Online Companion for

Robust Unit Commitment with Dispatchable Wind: An LP Reformulation of the Second-stage

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This document is an online companion† for [1].

1 LP Reformulation of the Second-stage Robust UC

1.1 The Second Stage Problem for uncertain Wind or Solar

This report shows the step-by-step procedure of eliminating some variables of the problem presented in [1] through the Fourier Motzkin procedure.

As stated in [1], the the second-stage max-min problem of the robust unit commitment with dispatchable wind, which leads to a bilinear problem, is equivalent to the following LP formulation:

\[
\begin{align*}
\min_{p, w, v^+, v^-} & \quad c^T p + d^T w \\
\text{s.t.} & \quad Hp + Jw \leq h \\
& \quad Bp + Cw \leq \tilde{g} \\
& \quad w + \mathbf{W}^T v^+ - \mathbf{W}^T v^- \leq 0 \\
& \quad v^- - v^+ \leq 1 \\
& \quad p, w, v^+, v^- \geq 0
\end{align*}
\]

where \( \mathbf{W} \) and \( \mathbf{W}^T \) are diagonal matrices containing the vectors \( \mathbf{w} \) and \( \mathbf{w}^T \) in the diagonal, respectively. Since \( \mathbf{W} \) and \( \mathbf{W}^T \) are diagonal symmetric matrices, then \( \mathbf{W} = \mathbf{W}^T \) and \( \mathbf{W}^T = \mathbf{W} \).

By applying the Fourier Motzkin elimination, we obtain the following equivalent formulation where

\footnote{Last update 2015-07-24}
variables $v^+$ and $v^-$ no longer appear:

$$
\begin{align*}
\min_{p, w} & \quad c^T p + d^T w \\
\text{s.t.} & \quad Hp + Jw \leq h \\
& \quad Bp + Cw \leq g \\
& \quad w \leq w \\
& \quad p, w \geq 0.
\end{align*}
$$

(6)

(7)

(8)

(9)

The following subsections show the step-by-step Fourier Motzkin procedure of eliminating variables $v^+$ and $v^-$.  

**1.2 Eliminating $v^+$**

To start, constraints (3)-(5) can be rewritten as

$$
\begin{align*}
v^+ & \leq \mathbf{W}^{-1} (\mathbf{W} v^- - w) \\
-1 + v^- & \leq v^+ \\
0 & \leq v^+ \\
w, v^- & \geq 0
\end{align*}
$$

(10)

(11)

(12)

(13)

where (10) is equivalent to (3) after multiplying (3) by $\mathbf{W}^{-1}$. Notice that $\mathbf{W}^{-1}$ is a positive matrix, then the inequality does not change its sign ($\leq$).

The Fourier Motzkin elimination consists in combining all the upper bounds of $v^+$ (10) with their lower bounds (11)-(12), hence the following equivalent set of inequalities can be obtained:

$$
\begin{align*}
-1 + v^- & \leq \mathbf{W}^{-1} (\mathbf{W} v^- - w) \\
0 & \leq \mathbf{W}^{-1} (\mathbf{W} v^- - w) \\
w, v^- & \geq 0
\end{align*}
$$

(14)

(15)

(16)

by multiplying both sides of (14)-(15) by $\mathbf{W}$, we obtain:

$$
\begin{align*}
-w + \mathbf{W} v^- & \leq \mathbf{W} v^- - w \\
0 & \leq \mathbf{W} v^- - w \\
w, v^- & \geq 0
\end{align*}
$$

(17)

(18)

(19)

Now, reorganizing:

$$
\begin{align*}
(\mathbf{W} - \mathbf{W}) v^- & \leq \mathbf{W} - w \\
\mathbf{W} v^- & \geq w \\
w, v^- & \geq 0
\end{align*}
$$

(20)

(21)

(22)

and multiplying (20) and (21) by $(\mathbf{W} - \mathbf{W})^{-1}$ and $\mathbf{W}^{-1}$, respectively:

$$
\begin{align*}
v^- & \leq (\mathbf{W} - \mathbf{W})^{-1} (\mathbf{W} - w) \\
\mathbf{W}^{-1} w & \leq v^- \\
0 & \leq v^- \\
w & \geq 0
\end{align*}
$$

(23)

(24)

(25)

(26)

\(^2\)Notice that $(\mathbf{W} - \mathbf{W})^{-1} \geq 0$ since $\mathbf{W} \geq \mathbf{W}$ by definition.
1.3 Eliminating $\nu^-$

Again, by applying the Fourier Motzkin elimination, we combine all the upper bounds of $\nu^-$ (23) with the lower bounds (24)-(25). Hence, the following set of constraints are equivalent to (23)-(26) where variables $\nu^-$ no longer appear:

\[
W^{-1}w \leq (\overline{W} - W)^{-1}(\overline{w} - w) \tag{27}
\]
\[
0 \leq (\overline{W} - W)^{-1}(\overline{w} - w) \tag{28}
\]
\[
w \geq 0 \tag{29}
\]

and multiplying (27) and (28) by $(\overline{W} - W)$

\[
(\overline{W} - W)W^{-1}w \leq \overline{w} - w \tag{30}
\]
\[
w \leq \overline{w} \tag{31}
\]
\[
w \geq 0 \tag{32}
\]

which by reorganizing becomes

\[
(\overline{W}W^{-1} - I + I)w \leq \overline{w} \tag{33}
\]
\[
w \leq \overline{w} \tag{34}
\]
\[
w \geq 0 \tag{35}
\]

where $I$ is the identity matrix.

Now, by multiplying both sides of (33) by $(\overline{W}W^{-1})^{-1}$, which is equivalent to $W^t \left(\overline{W}^t\right)^{-1} = \overline{W}W^{-1}$:

\[
w \leq \overline{W}W^{-1}w \tag{36}
\]
\[
w \leq \overline{w} \tag{37}
\]
\[
w \geq 0 \tag{38}
\]

Since $\overline{W}$ is a diagonal matrix with elements in its diagonal $\overline{w}_i \neq 0$, the inverse $\overline{W}^{-1}$ is also a diagonal matrix with elements in its diagonal $1/\overline{w}_i$. Then the matrix operation $\overline{W}^{-1}w$ is a column vector of ones (1). Therefore, (36)-(38) become

\[
w \leq w \tag{39}
\]
\[
w \leq \overline{w} \tag{40}
\]
\[
w \geq 0 \tag{41}
\]

where (40) is redundant because it is dominated by (39), which imposes a tighter upper bound to $w$.

Finally, after Fourier Motzkin elimination, constraints (3)-(5) are equivalent to

\[
w \leq \overline{w} \tag{42}
\]
\[
w \geq 0 \tag{43}
\]

therefore, the formulation (1)-(5) is equivalent to

\[
\min_{p, w} c^t p + d^t w \tag{44}
\]
\[
\text{s.t. } Hp + Jw \leq h \tag{45}
\]
\[
Bp + Cu \leq \tilde{g} \tag{46}
\]
\[
w \leq \overline{w} \tag{47}
\]
\[
p, w \geq 0. \tag{47}
\]
References