Analytic evaluation of single-pass cross-correlation response for the case of a sinusoidal displacement

R. Theunissen

Department of Aerospace Engineering, Bristol University, Bristol, United Kingdom
r.theunissen@bristol.ac.uk

ABSTRACT

This paper addresses the intrinsic response of the cross-correlation operator to a one-dimensional sinusoidal displacement field to assess the general assumption of cross-correlation behaving as a moving average (MA) filter. Analytic expressions are derived describing the correlation map, which further allow the identification of pivotal parameters influencing the response. Contrary to common belief, the analysis indicates the response not to generally behave as that of an MA filter, but instead to be non-linear and determined by the ratio between displacement amplitude and particle image diameter. This is shown to have implications for recursive image processing necessitating predictor-corrector filtering. In addition, readily available spectral filtering techniques are assessed in terms of effectiveness in minimization of correlation deterioration due to pixelization and displacement filtering inherent to particle image self-correlation. It is shown that the best possible correlation response in digital PIV is dictated by the convolution between particle image self-correlation and displacement distribution function, inevitably retaining particle image inherent amplitude modulation.

I. INTRODUCTION

Particle Image Velocimetry has reached a stage where basic image evaluation follows a tried and proven recipe involving the segmentation of consecutive flow recordings into so-called interrogation windows, followed by a statistical operator to define the motion of the captured particle image intensity distributions. Although various alternatives have been proposed such as for example optical flow [1], least squares matching [2], etc., extraction of particle image displacements between PIV images by means of cross-correlation has become main practice where the position of the highest correlation peak indicates a volume-averaged displacement [3]. While cross-correlation effectively yields unique and reliable displacement estimates in case of uniform velocity fields, Nogueira et al. [4] reported the response of the cross-correlation operator to resemble that of a moving averaging filter (MA). The achievable spatial resolution of such image processing routines is consequently prescribed by interrogation window size besides physical constraints imposed by tracer spacing [4]. This has led to the introduction of more complex recursive image analyses focused on the iterative reduction of correlation window sizes while preserving the particle image pairs contributing to the correlation peak[5]. Iterative image warping based on cross-correlation on the other hand has been shown to introduce the appearance of instabilities in displacement estimates. Such instabilities are typically ascribed to the negative amplitude response of cross-correlation analogue to that of a moving average and concomitantly obey wavelengths roughly 1.5 times the applied interrogation window size [6].

The MA model has successfully lead to methodologies reducing overall signal modulation and instabilities by adequate filtering during each of the involved iterative PIV image processing steps. Reduction of the modulation was achieved in [7] by introducing a varying filter kernel size based on adaptivity concepts [8] when updating the resulting velocity field. On the other hand filtering methods applied to the corrector displacement are able to avoid the spurious oscillations in the resulting displacement field [9].

The equivalence between the moving averaging filter and cross-correlation has become a generally adopted simplification as it captures correctly the low-pass filtering effect attenuating length scales smaller than the physical dimensions of the interrogation windows’ [10]. Moreover, sinusoidal displacements of 1 pixel amplitude and particle image sizes of 3 pixels are typically considered for which the correlation does behave as a MA filter. For larger amplitude to particle size ratios however the inherent correlation function has a more complex, non-linear behavior [11].

In the majority of PIV applications, particle image diameters are 3 pixels to avoid peak locking effects. The minimum separation between particle images $\lambda_p$ equals approximately one particle image diameter to ensure individual tracer intensity distributions to be resolved (cf. the Rayleigh criterion). The Nyquist sampling criterion then translates in an absolute minimum resolvable flow scale $\lambda_{fres}$ of at least $2 \lambda_p$ or a minimum of 6 pixels. Imposing an image density of 10 particle images per correlation area (neglecting loss-of particles) accordingly leads to minimum window sizes of approximately 9-by-9 pixels². In other words, each correlation window will contain only about two $\lambda_{fres}$ wavelengths.
Over the years several research papers have addressed the intrinsic correlation response. Lecuona et al. [11] observed the retrieved displacement in case of a sinusoidal displacement to be biased to the most frequent displacement. In the same line, the theoretical analysis in [12] on the effect of velocity gradients, the response of cross-correlation was shown to depend on the displacement distribution within the interrogation area. Young et al. [13] additionally indicated the correlation peak to be influenced by the higher particle image intensities, implying the need of displacement vector relocation schemes.

Is it not the objective of this paper to question the efficiency of multi-pass, iterative predictor corrector schemes or correlation routines adopting intensity weighting? At this stage, provision of alternative solutions to alleviate or minimize instabilities is neither intended. Instead, this paper challenges the above MA simplification by deriving a mathematical description of the correlation response in case of a one-dimensional sinusoidal displacement field. This functional has been selected as it is commonly used to investigate the spatial response of image processing routines. The aim is therefore to understand when the simplification is valid and which pivotal parameters influence the cross-correlation operator. The underlying image analysis considered is consequently a single-pass approach but proves nevertheless to be instructive in comprehending this fundamental correlation operator. From this perspective, displacement modulation attributed to particle image self-correlation is identified as a major contributor and common spectral filtering methodologies aimed at minimizing such inherent effects are examined.

II. IMAGE CORRELATION FROM A DIFFERENT PERSPECTIVE

The spatial cross-correlation of two continuous image intensity fields, \( I_1 \) and \( I_2 \), separated by a time delay \( \Delta t \) is defined by [14]

\[
R(s) = \int I_1(X)I_2(X+s)dX
\]  

(1)

where \( X \) defines the spatial coordinate in the image domain and vector \( s=(r,s) \) with shifts \( r \) and \( s \) in horizontal and vertical direction respectively. Sections of the image fields are expressed as

\[
I_k(X) = W_k(X - X_k) \sum_i I_{0k}(x)\tau_0(X - Mx)\delta(x - x_i(t))dx
\]

(2)

where spatial coordinates in the object plane (i.e. the flow) are given by \( x \). The weighting function \( W_k \) defines the interrogation window extents centred on \( X_k \) and is typically represented as a top-hat function. \( M \) is the magnification, \( \tau_0 \) is the point spread function of the imaging lens, which for a circular aperture equals the Airy function. The position of the \( i^{th} \) particle in the flow field at time \( t \) is prescribed by \( x_i(t) \). \( I_{0k}(x) \) refers to the illumination intensity profile, which for a common laser light sheet can be modelled as a Gaussian depending only on the coordinate normal to the sheet. After subtracting the mean image intensities, the correlation of the fluctuating image intensities, \( R_0 \), is obtained.

Clearly the velocity information is indirectly extracted from the locations of the particle images. Indeed particle images effectively constitute a random spatial sampling of the underlying displacement field. Correlation windows contain only a finite number of tracer images and peaks in \( R_0 \) consequently represent random realizations of the displacement field \( \Delta X(X) = M u(x,t) \Delta t \). As a result the ensemble average \( \langle R_0(s\mid u) \rangle \) is a collection of realizations and effectively represents the measured displacement distribution for a given flow field \( u \) [12];

\[
\langle R_D(s\mid u) \rangle \equiv K^* N_1 F_o(\Delta z) F_I(s) R_{DD}(s) \quad \text{with} \quad R_{DD}(s) = \left\{ F_I(s - \xi) F^*_N(\xi) d\xi \right\}
\]

(3)

where \( R_{DD} \) is the displacement correlation based on the continuous distribution function of the displacement field \( \Delta X \) symbolized by \( F_o \) and particle image self-correlation \( F_I \). \( K \) is a proportionality constant, \( N_1 \) denotes the image density, and \( F_o \) and \( F_I \) represent the out-of-plane and in-plane loss of correlation respectively [14] where

\[
F_I(s) = \int W_1(X)W_2(X+s)dX \int W_1(X)W_2(X)dX
\]

(4)

In equation (3) the light sheet intensity distribution is assumed to be dependent only on out-of-plane position \( \Delta z \), the optical axis to be normal to the light-sheet plane and both \( I_1 \) and \( I_2 \) to be recorded in the same plane.

A decreasing effective image density \( N_1 F_o F_i \) causes the correlation to skew towards smaller displacements [14], with the related error reducing with particle image size. Proper normalization of the correlation function [15] or interrogation window offset [5] tend to minimize this bias error allowing \( \langle R_0(s) \rangle \) to be prescribed by \( R_{DD} \). It should be noted that such techniques however do not affect the intrinsic correlation response \( R_{DD}(s) \); a uniform spatial offset \( \Delta \) of an interrogation window causes a mere translation \( F_i(s - \Delta) \) of the displacement distribution with a corresponding shift \( s_D \).

Alternatively the in-plane “loss of pairs” function \( F_o(s) \) can be altered by modifying the intensity distribution within an interrogation window viz. equations (2) and (4). Nogueira et al. [16] and Eckstein and Vlachos [17] have shown the use of higher order functionals for \( W_k \) to yield a reduction in amplitude modulation compared to standard top-hat functions. Intensity weighting essentially reduces the effective window width but does not affect the inherent response of the correlation operator either. For this reason windowing functions will be left out of consideration in the following to allow a proper assessment of the correlation operator’s response.
III. IMAGE CORRELATION RESPONSE

1. Continuous image correlation response to a sinusoidal displacement field

In the following analytic expressions for the correlation distributions $<R_D>$ and $R_{DD}$ are obtained on the basis of continuous intensity distributions i.e. negating the discrete nature of the imaging sensor. The particle image self-correlation introduced in (3) is prescribed by

$$\tau_o = \int_{\mathbb{R}^2} \rho(X) \rho(X+s)dX$$  \hspace{1cm} (5)$$

$t_o$ serves as a normalisation constant. The particle image intensity per unit of illumination $\tau_o$ based on the particle image diameter $d_{\tau}$ is assumed to be Gaussian.

$$\rho(x) = \frac{1}{\sqrt{2\pi} \alpha d_{\tau}} \exp\left(-\frac{x^2}{2\alpha^2 d_{\tau}^2}\right)$$  \hspace{1cm} (6)$$

allowing $F_{\tau}$ and its Fourier Transform, defined as

$$\mathcal{F}(f) = \int_{\mathbb{R}^2} f(x) \exp(-i\kappa x)dx,$$

to be expressed respectively as

$$F_{\tau}(s) = \int_{\mathbb{R}^2} F_{\tau}(s|\beta) p(\beta) d\beta\hspace{1cm} (7)$$

with $k=(m,k)$. So far the image diameter $d_{\tau}$ has been assumed to be identical for all particle images. If tracer image diameters follow a continuous distribution, the particle image self-correlation becomes

$$\mathcal{F}_{\tau}(s) = F_{\tau}(s)(\beta) = F_{\tau}(s|d_{\tau})$$

The displacement assumed hereafter will be a sinusoid in vertical direction representing velocity fluctuations of varying spectral content depending on wavelength $\lambda; \Delta X(X)=(\Delta X, \Delta Y)$ with $\Delta X=0$ and $\Delta Y=A_o \sin(2\pi X/\lambda)$. The distribution function for the displacement within a correlation window of size $W_S$ centered on $X_o$ and the corresponding Fourier transform are defined as

$$F_{\Delta}(s) = \delta(r) \frac{1}{W_S} \int_{X_o-W_S/2}^{X_o+W_S/2} \delta(s-\Delta Y(X))dX$$  \hspace{1cm} (8)$$

$$\mathcal{F}_{\Delta}(m,k) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \frac{1}{W_S} \int_{X_o-W_S/2}^{X_o+W_S/2} \exp(-ik\Delta Y(X))dX = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi} \alpha} \frac{\pi}{2} \exp(-iA_o \sin(W_S \eta + 2\pi X_o))d\eta$$  \hspace{1cm} (9)$$

with $\eta=2\pi(X-X_o)/W_S$. The following identity holds; $\exp\left(\frac{i}{2}a(b-b^{-1})\right) = \sum_{n=1}^{\infty} b^n J_n(a) = J_0(a)+\sum_{n=1}^{\infty} (b^n+b^{-n}) J_n(a)$. Hence by setting $a=-A \kappa$ and $b=\exp\left(i\{W_S \eta+2\pi X_o\}/\lambda\right)$;

$$\mathcal{F}_{\Delta}(m,k) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \frac{1}{\sqrt{2\pi}} \frac{\pi}{2\pi} \left[ \int_{-\pi}^{\pi} J_0(a) \sin(\frac{2\pi}{X_o} W_S \eta) + 2\pi X_o) d\eta \right]$$

$$+ \sum_{n=1}^{\infty} J_{2n}(a) 2i \int_{-\pi}^{\pi} \sin(\frac{(2\pi-1)}{X_o} W_S \eta) - \int_{-\pi}^{\pi} \sin(\frac{2\pi}{X_o} W_S \eta) \sin(\frac{2\pi}{X_o} 2\pi X_o) d\eta$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^2 \frac{1}{\sqrt{2\pi}} \frac{\pi}{2\pi} \left[ \int_{-\pi}^{\pi} J_0(a) + \sum_{n=1}^{\infty} J_{2n}(a) \cos(\frac{2\pi}{X_o} W_S \eta) \cos(\frac{2\pi}{X_o} 2\pi X_o) d\eta \right]$$

$$+ \sum_{n=1}^{\infty} J_{2n}(a) \left[ \int_{-\pi}^{\pi} \sin(\frac{2\pi-1}{X_o} W_S \eta) \cos(\frac{2\pi-1}{X_o} 2\pi X_o) d\eta + \int_{-\pi}^{\pi} \cos(\frac{2\pi-1}{X_o} W_S \eta) \sin(\frac{2\pi-1}{X_o} 2\pi X_o) d\eta \right]$$
\[
J_0(a) + \sum_{n=1}^{\infty} J_{2n}(a) \cos(2n \frac{2\pi x}{\lambda}) - \sum_{n=1}^{\infty} J_{2n-1}(a) \sin((2n-1) \frac{2\pi x}{\lambda})
\]

\[
\mathcal{F}_\Delta(m, k) = \left( \frac{1}{\sqrt{2\pi}} \right)^2 \left[ J_0(-A_\kappa k) + 2 \sum_{n=1}^{\infty} i^n J_n(-A_\kappa k) \frac{\sin(n \pi \frac{X_\kappa}{\lambda})}{\frac{n \pi X_\kappa}{\lambda}} \cos(n \frac{\pi}{2} - n \frac{2\pi x}{\lambda}) \right]
\]  

(10)

where \( J_0 \) refers to the Bessel function of the first kind. Expressions for \( F_\Delta \) and first statistical moments \( E(s) \) and \( E(s^2) \) are obtained by taking the inverse transform of (10) using [19]

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} J_n(x) \exp(-iax) \, dx = \frac{\sqrt{2\pi}}{\sqrt{\lambda}} (-1)^n (1 - a^2)^{-0.5} \cos(n \cdot \text{acos}(a)) \text{ for } a^2 < 1
\]

yielding

\[
F_\Delta(r, s) = \frac{\delta(r)}{\pi \alpha} (1 - \frac{r^2}{\alpha^2})^{-\frac{1}{2}} \lim_{N \to +\infty} \left[ 1 + 2 \sum_{n=1}^{N} (-1)^n \text{sinc}(\frac{W_\kappa}{\lambda}) \cos(n \frac{\pi}{2} - 2n \pi \frac{X_\kappa}{\lambda}) \cos(n \cdot \text{acos}(\frac{r}{\alpha})) \right]
\]  

(11)

\[
E(s) = \int s F_\Delta(s) ds = A_0 \sin(\frac{W_\kappa}{\lambda}) \sin(2\pi X_\kappa)
\]  

(12)

\[
E(s^2) = \int s^2 F_\Delta(s) ds = 0.5 A_0^2 (1 - \text{sinc}(\frac{W_\kappa}{\lambda}) \cos(4\pi X_\kappa))
\]  

(13)

Note that \( R_{DD}(s) \) is separable in \( r \) and \( s \) with \( R_{DD}(r,s) = \exp(-0.5 r^2 (zd_\kappa)^2) \) as a result of considering the one-dimensional displacement field. An explicit expression for the convolution integral \( R_{DD}(s) \) is obtained by transforming (3) to the frequency domain, \( 3R_{DD} = (2\pi)^3 \mathcal{F}_\Delta^2 \mathcal{F}_{\Delta^2} \), substituting (7) and (10) and converting back to the spatial domain;

\[
R_{DD}(s) = \frac{\alpha d}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} J_n(-A_\kappa k) \exp\left( -\frac{(kad)^2}{2} \right) \exp(iks) dk
\]

(14)

\[
= \int J_n(-A_\kappa k) \exp\left( -\frac{\alpha d^2 k^2}{2} + \frac{2is}{\sqrt{2\alpha d}} k + \frac{s^2}{2 \alpha d} \right) \exp\left( -\frac{s^2}{2 \alpha d} \right) dk
\]

\[
= \exp\left( \frac{1}{2} \frac{s^2}{\alpha d} \frac{A_\alpha^2}{\alpha^2 d_i^2} \right) \int J_n(-A_\kappa k) \exp\left( -\frac{(ad)^2 k^2}{2} - \frac{is}{\sqrt{2\alpha d}} k + \frac{s^2}{2 \alpha d} \right) \exp\left( -\frac{s^2}{2 \alpha d} \right) dk
\]

substituting \( \kappa = \frac{ad}{\sqrt{2} \alpha d} k - \frac{s}{\sqrt{2\alpha d}} \to dk = \frac{\sqrt{2}}{\alpha d} d\kappa, A_\kappa = \frac{\sqrt{2}}{\alpha d} \kappa + \frac{isA_\alpha^2}{A_\alpha^2 d_i^2} \)

\[
= \frac{\sqrt{2}}{\alpha d} \exp\left( -\frac{1}{2} \frac{s^2}{\alpha d} \frac{A_\alpha^2}{\alpha^2 d_i^2} \right) \int J_n(-\sqrt{2} A_\alpha \kappa - is A_\alpha^2) \exp(-\kappa^2) d\kappa
\]

\[
= \frac{\sqrt{2}}{\alpha d} \exp\left( -\frac{1}{2} \frac{s^2}{\alpha d} \frac{A_\alpha^2}{\alpha^2 d_i^2} \right) \sum_{p=-\infty}^{+\infty} \int_{-\infty}^{+\infty} J_p(-\sqrt{2} A_\alpha \kappa - is A_\alpha^2) \exp(-\kappa^2) d\kappa
\]

\[
= \frac{\sqrt{2}}{\alpha d} \exp\left( -\frac{1}{2} \frac{s^2}{\alpha d} \frac{A_\alpha^2}{\alpha^2 d_i^2} \right) \left\{ \sum_{p=-\infty}^{+\infty} \int_{-\infty}^{+\infty} J_p(-\sqrt{2} A_\alpha \kappa) J_{p-1}(-is A_\alpha) \exp(-\kappa^2) d\kappa \right\}
\]

\[
= \frac{\sqrt{2}}{\alpha d} \exp\left( -\frac{1}{2} \frac{s^2}{\alpha d} \frac{A_\alpha^2}{\alpha^2 d_i^2} \right) \left\{ \sum_{p=-\infty}^{+\infty} \left[ ((-1)^p J_{p+1}(-is A_\alpha) + J_{p-1}(-is A_\alpha) \right] \int_{-\infty}^{+\infty} J_p(-\sqrt{2} A_\alpha \kappa) \exp(-\kappa^2) d\kappa \right\}
\]

\[
+ J_0(-is A_\alpha) \int_{-\infty}^{+\infty} J_0(-\sqrt{2} A_\alpha \kappa) \exp(-\kappa^2) d\kappa
\]

\[
= \frac{\sqrt{2}}{\alpha d} \exp\left( -\frac{1}{2} \frac{s^2}{\alpha d} \frac{A_\alpha^2}{\alpha^2 d_i^2} \right) \left\{ \sum_{p=-\infty}^{+\infty} \left[ ((-1)^p J_{p+1}(-is A_\alpha) + J_{p-1}(-is A_\alpha) \right] \int_{-\infty}^{+\infty} J_p(-\sqrt{2} A_\alpha \kappa) \exp(-\kappa^2) d\kappa \right\}
\]

\[
+ J_0(-is A_\alpha) \int_{-\infty}^{+\infty} J_0(-\sqrt{2} A_\alpha \kappa) \exp(-\kappa^2) d\kappa
\]

\[
= \frac{\sqrt{2}}{\alpha d} \exp\left( -\frac{1}{2} \frac{s^2}{\alpha d} \frac{A_\alpha^2}{\alpha^2 d_i^2} \right) \left\{ \sum_{p=-\infty}^{+\infty} \left[ ((-1)^p J_{p+1}(-is A_\alpha) + J_{p-1}(-is A_\alpha) \right] \int_{-\infty}^{+\infty} J_p(-\sqrt{2} A_\alpha \kappa) \exp(-\kappa^2) d\kappa \right\}
\]

\[
+ J_0(-is A_\alpha) \int_{-\infty}^{+\infty} J_0(-\sqrt{2} A_\alpha \kappa) \exp(-\kappa^2) d\kappa
\]
The integral in the first term reduces to zero for every odd \( p \) allowing the overall expression to be rewritten

\[
= \sqrt{\frac{\pi}{a_d}} \exp\left(-\frac{1}{2}a_d^2 \right) \left\{ \sum_{p=1}^{+\infty} \left[ J_{q+2p}(\pm is_n A_{on}^2) + J_{q-2p}(\pm is_n A_{on}^2) \right] \int_{-\infty}^{+\infty} (-\sqrt{2} A_{on} \kappa) \exp(-\kappa^2) d\kappa \right\}
\]

Using the identity

\[
\int_{-\infty}^{+\infty} J_n(ax) x^m \exp(-hx^2) dx = \frac{(1+(-1)^n) a^n \Gamma\left(\frac{n+m+1}{2}\right)}{2^{(n+1)/2} \pi \Gamma\left(n+1\right)} K\left(\frac{n+m+1}{2}; n + l; \frac{-a^2}{4h}\right)
\]

where \( \Gamma \) and \( K \) are the Gamma and Kummer functions respectively, the sought for integral reduces to

\[
= \sqrt{\frac{\pi}{a_d}} \exp\left(-\frac{1}{2}a_d^2 \right) \left\{ \lim_{p \to -\infty} \sum_{p=1}^{+\infty} E_2 \int_{-\infty}^{+\infty} J_{q-2p}(\pm is_n A_{on} s_n) + J_{q+2p}(\pm is_n A_{on} s_n) \right\}
\]

with the coefficients \( E_2 \) and \( E_{2p} \) given by

\[
E_0 = \sqrt{\pi} K\left(\frac{1}{2}; \frac{-1}{2} A_{on}^2 \right)
\]

\[
E_{2p} = \frac{\Gamma\left(p+0.5\right)}{\Gamma\left(2p+1\right)} 2^{-p} A_{on}^2 K\left(p + \frac{1}{2}; 2p + 1; \frac{-1}{2} A_{on}^2 \right)
\]

Combining (14), (15) and (16) finally leads to an explicit expression for the convolution \( R_{DD} \), and consequently \( \langle R_d(s|u) \rangle \), in terms of normalized amplitude \( A_{on} = A_o/\alpha_d \) and normalized displacements \( r_n = r/A_o \) and \( s_n = s/A_o \).

2. Correlation as a moving average filter

To derive the similarity with the moving average filter, small values of \( A_{on} \) are assumed allowing the expression for \( R_{DD} \) to be further simplified by setting the Bessel functions to their asymptotic form \( J_q(x) \approx (x/2)^q/\Gamma(q+1) \) and neglecting higher order terms \( O(A_{on}^3) \) i.e. \( J_q(-is_n A_{on}^2) \approx 0 \) for \( |q| \geq 2 \) and \( E_{2p} \approx 0 \) for \( p \geq 1 \) yielding

\[
R_{DD}(s) \approx \frac{1}{\sqrt{\pi}} \exp(-A_{on}^2 s_n^2 / 2) E_0 \cdot [1 + \sin(\frac{W_s}{\lambda}) \sin(\frac{2\lambda s_n^2}{\lambda}) A_{on}^2 s_n]
\]

At this point the moving average filter response is obtained, albeit under stringent conditions.

3. Discrete image correlation

So far intensity distributions \( I_k \) have been assumed to be continuous. The effect of image sampling on the correlation is taken into account by convoluting the continuous correlation map \( \langle R_d(s|u) \rangle \) with the triangular transfer function \( \phi_{pp} \) of a pixel [20] and subsequently sampling at integer pixel values \( \xi_{cd} = (c,d) \)

\[
\langle R_D^*(\xi_{cd} | u) \rangle = \langle R_D^*(\xi | u) \rangle \cdot \delta(\xi - \xi_{cd}) \quad \text{where} \quad \langle R_D^*(\xi | u) \rangle = \int \phi_{pp}(\xi - s) \langle R_D(s | u) \rangle ds
\]

and \( \phi_{pp}(s) = \phi_{pp}(r,s) = \Lambda(r_n A_{on} \alpha_d/p_{pp}) \cdot \Lambda(s_n A_{on} \alpha_d/p_{pp}) \)

\( \Lambda \) is the triangular function defined as \( \Lambda(t) = 1-|t| \) provided \(|t| \leq 1 \) and zero for \(|t| > 1 \). The so-called pixel-fill ratio is symbolized by \( p_{pp} \). The transfer function \( \phi_{pp} \) is obtained by auto-correlating the pixel sensor as graphically presented in the figure below.

![Figure 1: Origin of the pixel transfer function.](image-url)
Because an analytic expression is missing, (18) is calculated numerically. Given the discrete correlation $<R_D(\xi_d \Delta m)>$ displacement estimates $s_\Delta$ with sub-pixel accuracy are then obtained by fitting the correlation values with dedicated functions of which the three-point Gaussian is most common [3].

IV. NUMERICAL VALIDATION

The analytical expressions for the continuous and digital ensemble average correlations presented in equations (14) and (18) are validated against Monte-Carlo simulations on the basis of computer generated PIV images. Sinusoidal displacements are imposed of varying amplitude with correlation windows centered on $X_o=\lambda/4$. Interrogation areas are rectangular in shape with aspect ratios of 4 to minimize effects of finite interrogation windows viz. $F_1(s)$. Particles are distributed randomly in a light sheet with Gaussian intensity profile adopting a mean concentration of 0.02 particles per pixel in each correlation window. Particle image diameters follow a normal distribution with unity standard deviation. The individual Gaussian shaped particle intensities are projected onto an ideal noise-free background and integrated pixel-wise applying a pixel-fill ratio of 0.7. The unbiased spatial cross-covariance $R_D(c,d)$ between two digital images $I_1^*$ and $I_2^*$ is estimated in a straightforward manner

$$R_D^*(c,d) = \frac{1}{N_{c,d}} \sum_{m=\max(1,1-c)}^{\min(N-c,N)} \sum_{n=\max(1,1-d)}^{\min(N-d,N)} (I_1^*(m,n) - \overline{I_1^*}) \cdot (I_2^*(m + c,n + d) - \overline{I_2^*})$$

where $\overline{I_k}$ is the mean image intensity over the interrogation area. After ensemble averaging 3000 discrete correlation maps, fractional displacements are retrieved by a three-point Gaussian peak fitting routine. An exemplary image and prescribed displacement distributions are depicted in figure 2 for clarity.

![Figure 2](image-url)

**Figure 2:** (a) Exemplary synthetic image ($W_S=81$ pixels, $d=3$ pixels). Normalized displacement distributions for $W_S/\lambda$. (b) 0.5 (c) 1 (d) 1.5.

![Figure 3](image-url)

**Figure 3:** Ratio between correlation response $s_\Delta$ (i.e. measured displacement amplitude) to a sinusoid displacement of amplitude $A_o$ in function of interrogation window size (centred on the peak) for different normalized amplitudes. Dashed black lines: based on analytic expression (14). Solid black or gray lines: based on (18). Symbols: (a) synthetic images evaluated with (19) (direct cross-correlation), (b) synthetic images evaluated with (21) (FFT). The solid blue line represents the MA response. (a) $p_f=0.7$, $d=3$ pixels (b) $p_f=0.7$, $d=3$ pixels, varying sampling frequency.
Figure 3-a presents the evolution in amplitude modulation with \( W_S/\lambda \) obtained from the theoretical prediction for both continuous and pixelized images with simulation results viz. equation (19) superimposed. Results for equation (18) were determined by numerical integration after substitution of equations (3) and (14). The perfect correspondence between analytical and numerical results indicates the appropriateness of the derived equations for the assessment of the statistical correlation-operator’s response to a one dimensional sinusoidal displacement field for PIV applications. Figure 3-b reveals that modulation introduced by image digitization is ascribed mainly to discretization of the correlation map; an increased sampling frequency \( f_s \) clearly yields a response of (18) nearly identical to (14). Effects of pixel fill ratio and image diameter can be found in [21] but have been left out the current paper as they do not aid understanding at this point.

V. DISCUSSION

1. Non-conformal correlation response and effect of recursion

While typical modulation-related studies assume the measured signal to resemble a sinusoid with reduced amplitude, figure 4-a shows this clearly not to be the case. Such effects will be detrimental in recursive predictor-corrector routines where displacement estimates of the previous iteration direct the image deformation. The resolution in discernible displacements and therefore spatial resolution will consequently, as suggested afore, be dependent on particle image diameter \( d_\tau \).

Figure 4: (a) Measured sinusoid in function of \( W_S/\lambda \) for \( A_{on}=6 \) from (14). The black solid line corresponds to the ideal \( A_{on}/d_\tau=6 \) in Figure 3. (b) Normalized correlation maps of \( <R_{dd}> \) and \( R_{dd} \) for \( W_S/\lambda=0.5 \) with varying \( A_{on} \) from (3), (14) and (17). Vertical lines indicate various correlation peak locations.

Following [22], in case of iterative image deformation the construction of the generic dense predictor in the \( k^{th} \) iteration \( u_k \) involves a modulation factor \( c \) as a result of the involved interpolation (Figure 5). Once deformed, the images contain the residual displacement. Correlation then yields the corrector \( u_{c_k} \) being the residual modulated by a factor \( M_k \). The evaluated displacement \( u_{k+1} \) finally consists of the sum between the weighted average of the dense predictor, introducing a modulation \( b \), and the corrector.

Figure 5: Consecutive displacement modulation in iterative image analysis incorporating image deformation.

Because the measured amplitude does not reveal how good an approximation the measured displacement field is, here the average representation error \( \epsilon_r \) is introduced. This error, defined below, can be considered as a heuristic for the level
of accuracy with which the underlying imposed sinusoidal displacement field is measured.

\[ e_r = \frac{1}{N_\lambda} \int_0^{N_\lambda} \left| A_\lambda \sin\left( \frac{2\pi}{\lambda} (X + X_0) \right) - u_i(X) \right| dX \]  \hspace{1cm} (20)

In evaluating the representation error, the approximation \( u_i \) essentially represents the dense predictor built from discrete samples spaced by \( h = (1 - \text{WOR}) \cdot W_s \) where \( \text{WOR} \) represents the Window Overlap Factor. Both linear and cubic interpolation has been considered when reconstructing the continuous signal. Discrete velocity estimates are obtained by numerically evaluating the Fourier transform of the displacement field \( \Delta Y \) observed within the correlation window with size \( W_s \) (cf. (9), term \( \mathcal{D} \)) and subsequently the inverse Fourier transform of the product between (9), (7) (term \( \mathcal{F} \)) and \( \phi_{pp} \) and \( F_{I} \) (term \( \mathcal{G} \)) to yield (18).

\[
\left\langle R_D^{*}(\xi|\Delta Y) \right\rangle = \frac{\text{ad} N_{F_{\xi}(\Delta Y)}}{W_s} \delta(\xi - \xi_c) \int_{-\infty}^{+\infty} \exp\left( -\frac{1}{2} ik_0 \xi^2 \right) \exp(iks) d\xi
\]

Note that this procedure allows to calculate the general response to any displacement field \( \Delta Y \). Ignoring any filtering (i.e. \( b = c = 1 \)), results are depicted in Figure 6 for two iterations, following the procedure depicted in Figure 5, and two normalized amplitudes \( \lambda_0 = 0.5 \) and \( \lambda_0 = 3 \). To avoid artificial errors introduced by signal truncation, for each combination of \( W_s/\lambda \) and \( \text{WOR} \) the error estimate is performed over \( N_\lambda \) wavelengths such that the ratio \( N_\lambda/h \) yielded an integer number. For an exact moving averaging filter the sinusoidal shape is retained with an amplitude \( M \cdot \lambda_0 \) where

\[ M = 1 - \text{sinc}(W_s/\lambda) \]

For any given normalized amplitudes \( \lambda_0/\alpha d_\lambda \), differences in representation error as a result of change in interpolation order are marginal. This seems to encourage the wide spread use of bi-linear interpolation in the image deformation step. In addition, provided \( W_s/\lambda < 1 \) the reconstruction error reduces with increasing window overlap ratio i.e. improved flow sampling. However, differences for \( \text{WOR} = 75\% \) and \( \text{WOR} = 95\% \) are negligible, favoring overlap ratios of 75\% given the accompanying lower computational effort. The behavior of \( e_r \) with varying \( \text{WOR} \) and iteration does change dramatically however with \( \lambda_0/\alpha d_\lambda \). For normalized amplitudes in the order of 0.5, the error in the first iteration tends to that of a moving averaging filter with increasing \( \text{WOR} \) suggesting the error to be caused predominantly by inherent amplitude modulation. For \( W_s/\lambda < 1 \) a recursive approach is beneficial in reducing error and can yield error levels below that of the MA. When normalized window sizes exceed unity, windows overlapping mutually by more than 25\% tend to increase the representation error as a direct result of the negative amplitude response. \( W_s/\lambda \) values corresponding to the
peak in \( \varepsilon \) are seen to approach the ratio at which \( \text{sinc}(W_S/\lambda) \) reaches the first minimum (\( W_S/\lambda = 1.43 \)). A strong erratic behavior however is observed for \( A_{on}>1 \) with strong dependencies on window overlap and iteration. Errors are now larger, any similarity with the moving averaging response has disappeared and for \( W_S/\lambda<1 \) the error decreases with window spacing whereas a strong increase can be observed with \( \text{WOR} \) in case \( W_S/\lambda>1 \). More importantly, the obtained results imply an iterative procedure to not necessarily reduce the error even for \( W_S/\lambda \) ratios smaller than unity. The window to wavelength ratio at which the error for the 2\(^{nd} \) iteration surpasses the 1\(^{st} \) can be roughly estimated by the Nyquist criterion i.e. \( h/\lambda=(1-\text{WOR})W_S/\lambda \leq 0.5 \), giving limits for \( W_S/\lambda \) of 0.5, 0.66 and 1 respectively for \( \text{WOR} \) ratios of 0\%, 25\% and 50\%. Although not depicted, simulations with higher iteration numbers showed a mutual crossing point. Beyond \( W_S/\lambda>1 \) the error always continues to increases with iteration.

Adequate predictor-corrector filtering such as second order polynomial least-squares regression [9] effectively minimizes instabilities for \( W_S/\lambda>1 \) and, although not explicitly mentioned or tested here, are expected to be capable of dampening or even canceling the observed increase in error with iteration for \( W_S/\lambda<1 \). Such schemes must therefore be considered a necessity in iterative routines.

2. Particle image self-correlation and pixelization filtering

Equation (3) implies \( F_t \) to act on \( F_\Delta \) as a smoothing filter, increasingly attenuating high frequencies with image diameter. Overall the correlation operator therefore tends towards displacements attributed to regions of minimal variance. Extents of this filtering effect are depicted in figure 4-a (for \( A_{on}/W_S<\frac{1}{4} \)). Instances where image diameters are large or displacement intervals are small compared to the variance of \( F_t \) both result in low values of \( A_{on} \). For sufficiently low \( A_{on} \) \( R_{DD} \) can be prescribed by (17) and the corresponding peak location \( s_0 \) approaches the first statistical moment of the displacement distribution \( F_\Delta \) annotated by (12). With increasing \( A_{on} \) differences between \( <R_{DD}(s)|u> \) and \( R_{DD}(s) \) become marginal and both distributions exhibit peak locations approaching the exact value \( s_n=1 \) (figure 4-b). Modulation is now proportional to the image diameter as a result of the inherent convolution with \( F_t \) and, more importantly, nearly independent of normalized window size (figure 2). The relevance of the latter and recurrence of the deviant sinusoidal response depicted in figure 4-a is demonstrated by considering a generic PIV analysis with correlation windows typically 32 pixels in size, displacement amplitudes of 8 pixels and particle images in the order of 3 pixels \( (\sigma_d/\tau)\approx0.78 \), yielding \( A_{on}=10 \).

Considering large scale PIV experiments whereby image recordings constitute cloudy smoke patterns contrary to individual particle images, equivalent particle image diameters are quite large. Given the ultimate dependency of the modulation on particle image diameter and the low pass filtering effect the particle image self-correlation \( F_t \) exerts on the displacement distribution, the obtained results further explain why cross-correlation must be considered unsuitable in the analyses of this type of images as it would only result in poor spatial resolution. Instead, it can be shown that approaches adopting optical flow principles are better suited. The latter is exemplified in Figure 7 where a synthetic image pair containing large features (obtained by consecutive Gaussian smoothing) and vertical sinusoidal displacement (1 pixel amplitude, \( A_{on}<0.1 \), and 100 pixels wavelength) is analyzed by means of typical cross-correlation

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**Figure 7:** Reconstruction error for varying window overlap ratio (colors), displacement interpolation order (hollow or filled symbol) and iteration number (square or circle) (a) \( A_{on}=0.5 \), (b) \( A_{on}=3 \). The dashed line represents the error related to the moving average filter.
and a routine based on optical flow. Amplitude modulation in cross-correlation is in agreement with the results presented aforesaid whereas a considerable improvement is obtained by the optical flow based algorithm. In addition, the results suggest the modulation to be quite insensitive to the final correlation interrogation area size. As a result of pixelization the convolution expressed in (18) will introduce a second smoothening, further distorting the continuous correlation function \( <R_D(\xi(u))> \). The magnitude and direction of the accompanying shift in peak location depends on the peak skewness of \( <R_D(\delta(u))> \) kernel extent of the pixel transfer function. For the case of a sinusoidal displacement, measured displacement amplitudes from digital images are therefore readily biased towards lower values as a mere result of pixelization ignoring any additional errors inherent to the sub-pixel displacement estimation.

VI. SPECTRAL FILTERING

In practice, when attempting to optimize \( A_\text{on} \), little can be gained by modifying the optical magnification as both displacement amplitude and particle image diameter are functions of magnification. Instead, spectral filtering is often opted for. Equation (19) can be implemented as a Fast Fourier Transform (FFT) to increase the computational efficiency of the double summation. Ignoring in-plane loss of particles, the Fourier-based continuous convoluted correlation \( <R_D(\xi(\mathbf{u}))> \) viz. (18), can be evaluated as

\[
<R_D^*(\xi(\mathbf{u}))> = \delta^{-1} \{\mathbf{u} \cdot 2\pi \cdot \mathbb{R}_D \} = \delta^{-1} \{\mathbf{u} \cdot 2\pi \cdot \mathbb{R}_D \} = \delta^{-1} \{\mathbf{u} \cdot 2\pi \cdot \mathbb{R}_D \}
\]

(21)

where \( I \) is the mean subtracted image intensity and \( \mathbb{3} \) the complex conjugate Fourier transform. Filtering functions \( \psi(\mathbf{k}) \) are commonly added to improve correlation robustness to noise. Among such filters, the Symmetric Phase-Only (SPOF) introduced in [23] and Phase Transform (PHAT) studied in [24] are most used in PIV image processing. Because spatial or temporal shifts between signals translate as phase lags in the Fourier domain both filters aim at accentuating the spectral phases rather than magnitudes. These filters have been shown to unite high accuracy with robustness for a variety of imaging conditions. The SPOF and PHAT filters relate as

\[
\psi_{\text{SPOF}}(\mathbf{k}) = (|\mathbf{S}| \cdot |\mathbf{I}|) \frac{1}{2} = \mathbf{S} \cdot |\mathbf{I}| \cdot |\mathbf{I}| \cdot |\mathbf{S}| \cdot |\mathbf{I}| \cdot |\mathbf{S}| \cdot |\mathbf{S}| \cdot |\mathbf{S}|
\]

(22)

Combining equations (1), (3) and (21), with the assumption of \( F_1 = 1 \), and noting that the Fourier transform of both the triangular function and particle image self-correlation are positive-definite, the expression for the filtered continuous correlation \( <R_D(\xi(\mathbf{u}))> \) can be rewritten as

\[
<R_D^*(\xi(\mathbf{u}))> = 2\pi \cdot \delta^{-1} \{\mathbb{R}_D \} = \delta^{-1} \{\mathbb{R}_D \} = \delta^{-1} \{\mathbb{R}_D \}
\]

(23)

where \( \varepsilon = \frac{1}{2} \) or 1, designating the SPOF and PHAT filter respectively. Interpretation of (23) highlights the advantage of spectral filters to minimize \( (\varepsilon = \frac{1}{2}) \) even completely remove \( (\varepsilon = 1) \) the influence of \( F_1 \) and \( \phi_{pp} \). Although particle self-correlation filtering persists in SPOF, the apparent image diameter is lowered by a factor \( \sqrt{2} \); thereby reducing the subsequent effects. Equation (10) implies the displacement distribution of a windowed sinusoid to consist of an infinite range of wavenumbers \( \mathbf{k} \) with highest magnitudes attributed to lower frequencies. With SPOF, energy spectral densities are rescaled non-linearly thus intensifying spectral magnitudes at higher wavenumbers. Due to the inherent normalization with magnitude, the PHAT on the other hand places equal emphasis on each frequency of \( F_1 \), retaining purely phase information. From a practical perspective, as the particle self-image correlation falls off to zero at higher wavenumbers the convolution in \( <R_D(\xi(\mathbf{u}))> \) becomes a surjective mapping impeding the idealistic and unique retrieval of \( 3F_1 \) as suggested in (23). Realistic PHAT performances will therefore remain dependent on \( A_\text{on} \) values.

Figure 8 compares the ideal correlation \( R_{\text{PID}} \) response (14) with results obtained from (21) and (23). Both SPOF and PHAT filters are analyzed in addition to regular FFT implementation (i.e. \( \psi(\mathbf{k}) = 1 \)). In all cases interrogation areas are zero-padded to avoid intensity wraparound aliasing and correlations are weighted by \( F_1 \) to eliminate velocity bias errors [15]. To improve sub-pixel displacement estimations and minimize peak-locking a Gaussian transform filter of 1 pixel diameter is utilized when PHAT filtering as proposed in [25].

In practice the evaluation of the Fourier transforms is based on discrete samples, which introduces nonconformities between (19) and approaches implementing the FFT algorithm. With the displacement field in the best case sampled every pixel, the Nyquist criterion imposes a maximum measureable frequency of \( 0.5 \text{pix}^{-1} \) while \( F_1 \) of a sinusoidal displacement spans an infinite frequency range. The unavoidable spectral leakage will ultimately affect the retrieved correlation when taking the inverse Fourier transform with discrepancies inversely proportional to \( A_\text{on} \). (Note that in section 3 discretization was identified as a dominant factor in amplitude modulation, cf. figure 3b.) The above suppositions are confirmed by the results in figures 3-b and 8. For a fixed image diameter of 3 pixels and sufficiently large \( A_\text{on} \), FFT performances are in very good agreement with those obtained by direct cross-correlation viz. (18). PHAT shows a response closely resembling the ideal correlation in absence of pixelization. At low \( A_\text{on} \) however all FFT-based correlation performances are inferior to the direct approach.
The Gaussian transform filter implemented in PHAT involves an additional convolution in the correlation defined in (21). Consequently, the equivalent image diameter is enlarged to \((\alpha d)^{eq} = (1 + (\alpha d \tau)^2)^{\frac{1}{2}}\) reducing the effective \(A_{on}\). Indeed analytic curves for \(A_{on}\) based on the equivalent image diameters \((A_{on,eq} = 0.62 \cdot A_{on}\) for \(d = 3\) pixels) clearly provide a better estimate of the PHAT performances (figure 7b) nearly identical to the ideal response absent of pixelization effects.

**Figure 8:** Ratio between correlation response \(s_0\) (i.e. measured displacement amplitude) to a sinusoid displacement of amplitude \(A_{on}\) in function of interrogation window size (centred on the peak) for different normalized amplitudes, \(d = 3\) pixels, \(p_f = 0.7\). Dashed or dotted black lines: ideal response based on (14). Gray lines: based on (18) (equivalent to (19)). Symbols: synthetic images evaluated with (a) FFT (open), SPOF (filled) (b) PHAT. The solid blue line represents the MA response.

**VII. CONCLUSIONS**

In this paper an analytic expression is derived describing the PIV image cross-correlation map for the case of a one-dimensional sinusoidal displacement field. Assessments using computer generated PIV images have shown very good agreement with analytical expressions validating the derived equations.

It is shown that contrary to widespread acceptance, the cross-correlation response is heavily influenced by the ratio between displacement amplitude and image diameter, referred hereafter by \(A_{on}\). Simulations support the latter by depicting the tendency of the retrieved displacement towards a square-like displacement field in case of large \(A_{on}\). Only for sufficiently small ratios does the correlation response tend to that of a moving average filter.

It is shown that the non-linear response leads to a deterioration in the quality of the measured displacement and can lead to instabilities in case of multi-pass algorithms even for normalized window sizes below \(~1.4\) depending on the imposed window overlap ratio. This analysis stresses the importance of proper flow sampling.

Spectral filtering techniques such as Phase Only Filtering and Symmetric Phase Only Filtering have been studied in view of their capacity to minimize or remove the displacement modulation. The symmetric phase-only filter has been shown to improve the response by minimizing particle image and pixelization-induced modulation. The inherent necessity in PHAT to convolute with a Gaussian as to minimize peak-locking effects only worsened the results.

The conducted analyses further imply the best possible correlation response to a sinusoidal displacement, even considering spectral filtering, to be defined by the convolution between the displacement distribution and particle image self-correlation. Amplitude modulation caused by the particle image self-correlation is consequently inevitable.

**REFERENCES**

