DISCUSSION OF BEDLOAD MOVEMENT FORMULAS OF KALINSKE, EINSTEIN, AND MEYER-PETER AND MULLER AND THEIR APPLICATION TO RECENT MEASUREMENTS OF BEDLOAD MOVEMENT IN THE RIVERS OF HOLLAND

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Note: This is a working translation prepared primarily for use within the Bureau of Reclamation. It should not be considered final, but practically all the ideas expressed by the original author can be obtained from it in its present form. All illustrations have been duplicated from the original article without complete translation of the descriptive material or reduction of all values to English units.
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1. Introduction.

There are many reasons why a great deal of effort should be expended toward establishing a good formula for calculating transported bed material.

In practice, comprehensive knowledge of the laws of bedload transport is necessary, among other things, to permit the study of the consequences of man's disturbing natural water courses.

For laboratory studies a good bedload formula is indispensable for calculation of scale ratios for movable bed models.

The formulas employed for transport of bed material may be divided into:

(1) Formulas principally derived by theory and

(2) Formulas primarily based on the results of measurement of transported materials in which the relationships between the coefficients governing transport are, for the most part, established empirically.

The recent formulas of KALINSKE (1947) and EINSTEIN (1950) may be considered as the most representative of the first group; the formula of MEYER-PETER and MULLER emanating from the Laboratory of E.T.H. at ZURICH has been chosen as an example from the group of empirical formulas.

In the following discussion the three are compared and examined on the basis of their applicability. A detailed criticism is not attempted because the formulas of KALINSKE and of MEYER-PETER and MULLER have already been covered in a previous publication (Reference 4)\textsuperscript{a}.

As a result of that study a formula is given for the proper approximation, within practical limits, which is based on the formulas discussed and also on a certain number of measurements made in the

\textsuperscript{a} Numbers refer to references at end of paper.
branches of the RHINE in Holland and in the Laboratory at Delft.

2. Comparison of the Formulas of Kalinske, Einstein and Meyer-Peter and Muller

a. Kalinske's formula (Reference 1)

This formula is principally based on the observations of White (Reference 5), concerning the critical shear along the bed for which a grain starts to move, and on the knowledge of the turbulent velocity distribution made near the bed.

In its dimensionless form, the formula is:

\[
\frac{T}{d \sqrt{g H \eta}} = 2.5 \theta \left( \frac{\tau_c}{\tau_0} \right) \tag{1}
\]

where \( T \) is the bedload expressed in units of volume per unit of time and per unit of width.\(^\#\)

\( d = \) the diameter of the grains
\( R = \) the hydraulic radius
\( I = \) the slope of the energy gradient
\( \tau_c = \) the critical bottom shear
\( \tau_0 = \) the mean value of shear on the bottom that is present

\( \theta \) is a complicated function of \( \frac{\tau_c}{\tau_0} \) that Kalinske has developed by considering the law of difference according to Gauss for the fluctuation of the velocity along the bed. A graph of the function is given in the publication by Kalinske.

According to the results of White, the quotient \( \frac{\tau_c}{\tau_0} \) may be written as follows if the Reynolds number for a grain is greater than 3.5:

\[
\frac{\tau_c}{\tau_0} = 0.12 \frac{\eta d}{RI} \tag{2}
\]

\(^\#\) The total volume of the grain is known. Account has not then been taken of the volume of the pores which must be added to \( T \) for finding the volume of a deposited mass, for example.
where \( \Delta = \frac{(\rho_1 - \rho_0)}{\rho} \), the relative density of the solid grains under water. By the aid of this formula, we may write:

\[
\frac{m}{d^{3/2}g^{1/2}A^{1/2}} = A_1 \left( \frac{\Delta d}{R_1} \right)^{-1/2} \left( \frac{E_0 d}{R_1} \right)^n \quad (1a)
\]

or

\[
X = F_1(Y) \quad (1b)
\]

\( A_1, B_1 \) are constants, \( X = T/d^{3/2}g^{1/2}A^{1/2} \), a dimensionless expression of transport, \( Y = \Delta d/R_1 \), a parameter which depends on the intensity of transport.

b. Einstein’s formula (Reference 2)

In the most general form, this formula is written:

\[
\frac{A_2 X}{1/B - A_2 X} = 1 \frac{1}{\sqrt{\pi}} \int_0^{\infty} Y_x - 1/n_0 \left( \frac{d}{\delta} \right)^2 d\delta
\]

in which \( n_0, A_2, B_2 = \) constants,

\( t = \) the variable of integration,

\( i_B = \) the fraction of the bedload having the size of grains considered,

\( i_b = \) the fraction of material of which the bed is constituted, having a size of grains considered,

\( Y = Y \prime f(\delta, d, K) \) or \( f(\delta, d, K) \) and is a function in which the size analysis of the mixture, and of the current near the bed is expressed. If the size of the grains is uniform and if the is hydraulically rough, \( Y_x \) is equal to \( Y' \).

\( R' \) is the part of the hydraulic radius, \( R \), which is due to the roughness of the grains. Einstein expresses also the influence of the irregularities of the bed. In paragraph 4, the manner in which \( R' \) is determined will be shown.

In the case where the bed is smooth, without ripples or bars and without lateral resistances, \( R' \) must be equal to \( R \).

The formula of Einstein is then:
\( X = F_2(Y) \)  

(3a)

For a mixture of grains of different diameters when the bed is in waves and the thickness of the laminar sublayer is not negligible, it is also necessary to apply a certain number of corrections which may be found from the four graphs. The equation, (3a), is also given in graphic form.

Einstein, in the derivation of this formula, has considered turbulence as a statistical problem.

c. Formula of Meyer-Peter and Muller (Reference 3)

This formula is written as follows:

\[
\frac{T}{d^{3/2} g^{1/2} A^{1/2}} = \left( \frac{R T'}{A d} - 0.047 \right)^{3/2}
\]  

(4)

I' represents the part of the slope, I, used for overcoming the roughness of the bed, while the remainder—that is to say, \( I - I' \)—will be used in overcoming the roughness of the walls that caused the ripples, etc.

Using the formula of Chezy, one is able to write consecutively:

\[
v = C \sqrt{R I}
\]  

and  

\[
v = C' \sqrt{R I'}
\]  

(5)

Then,

\[
I' = \left( \frac{C}{C'} \right)^2 I
\]  

(6)

C, the actual coefficient of Chezy, might be calculated with the Equation (5) from data obtained from the water course, while, with the aid of formulas for roughness, \( C' \) is determined from values of \( R, k \) (roughness of the bed) and eventually of \( \delta \), the thickness of the laminar sublayer. Meyer-Peter and Muller use the formula of Strickler:

\[
C' = 26 \left( \frac{R}{k} \right)^{1/6}
\]  

(7)
In the hydraulically rough ranges, this formula differs little from the logarithmic formula. (Reference 6)

\[ G' = 18 \log \left( \frac{12 R}{k + 3v/\mu} \right) \]  

(7a)

In our opinion, this formula should be preferred because it is valid for a greater range of conditions.

It appears, however, that the exponent 2 in Equation (6) is not in accordance with experiments made at Zurich. According to Meyer-Peter and Muller, an exponent 3/2—that is to say, 75 percent of the theoretical value gives better results. Equation (6), then, may be written:

\[ I' = \left( \frac{G'}{G} \right)^{3/2} I \]  

(6a)

Here also \( I' \) must be equal to \( I \) when the bed is smooth, without ripples or dunes, or, in general, when the transport is small.

Therefore, \( X = 8 \left( \frac{1}{Y} - 0.047 \right)^{3/2} \)  

(4a)

or \( X = F_3 (Y) \)  

(4b)

d. Comparison of the three formulas

In comparing the three formulas (1b), (3a) and (4b), it may be seen that for certain circumstances, each may be written as a ratio between the same dimensionless parameters. These functions are shown graphically in Figure 1.

It appears that the formulas of EINSTEIN and MEYER-PETER and MULLER agree quite well for values of "X" greater than 0.003, which is when the bedload is not very small. For smaller values of "X" the curve from ZURICH approaches asymptotically the horizontal line, \( Y = 21.3 \). This is because of the conception of a critical value of the mean boundary shear. This will be referred to in paragraph 5.

Figure 1 shows that the KALINSKE formula, when it is applied for uniform materials in which ripples are not formed, gives values
for "X" or for bedload transport of about 1/2 the values of the other two formulas. This comes from the fact that KALINSKE does not apply in his formula a special ripple coefficient for the reduction of the force due to friction caused by bed ripples. In paragraph 4, this will be examined again more closely.

3. Characteristic Diameters of Grains

The question arises—what is the value of the diameter of the grains to be used in the previously mentioned formulas when mixtures of materials of different grain diameters are involved instead of uniform materials? One must distinguish between the composition of the bed materials and the composition of the materials in movement which as Einstein has properly said are not always the same.

For the value of roughness of the grains on the bed (Equations 7 and 7a) MEYER-PETER and MULLER take d 90, which is the diameter of which 90 percent of the mixture is equal or smaller in grain size. EINSTEIN uses a value d65 which is the diameter of which 65 percent of the mixture is equal or smaller in grain size. In the Netherlands the best results have been attained with d 90, a value that was also the result of a special study.

EINSTEIN, in his latest publication, no longer applied the bedload transport formula on the mixture in its entirely, but on each fractional part, where consideration should be given to reciprocal influence. However in our opinion, this refinement is not in proportion with the precision that a calculation of transport is able to give now.

However, if it is applied to the entire mixture, it is preferable to express more or less the form of the curve of sieve size in the value of the diameter of the grains. We believe that the best means of expressing this is to introduce the value of KREY (Reference 7):
in which "d" is the mean diameter in a portion "p".

For mixtures of sand in the Holland rivers that value corresponds to d₆₀ to d₈₀.

MEYER-PETER and MULLER have also used the expression "dₘ" in developing the results of their observations. KALININSTE recommends the mean diameter d₅₀, while Einstein, in a preceding publication (Reference 8), employs d₃₅ to d₄₅ as a characteristic diameter.

4. Ripple Coefficients.

a. Transport of bed material and loss of energy

It is well known that the bedload transport is obstructed by the ripples on the bed or by the dunes. A part of the energy is expended by overcoming the resistance of the form of the ripples, since, behind the ripples, the current is detached from the bed, and her vortices are formed, with the consequent transformation of energy into turbulence and, finally, into heat. This transformation of energy takes place at such a distance from the bed in which the transport considered is produced. Therefore, it is considered to contribute little or nothing to that transport.

This is quite different from the loss of energy which is caused by the pure roughness of the grains and which is directly related to transport.

It is for this reason that it appears proper to divide the mean shear, \( \tau_o \), equal to \( \rho g R I \), into two parts.

For example, similar to the resistance in a pipe being divided into friction resistance and into form resistance caused by the entrance losses, sudden enlargement, and by the loss due to curves rather than a general expression of loss of head due to turbulent agitation.

The coefficient that indicates which part of \( \tau_o \) should be taken for calculating the bedload is called the ripple coefficient.
6. Coefficients on ripples according to Einstein and according to Meyer-Peter and Muller

Einstein introduced a ripple coefficient by reduction of the hydraulic radius, \( R \), while Meyer-Peter and Muller introduce the ripple coefficient by reduction of the slope \( I \). Since it is always the value \( pgR/1 \) which controls the bedload transport, it makes little difference which element is reduced. It is only necessary to point out that a reduction of the hydraulic radius is more difficult to imagine than a reduction of the slope.

The manner in which the coefficient of reduction is determined is nearly the same in both cases. One calculates mainly the hydraulic radius, \( R' \) (Einstein) or the slope, \( I' \) (Zurich), which results from formulas of resistance, for the velocity observed and at the introduction of \( k \) as a measure of roughness. This value of \( R' \) or, in the Swiss formula, \( I' \), which is smaller than the original value, is used Formulas (3) and (4) for calculation of bedload transport.

In order to permit a comparison of the different processes, there is represented below the forms of the formulas as well as the application of the resistance formula of Strickler which has the application of the logarithmic resistance formula. As has been said already, the two formulas give very nearly the same results for the hydraulically rough conditions.

\[
\begin{align*}
v &= 26k^{-1/6}n^{1/2}\frac{2/3}{1}^{1/2} \quad (9a) \\
v &= 26k^{-1/6}n^{1/2}\frac{2/3}{1}^{1/2} \quad (9b) \\
R' &= \left(\frac{k}{R}\right)^{1/4}R \quad (10)
\end{align*}
\]

\[
\begin{align*}
v &= 26k^{-1/6}n^{1/2}\frac{2/3}{1}^{1/2} \quad (11a) \\
v &= 26k^{-1/6}n^{1/2}\frac{2/3}{1}^{1/2} \quad (11b) \\
I' &= \left(\frac{k}{K}\right)^{1/3}I \quad (12)
\end{align*}
\]

In these equations, \( k \) represents the roughness of the grains \((d90)\) and \( K \), the equivalent roughness of the walls, and included the ripples, dunes, etc.

\[
\begin{align*}
k &= \left(\frac{k}{K}\right)^{6} \\
K &= \frac{k}{n} \quad \text{for} \quad K = \frac{k}{n} \quad \text{and} \quad k \quad \text{for} \quad k \quad \text{in} \quad \text{in}
\end{align*}
\]
As has been said in Paragraph 2, the experiments made at Zurich have demonstrated that the theoretical exponent $1/3$ in Equation (12) is too large: a value of $0.75 \times 1/3 = 1/4$, giving better results.

$$I' = \left( \frac{k}{K} \right)^{1/4} I$$

(13)

This gives the same reduction of the shear $gH$ as that which Einstein found in using the hydraulic radius (Equation 10).

In applying the logarithmic formula, the similarity of the bedload formulas is not quite so apparent.

$$v = 18 \sqrt{R'} \frac{k}{K} \log \left( \frac{12R'}{K} \right)$$

(14)

$$v = 18 \sqrt{R'H} \log \left( \frac{12R}{K} \right)$$

(15a)

$$v = 0 \sqrt{H}$$

(15b)

$$I' = \left( \frac{18 \log \left( \frac{12R}{k} \right)}{2} \right)^{1/2} I$$

(15)

Again, according to the experiments made at Zurich, the exponent 2 in Equation (15) deduced from the above consideration, should be modified by $0.75 \times 2$, which is equal to $3/2$.

If one introduces this reduction by means of Equation (10) or Equation (13) and takes this value of $K$ of the Equation (9b) or (11b), then one obtains the following result:

$$v' = \rho g R'I = \rho g R'I' = \rho g (v/2r)^{3/2} + 1/4 + 1/4$$

Apparently the hydraulic radius, $R$, is eliminated from the bedload formula if this reduction is applicable.

\[ v' = \rho g \left( \frac{k}{R} \right)^{1/2} \frac{R}{K}^{1/4} \]

\[ v' = \rho g \left( \frac{k}{R} \right)^{1/2} \frac{R}{K}^{1/4} \]

\[ v' = \rho g \left( \frac{k}{R} \right)^{1/2} \frac{R}{K}^{1/4} \]

\[ v' = \rho g \left( \frac{k}{R} \right)^{1/2} \frac{R}{K}^{1/4} \]
or

$$I' = \left(\frac{C}{18 \log\left(\frac{2R'}{K}\right)}\right)^{3/2} I$$  \hspace{1cm} (17)

In Equation (14) of Einstein, $R'$ appears implicitly as the only unknown. The value $R'$ must be found by trial. Einstein and Barbarossa (Reference 9) have determined for that formula the value of $R'$ for a certain number of American rivers, and they have given the results in graphic form. These curves show that $R'$ approaches $R$ when the discharges are large. This, the authors explained, in passing, that the rivers meander a great deal at small discharges. When the discharges are larger, and the bedload movement is increased, the river will taken a straighter course and encounter less supplementary resistance.

This is a case similar in principle to that in which bed ripples increase with discharge making the supplementary resistance larger and larger. Measurements on Holland rivers and in the laboratory demonstrate this fact.

c. Consideration in detail of the ripple coefficient.

With respect to the equality of hydraulic roughness of the logarithmic formula for resistance and the approximation according to Strickler, the concordance indicated above for Equations (10) and (13) must also exist for Equations (14) and (17). This appears to be perfectly true. The value of $R'/R$, according to Equation (14) is the same as that of $I'/I$, according to Equation (17). However, Equation (14) has the practical inconvenience that $R'$ does not appear explicitly. Equation (17) leads to simple calculations and is therefore preferable.

One can write $R'/R = I'/I = \mu$ , where $\mu$ is the ripple coefficient with which the mean shear, $\tau$, equal to $\rho g R I$ must be multiplied in order to eliminate, in the transport formula, the influence of the roughness of the walls and of the bed.
Then, according to Equation (17):

$$\mu = \left[ \frac{C}{18 \log(12R/k)} \right]^{3/2}$$  \hspace{1cm} (18)

with units in meters, kilograms, seconds, and

- $C = \sqrt{\frac{v}{f}}$, the apparent coefficient of Chezy,
- $R$ = the apparent hydraulic radius,
- $k$ = the roughness of the grains which, according to Paragraph 3, is equal to $d90$

From Equation (18) has been made a graph (Figure 2) which gives the value of $\mu$ as a function of $C$ and of $R/k$.

The inconvenience of Equation (18) is that $\mu$ must be calculated from the apparent value of $C$ as it presents itself in nature, which must then be calculated from the measurements of velocity, slope and cross section. A forecast of transport of sand in a new situation of a water course without introduction of other factors is then impossible.

However, as Einstein and Barbarossa (Reference 9) have stated, for banks of sand, the ripple height may, accordingly, be related to transport which, in turn, is a function of the quantity $Y/\mu = \Delta d/\mu R$.

Consequently, $\mu = f_1 (\Delta d/\mu R)$

and for the same reason $\mu = f_2 (\Delta d/\mu R)$ or $\mu = f_2 (\Delta d/K)$ \hspace{1cm} (19)

It appears that there is, in effect, a net relationship according to Equation (20), for a straight-water course where the influence of roughness of the lateral walls on the slope may be eliminated. In Figure 3-a, that relationship, as indicated according to the studies made at Delft, (see appendix), has been indicated by a straight line. When the values of $\Delta d/\mu R$ are large — that is to say, when the transport is small and the bed smooth — $\mu$ must approach unity because the roughness of the bed, $K$, is nearly equal to the roughness of the grains, $k$. 
For natural water courses with revetments, dikes, groins, and other artificial obstructions, the value of $u$, if $\Delta d/RI$ is large, must always be a little less than 1, because then the equivalent roughness, $K$, always remains larger than the roughness of the grains, $k$. Only in the case where one succeeds in eliminating the influence of the groins, etc., on the slope, could one again find the relationship of $u$ and $\Delta d/RI$ from laboratory studies.

For the Holland rivers, such a relationship appears to actually exist. In Figure 3-b, the values of have been drawn such that they follow the measurements of transport made in the Waal and in the Lower Rhine Rivers (see appendix).

With a similar relationship between $u$ and $\Delta d/RI$, it is equally possible to predict for a certain discharge the total roughness, and consequently the value of $C$ from Chezy.

a. Influence of ripples in Kalinske's formula

Kalinske did not apply corrections for taking into account the influence of ripples, etc., on bedload transport. Since, as the author has demonstrated, his formula is in concordance with a certain number of observations of different origin, the supposition must be accepted that the influence on ripples is implicitly contained in the formula. This is also possible for the two formulas discussed unless one accepts a constant relationship between the ripple coefficient, $\mu$, and the dimensionless parameter, $Y = \Delta d/RI$: for example, the straight line in Figure 3-a. Doing this, one takes into account only the natural formation of the ripples and does not have to consider all the other supplementary resistances such as sand and gravels, bars, groins, etc.

On the contrary, in admitting, for example, that the formula of Einstein must be exact without the influence of the ripples, one is able to determine from Figure 1, the above-mentioned relationship between $u$ and $Y$, which is implicitly contained in the formula of
Kalinske. Then it is a matter only of dividing the ordinates of the curves of Kalinske by those of Einstein to determine values of $Y$.

Figure 3c gives the results. There is an agreement with the straight line which follows the test made at Delft and at Zurich (Reference 3, Figure 12).

5. **Critical Velocity of Critical Shear for the Beginning of Transport**

In most of the formula derived in the past, one is able to find a critical value of velocity or the mean shear, $\tau_0$, which would be characteristic for the beginning of the transport. It is only in the formula based on the modern theories of turbulence, such as the formulas discussed of Kalinske and of Einstein, that one does not encounter such a value. This value is replaced by the probability of movement which always diminishes in proportion as $\tau_0$ becomes smaller.

This is accord with what one is able to see in laboratory studies concerning the commencement of transport. In effect, there are always some grains that are in movement because of a small amount of turbulence, and therefore an absolute repose of bed materials is not quite attainable. Also, the limits in the above-mentioned formulas are often arbitrary, and besides are difficult to determine. Meyer-Peter and Muller introduce the same two limits, one for the absolute repose and the other for commencement of transport.

Also, it is necessary in the bedload transport formulas to give preference to the probability of movement that is in accordance with nature. It is chiefly for the very small bedloads that one obtains the most exact results by the use of a critical value.

However, in certain cases it may be useful -- for example, when stable canals are proposed -- to be able to insert a value of velocity or shear for which there will never be any appreciable bedload transport.
It is remarkable that most of the experiments expressed that limit by the parameter, \( Y = \Delta d/RI \), which plays such an important role in the transport formulas discussed.

Tison (Reference 10) has assembled a certain number of data from different sources, and he has established the influence of \( R_g \), the Reynolds number for a grain:

\[
R_g = \frac{d \sqrt{g d}}{v}
\]  

(21)

When \( R_g \) has a value between 10 and 100, the value of \( Y \) for the practical beginning of transport appears to lie between 50 and 25; it has a value between 30 and 20.

Meyer-Peter and Muller (Reference 3) give the value of \( Y = 33 \) for the absolute beginning of movement.


The three formulas discussed above are all somewhat inconvenient for practical application.

Figure 1 shows, however, that the curves of Einstein and of Meyer-Peter and Muller may be approximated with sufficient accuracy by a simple formula:

\[
X = 5Y^{-1/2} e^{-0.27Y}
\]  

(22)

When the value of \( Y \) are large, this curve differs from the one developed at Zurich because in this range the smaller critical value does not enter in. At the same time one may find with that curve greater values of transport than by the use of Einstein's curve, which again corresponds to the results of observation (see Reference 4).

The middle part of the curve, for the value of \( Y \) between 3 and 15, is practically identical to both the Einstein and Zurich curves.
If the values of \( Y \) are smaller than 3, the approximate curve tends to give too small values of transport. In this region where bedload transport is very intensive, it cannot be said, with few observations available, these formulas give the exact bedload transport because some suspended load may be included.

As has been explained in Paragraph 4a, it is preferable at the DELFT laboratory to utilize the ripple factor of MEYER-PETER and MULLER when applying the formula of logarithmic resistance:

\[
\lambda = \left[ \frac{C}{18 \log \frac{12R}{d_G + 3 \sqrt{gRT}}} \right]^{3/2}
\]  

(18a)

Having taken into account the influence of the ripples, it then follows:

\[
X = 5 \left( \frac{Y}{\mu} \right)^{-1/2} e^{-0.27 \frac{Y}{\mu}}
\]

This equation may also be written

\[
\frac{T}{d_m \sqrt{g \mu RT}} = 5e^{-0.27d_m/A_\mu RT}
\]

(23)

This formula gives a simple straight line when the dimensionless term on the left is plotted as an abscissa on a logarithmic scale and the large value \( Y/\mu = \Delta d/\Delta t \mu \) as the ordinate on a linear scale (Figure 4).

7. Conclusions

a. It appears that each of the three formulas can be reduced to a ratio between dimensionless parameters which are the same for each formula. The formulas appear to be nearly the same.

b. Kalinske's formula can only be applied by means of a diagram because its construction is complicated.

c. For the same reason Einstein's formula can only be used with a diagram. Einstein applies a reduction of the boundary shear \( \tau_0 \) in order to obtain that part of \( \tau_0 \) which is characteristic for the transport material. The value of that coefficient is equal to that of MEYER-PETER and MULLER, whereas the form of the latter leads to simpler and more rapidly made calculations.
d. Einstein does not recommend the application of the transport formula on a mixture of grains in its entirety, because he found that it is not exact enough. According to him, the calculation of transport of each separate fraction of material, by applying the corrective coefficient for their mutual influence, leads to much greater accuracy. It is questionable if present knowledge of the phenomenon that is at our command justifies these complicated calculations. For instance, the uncertainty in the determination of the diameter of the grains leads one to believe that the attained precision is only apparent.

e. MEYER - PETER and MULLER use in their primarily empirical formula a limit of the mean tractive force that will be characteristic for the beginning of transport. Because of the effect of turbulence that is always found in water courses transporting materials, that limit is totally fictitious. The formula leads to false values for very small bed load transport.

Aside from this, the formula gives good results, and its construction is simple. The reduction of the mean tractive force is made in an acceptable manner.

f. For straight channels with only the resistance of the ripples and the roughness of the grains, the ripple coefficient appears to depend uniquely on the parameter $\frac{A_d}{RI}$. That function takes other values when the water course is curved and when there are bars or when there are other obstacles. The function is fundamentally another than that which Einstein has found for the American rivers, in which the bed with small discharges meanders between the banks, straightens at the high flows and becomes less rough.

g. The three formulas discussed agree very well. Besides they may be approximated with sufficient precision by the simple formula

$$\frac{1}{d_{r}} \sqrt{\frac{T}{\mu RI}} = 50 - 0.27 \frac{A_d}{RI}$$  (23a)
which covers for the most part the experimental curve of MEYER-PETER and MULLER derived from a large number of laboratory observations and which is in satisfactory accordance with a certain number of observations in the field.

**APPENDIX A.**

**Description of observations in the rivers of Holland.**

The measurements were made with the apparatus for measuring transport called ARCHEM (BTMA) which has already been described elsewhere (Reference 11 and 12). It is possible to measure with this equipment the matter carried in the first 0.1 meter (3.93 inches) above the bed.

Ten observations were made at the same point. Then the measurements were repeated at a point 10 meters (33 feet) upstream and 10 meters (33 feet) downstream for the initial point, so that the amount of transport at a single point in the transverse profile was determined by 30 observations, Figure 5. The transverse profile was measured completely with a distance of 40 meters between two points along the axis of the river. One complete observation took 2 days on the Lower Rhine and 3 days on the Waal.

The variation in the 10 measurements at a single point are somewhat large. Values from 0 to 500 percent of the mean value are found. The variation in the mean values for the three points located one behind the other are no more than 50 percent in the normal cases.

The mean transport over the entire rivers was calculated with the aid of a diagram, Figure 5. The values A, B, C, etc. are the mean values calculated from the 30 observations at hand. The area of the diagram divided by the effective width b, gives the value T, the mean bed load transport per unit of width for the cross-section.
This effective width is used because it is clear that the use of the total width B gives results that are not exact. It is known that in that value of b, determined from a diagram such as Figure 5, some uncertainty must be acknowledged. It was not possible to find a better method. Tables 1 and 2 give postwar data for the Lower Rhine and the Waal. In addition to the transport $T$, the following values were given: The mean depth $h$, the slope, the discharge $q$ per unit of width, the variation $h$ in the level of water during the observation, the effective width $b$, the calculated values of $T/d_m g \mu \text{RI}$ and $Y/\mu = \Delta d_m/\mu \text{RI}$, and the ripple coefficient.

During each measurement of transport, the cross-section was measured with an ultrasonic sounder. The depth $-h-$ is the mean depth along the effective width $b$, Figure 5.

The value given for the diameters of the grains are the mean of a very large number of measurements. During the observations of bed load transport, some samples from the bed were taken at each point of measurements: the value given of $d_{90}$ of all the samples.

All the samples taken with the BTMA (Sampler) have also been sifted: the value given for $d_m$ is likewise the mean of $d_m$ of all the samples.

**APPENDIX B.**

**Description of measurements made at the Hydraulic Laboratory at DELFT.**

A certain number of measurements of transport were made in a glass sided flume 15 meters (50 feet) long, 0.5 meter (1.65 feet) wide, and 0.6 meter (2 feet) deep, with light materials (ground pumice stone and ground bakelite). The main object of these studies was to examine these materials to determine their adaptability to use in models with movable beds.
The transported materials were recovered at the downstream end of the flumes in a deep pit. By weighing the recovered materials during each study, the amount of transport was determined. Tables 3 and 4 give the results of these observations.
LIST OF REFERENCES


4. Elzerman, J. J. et Frijlink, H. C., Present state of the investigations on bed-load movement in Holland, IX me Assemblee de l'U.G.C.I., Bruxelles, 1951


<table>
<thead>
<tr>
<th>Symbol</th>
<th>Dimension</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>( 1^2 t^{-1} )</td>
<td>Reynolds number for the grains</td>
</tr>
<tr>
<td>( \gamma )</td>
<td></td>
<td>Transformed material in units of volume per unit of width and per unit of time</td>
</tr>
<tr>
<td>( t )</td>
<td></td>
<td>Variable of integration</td>
</tr>
</tbody>
</table>
| \( x \) | | Dimensionless parameter \( \frac{\tau}{\rho_1 \Delta} \)
| \( Y \) | | Dimensionless parameter \( \frac{\rho_1 - \rho}{\rho} \)
| \( \Delta \) | | Relative density \( \frac{\rho_1 - \rho}{\rho} \)
| \( \eta_0 \) | | Constant |
| \( \phi \) | | Kalinske's function |
| \( \mu \) | | Ripple coefficient \( \mu = \frac{I'}{I} = \frac{R'}{R} \)
| \( \nu \) | \( 1^2 t^{-1} \) | Kinematic viscosity |
| \( \rho \) | \( ml^{-3} \) | Density of the transporting liquid |
| \( \rho_1 \) | \( ml^{-3} \) | Density of the transported grains |
| \( \tau_0 \) | \( ml^{-1} t^{-2} \) | Mean boundary shear \( \frac{g}{RI} \) |
| \( \tau_c \) | \( ml^{-1} t^{-2} \) | Critical boundary shear |
APPROXIMATION PROPOSÉE : 
\[ x = 5 \left( 1 - 0.47y \right) \]
FIG. 2. ABACUS FOR THE COEFFICIENT OF RIDES $\mu$. 

\[ \mu = \left[ \frac{C}{18 \log_{10} \frac{R}{k}} \right]^{\frac{3}{2}} \]
3a. MESURES AU LABORATOIRE HYDRAULIQUE DE DELFT
3b. MESURES DANS LES RIVIÈRES NÉERLANDAISES
3c. COMPARAISON DE RÉSULTATS

FIG. 3. RAPPORT ENTRE LE COEFFICIENT DES RIDES ET LA GRANDEUR $y = \Delta d/RI$. 
FIG. 4. SIMPLIFICATION PROPOSÉE DE LA FORMULE DE TRANSPORT.
RÉSULTATS DES MESURES.
FIG. 5. SCHEMA DE MESURES SUR LE WAHAL.