Seabed response to water waves

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1. INTRODUCTION

The development of offshore activities during the past and in the future have and will put a demand for an understanding and realistic description of wave-interaction with the seabed. For this reason the Technological University of Delft has carried out a research project, supported by the Netherlands Technology Foundation (STW), on the behaviour of the seabed, in particular near pipelines. In this project research focussed on phenomena relevant to the conditions on the Continental Shelf of the North Sea.

The final objective was to evaluate the stability of an offshore pipeline from a geotechnical point of view. On the Dutch part of the Continental Shelf pipelines are used for the transportation of oil and gas and utilities like glycol. The mean water-depth on this part of the North Sea varies from 20-40 meters. As this region is a major fishing area while in some places the pipelines cross the shipping lanes the authorities demand that pipelines are buried to a depth of 1.00 m below the surface of the seabed. In view of the high costs of the burying operation and because of some reported cases of instability an evaluation of the possibility of instability seems relevant.

In some cases regulations were released to lay pipelines unprotected on top of the seabed. In these cases it was supposed that the pipeline would be buried by natural processes. After installation the behaviour of these pipelines was monitored. Although this method was not successful in all cases it could be concluded that self-burial of a pipeline is possible. The morphological aspects of potential self-burial together with a study into the forces induced by the dynamic fluid pressures have been studied in separate projects, see [1.1] and [1.2].

In 1983 a field investigation was initiated by the State Supervision of Mines in order to obtain geotechnical data near two pipelines and relate the results to the observed self-burial behaviour. This project was executed by Delft Geotechnics while in the last stage during the evaluation a cooperation existed with the Netherlands Technology Foundation. Parts of the results of this project, which were published in separate reports [4.12]
will be published here.

Although clay is also found at the Continental Shelf, great parts of the North Sea bottom consist of loose sediments at least in the upper zones. The theoretical solutions derived here govern the behaviour of a seabed consisting of sand. The results may be converted in order to describe the behaviour of clay as well.

The average water depth on the Continental Shelf is such that these are "intermediate" with respect to the wave length. Because of the local depth a pressure variation is present at the top of the seabed which is conformal to the waves at the surface, but of smaller amplitude. The response of the seabed can be separated in two effects, which can be characterized as an instantaneous short-term effect and a time dependent long-term effect.

The short-term effect is the instantaneous response to water waves of a seabed consisting of a fluid saturated porous medium. The fundamental description of such a material in terms of a linear elastic continuum is furnished by the linear consolidation theory. In chapter 2, based on this theory, analytical solutions will be derived for some configurations e.g. a semi-infinite half-plane. It will appear that with the assumption of an incompressible pore fluid the pore pressure satisfies the simple Laplace equation in the case of a semi-infinite half-plane. It will be shown in chapter 3 that the "incompressible fluid" approximation can be used to develop analytical solutions for the hydrodynamical force on a pipeline for a variety of configurations.

The long-term effect is the time dependent generation of residual pore pressures as a result of the compaction induced by the cyclic shear stress variations. However, a rise in pore pressure decreases the effective soil stress. Sufficient pore pressure generation may lead to instability of the seabed when the intergranular stresses are reduced to zero. This situation is generally known as liquefaction. Much of the present knowledge on cyclic behaviour of sand was developed during research on the effects of earthquakes. In the Netherlands experience was gained.
during the design of the Eastern Scheldt Storm Surge Barrier. In chapter 4 an uncoupled method will be used to calculate the cyclic effects in the seabed. This procedure enables to develop an analytical solution for the pore pressure generation in case of a semi-infinite seabed. Within this concept the driving force to pore pressure generation, i.e. the short-term instantaneous wave response, is separated from the resulting compaction. A dominating factor for the response to cyclic shear stress variations is the stress history, known as preshearing. As a result of previous storms a different, mostly lower, response is found than during first loading. Here a constitutive relation is proposed which includes the effect of the stress history. It is supposed that the influence of a pipe-line may be neglected as the dimensions of a pipe are small compared to those of the wave-induced pressure field.

In practical situations sometimes a layered seabed is found instead of a homogeneous half-plane. The different stress distribution in a layered seabed may result in a stronger compaction and subsequent higher rate of pore pressure generation. In order to calculate this effect the instantaneous response of a finite layer overlaying a stiff impermeable base is determined in chapter 2. This configuration can be considered as the extreme of a layered seabed. Moreover different conditions describing the interface between layer and base are considered. The conditions that are studied here are the case of a perfectly smooth and the other extreme case of a perfectly rough interface. Two methods of solution will be used i.e. the strictly analytical method and a method based on a variational principle. With the last method approximate solutions will be derived for relative thin layers which yield good results compared to the more elaborate analytical solution.

Finally in chapter 5 the stability of offshore pipelines is discussed. It will be shown that the instantaneous wave-induced force on a pipeline is relative small. Furthermore a liquefaction criterion is derived for an infinite seabed. Results of laboratory tests on North Sea samples obtained in the field investigation and results from literature will be used in order
to present an evaluation of the liquefaction potential. Also the influence of a layered seabed is discussed.
2 INSTANTANEOUS WAVE RESPONSE

2.1 Introduction

In this chapter solutions will be derived for the instantaneous stresses and strains in the seabed induced by wave interaction. As explained in the previous chapter the effect of a layered soil profile on the stress distribution is emphasized here. The problem that will be considered is the wave response of a single layer resting on a stiff impermeable base. Basically the same methods, that will be derived here in order to obtain the response of a single layer, can be used for the case of a multi-layered seabed. Only then the number of mathematical manipulations will increase substantially.

It is supposed that the stress level remains within the elastic range. The fundamental description of the deformation of a (almost) completely saturated porous medium has been accomplished by Biot and is known as Biot's theory of (linear) consolidation or poro-elasticity. A complete description of the theory of consolidation is published in e.g. Verruijt [2.1].

Two methods of solution for the response of a finite layer to water waves will be presented. In the first method the problem is treated analytically in the classical way. By solving the differential equations a general solution can be derived. With the boundary conditions the solution is completely determined. The second method is based on a variational principle. Here the problem is formulated into an integral that replaces the set of basic differential equations. With the last approach approximate solutions will be derived for relatively thin layers. It will be shown that these solutions correspond well with the results from the first method within a specific range of the relative thickness of the layer. The theoretical basis of a variational principle and existing techniques of approximation can be found in Kantorovich and Krylov [2.2] and Schechter [2.3].
2.2 Basic equations

It is supposed that the seabed is loaded by simple two-dimensional harmonic waves. As a result the seabed is deformed under plane strain conditions. Following the concept of linear consolidation or poro-elasticity there exist three basic equations. From Terzaghi's principle of effective stress together with Hooke's law, defined by a bulk modulus $K$ and a shear modulus $G$, the equations describing equilibrium in $x$ and $y$ direction are:

\[ Gv^2u + (K + 1/3G) \frac{\partial e}{\partial x} + \frac{\partial p}{\partial x} = 0 \]  
\[ Gv^2v + (K + 1/3G) \frac{\partial e}{\partial y} + \frac{\partial p}{\partial y} = 0 \]  

The pore pressure is denoted by $p$ and the horizontal and vertical displacement by $u$ and $v$. The volume strain $e$ is defined as:

\[ e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \]  

The third equation, known as the storage equation, describes continuity for the pore fluid:

\[ -n\beta' \frac{\partial p}{\partial t} + \frac{\partial e}{\partial t} + k \frac{\partial^2 p}{\partial y^2} = 0 \]  

where $n$ is the porosity, $k$ the permeability of the solid matrix and $\gamma$ the density of the fluid. In principle the pore fluid consists of two phases: liquid (here: seawater) and air. According to Verruijt [2.4] the compressibility of a liquid-air mixture can be expressed depending on the degree of saturation $S$ and the absolute water pressure $p_a$ as:

\[ \beta' = \frac{S}{K_I} + \frac{(1-S)}{p_a} \]  

where $K_I$ is the bulk modulus of seawater in this case. Taking into account North-Sea conditions with mean water depths of about 30 metres combined with a low air content of only a few percent it follows that the compressibility of the pore fluid equals the compressibility of seawater alone. As the compressibility of water is very small compared to the compressibility of the solid matrix it is reasonable to assume that the pore fluid is incompressible. This results into a value of zero for $\beta'$. Defining an elastic coefficient $m$ as:

\[ m = \frac{1}{1-2\nu} \]
the basic equations can be written as:

\[ G \varphi^2 u + mG \frac{\partial \varepsilon}{\partial x} + \frac{\partial p}{\partial x} = 0 \]

\[ G \varphi^2 v + mG \frac{\partial \varepsilon}{\partial y} + \frac{\partial p}{\partial y} = 0 \]

\[ (m+1)G \frac{\partial \varepsilon}{\partial t} + c \varphi^2 p = 0 \]

where \( c \) is the coefficient of consolidation.

\[ c = \frac{k}{y} (m+1)G \quad (2.8) \]

2.3 Constitutive relation

Resulting from the assumption that the seabed can be considered as an isotropic homogeneous linear elastic medium stresses and strains are coupled by Hooke's law. According to Terzaghi's principle of effective stress the deformations in the solid matrix are determined by the difference of the total stress and the fluid stress. This stress difference, defined as the effective stress, characterises the contact forces between the individual soil grains.

The stress-strain relation is:

\[ \sigma_{xx} = 2G \frac{\partial u}{\partial x} + (m-1)G \varepsilon + p \]

\[ \sigma_{yy} = 2G \frac{\partial v}{\partial y} + (m-1)G \varepsilon + p \]

\[ \sigma_{xy} = G \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (2.9) \]

2.4 Boundary conditions

Here a finite layer overlaying a stiff impermeable base, see figure 2.1, is considered. Therefore at two levels boundary conditions have to be specified.
On the surface of the seabed, where \( y=D \), a wave loading is present. It is supposed that this load is a travelling two-dimensional harmonic wave, that for convenience will be written in complex form. The amplitude \( \hat{p} \) of the wave load is calculated from the waves on the sea surface using linear wave theory. Furthermore the wave number is denoted by \( \lambda \) and the wave frequency by \( \omega \).

It is assumed that the main phenomenon is a travelling pressure wave and that boundary effects near the surface of the seabed can be neglected. The wave-induced pressure field has a length scale which equals the wave length while the boundary layer effects are rapidly damped because of the small permeability of soil. As a result the shear stress \( \sigma_{xy} \) is zero at the surface of the seabed. Furthermore the vertical effective stress \( \sigma_{yy} \) is also zero at the seabottom because the seabed is loaded by waves in water only.

At the bottom of the layer, where \( y=0 \), a stiff impermeable base is supposed. From this assumption follows that there the vertical displacement \( v \) and the pressure gradient in \( y \)-direction, i.e. normal to the base, are zero. The last condition refers to the contact between the layer and the base.

In the case of a perfectly smooth interface the shear stress \( \sigma_{xy} \) is zero and horizontal displacements are possible along this boundary. If on the contrary the contact is perfectly rough a no slip condition exists along \( y=0 \). In that case the horizontal displacement \( u \) must be zero. It can be expected that generally the constraint (= no slip) condition will be a better description
of the contact between a soil layer and a base than a perfectly smooth interface.

Hence the boundary conditions are assumed to be:

at the surface of the seabed \((y=D)\)

\[
p = p \exp(i(\omega t - \lambda x))
\]
\[
\sigma_{xy} = 0
\]
\[
\sigma_{yy'} = 0
\]

at the bottom of the layer \((y=0)\)

\[
v = 0
\]
\[
\frac{\partial p}{\partial y} = 0
\]  
\[
(a) \sigma_{xy} = 0 \text{ perfectly smooth}
\]
\[
(b) u = 0 \text{ perfectly rough}
\]

2.5 Analytical method

2.5.1 General solution

The load on the surface of the seabed changes periodically in time and space. The process of consolidation is dominated by diffusion. Combined with a small permeability, which results into small fluid velocities, it is reasonable to assume a harmonic response of the seabed. Furthermore it can be supposed that the response does not depend on the frequency of the wave. The response of the displacements \(u\) and \(v\) and the pore pressure \(p\) has the following form:

\[
p = A \exp(ay) \exp(i(\omega t - \lambda x))
\]
\[
u = B \exp(ay) \exp(i(\omega t - \lambda x))
\]
\[
v = C \exp(ay) \exp(i(\omega t - \lambda x))
\]  
\[
(2.12)
\]

Substitution of this solution \((2.12)\) into the basic equations \((2.7)\) leads to a set of three homogeneous, linear equations in \(A, B\) and \(C:\)

\[
-i\lambda A + G(\alpha^2 - (m+1)\lambda^2)B - i\alpha\lambda mGC = 0
\]
\[
\alpha A - i\alpha\lambda mGB + G((m+1)\alpha^2 - \lambda^2) C = 0
\]  
\[
c(\alpha^2-\lambda^2)A + (m+1)G\alpha B + i(m+1)G\alpha C = 0
\]  
\[
(2.13)
\]

The basic solution of this system is found through the
determinant D. Equating D to zero results into the eigenvalues or roots of the equations (2.13).

\[
\text{Det} = (m+1) \, G(a^2 - \lambda^2)^2 \, (i\omega + c(a^2 - \lambda^2)) = 0
\]  

(2.14)

The eigenvalues are:

\[
\alpha = \pm \lambda \text{ (double)}, \quad \alpha^2 = \lambda^2 (1 + i\frac{\omega}{cl^2})
\]  

(2.15)

With the eigenvalues (2.15) the set of eigenvectors is obtained.

The number of coefficients can be minimized by substitution into the basic equations (2.7). This results into the general solution for the wave response of the seabed:

\[
p = (A_1 \exp(\lambda y) + A_4 \exp(-\lambda y) + A_5 \exp(\omega y) + A_6 \exp(-\omega y)) \exp(i(\omega t - \lambda x))
\]

\[
u = (B_1 \exp(\lambda y) + B_4 \exp(-\lambda y) + \frac{i}{2G\lambda} (A_1 \lambda y \exp(\lambda y) - A_4 \lambda y \exp(-\lambda y))
\]

\[
+ \frac{c\lambda}{(m+1)G\omega} (A_3 \exp(\omega y) + A_6 \exp(-\omega y)) \exp(i(\omega t - \lambda x))
\]

(2.16)

where \( \alpha = \lambda \sqrt{1 + i\frac{\omega}{cl^2}} \), \( \text{Re}(\alpha) > 0 \)

(2.17)

A non-dimensional parameter can be distinguished, defined as:

\[
\phi = \frac{\omega}{cl^2}
\]

(2.18)

2.5.2 Infinite homogeneous seabed

For a semi-infinite seabed only half of the general solution (2.16) is valid. At infinite depth all displacements and stresses must vanish. It is assumed that the seabed is homogeneous and occupies the lower part of the \( x, y \) plane where \( y < 0 \), see figure 2.2.
The part of the solution (2.16) with negative exponential powers does not conform to the conditions at infinite depth and therefore:

$$A_4 = A_5 = B_4 = 0$$ (2.19)

The boundary conditions at the surface of the seabed require that both the vertical effective stress $\sigma_{yy}'$ and the shear stress $\sigma_{xy}$ are zero. As a result $A_3$ and $B_1$ are zero. Finally the last coefficient $A_1$ is found with the boundary condition for the pore pressure, resulting in:

$$p = \hat{p} \exp(\lambda y) \exp(i(\omega t - \lambda x))$$
$$u = i \frac{\hat{p}}{2G\lambda} \lambda y \exp(\lambda y) \exp(i(\omega t - \lambda x))$$
$$v = \frac{\hat{p}}{2G\lambda} (\exp(\lambda y) - \lambda y \exp(\lambda y)) \exp(i(\omega t - \lambda x))$$
$$\varepsilon = 0$$ (2.20)

Although the displacements are non-zero the volume strain $\varepsilon$ appears to be zero. The pore pressure satisfies the Laplace equation. In this case the coupled consolidation equations uncouple and it is possible to study the fluid pressure separately. The fact that the volume strain $\varepsilon$ is zero implies that no consolidation is taking place and that the grain skeleton has a constant volume. The seabed reacts like an incompressible elastic material. This remarkable result must be due to the special circumstance of an incompressible pore fluid combined with the loading condition of a travelling wave in water only.
The solution for the response in case of a homogeneous poro-elastic semi-infinite seabed seems to have first been found by H.L. Koning (priv. comm. [2.5]), see also Yamamoto et al [2.6], Madsen [2.7] and Verruijt [2.8]. A laboratory verification of the theory is also found [2.6] which shows a good agreement between theory and test results. The elastic behaviour of the seabed has been used by Mei, Foda and Mynett see [2.9] to develop a boundary layer approximation. This approximation is applicable under normal sea conditions, considering wind-driven sea waves with a period of about 10s and a wave length of the order of 100 metres and assuming normal values for the sand. Mei and Foda [2.10] have applied this method to the problem of the response of a finite layer in case of a rough interface. The results of this solution correspond with the results of a numerical solution derived earlier by Yamamoto [2.11].

2.5.3 Finite layer "harmonic" solution

In the previous paragraph it was shown that in the case of a semi-infinite half-plane the equations appear to uncouple. The pore pressure satisfies the Laplace equation while the volume strain is zero although a displacement field varies conforming to the progressing waves at the surface of the seabed. It is not likely that the pore pressure will satisfy the Laplace equation also in regions different from a semi-infinite half-plane, e.g. a finite layer.

However it can be expected that for an increasing thickness of the layer the exact solution tends to the "harmonic" solution according to the Laplace equation. In this paragraph the "harmonic" solution for a finite layer will be derived.

As a consequence of the assumption that the pore pressure satisfies the Laplace equation it follows that only one eigenvalue ($\alpha=\lambda$) is valid. For the general solution for the pore pressure $p$ (2.16) this leads to:

$$A_3 = A_6 = 0 \quad (2.21)$$

The complete harmonic solution is determined from the condition
for the gradient of the pressure at the base and the conditions at the surface of the seabed.

\[
p = \frac{\cosh(\lambda y)}{\cosh(\lambda D)} \exp(i(\omega t - \lambda x))
\]

\[
u = \frac{\beta}{2G} \lambda (y-D) \frac{\sinh \lambda y}{\cosh \lambda D} \exp(i(\omega t - \lambda x))
\]

\[
v = \frac{\beta}{2G} \lambda (y-D) \frac{\cosh \lambda y}{\cosh \lambda D} \exp(i(\omega t - \lambda x))
\]

This solution is not exact. At the base, where \(y=0\), still vertical displacements exist. Moreover if a smooth base is considered the shear stress at the base does not vanish. For an increasing thickness of the layer the solution for the vertical displacement at the base tends to zero. It may be expected that for sufficiently thick layers the harmonic solution and the exact solution give similar results.

2.5.4 Finite layer over a perfectly smooth base

Taking the general solution (2.16) for the wave response together with the boundary conditions (2.10 and 2.11a) for a finite layer overlaying a perfectly smooth, stiff and impermeable base a set of six linear equations is obtained. In reality there are twelve equations involved because the coefficients are complex.

In this case the combination of boundary conditions at the base simplify the equations fairly. A simple relation between two coefficients follows from the condition for the vertical displacement \(v\) together with the condition for the shear stress \(\sigma_{xy}\) at the base.

at the base \((y=0)\): \(v\) \[v \mid_{y=0} = i(B_1 - B_4) + \frac{1}{2\lambda a} (A_1 - A_4) + \frac{i\alpha c}{(m+1)G} (A_3 - A_6) = 0 \]
\[\sigma_{xy} \mid_{y=0} = 2G \lambda (B_1 - B_4) + \frac{2ca}{(m+1)\omega} (A_3 - A_6) = 0 \]

From these equations simply follows:

\[A_1 = A_4\] \hspace{50mm} (2.24)
The condition for the gradient of the pore pressure at the base results into:

\[ \frac{\partial p}{\partial y} = \lambda (A_1 - A_4) + \alpha (A_3 - A_6) = 0 \]  \hspace{1cm} (2.25)

Substitution of equation (2.24) now lead to

\[ A_3 = A_6 \]  \hspace{1cm} (2.26)

and therefore

\[ B_1 = B_4 \]  \hspace{1cm} (2.27)

Combining these equations (2.24, 2.26, 2.27) with the boundary conditions at the surface of the seabed three linear equations remain which can be solved analytically by elimination. The remaining set of equations is:

\[
\begin{align*}
y &= D \quad \rho = \beta \exp(i(\omega t - \lambda x)) \\
A_1 (\exp(\lambda D) + \exp(-\lambda D)) + A_3 (\exp(\alpha D) + \exp(-\alpha D)) &= \beta \\
\sigma_{yy} &= 0 \\
2 \left( \frac{\gamma^2}{(m+1)\omega} - \frac{m-1}{m+1} \right) A_3 (\exp(\alpha D) + \exp(-\alpha D)) &= 0 \\
\sigma_{xy} &= 0 \\
2G\bar{A}_1 B_1 (\exp(\lambda D) - \exp(-\lambda D)) + iA_1 \lambda D (\exp(\lambda D) + \exp(-\lambda D)) \\
&\quad + \frac{2 \gamma^2 c \lambda}{(m+1)\omega} A_3 (\exp(\alpha D) - \exp(-\alpha D)) = 0
\end{align*}
\]  \hspace{1cm} (2.28)

2.5.5 Finite layer over a perfectly rough base

Principally the same procedure is applied for a finite layer overlaying a perfectly rough base. Substitution of the general solution into the boundary conditions (2.10 and 2.11b)
lead to six linear equations that can be solved by elimination.

For reasons of simplicity the general solution (2.16) is first written in terms of hyperbolic functions.

The set of equations can be written in the following form at the base, \( y = 0 \):

\[
\frac{\partial^2 p}{\partial y^2} = \lambda (A_1 - A_4) + \alpha (A_3 - A_6) = 0 \quad (a)
\]

\[
u |_{y = 0} = (B_1 + B_4) + \frac{c\lambda}{(m+1)G\omega} (A_3 + A_6) = 0 \quad (b) \quad (2.29)
\]

\[
u |_{y = 0} = i(B_1 - B_4) + \frac{1}{2G\lambda} (A_1 - A_4) + \frac{i\alpha c}{(m+1)G\omega} (A_3 - A_6) = 0 \quad (c)
\]

at the surface of the seabed, \( y = D \):

\[
p |_{y = D} = (A_1 + A_4) \cosh \lambda D + (A_1 - A_4) \sinh \lambda D + (A_3 + A_6) \cosh \lambda D + (A_3 - A_6) \sinh \lambda D = \beta \exp(i(\omega t - \lambda x)) \quad (d)
\]

\[
c_{yy} |_{y = D} = 2G\lambda i ((B_1 + B_4) \cosh \lambda D + (B_1 - B_4) \sinh \lambda D) - \lambda D ((A_1 + A_4) \cosh \lambda D + (A_1 - A_4) \cosh \lambda D) + \frac{2ic\alpha}{(m+1)} \frac{m-1}{m+1} (A_3 + A_6) \cos \lambda D + (A_3 - A_6) \sin \lambda D) = 0 \quad (e)
\]

\[
c_{xy} |_{y = D} = 2G\lambda ((B_1 + B_4) \sinh \lambda D + (B_1 - B_4) \cosh \lambda D) + i\lambda D ((A_1 + A_4) \cosh \lambda D + (A_1 - A_4) \sinh \lambda D) + \frac{2c\alpha}{(m+1)} \frac{m-1}{m+1} ((A_3 + A_6) \sin \lambda D + (A_3 - A_6) \cos \lambda D) = 0 \quad (f)
\]

Defining the combinations of coefficients as new constants e.g. \( A_5 : A_1 + A_4 \) and \( A_0 : A_1 - A_4 \) again a set of six linear equations remain for six unknown coefficients.

For this case solutions have been given by Yamamoto [2.11] and Mei & Foda [2.10].

2.5.6 General results of the analytical method

The general results of the response of a finite overlaying either a smooth or a rough layer are given in the figures 2.3-2.5 and 2.6-2.8. Figures are given for
1) the quotient of the amplitude of the pore pressure at the base and the surface (2.3 smooth, 2.6 rough)
2) the horizontal displacement at the surface and the base (2.4, 2.7)
3) the vertical displacement at the surface and the base (2.5, 2.8) as a function of the relative thickness of the layer \( \lambda D \).

In the calculations the value of the non-dimensional wave parameter was fixed to 100.

As was supposed earlier it can be concluded that for relative thick beds the solution tends to the harmonic solution.
smooth interface

\[ \frac{p_{\text{base}}}{\hat{\rho}} \]

harmonic
analytic

Fig. 2.3
\( \lambda \Omega \)

amplitude of pore pressure at the base

rough interface

\[ \frac{p_{\text{base}}}{\hat{\rho}} \]

analytic
harmonic

Fig. 2.6
\( \lambda \Omega \)

horizontal displacement

\( u / \frac{\hat{\rho}}{2GA} \)

\( u_{\text{base}} \)
\( u_{\text{surface}} \)

Fig. 2.4
\( \lambda \Omega \)

vertical displacement

\( v / \frac{\hat{\rho}}{2GA} \)

\( v_{\text{surface}} \)

Fig. 2.5
\( \lambda \Omega \)

Fig. 2.7
\( \lambda \Omega \)

Fig. 2.8
\( \lambda \Omega \)

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2.6 Variational method

2.6.1 Introduction

An alternative formulation of a problem in continuum mechanics is by following a variational principle. The basis of a variational principle is an integral, or functional, that replaces the balance equations. The variational theory describes the conditions for the integral to attain an absolute extremum with respect to all variables that are involved. This implies that the different derivatives must vanish in the same point. Generally such a point is called a stationary point. In the stationary point the set of balance equations is satisfied while at the same time it can be characterized by a condition of minimum entropy production.

In the theory of elasticity the functional describes the potential energy in the volume that is being considered. In the stationary point where the potential energy reaches an absolute minimum the balance equations are satisfied. In this case a state of balance corresponds to a state of minimum potential energy. In linear consolidation theory the same is true only in the absence of pore fluid. Then the coupled poro-elastic problem reduces to the deformation of an elastic continuum. For consolidation problems where pore pressures may be generated, the physical consistancy is postulated. The functional that is used here is mathematically defined in such a way that the stationary point conforms to a state of balance.

One particular field of application of variational principles is to develop approximate solutions. A general approximation procedure is to suppose a solution taken from a restricted class of functions. This solution is optimized with the functional. The theory and application of variational principles can be found in [2.2] and [2.3].

2.6.2 Variational principle

Unfortunately it is not possible to define a functional in the classical way, for time-dependent processes like consolidation of a poro-elastic medium or conduction of heat in solids.
This can be solved by introducing a time step. Solutions can be found numerically with e.g. the finite element method.

However the special circumstances here of cyclic loading enables to define a functional for the consolidation equations.

In paragraph 2.5.1 it is supposed that all variables have the following form:

\[ u = u(y) \exp(i(\omega t - \lambda x)) \quad \text{or} \quad u = \bar{u} \exp(i\omega t) \]  

(2.30)

where \( \bar{u} \) can be regarded as the average of a variable over a wave period. With the use of a Fourier transformation with respect to time or by just considering the average of all variables, the set of basic differential equations (2.7) can be transformed into:

\[
GV^2 \bar{u} + mG \frac{\partial \bar{e}}{\partial x} + \frac{3p}{\partial x} = 0
\]

\[
GV^2 \bar{v} + mG \frac{\partial \bar{e}}{\partial y} + \frac{3p}{\partial y} = 0
\]

(2.31)

\[ i\omega \bar{e} + cG^2 \bar{p} = 0 \]

The solution to a problem defined by the equations (2.31) can be replaced by a functional \( U \).

\[
U = \frac{1}{2} \int_0^L \int_0^D \left[ (K-2/3G) \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right)^2 + 2G \left( \frac{\partial \bar{u}}{\partial x} \right)^2 + \left( \frac{\partial \bar{v}}{\partial y} \right)^2 \right]
\]

\[
+ G \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right)^2 + 2p \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right)^2
\]

\[
- \frac{k}{i\omega} \left( \frac{\partial \bar{p}}{\partial x} \right)^2 + \left( \frac{\partial \bar{p}}{\partial y} \right)^2 \right) \, dx \, dy
\]

(2.32)

The total volume of soil that is being considered here is a layer with depth \( D \). Because the applied loading that will be considered is cyclic in \( x \)-direction, the length is chosen equal to the wave length \( L \).

2.6.3 Proof of the variational principle

According to the definition the functional \( U \) reaches a stationary value when the variation of \( U \) is zero. It is supposed that \( U \) reaches a stationary value, denoted by \( U_0 \), when from all possible functions the solutions for the displacements and
the pore pressure attain $\ddot{u}, \ddot{v}$ and $\ddot{p}$. In order to prove that for
$U=U_0$ indeed the balance equations (here 2.31) are satisfied
functions different from $\ddot{u}, \ddot{v}, \ddot{p}$ have to be considered.
Therefore suppose:

$$
\begin{align*}
\ddot{u}_\Delta &= \ddot{u} + \Delta \xi \\
\ddot{v}_\Delta &= \ddot{v} + \Delta \eta \\
\ddot{p}_\Delta &= \ddot{p} + \Delta \tau
\end{align*}
$$

(2.33)

where $\Delta$ is an arbitrary number and $\xi$ and $\eta$ are such that the
boundary conditions for the displacements are satisfied. The
value of $U$ corresponding to $\ddot{u}, \ddot{v}$ and $\ddot{p}$ is found by substituting
expression (2.33) into the functional (2.32). The result will
differ from $U_0$ and the difference is called the variation $\delta U$, defined by

$$
U = U_0 + \delta U
$$

(2.34)

Furthermore the variation can be written as:

$$
\delta U = A_1 \Delta + A_2 \Delta^2
$$

(2.35)

where

$$
A_2 = \frac{1}{2} \int_{0}^{L} \int_{0}^{D} \left( K - \frac{2}{3} G \right) \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right)^2 + 2G \left( \frac{\partial^2 \xi}{\partial x^2} \right)^2 + 2G \left( \frac{\partial^2 \eta}{\partial y^2} \right)^2 \\
+ \frac{k}{\gamma} \int \left( \frac{\partial \tau}{\partial x} \right)^2 + \left( \frac{\partial \tau}{\partial y} \right)^2 \right) dxdy
$$

(2.36)

and

$$
A_1 = \frac{1}{2} \int_{0}^{L} \int_{0}^{D} \left( K - \frac{2}{3} G \right) \left( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right)^2 + 2G \left( \frac{\partial \xi}{\partial x} \right)^2 + 2G \left( \frac{\partial \eta}{\partial y} \right)^2 \\
+ \frac{k}{\gamma} \left( \frac{\partial \tau}{\partial x} \right)^2 + \left( \frac{\partial \tau}{\partial y} \right)^2 + \left( \frac{\partial \tau}{\partial x} \right)^2 + \left( \frac{\partial \tau}{\partial y} \right)^2 \\
- \frac{k}{\gamma} \left( \frac{\partial \tau}{\partial x} \right)^2 + \left( \frac{\partial \tau}{\partial y} \right)^2 \right) dxdy
$$

(2.37)

The expression $A_1$ can be partially transformed by partial
integration into an expression that vanishes because for the
functions, $\ddot{u}, \ddot{v}$ and $\ddot{p}$ the differential equations (2.31) are full-
filled. The remaining part also equals zero because the boundary
conditions are satisfied.
Therefore

\[ A_1 = 0 \]  \hspace{1cm} (2.38)

and equation (2.35) falls to:

\[ \delta U = A_2 \Delta^2 \]  \hspace{1cm} (2.39)

The sign of the variation \( \delta U \) depends entirely on the sign of \( A_2 \). Apparently all values of \( U \) differ in the same way from \( U_0 \), meaning that \( U_0 \) indeed is an absolute extremum and a stationary value of \( U \). It can be concluded that the stationary value \( U_0 \) of \( U \) is attained in the point \( \bar{u}, \bar{v}, \bar{w} \).

The expression for \( A_2 \) contains apart from one only quadratic terms. Besides that the physical constants \( K, G, k, \gamma \) and \( \omega \) are such that all quadratic terms are positive.

A positive value for \( A_2 \) implies that \( U_0 \) is an absolute minimum of \( U \) because all variations from \( U_0 \) lead to a value greater than \( U_0 \). In the theory of elasticity the functional describes the potential energy of the observed medium and reaches an absolute minimum in the stationary point. The state of balance corresponds with minimum potential energy. The functional \( U \) describing consolidation (for cyclic variations in time) reduces to the functional for elasticity when there is no water in the pores. Concluding it seems likely that although the functional is defined in a way that the consolidation equations are satisfied also in this case \( A_2 \) is positive and consequently \( U_0 \) is an absolute minimum.

2.6.4 Method of approximation

The principle concept of approximation here is that solutions will be derived for relatively thin layers. Field investigations frequently prove the existence of layers with different properties at shallow depth. The soil profile in such cases may look like a sand layer of a few metres overlaying a thick impermeable clay stratum. Under those conditions the ratio of the thickness of the layer and the wave length is small (\( \approx 0.01-0.10 \)) and the layer can be considered as thin.
As an approximation criterion the shear stress $\sigma_{xy}$ will be estimated. In the case of a smooth interface between layer and base the shear stress is zero at the surface of the seabed as well as at the base. As the layer is supposed to be thin it can be assumed that the shear stress $\sigma_{xy}$ is zero throughout the whole layer.

For a rough condition at the base a similar approach is possible. At the surface of the layer the shear stress is zero and at the base it reaches a fixed value in order to satisfy the no-slip condition. As a first approximation a linear function for the shear stress can be assumed.

The validity of these assumptions is supported by the results of the analytical solution. In figure 2.9 the shear stress is shown for two cases:
- a thin layer ($\lambda D = 0.5$) overlaying a (stiff, impermeable) smooth base
- a thin layer ($\lambda D = 0.5$) overlaying a (stiff, impermeable) rough base

![Shear stress $\sigma_{xy}$ in case of a finite layer for a smooth (left) and rough (right) interface.](image)

It can be concluded that the shear stress is indeed small in case
of a smooth interface and almost linear for a rough interface.

Two different techniques will be used in order to arrive at the complete solution namely the Kantorovich and the Rayleigh-Ritz method.

The first method is generally based upon partial integration and is due to Kantorovich, see e.g. [2.2]. The fundamental idea behind the method is that part of the solution is set and the other part is left as an undetermined function. Here correspondingly to the supposed loading a cyclic behaviour in x-direction is assumed and the behaviour in y-direction is left undetermined. Next the estimated function for the shear stress \( \sigma_{xy} \) leads to a more specified displacement field. In order to find conditions for the arbitrary functions this displacement field can be substituted into the functional (2.32). According to the variational principle the governing balance equations are satisfied when the functional attains a stationary value. Starting from an approximate displacement field an optimum solution or "best fit" is obtained from the extremalization procedure.

Instead of the original balance equations (2.31) a set of simultaneous ordinary differential equations is obtained together with two additional conditions. These conditions will appear to be vertical equilibrium at the surface of the layer and at the base. These conditions are necessary to derive the complete solution. The method of partial integration will be applied to the problem of the wave response of a finite layer in paragraph 2.6.5 and 2.6.6.

Another way of deriving this set of differential equations is by substituting the supposed displacement field into the set of balance equations. Only then a choice has to be made which additional condition is used. Here vertical equilibrium seems obvious but with the use of the variational principle the equations and conditions are derived in a more consistent way.

In the second method the function describing the behaviour in y-direction of the displacements \( u, v \) and the pore pressure \( p \) are approximated by a series solution with a limited number of
terms. This method has been developed by Rayleigh and Ritz. The assumed condition for the shear stress is used in order to derive a relation between some of the coefficients.

The displacement field and the pore pressure are substituted into the functional. In the stationary point the functional attains an extremum and therefore the derivatives of the functional to the remaining coefficients must vanish. The result of the last procedure, taking the derivative of the functional to the remaining coefficients, leads to a set of linear equations from which the coefficients can be solved. The Rayleigh-Ritz method will be applied in paragraph 2.6.7 and 2.6.8.

Generally variational methods lead to approximate solutions that for a certain range coincide well with the exact solution. Only the results are as good as the assumptions about the displacement field. In the Rayleigh-Ritz method principally the undetermined function from the method of partial integration is approximated by a series expansion with a limited number of terms. In cases where such expansions are possible it can be expected that the Rayleigh-Ritz method will yield to less exact results because in the method of partial integration there are more admissible degrees of freedom. It will appear that also for the problems considered here variational methods are quite powerful.

2.6.5 Kantorovich technique for a finite layer over a perfectly rough base

In this paragraph the Kantorovich technique or method of partial integration will be applied to determine the response to water waves a (thin) finite layer overlaying a stiff impermeable base. The contact between layer and base is perfectly rough. For the response the following solution is assumed:

\[
\begin{align*}
\ddot{p} &= w(y) \exp(-i\lambda x) \\
\ddot{u} &= f(y) \exp(-i\lambda x) \\
\ddot{v} &= g(y) \exp(-i\lambda x)
\end{align*}
\]  

(2.40)

Conforming to the load the response is cyclic in \(x\)-direction. The functions, \(w, f\) and \(g\) are arbitrary functions in \(y\). Furthermore it
is supposed that in this case of a thin layer with perfectly rough conditions at the base the shear stress $\sigma_{xy}$ is linear. For the shear stress the following equation can be derived:

$$\sigma_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = G\left(f' - i\lambda g\right)\exp(-i\lambda x) = G(Ay + B)\exp(-i\lambda x) \quad (2.41)$$

where $A$ and $B$ are undetermined coefficients.

The assumed solution can be changed to:

$$\tilde{p} = w\exp(-i\lambda x)$$
$$\tilde{u} = f\exp(-i\lambda x) \quad (2.42)$$
$$\tilde{v} = \frac{1}{i\lambda}\left(f' - (Ay + B)\right)\exp(-i\lambda x)$$

In order to determine the variation of the functional $U$ it can be assumed that a stationary value $U_0$ is obtained for the solution $(2.42)$. Functions that differ from $w$ and $f$ lead to conditions for a stationary value. Suppose:

$$\tilde{p}_\Delta = w\exp(-i\lambda x) + \Delta \tau$$
$$\tilde{u}_\Delta = f\exp(-i\lambda x) + \Delta \xi \quad (2.43)$$
$$\tilde{v}_\Delta = \frac{1}{i\lambda}\left(f' - (Ay + B)\right)\exp(-i\lambda x) + \Delta \frac{\partial \xi}{\partial y}$$

Substitution of this solution into the functional leads to the variation $\delta U$ defined as

$$U = U_0 + \delta U \quad (2.44)$$

The variation $\delta U$ can be written as:

$$\delta U = \Delta A_1 + \Delta^2 A_2 \quad (2.45)$$

Similar to the procedure for the proof of the variational principle (par. 2.6.3) the term $A_1$ leads to a set of differential equations. Earlier the balance equations followed from the exact functional, the result here will be an approximation.
The term $A_1$ follows as:

$$A_1 = \int \int_0^L \int_0^D (K-2/3 \, G)(-i \lambda f e^{-i \lambda x} + \frac{1}{i \lambda} (f''-A)e^{-i \lambda x}) \left( \frac{\partial \xi}{\partial x} + \frac{1}{i \lambda} \frac{\partial^2 \xi}{\partial y^2} \right)$$

$$+ 2G \left( -i \lambda f e^{-i \lambda x} \frac{\partial \xi}{\partial x} + \frac{1}{i \lambda} (f''-A)e^{-i \lambda x} \frac{1}{i \lambda} \right)$$

$$+ G \left( (Ay+B)e^{-i \lambda x} \right) \left( \frac{\partial \xi}{\partial y} + \frac{1}{i \lambda} \frac{\partial^2 \xi}{\partial x \partial y} \right)$$

$$+ \left( we^{-i \lambda x} \frac{\partial \xi}{\partial x} - \frac{\partial \xi}{\partial x} - i \lambda f e^{-i \lambda x} \right) + \left( we^{-i \lambda x} \frac{1}{i \lambda} \frac{\partial^2 \xi}{\partial y^2} + \frac{1}{i \lambda} (f''-A)e^{-i \lambda x} \right)$$

$$- \frac{k/\gamma}{i \omega} \left( -i \lambda w e^{-i \lambda x} \frac{\partial \xi}{\partial x} + w e^{-i \lambda x} \frac{\partial \xi}{\partial y} \right) \, dx \, dy \quad (2.46)$$

Through partial integration this integral can partly be changed into $I_1$:

$$I_1 = \int \int_0^L \int_0^D \xi e^{-i \lambda x} \left( - \frac{(m+1)G}{\lambda^2} f(4) + (m+1)G \lambda^2 f + i \lambda w + \frac{1}{i \lambda} w'' - (m+1)GA \right)$$

$$+ \tau e^{-i \lambda x} \left( - i \lambda f + \frac{1}{i \lambda} (f''-A) + \frac{k/\gamma}{i \omega} (w'' - \lambda^2 w) \right) \, dx \, dy \quad (2.47)$$

As $U_0$ is an extremum of $U$ the integral $I_1$ must be equal to zero for all arbitrary functions $\xi$ and $\tau$. This leads to a set of simultaneous differential equations:

$$- \frac{(m+1)G}{\lambda^2} f(4) + (m+1)G \lambda^2 f + i \lambda w + \frac{1}{i \lambda} w'' = (m+1)GA \quad (2.48)$$

$$-i \lambda f + \frac{1}{i \lambda} (f''-A) + \frac{k/\gamma}{i \lambda} (w'' - \lambda^2 w) = 0$$

Another result of the partial integration are conditions at the surface and the bottom of the layer which are satisfied when the boundary conditions are satisfied. Furthermore additional conditions at the top and the bottom of the layer follow:

$$\left. \frac{(m+1)G}{\lambda^2} f(3) + (m-1)Gf' - G(Ay+B) - \frac{1}{i \lambda} \right|_{y=0} w' = 0 \quad (2.49)$$

These last equations (2.49) refer to vertical equilibrium at the surface and the bottom of the layer.

The general solution for the simultaneous linear ordinary differential equations is found by combining both and eliminate $w$ or $f$. For the function $f$ the following equation can be derived:

$$- \frac{(m+1)G}{\lambda^2} f(4) + \frac{i \omega}{k/\gamma} f'' + ((m+1)G \lambda^2 + \frac{i \omega}{k/\gamma}) f = ((m+1)G + \frac{i \omega}{k/\gamma}) A \quad (2.50)$$

\[26\]
This equation is an inhomogeneous ordinary linear differential equation of the fourth order that can be solved in the classical way. The characteristic equation is:

\[ a_{1,2}^2 = i\lambda^2 (\phi \pm \sqrt{(\phi)^2 - 1 - i\phi}) \]  

(2.51)

where \( \phi \) is the wave parameter (2.18) defined as:

\[ \phi = \frac{\omega}{\lambda^2} \]  

(2.52)

The particular solution \( f_p \) of the nonhomogeneous part of the equation is

\[ f_p = \frac{A}{\lambda^2} \]  

(2.53)

The general solution for \( f \) can be written as:

\[ f = A_1 \cosh \alpha_1 y + B_1 \sinh \alpha_1 y + C_1 \cosh \alpha_2 y + D_1 \sinh \alpha_2 y + \frac{A}{\lambda^2} \]  

(2.54)

The general solution for \( w \) is found by using the solution for \( f \) as a nonhomogeneous part of the differential equations and substitute expression (2.54) into one of the equations for instance (2.48b). The general solution for \( w \) follows as:

\[ w = C_2 \cosh \alpha_1 y + D_2 \sinh \alpha_1 y \]

\[ - \frac{\omega}{k/\gamma} \left( \frac{a_1^2 + \lambda^2}{\lambda a_1^2 - \lambda^2} (A_1 \cosh \alpha_1 y + B_1 \sinh \alpha_1 y) + \frac{a_2^2 + \lambda^2}{a_2^2 - \lambda^2} (C_1 \cosh \alpha_2 y + D_1 \sinh \alpha_2 y) \right) \]  

(2.55)

It is not surprising that one part of the general solution of \( w \) contains the characteristic eigenvalue \( \lambda \) for it represents the harmonic solution. For large values of the thickness \( D \) the other part of the solution must vanish because then the pore pressure satisfies the Laplace equation.

The total number of coefficients is eight \((A_1-D_2, A, B)\) while from the boundary conditions six equations are obtained. The remaining two conditions are the additional conditions for vertical equilibrium that followed from the variational method.

As already mentioned in paragraph 2.6.4. the same set of differential equations can be derived by substitution of the general solution (2.40) into the balance equations. Only the additional conditions necessary to derive the complete solution...
are lacking. In that case a choice has to be made which equation (hor., vert. equilibrium or continuity) is used; but not every choice will lead to the same result. Because the derived solution is an approximation it is not likely that in the end all balance equations will be satisfied. In this case where the applied load is cyclic in x-direction vertical equilibrium seems more relevant rather than horizontal. However with the variational principle the equations and conditions are derived in a more consistent way.

2.6.6 Kantorovich technique for a finite layer over a perfectly smooth base

In the previous paragraph an approximate solution has been derived for the response of a finite layer in case of perfectly rough conditions at the base. This solution can be used here. As described in paragraph 2.6.4 it is assumed for a perfectly smooth base that the shear stress $\sigma_{xy}'$ is zero throughout the whole layer. By making the coefficients determining the linear shear stress function in case a rough layer A and B zero the shear stress $\sigma_{xy}'$ vanishes. The supposed displacement field transforms into:

$$\begin{align*}
\tilde{p} &= w(y) \exp(-i\lambda x) \\
\tilde{u} &= f(y) \exp(-i\lambda x) \\
\tilde{v} &= \frac{1}{i\lambda} f'(y) \exp(-i\lambda x)
\end{align*}$$  \hspace{1cm} (2.56)

The general solution is found by simply substituting $A=0$ and $B=0$ in the solution (2.54, 2.55). The six remaining coefficients have to be determined. From the boundary conditions (2.10, 2.11a) only four equations are obtained because the two conditions for the shear stress drop out. The two additional conditions for vertical equilibrium yield the required equations. Also in the last expression $A=0$ and $B=0$ has to be inserted.

2.6.7 Rayleigh-Ritz technique for a finite layer over a perfectly smooth base

In the Rayleigh-Ritz technique part of the solution is written as a series expansion with a limited number of terms.
Here a series approximation is supposed for the behaviour in y-direction. Hence the function \( f(y) \) as supposed in the two previous paragraphs is, with this procedure, approximated by a series solution. For analytic functions it can be expected that the accuracy increases with the number of terms.

For convenience it is assumed that Poisson's ratio is zero. As a first approximation a solution with quadratic terms is supposed:

\[
\begin{align*}
\ddot{p} &= (p_0 + p_1 y + p_2 y^2) \exp(-i\lambda x) \\
\ddot{u} &= (u_0 + u_1 y + u_2 y^2) \exp(-i\lambda x) \\
\ddot{v} &= (v_0 + v_1 y + v_2 y^2) \exp(-i\lambda x)
\end{align*}
\]  

(2.57)

Some of the coefficients can be solved from the boundary conditions. At the bottom of the pipe, where \( y=0 \), the vertical displacement as well as the gradient in y-direction of the pore pressure are zero, therefore:

\[
\begin{align*}
v_0 &= 0 \text{ and } p_1 = 0
\end{align*}
\]  

(2.58)

Furthermore it is assumed that everywhere the shear stress \( \sigma_{xy} \) is zero:

\[
\sigma_{xy} = G \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = G (u_1 + 2u_2 y - i\lambda (v_1 y + v_2 y)) \exp(-i\lambda x) = 0
\]

(2.59)

From this condition the following equations can be derived:

\[
\begin{align*}
u_1 &= 0 \\
v_2 &= 0 \\
\frac{u_2}{2} &= i\lambda v_1
\end{align*}
\]  

(2.60)

It is supposed that Poisson's ratio is equal to zero. This simplifies the expressions for the stresses. At the surface of the layer, where \( y=D \), the effective stress is zero:

\[
\sigma_{yy} \bigg|_{y=D} = 2G \frac{\partial \ddot{v}}{\partial y} = 2Gv_1 \exp(-i\lambda x) = 0
\]

(2.61)

The final condition for the pore pressure at the surface of the
layer refers to the applied load and an equation follows:
\[
\bar{p}\big|_{y=0} = (p_0 + p_2 D^2) \exp(-i\lambda x) = \bar{p} \exp(-i\lambda x)
\]  (2.62)

As a result of these equations the displacement field can be simplified to:
\[
\begin{align*}
\bar{p} &= (\bar{p} + p_2 (y^2 - D^2) \exp(-i\lambda x) \\
\bar{u} &= u_0 \exp(-i\lambda x) \\
\bar{v} &= 0
\end{align*}
\]  (2.63)

Two coefficients are still undetermined. Next the displacement field is substituted into the functional \(U\) (2.32). A stationary value for \(U\) is obtained when:
\[
\frac{3U}{3p_2} = 0 \text{ and } \frac{3U}{3u_0} = 0
\]  (2.64)

These conditions lead to a set of two linear equations in \(p_2\) and \(u_0\):
\[
\begin{align*}
\frac{4}{3} i\lambda u_0 - \frac{k/\gamma}{i\omega} p_2 \left( \frac{8}{3} - \frac{16 i\lambda^2 D^2}{15} \right) - \frac{k/\gamma}{i\omega} \bar{p} \frac{4}{3} \lambda^2 &= 0 \\
- 2G i\lambda u_0 - \frac{2}{3} p_2 D^2 + \bar{p} &= 0
\end{align*}
\]  (2.65)

With these equations the approximation solution is completely determined.

2.6.8 Rayleigh-Ritz technique for a finite layer over a perfectly rough base

For a finite layer overlaying a perfectly rough base a similar approach is adopted as used in the previous paragraph for a layer over a rough base. For the response the same quadratic solution (2.57) is supposed.

Here at the base both the displacements in \(x\)- and \(y\)-direction together with the gradient of the pore pressure normal to the base is zero. Moreover it is supposed that the shear stress \(\sigma_{xy}\) is a linear function in \(y\)-direction while \(\sigma_{xy}\) at the surface of the layer is zero. Furthermore the vertical effective stress is zero at the surface and the pore pressure equals from the applied load there. With these conditions the response can be
reduced to
\[
\tilde{p} = (\tilde{p} + p_2(y^2 - D^2)) \exp(-i\lambda x)
\]
\[
\tilde{u} = u (y^2 - 2Dy) \exp(-i\lambda x)
\]
\[
\tilde{v} = 0
\]
(2.66)

This solution is substituted into the functional 2.32. A stationary value for \( U \) is obtained when:

\[
\frac{3U}{3p_2} = 0 \text{ and } \frac{3U}{3u_2} = 0
\]
(2.67)

These conditions lead to a set of two linear equations

\[
-11\lambda Du_2Di + \frac{k/y}{\omega} 8p_2(5-2\lambda^2D^2)i = -\frac{k/y}{\omega} 20\lambda^2\beta i
\]
\[
u_2(5-4\lambda^2D^2) - 11\lambda Dp_2Di = -20 \beta y i
\]
(2.68)

With these two equations the complete approximate solution can be derived.

2.6.9 General results of the variational method

The general results of the approximate solutions according to the Kantorovich method are shown for the same variables as has been done in paragraph 2.5.6 for the analytical method:

1) the quotient of the amplitude of the pore pressure at the base and at the surface of the layer (2.10 smooth, 2.13 rough)
2) the horizontal displacement at the surface and the base (2.11, 2.14)
3) the vertical displacement at the surface and at the base (2.12, 2.15)

as a function of the relative thickness of the layer \( \lambda D \). In the calculations the value of the non-dimensional wave parameter was fixed to 100. Also the results of the analytical solution are given in these figures. It can be concluded that in both cases the results of the Kantorovich method correspond with the analytical solution onto a specific value of the relative thickness. For a smooth interface this region is up to a relative thickness
of 1.0 and in case of a rough interface 1.5.

The results for the ratio of the amplitudes of the pore pressure according to the Rayleigh-Ritz method are given in figure 2.16 (smooth) and 2.17 (rough). As was supposed earlier is the general trend of these solutions good but the Kantorovich method yields better results.
smooth interface  
rough interface

\[ \frac{p_{\text{base}}}{\hat{p}} \]

Fig. 2.10  
Fig. 2.13  
\( \lambda \bar{D} \)

amplitude of pore pressure at the base

\[ \frac{u/\hat{p}}{2G\lambda} \]

Fig. 2.11  
Fig. 2.14  
\( \lambda \bar{D} \)

horizontal displacement

\[ \frac{v/\hat{p}}{2G\lambda} \]

Fig. 2.12  
Fig. 2.15  
\( \lambda \bar{D} \)

vertical displacement

--- analytical solution  
--- appr. "Kantorovich" solution
2.7 Results

Two methods of solution have been derived for the response to water waves of a finite layer overlaying a stiff impermeable base. For relative thin layers, smooth upto $\lambda D \approx 1.0$ and rough $\lambda D \approx 1.5$, the results according to the variational method with the Kantorovich technique correspond to the analytical solution. The solution of the amplitude of the pore pressure, the normal and shear stresses, the mean and deviatoric stresses and the displacements as a function of depth for a relative thickness $\lambda D = 0.5$ are given in figure 2.16-2.20 (smooth), 2.21-2.23 (rough). The value of Poisson’s ratio in these cases is zero. The same solutions only for a relative depth of $2.0$ and Poisson’s ratio $\nu = 1/3$ are given in figure 2.24-2.26 (smooth) and 2.27-2.29 (rough).

Important for possible pore pressure buildup, which will be discussed in chapter 4, is the effect on the deviatoric shear stress of the stiff impermeable base. In the figures 2.19, 2.22, 2.25 and 2.28 also the amplitude of the deviatoric shear stress for an semi-infinite half-plane is given. At the surface of a finite layer is the deviatoric shear stress not equal to zero which is due to the horizontal stresses at that level.

Fig.2.16 Result of Rayleigh-Ritz method for a smooth interface. Fig.2.17 Result of Rayleigh-Ritz method for a rough interface.
smooth interface, $\lambda D = 0.5$, $\nu = 0$

stresses:
- pore pressure $p$
- horizontal effective stress $G_{xx}$
- vertical effective stress $G_{yy}$
- shear stress $G_{xy}$

Fig. 2.18

dev. shear stress $G_d$

Fig. 2.19

 displacements:
- horizontal displacement $u$
- vertical displacement $v$
- volumetric strain $e$

Fig. 2.20

rough interface, $\lambda D = 0.5$, $\nu = 0$

stresses:
- pore pressure $p$
- horizontal effective stress $G_{xx}$
- vertical effective stress $G_{yy}$
- shear stress $G_{xy}$

Fig. 2.21

dev. shear stress $G_d$

Fig. 2.22

 displacements:
- horizontal displacement $u$
- vertical displacement $v$
- volumetric strain $e$

Fig. 2.23
smooth interface, $\lambda D = 1.5$, $\nu = 1/3$

\begin{align*}
\text{stresses:} & \\
\text{pore pressure} & \frac{p G_{xx}}{\beta} \\
\text{hor. eff. stress} & \frac{G_{yy}}{\beta} \\
\text{vert. eff. stress} & \frac{G_{xy}}{\beta} \\
\text{shear stress} & \frac{G_{xy}}{\beta} \\
\text{dev. shear stress} & \frac{G_d}{\beta} \\
\text{mean stress} & \frac{G_d}{\beta} \\
\text{displacements:} & \\
\text{horizontal} & \frac{u}{\sqrt{2G\lambda}} \\
\text{vertical} & \frac{v}{\sqrt{2G\lambda}} \\
\text{vol. strain} & \frac{\varepsilon}{\sqrt{2G\lambda}}
\end{align*}

rough interface, $\lambda D = 1.5$, $\nu = 1/3$

\begin{align*}
\text{stresses:} & \\
\text{pore pressure} & \frac{p G_{xx}}{\beta} \\
\text{hor. eff. stress} & \frac{G_{yy}}{\beta} \\
\text{vert. eff. stress} & \frac{G_{xy}}{\beta} \\
\text{shear stress} & \frac{G_{xy}}{\beta} \\
\text{dev. shear stress} & \frac{G_d}{\beta} \\
\text{mean stress} & \frac{G_d}{\beta} \\
\text{displacements:} & \\
\text{horizontal} & \frac{u}{\sqrt{2G\lambda}} \\
\text{vertical} & \frac{v}{\sqrt{2G\lambda}} \\
\text{vol. strain} & \frac{\varepsilon}{\sqrt{2G\lambda}}
\end{align*}
Pore pressures different from the harmonic solution are the result of consolidation. In a finite layer volume changes are induced by the waves whereas in the case of an infinite half-plane the volume change is zero. As in the consolidation theory volume strain is strongly coupled to pore pressure the solution of the pore pressure is different for the various configurations. In a finite layer with a smooth interface at the base are more degrees of freedom than in case of a rough interface. The result is that in a layer with a smooth interface the displacements and the volumetric strain are larger resulting in higher stresses. Moreover the higher stress level and larger displacements result in a rapid damping of the wave amplitude. However smaller displacements and a lower stress level in a layer with a perfectly rough interface result in a slow damping of the amplitude of the wave. In figure 2.30 the amplitude of the pore pressure is given for both conditions with a relative thickness $\lambda D = 0.5$.

![Pore pressure distribution](image)

Fig. 2.30 Damping of the wave amplitude.

This result seems to be confirmed by laboratory experiments. Clukey et al [2.12], who performed experiments on sand in a wave tank, found a more rapid attenuation in depth of the wave as predicted by the numerical solution from Yamamoto [2.11]. In this numerical solution the boundary conditions at the base are
perfectly rough. Clukey et al consider slip at the bottom as a possible reason for the discrepancy and refer to experiments by Yamamoto [2.13] on clay. In these last experiments a strong attenuation of the wave in a soft clay bed was found.

Another typical aspect of the wave response is the phase shift that occurs. The phase of the different variables changes in depth relative to the phase of the wave at the top of the layer. For instance the phase of the pore pressure is shown in figure 2.31. The phase of the wave pressure changes in depth and is partly "slow" and partly "fast" with respect to the wave at the surface of the layer. The relative thickness in this case is $\lambda D = 0.5$. It appears that the maximum phase shift is limited.

![Diagram](image)

Fig. 2.31 Phase shift of the pore pressure in case of a smooth (left) and rough (right) interface.

In the next chapter the special character of the response in the case of a semi-infinite half-plane will be used to derive an approximate solution for the wave-induced force on a buried pipeline.
3 WAVE-INDUCED PORE PRESSURES AROUND SUBMARINE PIPELINES

3.1 Introduction

The stability of an offshore pipeline that is completely or partly embedded in the seabed is influenced by the dynamic pore pressures and soil stresses resulting from the instantaneous wave interaction with the seabed. These stresses can be expressed in terms of a body force that follows by integration of the stresses along the surface of the pipe.

As was shown in paragraph 2.5.2 it appears that for a homogeneous semi-infinite seabed the pore pressure satisfies the Laplace equation. It is not likely that this will also be the case when a pipeline is present in or on top of the seabed. As a consequence of the boundary conditions at the surface of the pipe both the natural wave-induced deformation and pressure field are disturbed near the pipe.

The impermeable boundary of the pipe prevents fluid flow across the surface of the pipe. Furthermore displacements perpendicular to this boundary are excluded. The last condition relates to the contact between the pipe and the surrounding medium. In the case of a rough surface no displacements along the pipe surface are possible while in the case of a smooth pipe no shear stresses exist along the surface of the pipe. Together with the conditions at the surface of the seabed (2.10) the coupled poro-elastic problem is fully described. In the sense of a boundary value problem it resembles the problem of a finite layer overlaying a stiff impermeable base as discussed in the previous chapter. In that case it could be concluded that when the thickness of the layer is increased the exact solution for the displacements and the pore pressure tends to the harmonic solution according to the Laplace equation.

It is obvious here, that when the diameter of the pipe is reduced to zero the exact solution will be equal to the harmonic solution. In practical cases the pipe diameter has a maximum value of \( t \) 1.00 metre, while wind driven sea waves have a wave length in the order of 100 metres. Hence the ratio of the pipe diameter and the wave length is rather small (\( \% 0.01 \)) and it seems
reasonable to calculate the wave-induced pore pressures from the Laplace equation as a first approximation.

3.2 Method of analysis

Mathematically the problem is to find a function which describes the fluid pressure in the region bounded by the pipe and the surface of the seabed. As the fluid pressure must satisfy the Laplace equation this is a harmonic function.

The problem is solved by adding a so-called "disturbance" function \( \varphi \) to the solution of the "undisturbed" wave-induced pressure \( p_w \). The sum of the two functions \( p_w + \varphi \) is the solution to the total problem:

\[
p_{\text{tot}} = p_w + \varphi
\]

Assuming that the pressure field satisfies the Laplace equation, it follows that \( \varphi \) must be a harmonic function:

\[
\nabla^2 \varphi = 0
\]

The boundary conditions are such that on the seabed the pressure is known and equal to the wave loading, so there \( \varphi \) must be zero. At the pipe the fluid cannot flow across the surface. Therefore at the surface of the pipe the gradient normal to the pipe is a prescribed function. The boundary conditions for \( \varphi \) are:

at the seabed \( \varphi = 0 \)

at the pipe

\[
\frac{\partial p_{\text{tot}}}{\partial n} = 0 \quad \text{hence,} \quad \frac{\partial \varphi}{\partial n} = - \frac{\partial p_w}{\partial n} = g(s)
\]

In the following sections this boundary value problem will be solved for a variety of configurations using complex function theory and the technique of conformal mapping. In section 3.6 the hydrodynamic force resulting from the fluid pressures around the pipe will be calculated.
3.3 Partially buried pipeline

For a pipeline on the surface of the seabed or partially buried in the seabed (see figure 3.1) the physical plane is a simply connected region. The upper boundary of that region includes the seabed and the buried part of the surface of the pipe. This boundary is one with mixed boundary conditions. In this case it is most convenient to use a transformation onto a semi-infinite strip (see figure 3.2) where the mixed type boundary is separated. The boundary along which \( \phi = 0 \) can be mapped on the vertical boundaries and the boundary of the pipeline can be mapped on the part of the \( \xi \)-axis where \(-\pi/2 < \xi < \pi/2\).

For example, in the case of a half-embedded-half exposed pipeline the proposed transformation can be established by the use of two relatively simple functions. First the semicircle can be removed from the \( x,y \)-plane and mapped upon a part of the \( u \)-axis in the \( w \)-plane (see figure 3.3), using the transformation:

\[
 w = z + \frac{R^2}{z} \tag{3.4}
\]

where \( R \) is the radius of the pipe.

From this the lower half-plane in the \( w \)-domain can be mapped upon a semi-infinite strip in the \( \xi \)-domain (see figure 3.4) by the
trigonometric function:

\[ \frac{w}{2R} = \sin \zeta \]  

(3.5)

After this transformation the problem now is to find a function \( \phi \) that is harmonic upon a semi-infinite strip. Along the vertical boundaries \( \phi \) must be zero and along the horizontal boundary the gradient of \( \phi \) normal to this boundary is a prescribed function. This function, apart from the negative sign (see equation 3.3), is the transformed function of the derivative normal to the pipe surface of the "undisturbed" fluid pressure \( p_w \). The maximum force occurs when a wave top (or trough) is above the pipe, therefore time can be fixed to a quarter (or three quarters) of the wave period.

![Fig. 3.3 w-plane with semi-circle removed.](image)

![Fig. 3.4 \( \zeta \)-domain.](image)

For the general solution of the wave interaction with the seabed (2.10) the load was written in terms of a complex exponential function. The function for the physical wave at the seabed is obtained by taking for instance the imaginary part. In that case the undisturbed wave pressure is simplified to:

\[ p_w = \bar{\beta} \exp(\lambda y) \cos(-\lambda x) \]  

(3.6)

Transforming this function to the \( \zeta \)-domain the derivative normal to the pipe surface in case of a half embedded-half exposed
pipeline is:

\[ \frac{\partial p_w}{\partial n} = \lambda R \bar{\rho} \exp(\lambda R \cos \xi) \left[ \cos \xi \cos(\lambda R \sin \xi) - \sin \xi \sin(\lambda R \sin \xi) \right] = -g(\xi) \]  \hspace{1cm} (3.7)

Now the boundary conditions are completely determined and the solution to the problem is found by the method of separation of variables. Together with the boundary condition at the vertical boundaries this results in the general solution for \( \phi \).

\[ \phi = \sum_{m=1,3,5}^{\infty} A_m \cos(m \xi) \exp (mn) \] \hspace{1cm} (3.8)

The coefficients \( A_m \) have to be determined from the boundary condition (equation 3.3) at the horizontal boundary which leads to:

\[ \sum_{m=1,3,5}^{\infty} m A_m \cos(m \xi) = g(\xi) \text{ for } -\frac{\pi}{2} \leq \xi \leq \frac{\pi}{2} \] \hspace{1cm} (3.9)

This is a standard problem of Fourier series analysis, the solution of which is:

\[ A_m = \frac{2}{m \pi} \int_{-\pi/2}^{\pi/2} g(\xi) \cos(m \xi) d\xi \] \hspace{1cm} (3.10)

In the case of a half embedded-half exposed pipeline the function \( g(\xi) \) is expressed by equation 3.7. The Fourier integral (eqn. 3.10) can be evaluated using formulae (337, 13a and 13b) of Gröbner and Hofreiter [3.1]:

\[ A_m = \frac{2}{mn} \int_{-\pi/2}^{\pi/2} -\lambda R \bar{\rho} \exp(\lambda R \cos \xi) \left[ \cos \xi \cos(\lambda R \sin \xi) \right. \left. - \sin \xi \sin(\lambda R \sin \xi) \right] \cos m \xi \ d\xi \\
= - \frac{\bar{\rho}(\lambda R)^m}{m!} \] \hspace{1cm} (3.11)

Substitution of this result in the general expression for \( \phi \) [3.8] gives:

\[ \phi = -\bar{\rho} \sum_{m=1,3,5}^{\infty} \frac{(\lambda R)^m}{m!} \cos(m \xi) \exp (mn) \] \hspace{1cm} (3.12)

The complete solution for the pressures around a half embedded-half exposed pipeline is determined as the sum of the function \( \phi \) and the expression for the "undisturbed" wave-induced pressure field.
Fig. 3.5 Streamlines for the natural wave-induced flow (right) and the flow around a half-embedded-half exposed pipeline (left).

In fig. 3.5 streamline patterns are drawn for the undisturbed flow as well as the flow around a half embedded-half exposed pipeline. The pressure around the pipe is calculated using a three-term approximation for the solution of $\phi$. The pipe is indeed seen to act as an impermeable boundary.

3.4 Flat plate on top of the seabed

Also in the case of a flat plate on top of the seabed the disturbance function $\phi$ can easily be derived. In fact in this case the physical $x,y$-plane is represented by the $u,v$-plane in figure 3.3. This solution can also be considered as the flow around a pipeline where the geometry of the pipe is reduced to a plate. The Fourier integral (3.10) is reduced to a simple expression that can be evaluated using formula 334, 54b of Gröbner and
Hofreiter [3.1].

\[
A_m = \frac{2}{m\pi} \int_{-\pi/2}^{\pi/2} -\frac{b}{2} \cos \xi \cos \left(\frac{b\lambda}{2}\right) \sin \xi \cos m\xi d\xi
\]

\[
= -\frac{\beta b\lambda}{m\pi} \frac{\pi}{2} \left[ J_{m+1} \left(\frac{b\lambda}{2}\right) - J_{m-1} \left(\frac{b\lambda}{2}\right) \right]
\]  

(3.13)

or, with formula (9.1.27) of Abramowitz and Stegun [3.2]:

\[
A_m = -2\beta J_m \left(\frac{b\lambda}{2}\right)
\]

(3.14)

where \( b \) is the plate width.

3.5 Completely buried pipeline

In the case of a completely buried pipeline the region in the physical plane is multiply connected, for instance a half-plane with a circular hole (see figure 3.6). In this case two boundaries exist with uniform boundary conditions. Hence a transformation onto a region bounded by two concentric circles in the \( \xi, \eta \)-plane seems most appropriate. The pipe is mapped on the inner circle \( C_2 \) with radius \( R_2 \) and the seabottom defined as the \( x \)-axis is mapped on the unit circle \( C_1 \) (see figure 3.7).

Fig. 3.6 Completely buried pipeline.

Fig. 3.7 Transformed z-plane in case of a completely buried pipeline
The transformation is according to:

\[ \zeta = \frac{z+is}{z-is} \]  

(3.15)

and the inverse transformation:

\[ z = -is \frac{1+\zeta}{1-\zeta} \]  

(3.16)

where

\[ s = d^2 - R^2 \]  

(3.17)

The radius \( R_2 \) of the image of the pipe is determined by

\[ R_2 = \frac{d-s}{R} \]  

(3.18)

In case of an infinite small piperadius \( R \) the \( x,y \)-plane transforms to a semi-infinite half-plane, while it is mapped upon the interior of the unit circle \( C_1 \) in the \( \xi,\eta \)-plane. Then the solution according to the Laplace equation corresponds with the exact solution. In practical cases pipelines are frequently found with a diameter of 0.25 m that are buried approximately 1.00 metre below the seabed. The ratio of the pipe radius to the wave length is in the order of 0.00125 while the radius \( R_2 \) of the pipe in the \( \xi,\eta \) plane follows as

\[
\begin{align*}
R &= 0.125 \text{ m} \\
R/L &\approx 0.00125 << 1 \\
d &= 1.25 \text{ m} \\
s &= 1.244 \text{ m} \\
R_2 &= 0.05 << 1
\end{align*}
\]

In this case the radius of the pipe in the \( \xi,\eta \)-plane is small compared to the unit circle.

It can be concluded that the pipe is a small disturbance. Moreover relative to the dimensions of the wave-induced deformation field the pipe is small. Therefore the harmonic solution seems a good approximation.

The problem of the wave-induced flow around a completely buried pipeline is treated strictly as a potential flow problem. Because the pressure satisfies the Laplace equation a potential \( \phi \) can be defined:

\[ \phi = k\phi \]  

(3.19)
where $\phi$ is the head and follows from the fluid pressure $\phi = p/\gamma$, $k$ is the permeability of soil. The same potential function describes the groundwater flow through a homogeneous porous but incompressible medium (see e.g. [3.3]).

The equations for the specific discharges are governed by Darcy's law and follow from the potential function.

\[
q_x = -k \frac{\partial p}{\partial x} = -\frac{\partial \phi}{\partial x} \\
q_y = -k \frac{\partial p}{\partial y} = -\frac{\partial \phi}{\partial y}
\]  

(3.20)

As the real part of an analytic function is harmonic, $\phi$ can be defined as the real part of a complex potential function $\Omega(z)$. The imaginary part of $\Omega(z)$, $\Psi$, is known as the streamfunction. This function is defined by the equations:

\[
\frac{\partial \Psi}{\partial x} = q_y \\
\frac{\partial \Psi}{\partial y} = -q_x
\]  

(3.21)

The streamfunction $\Psi$ has the property to be constant along a streamline.

For the undisturbed wave-induced flow the pore pressure is given in paragraph 2.5.2, the potential function $\phi_w$ for that flow is

\[
\phi_w = \phi_o \exp(\lambda y) \exp(i(\omega t - \lambda x))
\]  

(3.22)

where

\[
\phi_o = \frac{k\phi}{\gamma}
\]  

(3.23)

The streamfunction $\Psi_w$ can be derived with the equations (3.21).

\[
\Psi_w = -\phi_o \exp(\lambda y) \exp(i(\omega t - \lambda x))
\]  

(3.24)

Combination of the potential and the stream function yields to the complex potential $\Omega_w$.

\[
\Omega_w(z) = \phi_w + i\Psi_w = -i\phi_o \exp(i\omega t) \exp(-i\lambda z)
\]  

(3.25)
With the inverse transformation this function can be transformed to the $\xi, \eta$-plane.

$$\Omega_w(\xi) = -i\phi_0 \exp(i\omega t) \exp(-i\lambda s \frac{L+e}{L-e})$$  \hspace{1cm} (3.26)

The solution for the total problem of the flow around an embedded pipeline $\Omega_t$ is found by adding a so-called "disturbance" function $\Omega_d$ to the function of the undisturbed flow $\Omega_w$.

$$\Omega_t = \Omega_w + \Omega_d$$  \hspace{1cm} (3.27)

Also the disturbance function $\Omega_d$ is an analytic function as the real and the imaginary part have to be harmonic. Because this function is analytic on a region between two concentric circles in the $\zeta$-plane it can be written as a Laurent series.

$$\Omega_d(\xi) = \exp(i\omega t)\left( \sum_{n=0}^{\infty} c_n \zeta^n + \sum_{n=1}^{\infty} m_n \zeta^{-n} \right)$$  \hspace{1cm} (3.28)

Corresponding to the undisturbed flow time is put in as an independent variable. The coefficients $c_n$ and $m_n$ are complex and must be determined from the boundary conditions.

$$c_n = a_n + ib_n$$
$$m_n = p_n + iq_n$$  \hspace{1cm} (3.29)

For convenience a transformation to polar coordinates $r, \theta$ in the $\xi$-plane is executed. This transforms the general form of the disturbance function to a Fourier series:

$$\Omega(r, \theta) = \exp(i\omega t)\left( \frac{1}{2} c_0 + \sum_{n=1}^{\infty} c_n r^n (\cos\theta + i\sin\theta) + \sum_{n=1}^{\infty} m_n r^{-n} (\cos\theta - i\sin\theta) \right)$$  \hspace{1cm} (3.30)

The constant $c_0$ is an arbitrary constant that is chosen zero here. At the surface of the seabed, represented by the unit circle $r=1$, the potential $\Phi_t$ is equal to the potential of the undisturbed flow $\Phi_w$. For the disturbance function $\Omega_d$ follows that the real part of the function must vanish for $r=1$. The following equations can be derived.

$$(a_n + p_n) \cos\omega t = (b_n + q_n) \sin\omega t$$
$$(q_n - b_n) \cos\omega t = (a_n - p_n) \sin\omega t$$  \hspace{1cm} (3.31)
The surface of the pipe, where \( r=R_2 \), is a streamline, there the streamfunction \( \Psi_t \) is constant and the derivative of \( \Psi_t \) along the boundary zero.

\[
\left. \frac{\partial \Psi_t}{\partial \theta} \right|_{r=R_2} = \left. \frac{\partial \Psi_s}{\partial \theta} \right|_{r=R_2} + \left. \frac{\partial \Psi_d}{\partial \theta} \right|_{r=R_2} = 0 \quad (3.32)
\]

The streamfunction \( \Psi_t \) is the sum of the streamfunctions of the undisturbed flow and the disturbance function. As a result the imaginary part of the disturbance function is a prescribed function which apart from the negative sign is equal to the imaginary part of the undisturbed flow. This condition corresponds to equation (3.3).

\[
\left. \frac{\partial \Psi_d}{\partial \theta} \right|_{r=R} = \left. \frac{\partial \Psi_s}{\partial \theta} \right|_{r=R} = \mathrm{Im} \left\{ - \frac{\partial \Omega_s(\zeta)}{\partial \theta} \right\} \quad (3.33)
\]

With the general form for the disturbance function (3.30) an equation can be derived with the condition at the pipe.

\[
\left. \frac{\partial \Psi_d}{\partial \theta} \right|_{r=R_2} = \sum_{n=1}^{\infty} A_n \sin n \theta + \sum_{n=1}^{\infty} B_n \cos n \theta = \mathrm{Im} \left\{ - \frac{\partial \Omega_s(\zeta)}{\partial \theta} \right\} \quad (3.34)
\]

where \( A_n \) and \( B_n \) are defined as:

\[
A_n = (a_n \sin \omega t + b_n \cos \omega t) (R_2^n + R_2^{-n}) \quad (3.35)
\]

\[
B_n = (a_n \cos \omega t - b_n \sin \omega t) (R_2^n + R_2^{-n})
\]

This is a standard problem of Fourier series analysis, the solution of which is:

\[
A_n = - \frac{1}{n \pi} \int_0^{2\pi} \mathrm{Im} \left\{ - \frac{\partial \Omega_s(\zeta)}{\partial \theta} \right\} \sin n \theta \, d\theta \quad (3.36)
\]

\[
B_n = - \frac{1}{n \pi} \int_0^{2\pi} \mathrm{Im} \left\{ - \frac{\partial \Omega_s(\zeta)}{\partial \theta} \right\} \cos n \theta \, d\theta
\]

These Fourier integrals in the \( \zeta \)-plane are in fact a contour integral along the closed contour of the surface of the pipe. An integral along a closed contour in the complex \( \zeta \)-plane of an analytic function is zero while in case of singularities the residues are calculated with the Residue-theorem.
As the integral of the summation of two functions equals the sum of the integrals of each function, the contour integration may be completed before separating the imaginary part.

\[ A_n = \text{Im} \left\{ -\int_0^{2\pi} \frac{\partial \Omega_w(\zeta)}{\partial \theta} \sin n \theta d\theta \right\} \] (3.37)

With the solution for \( \Omega_w \) the integral can be evaluated and it appears that the integrant contains a pole of the order \( n \). The Fourier coefficients \( A_n \) and \( B_n \) are determined by

\[ A_n = -\frac{2}{n} \lambda s \phi \left( \frac{R_2}{O_2} \right)^n \lim_{\zeta \to 0} \frac{1}{(n-1)!} \frac{d^{(n-1)}(n-1)}{d\zeta^{(n-1)}} \frac{\exp(-\lambda s \frac{1+\zeta}{1-\zeta})}{(1-\zeta)^2} \cos \omega t \]

\[ B_n = -\frac{\sin \omega t}{\cos \omega t} A_n \] (3.38)

With these equations the Laurent coefficient \( c_n \) and \( m_n \) can be derived

\[ c_n = a_n + ib_n = ib_n \]

\[ m_n = p_n + iq_n = ib_n \exp(-2i\omega t) \] (3.39)

where

\[ b_n = -\frac{2\lambda s \phi \left( \frac{R_2}{R_2^2 + R_2^{-2}} \right)^n}{(R_2^n + R_2^{-n})} \lim_{\zeta \to 0} \frac{1}{(n-1)!} \frac{d^{(n-1)}(n-1)}{d\zeta^{(n-1)}} \frac{\exp(-\lambda s \frac{1+\zeta}{1-\zeta})}{(1-\zeta)^2} \] (3.40)

The total solution for the disturbance function can be simplified to:

\[ \Omega_d(\zeta) = \exp(i\omega t) \sum_{n=1}^{\infty} ib_n \zeta^n + \exp(-i\omega t) \sum_{n=1}^{\infty} ib_n \zeta^{-n} \] (3.41)

With the transformation (3.16) the solution can be transformed to the \( z \)-plane.

Basically this leads to the same solution as was found by McPherson in 1978 [3.4], who derived a solution using the bipolar coordinate transformation and the method of separation of variables.
3.6 Hydrodynamic force on a pipeline

The force resulting from the pressure distribution around the pipe can be divided into a vertical and a horizontal component. Each of them is equal to the surface integral of the pressure gradient over the cross-section of the pipe. These integrals can be transformed into a contour integral along the surface of the pipe:

\[
F_x = - \int \int_\gamma \frac{\partial p_{tot}}{\partial x} \, dx \, dy = - \oint p_{tot} \, dy
\]

\[
F_y = - \int \int_\gamma \frac{\partial p_{tot}}{\partial y} \, dx \, dy = + \oint p_{tot} \, dx
\]

These components can be combined to the total force:

\[
F = F_x + iF_y = i \oint p_{tot} \, dz
\]

With the mapping function the total force can be calculated in the \( \xi \)-domain.

The total force follows as the sum of the contour integrals of the transformed undisturbed wave pressure and the disturbance function.

The solution of the undisturbed wave-induced pressure is a harmonic function (see 3.6). A contour integral along a closed contour of such a function is zero. The contour integral of the disturbance function follows from the singularities inside the contour.

3.6.1. Partially buried pipeline

In the case of a half embedded-half exposed pipeline the contour in the \( \xi \)-plane is the part of the \( \xi \)-axis where \( |\xi|<\pi/2 \).

The disturbance function \( \phi \) is a Fourier cosine series (see 3.12). The contour integral in this case is:

\[
F = i \int_{-\pi/2}^{\pi/2} \sum_{m=1,3,5} A_m \cos(m\xi) R(\cos\xi - i\sin\xi) \, d\xi = i R A_1 \pi/2
\]

The results show that in this case the expression for the force
only contains the first term that can be calculated with (3.11):

\[ F = -\frac{i}{2} \pi \lambda R^2 i \]  
(3.46)

or

\[ \frac{F}{F_o} = -\frac{\lambda R}{2} i \text{ where } F_o = \pi R \tilde{\rho} \]  
(3.47)

From the negative sign it follows that the force is directed downwards which corresponds with the supposed loading situation of a wave top above the pipe. Instead of considering only the buried part of the pipe where the wave response is valid, the force can also be related to the buoyancy of the total pipe:

\[ \frac{F}{F_b} = \frac{h \lambda}{2} i \]  
(3.48)

where \( h \) is the wave amplitude at the seabottom. This ratio is strongly dependent on the wave steepness. Introducing a steepness of 0.07 this ratio has a theoretical maximum of 10%.

3.6.2. Flat plate on top of the seabed

Also in the case of a flat plate on the top of the seabed the expression for the force appears to be very simple, using (3.14):

\[ F = \int_{-\pi/2}^{\pi/2} -2\pi J_1 \left( \frac{b \lambda}{2} \right) \cos \xi \frac{b}{2} \cos \xi \, d\xi = -\frac{b \pi \tilde{\rho}}{2} J_1 \left( \frac{b \lambda}{2} \right) i \]  
(3.49)

or

\[ \frac{F}{F_o} = -\frac{i}{2} \pi J_1 \left( \frac{b \lambda}{2} \right) \text{ with } F_o = b \tilde{\rho} \]  
(3.50)

When the plate width is small in relation to the wave length this expression can be simplified to:

\[ \frac{F}{F_o} = -\frac{i}{2} \pi \frac{b \lambda}{2} i = -i \pi b \lambda / 8 \quad \text{for } b \lambda << 1 \]  
(3.51)

3.6.3 Completely buried pipeline

For a completely buried pipeline the total solution is
known in terms of the potential function $\phi$, which is defined as the real part of the complex potential $\Omega_t$.

$$\Omega_t = \phi_t + i\psi_t$$ (3.52)

The force is calculated from a contour integral of the pressure along the surface of the pipe. The relation between the pressure and the potential $\phi$ is

$$p = \frac{\gamma}{k} \phi$$ (3.53)

Because the surface of the pipe is a streamline the stream-function $\Psi_t$ is constant there. As the contour integral of that part is zero, is the contour integral of the potential $\phi_t$ equal to the integral of the complex potential $\Omega_t$. In the $\zeta$-plane the surface of the pipe is represented by the circle with radius $R_2$.

$$F = i \frac{\gamma}{k} \oint_{R=R_2} \phi_t(\zeta) d\zeta = i \frac{\gamma}{k} \oint_{R=R_2} \phi_t(\zeta) d\zeta$$ (3.54)

The total solution $\Omega_t$ is the sum of the undisturbed flow $\Omega_w$ and the disturbance function $\Omega_d$. As the undisturbed flow is described by an analytic function this part of the solution does not contribute to the force. The general solution for the disturbance function is given by expression (3.28) as a series expansion. A contour integral of a summation is equal to the sum of the integrals.

$$F = i \frac{\gamma}{k} \sum_{k=1}^{\infty} \Omega_d(\zeta) d\zeta$$

$$= i \frac{\gamma}{k} \sum_{k=1}^{\infty} \left\{ \exp(i\omega t) \oint_{R=R_2} \phi \phi b_n \zeta^n + \exp(-i\omega t) \oint_{R=R_2} \phi \phi b_n \zeta^{-n} \right\}$$ (3.55)

The first part of expression (3.55) with a positive of $\zeta$ is an analytic functional therefore does not contribute to the force. The second part with negative powers in $\zeta$ holds a pole($\zeta=0$) of the order $n$ inside the contour. The residue of this Cauchy type singularity is calculated with the Residue theorem.

53
The solution for the force can be written:

\[
F = \sum_{n=1}^{\infty} \frac{8\pi \lambda s^2 R_n^2}{R_2 + R_2} \beta_n a_n \exp(-i\omega t)
\]  

(3.56)

where

\[
a_n = \lim_{\zeta \to 0} \frac{1}{(n-1)!} \frac{d^{n-1}}{d\zeta^{n-1}} \exp(-\lambda s \frac{1+\zeta}{1-\zeta}) \frac{1}{(1-\zeta)^2}
\]

(3.57)

The solution can be approximated for instance by the first term. In figure 3.8 relationships are given of the ratio of the wave-induced force and the buoyancy of the pipe versus the water depth and versus the pipe diameter together with the first term approximation of the solution derived by McPherson (1978) [3.4].

![Graph](image)

**Fig. 3.8** Ratio of the cyclic wave-induced force and the buoyancy of the pipe for a completely buried pipeline against waterdepth (left) and pipe diameter (right).

### 3.7 Results

For a number of configurations an analytical solution has been derived for the hydrodynamical force on a pipeline. The ratio of this force and the buoyancy of the pipe will be in the order of 10-30% depending on the maximum wave load and the buried
These solutions are based on the assumption that similar to the response of a semi-infinite half-plane the pore pressure satisfies the Laplace equation. Although the derived solutions are not exact the obtained results seem a reasonable approximation because of the relative dimensions of an offshore pipeline.

In section 5.1 the influence of the wave-induced force on the overall stability of a buried submarine pipeline will be discussed. The next chapter deals with the time-dependent long-term behaviour of the seabed as a result of a cyclic loading effects.
4.1 Introduction

Apart from the instantaneous wave response, resulting into stresses and strains in the seabed (see chapter 2) or forces on pipelines (see chapter 3), the dynamic stress variations may develop a non-linear time dependent behaviour of the seabed. A phenomenon that is commonly recognized is, that in dry sands cyclic shear stress variations generally lead to volumetric compaction. In saturated sands changes of volume are strongly coupled to the pore pressure. There the process of compaction is retarded because the generated pore pressures have to drain first before grain contraction is possible. Depending on the drainage, determined by the permeability, the geometry together with the conditions at the boundaries of the considered soil volume, and the rate of pore pressure generation a residual pore pressure may be generated after a full cycle of shear stress application.

However a rise in pore pressure decreases the intergranular forces expressed as the effective stress. As a result of a series of successive stress cycles the pore pressure generation can eventually reduce the effective stress to zero. In that case the soil structure breaks down and the sand behaves like a dense fluid. This phenomenon is called liquefaction or cyclic mobility in case of limited strain. The last term refers to situations where at the onset of failure the soil is in a dilating condition. In that case the large strains imposed by instability will cause volume expansion and subsequent redistribution of pore fluid. Nevertheless also in such cases a situation of large plastic deformation can occur as a result of pore pressure generation.

The driving force to pore pressure generation i.e. the compaction is strongly non-linear. Shear deformations arise regardless of the direction of the stress, while cyclic stresses result into a negative volume change. This in contrast to monotonic shear loading where the response depends on the initial density or porosity of the sample. For instance in a (monotonic) triaxial test loosely packed samples show grain
contraction while on dense samples dilation is observed. Furthermore the rate of compaction is determined not only by the applied load conditions but also by the stress history. The possibility for further compaction diminishes as the densification is limited. The result is a gradually increasing stiffness of the specimen. This implies that while compaction progresses a hardening effect develops resulting in a gradually decrease of the pore pressure generation. Moreover a specimen that is subjected to renewed cyclic loading will show a lower response to cyclic loading than a virgin sample.

An essential restriction in the linear consolidation theory is that shear stresses do not lead to any volume change. In order to incorporate plastic behaviour into a continuum approach a constitutive relation is developed that assumes a residual volume contraction after each loading cycle. This behaviour is generally found in experiments, see e.g. Silver and Seed [4.1]. For the analysis of pore pressure generation due to soil contraction the so-called uncoupled approach is adopted where pore pressure generation is separated from the instantaneous wave response. This concept has already been used by many and seems mathematically justified.

For the stability of buried pipelines cyclic loading effects are of great importance. If indeed the seabed would liquefy the seabed reaction would be that of a dense fluid with a specific density in the order or 2. As most offshore pipelines have a lower specific density instability can occur. However a pipeline is relatively small compared to the wave-induced displacement field and as a result the influence of the pipe on the stress distribution is small. The scale of the wave-induced displacement field is equal to the wave length \( L = 100 \text{ m} \) while the disturbance of the pipe is of the order of the pipediameter \( D = 1 \text{ m}, D/L = 0.01 \). It is assumed that as a first approximation the effect of an embedded pipeline can be neglected. Moreover it will be shown that for the pore pressure generation due to waves a one-dimensional model can be adopted.
4.2 Constitutive relation

For the calculation of pore pressure generation the uncoupled approach is adopted here. It is supposed that the interrelated processes i.e. the instantaneous wave response and the subsequent compaction can be separated. Although this is a crude simplification still a reasonable estimate of the generated pore pressures can be obtained. Moreover the mathematical solution methods are simple and in this case enable to develop analytical solutions. In stead of describing the volume changes completely, here in the uncoupled approach the effect of compaction is incorporated by assuming a residual volume strain after each loading cycle. The residual volume contraction is the net result of the volume changes during loading and unloading. The mathematical procedures of the pore pressure generation due to waves are treated in paragraph 4.5.1.

This concept has already been used by many see e.g. Koning[4.2] or Smits[4.3]. For the pore pressure generation due to waves a numerical solution was derived in this way by Seed & Rahman[4.4] and Barends & Calle[4.5] investigated the random aspect of the wave loading. Results of different solution procedures were given by Gudehus[4.6].

The plastic volume strain (contraction) produced by a series of N cycles of shear stresses may be written in the following form:

\[ \varepsilon_c = -F(T, \sigma) \cdot f(N) \tag{4.1} \]

The shear stress amplitude is denoted by \( T \) and the isotropic effective stress by \( \sigma \).

The shear stress amplitude can be determined from a Mohr’s stress analysis as the deviatoric shear stress. As a first approximation the contraction can be related to the stress level defined as the quotient of the shear stress amplitude and the isotropic stress. From test results it can be concluded that the stress level is an important parameter. In addition it can be considered that all frictional behaviour of a granular material generally is expressed in terms of \( T/\sigma \).

The function \( f(N) \) can describe the dependency of the compaction
to the number of waves. In this constitutive relation the variation in time and the stress level are separated. Therefore the effect of pore pressure generation on the isotropic effective stress is not included. However due to the pore pressure generation the effective stress will be reduced and as a consequence the stress level increases. Therefore here the stress level refers to the stress state at the beginning of the test.

As a first approximation linearity to the stress level as well as to the number of cycles can be assumed

$$\varepsilon_c = - D \frac{\varepsilon}{\sigma} N$$  \hspace{1cm} (4.2)

where D represents a material parameter, see figure 4.1.

When sand is continuously compacted the possibility for further densification gradually fades away as the density approaches the maximum density. Results from laboratory experiments performed by Youd [4.7] during cyclic strain controlled tests show that specimens tend to a maximum density that does not depend on the applied strain amplitude.

A more realistic constitutive relation is to suppose that the volumetric strain produced by a single load cycle decreases with the total number of cycles. Here a reduction according to a negative exponential function is proposed, see figure 4.1.

$$\varepsilon_c = - D \frac{\varepsilon}{\sigma} N_t \left(1 - \exp\left(-\frac{N}{N_t}\right)\right)$$  \hspace{1cm} (4.3)
The number of cycles $N$ is related to the total number of cycles $N_t$ required to achieve the maximum density or a certain rate. For large values of $N_t$ this model with limited densification (4.3) falls to the linear model (4.2). For large values of $N$ the volumetric strain attains a constant value meaning that there is no further compaction. This is also shown by the volumetric strain per loading cycle that vanishes for large values of $N$ and is obtained by differentiation of formula 4.3 to $N$.

$$\frac{\partial \varepsilon_v}{\partial N} = - D \frac{t}{\sigma} \exp\left(-\frac{N}{N_t}\right)$$

(4.4)

Under saturated conditions the volumetric strain is coupled to the pore pressure. As a result of the imposed compaction pore pressures are generated while drainage develops at the same time. Because pore pressures cannot drain instantaneously the process of densification is retarded here.

4.3 Preshearing

An important factor that determines the response to cyclic loading of sand is the stress history. Due to the imposed compaction the sand matrix gets stiffer and the possibility for further densification gradually diminishes. This implies that during cyclic loading a hardening effect develops as the density gradually tends to the maximum density. As a consequence specimens that are subjected to renewed cyclic loading show a different, mostly lower, response than virgin samples. This hardening effect is generally known as preshearing.

In the model a gradually increasing stiffness is included by introducing a decaying function in time for the volumetric strain, see equation (4.3). The effect during reloading of previous loads can be inserted by supposing a certain number of preshearing load cycles $N_o$. The expression for the volumetric strain then is:

$$\varepsilon_v = - D \frac{t}{\sigma} t N_t (1 - \exp(-\frac{(N+N_o)}{N_t}))$$

(4.5)

As a result of imposed compaction preshearing generally
results in a lower response as illustrated by the lower curve in figure 4.2. Subsequently lower pressures will be generated in undrained tests. Furthermore the hardening is not the consequence of the densification only. The increased stiffness is also the result of the development of a certain grain arrangement or fabric, see [4.8] and [4.9].

Fig. 4.2 Effect of preshearing.

On the other hand if the load condition exceeds a certain limit a presheared sample reacts as in first loading as indicated by the upper line in figure 4.2. In such cases particles were rearranged and the fabric or strain history destroyed. With respect, to the threshold beyond which particles are rearranged Ishihara [4.10] refers to a line of phase transformation.

In the next section the exactness of the proposed relation (4.3) will be evaluated. Results from standard laboratory tests will be compared to predictions based on the model. In section 4.5 the proposed general relation (4.3) is applied to the calculation of pore pressure build-up in the seabed due to waves.
4.4 Comparison with results from experiments

4.4.1 Undrained cyclic loading tests

One of the tests that is commonly used to study cyclic loading effects is the undrained test, e.g. the cyclic simple shear or triaxial test. When specimens are subjected to cyclic loading under undrained conditions the volume of the specimen remains (almost) constant. The cyclic imposed plastic compaction is compensated by an elastic deformation. Because the total stresses do not change this results in an increasing pore pressure. As the volume remains constant and therefore compaction does not occur the tendency of the specimen to contract will remain. Actually a linear rising mean pore pressure is observed in these experiments and even the theory predicts such behaviour.

The volumetric strain consists of the plastic part $\varepsilon_c$ induced by the cyclic load and an elastic part $\varepsilon_e$ in order to insure that the total strain is zero.

\[ \varepsilon_e + \varepsilon_c = 0 \]  \hspace{1cm} (4.6)

The plastic part follows from the proposed constitutive relation whereas in this case compaction according to the linear model is supposed. The elastic part is determined by Hooke's law.

\[ \frac{\Delta \bar{\sigma}}{K} - D \frac{\bar{\varepsilon}}{\bar{\sigma}} \cdot N = 0 \]  \hspace{1cm} (4.7)

The bulk modulus $K$ of the soil depends on the isotropic stress level. Here the elastic bulk modulus is not a constant but a tangent stiffness modulus, which is more or less linear to $\sigma$. The elastic parameters of soil are determined for instance in a compression test.

In the one-dimensional compression or oedometer test strains develop only in vertical direction. Therefore the volume strain $\varepsilon$ equals the vertical strain $\varepsilon_{yy}$.

When elastic soil behaviour is supposed the following equations
for the strains can be derived

\[ \epsilon_{xx} = \frac{1}{E} (\sigma'_{xx} - \nu (\sigma'_{yy} + \sigma'_{zz})) = 0 \]
\[ \epsilon_{yy} = \frac{1}{E} (\sigma'_{yy} - \nu (\sigma'_{xx} + \sigma'_{zz})) = \epsilon \]  (4.8)
\[ \epsilon_{zz} = \frac{1}{E} (\sigma'_{zz} - \nu (\sigma'_{xx} + \sigma'_{zz})) = 0 \]

where the elastic parameters \( E \) and \( \nu \) are Young's modulus and Poisson's ratio.

Combination of the equations for the strains in x- and z-direction (4.7a and 4.7c) yields to

\[ \sigma'_{xx} = \sigma'_{zz} = \frac{\nu}{1-\nu} \sigma'_{yy} \]  (4.9)

The isotropic effective stress \( \sigma \) follows as

\[ \sigma = \frac{1}{3} \left( \frac{1+\nu}{1-\nu} \right) \sigma'_{yy} \]  (4.10)

The vertical effective stress \( \sigma_{yy}' \) is calculated with Hooke's law while now a bulk modulus \( K \) and shear modulus \( G \) are introduced.

\[ \sigma'_{yy} = 2G \epsilon_{yy} + (K - 2/3G) \epsilon = (K + 4/3G) \epsilon \]  (4.11)

With the basic relations between \( K \) and \( G \) the last equation can be written in terms of \( K \) only.

\[ \sigma'_{yy} = \frac{3(1-\nu)}{1+\nu} K \epsilon \]  (4.12)

According to Terzaghi's law of one-dimensional compression the relation between vertical stress and strain is a logarithmic function of the initial vertical effective stress \( \sigma_{yy}' \) and the change in vertical stress \( \Delta \sigma_{yy}' \). For convenience a natural log will be used here

\[ \epsilon_{yy} = \frac{1}{C_{10}} \log \left( \frac{\sigma'_{yy} + \Delta \sigma'_{yy}}{\sigma'_{yy}} \right) = \frac{1}{C_{10}} \ln \left( \frac{\sigma'_{yy} + \Delta \sigma'_{yy}}{\sigma'_{yy}} \right) \]  (4.13)

where \( C_{10} \) is the coefficient of compression.

In case of small stress variations the logarithmic function can be linearized.
According to an elastic analysis this vertical deformation can be calculated from equation (4.11).

\[
\varepsilon = \frac{1}{c_{10} \ln 10} \frac{\Delta \sigma_{yy}'}{\sigma_{yy}'}
\]

(4.14)

It can be concluded that the bulk modulus \( K \) is linear to the initial vertical stress \( \sigma_{yy}' \).

With expression (4.9) a conversion can be made to the isotropic effective stress \( \sigma \) which simply yields to

\[
K = \frac{1}{c_{10} \ln 10} \sigma
\]

(4.16)

Sofar only one-dimensional compression has been considered. According to critical state soil mechanics [4.11] a similar relation has been observed in isotropic compression.

\[
K = \frac{1+e}{\lambda} \sigma
\]

(4.17)

where \( \lambda \) is the slope of the normal consolidation line.

When the expression for the bulk modulus 4.16 is substituted in equation 4.7 it follows

\[
\Delta \sigma = Dc_{10} \ln 10 \overset{?}{=} N
\]

(4.18)

or

\[
\Delta \sigma = D \frac{1+e}{\lambda} \overset{?}{=} N
\]

(4.19)

when expression 4.17 is used.

Except from the cyclic load there are no other changes of the loading conditions therefore a change of the effective stress is compensated by a pore pressure.

\[
\Delta p = \Delta \sigma = Dc_{10} \ln 10 \overset{?}{=} N
\]

(4.20)

The same response has been observed in experiments.

A typical result of a cyclic triaxial test is shown in figure 4.3. This test is taken from an investigation initiated by the Dutch State Supervision of Mines and executed by Delft
In the first cycles a rapid buildup of pore pressure occurs followed by a stage of linear increasing pore pressure. When failure is nearly reached pore pressures are generated progressively until the pore pressure equals the cell pressure.

![Graph](image)

**Fig. 4.3** Typical result of an undrained cyclic triaxial test [4.12].

Bjerrum [4.8] reports similar results of cyclic simple shear tests and proposed the following expression for the excess pore pressure

\[ p = \sigma_1 N \tan \beta \]  

(4.21)

Here \( \sigma_1 \) is the initial effective stress and \( \tan \beta \) the slope of the linear path. As mentioned in paragraph 4.2 in this equation the stress level at the beginning of the test appears. This equation complies well with the theoretical behaviour according to expression 4.20. The constant \( \tan \beta \) simply yields as
\[
\tan \beta = \frac{D C_{10} \ln 10 \ \hat{\tau}}{\sigma_1}
\]

or

\[
\tan \beta = \frac{1 + e^{-D \hat{\tau}}}{\lambda} \frac{\hat{\tau}}{\sigma_1}
\]

(4.23)

In paragraph 4.2 it was supposed that the cyclic induced plastic strain depends linear on the applied stress level \(\tau/\sigma\). In reality there exists a non-linear response. This can be concluded from the results of the cyclic triaxial tests mentioned above [4.12] which were performed at different stress levels. In figure 4.4, the measured \(\tan \beta\) is given as a function of the applied stress level. In this case both \(\tan \beta\) and the stress level are drawn on a logarithmic scale. In addition results from literature (Andersen & Moussa see Bjerrum [4.8] and Smits [4.3]) are drawn in this figure. From these results it can be concluded that an exponential function will be a more realistic description.

In figure 4.4 also predictions according to suitable exponential functions are given.

---

**Fig. 4.4 Influence of the applied stress level on \(\tan \beta\).**

A seabed loaded by travelling waves in water is loaded...
under special conditions that differ from standard laboratory tests like the simple shear and the triaxial test. A stress analysis (see paragraph 4.5.2) shows that at a certain depth stresses rotate continuously while the magnitude of the deviatoric shear stress remains constant. It can be expected that this loading situation yields a more efficient compaction than during other tests. In a simple shear or a triaxial test stresses do not rotate continuously but alternately change in direction.

As a result of a higher rate in compaction pore pressure generation will be stronger. Therefore the tan \( \beta \) measured during tests with circular stress rotation will be higher than a tan \( \beta \) measured in e.g. a cyclic triaxial test.

Ishihara and Towhata [4.13] have shown that also under loading conditions with circular stress rotation pore pressures are generated and that these conditions indeed result into a higher tan \( \beta \). Strain measurements showed that during the process of pore pressure generation strains develop elastic while beyond a certain level plastic behaviour is observed rapidly leading to failure.

At present only a few results of undrained cyclic tests with circular stress rotation are available. Therefore only the results of simple shear and triaxial tests will be used here.

4.4.2 Drained cyclic loading tests

When during cyclic loading tests drainage is allowed there will be no pore pressure generation. Instead the sample is compacted.

In 1972 Youd [4.14] published results of drained strain controlled cyclic simple shear tests. Samples with the same initial void ratio were subjected to different strain amplitudes varying from 0.10% - 8.45%. In figure 4.5 the original test results are shown of the relation of the void ratio \( e \) against the number of cycles \( N \).

It was concluded that eventually all samples arrived at the same minimum void ratio independent of the applied strain amplitude. The final void ratio was lower than the minimum void ratio
determined according to ASTM standards. This results implies that a relative density greater than 100% was obtained and apparently cyclic shear stresses are a more efficient way of compacting. Similar results have already been reported earlier by Youd [4.7]. Also during a field investigation at the North Sea [4.12] in situ density measurements resulted in an in-situ density greater than the maximum density determined in the laboratory.

Fig. 4.5 Void ratio \( e \) as a function of the number of cycles, Youd 1972 [4.14].

In order to compare the experimental data with the predicted behaviour according to the model a conversion of the model is needed from volumetric strain to void ratio. The volumetric strain rate with respect to the number of cycles \( N \) is defined as

\[
\frac{dc}{dN} = \frac{1}{V} \frac{dV}{dN}
\]

(4.24)

The total soil volume \( V \) is the sum of the volume of grains and air when dry sand is considered. In this case pore pressures drain almost instantaneous and therefore the pore fluid can be excluded here. Because the volume of soil grains \( Vg \) is constant, the derivative to time or the number of cycles \( N \) is zero. The
volume of soil grains is determined by the total volume of soil and the void ratio, a derivative with respect to the number of cycles contains both variables.

\[
\frac{dV}{dN} = \frac{d}{dN} \left( \frac{V}{1+e} \right) = \frac{1}{1+e} \frac{dV}{dN} - \frac{1}{(1+e)^2} \frac{de}{dN} \quad V = 0
\]

(4.25)

With this result the derivative of the volumetric strain (4.24) can be written as

\[
\frac{de}{dN} = \frac{1}{1+e} \frac{de}{dN}
\]

(4.26)

When this expression is substituted into the constitutive relation (4.3), a differential equation for the void ratio is found.

\[
\frac{1}{1+e} \frac{de}{dN} = -D \frac{\sigma}{\sigma} \exp\left( -\frac{N}{N_t} \right)
\]

(4.27)

With the void ratio \(e_0\) at the beginning of the test this equation has the following solution

\[
\ln \left( \frac{1+e}{1+e_0} \right) = N_t D \frac{\sigma}{\sigma} \left( 1 - \exp\left( -\frac{N}{N_t} \right) \right)
\]

(4.28)

or

\[
\ln \left( \frac{1+e}{1+e_0} \right) = A \left( 1 - \exp\left( -\frac{N}{N_t} \right) \right)
\]

(4.29)

where the parameter \(A\) describes the loading conditions. In order to correspond to the test results \(A\) should be a constant.

The test results are compared to the model by choosing a fixed final void ratio \(e_f\), here \(e_f = 0.420\) is chosen, and determine the total number of cycles \(N_t\) for each strain amplitude (see dotted lined in figure 4.5). In figure 4.6 the original test data have been transformed according to the proposed model (4.28). With the total number of cycles \(N_t\) and the initial void ratio \(e_0\) intermediate points on the measured curves have been transposed. Also predictions for different values of \(A\) are given.
A better agreement between theory and experiments is obtained when the first cycles are neglected. Corresponding to the results of undrained tests where during the first cycles a rapid buildup of pore pressure was measured, here a strong compaction occurs in the beginning. In figure 4.7 the test results are shown transposed in the same way as in figure 4.6 only here the compaction during the first 10 cycles is omitted. This number of ten cycles is chosen arbitrary.

Fig. 4.6 Experimental data from Youd and theoretical behaviour

Fig. 4.7 Experimental data from Youd after omitting the first 10 cycles and theoretical behaviour.
Lines for each strain amplitude start at different points because each test starts at a different initial void ratio. In this case a reasonable agreement between the model and the experimental data is obtained.

The effect of preshearing on undrained tests is generally illustrated by performing successive tests on one sample. First a virgin sample is loaded until a pore pressure generation has developed of about 50 percent of the initial effective stress i.e. 50 percent of the cell pressure in case of a cyclic triaxial test. During the next step the valve to the sample is opened and drainage of the residual pore pressure is allowed. When after settling of the sample the valve is closed again and the sample is subjected to renewed cyclic loading a lower tan $\beta$ is measured than during the first test.

Results of triaxial tests with several intermittent stages of drainage have been published by Smits et al [4.15]. During these experiments the mean cyclic stress was not equal to zero in order to investigate the effect of an external load e.g. a construction. In figure 4.8 the original experimental data are presented of the measured tan $\beta$ versus the change after settling of the porosity $\Delta n$.

![Fig. 4.8 Reduction of tan $\beta$ with decreasing porosity by cyclic loading, from Smits [4.15].](image)
From these experiments an exponential relation between the actual \( \tan \beta \) and the compaction \( \Delta n \) can be derived. The initial \( \tan \beta \) measured during the first test when the sample had a porosity of \( n_0 \) is denoted by \( \tan \beta_0 \)

\[
\tan \beta = \tan \beta_0 \exp(-c\Delta n)
\]  

(4.30)

where \( \Delta n \) is the absolute value of the change in porosity.

The constitutive relation for renewed cyclic loading after a number of preshearing can be derived from the constitutive relation (4.1)

\[
\varepsilon_c(N>/N_0) = \left( \frac{\partial \varepsilon_c}{\partial N} \right)_{N=N_0} \ast f(N)
\]  

(4.31)

With the constitutive relation (4.3) the relation for renewed cyclic loading follows as

\[
\varepsilon_c(N>/N_0) = -D \frac{\varepsilon}{\sigma} \exp\left(-\frac{N}{N_t}\right) \left(1 - \exp\left(-\frac{N}{N_t}\right)\right)
\]  

(4.32)

The model based on linear contraction which covers the behaviour during an undrained test is the tangent to the function describing fully drained behaviour. With the derived expression for \( \tan \beta \) the theoretical relation between the actual \( \tan \beta \) after preshearing and the initial \( \tan \beta \) follows as

\[
\tan \beta = \tan \beta_0 \exp\left(-\frac{N}{N_t}\right)
\]  

(4.33)

During the tests with intermittent stages of drainage the induced compaction was rather small. The change in porosity was in the order of 0.1%. A theoretical solution has been derived for the void ratio \( e \): expression (4.28). A similar solution can easily be derived for the porosity \( n \). This solution is:

\[
\ln \left( \frac{1-n}{1-n_0} \right) = D \frac{\varepsilon}{\sigma} N_t \left(1 - \exp\left(-\frac{N}{N_t}\right)\right)
\]  

(4.34)

where \( n_0 \) is the initial porosity.

In case of small changes \( \Delta n \) of the porosity the logarithmic function can be linearized. Furthermore a small number of cycles is assumed leading to densification linear to the number of
waves.
\[
\frac{\Delta n}{(1-n_0)} = - D \frac{\dot{\epsilon}}{\sigma} \frac{\Delta n}{N}
\]  
(4.35)

Here the change in porosity is negative which confirms to densification. The calculated number of preshearing cycles \( N_0 \) depends linear on the imposed change in porosity.

\[
N_0 = \frac{\Delta n}{(1-n_0)D \frac{\dot{\epsilon}}{\sigma}}
\]  
(4.36)

In order to comply with the experiments the absolute value of the change in porosity is considered here.

When this result is inserted in the theoretical relation for the actual \( \tan \beta \) a relation similar to the experimental result is found

\[
\tan \beta = \tan \beta_0 \left( - \frac{1}{(1-n_0)D \frac{\dot{\epsilon}}{\sigma} N_t} \right) \Delta n
\]  
(4.37)

According to the experimental data the coefficient \( c \) is constant for the applied stress levels. This result confirms to the experiments of Youd [4.14] as in the theoretical expression for \( c \) the parameter \( A \) appears. As mentioned before this parameter describes the loading conditions and should be constant in the theoretical model in order to correspond to the experiments by Youd [4.14].

\[
c = \frac{1}{(1-n_0)D \frac{\dot{\epsilon}}{\sigma} N_t} = \frac{1}{(1-n_0)A}
\]  
(4.38)

Besides the parameter \( A \) the expression for \( c \) contains the initial porosity \( n_0 \) which is a constant.

The experimental data from Smits et al [4.15] show that the cyclic induced densification can be small, here not more than one percent change in porosity. However the cyclic shear loading results in a considerable reduction of \( \tan \beta \). Furthermore it can be considered that the \( \tan \beta \) of virgin samples that differ only one percent in initial porosity does not vary as much as measured in the cited tests. As mentioned before is the increased stiffness not only due to densification but also to the development of a certain grain arrangement or fabric.
4.5 Calculation of pore pressure buildup due to sea waves

4.5.1 Mathematical model

In the uncoupled method the interrelated processes of the instantaneous wave response and subsequent compaction are separated based on the assumption that the mutual influence of both phenomena is small. The two phenomena run at different characteristic time scales. For the instantaneous wave response the characteristic time interval is the wave period which is in the order of 10 sec while the pore pressure generation develops over a period of several hours i.e. a few thousand load cycles. As a result of one load cycle a residual pore pressure is generated which is small compared to the amplitude of the instantaneous pore pressures induced by the waves. Therefore the phenomenon of pore pressure generation can be incorporated as a linearly rise of the mean pore pressure, see figure 4.9.

![Graph showing relative time scales.](image)

Fig. 4.9 Relative time scales.

However a residual pore pressure reduces the effective stress. After a sufficient long series of waves the residual pore pressure predominates and instability occurs when the effective stress is reduced to zero.

The possibility of liquefaction for a certain location depends on the conditions of the seabed, i.e. the permeability, the rate of pore pressure generation, possible preshearing, and the loading conditions determined by the wave height and the total number of
waves.

The effect on the instantaneous wave response of the increased stiffness, as a result of the compaction, resulting in higher strength parameter of the seabed, is neglected here. This is a consequence of the uncoupled approach. In a semi-coupled method the load is applied in groups of cycles and during each interval the response is calculated with updated strength parameters.

Additionally the phenomenon of pore pressure generation is simplified to a one-dimensional model. The variation in pore pressure generation is mainly determined by the total number of load cycles.

![Diagram](image)

Fig. 4.10 Variation of pore pressure buildup in horizontal direction.

In a certain point B, see figure 4.10 the pore pressure generation is "slow" with respect to a point A at the same level but closer to the origin. The characteristic length in this case is the wave length. Considering the stability of a pipeline section the scale of the area interest is of the same order. Therefore it can be assumed that the variation in horizontal direction of the pore pressure due to compaction is small compared to the variation in vertical direction. As a result drainage occurs only in vertical direction.

For realistic pipe dimensions, where the pipediameter is small compared to the wave length, the effect of the pipe on the wave-induced stress field is small. It is supposed that a
stronger compaction near the pipe, due to a higher stress level close to the pipe, can be neglected. In order to judge the liquefaction potential for a certain location the stability of the seabed alone is investigated here.

4.5.2 Differential equation for pore pressure buildup

In order to analyse the pore pressure generation due to wind driven sea waves the actual function for the stress level has to be inserted into the constitutive relation (4.3):

\[ \varepsilon_c = - D \frac{p}{\sigma} N_t \left(1 - \exp\left(-\frac{N}{N_t}\right)\right) \]  

(4.39)

Here a semi-infinite isotropic homogeneous half-plane is considered that occupies the part of the \( x,y \)-plane where \( y > 0 \).

The wave-induced stresses needed to calculate the deviatoric shear stress are derived in paragraph 2.5.2. For a physical wave the solutions for the normal stresses and the shear stress are given by:

\[ \sigma'_{xx} = - \sigma'_{yy} = \bar{\sigma} \lambda y \exp(-\lambda y) \cos(\omega t - \lambda x) \]
\[ \sigma'_{xy} = \bar{\sigma} \lambda y \exp(-\lambda y) \sin(\omega t - \lambda x) \]

(4.40)

The stresses depend mainly on the amplitude of the wave pressure at the seabed \( p \) and the wavenumber \( \lambda \).

The deviatoric shear stress \( \tau \) is calculated according to a Mohr's stress analysis.

\[ \tau = \sqrt{\left(\frac{\sigma'_{yy} - \sigma'_{xx}}{2}\right)^2 + (\sigma'_{xy})^2} = \bar{\sigma} \lambda y \exp(-\lambda y) \]

(4.41)

The principle stress direction follows from

\[ \tan(2\beta) = \frac{2 \sigma'_{xy}}{\sigma'_{yy} - \sigma'_{xx}} = \tan(\omega t - \lambda x) \]

(4.42)

Hence the principle stress angles are given by

\[ \beta = \frac{\omega t - \lambda x}{2} \pm \frac{\pi}{2} \]

(4.43)

Along a horizontal line stresses rotate 180 degrees with an interval equal to the wave length while in a certain point stresses rotate continuously in time.
The isotropic effective stress \( \sigma \) increases linear in depth while a coefficient of lateral earth pressure \( K_o \) determines the relation the vertical and horizontal effective stress. As the sum of the vertical and horizontal effective stress induced by the waves is zero the initial effective stress is not affected by the instantaneous wave response. The number of waves follows with the wave period \( T \) or frequency \( \omega \). The expression for the residual volume contraction can be written as

\[
\varepsilon_c = -D \frac{\beta \lambda y \exp(-\lambda y)}{\frac{1}{3}(1+2K_o)(\gamma_s - \gamma)y} N_t \left(1 - \exp\left(-\omega(t + t_o) / 2\pi N_t \right)\right) (4.44)
\]

where \( t_o \) determines the number of preshearing waves. The densities of soil and water are denoted by \( \gamma_s \) and \( \gamma \) and \( K_o \) is the coefficient of lateral earth pressure.

In order to obtain the total volumetric strain, the cyclic induced plastic strain \( \varepsilon_c \) has to be added to the elastic part \( \varepsilon_e \).

\[
\varepsilon = \varepsilon_e + \varepsilon_c \quad (4.45)
\]

The contraction is given by expression (4.44). According to the principle of effective stress the elastic strains follow from the difference of the surface load \( \sigma_w \) and the pore pressure. The compressibility of the solid matrix is denoted by \( \alpha \).

\[
\varepsilon = \alpha(\sigma_w - p) - D \frac{\beta \lambda \exp(-\lambda y)}{\frac{1}{3}(1+2K_o)(\gamma_s - \gamma)y} N_t \left(1 - \exp\left(-\omega(t + t_o) / 2\pi N_t \right)\right) (4.46)
\]

The pore pressure \( p \) is the sum of the instantaneous wave-induced pressure and the pore pressure generated by the compaction. In case of a semi-infinite half-plane the volumetric strain due to the instantaneous wave response is constantly zero, see paragraph 2.5.2, expression (2.20). Therefore the surface load and the resulting pore pressure can be excluded from the equations here, leaving the pore pressure due to compaction only.

The additional equation is equation (2.4) which describes continuity of the pore fluid. With this equation the effect of drainage is incorporated. The pore fluid is assumed to be incompressible. As has been explained in the previous paragraph 4.5.1 the consolidation due to cyclic compaction can be simpli-
fied to a one-dimensional model supposing that the variation in horizontal direction of the residual pore pressure is small. The continuity or storage equation simply reduces to:

\[
\frac{\partial e}{\partial t} = \frac{k}{\gamma} \frac{\partial^2 p}{\partial y^2}
\]  

(4.47)

Taking the derivative with respect to time of the volumetric strain equation (4.46), the result can be substituted into equation (4.47).

\[
\frac{\partial p}{\partial t} = c_v \frac{\partial^2 p}{\partial y^2} + B \exp\left(-\left(\lambda y + \omega t/2\pi N_t\right)\right)
\]  

(4.48)

The first part of the governing differential equation (4.48) for the residual pore pressure is the well-known diffusion equation describing one-dimensional consolidation or heat diffusion. The last term represents the cyclic effect and has the character of a source. As this term produces pore pressure, as if pore fluid is pumped into the seabed, it is also referred as the "pumping term", see [4.6].

The boundary conditions for the residual pore pressure are that at the seabed the pore pressure is zero and at great depth the fluid velocity must vanish. Furthermore at the beginning of the wave loading the residual pore pressure is zero everywhere.

The consolidation coefficient \(c_v\), determined by the permeability \(k\) and the compressibility \(\alpha\), is defined as

\[
c_v = \frac{k/\gamma}{\alpha}
\]  

(4.49)

and the coefficient \(B\)

\[
B = \frac{D}{\alpha} \frac{3}{1+2 K_o} \frac{\beta \lambda}{(\gamma_s - \gamma)} \frac{\omega}{2\pi} \exp\left(-\omega t_o/2\pi N_t\right)
\]  

(4.50)

The magnitude of the coefficient \(B\) depends largely on parameters known from soil tests and sea wave conditions. The cyclic parameter \(D\) can be derived from the tan \(\beta\) measured in undrained tests.

The characteristic time interval here is the wave period \(T\). The pressure variation \(\Delta p\) during that interval is determined by

\[
\frac{\Delta p}{T} = \frac{D}{\alpha} \frac{3}{1+2 K_o} \frac{\beta \lambda}{(\gamma_s - \gamma)} \frac{1}{T^2} \exp\left(-\omega t_o/2\pi N_t\right)
\]  

(4.51)
Related to the amplitude of the wave pressure $\hat{p}$ a ratio can be distinguished. When the number of preshearing cycles is zero this ratio has a maximum value.

$$\frac{\Delta p}{\hat{p}} = \frac{D}{\alpha} \frac{3}{1+2K_o} \frac{1}{Y_s - \gamma} \frac{\lambda}{t^2} \quad (4.52)$$

This ratio has a magnitude in the order of $10^{-b}$ which shows that during a wave period the pressure variations due to compaction are indeed small compared to the pressure variations due to the waves. Therefore the uncoupled approach is mathematically justified.

If instead of the model with limited densification linear densification is considered the differential equation is rather simplified. With the linear constitutive relation (4.2) the expression for the total volumetric strain (4.46) changes to

$$\varepsilon = a(\sigma_w - p) - D \frac{\beta \lambda}{3(1+2K_o)} \frac{\exp(-\lambda y)}{(Y_s - \gamma)} \frac{\omega t}{2\pi} \quad (4.53)$$

When the derivative with respect to time of the volumetric strain according to equation (4.53) is inserted in the storage equation (4.47) the result is the differential equation describing pore pressure generation in case of linear densification.

$$\frac{\partial p}{\partial t} = c_v \frac{\partial^2 p}{\partial y^2} + B_1 \exp(-\lambda y) \quad (4.54)$$

As the pumping term $B_1$ is constant in time no preshearing effect is included or it should be done by defining a lower value for $D$ resulting from a lower tan $\beta$.

$$B_1 = \frac{D}{\alpha} \frac{3}{1+2K_o} \frac{\beta \lambda}{(Y_s - \gamma)} \frac{\omega}{2\pi} \quad (4.55)$$

For the solution the same boundary conditions have to be satisfied as for the model with limited densification. When the number of preshearing waves is zero the coefficient $B$ of the model with limited densification is equal to $B_1$. 

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4.5.3 Analytical solutions

The mathematical problem is to find a function $p(y, t)$ which is the solution of the differential equation (4.48).

$$p(y, t) : \frac{\partial p}{\partial t} = c_v \frac{\partial^2 p}{\partial y^2} + B \exp\left(-\frac{\lambda y + \omega t}{2\pi N_t}\right)$$ (4.56)

Together with the boundary conditions the problem is completely defined in the sense of a boundary value problem

$$y = 0 \quad p(0, t) = 0$$
$$y \to \infty \quad \frac{\partial p(y, t)}{\partial y} = 0$$
$$t = 0 \quad p(y, 0) = 0$$ (4.57)

Here an analytical solution is found with methods from operational mathematics using an integral transformation method. The general theory and applications of these methods can be found in e.g. Churchill [4.16].

The transformation introduced here is the Laplace transformation defined as:

$$\mathcal{L}(p) = L(p) = \int_0^\infty e^{-st} p(y, t) \, dt$$ (4.58)

where $L$ is the Laplace operator.

With the Laplace transformation the partial differential equation (4.56) in $y$ and $t$ is transformed to an ordinary differential equation in $y$ and $s$.

$$sp - p(y, o) = c_v \frac{\partial^2 p}{\partial y^2} + \frac{B \exp(-\lambda y)}{s + \omega / 2\pi N_t}$$ (4.59)

The boundary conditions do not change under the transformation. Furthermore the initial value $p(y, o)$ of the pressure distribution reduces the equation (4.59) to an ordinary inhomogeneous differential equation. This equation can easily be solved.

$$sp - c_v \frac{\partial^2 \tilde{p}}{\partial y^2} = \frac{B \exp(-\lambda y)}{s + \omega / 2\pi N_t}$$ (4.60)

$$y = 0 \quad \tilde{p}(0, s) = 0$$
$$y \to \infty \quad \frac{\partial \tilde{p}(y, s)}{\partial y} = 0$$
The solution of the homogeneous part of the equation is found by supposing a solution according to an exponential function. With the characteristic equation the eigenvalues or roots are found. With Darcy's law the fluid velocity depends linear on the spatial derivative of the fluid pressure. As the fluid velocity must vanish at great depth only half of the homogeneous solution $p_h$ is valid here.

$$p_h = C_1 \exp(-\gamma \sqrt{\frac{s}{C_V}}) \quad (4.61)$$

The particular solution $p_p$ of the inhomogeneous equation is found by supposing a solution according to

$$p_p = C_2 \exp(-\lambda y) \quad (4.62)$$

The coefficient $C_2$ follows by substitution of the particular solution into the differential equation.

$$C_2 = \frac{B}{(s-c_V \lambda^2)(s+\omega/2\pi N_t)} \quad (4.63)$$

With the boundary conditions at the surface of the seabed the remaining coefficient $C_1$ is found and the solution is completely determined.

$$p = \frac{B}{(s-c_V \lambda^2)(s+\omega/2\pi N_t)} \left\{ \exp(-\lambda y) - \exp(-\gamma \sqrt{\frac{s}{C_V}}) \right\} \quad (4.64)$$

The inverse transformation of this solution is the required solution for $p(y,t)$

$$p = L^{-1}(\tilde{p}) \quad (4.65)$$

The inverse function is found with standard methods from Laplace transformation analysis. For a large number of functions the transformation formulae are listed in tables, see e.g. [4.16]. In case of the product of transforms the inverse function can be found with the convolution theorem.

The inverse transformation of the first part of the
solution is a summation of exponential functions.

\[
L^{-1} \left\{ \frac{B \exp(-\lambda y)}{(s-c_v \lambda^2)(s+\omega/2\pi N_t)} \right\} = \frac{B \exp(-\lambda y)}{c_v \lambda^2 + \omega/2\pi N_t} (s-c_v \lambda^2)(s+\omega/2\pi N_t) \exp(-\omega t/2\pi N_t) \tag{4.66}
\]

The second part of the solution is the product of two functions. The inverse transformation of both functions are standard solutions. One of them is given by expression (4.66) and the other is given as:

\[
L^{-1} \left\{ \exp(-y \sqrt{\frac{s}{c_v}}) \right\} = \frac{y}{2 \sqrt{\pi c_v t^3}} \exp\left(-\frac{y^2}{4c_v t}\right) \tag{4.67}
\]

According to the convolution theorem, the inverse transformation of the product of two functions is equal to the so-called convolution integral

\[
L^{-1} \{ f(t) g(t) \} = \int_0^t f(t) g(t-\tau) d\tau \tag{4.68}
\]

In this case, one of the inverse function is the earlier used summation of exponential functions. Therefore, this part of the solution contains two convolution integrals.

\[
L^{-1} \left\{ \frac{B}{(s-c_v \lambda^2)(s+\omega/2\pi N_t)} \right\} = \frac{B}{c_v \lambda^2 + \omega/2\pi N_t} \int_0^t \exp(c_v \lambda^2(t-\tau)) \frac{y}{2 \sqrt{\pi c_v t^3}} \exp\left(-\frac{y^2}{4c_v t}\right) d\tau \\
+ \int_0^t \exp(-\omega(t-\tau)/2\pi N_t) \frac{y}{2 \sqrt{\pi c_v t^3}} \exp\left(-\frac{y^2}{4c_v t}\right) d\tau \tag{4.69}
\]

These integrals can be evaluated with formulae 7.4.33 resp. 7.4.34 from Abramowitz & Stegun [3.2].

After summation of the inverse transformations of both parts of
the solution the function for \( p(y, t) \) is completely determined.

\[
p(y, t) = B \frac{c_v \lambda^2 + \omega}{2\pi N_t} \left\{ - \exp\left(-\left(t_2 + y_1\right)\right) \right. \\
+ \frac{1}{2} \exp(t_1 - y_1) \text{erfc} \left( \sqrt{t_1} - \frac{y_1}{2\sqrt{t_1}} \right) \\
- \frac{1}{2} \exp(t_1 + y_1) \text{erfc} \left( \sqrt{t_1} + \frac{y_1}{2\sqrt{t_1}} \right) \\
+ \frac{1}{2} \exp(-y_1^2) \left( w(\sqrt{t_2} + i \frac{y_1}{2\sqrt{t_1}}) + w(-\sqrt{t_2} + i \frac{y_1}{2\sqrt{t_1}}) \right) \right\}
\]

For convenience the following relative variables have been introduced:

\[
\begin{align*}
Y_1 &= \lambda y \\
t_1 &= c_v \lambda^2 t \\
t_2 &= t/2\pi N_t
\end{align*}
\]

In these expressions transcendental functions like the error and the \( w \)-function appear, defined as:

\[
\begin{align*}
erf(z) &= 1 - \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\zeta^2) d\zeta \\
w(z) &= \exp(-z^2) \left\{ 1 + \frac{2i}{\sqrt{\pi}} \int_0^z \exp(\zeta^2) d\zeta \right\}
\end{align*}
\]

where \( i \) is the imaginary unit.

Because of the symmetry relations of the \( w \)-function given by formulae 7.1.12 from Abramowitz & Stegun [3.2] the last part of expression (4.70) is real.

When the model with linear volumetric contraction is considered the differential equation describing pore pressure buildup reduces to a simpler equation given by (4.54).

\[
\frac{3p}{3t} = c_v \frac{3^2p}{3y^2} + B_1 \exp(-\lambda y)
\]

The boundary value problem is treated in the same way as for the model with limited densification.

The complete inverse solution in this case contains the same but
simplified terms.

\[
p = \frac{B_1}{(s-c_v \lambda^2) \cdot s} \left\{ \exp(-\lambda y) - \exp(-\sqrt{\frac{s}{c_v}}) \right\}
\]  

(4.74)

Also the inverse transformation proceeds in the same way. The second part of the solution again transforms into a convolution integral.

\[
L^{-1} \left\{ \frac{B}{(s-c_v \lambda^2) \cdot s} \right\} = B \int_0^t \exp(c_v \lambda^2 (t-\tau)) (1-\text{erf} \left( \frac{\tau}{2\sqrt{c_v \tau}} \right)) d\tau
\]  

(4.75)

In this case the integral can be evaluated with formula 7.4.37 from Abramowitz & Stegun [3.2]. The complete solution \( P_1(y, t) \) for the model with linear densification follows, as

\[
p_1(y, t) = \frac{B_1}{c_v \lambda^2} \left\{ 1 - \text{erf} \left( \frac{y_1}{2\sqrt{t_1}} \right) - \exp(-y_1) \right. \\
+ \frac{1}{2} \exp(t_1 - y_1) \text{erfc}(\sqrt{t_1} - \frac{y_1}{2\sqrt{t_1}}) \\
\left. - \frac{1}{2} \exp(t_1 + y_1) \text{erfc}(\sqrt{t_1} + \frac{y_1}{2\sqrt{t_1}}) \right\}
\]  

(4.76)

where the relative variables defined by (4.71) are used

4.5.4 Behaviour at infinite time \((t \to \infty)\)

For large values of \( t \) the transient solution for the linear model as given by expression (4.76) changes into a steady state solution.

\[
p_1(y, \infty) = \frac{B_1}{c_v \lambda^2} \left\{ 1 - \exp(-\lambda y) \right\}
\]  

(4.77)

This steady state solution itself can be derived rather easily from the original differential equation assuming the existence of such behaviour and as a consequence supposing that the derivative to time of the pore pressure tends to zero for large values of \( t \).
This assumption is reasonable from the character of the consolidation equation and the behaviour of the source function describing the cyclic effect. When a steady state solution is supposed the differential equation (4.54) transforms into:

\[
\frac{\partial^2 p}{\partial y^2} + B \exp(-\lambda y) = 0
\]

The steady state solution is simply found by integration.

In the model with limited densification the driving force to pore pressure generation gradually disappears for the contraction per load cycle vanishes as the number of cycles increases. As a result at infinite time all generated pore pressure will be dissipated. This can be shown by the behaviour of the transformed solution \( \tilde{p}(y, s) \). A limit of the function \( p(y, t) \) for infinite large values of \( t \) is equivalent to a limit of the transformed solution \( \tilde{p}(y, s) \) multiplied by \( s \) for infinite small values of \( s \). When this limit is evaluated it appears that the transformed solution, and consequently the function \( p(y, t) \), is zero.

\[
\lim_{t \to \infty} p(y, t) = \lim_{s \to 0} s \tilde{p}(y, s) = \lim_{s \to 0} \frac{B}{s - c_{v} \lambda^2 / (s + \omega / 2\pi N_t)} (\exp(-\lambda y) - \exp(-y \frac{s}{c_{v}})). s = 0
\]

4.6 Results

Numerical values of the error function and the \( w \)-function can be obtained with a polynomial approximation, see [3.2]. For large values of the argument up to infinity an asymptotic series can be used.

Using realistic values for the soil properties, the pressure distribution after eight hours is given in figure 4.11.
The pressure distribution according to the model with limited densification is given for different values of the parameter $N_t$. For large values of $N_t$, the two models give comparable results. In these cases preshearing is excluded ($t_0 = 0$).

As was shown in the last paragraph, the pore pressure reaches a steady state according to the linear model. In figure 4.12 the pore pressure generation for both models at certain depth is given as a function of time.
The maximum pore pressure generation as a function of depth is important for the evaluation of the liquefaction potential. For the linear model this is given by the steady state solution. The maximum for the model with limited densification can be determined numerically. At a certain depth the maximum is reached when the derivative to time is zero. Here the maximum was determined using the Newton-Raphson interpolation technique. In figure 4.13 the maximum pore pressure generation according to both models is given a low value for $N_t$ being assumed for the second model.

![Graph showing maximum pore pressure buildup](image-url)
It can be concluded that according to the model with limited densification only a limited zone of the seabed is affected. At great depth no pore pressures are generated. This result seems reasonable and can easily be understood. At great depth pore pressures would only be generated after many waves while at large values of $t$ the densification process i.e. the driving force to pore pressure generation is already exhausted.

A general conclusion is that, based on the proposed model describing cyclic effects, analytical solutions can be derived for the pore pressure build-up in a semi-infinite seabed. The constitutive relation itself is in good agreement with the behaviour of sand as measured in laboratory tests.

In section 5.2 a liquefaction criterion will be derived based on the linear model. Practical values for the parameters will be used to show the results for North Sea conditions. Furthermore the influence of the stress history and a finite thickness of the seabed will be discussed.
5 APPLICATION OF THE DERIVED THEORY

5.1 Stability of a submarine pipeline

The stability of a submarine pipeline that is completely buried or partially embedded in the seabed is influenced in different ways.

The wind driven waves on the sea surface generate instantaneous pore pressures and effective stresses around the pipe. As has been discussed in chapter 3 the hydrodynamical force resulting from the dynamic pore pressures can be calculated from the Laplace equation. For a number of configurations an analytical solution has been derived for this force. Taking into account the dimensions of the pipe relative to those of the wave-induced pressure field these solutions seem a reasonable approximation of the total force on a pipeline. The ratio of the hydrodynamic force and the buoyancy of the pipe is in the order of 10-30\% depending on the maximum wave load and the buried depth.

The weight of offshore pipelines, expressed in terms of a density, varies in order of magnitude from 1,25 to 4 of the density of water. Small diameter pipelines (D\approx 0,1m) are used for utilities like glycol. These pipelines are relatively heavy (density 3-4). In such cases the wave forces are unimportant. Pipelines that are used for the transportation of oil and gas are much lighter because of the dimensions. The density of these pipe is in the order of 1,25.

It can be concluded that the overall stability of a completely buried pipeline is not endangered. When the hydrodynamic force is directed upward only small stresses are imposed on the overburdening soil. Moreover when the wave-induced force is compared to the static break out force the dynamic force is small.

A lower bound of the static breakout force can be obtained by considering straight vertical slipslips, see figure 5.1. The pipeline is buried to a depth \(d\) and the diameter of the pipe is denoted by \(D\).
Fig. 5.1 Forces on a buried pipeline.

The break out force $F_b$ per unit length of the pipeline is determined by the weight $W$ of the overburdening soil and the shear force $S$ resulting from the shear stresses along the slip-lines. The weight of the soil can be approximated by

$$W = (\gamma_s - \gamma)dD$$  \hspace{1cm} (5.1)

The shear force follows as the sum of the shear stresses.

$$S = (\gamma_s - \gamma)d^2\sin\phi$$  \hspace{1cm} (5.2)

where $\phi$ is the angle of friction.
For the large transportation pipes it can be assumed that the buried depth and the diameter are of the same order of magnitude \(d \approx D \approx 1\) metre. Furthermore the density of these pipes is of the same order as the density of water. As the density of soil is about twice as much as the density of water the buoyancy \(B\) of the pipe can be approximated as

\[
B = \frac{2}{\pi} (\gamma_s - \gamma)D^2
\]  

(5.3)

When a friction angle of 30° degrees is assumed the ratio of the static breakout force and the buoyancy of the pipe can be calculated. When the assumption of the buried depth is used a simple expression for this ratio is obtained.

\[
\frac{F_b}{B} = \frac{W + S}{B} = \frac{(\gamma_s - \gamma)D^2(1 + \sin \phi)}{(\gamma_s - \gamma)D^2 \frac{2}{\pi}} = \frac{1 + \sin \phi}{\frac{2}{\pi}} = 2.0
\]

(5.4)

This ratio will generally be a lower bound. The assumption of straight vertical slip-lines is valid for loose soils only. Due to dilation slip-lines in a dense soil will meet a vertical axis under an angle. This will increase the breakout force as the weight of the overburdening soil increases. It can be concluded that compared to the static breakout force the hydrodynamic force is small.

Furthermore it can be considered that when lifted pore pressures will be generated under a buried pipeline. These pressures will apply suction to the pipe and as a result of that a restoring force will be present on the pipe, see (5.1).

The possibility of floatation of a pipeline as a result of liquefaction is mainly determined by the properties of the seabed and the wave characteristics. As the dimensions of a submarine pipeline compared to the wave length are small the influence of the pipe can be neglected as a first approximation. The subject of pore pressure buildup due to wave interaction has been discussed in chapter 4.

In this way the stability of a pipeline is determined by the initial state of the seabed and the method of installation of the pipeline. For a pipeline that is laid unprotected on top of the seabed the liquefaction potential depends on the initial
state of the seabed. When as a result of a large preshearing during a long history, combined with a low sedimentation rate, the seabed has attained a dense state, the response to cyclic loading will be weak. In such cases floatation of the seabed is unlikely to happen. If on the other hand the pipeline is mechanically buried for instance by fluidisation then the initial properties of the seabed have been changed. The material around the pipe may have become looser and subsequently more susceptible to cyclic loading. A similar configuration is obtained when the pipe is laid in a trench and later covered artificially or left to natural backfilling. An upper bound of this situation is found when the liquefaction potential of a homogeneous seabed is considered, which is also relevant to pipelines that are laid on top of a seabed where high pore pressures are generated e.g. freshly deposited sediments. Liquefaction of the seabed in such case result in vertical displacements of the pipe only if the pipe is heavy enough. However a pipeline with a density greater than 2 might sink into the liquefied bed depending on the properties of the bed and the area that is liquefied. The liquefaction potential of a homogeneous seabed is treated in the following section.

5.2 Liquefaction potential of a homogeneous seabed

The seabed will reach a state of instability when the effective soil stresses are reduced to zero. This implies that the generated pore pressure is equal to the initial effective stress.

The phenomenon of pore pressure generation was discussed in chapter 4. There it was found that an absolute maximum for the pore pressure is given by the solution for infinite time in case of linear compaction. This solution is a steady state function. Although there is still compaction resulting in pore pressure generation there is no further increase of the pore pressure. In reality the compaction decreases in time. The maximum pressure distribution is given by expression (4.77)

\[ p(y,w) = \frac{B_1}{c_v^{\lambda^2}} (1 - \exp(-\lambda y)) \]  

(5.5)

It can be expected that instability will occur only over a
limited zone near the surface of the seabed. As the depth of this zone will be small compared to the wave length the exponential function can be linearized

$$p(y,\omega) = \frac{B_1}{c_0 \lambda^2} \lambda y \quad \lambda y \ll 1$$  (5.6)

The maximum pore pressure generation is determined by the wave length, the compressibility of soil and the parameter $B_1$ which is defined as

$$B_1 = \frac{D}{\alpha} \frac{3}{1+2K_0} \frac{\beta^\lambda}{(\gamma_s - \gamma)} \frac{\omega}{2\pi}$$  (5.7)

The cyclic parameter $D$ can be determined from undrained cyclic loading tests. Although in reality the stresses rotate circular and therefore lead to a higher $\tan \beta$ the result of cyclic triaxial (or simple shear) tests will be used here. According to the derived relation (4.22) the parameter $D$ follows from the measured $\tan \beta$ as

$$D = \frac{\tan \beta}{C_1 \ln 10 \gamma/\sigma_i}$$  (5.8)

The ratio of the amplitude $\tau$ of the applied shear stress variation and the initial effective stress $\sigma_i$ has been defined as the stress level. According to the results of cyclic triaxial tests [4.12] which have been shown in figure 4.4, depends the value of $\tan \beta$ on the applied stress level. The experimental data can be fitted with an exponential function.

$$\tan \beta = \beta_0 \left(\frac{\tau}{\sigma}\right)^a$$  (5.9)

As a result the parameter $D$ depends also on the stress level

$$D = \frac{\beta_0}{C_1 \ln 10 \left(\frac{\tau}{\sigma}\right)^{a-1}}$$  (5.10)

When the function for the cyclic parameter $D$ is inserted in the constitutive relation (4.3) the same differential equation is found as was derived in paragraph 4.5.2 for linear compaction, see equation 4.5.4. Only here a stress parameter $a$ appears in
the equation
\[ \frac{3p}{3t} = c_v \frac{3^2 p}{3y^2} + B_1 \exp(-a\lambda y) \]  
\[ (5.11) \]

The strength of the "pumping term" is determined by the parameter \( B_1 \) defined as
\[ B_1 = \frac{b_o}{c_{10} \ln 10} \left( \frac{3}{1+2 K_o} \frac{\beta \lambda}{(\gamma_s - \gamma)} \right)^a \frac{\omega}{2\pi} \]  
\[ (5.12) \]

Basically has the differential equation the same solution as derived in 4.5.3. This applies also to the model with limited densification.

At infinite time the solution for the linear model reduces to a steady state function
\[ p(y,\infty) = \frac{B_1}{c_v a^2 \lambda^2} (1 - \exp(-a\lambda y)) \]  
\[ (5.13) \]

As \( a \) is greater than one the effect of the exponential function for the stress level is that the penetration depth decreases. The linearized solution of the pore pressure is valid over a smaller range.

\[ p = \frac{B_1}{c_v a^2 \lambda^2} a \lambda y \quad a\lambda y < 1 \]  
\[ (5.14) \]

Instability of the solid matrix occurs when the pore pressure equals the initial effective soil stress. Here the isotropic effective stress is considered which can be calculated with the coefficient of lateral earth pressure \( K_o \).

\[ \sigma = \frac{1}{3} (1+2 K_o) (\gamma_s - \gamma) \gamma \]  
\[ (5.15) \]

The ratio of the maximum pore pressure and the initial effective stress can be considered as a liquefaction criterion. As both functions change linear in depth the ratio is constant. This means that if instability occurs it will occur practically homogeneous over the region where the linearized function of the pore pressure is valid.

\[ \frac{p}{\sigma} = \frac{B_1}{c_v a^2 \lambda^2} \frac{a\lambda}{\frac{1}{3}(1+2 K_o) (\gamma_s - \gamma)} \geq 1 \]  
\[ (5.16) \]
With the definitions of the parameter $B_1$ (5.13) and the consolidation coefficient $c_v$ (4.49) the criterion is completely determined. Furthermore the variables can be separated in a group determined by the wave characteristics and a group determined by soil conditions only. The wave steepness appears to be a dominant factor.

$$\frac{\beta_0}{2\pi kC_{10} \ln 10} \left( \frac{3}{1+2 K_0} \right)^{a+1} \left( \frac{\gamma}{\gamma_s - \gamma} \right)^{a+1} \geq \frac{\alpha \lambda/\omega}{H_s^{a} \left( \frac{2 \cosh \lambda h}{a} \right)} \quad (5.17)$$

where $h$ is the local waterdepth.

For restricted areas an experimental relation can be determined between the wave height $H$ at the surface and the wave period $T$. According to v. Heteren-Bruinsma [5.2] the following relation is valid for the southern part of the North Sea.

$$H = 0.06 T^{2.2} \quad (5.18)$$

where $H_s$ is the significant wave height.

The wave height at the top of the seabed $H_b$ and the wave length $L$ are calculated from linear wave theory. For a certain location with a specific waterdepth $h$, the average waterdepth on the Continental Shelf of the North Sea varies from 20-40 metres, the right part of the liquefaction criterion can be calculated as a function of the significant wave height $H_s$. Moreover a value for the stress level parameter $a$, which followed from undrained tests has to be included. In figure 5.2 these curves are given for a waterdepth of 20 and in figure 5.3 for a waterdepth of 30 metres.

From the level of the left part of the inequality follows the significant wave height at which instability occurs. This value depends largely on the response to cyclic loading determined by $\beta_0$ together with the parameter $a$, the permeability and the compressibility of the seabed. The value of $\beta_0$ depends largely on the strain history as can be concluded from the results of Smits (4.15) and figure [4.8]. Values for the different variables are listed in table 1.
Fig. 5.2 Liquefaction criterion
Waterdepth $h = 30$ m.

Fig. 5.3 Liquefaction criterion
From these results it can be concluded that for a depth of 20 metres instability might be possible for a significant wave height of 5.50 m. An important factor is the coefficient of lateral earth pressure \( K_0 \). Moreover it can be concluded that the effect of the stress level parameter \( a \) on the critical significant wave height is small. However the depth of the unstable zone will decrease for higher values of \( a \).

As a result of preshearing the critical wave height increases considerable. For instance when the order of magnitude of \( \beta_0 \) is ten times lower the critical wave height for a waterdepth of 20 m (30 m) increases to 8.50 m (>10 m).

Furthermore the effect of preshearing during loading has been studied by introducing a decaying function in time for the volumetric contraction. It was concluded that the maximum pore pressure generation can considerably decrease even for a slowly decaying function.

In chapter 2 the influence of a stiff impermeable base on the stresses and displacements in a single layer was studied. Special attention was given to the effect of different boundary conditions at the base which result from a different interface between layer and base. The results show that for relatively thin layers the stress distribution differs largely from the stresses in a semi-infinite half-plane. Furthermore it can be concluded

<table>
<thead>
<tr>
<th>stress level parameter ( a ) (fig. 4.4)</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_0 ) after preshearing (fig. 4.8)</td>
<td>300</td>
<td>3000</td>
</tr>
<tr>
<td>permeability ( K )</td>
<td>10^{-5} \text{ m/s}</td>
<td></td>
</tr>
<tr>
<td>compressibility ( C_{10} )</td>
<td>20-200 (dense)</td>
<td></td>
</tr>
<tr>
<td>( \gamma )</td>
<td>10^4 \text{ kN/m}^3</td>
<td></td>
</tr>
<tr>
<td>( \gamma_s )</td>
<td>2 \times 10^4 \text{ kN/m}^3</td>
<td></td>
</tr>
<tr>
<td>( K_0 )</td>
<td>0.5-1.0</td>
<td></td>
</tr>
</tbody>
</table>

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that in case of a perfectly smooth interface the stress level is generally higher than in case of a perfectly rough interface. The reason for this is that in case of a smooth interface the system has more degrees of freedom which results in larger displacements. These larger displacements in a layer with a smooth interface result in a higher stress level but a more rapid damping of the wave amplitude.

Important for the liquefaction potential is the effect on the deviatoric shear stress of a stiff impermeable base. In figure 5.4 the deviatoric shear stress is given for both a smooth and a rough interface together with the deviatoric shear stress in a semi-infinite half-plane. The relative thickness of the layer is 0.5.

![Fig. 5.4 Deviatoric shear stress in different configurations.](image)

It can be concluded that in a layer with a perfectly smooth interface the deviatoric shear stress is considerably larger than in a layer with a perfectly rough interface. In the last case the deviatoric shear stress is almost equal to the stress in a semi-infinite half plane.

In practise the soil profile may look like a relative thin layer of sand overlaying an impermeable clay stratum. In this
case the interface between sand and clay may be rough and therefore displacements of sand relative to clay can be neglected at the level of the interface. However the condition at the base of the layer is not perfectly rough as in the clay stratum displacements will be induced. Therefore displacements will arise at the level of the base. The exact condition at the interface will be in between the considered limits. Therefore it can be expected that the deviatoric shear stress is higher in a finite layer than in a semi-infinite seabed.

Direct application of the derived solution is not possible as here an elastic soil model was used. As a result horizontal stresses are possible at the top of the layer while in reality in sand the horizontal stress must vanish at this level. At the top of the layer where the vertical effective stress is zero the stiffness, e.g. expressed in a bulk or shear modulus, is zero as it depends on the stress level. This is show for instance by relation (4.16) or (4.17). It can be expected that the same behaviour is found when a physical more realistic soil model is used for instance an elastic model but with a bulk and shear modulus that increase linear in depth.

As has been shown a rather simple criterion can be derived as a first approximation in order to evaluate the liquefaction potential. For a certain location much will depend on the actual values of the soil parameters and the wave characteristics. The soil parameters have to be determined during a field investigation together with laboratory tests.
PRINCIPAL NOTATIONS

$c_v$  coefficient of consolidation
$D$  cyclic property of soil
$e$  void ratio
$E$  Young's modulus of soil
$G$  shear modulus of soil
$h$  water depth
$H$  wave height
$K$  permeability of soil skeleton
$K$  bulk modulus of soil
$K_l$  bulk modulus of water
$K_o$  coefficient of lateral earth pressure
$n$  porosity of soil
$p$  excess water pressure
$P_a$  absolute water pressure
$p$  amplitude of wave pressure at the seabed
$R$  pipeline radius
$S$  degree of saturation
$t$  time
$x$  horizontal coordinate
$y$  vertical coordinate

$\alpha$  compressibility of soil
$\beta'$  apparent compressibility of water
$\gamma$  density of water
$\gamma_s$  density of soil
$\epsilon$  volume strain
$\lambda$  wave number
$\nu$  Poisson's ratio
$\sigma$  isotropic effective stress
$\sigma_{ij}$  effective stress (tensor)
$\tau$  amplitude of cyclic shear stress
$\omega$  wave frequency

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SAMENVATTING

Bij de winning van olie en gas op het Nederlandse deel van het Continentale Plat worden voor het transport tussen de platforms en het transport naar de kust pijpleidingen gebruikt. Thans ligt er op dit deel van de bodem van de Noordzee totaal zo'n duizend kilometers pijpleiding. Bij de keuze van het tracé spelen belangen van de overheid, de olieindustrie, de visserij en de scheepvaart. Om het risico van schade door scheepsankers en vistuig zo klein mogelijk te houden worden pijpleidingen ingegraven in de zeebodem. Een belangrijke vraag is of een eenmaal begraven pijpleiding zichzelf omhoog kan werken onder invloed van de golfwerking. In dit proefschrift worden de grondmechanische aspecten van deze problematiek beschouwd.

De golven aan het zeeoppervlak oefenen een cyclisch in de tijd varierende kracht uit op een pijpleiding. De grootte van deze kracht hangt af van de ingraafdiepte, de pijpdiameter, de eigenschappen van de grond en de golfkarakteristieken uitgedrukt in de golfhoogte, de golflengte en de lokale waterdiepte. Uitgaande van de respons van de zeebodem op cyclische drukgolven in water aan de het oppervlak zijn analytische oplossingen uitgewerkt voor de kracht op een pijpleiding bij verschillende configuraties. Hierbij is gebruik gemaakt van een benaderingsmethode welke toepasbaar is vanwege de afmetingen van een pijpleiding t.o.v. die van de golven. De zeebodem is geschematiseerd tot een lineair elastisch homogeen isotroop poreus medium belast door enkelvoudig harmonische tweedimensionale golven in water. Uit de resultaten volgt dat de golfkracht op een volledig begraven pijpleiding klein is, klein in verhouding tot de statische kracht welke benodigd is het bovenliggende pakket te doen bezwijken.

Naast de instantane golfrespons treedt in de zeebodem ook een tijdsafhankelijk proces op. Als gevolg van de door de golven opgewekte schuifspanningswisselingen in de zeebodem heeft het korrelskellet de neiging te verdichten d.w.z. een afname van het totale volume. Echter in met water verzadigde grond zijn volumeveranderingen van invloed op de waterspanning. Het resultaat is dat na passage van een golf de waterspanning is toegenomen. De
mate hangt af van de gevoeligheid van zand voor cyclische belastingen en de mogelijke afstroming. Onder extreme condities tijdens een stormperiode is het mogelijk dat de druk in het water zover oploopt dat de grond bezwijkt. Dit verschijnsel noemt men liquefaction en een begraven pijpleiding wordt in zo'n geval instabiel. Voor de berekening van waterspanningssopbouw is een rekenmodel ontwikkeld. De resultaten hiervan voor een standaardproef zijn in goede overeenstemming met resultaten laboratoriumproeven op zand. In de berekening van de waterspanningssopbouw in de zeebodem wordt het proces van waterspanningssopbouw apart bestudeerd van de instantane golfrispons gebruikmakend van een ontkoppelde methode. Hierdoor wordt het probleem wiskundig vereenvoudigd en is het mogelijk een analytische oplossing af te leiden. Daarnaast is het effect van een begraven pijpleiding op dit proces verwaarloosd gezien de afmetingen van een offshore pijpleiding. Met de resultaten van deze berekening is een criterium afgeleid waarmede bepaald kan worden onder welke omstandigheden in het bijzonder bij welke significante golfhoogte instabiliteit van de bovenste zone van de zeebodem optreedt. Voor Noordzee omstandigheden is dit uitgewerkt waarbij gebruik is gemaakt van grondmechanisch onderzoek op monsters van de zeebodem. Het blijkt dat instabiliteit mogelijk is echter dat veel afhangt van de locale omstandigheden zoals de waterdiepte en de invloed van voorafgaande stormen op de zeebodem.

De aandrijvende kracht tot verdichting is de instantane golfrispons van de zeebodem in het bijzonder de door de golven opgewekte schuifspanningswisselingen. Door een eventuele gelaagdheid is de spanningstoestand in de zeebodem anders dan in een volledig homogeen pakket. Het effect hiervan is onderzocht aan de hand van de golfrispons van een enkele laag. De zeebodem is hierbij geschematiseerd tot een laag bestaande uit een lineair elastisch homogeen isotroop poreus medium boven een stijf ondoorlatend massief. Speciale aandacht is besteed aan de invloed van het contact tussen de laag en het onderliggend massief. Daarbij zijn twee uiterste situaties beschouwd n.l. een volledig glad en een volledig ruw contact. Oplossingen zijn ontwikkeld m.b.v. een analytische methode en een benaderende methode op basis van een
variatieprincipe. De resultaten van de benaderende oplossing stemmen tot een bepaalde waarde van de dikte van de laag goed overeen met de resultaten van de analytische methode. Geconcludeerd kan worden dat de voor verdichting maatgevende schuifspanning beïnvloed wordt door het contact tussen de laag en het onderliggende pakket. In een laag met een volledig glad contact is de schuifspanning veel hoger dan in een laag met een ruw contact of een half-oneindig pakket. Instabiliteit zal zo'n geval eerder optreden.