pi}} \]  \hspace{1cm} (8.1)

A slightly more detailed analysis, which is based on the solution of the fluid dynamic equations for inviscid flow, yields a small correction w.r.t. this frequency, and predicts a frequency, \( F \):

\[ F = \left\{ a \sqrt{\frac{0}{L \left( \frac{1}{V_1} + \frac{1}{V_2} \right)/2\pi}} \right\} \cdot \frac{1}{\sqrt{1 + \frac{L_1}{L} \frac{0}{30_1} + \frac{L_2}{L} \frac{0}{30_2}}} \]  \hspace{1cm} (8.2)

In many cases, the difference between these two frequencies is negligible and in the analysis of the H.R. with burning propellant and a nozzle, this correction has been omitted. However, it is no problem to account for this correction, as Eq. (8.2) indicates.

If the HR is provided with propellant in the primary cavity and a supersonic nozzle at the secondary cavity, the device may act either as a conventional \( L^* \)-burner, or as a Helmholtz resonator.

If the device acts as a conventional \( L^* \)-burner, the basic frequency corresponds to \( k_{01} = 0 \) and the actual frequency corresponds to

\[ k = \frac{-iRT}{L^*} \left\{ \gamma_{R_b} - \frac{\gamma+1}{2} \right\} \]  \hspace{1cm} (8.3)

Now, the frequency equals:

\[ F = \frac{\Gamma RT}{2\pi L^*} \frac{i}{\gamma_{R_b}} \]  \hspace{1cm} (8.4)

and the growth of the amplitudes is

\[ \frac{iRT}{L^*} \gamma \left( \frac{R_b^f - \frac{\gamma+1}{2}}{\gamma} \right) \]

\[ p' = p'(t=0) e \]  \hspace{1cm} (8.5)

This is in full agreement with what has been found previously for the \( L^* \)-burner.
In the case that the HR-burner acts as an L-burner, the pressure fluctuations in both cavities are about in phase. In that case, the gas in the neck does not act as a vibrating piston, but merely varies its pressure in agreement with the gas in the cavities. If the HR-burner acts as a Helmholtz resonator, on the other hand the pressure fluctuations in both cavities are about \( \pi \) radians out of phase.

In the case that the Helmholtz Resonator is tuned to its basic frequency, the analysis of the measured pressure signals, to obtain the propellant response function needs some more elaboration. In the present studies, it has been assumed that pressure is measured at the nozzle end plate (\( \chi = \lambda_0 \)) and behind the propellant (\( \chi = 0 \)).

The expected pressure fluctuations are given by the Eqs. (7.47) and Eq. (7.48).

These two equations contain four different integration constants: \( A_1 \), \( A_2 \), \( B_1 \) and \( B_2 \). These integration constants depend on each other. The relation between these integration constants is given by the Eqs. (7.32) and (7.33).

To simplify the analysis, small order terms can be neglected. Therefore, recall that \( M = 0 \{ 0.01 \} \),

\[ \lambda = 0 \{ 0.1 \} \]

and \( \tau = k t = 2\pi f t \).

If not more than 15 successive oscillations are taken for the analysis of the data, then \( \tau = 2\pi \times 15 \approx 100 \). If more successive oscillations are found, the analysis must be made by analyzing successive groups of oscillations, each containing a few successive oscillations, such that \( M. \tau \sin \lambda \) remains small w.r.t. 1.

From Eq. (7.32) we have

\[
O_2 A_2 = O_1 A_1 \frac{-i\lambda_1 + M_1 [(\gamma R_b - 2) e^{+i\lambda_1} + \tau \sin \lambda_1]}{i\lambda_1 + M_2 \left[ \frac{\gamma - 3}{2} e^{-i\lambda_2} - \tau \sin \lambda_2 \right]} e^{i\lambda_1 t}.
\]

\[
O_2 A_2 = O_1 A_1 \frac{-i\lambda_1 - iM_1 [(\gamma R_b - 2)(1+i\lambda_1) + \tau \lambda_1]}{i\lambda_1 - iM_2 \left[ \frac{\gamma - 3}{2} (1-i\lambda_2) - \tau \lambda_2 \right]} e^{i\lambda_1 t}.
\]

\[
\approx \frac{-i\lambda_1 - iM_1 \tau \lambda_1}{i\lambda_1 + iM_2 \tau \lambda_2} e^{i\lambda_1 t}.
\]

(8.6)

Here \( (\gamma R_b - 2)(1+i\lambda_1) \) has been neglected with respect to \( \tau \lambda_1 \), and similarly for the nozzle response terms in the denominator. Hence,
\[ O_2 A_2 = -O_1 A_1 e^{-\frac{1}{L_2} + \frac{1}{1+M_2 \tau}} \text{, or} \]

\[ V_2 A_2 = -V_1 A_1 e^{i\lambda t - (M_1 + M_2)kt} \tag{8.7} \]

In a similar way we find for a relation between the "constants" \( B_1 \) and \( B_2 \):

\[ O_2 B_2 = -O_1 B_1 e^{-\frac{1}{L_2} + \frac{1}{1+M_2 \tau}} \text{, or} \]

\[ V_2 B_2 = -V_1 B_1 e^{-\frac{1}{L_2} + \frac{1}{1+M_2 \tau}} \text{, or} \]

\[ V_2 B_2 = -V_1 B_1 e^{-i\lambda t + (M_1 - M_2)kt} \tag{8.8} \]

So:

\[
\begin{align*}
A_2 &= -\frac{V_1}{V_2} A_1 e^{i\lambda t - (M_1 + M_2)kt} \\
B_2 &= -\frac{V_1}{V_2} B_1 e^{-i\lambda t + (M_1 - M_2)kt} 
\end{align*} \tag{8.9}
\]

By substituting these expressions for \( A_2 \) and \( B_2 \) into the expression for the pressure fluctuation at the nozzle endplate,

\[ p_2' = 2A_2 e^{i\tau (M_2 + 1) - \lambda t} - \frac{Y - 1}{2} M_2 + \\
+ 2B_2 e^{-i\tau (M_2 + 1) - \lambda t} - \frac{Y - 1}{2} M_2 , \]

we obtain

\[ p_2'(x = \lambda t) = -2\frac{V_1}{V_2} A_1 e^{i(1-M_1)\tau - \frac{Y - 1}{2} M_2} - 2\frac{V_1}{V_2} B_1 e^{-i(1+M_1)\tau - \frac{Y - 1}{2} M_2} . \]

As now \( \frac{Y - 1}{2} M_2 \approx 0(0,001) \) and \( M_1 \ll 1 \) this can very well be approximated by

\[ p_2'(x = \lambda t) = \frac{V_1}{V_2} \left[ -A_1 e^{ikt} - B_1 e^{-ikt} \right] . \tag{8.10} \]
Similarly for the pressure fluctuations which are measured behind the propellant, we obtain,

\[ P'_1(\chi = 0) = 2A'_1 e^{ik \tau} + 2B'_1 e^{-ik \tau}. \quad (8.11) \]

From the Eqs (8.10) and (8.11) it is obvious that the pressure fluctuations in the primary and secondary chamber indeed are in phase out of phase. Moreover the amplitude of the pressure oscillations differs by a factor \( V_1/V_2 \) which is the ratio of the volumes of the cavities.

We will consider here the case that the pressure is measured at the outer ends of the cavities, i.e. behind the propellant (\( \chi = 0 \)) and at the nozzle end-plate, (\( \chi = \lambda_t \)). Then the pressure fluctuation is described by:

\[ P'_1 = \bar{A}'_1 e^{ik \tau} + \bar{B}'_1 e^{-ik \tau}. \quad (8.12) \]

Here \( i \) stands for the cavity 1 or 2, \( \bar{A}'_1 = 2\bar{A}_1, \bar{A}'_2 = -2V_1/V_2 \bar{A}_1 \) etc.

The complex frequency \( k \) is given by Eq. (7.45) for the case that \( k_0 = 0 \), i.e. in the case that the HR-burner acts as an L^*-burner. We will not elaborate on this case as this has been done elsewhere (16).

If the HR-burner is in the Helmholtz mode, the complex frequency \( k \) is obtained from Eq. (7.44) or Eq. (7.46). This complex frequency, \( k \), in all cases may be written as

\[ k = k^r + ik^i \quad (8.13) \]

The pressure fluctuation that is measured by a pressure transducer is, of course, only the real part of the pressure fluctuation as given by Eq. (8.12). The integration constants \( A'_1 \) and \( B'_1 \) may be complex too, i.e.:

\[ \bar{A}'_1 = \bar{A}_1^r + i\bar{A}_1^i, \]

\[ \bar{B}'_1 = \bar{B}_1^r + i\bar{B}_1^i. \]

So the measured signal corresponds to:
\[ p'_1 = \text{Re} \left[ \left( \tilde{\alpha}^R_1 + i \tilde{\alpha}^i_1 \right) e^{-i\kappa^i_1 x_1 t} + \left( \tilde{\beta}^R_1 + i \tilde{\beta}^i_1 \right) e^{i\kappa^i_1 x_1 t} \right] \]

\[ = e^{-\kappa^i_1 t} \left[ \frac{a}{b_1^c} \frac{b_2^c}{b_1^c} \cos(k^c x_1 t) - \frac{a}{b_1^i} \frac{b_2^i}{b_1^i} \sin(k^i x_1 t) \right] + e^{\kappa^i_1 t} \left[ \frac{a}{b_2^c} \frac{b_2^c}{b_2^i} \cos(k^c x_1 t) + \frac{a}{b_2^i} \frac{b_2^i}{b_2^i} \sin(k^i x_1 t) \right]. \] (8.14)

Initially, the pressure oscillations will be small. Depending on the value of \( k^i_1 \), either one of the exponentials will grow in time while the other one will dampen out.

To investigate what can be expected, consider the imaginary part of the complex frequency, as it is given by Eq. (7.46), where it should be recalled that also the response function \( R_b \) is complex.

\[ k^i_1 = \frac{\Gamma a}{2 \sqrt{\gamma}} \left\{ \frac{\gamma - 3}{2} \frac{V_1}{V_2} - (\gamma R_b^x - 2) \frac{V_2}{V_1} \right\}. \] (8.15)

In fact this equation expresses the difference between energy gains and losses in the HR-burner. The gains are represented by the real part of the response function, the "losses" by the remaining parts. To see this consider the case of a burning propellant that is not sensitive to pressure fluctuations, i.e. a pressure fluctuation is not accompanied by a fluctuation in the burning rate. In that case \( R_b = 0 \). The other case is that a pressure fluctuation indeed is accompanied by a fluctuation in the burning rate.

The energy of a unit volume of gas equals

\[ e = \rho c_v T + \frac{1}{2} \rho \rho_v^2 = \frac{\rho}{\gamma - 1} + \frac{1}{2} \rho \rho_v^2. \]

If there is a pressure fluctuation \( p'_1 \), accompanied by a fluctuation in the density, gas velocity, and burning rate, the fluctuation of the energy content of the volume, \( e' \), amounts to

\[ e' = \frac{\rho'}{\gamma - 1} + \rho^2 \left( \frac{1}{2} \rho \frac{\gamma}{\rho} + \nu^2 / \nu \right), \] (8.16)

and by replacing the velocity perturbation by the response function with the help of Eq. (7.1), one obtains:

\[ e' = \frac{\rho'}{\gamma - 1} + \rho^2 \left( R_b - \frac{1}{2} \frac{g^i_1}{\rho} \right) \]

In the case that there is no burning rate fluctuation, \( R_b = 0 \) and in that case
\[ e'_{R_b=0} = \frac{p'}{\gamma-1} - \frac{\rho v^2 p'}{2\gamma p} \]

So, due to the burning rate fluctuation of the propellant an additional amount of energy,

\[ \Delta e' = \rho v^2 \frac{p'}{p} R_b \quad (8.17) \]

is added to the unit volume of gas. For negative values of \( R_b \), energy is "removed" from the system. Returning now to Eq. (8.15) we note that driving of oscillations may occur for \( R_b > 0 \) and hence for \( k^i < 0 \).

So if growing oscillation amplitudes are observed, these are well described by

\[ p'_1 = e^{-k^i t} \left[ \bar{A}_1 \cos(k^R t) - \bar{A}_1 \sin(k^R t) \right] \]

\[ = e^{-k^i t} \bar{A}_1 \cos(k^R t + \varphi) \quad (8.18) \]

as long as (non-linear) damping is negligible. Viscosity and heat transfer, which have been neglected in this analysis, will dampen the oscillations.

Therefore, measures may have to be taken to keep the effects of heat transfer and viscosity to a minimum. The phase angle \( \varphi \) in Eq. (8.18) is determined by the origin of the time scale, i.e. by the point where \( t = 0 \) is defined. It seems logical to take \( t = 0 \) at a maximum of a pressure fluctuation, in which case \( \varphi = 0 \).

From the expected exponential growth of the pressure amplitude \( \tilde{p}'_1 \), the imaginary part of the complex frequency is determined:

\[ \tilde{\omega} = |\bar{A}_1| e^{-k^i t} \quad (8.19) \]

If the pressure amplitude \( \tilde{p}' \) is measured at the times \( t_1 \) and \( t_2 \), then,

\[ k^i = \frac{\ln \left[ \tilde{p}'(t_2)/\tilde{p}'(t_1) \right]}{t_2 - t_1} \quad (8.20) \]

The imaginary part of the frequency may also be interpreted as the negative of the slope of the logarithm of the amplitude. It therefore is the negative of the growth constant.

After that \( k^i \) has been determined, the real part of the propellant response function follows from:
\[ R_b = \frac{Y - 3}{2Y} \left( \frac{V_1}{V_2} \right)^2 + \frac{2}{Y} - \frac{2kL}{\sqrt{Y} \Gamma a} \frac{V_1}{V_2} \]  

(8.21)

One may in fact replace the imaginary part of the frequency by the growth constant \( \alpha \)

\[ \alpha = -k^i \]  

(8.22)

If viscous damping is present, an estimate of the magnitude of the viscous damping may be made in the following way. It can be expected that after burnout, the pressure oscillations will decay exponentially. This is partly due to the first two terms in Eq. (8.21) which are damping terms, the remainder may be attributed to damping mechanisms that have not been accounted for specifically in these studies. A difficulty, of course, is that in the meantime, the mean pressure also falls off. Experiments will have to show whether it is possible to estimate the effects of damping mechanisms in this way.

The HR-burner allows to repeat testruns at the same resonance frequency, but with different values for the ratio \( V_1/V_2 \). By repeating testruns at different volumetric ratios again additional information may be obtained on the effect of other damping mechanisms. Moreover, as the response function is an independent variable, the growth constant \( \alpha \), should be different for testruns with different volumetric ratios.

An accurate determination of the imaginary part of the propellant response function does not seem to be well possible.

According to Eq. (7.46) there will be a small frequency shift caused by the imaginary part of the propellant response function. This frequency shift, \( \Delta k \), amounts to

\[ \Delta k = k - k_o = \frac{V_2}{V_1} \frac{\Gamma}{\sqrt{Y}} \frac{a}{2L^{1/2}} R^i_D \]  

(8.23)

If one assumes \( R^i_D = 4 \), which is extremely large, \( L^{1/2} = 70 \) m, and \( a = 1100 \) m/s than \( \Delta k = 22 \cdot V_2/V_1 \). Assuming \( V_2 \) and \( V_1 \) to be of the same order, the frequency shift is of the order of 4 Hz. Due to uncertainties in the velocity of sound, \( a \), the resonance frequency usually will not be known to such an accuracy.

It therefore seems that the HR-burner may be developed to become a suitable tool for the determination of the real part of the propellant response function, but for the accurate determination of the imaginary part the device looks less promising.
9. Other Possible Forms of the HR-Burner

The performance of an HR-burner with propellant in the primary cavity and a nozzle at the end of the secondary cavity has been analyzed. For the analysis, it has been assumed that the propellant, in the shape of a flat disk, is located at the far end of the primary cavity, opposing the entrance of the neck. Such an configuration by no means is the only possible one. There are many variations possible and it will have to be investigated experimentally which configuration is most suitable for the determination of the propellant admittance or response function. First of all, neither the cavities, nor the neck have to be cylindrical. In general, any two cavities of arbitrary shape connected by a small neck of arbitrary cross-section may serve the purpose. However, for practical reasons one will usually prefer cylindrical cavities and a cylindrical neck. Second, it is not necessary that the HR-burner has a line or a plane of symmetry. Again for practical reasons, there usually will be one or more lines and planes of symmetry which may simplify interpreting the data, but it is conceivable that in some cases a-symmetry has some beneficial effects.

In the previous Sections it has also been assumed that the propellant disk in the primary cavity had the same diameter as the internal diameter of the primary cavity. To save on the amount of propellant, a smaller diameter disk may be employed. As the mass flow from the propellant then is reduced accordingly, a smaller nozzle, and hence a smaller neck may be used for the same mean pressure level. As the neck cross-sectional area is smaller now, the length of the HR-burner may be reduced for the same resonance frequency, or vice versa, for the same HR-burner dimensions, the resonance frequency is lowered. An HR-burner with a reduced size propellant disk is shown in Fig. 3.

If a reduced size propellant disk is used in the primary cavity, there is a smaller amount of propellant available to pressurize the same HR-burner volume. If heat transfer losses become large, the rise in pressure may become very slow. This may be overcome by putting a second disk of propellant in the secondary cavity. As the size of the cross-sectional area of the neck only depends on the mean mass flow from the primary cavity, and is not affected by the burning propellant in the secondary cavity, the neck-diameter can be kept small. This again leads to small HR-burner dimensions. This configuration is depicted in
Fig. 4. Figure 5 shows a configuration with only propellant in the secondary cavity. A possible advantage is the absence of mean flow through the neck, which again may contribute to small neck dimensions, and hence small HR-burner dimensions.

Though it has been assumed in the analysis that flat propellant disks were used, this is by no means a restriction for an HR-burner. In fact one might consider the use of any suitably shaped propellant grain. For example, if tubular grains are used, the (cold) cylindrical walls are isolated from the hot combustion products, thus reducing heat transfer effects. Such a configuration is depicted in Fig. 6, where a tubular grain is mounted in the secondary cavity. It is, of course, also possible to provide the primary cavity, or both cavities with tubular grains. In fact any arbitrary grain that isolates the cavity walls from hot combustion products might be used. Difficulties that may arise are the manufacturing (and mounting) of the grain, a reliable analysis of the resulting (non-steady) flow field and the determination of the instantaneous burning surface which is necessary to achieve a reliable data reduction.

A different approach is to pre-pressurize the HR-burner. In that case the mean pressure is not a result of the burning propellant but due to an external pressure source, such as a large pressurized surge tank. Such an configuration is shown in Fig. 7. Here, the secondary cavity has been replaced by a large surge tank. To prevent a substantial pressure build-up during a test run, the volume of the surge tank has to be large in comparison with the amount of combustion products. In that case the pressure fluctuations in the surge tank, which acts as the secondary cavity, will only be very small. An advantage may be that nozzle damping has been eliminated. Another advantage obviously applies for laboratories that are equipped for T-burner test runs. One may use the same surge-tank that is used in T-burner test runs, while the HR-burner may be put together partly from already existing T-burner parts. One would only have to add a new end section and neck to one half of a T-burner. In fact one may expect Helmholtz oscillations in a T-burner and these seem to have been observed occasionally. As the flow near the (central) vent of the T-burner turns 90°, this will affect the resonance frequency, as the "end-corrections" strongly depend on
the flow patterns near the exit and entrance of the neck. If half
a T-burner is modified with a different end section and turned 90°
the classical end corrections\textsuperscript{13,14} apply, and the Helmholtz fre-
quency might be predicted more accurately. A possible T-burner modi-
fication is suggested in Fig. 8.

Another modification has successfully been tried by Price\textsuperscript{17} in the
"remote" past. This device, that colloquially became known as the
Dumbbell resonator, consists of two cavities, both containing prop-
ellant, and connected by a neck. The main difference with the
other configurations is the location of the vent in the middle of
the neck. The vent again is connected to a pressurized surge tank.
As the pressure fluctuations in the surge tank may be expected to
be negligible, this device should act as a double Helmholtz resonator.
Though the Dumbbell resonator did not mature into an operational tool
for determining propellant response or admittance functions, Price
has mentioned satisfactory operational results. Figure 9 provides a
schematic of the Dumbbell resonator.

All configurations, discussed above, may be expected to generate the
Helmholtz oscillation mode. Such devices may be used over a large
frequency range, where the lower frequency limit is expected to lie
substantially below the practical lower frequency limit of the
T-burner. The possibility to experiment at these low frequencies is
mainly due to the reduced dimensions of the HR-burner as compared
to the T-burner. At higher frequencies the HR-burner allows for an
additional technique to the already existing ones. The results may
be used to compare with e.g. T-burner, Rotating valve device, Micro-
wave, and Impedance tube data, which is important as there still seem
to be discrepancies between response function measurements as obtained
by the various existing techniques. An additional measuring technique
might help to clarify some of these discrepancies.
10. Conclusions

The HR-burner is a very simple and inexpensive tool, that may use
T-burner components, and that should be very simple to operate.

The HR-burner provides an additional measuring technique for the
response, or admittance function in a large frequency range. In the
higher portion of the frequency range, comparison with data from
already existing techniques may perhaps help to explain discrepancies
between these data.

The elementary analysis of the Helmholtz Resonator burner shows this
device to be a possible practicable tool for the determination of the
real part of the propellant response function. As its dimensions, at
the same frequency, are smaller than T-burner dimensions, and as it is
a tuned device, so that the resonance frequency is only slightly
affected by the imaginary part of the response function, this device
allows for systematic response function measurements in the low fre-
quency range.

If the device is provided with a supersonic nozzle, the fluctuations
are damped by the nozzle flow, and moreover the HR-burner has to
pressurize itself.

If the device is pressurized from an external source, such as a surge
tank, these effects are absent.
11. References


Fig. 1: The principle of the Helmholtz resonator

Fig. 2: Schematic of the HR-burner.
Fig. 3: HR-burner with a reduced size propellant disk

Fig. 4: HR-burner with propellant in both cavities
Fig. 5: HR-burner with propellant only in the secondary cavity

Fig. 6: HR-burner with a tubular propellant charge in the secondary cavity

Fig. 7: HR-burner connected to a pressurized surge tank
Fig. 8: A possible T-burner modification for its adaption as an HR-burner

Fig. 9: Schematic of the Dumbbell Resonator as Used by Price (ref. 17)