REGULARITY OF PAIRS OF POSITIVE OPERATORS

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0. Introduction

In this paper, we consider a pair \((A, B)\) of closed operators on a Banach space \(X\) with domain \(D(A)\) and \(D(B)\). The pair \((A, B)\) is called regular if for every \(f \in X\), the problem \(Au + Bu = f\) possesses one and only one solution.

Related to the notion of coercively positive pair of operators, introduced in [S], we also consider the existence of a solution to the problem \(\lambda Au + Bu = f\) for all \(\lambda > 0\), with some uniformity in \(\lambda\). This stronger property is called \(\lambda\)-regularity.

These notions of regularity and \(\lambda\)-regularity naturally arise in vector-valued Cauchy problems; see [G], [DG], [S] and also [CD]. The uniformity in \(\lambda\), given by the \(\lambda\)-regularity, is often useful in certain applications to partial differential equations.

In [G], under the hypothesis that \(0 \in \rho(B)\) and in [DG], some sufficient conditions are given to ensure the regularity of a pair \((A, B)\) on certain subspaces of \(X\), related to the operator \(B\). These subspaces, denoted by \(D_B(\theta, p)\), are real interpolation spaces between \(D(B)\) and \(X\) (Theorem 1.2).

It was observed in [S] that if \(0 \in \rho(A) \cap \rho(B)\), then the pair is \(\lambda\)-regular on \(D_B(\theta, p)\).

In this paper, we prove the \(\lambda\)-regularity of this pair \((A, B)\), considered in [G], on \(D_B(\theta, p)\) under the weaker assumption that \(0 \in \rho(B)\) only (Theorem 2.1). Note that if \(B\) is bounded, then the pair is \(\lambda\)-regular on \(X\).

We construct an example of a regular pair \((A, B)\) of operators in a Hilbert space, with \(B\) bounded, satisfying the assumptions of the theorem of Grisvard [G], which is not \(\lambda\)-regular (Example 2.2).

1. Preliminaries

In this section we give precise definitions of regularity and \(\lambda\)-regularity of a pair of operators. Then, for the sake of completeness, we recall a result of Da Prato and Grisvard [DG] (see also [CD]), which is the starting point of our results.

Let \(X\) be a Banach space and \(A\) and \(B\) be two closed operators in \(X\).

**Definition 1.** The pair \((A, B)\) is called regular, if for all \(f \in X\), there exists a unique \(u \in D(A) \cap D(B)\) such that \(Au + Bu = f\).
If the pair \((A, B)\) is regular, it follows from the Banach theorem that

\[
\|u\| + \|Au\| + \|Bu\| \leq M\|Au + Bu\|
\]

for some \(M \geq 1\) and for all \(u \in D(A) \cap D(B)\).

It is easy to verify the following lemma.

**Lemma 1.0.** Let \(A\) and \(B\) be two closed operators in \(X\). Then the pair \((A, B)\) is regular if and only if

1. \((1.0)\) holds and
2. \(R(A + B)\) is dense in \(X\).

Moreover, if \(0 \notin \rho(A)\) or \(\rho(B)\) (where \(\rho(.)\) denotes the resolvent set of an operator), then \((1.0)\) is equivalent to

\[
\|Au\| + \|Bu\| \leq M\|Au + Bu\|
\]

for some \(M \geq 1\) and for all \(u \in D(A) \cap D(B)\).

**Remark 1.** The operator \(A + B\) is closed if and only if

\[
\|u\| + \|Au\| + \|Bu\| \leq M(\|Au + Bu\| + \|u\|)
\]

for some \(M \geq 1\) and for all \(u \in D(A) \cap D(B)\).

In particular, if the pair \((A, B)\) is regular, \(A + B\) has to be closed.

A regular pair of operators \((A, B)\) is called coercive in \([S]\).

Also, the stronger notion of coercively positive pair is introduced in \([S]\), which motivates our Definition 2.

**Definition 2.** The pair \((A, B)\) is called \(\lambda\)-regular in \(X\), if for all \(f \in X\) and for all \(\lambda > 0\), there exists a unique \(u \in D(A) \cap D(B)\) such that \(\lambda Au + Bu = f\) and moreover, for all \(\lambda > 0\),

\[
\|\lambda Au\| + \|Bu\| \leq M\|\lambda Au + Bu\|
\]

for some \(M \geq 1\), independent of \(\lambda\) and for all \(u \in D(A) \cap D(B)\).

**Remark 2.** Clearly if \((1.1)\) holds, then the inequality

\[
\lambda\|Au\| + \mu\|Bu\| \leq M\|\lambda Au + \mu Bu\|
\]

holds for some \(M \geq 1\), for all \(\lambda, \mu > 0\) and \(u \in D(A) \cap D(B)\), which shows that the definition of \(\lambda\)-regularity is symmetric in \(A\) and \(B\).
It is also clear that this inequality is equivalent to the following ones:

\[ \|Au\| \leq M \|Au + \lambda Bu\|, \]

for some \( M \geq 1 \) and all \( \lambda > 0 \) and \( u \in D(A) \cap D(B) \), and

\[ \lambda \|Bu\| \leq M \|Au + \lambda Bu\| \]

for some \( M \geq 1 \) and all \( \lambda > 0 \) and \( u \in D(A) \cap D(B) \).

**Lemma 1.0.λ.** Let \( A \) and \( B \) be two closed operators in \( X \) (not necessarily densely defined). If \( 0 \in \rho(A) \), then the pair \((A, B)\) is \( \lambda \)-regular if and only if:

1. \((1.1)_\lambda\) holds for all \( \lambda > 0 \);
2. There exists \( \lambda_0 > 0 \) such that \( R(\lambda_0 A + B) \) is dense in \( X \).

**Proof.** Clearly, it is enough to prove that conditions (1) and (2) imply that the pair \((A, B)\) is \( \lambda \)-regular.

First observe that conditions (1) and (2) together with Lemma 1.0, where \( A \) is replaced by \( \lambda_0 A \), and the fact that \( 0 \in \rho(A) \), imply that the pair \((\lambda_0 A, B)\) is regular. Thus, in particular, \( 0 \in \rho(\lambda_0 A + B) \).

Next we show that if \( 0 \in \rho(\lambda A + B) \) for some \( \lambda_1 > 0 \), then \( 0 \in \rho(\lambda A + B) \) for all \( \lambda > 0 \) such that

\[ \frac{\lambda}{\lambda_1} \in \left( \frac{M}{M + 1}, \frac{M}{M - 1} \right) \text{ if } M > 1 \text{ and } \left( \frac{M}{M + 1}, \infty \right) \text{ if } M = 1. \]

Indeed, problem \( \lambda Au + Bu = f \) is equivalent to

\[ \lambda_1 Au + Bu = \left( 1 - \frac{\lambda_1}{\lambda} \right) Bu + \frac{\lambda_1}{\lambda} f. \]

Setting \( v = \lambda_1 Au + Bu \), we have

\[ v = \left( 1 - \frac{\lambda_1}{\lambda} \right) B(\lambda_1 A + B)^{-1} v + \frac{\lambda_1}{\lambda} f. \]

From \((1.1)_\lambda\), it follows that

\[ \|B(\lambda_1 A + B)^{-1}\| \leq M. \]

Under assumption \((*)\), by the Banach fixed point theorem, it is clear that there exists one and only one \( v \in X \) satisfying \((**)\) and hence \((\lambda A, B)\) is a regular pair for such \( \lambda \). Noting that \( \|B(\lambda_1 A + B)^{-1}\| \leq M \) also holds for \( \lambda \) in this interval, we can repeat this argument and, since \( \frac{M}{M + 1} < 1 \) and \( \frac{M}{M - 1} > 1 \), show by induction that the pair \((\lambda A, B)\) is regular for all \( \lambda > 0 \), which together with \((1.1)_\lambda\) implies that the pair \((A, B)\) is \( \lambda \)-regular. This finishes the proof of Lemma 1.0.λ. \( \Box \)
Let us recall classical definitions on closed operators: A closed linear operator $A : D(A) \subset X \to X$ (not necessarily densely defined) is called \textit{positive} in $(X, \| \cdot \|)$ [Tr] if there exists $C > 0$ such that

\begin{equation}
\| u \| \leq C \| u + \lambda Au \|, \text{ for every } \lambda > 0 \text{ and } u \in D(A),
\end{equation}

and if $R(I + \lambda A) = X$ for some $\lambda > 0$, equivalently for all $\lambda > 0$.

\textbf{Remark 3.} In [Tr], an operator $A$ is called positive if it is positive and satisfies the additional assumption that $0 \in \rho(A)$. In this paper, it is convenient to relax this extra condition.

Observe also that $A$ is positive if and only if the pair $(A, I)$ is $\lambda$-regular.

If $A$ is positive, injective and densely defined, it is easy to prove that $A^{-1}$ is also positive.

If $X$ is reflexive and $A$ is positive, then $A$ is densely defined [K].

Let $\Sigma_\sigma := \{ \lambda \in \mathbb{C} \setminus \{ 0 \}; |\arg \lambda| \leq \sigma \} \cup \{ 0 \}$, for $\sigma \in [0, \pi)$. If $A$ is positive, there exists $\theta \in [0, \pi)$ such that (1.3) holds, [K p. 288]:

\begin{equation}
(\text{i}) \sigma(A) \subseteq \Sigma_\sigma \quad \text{and} \quad (\text{ii}) \text{ for each } \theta' \in (\theta, \pi], \text{ there exists } M(\theta') \geq 1 \text{ such that } \| \lambda(\lambda I - A)^{-1} \| \leq M(\theta'), \text{ for every } \lambda \in \mathbb{C} \setminus \{ 0 \} \text{ with } |\arg \lambda| \geq \theta'
\end{equation}

where $\sigma(A)$ denotes the spectrum of $A$.

The number $\omega_A := \inf\{ \theta \in [0, \pi); (1.3) \text{ holds}\}$ is called the \textit{spectral angle} of the operator $A$. Clearly $\omega_A \in [0, \pi)$.

An operator $A$ is said to be of \textit{type} $(\omega, M)$ [Tan], if $A$ is positive, $\omega$ is the spectral angle of $A$ and

\[ M := \inf\{ C \geq 0; (1.2) \text{ holds } \} = \min\{ C \geq 0; (1.2) \text{ holds } \}. \]

Note that $M$ is also the smallest constant in (1.3) ii) for $\theta' = \pi$.

Two positive operators $A$ and $B$ in $X$ are said to be \textit{(resolvent) commuting} if the bounded operators $(I + \lambda A)^{-1}$ and $(I + \mu B)^{-1}$ commute for some $\lambda, \mu > 0$, equivalently for all $\lambda, \mu > 0$.

If $A$ and $B$ are commuting positive operators then $A + B$ (with domain $D(A) \cap D(B)$) is closable [DG].

The following theorem, which is a consequence of a theorem of Da Prato-Grisvard [DG] and of Grisvard [G] will be essential in the sequel.

\textbf{THEOREM 1.1.} \textit{Let $A$ and $B$ be two commuting positive operators in $X$ such that}

\begin{enumerate}
  \item[(i)] $D(A) + D(B)$ is dense in $X$,
  \item[(ii)] $\omega_A + \omega_B < \pi$.
\end{enumerate}
Then the closure of $A + B$ is of type $(\omega, M)$ with $\omega \leq \max(\omega_A, \omega_B)$. If moreover

(iii) $0 \in \rho(A)$ or $\rho(B)$ (resolvent set of $A$ or $B$), then
(a) there exists $M \geq 1$ such that

(1.4) \[ \|u\| \leq M\|Au + Bu\|, \text{ for all } u \in D(A) \cap D(B), \]

and $0 \in \rho(A + B)$,
(b) $R(A + B) \supseteq D(A) + D(B)$,
(c) $A + B$ is closed if and only if $R(A + B) = X$ if and only if (1.1) holds,
(d) the inverse of $A + B$ is given by

\[ (A + B)^{-1}x = \frac{1}{2\pi i} \int_{\gamma} (A + z)^{-1}(B - z)^{-1}x \, dz, \]

where $\gamma$ is any simple curve in $\rho(B) \cap \rho(-A)$ from $\infty e^{-i\theta_0}$ to $\infty e^{i\theta_0}$, with $\omega_B < \theta_0 < \pi - \omega_A$.

**Remark 4.** (1) Under hypotheses (i)–(iii) of Theorem 1.1, assumption 2) of Lemma 1.0 is always satisfied. Therefore, in order to prove the regularity of a pair $(A, B)$, it is sufficient to verify inequality (1.1), which means that $A(A + B)^{-1}$ is a bounded operator.

(2) Similarly, under hypotheses (i)–(iii) of Theorem 1.1, assumption (2) of Lemma 1.0 is always satisfied. Therefore, in order to prove the $\lambda$-regularity of a pair $(A, B)$, it is sufficient to verify inequality (1.1)$_\lambda$, which means that $\lambda A(\lambda A + B)^{-1}$ is a uniformly bounded operator for all $\lambda > 0$.

In this paper, we shall always be in the situation of (i)–(ii) of Theorem 1.1, which means that we will consider the following three hypotheses for a pair of positive operators $A$ and $B$ in $X$ of type respectively $(\omega_A, M_A)$ and $(\omega_B, M_B)$:

$H_0$: $D(A) + D(B)$ is dense in $X$.
$H_1$: $A$ and $B$ are resolvent commuting.
$H_2$: $\omega_A + \omega_B < \pi$.

In order to obtain results on the regularity and the $\lambda$-regularity of a pair of operators, we need to introduce the interpolation spaces $D_A(\theta, p)$, associated with a closed operator $A$, for $\theta \in (0, 1)$ and $p \in [1, +\infty]$. These spaces are subspaces of $X$ which are dense in $X$ for the norm $\|\|\|$ whenever $A$ is densely defined.

For $\theta \in (0, 1)$ and $p \in [1, +\infty)$, $D_A(\theta, p)$ is the subspace of $X$ consisting of all $x$ such that

\[ \|t^\theta A(A + t)^{-1}x\| \in L^p_x, \]
where $L^p_+$ is the space of $p$-integrable Borel functions on $(0, +\infty)$ equipped with its invariant measure $dt/t$.

For $\theta \in [0, 1]$, $D_A(\theta, \infty)$ is the subspace of $X$ consisting of all $x \in X$ such that

$$\sup\{\|t^\theta A(A + t)^{-1} x\| \mid t \in (0, +\infty)\} < +\infty.$$ 

When $0$ belongs to $\rho(A)$, $D_A(\theta, p)$ equipped with the norm

$$\|x\|_{D_A(\theta, p)} = \|t^\theta A(A + t)^{-1} x\|_{L^p_+}$$ 

becomes a Banach space.

When $0 \in \rho(A)$ and $A$ is bounded, $\|\cdot\|_{D_A(\theta, p)}$ is equivalent to the norm of $X$.

The following fundamental result, due to Grisvard (Theorem 2.7 of [G]) is the starting point of this paper.

**THEOREM 1.2.** Let $X$ be a complex Banach space, and let $A$ and $B$ be two positive operators in $X$, of type $(\omega_A, M_A)$ and $(\omega_B, M_B)$ respectively, satisfying hypotheses $H_0$, $H_1$, $H_2$.

If $0 \in \rho(B)$, the pair $(A, B)$ is regular in $D_B(\theta, p)$.

**2. Results**

The first result of this paper is the following theorem which is an extension of Theorem 1.2 to the case of $\lambda$-regularity.

**THEOREM 2.1.** Let $X$ be a complex Banach space, and let $A$ and $B$ be two positive operators in $X$, of type $(\omega_A, M_A)$ and $(\omega_B, M_B)$ respectively, satisfying hypotheses $H_0$, $H_1$, $H_2$. If $0 \in \rho(B)$, the pair $(A, B)$ is $\lambda$-regular in $D_B(\theta, p)$ for every $0 < \theta < 1$ and $1 \leq p \leq \infty$.

**Remark 5.** If moreover $B$ is bounded, it is clear that the pair $(A, B)$ is $\lambda$-regular in $X$.

The next example shows that in particular, even if $X$ is a Hilbert space, the hypothesis $0 \in \rho(B)$ cannot be omitted in Theorem 2.1.

**Example 2.2.** There exists a Hilbert space $G$ and there exist two positive operators $A$ and $B$ in $G$ satisfying hypotheses $H_0$, $H_1$ and $H_2$, with $B$ bounded, such that the pair $(A, B)$ is regular, but not $\lambda$-regular in $G$.

**Remark 6.** In [L, Theorem 2.4] (see also [CD]), another example is given, where $A$ is the derivative acting on $L^p([0, T]; Y)$ for some non reflexive space $Y$, such that the pair $(A, B)$ is not $\lambda$-regular in $D_A(\theta, p)$. 

Proof of Theorem 2.1. Fix \( \lambda > 0 \). By Theorem 1.2, we know that the pair \((A, \lambda B)\) is regular in \( D_B(\theta, p) \). In particular, for all \( x \in D_B(\theta, p) \),

\[
y_\lambda = (A + \lambda B)^{-1}x \in D(A) \cap D(B)
\]
and we have \( B y_\lambda \in D_B(\theta, p) \) together with the inequality

\[
\| \lambda B y_\lambda \|_{D_B(\theta, p)} \leq C \| x \|_{D_B(\theta, p)}.
\]

We shall show that \( C \) is independent of \( \lambda \). For this, we are going to use equality (\( \ast \)) of Theorem 1.1, applied to \( A \) and \( \lambda B \). Without loss of generality, since \( 0 \in \rho(B) \), we can suppose that \( \gamma \) consists of the half line \((\infty e^{-\theta_0}, \infty e^{i\theta_0})\), the arc of the circle \( C_\varepsilon = \{ z : |z| = \varepsilon, |\arg(z)| \leq \theta_0 \} \) and the half line \([\varepsilon e^{i\theta_0}, \infty e^{i\theta_0})\), for some fixed \( \theta_0, \omega_B < \theta_0 < \pi - \omega_A \) and for sufficiently small \( \varepsilon \) in order to insure that \( \gamma \) is in \( \rho(-A) \cap \rho(\lambda B) \). Since \( A \) is of type \((\omega_A, M_A)\), by (1.3) there exists \( M'_A \) such that for all \( z \) such that \( |\arg z| \leq \theta_0 \),

\[
\| (A + z)^{-1} \| \leq \frac{M'_A}{|z|}.
\]

As in the proof of Theorem 3.11 of [DG], for every \( t > 0 \) we can write

\[
(\lambda B + t)^{-1}y_\lambda = (\lambda B + t)^{-1}(A + \lambda B)^{-1}x
\]

\[
= \frac{1}{2\pi i} \int_y (A + z)^{-1}(\lambda B + t)^{-1}(\lambda B - z)^{-1}x \, dz
\]

by (\( \ast \)) and \( H_1 \)

\[
= \frac{1}{2\pi i} \int_y (A + z)^{-1}(\lambda B - z)^{-1}x \frac{dz}{t + z} - \frac{1}{2\pi i} \int_y (A + z)^{-1}(\lambda B + t)^{-1}x \frac{dz}{t + z}
\]

\[
= \frac{1}{2\pi i} \int_y (A + z)^{-1}(\lambda B - z)^{-1}x \frac{dz}{t + z} - (\lambda B + t)^{-1} \frac{1}{2\pi i} \int_y (A + z)^{-1}x \frac{dz}{t + z}
\]

by analyticity of the function \( \frac{(A+z)^{-1}}{t+z} \) and the fact that \( \| \frac{(A+z)^{-1}}{t+z} \| \leq \frac{M'_A}{|z(t+z)|} \) for \( |\arg z| \leq \theta_0 \).

Hence

\[
\lambda B(\lambda B + t)^{-1}y_\lambda = y_\lambda - t(\lambda B + t)^{-1}y_\lambda = (A + \lambda B)^{-1}x - t(\lambda B + t)^{-1}(A + \lambda B)^{-1}x
\]
\[
\lambda B(\lambda B + t)^{-1} y_\lambda = \frac{1}{2\pi i} \int_{C} \frac{z}{z + t} (A + z)^{-1}(\lambda B - z)^{-1} x \, dz.
\]

First, we claim that
\[
\lim_{\varepsilon \to 0^+} \int_{C'} \frac{z}{z + t} (A + z)^{-1}(\lambda B - z)^{-1} x \, dz = 0.
\]

Since \(B\) is invertible, \(\|(\lambda B - z)^{-1}\|\) is uniformly bounded with respect to \(z\) in a neighborhood of the origin. So there exists \(\varepsilon_0\) such that \(\|(\lambda B - z)^{-1}\| \leq 2\|(\lambda B)^{-1}\|\) for \(|z| \leq \varepsilon_0\). We can suppose that \(\varepsilon_0 \leq \frac{1}{2}\). Then, for \(\varepsilon \leq \varepsilon_0\) we have
\[
\left\| \int_{C'} \frac{z}{z + t} (A + z)^{-1}(\lambda B - z)^{-1} x \, dz \right\| \\
\leq \int_{C'} \frac{|z|}{|z + t|} \|(A + z)^{-1}\| \|(\lambda B - z)^{-1}\| \|x\| |dz| \\
\leq 2M_A'\|(\lambda B)^{-1}\|\|x\| \varepsilon \int_{-\theta_0}^{\theta_0} \frac{d\theta}{t + \varepsilon \cos \theta} \leq \frac{8M_A'\|(\lambda B)^{-1}\|\|x\|\varepsilon\theta_0}{t}
\]
which tends to zero when \(\varepsilon \to 0^+\). The claim is proved; hence we have
\[
\lambda B(\lambda B + t)^{-1} y_\lambda = \frac{1}{2\pi i} \int_{\gamma_0} \frac{z}{z + t} (A + z)^{-1}(\lambda B - z)^{-1} x \, dz
\]
where \(\gamma_0\) consists of the half-lines \(\{z : \arg(z) = -\theta_0\}\) and \(\{z : \arg(z) = \theta_0\}\).

By hypotheses \(H_1\) and \(H_2\),
\[
\lambda B(\lambda B + t)^{-1} \lambda B y_\lambda = \frac{1}{2\pi i} \int_{\gamma_0} \frac{z}{z + t} (A + z)^{-1}\lambda B(\lambda B - z)^{-1} x \, dz
\]
and so
\[
\|\lambda B(\lambda B + t)^{-1} \lambda B y_\lambda\| \\
\leq \frac{1}{2\pi} \int_{\gamma_0} \frac{|z|}{|z + t|} \|(A + z)^{-1}\| \|\lambda B(\lambda B - z)^{-1} x\| |dz| \\
\leq K \int_{0}^{+\infty} \frac{r}{\sqrt{t^2 + r^2 + 2t\varepsilon\cos \theta_0}} \phi_\lambda(r) \frac{dr}{r}
\]
where $K$ is a constant depending only on $A$ and $B$, and

$$
\phi_\lambda(r) = \max\{\|\lambda B(\lambda B - re^{i\theta_0})^{-1} x\|, \|\lambda B(\lambda B - re^{-i\theta_0})^{-1} x\|\} = \phi_\lambda \left( \frac{r}{\lambda} \right).
$$

The hypothesis $x \in D_B(\theta, p)$ means that $r^\theta \phi_\lambda(t) \in L^p_\pm(R^+)$ (see [DG]); thus we have

$$
t^\theta \|\lambda B(\lambda B + t)^{-1} \lambda B y_\lambda\| \\
\leq K \int_0^{+\infty} \frac{r t^\theta}{\sqrt{1 + r^2 + 2tr\cos\theta_0}} \phi_\lambda(r) \frac{dr}{r} = K \int_0^{+\infty} \frac{(rt^{-1})^{1-\theta}}{\sqrt{1 + (rt^{-1})^2 + 2rt^{-1}\cos\theta_0}} r^\theta \phi_\lambda(r) \frac{dr}{r}
$$

where

$$
f(t) = \frac{t^{1-\theta}}{\sqrt{1 + t^2 + 2t\cos\theta_0}} \in L^\theta_\pm(R^+)
$$

$$
g(t) = t^\theta \phi_\lambda(t) \in L^p_\pm(R^+)
$$

By Young’s theorem, we can write

$$
\|t^\theta \lambda B(\lambda B + t)^{-1} \lambda B y_\lambda\|_{L^p_\pm(R^+)} \\
\leq K \|f\|_{L^\theta_\pm(R^+)} \|g\|_{L^p_\pm(R^+)} \\
\leq K' \left( \int_0^{+\infty} (r^\theta \phi_\lambda(r))^p \frac{dr}{r} \right)^{1/p} \\
= K' \lambda^\theta \left( \int_0^{+\infty} (r^\theta \phi_1(r))^p \frac{dr}{r} \right)^{1/p} \\
\leq K'' \lambda^\theta \|x\|_{D_B(\theta, p)}
$$

where $K''$ is a constant depending only on $A$ and $B$, see [DG]. On the other hand,

$$
\|t^\theta \lambda B(\lambda B + t)^{-1} \lambda B y_\lambda\|_{L^p_\pm(R^+)} \\
= \left( \int_0^{+\infty} (t^\theta \|\lambda B(\lambda B + t)^{-1} \lambda B y_\lambda\|)^p \frac{dt}{t} \right)^{1/p} \\
= \lambda^{1+\theta} \left( \int_0^{+\infty} (t^\theta \|B(B + t)^{-1} B y_\lambda\|)^p \frac{dt}{t} \right)^{1/p} \\
= \lambda^{1+\theta} \|B y_\lambda\|_{D_B(\theta, p)};
$$
hence
\[ \lambda^0 \| \lambda B y_\lambda \|_{D_B(\theta, p)} \leq K'' \lambda^0 \| x \|_{D_B(\theta, p)} \]
or
\[ \| \lambda B (A + \lambda B)^{-1} x \|_{D_B(\theta, p)} \leq K'' \| x \|_{D_B(\theta, p)}. \]

This is the inequality that we wanted. It implies that
\[ \| \lambda B (A + \lambda B)^{-1} \|_{D_B(\theta, p)} \leq K'', \]
which shows the \( \lambda \)-regularity of the pair \((A, B)\) on \( D_B(\theta, p) \) by Remark 4.2. \( \square \)

Let us mention another case of \( \lambda \)-regularity which is a consequence of Theorem 1.2 applied in the context of [DV], namely when \( B^s \) is bounded for all \( s \in [-1, +1] \):

**Corollary 2.3.** Let \( H \) be a Hilbert space and let \( A \) and \( B \) be two positive operators in \( H \) satisfying \( H_0 \), \( H_1 \) and \( H_2 \). If \( 0 \leq p(B) \) and \( \sup \{ \| B^s \| \mid |s| \leq 1 \} < +\infty \), then the pair \((A, B)\) is \( \lambda \)-regular in \( H \).

**Proof of Corollary 2.3.** As mentioned in [DV], under the hypothesis that \( \sup \{ \| B^s \| \mid |s| \leq 1 \} < +\infty \), \( D_B(\theta, 2) = D(B^\theta) \). Thus Theorem 2.1 implies that \((A, B)\) is a \( \lambda \)-regular pair in \( D(B^\theta) \). Then Dore and Venni show that, under the hypothesis of Corollary 2.3, \((A, B)\) is a regular pair in \( H \). An adaptation of their proof can be done to prove that in fact, the pair is \( \lambda \)-regular. Indeed, for \( x \in H \), by Theorem 2.1, observing that \( B^{-\theta} x \in D_B(\theta, 2) \), we have
\[ \| \lambda B (A + \lambda B)^{-1} x \| = \| B^\theta \lambda B (A + \lambda B)^{-1} B^{-\theta} x \| \leq C \| B^\theta B^{-\theta} x \| = C \| x \| \]
where \( C > 0 \) is independent of \( \lambda > 0 \). \( \square \)

**Construction of Example 2.2.** Let \( G \) be a complex Hilbert space and let \( A \) and \( B \) be two positive operators with \( B \) bounded, satisfying hypotheses \( H_1 \) and \( H_2 \). Observe that since \( B \) is bounded, \( H_0 \) is also satisfied. If moreover \( 0 \in \rho(A) \), then by Theorem 1.1, the pair \((A, B)\) is regular and \( G = D_B(\theta, p) \) for every \( \theta \in (0, 1) \) and \( p \in [1, \infty] \). Hence if the pair \((A, B)\) is not \( \lambda \)-regular, we are done.

In order to construct such a pair, we consider, as in [BC], the space
\[ G = \ell_2(H) = \left\{ x = (x_k)_{k \in \mathbb{N}} \mid x_k \in H \text{ and } \| x \|^2 = \sum_{k=1}^{+\infty} \| x_k \|^2 < +\infty \right\} \]
where \((H, \|\cdot\|)\) is a complex Hilbert space. A family \((A_k)_{k \in \mathbb{N}}\) of bounded operators on \(H\) defines the following closed densely defined operator \(A\) on \(G:\)

\[
D(A) := \{ x = (x_k)_{k \in \mathbb{N}} , \ x_k \in H , \ \sum_{k \in \mathbb{N}} \|A_k x_k\|^2 < \infty \}
\]
\[
(A x)_k := A_k x_k , \ k \in \mathbb{N} \text{ for } x = (x_k)_{k \in \mathbb{N}} \in D(A).
\]

Moreover \(A\) is bounded if and only if \(\sup_{k \in \mathbb{N}} \|A_k\| < \infty\) and if this is the case, we have \(\|A\| = \sup_{k \in \mathbb{N}} \|A_k\|\).

If \(0 \in \rho(A_k)\) for all \(k \in \mathbb{N}\) and \(\sup_{k \in \mathbb{N}} \|A_k^{-1}\| < \infty\), then \(0 \in \rho(A)\). As in [BC], we shall say that the family of positive operators \((A_k)_{k \in \mathbb{N}}\) of type \((0, M_k)\) satisfies property \((P)\) if for every \(k \in \mathbb{N},\)

(i) \(\sigma(A_k) \subset [0, \infty)\) and

(ii) for every \(\theta \in [0, \pi[,\) there is \(M(\theta)\), independent of \(k\), such that \(\|(I + zA_k)^{-1}\| \leq M(\theta)\), for every \(z \in \Sigma_\theta\).

We will need the following slight extension of Lemma 4.1 of [BC], which we state without proof.

**Lemma 2.4.** Let \((A_k)_{k \in \mathbb{N}}, (B_k)_{k \in \mathbb{N}}\) be two families of bounded positive operators on \(H\), satisfying property \((P)\) and such that \(A_k B_k = B_k A_k\) for all \(k \in \mathbb{N}\). Then the operators \(A\) and \(B\) defined by (2.1) are densely defined and of type \((0, M_A)\) and \((0, M_B)\) respectively. Moreover, the pair \((A, B)\) satisfies hypotheses \(H_0, H_1, H_2\).

Now suppose that \((A_k)_{k \in \mathbb{N}}\) and \((\tilde{B}_k)_{k \in \mathbb{N}}\) are two families of operators in \(H\) as in Lemma 2.4 satisfying (2.2) and (2.3):

\[
0 \in \rho(A_k) \text{ for every } k \in \mathbb{N} \text{ and } \sup_{k \in \mathbb{N}} \|A_k^{-1}\| < \infty
\]

\[
\forall l \geq 1 , \ \exists x_l \in H , \ \|x_l\| = 1 , \ \text{ such that } l \|A_l x_l + \tilde{B}_l x_l\| \leq \|A_l x_l\|.
\]

Set \(B_k = \mu_k \tilde{B}_k\), with \(\mu_k > 0\), \(k \in \mathbb{N}\) such that \(\|B_k\| \leq 1\) for all \(k \in \mathbb{N}\). Then the families \((A_k)_{k \in \mathbb{N}}\) and \((B_k)_{k \in \mathbb{N}}\) also satisfy the assumptions of Lemma 2.4. The pair \((A, B)\) defined by (2.1) satisfies \(H_0, H_1, H_2\). Moreover \(0 \in \rho(A)\) by (2.2) and \(B\) is bounded with \(\|B\| \leq 1\).

We claim that the regular pair \((A, B)\) is not \(\lambda\)-regular. Clearly for every \(\lambda > 0\), the pair \((A, \lambda B)\) is regular and if \((A, B)\) is \(\lambda\)-regular, then there exists \(M \geq 1\), independent of \(\lambda\) such that for all \(y \in G\),

\[
\|A(A + \lambda B)^{-1} y\| \leq M \|y\|
\]

Choose \(y = y^{(l)} = (y_k^{(l)})_{k \in \mathbb{N}}\) with

\[
y_k^{(l)} = 0 \text{ for } k \neq l
\]

\[
y_l^{(l)} = (A_l + \tilde{B}_l)x_l, \ l \in \mathbb{N}.
\]
Hence with $\lambda = \mu_l^{-1}$, from (2.4) we obtain

\begin{equation}
M \| (A_l + \tilde{B}_l) x_l \| \geq \| A_l x_l \| \geq l \| (A_l + \tilde{B}_l) x_l \|
\end{equation}

for every $l \in \mathbb{N}$, a contradiction since $\| (A_l + \tilde{B}_l) x_l \| \neq 0$.

It remains to construct the operators $A_l$ and $\tilde{B}_l$. For this purpose, we shall need the following lemma, which can be essentially found in [BC].

**Lemma 2.5.** Let $H$ be a complex separable Hilbert space with a Schauder basis $(e_n)_{n \in \mathbb{N}}$ and let $(e_n^*)_{n \in \mathbb{N}}$ be the corresponding coordinate functionals. Let $(c_n)_{n \in \mathbb{N}}$ be a nondecreasing sequence of positive real numbers and let $C_k$ be the linear operators defined by

\begin{equation}
C_k x := \sum_{l=0}^{N_l} c_l e_l^* (x) e_k
\end{equation}

where $N_k \in \mathbb{N}$ for all $k \in \mathbb{N}$.

Then the operators $C_k$ are bounded positive operators of type $(0, M_k)$ satisfying property (P). Moreover, $0 \in \rho(C_k)$ for all $k \in \mathbb{N}$ and $\sup_{k \in \mathbb{N}} \| C_k^{-1} \| < \infty$.

In view of this lemma, if $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are two nondecreasing sequences of positive numbers and $A_k$, $\tilde{B}_k$ are defined by (2.6) where $(N_k)_{k \in \mathbb{N}}$ is an arbitrary sequence of natural numbers, then the operators $A_k$, $\tilde{B}_k$ satisfy all required properties except (2.3). In order to satisfy this condition, we choose for $(e_n)_{n \in \mathbb{N}}$ a conditional basis of $\ell_2$ as in [BC] and we choose for $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ the sequences denoted by $f(n)$ and $g(n)$ in [BC], having the property that

$$\sup_{x \in G_0, \| x \|=1} \left\| \sum_{k=0}^{\infty} \frac{a_k}{a_k + b_k} e_k^* (x) e_k \right\| = \infty$$

where $G_0 = \text{span}\{e_n \mid n \in \mathbb{N}\}$. It follows that for every $l \in \mathbb{N}$, there exists $N_l \in \mathbb{N}$ and $\alpha_{k,l} \in \mathbb{C}$ for $0 \leq k \leq l$ such that

$$\left\| \sum_{k=0}^{N_l} \frac{a_k}{a_k + b_k} e_k^*(y^{(l)}) e_k \right\| \geq l$$

where $y^{(l)} = \sum_{k=0}^{N_l} \alpha_{k,l} e_k$. Setting

$$\begin{cases}
A_k x = \sum_{m=0}^{N_l} a_m e_m^* (x) e_m \\
\tilde{B}_k x = \sum_{m=0}^{N_l} b_m e_m^* (x) e_m
\end{cases}$$

we obtain

$$\| A_l (A_l + \tilde{B}_l)^{-1} y^{(l)} \| \geq l \| y^{(l)} \|$$
or equivalently
\[ \| A_j \tilde{x}^{(i)} \| \geq I \|(A_j + \tilde{B}_j) \tilde{x}^{(i)} \| \]
where \( \tilde{x}^{(i)} = (A_j + \tilde{B}_j)^{-1}y^{(i)} \neq 0. \) Setting
\[ x^{(i)} = \frac{\tilde{x}^{(i)}}{\|\tilde{x}^{(i)}\|} \]
we obtain (2.3). This concludes the construction of Example 2.2. □

**Remark 7.** In this construction, we can obtain a bounded operator \( A' \) by defining
\[ A'_k = \nu_k A_k \text{ with } \nu_k > 0, \ k \in \mathbb{N} \]
in order to ensure that \( \| A'_k \| \leq 1 \). Then, similar arguments show that the pair \( (A', B) \)
does not satisfy (1.1)_0 although it satisfies (1.1).

It follows from Theorem 2.1 that \( 0 \notin \rho(A') \cup \rho(B) \). Hence one cannot assert as in Example 2.2 that the pair \( (A', B) \) is regular.

**REFERENCES**


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