Opdrachtgever:
DG Rijkswaterstaat
Rijksinstituut voor Kust en Zee, RIKZ

SWAN fysica plus

Project HR-ontwikkeling

Samenwerkingsverband
WL | Delft Hydraulics - Alkyon Hydraulic Consultancy & Research

September 2002

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SAMENVATTING:

In this report results of the project 'SWAN fysica plus' are described. Existing formulations of some dominant processes in coastal regions have been either investigated, validated or improved for application in the wave prediction model SWAN. The benefit for the client is evident with respect to the determination of boundary conditions at the coast. In particular, we have aimed at improving the prediction of the wave height and wave period near the coast line.

The dominant processes that have been investigated, entirely or partially, are depth-induced wave breaking, three (triplets) and four (quadruplets) wave-wave interactions and whitecapping. Three alternatives for the breaker formulation of Battjes and Janssen have been validated using an extensive test set. An alternative formulation for white-capping has been proposed and validated. Furthermore, a phenomenological study has been carried out to find out under which circumstances triads are important in nature, and in SWAN in particular. An extensive study on quadruplets resulted in a number of useful formulations ranging from accurate and expensive in computational time to less accurate but suitable for application in numerical models, such as SWAN.

Furthermore, a preprocessing tool has been developed for the modeling of wave reflection. Finally, the convergence and iteration behavior of SWAN has been analyzed, taking into account alternative formulations for the presently used limiter in the numerical solution procedure.

The entire study has resulted in a number of suggestions for improving wave modeling using spectral wave models in general. These suggestions are useful for application in SWAN. Furthermore, some useful functionalities have been developed for SWAN and recommendations for improving the convergence behavior in SWAN are given. The overall conclusion of this study is that major improvements to the various source terms have been achieved, but that the implications to improve the predictive capabilities of the SWAN model could not be determined due to various problems in the numerical scheme.

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Summary

In this report results of the project 'SWAN fysica plus', being part of the product "HR-ontwikkeling" (formerly TCRAND), are described. The aim of the present project was to improve the formulations of some of the dominant processes in the Dutch coastal region for application in the wave prediction model SWAN. In particular, we have aimed at improving the prediction of the wave period and wave height near the coast line. The benefit for RIKZ is evident with respect to the determination of boundary conditions at the coast.

The aspects that have been investigated, are:

1. Depth-induced wave breaking;
2. Convergence and iteration behaviour;
3. Reflection;
4. Quadruplet interactions;
5. Triad interactions;
6. Whitecapping.

Depth-induced wave breaking

For the modelling of depth-induced wave breaking alternative formulations have been considered. In this study we compared the results of recently developed breaker formulations proposed by Baldock et al. (1998), Vink (2001) using the Battjes and Stive model, and Hurdle and Van Vledder (2000) with measurements and with the results of the Battjes-Janssen formulation. The SWAN computations were all carried out in the stationary mode. Based on practical and generic possibilities for application, robustness and consistency, these formulations have been compared with each other.

The major conclusions that can be drawn are the following. Firstly, we confirm the conclusion of Vink (2001) that the Rayleigh model of Black et al. (1998) proved to be an improvement on steep slopes. Secondly, SWAN including the Battjes-Stive formulation results in the best prediction of the trend that is shown by the wave decay towards the shoreline. The local information that is being put in this model by means of the local wave steepness seems to be an improvement. Thirdly, strong variation in wave height over short distances are very well predicted by SWAN using either the Battjes-Janssen formulation or the Black formulation. Finally, the breaker formulation of Hurdle and Van Vledder (2000) does not seem to give better predictions for wave energy spectra and integral wave parameters for the wide range of test cases considered in this study. This is not surprising since this formulation was calibrated based on one set of flume measurements for slopes higher than 1:50.

The following recommendations are made: First of all, new formulations of the breaker formulation should be implemented in the experimental version 40.16 of SWAN, to avoid convergence problems due to switching on and off triad wave interactions. Alternatively, this functionality should be implemented in the operational SWAN version. Secondly, the Battjes-Stive breaker formulation with the breaker parameter depending on the local wave steepness (according to Vink, 2001) should be implemented in an operational version of SWAN. A further calibration of the breaker parameter is recommended. The trends in wave decay are correctly predicted, but the amount of dissipation is too low. Thirdly, the formulation of Black et al. (1998) should be implemented in an operational version of SWAN, either stand alone, or in combination with a breaker parameter that depends on the
local wave steepness (Vink, 2001). Finally, a more accurate formulation of the triad wave interactions should be implemented. Both triad wave interaction and depth-induced wave breaking are the dominant processes in the surf zone. As long as the triads are not better approximated, the inaccuracies due to wave breaking can hardly be isolated. It will then be hard to get a complete understanding of the effect of wave breaking on the wave energy spectrum.

**Convergence and iteration behavior**

It is well-known that the SWAN model does not reach convergence within an acceptable number of iterations in a number of situations. Furthermore, SWAN fails to reproduce some relevant physical features of the energy spectra in some cases. A number of techniques have been suggested to improve the iterative behaviour. Some techniques were already implemented in recent (test-)versions of SWAN. Haagsma and Otta (2001) have applied a limiter only for the quadruplet source terms, appearing not to be a viable alternative. Here a number of techniques attempting to solve the poor convergence and iteration behaviour in SWAN have been (re-)tested and discussed: A smoother for the action density spectrum, a combined onset of the triad and quadruplet source term, a limiter over the deep water source terms and underrelaxation of the action density update during the stationary iteration. All techniques were investigated individually and in combination. Test runs were made for both deep and shallow water cases.

From the study presented in this report we draw the following conclusions. First of all, source term limiters (for the quadruplet term and for all the deep water source terms) do not seem to be effective. They do not stabilise the convergence behaviour. In many cases, the old action density limiter should be preferred. Secondly, the main effect of the combined onset is an enhanced triad interaction resulting in a reduced $T_{m0,1}$. Consequently earlier found biases for coastal applications are enhanced. Furthermore, the convergence in the combined onset is considerably slower. Thirdly, a first study of underrelaxation of the pseudo time step as an alternative for the use of limiters showed that underrelaxation may provide more realistic solutions, but reduces the speed of convergence. Finally, the impact of an action density smoother on the convergence behaviour is small. Even in combination with other convergence techniques, the effect of a smoother on the convergence is small.

A number of recommendations have been made. Firstly, an inventory of the entire numerical solution procedure should be made to obtain insight in the numerical iterative systems. The numerical solution procedures used in SWAN are not well documented. Secondly, it does not seem worthwhile to study source term limiters at this time. It is uncertain whether they may ever function adequately. At the moment the understanding of underlying iterative system is too poor. Therefore, it seems more appropriate to gain insight in the underlying iterative system, or more generally, the entire numerical implementation. Thirdly, the SWAN code needs to be tidied up. A cleaner code will help the implementation of alternative functionality and will enhance logical specification of numerical experiments. It should be considered to rewrite substantial parts of the code. Finally, a good solution for expert users of SWAN to avoid the unwanted side-effects of limiters is to implement an option for a reduced (pseudo) time step which facilitates a slow but steady convergence of the model without limiters. Further research is needed to fully evaluate the benefits and drawbacks of using the underrelaxation of the pseudo time step as an alternative for limiters. A next step would be to implement the underrelaxation of the pseudo time step in a version of SWAN with a smooth onset of the quadruplet and triad source terms.
Reflections

The present implementation of the treatment of reflection and transmission in SWAN 40.11 has some restrictions with respect to the use of obstacle polygons. These restrictions comprise the relation between the individual line pieces of obstacle polygons and the computational grid. Another restriction comprises the relation between the obstacle polygons, the bottom grid and exception values. These restrictions are, in our opinion, too briefly described in the user manual. In the present study an analysis has been made of these restrictions to obtain more insight into them, which might benefit users of SWAN. Based on this analysis the following conclusions and recommendations are proposed. There must be a balance between the resolution of the grid and the resolution of the polygons used to represent reflection and transmission obstacles. Users must be made aware of the fact that reflection and transmission lines are ignored when they lie next to grid points that are defined as exception points. The definition of the reflection and transmission coefficients should be modified such that by default the amount of reflected wave energy only depends on the reflection coefficient, and not also on the transmission coefficient. The SWAN pre-processing module should be extended with a check on the consistency and validity of the coefficients for reflection and transmission. The SWAN pre-processing module should be extended with a check on the relation between the computational grid and the obstacle/transmission polygons.

Quadruplets

A series of studies has been performed to improve the computation of the non-linear four-wave interactions. This work has been summarised and continued in this study. First, the deficiencies of the presently used approximation (viz. the Discrete Interaction Approximation) to compute these interactions have been analysed. Secondly, an accurate method to compute this transfer rate has been established. This method is based on Webb/Resio/Tracy (WRT) method and needed to verify any improvement to the DIA. Thirdly, the DIA has been re-derived resulting in the Generalised Discrete Interaction Approximation (GDDIA). The main advantages of the GDDIA with respect to the original DIA are that it includes finite depth effects, it uses a general wave number configuration, it can include multiple interacting wave number configurations, and it makes the presently used WAM depth scaling obsolete. In addition a simple computational technique has been derived to compute the nonlinear transfer rate. This technique reduces the computation of the nonlinear transfer rate to the computation of a finite sum of weighted triple products of energy densities at discrete spectral grid points.

The accurate WRT and the Generalized DIA method have been implemented in a test version of the SWAN model and various tests have been performed to its effects on the source term balance. Based on the results obtained the following conclusions have been formulated. Replacing the DIA with an accurate method improves the prediction of spreading measures, i.e. the directional spreading becomes smaller and the frequency spectra become more peaked. Replacing the WAM depth scaling with a direct method to account for finite depth effects yields different results for wave height and period measures. However, no conclusions can yet be drawn whether or not the predictive capabilities of the SWAN model improve. Replacing the DIA with a more accurate method implies that the source term balance will change. Since shortcomings of the DIA are now masked by the use of various empirically derived coefficients, replacing the DIA will initially lead to a lower performance of the wave model, however good the new formulation. Results obtained with the WRT method have been compared with results obtained with the method of
Masuda/Hashimoto and it was found that the results are similar in shape but that the magnitude of the non-linear transfer rate differs by a factor of about 3.

For this part of the study, our recommendations are as follows. The main recommendation is that further tests should be performed to assess the implications of replacing the DIA with more accurate methods to compute the quadruplet wave-wave interactions. Such tests should focus on combined swell-wind sea wave systems in situations that are characteristic for the Dutch coastal waters. Both 1D and 2D situations should be considered in both deep and shallow water. The aim of such studies is to find a balance between the computational requirements and the required accuracy to compute the quadruplets. It is therefore advisable to pay further attention to the optimisation of the WRT method. In addition, the optimal extension of the Generalised DIA should receive further attention, because it provides a general framework to improve the DIA to any desired accuracy. The triplet method might be an efficient computational technique to speed up the computations. Since the Japanese method to compute the non-linear interactions has also been implemented in a test version of SWAN, it is strongly recommended that the origin of the discrepancy between this method and the WRT method should be determined. Apart from this particular mismatch, it is advised to perform a wider verification study of other available codes for computing the non-linear transfer rate. In such a study attention should be paid to aspects including the treatment of symmetries, numerical integration techniques, limited accuracy of integration techniques. Finally, since replacing the DIA with more accurate methods affects the total source balance, the SWAN model should be re-calibrated.

**Triads**

A phenomenological study on triad wave interactions has been carried out. Therefore we have used physical model results and model results obtained with the phase-resolving Boussinesq-type wave model TRITON and the phase-averaging model SWAN. The aim of this part of the study was to find under which circumstances triad wave interactions play a role and to give a qualitative indication of the effect of triads on the spectrum. We have identified the deficiency of the LTA formulation in SWAN. TRITON has been used as a benchmark model, under the assumption that non-linear interactions are determined correctly by this model.

A number of conclusions have been drawn from this study. Firstly, SWAN overestimates the (spatial) region in which triads play a role. Secondly, due to triad wave interactions wave energy from the peak frequency is shifted towards higher frequencies. SWAN strongly overpredicts the amount of wave energy that is transported from the peak frequency to the first harmonic, resulting in an underestimation of the mean wave period. Thirdly, nonlinear wave interactions (also triad wave interactions) are responsible for the generation of long waves. In very shallow regions the shift of wave energy towards low frequencies is significant, causing the mean wave period to increase. In SWAN the LTA formulation neglects the triads for the difference frequencies and is therefore not able to shift energy to low frequencies, resulting in a further underestimation of the mean wave period.

For this part of the study we have several recommendations. Most important is that the modelling of triads in SWAN should be improved. Not only self-self interaction should be considered, but attention should be paid to the interaction of one wave with the entire spectrum of waves. Furthermore, long wave generation should be made possible through the triad formulation to be developed and/or implemented. Secondly, the interaction between wave breaking and triad interaction should be investigated. Both mechanisms generate long waves. Furthermore, triad wave interactions generate high frequency waves,
which might be more sensitive to breaking. Also for the implementation in SWAN it is necessary to have more insight in whether the two processes work sequentially or whether there is a continuous interplay between them in nature. Field experiments would be helpful. Thirdly, TRITON or any other phase-resolving wave model should be extended with its underlying energy equation. As such, these models can be used to gain more insight in the effect of physical processes on the spectral distribution of wave energy. Furthermore, they might be useful as verification tool for newly developed functionalities in e.g. SWAN. Finally, for the prediction of wave fields in very shallow water, e.g. in coastal areas, harbors or near constructions the advantages and disadvantages of SWAN and TRITON should be combined in a natural way by coupling these two models.

Whitecapping

Based on the findings described in this chapter the following conclusions are given with respect to the modelling of whitecapping dissipation. The widely used Komen formulation for whitecapping dissipation produces unrealistic results for multi-peaked wave spectra since it depends on wave steepness computed based on a mean wave period. Consequently, it under-predicts the dissipation of the wind-sea when a small amount of low frequency energy is added, and it over-predicts the dissipation of swell waves when a wind sea is present. Both of these effects in the Komen method lead to an under-prediction of wave period measures. Since integral quantities of the spectrum depend on the shape of the spectral tail, it is important that the spectral decay with frequency is properly described. The extended Komen formulation depends on integral quantities of the spectrum and relates the dissipation at a certain frequency with the occurrence of wave energy at higher frequencies. This implies that the dissipation predicted by the extended Komen method is sensitive to the description of the spectral tail.

The cumulative steepness method does not depend on mean quantities of the spectrum, and is therefore independent of the shape of the spectral tail. In the cumulative steepness method the effect of adding low frequency wave energy (slightly) enhances the dissipation at higher frequencies, probably leading to an increase in wave period measures. The cumulative steepness method predicts much lower dissipation rates at lower frequencies than the (modified) Komen method. This may solve some of the reported problems with the over-prediction of swell dissipation. But, in combination with the numerical scheme in SWAN, implementation of the scheme gives rise to stability problems. This might be a reason for the occurrence of unrealistic spurious peak at low frequencies in directions opposite to the wind direction.

The inclusion of directional effects in the cumulative steepness method reduces the amount of dissipation by whitecapping. However, one of the parameters in the formulation is a constant of proportionality that has to be obtained by calibration, so this has little practical effect. The Komen and extended Komen formulations do not include directional effects on the dissipation rate.

For this part the following recommendations are given. The directional dependence of the cumulative steepness method should be investigated, not only with numerical experiments but also by analysing field data. A particular point of interest is whether the straining mechanism has the same effect for following waves or for opposing waves. Detailed field data might help to determine the effect of lower frequency energy on the dissipation of wave energy at higher frequencies. The reason for the occurrence of the reported instabilities with the cumulative steepness method should be found and possibly corrected. The use of a wind input source function, with negative input for opposing winds, should
also be considered. This may have a positive effect on the numerical stability of growth curve experiments performed with the SWAN model. Finally, further 1D and 2D tests are needed to assess the implications for growth curves of wave height, wave period and directional spreading.

Calibration

In this study attention has been paid to the development of new source terms for non-linear quadruplet wave-wave interactions, white-capping dissipation and depth-induced wave breaking. Versions of these source terms have been implemented in a test version of SWAN 40.11 and various tests and calibrations have been performed with this modified SWAN model. To isolate the effects of each of these improvements or modifications to the source terms, separate calibration and verification calculations have been performed, one for each source term.

The implementation and testing of the various new source terms was hampered by various numerical problems, which could not be solved in the framework of this project. Most of these problems are related to the convergence behaviour of SWAN and to the idiosyncrasies of the implementation of the various limiters, especially in shallow water. This situation forced us to limit the number and types of tests for this study. It was therefore decided that extensive tests are of limited value until the numerics of the SWAN model are improved considerably.

Based on the results of the limited calibration of the SWAN model with new source terms for quadruplet wave-wave interactions and whitecapping dissipation, the following conclusions can be drawn. The results of the implementation and calibration of the newly developed source term for non-linear quadruplet wave-wave interactions indicate that replacing the DIA with an accurate method improves the direction and frequency spreading. It is also found that replacing the WAM depth scaling with a direct computation of the non-linear transfer rate affects the prediction of the mean wave period (it increases) and other spreading measures. The results of the computations for the academic test spectrum indicate that the finite depth DIA leads to a better representation of shallow water effects on the non-linear transfer rate compared to the present depth scaling. Still, a mismatch in the non-linear transfer rate exists in comparison with results of exact computations. Initial experiments with the cumulative steepness in SWAN indicate that the dissipation of low frequency waves is much smaller than with the Komen formulation and that there is an accumulation of energy at those frequencies. This accumulation increases with iteration number. Any (small) amount of energy transfer (by quadruplet interactions) is therefore not dissipated fast enough to avoid this accumulation.

For this part the following recommendations are given. The SWAN model should be re-calibrated when major improvements in the numerics of the SWAN model are realised. This re-calibration should be performed with all new source terms activated, and not per individual source term. During this re-calibration attention should be given to the interaction between the various source terms, the integration scheme and the role of limiters.

The overall conclusion of this study is that major improvements to the various source terms have been achieved, but that the implications to improve the predictive capabilities of the SWAN model could not be determined due to various problems in the numerical scheme.
1 Introduction

In this report results of the project 'SWAN fysica plus' (contract no. RKZ 1018A), which is a part of the product "HR-ontwikkeling" (formerly TCRAND), are described. The aim of the present project was to improve the formulations of some of the dominant processes in the Dutch coastal region for application in the wave prediction model SWAN. In particular, we have aimed at improving the prediction of the wave period and wave height near the coast line. The benefit for RIKZ is evident with respect to the determination of boundary conditions at the coast.

The dominant processes that have been investigated, entirely or partially, are:

1. Depth-induced wave breaking;
2. Convergence and iteration behavior;
3. Reflection;
4. Quadruplets;
5. Triads;

The aim of the project was to improve the present formulations of the processes mentioned above. For a number of processes alternative formulations have been investigated and validated using well-documented data sets and recommendations for implementation in SWAN have been made. Part of these data sets is included in the RIKZ test bed. For the other processes both the physical mechanism and the modeling of those mechanisms have been investigated. This has led to more insight in the importance of the particular source terms.

WL | Delft Hydraulics and Alkyon Hydraulic Consultancy & Research carried out this study as a joint venture. WL | Delft Hydraulics acted as contact person. Mr. J.H. Andorka-Gal, Mr. J.J. Jacobs, Ms. A.T.M.M. Kieftenburg and Mr. C.G. Israël were involved for RIKZ. Dr M. Bottema and Ms. E. Claessens of RIZA provided all participants with useful advise. The study described in this report was performed by Dr G.Ph. van Vledder and Mr. D. Hurdle of Alkyon and Dr J. Groeneweg and Dr J.G. Bonekamp of WL | Delft Hydraulics.

For this study an experimental version of SWAN has been made available by Delft University of Technology, which includes implementations of the breaker formulations of Baldock et al. (1998) and Battjes and Steve (1985). Furthermore, RIZA provided the experimental version 40.16 of SWAN (approved by Delft University of Technology) in which triads and quadruplets may be determined independently. The joint venture of WL | Delft Hydraulics and Alkyon are grateful to both Delft University of Technology and RIZA.

The outline of this report follows the six topics mentioned above, which are described in the chapter 2 to 7. In chapter 8 the calibration of the model is described, based on the newly developed formulations of the quadruplets and whitecapping.
2 Depth-induced wave breaking

2.1 Introduction

The wave transformation process on shallow foreshores is governed by wave breaking. This dissipation mechanism determines the amount of wave energy that reaches the beach or any type of coastal structure. Design criteria for coastal structures strongly depend on the amount of wave energy approaching the structure. If the wave conditions near the structure are determined with a numerical wave model such as SWAN, accurate modeling of the wave breaking process is important.

Models for the energy dissipation due to wave breaking can roughly be divided into two classes: parametric and wave-by-wave models. In models that are based on the wave-by-wave approach the incident wave height distribution is divided into a number of discrete classes. According to Roelvink (1993), it is assumed that each class behaves like a periodic subgroup, that propagates shorward independently of the others. In the parametric class of models, a shape of the breaking wave height distribution that is based on locally defined parameters is assumed a priori. The local, time-averaged dissipation of wave energy by breaking is used in the energy flux balance, which is applied across the surf zone to determine the wave transformation.

In SWAN the breaker formulation of Battjes and Janssen (1978) has been implemented. This breaker model is a typical example of a parametric model. Several refinements of the model of Battjes and Janssen have been proposed in literature. Whereas Battjes and Janssen used a constant breaker parameter $\gamma$, which indicates a breaker height-to-depth ratio, Battjes and Stive (1985) found that there was a relation between the breaker parameter and the incident wave steepness evaluated at the peak frequency of the spectrum. Very recently Vink (2001) presented some results of the latter model in which the incident wave steepness was replaced by the local wave steepness in the expression for the breaker parameter $\gamma$. In its present formulation the breaker parameter increases if the wave steepness increases. Consequently, when waves become steeper, the breaker parameter increases and so does the maximum wave height. This is not to be expected according to physical logics. Calibration and validation using more suitable datasets should lead to more insight in the present formulation and the physics involved.

Battjes and Janssen (1978) assumed that the wave height distribution could be modeled with a Rayleigh distribution truncated at a maximum depth-limited breaker height. Baldock et al. (1998) have developed a dissipation model, based on the model of Battjes and Janssen, but adopting a full Rayleigh distribution. In accordance with measurements of Thornton and Guza (1983), the model of Baldock et al. (1998) provides a better estimation of the wave heights in the nearshore zone.

The yet unpublished model of Hurdle and Van Vledder was derived based on a parametric formulation presented in Hurdle and Van Vledder (2000). This publication gives relationships between the wave height on a constant slope (of 1:10, 1:20, 1:30 en 1:50) and a nominal quantity called "the shoaled wave height", denoting the wave height obtained at a given water depth due to shoaling from deep water, but excluding all other propagation effects. The given relationships are used to compute the dissipation due to breaking by
making the reasonable assumption that wave breaking is the only other term influencing the wave height at these water depths. A first verification of this model in a one-dimensional setting lead to good results.

The aim of this part of the study is to validate the recently developed breaker formulations proposed by Baldock et al. (1998), Vink (2001) using the Battjes and Stive model, and Hurdle and Van Vledder (2000) and compare the results with those using the Battjes-Janssen formulation. A number of suitable datasets are used. The SWAN computations are all carried out in the stationary mode. Based on practical and generic possibilities for application, robustness and consistency, these formulations will be compared with each other.

In Section 2.2 the breaker formulations of Baldock et al (1998), Vink (2001) (based on Battjes and Stive model) and Hurdle and Van Vledder (2000) are briefly described. The validation test sets are outlined in Section 2.3. The results of the validation of the breaker formulations are given in Section 2.4. Finally, conclusions and recommendations are given in Section 2.5.

### 2.2 Wave breaking models

In this section the three breaker formulations mentioned above are described in more detail. The formulations of Baldock et al. (1998) and Battjes and Stive (1985) are based on the model of Battjes and Janssen (1978), whereas Hurdle and Van Vledder (2000) showed a new approach.

The Battjes and Janssen model consists of two important elements: the probability of breaking and the dissipation of energy in a single breaking wave. The total energy dissipation is given by the product of the dissipation in a breaking wave \( D_b \), and the probability of the occurrence of a breaking wave \( Q_b \). The energy dissipation in a breaking wave is modeled after analogy with a bore of corresponding height:

\[
D_b = \frac{1}{4} \rho g \frac{H_b^3}{h},
\]

with \( H_b \) the breaker height and \( h \) the water depth. In application to random waves, the mean frequency \( \bar{f} \) of the wave energy spectrum is used as a representative value of the frequency \( f \).

In order to determine the probability of breaking, use is made of the expression for the maximum breaker height based on Miche’s criterion for the maximum wave height of periodic waves of a constant form. To avoid influence of the bottom slope on the deep-water limit, the following form was finally adopted:

\[
H_m = 0.88 k^{-1} \tanh(\gamma kh / 0.88),
\]

which in shallow water reduces to

\[
H_m = \gamma h.
\]
The wave height distribution is assumed to be a Rayleigh distribution, truncated at a maximum (depth-limited) height, such that all breaking waves have a height equal to $H_m = H_b$. Under this assumption the probability that the wave height equals $H_m$ (i.e. $Q_b$) can be derived, leading to the following implicit equation:

$$\frac{1 - Q_b}{\ln Q_b} = -\left(\frac{H_{rms}}{H_m}\right)^2,$$

(2.4)

with $H_{rms} = \sqrt{8m_0}$, and $m_0$ the zero-th moment of the spectrum, which is equal to the total variance. The mean energy dissipation per unit area is given by

$$D_{sw} = \frac{\alpha}{4} Q_b \bar{f} \rho g H_m^2,$$

(2.5)

in which $\alpha$ is a proportionality coefficient of order one. Together with the breaker parameter $\gamma$ the proportionality parameter control the level of energy dissipation in a breaker and the fraction of breaking waves.

### 2.2.1 The dissipation model of Baldock et al.

Baldock et al. (1998) adjusted the dissipation rate proposed by Battjes and Janssen (1978), by applying a full Rayleigh distribution across the entire surf zone. Consequently, a greater dissipation from the highest waves is included in the model. From a physical point of view, the approach is more realistic; higher waves will dissipate more energy. Using a full Rayleigh distribution, a different estimate for the fraction of breaking waves is obtained:

$$Q_b = \int_{H_m}^{\infty} p(H)dH = \int_{H_m}^{\infty} \frac{H}{4m_0} \exp\left[-\frac{H^2}{8m_0}\right]dH = \exp\left[-\left(\frac{H_b}{H_{rms}}\right)^2\right].$$

(2.6)

The Rayleigh model includes a greater rate of energy dissipation from the largest waves in the Rayleigh probability distribution function. The total energy dissipation rate is found by multiplying the energy dissipation due to each broken wave (with height $H$) by the probability of that wave height occurring. Thus, by combining (2.6) and (2.5) Baldock et al. (1998) obtained:

$$D_{sw} = \frac{\alpha}{4} \bar{f} \rho g \int_{H_m}^{\infty} H^2 p(H)dH = \frac{\alpha}{4} \bar{f} \rho g \exp\left[-\left(\frac{H_b}{H_{rms}}\right)^2\right] \left(H_b^2 + H_{rms}^2\right).$$

(2.7)

The breaker height is defined by the breaker parameter as in the model of Battjes and Janssen (1978), i.e. $\gamma = H_b / h$. The model of Baldock et al. (1998) has the same adjustable parameters, i.e. the proportionality parameter $\alpha$ and the breaker parameter $\gamma$.

### 2.2.2 Adaptation of Vink (2001) of Battjes-Stive model

Battjes and Stive (1985) proposed an expression for the breaker parameter including the deep-water steepness according to:
\[ \gamma = a + b \tanh(cs_0) \]  
(2.8)

where \( s_0 \) is the deep-water steepness based on \( H_{rms} \) and the peak frequency \( f_p \). By calibrating with both laboratory and field data, Battjes and Stive found the best fit of (2.8) through the data with \( a = 0.5, \ b = 0.4, \ c = 33 \).

Expression (2.8) was implemented in an experimental version of SWAN by Baaijens (1999), who demonstrated that using this expression the overall model performance improved for the Delilah data set of the DUCK experiment. In general applications the deep-water steepness is not suitable, because this value is not always well-defined, e.g. when the model boundary is already in shallow water.

Vink (2001) replaced in expression (2.8) the deep-water steepness by the local wave steepness \( s_{loc} \). The definition of the local wave steepness is chosen according to

\[ s_{loc} = \frac{H_{rms} k_{mn}}{2\pi} \]  
(2.9)

The (local) mean wave number is used because this is a more convenient measure than the wavenumber at the peak frequency, in the case of double-peaked spectra. The mean wave number that is used by Vink (2001) is given by

\[ k_{mn} = \left( \frac{\iint \sigma E(\sigma, \theta) \sigma \sigma d\sigma d\theta}{\iint k^2 E(\sigma, \theta) \sigma \sigma d\sigma d\theta} \right)^2 \]  
(2.10)

The lower wave numbers are more emphasized than the higher wave numbers. Therefore, (2.10) is a robust measure for the (local) mean wave number. In his computations Vink used the same parameters as the original parameters resulting from the calibration by Battjes and Stive (1985): \( a = 0.5, \ b = 0.4, \ c = 33 \).

### 2.2.3 Breaker model of Hurdle en Van Vledder

The breaking model due to Hurdle and van Vledder has not yet been published in literature. In this section a brief summary is given of the method used to derive the model and the formulation derived, implemented and applied in SWAN 40.11 is given.

The model was derived based on a parametric formulation presented in Hurdle and Van Vledder (2000). This publication gives relationships between the wave height on a constant slope and a nominal quantity called “the shoaled wave height”, the wave height obtained at a given water depth due to shoaling from deep water, but excluding all other propagation effects. These relationships are given for slopes of 1:10, 1:20, 1:30 en 1:50. They can be used to compute the dissipation due to breaking by making the reasonable assumption that wave breaking is the only other term influencing the wave height at these water depths. The derivation of the Hurdle and Van Vledder dissipation model is carried out as follows:
• Compute the wave height to water depth ratio and use the parametric relationship to find the corresponding shoaled wave height.

• Subtract a small increment to the water depth and compute the shoaled wave height at the new water depth. The corresponding increment in distance can be computed from the slope.

• Use the relationship to obtain the actual significant wave height at this water depth.

• Compute the rate of change in the wave height. This consists of two contributions: the term for shoaling and the term for breaking. The term for breaking can simply be obtained by subtracting the term for shoaling from the rate of change of in the wave height.

This can be expressed mathematically as:

\[
\frac{dH_{m0b}}{dx} = \frac{dH_{m0}}{dx} - \frac{dH_{m0s}}{dx}
\]  

(2.11)

where

\[
\frac{dH_{m0}}{dx} = \frac{H_{m0}(h - \delta h) - H_{m0}(h)}{\delta x}
\]  

(2.12)

Here \(H_{m0}(h)\) is the wave height at depth \(h\), \(\delta h\) is the increment in the water depth, \(\delta x\) is the increment in distance \(x\), \(dH_{m0b}/dx\) is the rate of change of wave height with distance due to breaking and \(dH_{m0s}/dx\) is the rate of change of wave height with distance due to shoaling.

The rate of change of wave height with distance due to breaking, \(dH_{3b}/dx\), was computed for a range of wave heights and constant bottom slopes. It was found that, to an acceptable degree of accuracy, \(dH_{3b}/dx\) did not depend on the slope but only on the wave period and on the local wave height to depth ratio. These dependencies were subsequently parameterized and converted to a form suitable for application with wave energy, resulting in the formulation given below. This derivation is not elegant and should the formulation for wave breaking prove very promising, the authors are planning to make a better derivation and try to simplify the formulation.

The formulation is as follows:

\[
\frac{dE_B}{dt} = \frac{1}{8} C_g \cdot H_{m0} \frac{dH_{m0b}}{dx}
\]  

(2.13)

where

\[
\frac{dH_{m0b}}{dx} = \left( C_1 \sqrt{\tanh(C_2 \left[ \frac{H_{m0}}{h} \right]^C_3)} \frac{dH_{m0s}}{dh} + \left( C_4 \sqrt{\tanh(C_5 \left[ \frac{H_{m0}}{h} \right]^C_3)} \right. \right)
\]  

(2.14)

and with

\[
\frac{dH_{m0s}}{dh} = \frac{dH_{m0s}}{dx} \cdot \frac{dx}{dh}
\]  

(2.15)
The coefficients $C_1$ to $C_6$ were obtained by implementing the above formula in SWAN and optimizing them to obtain the best fit for the wave height when compared to a number of values obtained from the parametric relationship. This was done using an optimization routine in Matlab. In each iteration of the optimization process, Matlab wrote new coefficients to the input file used by SWAN and start a new series of SWAN computations. Matlab was then used to read the results of the computations and compute the error. Note that for the purposes of this exercise the number of iterations carried out by SWAN itself was fixed to avoid problems with iterating towards the optimal settings. The estimated coefficients are given in Table 2.1.

<table>
<thead>
<tr>
<th>$C_1$</th>
<th>0.0655</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_2$</td>
<td>2.0821</td>
</tr>
<tr>
<td>$C_3$</td>
<td>6.1635</td>
</tr>
<tr>
<td>$C_4$</td>
<td>-0.0682</td>
</tr>
<tr>
<td>$C_5$</td>
<td>1.6068</td>
</tr>
<tr>
<td>$C_6$</td>
<td>7.7968</td>
</tr>
</tbody>
</table>

Table 2.1 Coefficient in breaker formulation of Hurdle and Van Vledder

To avoid having to make large changes in the code of SWAN, the source term due to shoaling was not taken as computed by SWAN but was computed separately based on the wave height $H_{m0}$. This was done using the approximate assumption that all the spectral energy was concentrated at the frequency corresponding to the spectral wave period, $T_{m-1.0}$. This may mean that the implementation of the formulation performs less well than it could, particularly for some cases with double peaked spectra.

### 2.3 Test cases for validation

A couple of data sets have been considered for the validation of the three breaker formulations mentioned above. In the next sections the following datasets will be described:

1. 1D laboratory experiment of Battjes and Janssen (1978);
2. Hiswa basin experiment of Dingemans (1987);
3. Petten field cases described in Andorka *et al.* (1998);
4. 1D laboratory experiment of Van Gent (1999);
5. 1D laboratory experiment of Baldock *et al.* (1998);

The first three experiments are taken from the test bank of RIKZ (WL | Delft Hydraulics, 2000).

#### 2.3.1 1D laboratory experiment of Battjes and Janssen

In the laboratory experiment of Battjes and Janssen (1978) random, uni-directional waves (in a flume) propagated towards a bar-trough beach profile, accompanied by depth-induced wave breaking over the bar (see Figure 2.1). Currents and wind are absent.
The SWAN simulations are executed in the stationary 1D-mode. The offshore boundary is located at $x = 7.4\text{m}$ (station 1 in Figure 2.1), approximately at 30.8m from the shoreline. The length of the computational grid is 30m. The resolution is equal to $\Delta x = 0.1\text{m}$. The spectral directions cover a sector from $-10$ to $+10$ degrees in 40 directional bins ($\Delta \theta = 0.5^\circ$). The frequency space is divided into 30 frequency bins, covering the frequency interval from $f_{\text{low}} = 0.2485\text{Hz}$ to $f_{\text{high}} = 3.5714\text{Hz}$.

The computations are executed in the third-generation mode. Besides the physical processes that are activated in the default option (wind growth, white-capping, refraction) depth-induced wave breaking, bottom friction, triad wave interaction and refraction are activated as well. Quadruplets are switched off. Setup is not set either.

Test case 111wav01 is a simulation of mildly breaking waves (i.e. run 13 of Battjes and Janssen, 1978). Violently breaking waves (i.e. run 15 of Battjes and Janssen, 1978) are studied in case 111wav02. In both cases, the same bottom file is used, but the difference is made by using different water levels, wave heights and characteristic wave periods. Maximum water depth is $0.762\text{m}$ in test case 111wav01 and $0.615\text{m}$ in test case 111wav02.

Input wave fields are of the Jonsswap type and the directional distribution is equal to $\cos^{500}$. In test case 111wav01 the imposed significant wave height equals $H_s = 0.147\text{m}$ and the peak period $T_p = 2.012\text{s}$. In test case 111wav02 $H_s = 0.2022\text{m}$, $T_p = 1.886\text{s}$.

### 2.3.2 HISWA basin experiment

A 2D experiment has been carried out by Dingemans (1987) and has become known as “the Hiswa basin experiment”. The dimensions of the basin are $34.0\text{ m} \times 26.4\text{ m}$ (see Figure 2.2). A wave maker is located along one of the short ends. Waves propagate over a horizontal bottom, across a submerged breakwater that extends over half the basin width, to a beach at the other end of the basin. The waves travel across the breakwater with a significant loss of energy and the generation of a relatively large high-frequency spectral peak. The breaking waves generate a mean current in the basin.
From all cases considered by Dingemans (1987), the test case in the testbank of RIKZ (WL | Delft Hydraulics, 2000) has been used, i.e. test case with number ME35, in the testbank referred to as I51his01. For this test case, SWAN is run in the 2D-mode. The computational grid is oriented perpendicularly to the submerged bar. The spectrum that is generated by the wave maker is a Jonsnap spectrum with a peak enhancement factor $\gamma_0 = 3.3$. The observed width of the directional energy distribution is $25^\circ$. The significant wave height and the peak frequency at the wave maker are 0.1m and 0.805Hz, respectively.

The SWAN simulations are executed with a resolution equal to $\Delta x = \Delta y = 0.5m$. The spectral directions cover the full directional circle, divided in 36 directional bins ($\Delta \theta = 10^\circ$). The frequency space is divided into 24 frequency bins, covering the frequency interval from $f_{low} = 0.317Hz$ to $f_{high} = 3.125Hz$.

The physical processes of depth-induced wave breaking, bottom friction, triads, whitecapping, refraction and frequency shift due to currents are activated.

The output is generated at the locations 8, 13 and 16 given in Figure 1 (i.e. stations 32, 33, and 34 in Dingemans, 1987), which are located along a section across the head of the breakwater and locations 2, 9, 14, 18 and 25 (i.e. stations 30, 12, 13, 14 and 39 in Dingemans, 1987) on a section across its main body. The computed significant wave height $H_s$ and mean wave period $T_{m0,1}$ are given along an output curve extending over the top of the submerged breakwater (line through points 2, 6, 9, 14, 18, 21 and 25 in Figure 2.2).

2.3.3 Petten field cases

The field situation of the shallow foreshore near to the Petten Sea Defence is dominated by the presence of a long system of shoals (see Figure 1). Shorterested waves approach the shore almost perpendicularly. Currents are assumed to be absent. From the measurement data, six different time levels have been selected for the cases:

1. f91pet01: 01-01-1995/ 15.40 UTC \( U_{10} = 17.3 m/s, \theta_{w} = 293^\circ, d = 2.10m \);
2. f91pet02: 01-01-1995/ 17.00 UTC \( U_{10} = 19.1 m/s, \theta_{w} = 283^\circ, d = 2.01m \);
3. f91pet03: 02-01-1995/ 04.00 UTC \( (U_{10} = 17.3 \text{ m/s}, \, \theta_u = 316^\circ, \, d = 2.18 \text{m});
\)
4. f91pet04: 02-01-1995/ 05.40 UTC \( (U_{10} = 18.3 \text{ m/s}, \, \theta_u = 318^\circ, \, d = 1.64 \text{m});
\)
5. f91pet05: 02-01-1995/ 16.20 UTC \( (U_{10} = 11.5 \text{ m/s}, \, \theta_u = 350^\circ, \, d = 1.60 \text{m});
\)
6. f91pet06: 10-01-1995/ 11.00 UTC \( (U_{10} = 13.6 \text{ m/s}, \, \theta_u = 274^\circ, \, d = 2.00 \text{m}).
\)

The water level \( d \) varies from 1.60\text{m} in case 5 to 2.18\text{m} in case 3. For each case, the wind speed \( U_{10} \) and wind direction \( \theta_u \) are assumed to be constant over the area considered. For more information see e.g. Andorka-Gal et al. (1998).

The computational grid covers an area of 1 km by 4 km, under an angle of -20 degrees relative to the positive x-axis. Since the waves propagate almost in cross-shore direction, the shore normal resolution is taken much finer than the longshore resolution: \( \Delta x = 10 \text{m}, \, \Delta y = 100 \text{m} \). The directional circle is divided in 36 equidistant directional bins of 10 degrees. In frequency space the interval between 0.04Hz and 0.5Hz is divided in 27 frequency bins. The discretisation is rather coarse, but in accordance with the setting that have been used in the test bed (WL | Delft Hydraulics, 2000).

Observations are available at three observation stations, MP3, MP5 and MP6 (see Figure 2.3). For all test cases, the 2D-mode of SWAN is activated. All physical processes are activated in SWAN except wave-induced setup. Since no obstacles are defined, wave reflection and transmission are not active either. At the western boundary the spectrum measured at MP3 is imposed.
2.3.4 Shallow foreshore

Van Gent (1999) conducted an experimental study in the Scheldt flume of WL | Delft Hydraulics. The effect of wave height and spectral distributions on wave runup was studied. Two different foreshores with slope of 1:100 and 1:250 have been considered. At the end of the foreshore a steep slope (1:2.5) was placed, on which the run-up was measured. Wave height meters were placed at the foreshore at five locations in groups of three wave height meters. At all five locations the incoming and reflected wave have been separated using the method described by Mansard and Funke (1980). The method requires signals from three wave gauges relatively close to each other, consequently leading to a lower accuracy for the energy density in the lower frequencies. These techniques assume linear wave theory which is a rather rough assumption in positions where severe breaking occurs.
In total 59 configurations have been considered. Both single and double-peaked wave energy spectra were imposed at the boundary. Furthermore, the water depth and the wave steepness have been varied. Out of all these test cases we have selected four cases, all of them with a single-peaked JONSWAP spectrum:

1. shallf01: \( H_s = 0.141 \text{m}, \quad T_p = 2.41 \text{s}, \quad d_{TOE} = 0.188 \text{m} \) (intermediate)
2. shallf02: \( H_s = 0.142 \text{m}, \quad T_p = 1.64 \text{s}, \quad d_{TOE} = 0.188 \text{m} \) (intermediate)
3. shallf03: \( H_s = 0.143 \text{m}, \quad T_p = 2.48 \text{s}, \quad d_{TOE} = 0.047 \text{m} \) (very shallow)
4. shallf04: \( H_s = 0.137 \text{m}, \quad T_p = 1.64 \text{s}, \quad d_{TOE} = 0.047 \text{m} \) (very shallow)

Here \( d_{TOE} \) denotes the (still-water) depth at the toe of the steep slope. In all cases the slope of the shallow foreshore is 1:250. See Figure 2.4 for the setup of the experimental facility.

![Figure 2.4 Model set-up of Scheldt flume with shallow foreshore and steep slope. Figure 3 in Van Gent (1999)](image)

The SWAN simulations are run in the stationary 1D-mode in a similar way as described in subsection 3.1 for the Battjes and Janssen experiment. The offshore boundary is located at \( x = -40 \text{m} \), the outflow boundary is located at \( x = 0 \text{m} \) (see Figure 1). Furthermore, \( \Delta x = 0.1 \text{m}, \quad \Delta \theta = 0.5^\circ \) (for a sector between -10 and +10 degrees). The frequency interval from \( f_{low} = 0.2 \text{Hz} \) to \( f_{high} = 2.0 \text{Hz} \) (or from \( f_{low} = 0.25 \text{Hz} \) to \( f_{high} = 3.2 \text{Hz} \) for cases 2 and 4) is divided in 30 frequency bins.

As for the Battjes and Janssen case the computations are executed in the third-generation mode. From all physical processes only quadruplets and setup have been switched off. As already mentioned, the input wave fields are of the JONSWAP type and the directional distribution is equal to \( \cos^{500} \) in all test cases, resulting in effectively no directional spreading.
2.3.5 Breaking on steep slope

In order to examine the behavior of the breaker formulations on steep slopes, we included the experimental study of Baldock et al. (1998). The experiments were carried out in a 50m long wave flume, with a bottom slope of 1:10 in the surf zone. The still-water depth was 0.9m. In the surf zone wave gauges were placed at a distance of 150mm from each other. Video analysis was used to obtain the fraction of breaking waves.

The SWAN simulations are run as in the previously mentioned 1D cases. The computational domain of 46.5m has a resolution of $\Delta x = 0.05m$. The directional sector from -15 to +15 degrees uses 60 equidistant directional bins. The frequency interval from 0.25Hz to 3.0Hz is divided in 62 bins. Here we consider case J2 of Baldock et al. (1998), which will be referred to as baldo01 in this study. A Jonswap spectrum with $H_s = 0.105m$, $T_p = 1.5s$ is imposed at the offshore boundary.

The SWAN computations were carried out with the same model settings that were used in the cases described in subsections 3.1 and 3.4. According to Vink (2001), if the settings with triad wave interactions were activated convergence problems occurred. Therefore, they have been switched off for this computation as well.

2.3.6 Flume experiment at HR Wallingford

A series of physical model studies (conducted at a nominal scale of 1:20) have been carried out in a wave flume at HR Wallingford. This research was reported by Coates et al. (1998) and by Hawkes et al. (1998). Wave conditions and wave overtopping were measured for two or three water levels over different bed slopes of 1:50, 1:30, 1:20 and 1:10. In a following study HR Wallingford, together with WL | Delft Hydraulics, Alkyon and Infram carried out a re-analysis of the data files to obtain various time and frequency domain parameters. Furthermore, they validated the SWAN model and analyzed wave breaking formulations.

From the 210 datasets we have selected the following six test cases in which bottom slope, water depth and spectral shape (uni-modal, bi-modal) of the incoming wave field, characterized by the integral wave parameters $H_s$ and $T_p$ at the toe of the slope, are given. In the case of bi-modal waves the integral parameters for the wind waves and swell waves are given separately (with subscript w and s):

1. s10d10b_1c: slope 1:10, $d = 10m$, $H_{s,w} = 2.5m$, $T_{p,w} = 7.0s$, $H_{s,s} = 2.5m$, $T_{p,s} = 11.0s$
2. s10d10u_012: slope 1:10, $d = 10m$, $H_s = 4.0m$, $T_p = 13.0s$
3. s20d10u_pv: slope 1:20, $d = 10m$, $H_s = 2.82m$, $T_p = 5.0s$
4. s20d12u_hv: slope 1:20, $d = 12m$, $H_s = 2.57m$, $T_p = 12.7s$
5. s50d14u_06g: slope 1:50, $d = 14m$, $H_s = 4.4m$, $T_p = 8.0s$
6. s50d14u_02g: slope 1:50, $d = 14m$, $H_s = 4.5m$, $T_p = 13.8s$

The tests on the 1:30 slope have not been considered, since these tests have been conducted in a wave basin, in contrast to the tests on the other slopes. For more information about the tests we refer to the extensive report of Van der Meer et al. (2000).
In the SWAN computations, the computational domain is 660m long, divided in 132 equidistant gridcells. The directional sector covers the interval from -5 to +5 degrees in 60 bins. The frequency interval from 0.03Hz to 0.607Hz is divided in 100 bins. At the boundary the (scaled) analyzed measured spectra at the deep water boundary are imposed.

The SWAN simulations are executed in the stationary 1D-mode. The quadruplets have been turned off. Bottom friction is included using the default Jonsvap formulation, although Van der Meer et al. (2000). concluded that bottom friction hardly played a role. Dominant effects such as depth-limited wave breaking, triad wave interactions and white-capping have been taken into account.

2.4 Validation

The validation of the three alternative breaker formulations for the Battjes and Janssen model, which have been described in section 2.2, is carried out in this section.

2.4.1 Convergence problems

For a number of 1D tests convergence has not been obtained. In the SWAN computations the standard convergence criterion should be satisfied in 99% of all grid points. If not, the computation stops after a number of iterations (maxiter), which is to be specified by the user in the SWAN input file. For all tests that have been carried out, the percentage of grid points in which the convergence criterion is satisfied, is given in Table 2.2. If convergence is obtained, the label ‘ok’ is given.

For the tests 111wavbr (Battjes and Janssen, 1978) and shallf0r (Van Gent, 1999) only the breaker formulation of Baldock et al. (1998) causes convergence problems. From Table 2.2 it is obvious that increasing the maximum number of iterations, does not lead to convergence. For closer inspection of a non-converging test, we have added Figure 2.1 to Appendix A. In this figure for all breaking models, the spatial distribution of the significant wave height $H_{m0}$, mean wave period $T_{m0}$ and breaker fraction $Q_b$ for test shallf03 have been given. Also the measured values for wave height and wave period have been plotted at 5 locations. Over the last 30m towards the shoreline the significant wave height obtained with SWAN including the Baldock breaker formulation is in mismatch with the measurements and the other computationally obtained results.

In Figure 2.1b the measured and computed spectra at the 5 locations are shown. In location 1 and 2 the secondary peak at twice the peak frequency is strongly overpredicted by SWAN. Furthermore the breaker models are not able to change the shape of the spectrum. The double-peaked structure of the wave energy spectrum is maintained. SWAN with the Baldock breaker formulation does not converge at locations 3, 4 and 5. A strong third peak is predicted, since the action density equation has not reached equilibrium at these locations. This causes not only an overestimation of the significant wave height, but also an underprediction of the mean wave period.

For the Wallingford cases there is even one case (s10d10u_0f2) for which SWAN does not converge with either of the four breaker formulations. In Figure 2.2 in Appendix A the results for this test case have been given. Over the last 100m the predicted spectra are useless (Figure 2.2c) and lead to unrealistic values for the integral wave parameters (Figure 2.2a).
Table 2.2  Percentage of grid points in which convergence is obtained (with triads activated)

In SWAN (Version 40.11) triads and quadruplets are not modeled simultaneously. Depending on the value Ursell number triads are switched on and quadruplets switched off, or vice versa (see chapter 3 and 6). Since the quadruplets are not taken into account, the only nonlinear wave-wave interaction is by means of triads. Nevertheless, if the Ursell number becomes less than 0.1 the triad source term will not be taken into account in the action balance equation. It is well known that switching the triads on and off may lead to convergence problems. In testcase baldo01 the triads have not been modeled (by switching them off permanently in the SWAN input file) and for this test the convergence problems do not occur.

Bottema (personal communications) found out that on the top of the slope the Ursell number increased rapidly, causing the triad wave interactions to become very strong. Even so strong that they switch themselves off one grid point further, due to the strongly decreased wave period. From the wave spectra he learned that the triads transported much energy to frequencies higher than $f_{max}$. Furthermore, parts of the spectrum have zero action density. These holes in the spectrum will certainly lead to convergence problems. Using the experimental version 40.16 of SWAN, in which the quadruplet and triad source terms are calculated simultaneously, Bottema succeeded in obtaining converged solutions for test s10d10u_0f2. Although $Q_b = 1$ at some locations, the convergence behavior and the predicted spatial distribution was correct.

The alternative breaker formulations have been implemented in SWAN 40.11. In the experimental version 40.16 only the standard Battjes and Janssen formulation can be used.
In order to obtain converged solutions we have chosen not to model triad wave interactions in any of the one-dimensional problems. Thus, in the testcases based on the experiments by Baldock et al. (1998, see section 2.3.5), Battjes and Janssen (1978, see section 2.3.1), Van Gent (1999, see section 2.3.4) and HR Wallingford (1998, see section 2.3.6) the triads have been switched off explicitly by means of the SWAN input file. In the two two-dimensional testcases (Hiswa basin, Petten field case) the triads have been taken into account.

### 2.4.2 Model results

For all one-dimensional tests it is not realistic to compare the measured and computed results in shallow areas where triad wave interactions are strong, since the triad interaction are not taken into account, but play a significant role in the measurements. Nevertheless, the total wave energy and thus the significant wave height will not be seriously affected, assuming that the breaking process is more or less frequency independent. Comparisons between measured and computed mean wave periods are useless, since the computed wave energy spectra are not suitable. On the other hand the behavior of the breaker formulations can now be investigated without being disturbed by the triads.

In Figure 2.3 in Appendix the results for the Baldock test are given. Only for this test the breaker fraction has been measured. On the 1:10 slope the breaker fraction is predicted correctly by SWAN using the Baldock breaker formulation. With the other breaker formulations SWAN strongly overpredicts this value. At \( x=46.0m \) the breaker fraction even becomes 1, which in general may result in unrealistic values for the maximum wave height. In regions where \( Q_s = 1 \) the amount of dissipation, predicted by SWAN is not reliable.

Nevertheless, the significant wave height is predicted more or less correct with all breaker formulations. The Battjes and Janssen formulation predicts a dissipation rate that is slightly too high in the shallowest region (\( x>45.5m \)), compared to both measurements and the other breaker formulations. The value for \( Q_s \) is too high in this region as well, resulting in a large dissipation rate. This can also be seen in the spectra in Figure 2.3b in Appendix A. The wave energy at the peak frequency is lower than predicted by the other breaker formulations. On the other hand, the onset of breaking, predicted by SWAN using the Battjes and Stive breaker formulation, is closer to the shoreline. Furthermore, the breaker parameter \( \gamma \) is larger (0.86-0.89) than for the Battjes-Janssen (or Baldock) breaker formulation (\( \gamma=0.73 \)). This implies that the maximum wave height is larger. Consequently, the wave dissipation rate is smaller and the significant wave height will be larger. This explains why the significant wave height at any location is larger than computed by the other breaker formulations.

The two experiments of Battjes and Janssen (1978) represent mild and severe wave breaking. The results have been plotted in Figure 2.4 (111wav01) and Figure 2.5 (111wav02) in Appendix A. For both cases the breaker parameter and thus the maximum wave height is too high for the Battjes and Stive formulation. At the onset of the breaking of waves the amount of wave energy dissipation is too small. This behavior is more pronounced in the severe breaking test (111wav02). The model is capable of following the spatial variation of the total wave energy. The strong spatial variation in significant wave height on the bank (between \( x=14m \) and \( x=18m \)) in testcase 111wav02 is very well computed with both the Battjes-Janssen and Baldock breaker formulations.

Note that the predicted total amount of dissipation in a breaking wave is different for the latter two models. On the bank the wave energy dissipation is the same, which can be
concluded from the spatial variation in significant wave height predicted by the two models. However, the breaker fraction computed by Baldock's breaker formulation is only half the breaker parameter from Battjes and Janssen's formulation (see Figure 2.4a in Appendix A).

Due to the omission of triads, there is no energy shift towards higher frequencies. Consequently, the mean wave period $T_{m0.1}$ does not decrease towards to shoreline in the computations. Unfortunately, measured wave periods are not available.

In Figure 2.6-2.9 in Appendix A the wave spectra and integral wave parameters are shown for the test shallf01 in which wave breaking occurs on a shallow foreshore (Van Gent, 1999). The four situations differ in water depth (intermediate, shallow) and wave steepness. In general the breaker formulation of Hurdle and Van Vledder dissipates too much energy over the foreshore. The Battjes-Stive model predicts the correct trend of wave height decay towards the shoreline. Apparently, the wave steepness-dependent breaker parameter is more realistic in the present situation. As in the previous tests the onset of breaking is too much in shoreward direction. For steep waves the total amount of wave energy is also underpredicted using the breaker formulations of Battjes and Janssen and Baldock et al.. This is mainly manifest in intermediate water depth, where in nature (flume experiment) triad wave interactions are not dominant. In shallow water the strong decay in significant wave height at the start of the shallow foreshore is very well predicted with the breaker formulations of Battjes and Janssen and Baldock et al..

Once again, in shallow water the comparison between measured wave spectra and spectra obtained by SWAN is not useful for the one-dimensional tests considered in this study. Triad wave interactions tend to shift wave energy from the peak frequency to both the higher and the lower frequencies. Low frequency energy is also generated, due to the time variation of the breaking process. In the SWAN computations neither of the two mechanisms is included. For the steep waves in intermediate water depth (case shallf02, Figure 2.7 in Appendix A) the prediction of the mean wave period is correct. However, Figure 2.7b shows that the wave energy spectra maintain their shape towards the shoreline. Compared to the measured wave spectra, SWAN predicts too much energy at the peak frequency and cannot shift energy too lower and higher frequencies.

The results for the Wallingford experiments are given in Figures 2.10-2.15 in Appendix A. Due to the lack of nonlinear interactions in the wave modeling, we will only focus on the significant wave height distribution. In all tests there is hardly a difference between the model predictions with the Baldock or Battjes-Janssen breaker formulations. Despite a significantly lower value for the breaker fraction the Baldock formulation leaves a higher amount of wave energy dissipation than the Battjes-Janssen formulation, although the difference is very small.

Once again, the Battjes-Stive formulation leads to a good prediction of the trend of the wave height decay, but the total amount of wave energy dissipation is too low, due to a large breaker parameter (and thus large maximum wave height).

At the mildest slope (1:50), the breaker formulation of Hurdle and Van Vledder provides good agreement with the measured significant wave heights, see Figures 2.14a and 2.15a in Appendix A. In these figures the formulations of Baldock and Battjes and Janssen lead to too much energy dissipation. On the steep slopes (also in relatively shallower water) the difference is less pronounced. Compared to the measurements, the significant wave heights predicted with the Battjes-Janssen formulation or the Baldock formulation are in closer agreement than those obtained with the breaker formulation of Hurdle and Van Vledder.
Especially for the steep waves in test case s20d10u_pv (Figure 2.12a) there is a significant overprediction of the significant wave height at the top of the slope.

In the two-dimensional test of the Hiswa basin triad wave interactions have been taken into account in SWAN. On the transect y=0m (through point 2, 6, 9, etc., see Figure 2.2) Figure 2.16a in Appendix A shows hardly any difference in the significant wave height and mean wave period that have been computed with the Battjes-Janssen, Baldock, and Hurdle and Van Vledder formulation. The significant wave energy is predicted accurately up to the top of the construction. Behind the top the significant wave height is underestimated by approximately 1cm (10%). On the other hand, with the Battjes-Janssen formulation the measured wave height is predicted correctly over the entire transect, except on top of the construction. Similarly, the mean wave period is predicted more accurate with the Battjes-Janssen formulation. In the spectra at the locations 14, 18 and 25 we observe more energy in both the primary and secondary peak.

The formulation of Battjes and Stive dissipates less wave energy, especially at the higher frequencies, as can be concluded from the fact that higher significant wave heights are predicted whereas the computed mean wave period is lower.

At several locations wave spectra have been measured. At these locations we compare them with the computed wave energy spectra, see Figure 2.16b. In general the wave energy spectra computed with SWAN using either the Battjes-Janssen formulation, the Baldock formulation or the breaker formulation of Hurdle and Van Vledder are almost similar. Compared with the measured spectra SWAN underestimates the amount of wave energy at the peak frequency. With the Battjes-Stive formulation more energy is left at the peak frequency. At locations 14, 18 and 25 a significant second peak is present in the measured wave energy spectrum. With the Battjes-Stive formulation also at the second peak more wave energy is predicted, which is not an improvement for all locations.

Since nonlinear wave-interactions and wave-current interaction are important phenomena in this test, it is hard to isolate the process of wave breaking and to draw conclusions for any of the breaker formulations.

The second two-dimensional testcase that has been considered here, is the Petten field case. In general the wave energy spectra, and thus the integral wave parameters, are very well predicted by SWAN, see Figures 2.17-2.22 in Appendix A. The differences in results obtained with the four breaker formulations is not pronounced. The breaker formulation of Battjes and Stive provides less wave energy dissipation than the other formulations. In the six cases considered the significant wave height in MP5 (in the trough between the bar and the shoreline) is overpredicted by the Battjes-Stive formulation. However, in the shallower location MP6 (on the slope towards the beach), the best prediction of the significant wave height is obtained with the latter formulation in all six cases.

The formulation of Baldock et al. provides slightly lower estimates for \( H_{m0} \) than the Battjes-Janssen formulation. On the other hand, in all cases higher estimates for the mean wave period \( T_{m0,1} \) are obtained by the Baldock formulation, especially in shallow areas where breaking occurs. Apparently, relatively more energy is taken away from the higher frequencies, which is physically realistic. The best results for \( T_{m0,1} \), compared to the measurements are obtained with the Battjes-Stive formulation, despite a rather strong overprediction in the shallow area (MP6). The measured wave spectra at this location show relatively much wave energy at higher frequencies.
2.5 Conclusions and recommendations

Based on the validation study described in this chapter the following conclusions and recommendations can be given with respect to the modeling of depth-induced breaking in SWAN.

Conclusions

- Modeling of triad wave interactions in one-dimensional cases has occasionally led to convergence problems. Switching triads on and off within a distance of one or two grid cells and every one or two iterations may cause holes in the spectrum. Therefore, all one-dimensional tests have been carried out without taking into account wave triad interactions.
- Switching off triads leads to incorrect predictions of the mean wave period in shallow water. Upon the one-dimensional tests we can draw no conclusions about the effect of the various breaker formulations on the mean wave period. The two-dimensional tests show a similar behavior of the spatial evolution of the mean wave period for all breaker formulations. This was to be expected in all breaker formulations the dissipation rate is evenly distributed over all frequencies.
- Despite the more realistic Rayleigh wave height distribution in the surf zone, applied in the Baldock formulation, there is hardly any difference with the results obtained with the conventional Battjes-Janssen formulation on flat bottoms. However, we confirm the conclusion of Vink (2001) that the Rayleigh model of Baldock et al. (1998) proved to be an improvement on steep slopes.
- With the present constants in the expressions for the breaker parameter in the Battjes-Stive formulation, the breaker parameter itself reaches values (depending on the local wave steepness) that are too high. High values for the breaker parameter allow higher maximum waves, leading to less dissipation. Consequently, the initiation of breaking is predicted at a location closer to the coastline, leading to an overestimation of the significant wave height. However, SWAN including the Battjes-Stive formulation results in the best prediction of the trend that is shown by the wave decay towards the shoreline. The local information that is being put in this model by means of the local wave steepness seems to be an improvement. Furthermore, in shallow areas the significant wave height is in close agreement with the measurements.
- Strong variation in wave height over short distances are very well predicted by SWAN using either the Battjes-Janssen formulation or the Baldock formulation.
- One-dimensional simulations that converged with triads taken into account, showed similar significant wave height distributions as those simulations computed without triads, although the spectra were entirely different. The total amount of wave energy dissipation seems to be almost insensitive to the wave energy distribution in frequency space. This is in line with the conclusions in Battjes and Beij (1992, p.49).
- The breaker formulation of Hurdle and Van Vledder (2000) does not seem to give better predictions for wave energy spectra and integral wave parameters for the wide range of test cases considered in this study. Especially on shallow foreshores the predicted energy dissipation rate is too high. This is not surprising, since the model is calibrated and validated for slopes steeper than 1:50.

Recommendations

- To avoid convergence problems due to switching on and off triad wave interactions, new formulations of the breaker formulation should be implemented in the experimental
version 40.16 of SWAN. In that version triads and quadruplets can be computed permanently, such that the switching is avoided. Alternatively, this functionality should be implemented in the operational SWAN version.

- Convergence problems should be analysed thoroughly.
- The Battjes-Stive breaker formulation with the breaker parameter depending on the local wave steepness (according to Vink, 2001) should be implemented in an operational version of SWAN. A further calibration of the breaker parameter is recommended. The trends in wave decay are correctly predicted, but the amount of dissipation is too low. Furthermore, by definition of the presently implemented breaker parameter, steeper waves lead to a larger breaker parameter, which implies a higher maximum wave height and consequently, less dissipation. This is physically incorrect.
- The formulation of Balduck et al. (1998) should be implemented in an operational version of SWAN, either stand alone, or in combination with a breaker parameter that depends on the local wave steepness (Vink, 2001).
- A more accurate formulation of the triad wave interactions should be implemented. Both triad wave interaction and depth-induced wave breaking are the dominant processes in the surf zone. As long as the triads are not better approximated, the inaccuracies due to wave breaking can hardly be isolated. It will then be hard to get a complete understanding of the effect of wave breaking on the wave energy spectrum.
3 Convergence and iteration behavior

3.1 Introduction

In several experiments (e.g. Bottema, 2000) it was noted that the SWAN wave model does not reach convergence within an acceptable number of iterations. In addition, it has been reported that SWAN, in some cases, fails to reproduce some relevant physical features of the energy spectra. The reasons for these drawbacks in performance are not fully understood. However, a number of techniques have been suggested to improve the iterative behavior. Some techniques have been already implemented in the more recent (test-) versions of SWAN. For example, Haagsma and Otta (2001) have applied a limiter only for the quadruplet source terms. They concluded that this limiter is not a viable alternative. Other possible method they suggested were not implemented, but may prove to be more successful. Various alternatives will be studied here.

In this chapter a number of techniques attempting to solve the poor convergence and iteration behavior in SWAN are (re-)tested and discussed. The techniques are:

- A smoother for the action density spectrum.
- The combined onset of the triad and quadruplet source term.
- Alternative limiters, including a limiter over the deep water source terms.
- Underrelaxation of the action density update during the stationary iteration.

All techniques are investigated individually and in combination. Test runs are made for both deep and shallow water cases. In section 3.2 the SWAN experiments are specified with a coding of the runs. Section 3.3 briefly presents results of the smoother. In Section 3.4 the combined onset of the triad and quadruplet source terms is discussed. Section 3.5 describes in detail the findings about the source term limiters and the underrelaxation. This is the main part of this chapter. Some remarks on the combined applications of the described techniques are given in Section 3.6. Finally, in Section 3.7 we conclude on this work and present several recommendations for future SWAN development.

3.2 Deep and shallow water test cases

The techniques attempting to solve the poor convergence, except for the underrelaxation, are all implemented in a unreleased dedicated research version of SWAN which is called 4016N. The unreleased dedicated research version with underrelaxation is called 4011R. Test runs are made for a one-dimensional fetch-limited growth case. All runs are executed for both deep water (10,000 km) and shallow water (5 m). In fact, the F27 experiments of the SWAN test bed WL | Delft Hydraulics (2000) which consist of a wind input of 30 m/s and a wave growth over 25 km, are rerun (see Table 3.2). In these experiments the wave energy sink term of wave breaking is switched off. This choice may give, in some cases, unrealistic results, but it will not hamper conclusions on the techniques aiming at convergence improvements. All experiments are run in stationary mode.

Firstly, to order the test runs, a three character code is introduced to indicate the techniques which are applied to influence the convergence behavior. Table 3.1 explains the meaning of
the different characters. For example, the three letter code RUA corresponds to the no smoother (R=rough), \( U_{r_{quad}} = U_{r_{triad}} =0.1 \) (U), and Action density limiter (A) case and hence is equivalent to a run with SWAN version 40.11. The exact meaning of the settings will be clear from the context of the next sections.

<table>
<thead>
<tr>
<th>Technique</th>
<th>Symbol</th>
<th>Explanation</th>
</tr>
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<tbody>
<tr>
<td>smoother</td>
<td>S</td>
<td>smoother</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>no smoother</td>
</tr>
<tr>
<td>onset of quadruplets and triads</td>
<td>U</td>
<td>( U_{r_{quad}} = U_{r_{triad}} =0.1 )</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>combined onset</td>
</tr>
<tr>
<td>limiter</td>
<td>A</td>
<td>Action Density limiter</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>quadruplet source term limiter</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>Deep water source term limiter</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>Relaxed Pseudo time step (no limiter)</td>
</tr>
</tbody>
</table>

Table 3.1  Coding of the techniques to influence the iterative behavior of SWAN

Secondly, numerical codes (Table 3.2) are added to the test runs to indicate the number of iterations and the reference depth. The convergence criterion is based on the relative changes in significant wave height and mean wave period by setting the accuracy parameters to \( drel = dhoval = dtmpl = 0.001 \) and \( npnts = 100 \) (see swan user manual Holthuijsen et al., 2000). Effectively, these relative changes are so small that convergence within the number of iterations mentioned in Table 3.1, is not reached. For each experiment the number of iterations equals the maximum number of iterations.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Depth</th>
<th>Max number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>02</td>
<td>10^-7 m</td>
<td>5</td>
</tr>
<tr>
<td>03</td>
<td>10^-7 m</td>
<td>15</td>
</tr>
<tr>
<td>04</td>
<td>10^-7 m</td>
<td>50</td>
</tr>
<tr>
<td>05</td>
<td>10^-7 m</td>
<td>100</td>
</tr>
<tr>
<td>12</td>
<td>5 m</td>
<td>5</td>
</tr>
<tr>
<td>13</td>
<td>5 m</td>
<td>15</td>
</tr>
<tr>
<td>14</td>
<td>5 m</td>
<td>50</td>
</tr>
<tr>
<td>15</td>
<td>5 m</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 3.2  Numerical code for the deep and shallow water experiments from the WL SWAN test bed (F27) indicating bot depth and number of iterations

Finally, for every run the ‘iterative evolution’ of the main integrated wave quantities (e.g. \( H_{m0} \), \( T_{m0.1} \)) and the source terms are stored. Additional output is generated at three downwind locations, that is, at 0 m (location 1), 12.5 km (location 2) and 25 km
(location 3). Scripts are set-up to loop automatically over the described runs, changing only the key parameters.

### 3.3 Smoother

#### 3.3.1 Formulation of the smoother

The approximation of the quadruplet interactions (DIA) is very sensitive for the small perturbations. A smoothing mechanism has been developed by van Vledder (1999) which suppresses high-frequency wiggles. Therefore, the smoother may improve the convergence behavior.

#### 3.3.2 Effect of smoother

Figure 3.1 in Appendix A shows the quadruplet source term for tests on deep water performed with (top panel) and without (bottom panel) smoother. The overall iterative behavior is fairly similar. The smoother does not provide a significant speed-up of the convergence. The main difference for the quadruplet term is a stronger loss of energy in the 0.3-0.4 Hz frequency band (see Figure 3.1A). This loss of energy at higher frequencies does not completely suppresses the wiggles in the magnitude of the quadruplet dissipation seen as a function of iteration.

Figure 3.2 in Appendix A presents the results of smoothed and unsmoothed run for \( H_{m0} \) and \( T_{m0,1} \) after 100 iterations. Differences occur at the begin of the fetch (< 5km). The smoothed version gives an unrealistic wiggle in both quantities. Figures 3.2a and Figure 3.2b in Appendix A contain respectively the source terms in the extremes of the wiggle, that is at 2.5 km and 4.0 km. In the first extreme (the ‘dip’) we observe a shift of the peak frequencies to higher values for all deep water source terms and decreased magnitudes for wind input and white capping. In the second extreme (the ‘peak’) we do not observe such a frequency shift but the magnitudes of all source terms are increased in this case. It seems that the smoother interacts with the quadruplets in a peculiar way. At the end of the fetch, differences in the values of \( H_{m0} \) and \( T_{m0,1} \) are negligible, according to Figure 3.2. The results for the shallow water cases do not contain any notable other impacts of the smoother on the convergence.

### 3.4 Onset of nonlinear source terms

#### 3.4.1 Implementation

In the standard version of SWAN (version 40.11) the source term for the quadruplets is deactivated as soon as the triad source term is activated. Especially in situations in which these terms are of comparable magnitude, the separated onset and offset of these source terms have shown to cause unacceptable shocks in the solution procedure (in x-y space as well as in the iteration process).

In the newer test versions of SWAN (4016, 4016N), a relaxation procedure is implemented allowing the combined onset of the quadruplets and triads source terms. In fact, individual
threshold values for the Ursell number $ Ur $ are introduced. These bounds are $ Ur_{quad} $ and $ Ur_{triad} $, respectively. Hence,

$$ \begin{cases} 
S_{n3} \text{ active,} & Ur \geq Ur_{triad} \\
S_{n4} \text{ active,} & Ur \leq Ur_{quad} 
\end{cases} \quad (3.1) $$

where $ Ur_{triad} $ is generally less than $ Ur_{quad} $. In the test cases of this chapter $ Ur_{quad} $ and $ Ur_{triad} $ are set to 0.5 and 0.02, respectively, whereas the original (SWAN version 40.11), threshold Ursell number is 0.1. This range of combined onset will generally have a strong impact on the results. Technically, $ Ur_{quad} $ is set in the LIMITER statement and $ Ur_{triad} $ is set in the TRIAD statement of the input file (see Holthuijsen et al., 2000 and Haagsma and Otta, 2001). Note that this may create confusion for the SWAN user.

For the deep water case the results with the combined onset are not different from those of Ursell equal to 0.1. The depth is too large to have any influence due to triads.

For the shallow water case, however, significant differences occur. In Figure 3.3 in Appendix A, the source terms as a function of frequency are plotted. In the RCA case triads are activated in contrast to the RUA case. It results in an enhanced transfer of energy from the lower frequencies to the higher frequencies. Dissipation by friction (middle right panel) is shifted towards higher frequencies and waves are thus dissipated more effectively. Wave breaking is switched off in these tests and the main dissipation is due to the bottom friction. The wind input (top left panel) is less energetic and so is the white capping (middle left panel). The absence of any triads in the RUA case suggests that Ursell equal to 0.1 is indeed a too strict onset threshold for the triads.

Figures 3.4 and 3.4a (spectra) in Appendix A show the effects on $ H_{m0} $ and $ T_{m0.1} $. Clearly, the combined onset of the nonlinear wave-wave interactions results in a reduced $ H_{m0} $ after 100 iterations. An additional run with 300 iterations is made to check the convergence. Apparently, the convergence in the combined onset is considerably slower, because after 300 iterations we get a different, more realistic solution (the $ H_{m0} $ at 25km are nearly equal, see Figure 3.4b). Figure 3.4c contains the corresponding source terms. The triad activity is further enhanced. The quadruplet activity is reduced. Note that in the RUA case strange quadruplet activity appears for the higher frequencies.

Finally, in the current versions the triad source term is switched off as soon as the Ursell number exceeds a threshold value. This step may also cause undesired effects with respect to the convergence behavior (see section 2.4.1).

### 3.5 Limiter

Most third generation wave models use a limiter on the action density spectrum increment to force numerical stability while iterating with a large time step. In this way convergence can be reached faster than when using a smaller time step. Additional arguments for justification are that in many cases the limiter is not active when convergence is nearly
reached or that the influence of limiter is small for the part of the spectrum of the main interest, that is, for deeper water, the lower frequencies (Hersbach and Janssen, 1999).

The SWAN model (version 40.11) also uses an action density limiter for both non-stationary and stationary cases. In the latter cases, the limiter assures convergence of the iteration procedure solving the stationary equations. The iteration can be seen as a time stepping procedure with a pseudo time step. Though not its prime role (as is suggested in the introduction of Haagsma and Otta, 2001), the action density limiter ‘corrects’ oscillations originating from the approximation of the quadruplets (DIA) enhancing further the stability of iteration.

It has been reported (e.g. Bottema, 2000) however that, the action density limiter significantly suppresses realistic physical processes in cases of shallow water. Examples of such processes are the increase of a second peak in the energy density spectrum at two times the peak frequency due to triad wave interactions, and the energy dissipation due to depth-induced wave breaking Haagsma and Otta (2001). In fact in this chapter, the RUA test runs show no triad activity in clearly shallow water, see Figure 3.3 in Appendix A. The limiter forces the Ursell number to remain smaller than 0.1, causing the quadruplets to be active and the triads to be switcheed off. This non-physical effects should be avoided by a less stringent functioning of the limiter.

The influence of the limiter on the linear and nonlinear processes has not been studied in great detail. Such a study would be a difficult task. It is a neat task to trace and understand the updates of source terms and the action densities during the iterations and the sweeps and make an stability analysis. As an alternative, other limiters can be tried out. At first sight, a promising approach, which is not well explored in the literature, but may relieve some of the side-effects of the action density limiter, is to limit the changes in parts of the source terms. This means that the limiter is used in a more restricted way. Here, we call these limiters source-term limiters. Haagsma and Otta (2001) have tested a source-term limiter, which works on the quadruplet source term only (SWAN test version 40.16). This limiter is discussed in more detail in the next section. In this report reruns of the tests of Haagsma and Otta (2001) are made for the F27 problem in deep and shallow water. In addition, following the recommendation of Haagsma and Otta (2001), the limiter is extended to be applied for the deep water source terms, that is, for the wind input, for the white capping and for the quadruplets.

Another alternative is to avoid the use of limiters altogether by using an adjusted (smaller) time step (or pseudo time step in stationary cases). In the wave model WaveWatch (Tolman, 1999) for example, this approach is implemented by calculating an optimal time step. The computational burden to reach the solution may be much larger, but the results may be more trustworthy from a physical point of view.

### 3.5.1 Formulation of the quadruplet source term limiter

First we describe the source term limiter of Haagsma and Otta (2001). Rearranging the BSBT finite difference equations, two successive iterations in SWAN can be represented by

\[
\begin{align*}
\left\{ \begin{array}{l}
    a_i \ N_i - bN_{i-1} = S_i + D_i, \\
    a_{i-1}N_{i-1} - bN_{i-2} = S_{i-1} + D_{i-1},
\end{array} \right.
\] (3.2)
where
- \( a_i = 1 / \Delta t + \sum \text{"factors related to } N_i\" \)
- \( b = 1 / \Delta t \)
- \( D_i = \sum \text{"shifts of } N_i\" \).

In fact, the finite differences are split up into central terms contributing to \( a_i \) and non-central terms contributing to \( D_i \). Substitution and elaboration of the expressions, e.g.,

\[
\begin{align*}
\{ a_i \Delta N + (a_i - b)N_{i-1} &= S_i + D_i, \quad \Delta N = N_i - N_{i-1}, \\
N_{i-1} &= (S_{i-1} + D_{i-1} + bN_{i-2}) / a_{i-1},
\end{align*}
\]

lead to the expression

\[
\Delta N = [(b - a_i) / a_i, a_{i-1}] (S_{i-1} + D_{i-1} + bN_{i-2}) + (S_i + D_i) / a_i, \quad (3.3)
\]

which corresponds to Haagsma and Otta (2001), equation 2.1. Building a limiter on the source terms only, the following steps are followed

- ignore \( D_i \) and \( D_{i-1} \)
- ignore \( bN_{i-2} \)

\[
\Delta N = (b - a_i)S_{i-1} / a_{i-1}a_i + S_i / a_i, \quad (3.5)
\]

which corresponds to Haagsma and Otta (2001), equation 2.2. Equation (3.5) is used to derive the limiter. The source term is adapted from the Phillips limiter by

\[
|\Delta N| = |(b - a_i)S_{i-1} / a_{i-1}a_i + S_i / a_i| \leq C_{\text{Limit}e} N_{\text{Phillips}}. \quad (3.6)
\]

Accordingly, for \( a_i > 0 \)

\[
-a_iC_{\text{Limit}e} N_{\text{Phillips}} \leq (b - a_i)S_{i-1} / a_{i-1} + S_i \leq a_iC_{\text{Limit}e} N_{\text{Phillips}}. \quad (3.7)
\]

and consequently
\[
\begin{align*}
S_i & \leq a_i C_{\text{Limiter}} N_{\text{Phillips}} + (a_i / a_{i-1}) S_{i-1} - (b / a_{i-1}) S_{i-1}, \\
S_i & \geq -a_i C_{\text{Limiter}} N_{\text{Phillips}} + (a_i / a_{i-1}) S_{i-1} - (b / a_{i-1}) S_{i-1}.
\end{align*}
\] (3.8)

Inequalities (3.8) correspond to Haagsma and Otto (2001) inequalities (2.3), except for a minus sign in front the last term! There seems to be a error in the report and, importantly, also in the implementation (version 40.16) of the limiter. This has consequences if the model is run for non-stationary cases. The consequences are not further explored, because we restrict ourselves to stationary cases (see Chapter 1). A further approximation is made by assuming \( a_i = a_{i-1} \). This results in

\[
\begin{align*}
S_i & \leq (1 - b / a_i) S_{i-1} + a_i C_{\text{Limiter}} N_{\text{Phillips}}, \\
S_i & \geq (1 - b / a_i) S_{i-1} - a_i C_{\text{Limiter}} N_{\text{Phillips}}.
\end{align*}
\] (3.9)

In case stationary cases, \( b = 0 \), the sign error in the derivation has no effect and the Haagsma and Otto (2001) limiter effectively reduces to

\[
|S_i - S_{i-1}| \leq a_i C_{\text{Limiter}} N_{\text{Phillips}}
\] (3.10)

This limiter still seems consistent, since near the stationary solution \( S_i \to S_{i-1} \) and the limiter correctly resorts no effect. However, this limiter raises a few questions, which cannot be answered shortly in this context of this project:

- For different experiments, different \( a_i \) occur due to differences in propagation speeds. Is it possible to make a proper choice of \( C_{\text{Limiter}} \) in the general case?
- What is the argument to use the Phillips spectrum \( N_{\text{Phillips}} \) as a limiting shape function? There is no clear relationship, especially in the case when \( S_i \) only corresponds to the quadruplet term.

### 3.5.2 Formulation and implementation of the deep water source term limiter

The deep water limiter explored in this report is essentially an extension of the quadruplet source term limiter. The limited part of the source term is replaced by

\[
S_i = S_{\text{wind},i} + S_{\text{nlk},i} + S_{\text{weap},i}
\] (3.11)

Again the Phillips spectrum is used to scale the limiter. In this case (assuming small variations in \( a_i \), see equation (3.10)) this choice seems to be more reasonable since the Phillips spectrum originates from deep water physics.
In the research version of SWAN (SWAN4016N) this limiter is implemented in the subroutine SOURCE.FOR (see Holthuijsen et al., 2000). The main reason is that the white-capping source term influences both diagonal (implicit) and the right-hand-side (explicit) of the linear system solved in every sweep (see Holthuijsen et al., 2000). In fact, the increment in the implicit parts of white capping source term is added to increment in the quadruplet and wind input term by multiplying the diagonal terms with $N_i$ before applying the limiter.

### 3.5.3 Validation of source term limiters

In Figures 3.5 en 3.6 in Appendix A, $H_{m0}$ and $T_{m0,1}$ results with the deep water source term limiter, the quadruplet source term limiter of Haagsma and Otta (2001) and the action density limiter are shown for the deep water case. The results for the deep water source term limiter are dramatic, $H_{m0}$ and $T_{m0,1}$ are quickly out of range. The left panels show the solutions for $H_{m0}$ and $T_{m0,1}$ after 15 iterations, whereas in the right panels the solution after 100 iterations is given. We have also checked the solution after 25 iterations, which is exactly the same as the one obtained by Haagsma and Otta (2001) for the quadruplet source term limiter. However, they stopped after 25 iterations and did not show solutions after 100 iterations. The right panels in Figure 3.6 clearly shows the lack of convergence, also in the quadruplet source term limiter case. This also causes the hump in the integral wave parameters at $x=10$ km. The blue lines indicating the RUD case are out of bounds of this plot.

In Figures 3.7 (deep water) and 3.8 (shallow water) in Appendix A, the source terms after 100 iterations of the deep water and shallow water case are plotted for the deep water source term limiter. In the deep water case, all the deep water source terms have grown out of bounds, demonstrating the malfunctioning of this limiter. For the shallow water case, however, the behavior is totally different. The wave growth is limited. The bottom friction also hinders the waves from growing infinitely.

Figure 3.9 in Appendix A shows the development of the triad source term for the action density limiter (top panel) and the deep water source term limiter (bottom panel). The action density limiter does not guarantee a smooth convergence behavior, because it shows an intermittent behavior of the triads. The latter limiter succeeds in being less restrictive for the triads, however results are still bumpy and without an acceptable convergence behavior.

### 3.5.4 Underrelaxation of the pseudo time step

As explained in the above the prime role of the limiter is to stabilize model runs with a larger (and thus faster) pseudo time step. The source term limiters of the previous section do not seem to be effective.

An alternative approach, which will compromise the rate of convergence, is to make smaller updates. In SWAN version 40.11 the pseudo time step for stationary cases is fixed. However, a smaller pseudo time step can be mimicked by an underrelaxation of the action density update with every iteration. In brief, this alternative can be described as follows:

Suppose the stationary update of SWAN is

$$N_{i+1} = N_i + S_i$$

(3.12)
Here the pseudo time step equals one. The underrelaxation implemented in the test version is given by

\[ N_{i+1} = \alpha(N_i + S_i) + (1 - \alpha)N_{i-1}, \quad \alpha < 1 \]  

(3.13)

When \( N_i \) is ‘near’ \( N_{i-1} \), \( N_{i+1} \approx N_i + \alpha S_i \), \( \alpha < 1 \), and \( \alpha \) is the new reduced pseudo time step. Shown in this report are results for \( \alpha = 0.1 \).

In Figure 3.10 in Appendix A, the results of the underrelaxation are shown in comparison with the action density limiter. The differences in \( H_{m0} \) and \( T_{m0,1} \) are very small, indicating that, for deep water, reliable results can be obtained without a limiter.

For the shallow water case, the convergence behavior is more problematic. The left panels of Figure 3.11 in Appendix A show still a stable solution after 25 iterations. Figure 3.12 in Appendix A shows the corresponding source terms. As expected, this solution lags that of the action density limiter case. After more iterations (see right panels of Figure 3.11 in Appendix A) the underrelaxation solution has become unstable. Figure 3.13 contains triad and quadruplet source terms for the underrelaxation case (RUR). For the first 20 iterations the triads (top panel) are much stronger (yet decaying) than in the action density case (RUA), see Figure 3.9 in Appendix A. This suggests that suppression of triad interactions by the action density limiter might be avoided by the underrelaxation. However, the bottom panel show an abrupt growth of the quadruplets likely due to the onset of this source term by exceeding the threshold Ursell number of 0.1. This abrupt growth has irrevocably destabilized the solution. A stable behavior is obtained by taking a smaller pseudo timestep. The determination of the optimal choice for this timestep is out of the scope of this study.

### 3.6 Combined application of techniques

The mentioned techniques to improve the convergence of SWAN are also systematically tested in combinations. These tests are not described in this report. In general, the dramatic impact of the limiter is dominant over the more modest influences due to the combined onset of triads and the quadruplets and/or due to the smoother. In other words, the combined tests do not lead any further understanding or conclusions.

Exceptions to the above are the tests with underrelaxation of the pseudo time step. These tests are implemented in a very ad hoc manner, frustrating, so far, a quick combination with the other improvement techniques. Figure 3.13 in Appendix A suggests that the underrelaxation of the pseudo time step in combination with the combined, and possibly less abrupt, onset of the triads and quadruplets might avoid destabilization.

### 3.7 Conclusions and recommendations

From the study presented in this report we draw the following conclusions and make a number of recommendations with respect to the convergence and iteration behavior.
Conclusions

- The limiter on the quadruplet source term as stated in Haagsma and Otta (2001) and as implemented in SWAN test version 40.16 contains an error. This error causes effects only for non-stationary cases. Therefore, the results of Haagsma and Otta (2001) can still be considered as authentic.
- In this study source term limiters (for the quadruplet term and for all the deep water source terms) are described and tested. These limiters do not seem to be effective. They do not stabilize the convergence behavior. In many cases, the old action density limiter should be preferred.
- In this study the combined onset of quadruplet and triad source terms in SWAN is studied. The main effect of the combined onset is an enhanced triad interaction, resulting in a reduced $H_{n0}$ and $T_{m0,1}$, consequently enhancing earlier found biases for coastal applications. The convergence in the combined onset is considerably slower.
- A first study has been made testing an underrelaxation of the pseudo time step as an alternative for the use of limiters. Though compromising the speed of convergence, this underrelaxation may provide more realistic solutions.
- The sole impact of an action density smoother on the convergence behavior is small. Also in combination with other convergence techniques the effect of a smoother on the convergence is small.

Recommendations

- In this chapter, sensitivity studies and ‘trial and error’ have been applied to obtain some insight. Real progress, however, requires careful numerical analyses of the iterative system. The numerical solution procedures used in SWAN are not well documented. To obtain insight in the numerical iterative systems, an inventory of the entire numerical solution procedure must be made.
- It does not seem worthwhile to study source term limiters at this time. It is uncertain whether they may ever function adequately. At the moment the understanding of underlying iterative system is too poor. Therefore, it seems more appropriate to gain insight in the underlying iterative system, or more generally, the entire numerical implementation.
- The SWAN code needs to be tidied up. A cleaner code will help the implementation of alternative functionality and will enhance a logic specification of numerical experiments. For example, to set $U_{r_{quad}}$ in the LIMITER statement and $U_{r_{tria}}$ in the triad statement may confuse the user. When more drastic revisions of SWAN are desired, it should be considered to rewrite substantial parts of the code.
- A good solution for expert users of SWAN to avoid the unwanted side-effects is to implement an option for a reduced (pseudo) time step which facilitates a slow but steady convergence of the model without limiters. Further research is needed to fully evaluate the benefits and drawbacks of using the underrelaxation of the pseudo time step as an alternative for limiters. A next step would be to implement the underrelaxation of the pseudo time step in a version of SWAN (e.g., 40.16) with a smooth onset of the quadruplet and triad source terms.
4 Reflection

4.1 Introduction

The present implementation of the treatment of reflection and transmission in SWAN 40.11 has some restrictions with respect to the use of obstacle polygons. These restrictions comprise the relation between the individual line pieces of obstacle polygons and the computational grid. Another restriction comprises the relation between the obstacle polygons and the bottom grid. These restrictions are, in our opinion, too briefly described in the user manual. It is therefore difficult for users to take care of these restrictions. Another problem concerns the definition of the reflection coefficient in relation with transmission.

To improve this situation, suggestions for a better description of the use of the obstacle command are given, and a computational procedure has been developed to check the relation between obstacle polygons and the computational grid.

Information about the implementation of obstacles in SWAN has been obtained from Annette Kieftenburg, formerly of the SWAN development group of Delft University of Technology.

4.2 Restrictions to obstacle polygons

For a correct use of obstacle polygons in SWAN the following criterions should be met:

1. Each line piece of an obstacle polygon should have at least one crossing with a line segment of the computational grid. If this condition is not met, SWAN does not resolve this line segment.
2. Each line segment between 2 grid points of the computational grid should have at most one crossing with an obstacle polygon. If more than one crossing point is found, SWAN takes only the last encountered crossing point into account, thereby neglecting the other(s).
3. Line pieces of an obstacle polygon should not cross other pieces of an obstacle polygon. If this condition is not met, it is probable that criterion 2 is also not met.
4. Line pieces of an obstacle polygon that lie between an active (wet or dry) point and an exception point, are not taken into account. The reason for this behaviour is that grid points that are assigned as exception points do not have a 'position'. Since their position is unknown, the reflection mechanism in SWAN cannot compute the point of reflection. The effect is that the obstacle is ignored and the computational results give the impression that no reflection occurs.

Despite the fact that these criteria are partly described in the user's manual, it is difficult in practice to adhere to these criteria. For instance, obstacle polygons are often generated by GIS systems in which lines are generated with a fixed step size of, say, 1 m, even if most of such points lie on a straight line. In that case the line segments are so small that most of the obstacle is not resolved by the reflection mechanism. An example of such a segmented line is shown in the upper panel of Figure 4.1 in Appendix A. The line segments that are detected by SWAN are visualised as thick red lines.
Another problem might occur if the computational grid size changes, e.g. for reasons of accuracy or computational efficiency. If a smaller grid size is chosen, it is possible that criterion 1 will be violated. This situation is shown in the middle panels of Figure 4.1. The arrow indicates the relation between the two grids. If, on the other hand, the grid size increases, it is possible that criterion 2 will be violated. This situation is illustrated in the lower panels of Figure 4.1. The problematic line segments are visualized as thick red lines.

Effects of various reflection line configurations on the incoming and reflected discrete spectra are shown in the Figures 4.2 to 4.6 in Appendix A. The layout of these figures is as follows. The lower right panel shows a discrete spatial grid with a fixed spacing of 10 m. The arrows on the lower and left boundary represent the incoming wave boundary conditions. The bold dots on the two right columns are dry points. The red circles with red crosses are test output locations and the thick red line indicates the discrete reflection line. The numbers near the output points are the location numbers and they reflect the column and row in the computational grid. The other 5 panels show the normalized polar energy density spectra in the output points.

Figure 4.2 in Appendix A shows the effect of a small reflection obstacle, which crosses the computational grid. It can be seen that with increasing distance to the obstacle, the width of the directional sector of the reflected spectrum decreases. This is due to propagation effects, such that with increasing distance fewer reflected (directional) wave components can reach these points. The effect of reducing the size of the obstacle is shown in Figure 4.3 in Appendix A. Compared to Figure 4.2 it can be seen that the width of the reflected directional sector is smaller. Figure 4.4 in Appendix A shows the situation that the size of the obstacle is as long as in Figure 4.2, but now shifted a little bit downwards, such that it has no crossing point with a grid line. It can be clearly seen that the obstacle is not resolved by the SWAN reflection mechanism. Figure 4.5 in Appendix A shows the effect of a large obstacle positioned in front of dry land points. The effects on the spectra are as expected. Defining the land points as exception points leads to the effect shown in Figure 4.6 in Appendix A. The reflection line is not resolved by the SWAN reflection mechanism.

To ensure that all criteria are met, a computational procedure has been developed which checks the relation between all obstacle polygons, and the computational grid and bottom grid. In first instance this procedure was developed to run outside the SWAN model. Later, it might be included in the SWAN pre-processor.

### 4.3 External check mechanism

The functionality of the procedure to check the relation between the obstacle polygons and the computational grid contains the following elements:

1. Scan the SWAN input file to retrieve the characteristics of the computational grid, bottom grid, exception values and all obstacle polygons.
2. Check if each individual line piece of the obstacle polygons has one or more crossings with the lines connecting the grid points of the computational grid.
3. Check if all individual lines connecting the grid points of the computational grid have at most one crossing per line piece of an obstacle polygon.
4. Check if obstacle polygons do not cross themselves or cross other obstacle polygons.
5. Check the relation between obstacle polygons and the properties of the depth values on the computational grid to determine if obstacle polygons do not lie next to an exception point.
6. Check is straight lines are composed of too small individual line pieces (cf. upper panel of Figure 4.1). If this is the case, a message should be issued to the user.

For the present report, the elements 2, 3 and 4 have been implemented in the computational procedure CHECK_OBSTACLE. Input elements to this procedure are the numerical characteristics of the computational grid and the name of an obstacle polygon. This procedure has been tested with a hypothetical computational grid and obstacle polygon. The characteristics of the polygon have been selected in such a way that it produces error messages for the violation of the criteria 1, 2 and 3. In addition a MATLAB script has been developed to visualise the errors. An example thereof is shown in Figure 4.7 in Appendix A in which the problematic elements of the obstacle polygon are indicated in red.

### 4.4 Definition of reflection and transmission coefficient

In the present implementation, the coefficient of reflection is applied on that part of the wave energy that is not transmitted. So, the amount of reflected energy depends as follows in the incoming wave energy \( E_{\text{inc}} \):

\[
E_R = C_R \times (1 - C_T) E_{\text{inc}}
\]  

(4.1)

In this equation, \( C_R \) is the reflection coefficient, \( C_T \) the transmission coefficient, \( E_{\text{inc}} \) the amount of incoming energy and \( E_R \) the amount of reflected energy. This implementation implies that the amount of reflected energy depends on two coefficients, viz. the coefficient of reflection and transmission. However, in our opinion it is more logical that the amount of reflected energy depends on one coefficient only, and on the amount of incoming wave energy.

In general, incoming wave energy is either reflected, transmitted or absorbed, or a combination of these.

\[
E_{\text{inc}} = E_R + E_T + E_A
\]  

(4.2)

in which

\[
E_R = C_R \times E_{\text{inc}}
\]

\[
E_T = C_T \times E_{\text{inc}}
\]

\[
E_A = C_A \times E_{\text{inc}}
\]

(4.3)

Conservation of energy implies that \( C_R + C_T \leq 1 \), but also that \( 0 \leq C_R \leq 1 \) and \( 0 \leq C_T \leq 1 \). It is advised that these criteria are included in the pre-processing part of the SWAN model.

### 4.5 Conclusions and recommendations

Based on the analysis of the reflection mechanism in the SWAN model the following conclusions and recommendations are formulated:

- There must be a balance between the resolution of the obstacles and the resolution of the polygons, which are used to represent reflection and transmission lines.
• Users must be aware of the fact that reflection and transmissions line are ignored when they lie next to grid points that are defined as exception points.
• The definition of the reflection and transmission coefficients should be modified such that the amount of reflected wave energy depends on only one coefficient.
• The SWAN pre-processing module should be extended with a check on the consistency and validity of the coefficients for reflection and transmission.
• The SWAN pre-processing module should be extended with a check on the relation between the computational grid and the obstacle/transmission polygons (e.g. by inclusion of the program CHECK_OBSTACLE).
5 Quadruplets

5.1 Introduction

5.1.1 The role of nonlinear quadruplet wave-wave interactions

This report presents various methods to improve the quality and computational efficiency of the non-linear quadruplet wave-wave interactions in discrete spectral operational wind wave models, like SWAN, WAVEWATCH and WAM. All of these models now use the Discrete Interaction Approximation (DIA). It is generally known that this approximation is in poor agreement with exact computations for this transfer. One of the consequences is that the source term balance in wind-wave models is incorrect. As long as this imbalance exists it is of limited use to develop and calibrate other source terms. Therefore, the DIA needs to be replaced with improved methods.

Non-linear wave-wave interactions between pairs of four wave components, so-called quadruplets, play an important role in the evolution of wind generated waves (Phillips, 1981; Young and Van Vledder, 1993). Hasselmann (1962) developed the theoretical framework for these interactions. He formulated an integral expression for the computation of these interactions, which is known as the Boltzmann integral for surface gravity waves. Hasselmann (1962) found that a set of four waves, called a quadruplet, could exchange energy when the following resonance conditions are satisfied:

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$

(5.1)

(5.2)

in which $\omega_j$ the radian frequency and $\vec{k}_j$ the wave number ($j=1,\ldots,4$). The linear dispersion relation relates the frequency and the wave number:

$$\omega^2 = gk \tanh(kh).$$

(5.3)

Here, $g$ is the gravitational acceleration and $h$ the water depth. Hasselmann (1962, 1963a/b) describes the non-linear interactions between wave quadruplets in terms of their action density $n$, where $n(\vec{k}) = E(\vec{k})/\omega$. The rate of change of action density at a wave number $\vec{k}_1$ due to all quadruplet interactions involving $\vec{k}_1$ is:

$$\frac{\partial n_1}{\partial t} = \iiint G(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) \times \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$$

$$\times \left[ n_1 n_2 (n_3 + n_4) - (n_1 + n_2) n_3 n_4 \right] d\vec{k}_2 d\vec{k}_3 d\vec{k}_4$$

(5.4)
Where \( n_i = n(k_i) \) is the action density at wave number \( \bar{k}_i \) and \( G \) is a complicated coupling coefficient (Hasselmann 1962, 1963a/b; Herterich and Hasselmann, 1980). The delta functions in (5.4) ensure that contributions to the integral only occur for quadruplets that satisfy the resonance conditions. The integral expression (5.4) is also known as the Boltzmann integral for wind waves. Zakharov (1968) developed a similar expression, known as the kinetic equation, which basically gives the same results.

The computation of the Boltzmann integral is rather complicated and very time consuming since it requires the solution of a 3-fold integral. As can be seen in Eq. (5.4) the integration variables are the three two-dimensional wave number vectors \( \bar{k}_2, \bar{k}_3, \) and \( \bar{k}_4 \). Because of this complexity it is (still) not feasible to include the full solution of the Boltzmann integral in operational wave models. To overcome this shortcoming Hasselmann et al. (1985) developed the Discrete Interaction Approximation (DIA). Hasselmann et al. (1985) show that the DIA preserves a few but important characteristics of the full solution, such as the slow downshifting of the peak frequency and shape stabilisation during wave growth. The development of the DIA triggered the development of third generation wave prediction models, like the WAM model (WAMDI, 1988), WAVEWATCH (Tolman, 1991), TOMAWAC (Benoit et al., 1996) and the SWAN model (Booij et al., 1999). The DIA was initially developed for deep water. The WAM group (WAMDI, 1988) introduced a scaling technique to compute the non-linear transfer for an arbitrary water depth. This depth scaling has some deficiencies, which are described in section 5.4.4 of this report and in Appendix C.

### 5.1.2 Deficiencies of the Discrete Interaction Approximation

In the last few years it became evident that the DIA shows some deficiencies (cf. Van Vledder et al., 2000). The DIA is not able to properly represent the non-linear transfer rate in comparison with exact solutions of the Boltzmann integral. This is illustrated in the Figures 5.1 through 5.11 in Appendix A for a number of test spectra using results of computations in which the nonlinear transfer rate is computed using the 'exact' wright-method (described in Section 5.3) and the Discrete Interaction Approximation (described in Section 5.4). All of these figure have 6 panels. The upper left panel shows the frequency spectrum, and the upper right panel shows a polar plot of the normalized 2d energy density spectrum. The middle left panel shows a comparison between the directionally integrated nonlinear transfer rate, and the lower left panel show the frequency integrated nonlinear transfer rate. The middle right panel shows a polar plot with the 2-dimensional exact nonlinear transfer rate and the lower right panel show a polar plot with the 2-dimensional nonlinear transfer rate as computed with the DIA. The contour levels in the normalized polar plots are 0.9, 0.5, 0.1, 0.01, 0.001, 0.0001, and 0.00001. Yellow and red colors refer to positive values, whereas blue and green colors refer to negative values.

Figure 5.1 shows a comparison between the nonlinear transfer rate for a JONSWAP spectrum with \( f_s = 0.1 \) Hz, \( \alpha = 0.0175 \), \( \gamma = 3.3 \) and a \( \cos^2(\theta) \)-directional spreading as input. The frequency range was 0.05 Hz - 0.8 Hz, with 40 frequency bins geometrically spaced. The directional step 10° over the full circle. The DIA was applied with \( \lambda = 0.25 \). Comparison of the results produced by the DIA with the results of the exact computations shows that the DIA produces a negative lobe that is too large, that the frequency of the position of minimum of this lobe is over-predicted, and that the second positive lobe is over-predicted. Further, the frequency of the first zero-crossing of the transfer rate is much higher than the peak frequency, whereas it is expected that the first zero-crossing is at the peak frequency. Inspection of the directional distributions indicates that the DIA transfers too much energy to
side-bands. This is one of the reasons that the DIA produces spectra that are wider in directions than those computed with models using an exact method to compute the nonlinear transfer rate. Further differences become evident from the 2-dimensional distributions, such as the double peak in the lowest positive lobe.

Figure 5.2 shows the effect of choosing another value of the DIA shape parameter, $\lambda=0.19$. As can be seen the agreement between the DIA and exact results increases considerably for the frequency or directionally integrated nonlinear transfer rate. This agreement for a JONSWAP spectrum is one of the reasons that in Japan (cf. Hashimoto and Kawaguchi, 2001) $\lambda=0.19$ is chosen as the default value for the DIA in operational wave prediction models. However, the 2-dimensional transfer rate shows some differences, mainly in the directional distribution as a function of frequency. The reason that the WAM, SWAN and WAVEWATCH model use $\lambda=0.25$ is due to the fact that Hasselmann et al. (1985) performed numerical growth curve experiments to determine the optimal value of the DIA shape parameter $\lambda$.

Figure 5.3 show a comparison a Pierson-Moskowitz spectrum. The agreement for the frequency and directionally integrated transfer rates is good, although some differences are visible for the 2-dimensional transfer rate. Figure 5.4 shows the results for a very peaked JONSWAP spectrum with $\gamma=10$. The frequency integrated transfer rate show good agreement, but the directionally integrated transfer rate shows a mismatch in the position of the first positive lobe and the first negative lobe. However, significant differences are visible in the 2-dimensional transfer rate.

The Figures 5.5 and 5.6 show the effect of various directional distributions on the nonlinear transfer rate. Figure 5.5 shows the results of a wide spectrum with a $\cos(0)$-distribution and Figure 5.6 shows the results of a very narrow spectrum with a $\cos^{40}(0)$-distribution. The results for the wide directional distribution are comparable with those shown in Figure 5.1. The results for the narrow directional distribution show large differences. Especially, the directional distribution shows large differences.

Figure 5.7 shows the results for a depth modulated spectrum. Here, the TMA scaling (Bouws et al., 1985) is applied with a scaling depth of 10 m. This depth is only used to modify the shape of the spectrum. For the computations of the nonlinear transfer rate deepwater is assumed.

Figure 5.8 shows the results for a JONSWAP spectrum but now with an $f^4$ spectral tail. The DIA has been applied with $\lambda=0.19$ and the results should be compared with those in Figure 5.2. As can be seen in these figures, the shapes of the nonlinear transfer rate are very different, indicating that the nonlinear transfer rate strongly depends on the power of the spectral tail. This result also implies that the Japanese preference of $\lambda=0.19$ is only valid for a mean JONSWAP spectrum with an $f^4$ spectral decay.

Figure 5.9 shows the results for a double peaked JONSWAP spectrum, with peaks at 0.1 Hz and 0.13 Hz. As can be seen the DIA produces a number of wiggles in the frequency distribution of the nonlinear transfer rate, especially at higher frequencies.

Figure 5.10 shows the results for a double peaked JONSWAP spectrum in which both peaks have a peak frequency of 0.1 Hz, but the peaks are separated in direction by 90°. It can be seen that the DIA produces a number of wiggles in the 2-dimensional transfer rate that do not appear in the exact solution. Another relevant difference in the nonlinear transfer rates appears at the higher frequencies around the bi-section angle of 45°.
Figure 5.11 shows the results for a JONSWAP spectrum represented by a very fine frequency distribution, 100 frequencies in the range 0.05-0.8 Hz. As can be seen the DIA produces a double peaked positive lobe. This behaviour was studied by Van Vledder et al. (2000) who showed that choosing a very fine frequency resolution produces unphysical effects on fetch-limited wave evolution. The results of the exact transfer rate do not show this behaviour. This test indicates that the DIA is only suited for a frequency resolution of about 10% (i.e. succeeding frequencies are related as $f_{i+1} = 1.1f_i$) and a directional step of about 10° to 30°.

The above presented discrepancies distort the source term balance of wind wave spectra. One of the most prominent effects is that spectra computed with a wave model incorporating the DIA are broader, both in frequency and direction, than computed with a wave model incorporating an exact solution of the Boltzmann integral. Another prominent feature of the DIA is that it often transfers too much energy towards higher frequencies contributing to the under-prediction of period measures and contributing to instabilities in numerical integration schemes.

Despite these deficiencies the DIA has been proven rather useful for the successful application of third-generation wave prediction models, since deficiencies in the DIA were masked by deficiencies of other source terms. With improvement in the modelling of other physical processes this argument cannot be supported anymore. Since these deficiencies affect the total source term balance the further development of source terms for other physical processes is of limited value as long as the present DIA is part of a wave model under consideration. To overcome the above shortcomings of the DIA, coefficients in the source terms for wind input and white-capping dissipation in WAM-type models are heavily tuned to compensate for the mismatch in the DIA.

Another point of concern is the computational requirement of the DIA in operational wave models. In comparison with other source terms it takes most of the required CPU. The relative time spent in the DIA also depends on the complexity of the numerical scheme of the host model. In the WAM model the DIA takes about 50% of the computational time. In the SWAN model the DIA only takes about 20% of the total time. The reason for this difference is that the SWAN model uses relatively more CPU time in additional source terms, data management and the numerical scheme than the WAM model.

To improve the overall model performance of third generation wave prediction models, at least from the viewpoint of quadruplet wave-wave interactions, attention is paid to the improvement of the quality of the non-linear quadruplet interaction source term and to the development of faster routines for the calculation of these interactions.

## 5.2 Methodology

To improve the DIA a number of methods have been developed. First, an exact benchmark model was established to be able to test any improved or extended DIA. To that end, the Webb/Resio/Tracy (WRT) model was chosen as the benchmark model. Since the original WRT code was not suited for that purpose, the WRT method was analysed and reprogrammed in Fortran 90. In the end, total agreement was found between the two methods. The analysis of the WRT also resulted in a good understanding of the Boltzmann integral. This knowledge has also been used to developed approximate methods of the Boltzmann integral.
Secondly, a general methodology was developed to extend the DIA. This work was carried out jointly with Rasmussen from Denmark. In Rasmussen (1995) a detailed derivation was given of the DIA. Based on that information Van Vledder (2001) found a general way to extend the DIA with additional wave number configurations. Recently Rasmussen and Van Vledder joined their efforts to derive the mathematical basis of a general DIA, which has the potential to approximate the exact solution of the WRT method with the same computational requirements.

Thirdly, a new simple technique was developed to compute the Boltzmann integral. This method is known as the triplet method and it resembles ideas originally proposed by Snyder et al. (1993).

The structure of this Chapter is as follows. In Section 5.3 a description is given of the WRT method for the evaluation of the Boltzmann integral. Section 5.4 contains the derivation of the Generalized Discrete Interaction Approximation, and the triplet method is presented in Section 5.5. Finally, conclusions and recommendations are given in the Sections 5.6 and 5.7.

### 5.3 The Webb-Resio-Tracy method

#### 5.3.1 Introduction

The WRT method is based on a number of transformations of the Boltzmann integral, originally proposed by Webb (1978). This method was programmed by Tracy and Resio (1982) for deep water. Later, the WRT-method was extended to include finite depth effects. Although this extension was made in the mid-eighties, no paper or report was published showing the equations. The only recent references on certain aspects of the finite depth WRT method are Van Vledder (2000) and Resio et al. (2001). In this chapter a full description is given of the finite depth WRT method.

In the method of Webb-Resio-Tracy a number of transformations are made to remove the delta functions in expression (5.4). A key element in the WRT method is to consider the integration space for each \((\vec{k}_1, \vec{k}_3)\) combination:

\[
\frac{\partial n_i}{\partial t} = \int d\vec{k}_2 T(\vec{k}_1, \vec{k}_3)
\]

in which the function \(T\) is given by:

\[
T(\vec{k}_1, \vec{k}_3) = \int d\vec{k}_2 d\vec{k}_4 \times G \times \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \\
\times \delta(\omega_1, \omega_2, \omega_3, \omega_4) \times \left[ n_i n_3 (n_i - n_3) + n_2 n_4 (n_2 - n_4) \right]
\]

For symmetry reasons the integration space in Eq. (5.6) can be reduced by a factor 2. Since the Boltzmann integral is symmetric with respect to interchanging the variables \(\vec{k}_1\) and \(\vec{k}_2\), or \(\vec{k}_3\) and \(\vec{k}_4\), we can omit part of the integration space where the wave number \(\vec{k}_1\) is closer
to wave number $\vec{k}_4$ than to wave number $\vec{k}_3$. Mathematically, this is achieved by using the Heaviside function $\theta(x)$:

$$\theta(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x < 0 
\end{cases} \quad (5.7)$$

and

$$x = |\vec{k}_1 - \vec{k}_4| - |\vec{k}_1 - \vec{k}_3| \quad (5.8)$$

The reduction in integration space is compensated by a factor 2 in the integral expression. Thus:

$$T(\vec{k}_1, \vec{k}_3) = 2 \int d\vec{k}_2 d\vec{k}_4 \times G \times \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4)$$

$$\times \delta(\omega_1, \omega_2, \omega_3, \omega_4) \times \theta\left(|\vec{k}_1 - \vec{k}_4| - |\vec{k}_1 - \vec{k}_3|\right)$$

$$\times \left[n_1 n_3 (n_4 - n_2) + n_2 n_4 (n_3 - n_1)\right] \quad (5.9)$$

In the following, it is assumed that the integration space is restricted by the criteria (5.7) and (5.8). The delta-function over the wave numbers can be eliminated by writing $\vec{k}_4$ as

$$\vec{k}_4 = \vec{k}_1 + \vec{k}_2 - \vec{k}_3 \quad (5.10)$$

The transfer integral $T(\vec{k}_1, \vec{k}_3)$ then becomes

$$T(\vec{k}_1, \vec{k}_3) = 2 \int d\vec{k}_2 \times G \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \times \left[n_1 n_3 (n_4 - n_2) + n_2 n_4 (n_3 - n_1)\right] \quad (5.11)$$

Before eliminating the remaining delta-function the integration space is considered. For a given combination of the wave numbers $\vec{k}_1$ and $\vec{k}_3$ the integration space consists of two coupled closed lines, one for the wave number $\vec{k}_2$ and one for $\vec{k}_4$. An example of such a line is given in Figure 5.12 in Appendix A. Such a line is also referred to as locus. The loci have a symmetry axis parallel to the difference vector of $\vec{k}_1$ and $\vec{k}_3$.

The crux of the Webb model consists of using a local co-ordinate system $(s, n)$ along the $\vec{k}_2$-locus. To that end the $(k_{2,n}, k_{3,n})$ co-ordinate system is replaced by a local $(s,n)$ co-ordinate system along the locus. We then get

$$T(\vec{k}_1, \vec{k}_3) = 2 \int ds dn \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$$

$$\times G \times \left[n_1 n_3 (n_4 - n_2) + n_2 n_4 (n_3 - n_1)\right] \quad (5.12)$$
It is noted that the variables \( \eta_1 \) refer to action density at wave number \( \bar{k}_1 \). Since \( \omega_1 \) and \( \omega_3 \) are fixed, \( \omega_2 \) is a function of the wave number \( \bar{k}_2 \), the remaining radian frequency \( \omega_4 \) follows from the wave number \( \bar{k}_4 = \bar{k}_1 + \bar{k}_2 - \bar{k}_3 \). Thus, the delta function can be written as:

\[
\delta \left( \omega_1 + \Omega \left( \bar{k}_2 \right) - \omega_3 - \Omega \left( \bar{k}_1 + \bar{k}_2 - \bar{k}_3 \right) \right) \tag{5.13}
\]

The term inside the delta-function are a function of the wave number \( \bar{k}_2 \). Writing this function as:

\[
W \left( \bar{k}_2 \right) = \omega_1 + \Omega \left( \bar{k}_2 \right) - \omega_3 - \Omega \left( \bar{k}_1 + \bar{k}_2 - \bar{k}_3 \right) \tag{5.14}
\]

The change of co-ordinate system results in:

\[
T \left( \bar{k}_1, \bar{k}_3 \right) = 2 \iint ds \, dn \, \delta \left( W \left( s, n \right) \right) \times G \times \left[ n_1 n_2 \left( n_4 - n_2 \right) + n_2 n_4 \left( n_3 - n_1 \right) \right] \tag{5.15}
\]

To integrate over the delta-function a special property of the delta-function is considered:

\[
\delta \left( f \left( x \right) \right) = \frac{\delta \left( x - x_0 \right)}{df \, dx} \tag{5.16}
\]

where \( f \left( x_0 \right) = 0 \) and \( \delta \left( f \right) df = \delta \left( x \right) dx \). Application of this property on \( \delta \left( W \left( s, n \right) \right) \) yields:

\[
\delta \left( W \left( s, n \right) \right) = \left. \frac{\delta \left( n - 0 \right)}{\delta W \left( s, n \right)} \right|_{\delta n} \tag{5.17}
\]

Using (5.17) the transfer function (5.15) can be written as:

\[
T \left( \bar{k}_1, \bar{k}_3 \right) = 2 \iint ds \, dn \, \left. \frac{\delta \left( n - 0 \right)}{\delta W \left( s, n \right)} \right|_{\delta n} \times G \times \left[ n_1 n_2 \left( n_4 - n_2 \right) + n_2 n_4 \left( n_3 - n_1 \right) \right] \tag{5.18}
\]

The term \( \left| \frac{\partial W}{\partial n} \right|^{-1} \) is denoted as the inverse of a gradient term. In this note it will be denoted with the symbol \( R \) (after gRadian and Resio) since the symbol \( G \) is already in use.
\[ R = \left| \frac{\partial W(\tilde{s}, \tilde{n})}{\partial n} \right| \quad (5.18b) \]

The normal derivative in the denominator is the magnitude of a gradient. Therefore, \( R \) can be rewritten in terms of the x- and y-components of the wave number \( k_2 \):

\[ R = \left| \frac{\partial W(s, n)}{\partial n} \right| = \left| \nabla W(k_{2,x}, k_{2,y}) \right| \quad (5.19) \]

In Tracy and Resio (1982) the term \( R^{-1} \) is referred to as the phase term. It is noted here that the term \( R \) is similar to the term \( S \) (from singularity) in Masuda (1980) and Hashimoto et al. (1998). This term is given by (as simplified by Gorman, 2000):

\[ S = \left| 1 - \frac{c_{g,2}}{c_{g,4}} \cos(\theta_2 - \theta_4) \right| \quad (5.19b) \]

Note that this term is not symmetric with respect to the group velocities.

The delta functions in Eq. (5.4) determines a region in wave number space along the integration should be carried out. For a given \((\tilde{k}_1, \tilde{k}_3)\) wave number combination the resonance condition for the radian frequencies can be written as:

\[ W(\tilde{k}_2) = 0 = \omega_1 + \Omega(\tilde{k}_2) - \omega_3 - \Omega(\tilde{k}_1 + \tilde{k}_2 - \tilde{k}_3) \quad (5.20) \]

Following Tracy and Resio (1982) the variables \( Q \) and \( \tilde{P} \) are introduced:

\[ Q = \omega_1 - \omega_3 \quad (5.21) \]

\[ \tilde{P} = \tilde{k}_1 - \tilde{k}_3 \quad (5.22) \]

Then the wave number \( \tilde{k}_4 \) can be written as:

\[ \tilde{k}_4 = \tilde{k}_1 - \tilde{k}_3 + \tilde{k}_2 = \tilde{P} + \tilde{k}_2 \quad (5.23) \]

The locus equation is obtained in terms of the wave number \( \tilde{k}_2 \) by solving the so-called locus equation:

\[ W(\tilde{k}_2, d) = Q + \Omega(\tilde{k}_2) - \Omega(\tilde{P} + \tilde{k}_2, d) = 0 \quad (5.24) \]

in which \( \omega_j = \Omega(\tilde{k}_j, d) \) is the general expression for the dispersion relation. In finite depth \( \Omega(\tilde{k}_j, d) \) is given by the linear dispersion relation:
\[ \omega^2 = g k \tanh(kd) \]  

(5.25)

The solution of Eq. (5.24) results in an egg-shaped contour for all possible \( \mathbf{k}_2 \) wave numbers in wave number space. Using the resonance conditions, all possible \( \mathbf{k}_4 \) wave numbers can be found by a simple translation formation over the vector \( \mathbf{P} \):

\[ \mathbf{k}_4 = \mathbf{k}_2 + \mathbf{P} \]  

(5.26)

An example of a locus for \( \mathbf{k}_2 \) and \( \mathbf{k}_4 \) for \( \mathbf{k}_2 = (0.15, 0) \) and \( \mathbf{k}_3 = (0.2, 0.1) \) is given in Figure 5.12 in Appendix A.

For the case that \( \omega_1 = \omega_3 \) the solution of the resonance conditions gives two parallel straight lines for the wave numbers \( \mathbf{k}_2 \) and \( \mathbf{k}_4 \). This is illustrated in Figure 5.13 in Appendix A, for \( k_1 = k_3 = 0.1 \ (1/\text{m}) \), \( \theta_1 = 30^\circ \) and \( \theta_3 = 60^\circ \).

The solution for \( \mathbf{k}_2 \) can be parameterised as a straight line. The direction of this line is given by the bisection angle

\[ \alpha = \frac{1}{2} (\theta_1 + \theta_3) \]  

(5.27)

The distance from the locus-lines for \( \mathbf{k}_2 \) and \( \mathbf{k}_4 \) is given by the variable \( k_0 \) which is defined as the projection of the wave number \( \mathbf{k}_3 \) perpendicular to the locus:

\[ k_0 = \sin \left( \frac{1}{2} \Delta \theta_3 \right) k_1 \]

\[ = \sin \left( \frac{1}{2} \Delta \theta_3 \right) k_3 \]  

(5.28)

in which \( \Delta \theta_3 \) is the difference between the angles \( \theta_1 \) and \( \theta_3 \). The point on the \( k_2 \)-line that is closest to the origin is given by:

\[
\begin{pmatrix}
  k_{x,0} \\
  k_{y,0}
\end{pmatrix} =
\begin{pmatrix}
  -\sin(\alpha) \\
  \cos(\alpha)
\end{pmatrix} k_0
\]  

(5.29)

and the corresponding point on the \( k_4 \)-line is given by:

\[
\begin{pmatrix}
  k_{x,0} \\
  k_{y,0}
\end{pmatrix} = k_0
\begin{pmatrix}
  \sin(\alpha) \\
  -\cos(\alpha)
\end{pmatrix}
\]  

(5.30)

Then, the \( k_2 \)-line is given by:

\[
\begin{pmatrix}
  k_{x} \\
  k_{y}
\end{pmatrix} =
\begin{pmatrix}
  -\sin(\alpha) \\
  \cos(\alpha)
\end{pmatrix} k_0 + \lambda
\begin{pmatrix}
  \cos(\alpha) \\
  \sin(\alpha)
\end{pmatrix}
\]  

(5.31)

Since \( \mathbf{k}_4 = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 \) it follows that the solution for \( \mathbf{k}_4 \) is also a straight line, given by:
\[
\vec{k} = \begin{pmatrix} k_x \\ k_y \end{pmatrix} = k_0 \begin{pmatrix} \sin(\alpha) \\ -\cos(\alpha) \end{pmatrix} + \lambda \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}
\] (5.32)

### 5.3.2 Geometric solution

The solution of the resonance conditions can also be found using geometric arguments. Therefore two circles are constructed and the crossing point is the solution of the resonance conditions. The first circle has origin \( \vec{k}_1 \) and radius \( k_2 \). The second circle has origin \( \vec{k}_3 \) and radius \( k_4 \). The general case has been described in Van Vledder (2000). The singular case, where \( k_1 = k_3 \) is reported here. For the singular case we also have \( k_2 = k_4 \) and we can use the symbol \( k \) to indicate their magnitude. Then the following system of equations can be set up:

\[
\vec{k}_1 + k \begin{pmatrix} \cos(\theta_2) \\ \sin(\theta_2) \end{pmatrix} = \vec{k}_3 + k \begin{pmatrix} \cos(\theta_4) \\ \sin(\theta_4) \end{pmatrix}
\] (5.33)

which can be written as:

\[
\vec{P} = \vec{k}_1 - \vec{k}_3 = k \begin{pmatrix} \cos(\theta_2) - \cos(\theta_4) \\ \sin(\theta_2) - \sin(\theta_4) \end{pmatrix}
\] (5.34)

squaring both components and summing gives:

\[
P^2 = k^2 \left( 2 - 2 \left[ \cos(\theta_2) \cos(\theta_4) + \sin(\theta_2) \sin(\theta_4) \right] \right)
= 2k^2 \left( 1 - \cos(\theta_2 - \theta_4) \right)
\] (5.35)

This gives:

\[
\cos(\theta_2 - \theta_4) = 1 - \frac{p^2}{2k^2}
\] (5.36)

Since

\[
\frac{1}{2} (\theta_2 + \theta_4) = \theta_i
\] (5.37)

The angle \( \theta_4 \) can be eliminated, such that

\[
\theta_2 = \theta_i + \frac{1}{2} \arccos \left( 1 - \frac{p^2}{2k^2} \right)
\] (5.38)

and similarly
\[ \theta_4 = \theta_s - \frac{1}{2} \cos \left(1 - \frac{P^2}{2k^2}\right) \]  

(5.39)

The other crossing points (for \( i=2,4 \)) are found from the consideration:

\[ \frac{\theta_1 + \theta_i'}{2} = \theta_i + \pi \]  

(5.40)

yielding

\[ \theta_2' = 2\theta_s + 2\pi - \theta_2 \]

\[ \theta_4' = 2\theta_s + 2\pi - \theta_4 \]  

(5.41)

The crossing points of the circles on both sides of the central point are shown as small circles in Figure 5.14 in Appendix A. Solutions of the resonance conditions for the wave number \( \vec{k}_2 \) can be found by connecting the end point of the wave number vector \( \vec{k}_1 \) to each small circle. The corresponding solution for the wave number \( \vec{k}_4 \) can be found by connecting the end point of the wave number vector \( \vec{k}_3 \) to this small circle.

For \( k \to \infty \) the angles \( \theta_2 \to \theta_s \) and \( \theta_4 \to \theta_s \). The shortest value for \( k \) is found when \( \theta_2 \) and \( \theta_4 \) have opposite directions. Thus \( \theta_2 = \theta_4 + \pi \). Then using (5.36) we find:

\[ \cos(\pi) = 1 - \frac{P^2}{2k^2} \]  

(5.42)

upon solving for \( k \)

\[ k = \frac{1}{2} P \]  

(5.43)

Thus, the smallest value of the wave numbers \( k_2 \) and \( k_4 \) is \( k_{\min} = \frac{1}{2} P \). The value of \( P \) follow from geometric considerations as:

\[ P = 2k \sin \left( \frac{1}{2} \Delta \theta_{13} \right) \]  

(5.44)

Using geometric considerations Eq. (5.44) can be written as:

\[ P^2 = 2k^2 - 2k^2 \cos(\Delta \theta_{13}) \]

\[ = 2k^2 \left[1 - \cos(\Delta \theta_{13})\right] \]  

(5.45)

Taking the square of (5.44) yields:

\[ P^2 = 4k^2 \sin^2 \left( \frac{1}{2} \Delta \theta_{13} \right) \]

\[ = 2k^2 \left(1 - \cos(\Delta \theta_{13})\right) \]  

(5.46)
The equations for the wave numbers \( \tilde{k}_2 \) and \( \tilde{k}_4 \) are given by:

\[
\tilde{k}_2 = \tilde{k}_0 + k_2 \begin{pmatrix} \cos \left( \theta_s + \frac{1}{2} \cos \left[1 - \frac{P^2}{2k^2} \right] \right) \\ \sin \left( \theta_s + \frac{1}{2} \cos \left[1 - \frac{P^2}{2k^2} \right] \right) \end{pmatrix} \]

\[
\tilde{k}_4 = \tilde{k}_0 + k_4 \begin{pmatrix} \cos \left( \theta_s - \frac{1}{2} \cos \left[1 - \frac{P^2}{2k^2} \right] \right) \\ \sin \left( \theta_s - \frac{1}{2} \cos \left[1 - \frac{P^2}{2k^2} \right] \right) \end{pmatrix} \begin{pmatrix} \cos \left( \theta_s - \frac{1}{2} \cos \left[1 - \frac{P^2}{2k^2} \right] \right) \\ \sin \left( \theta_s - \frac{1}{2} \cos \left[1 - \frac{P^2}{2k^2} \right] \right) \end{pmatrix} \]

(5.47)

(5.48)

Using geometric considerations the start point, i.e. the point with the minimum value of \( k \), is given by:

\[
\tilde{k}_0 = \frac{1}{2} P \begin{pmatrix} \cos \left( \theta_s \pm \frac{1}{2} \pi \right) \\ \sin \left( \theta_s \pm \frac{1}{2} \pi \right) \end{pmatrix} \]

(5.49)

This can also be obtained from (5.48) by substitution of \( k_{\text{min}} = \frac{1}{2} P \). Using the equations of the mirror image the lines can be constructed directly. An example of a solution for the singular case is shown in the Figure 5.15 in Appendix A. Additional information about this solution is shown in Figure 5.16 of Appendix A. The upper left panel shows the solution for the wave numbers \( \tilde{k}_2 \) and \( \tilde{k}_4 \). The upper right panel shows the variation of the angles \( \theta_s \) and \( \theta_t \) of these wave numbers, the lower left panel shows the variation of the coupling coefficient \( G \) and the lower right panel shows the variation of the inverse of the gradient term \( R \). It can be seen that these angles converge to the same value at large distance from the starting point. It can also be seen that the coupling coefficient and gradient term increase monotonically with increasing distance from the start point.

### 5.3.3 The gradient term

The gradient term from the WRT method has been written by Tracy and Resio (1982) in Cartesian coordinates for deep water only. In this Section the gradient term is derived for finite depth. As shown in Eq. (5.19) the gradient term can be expressed as:

\[
\left| \nabla W \right| = \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right)^{1/2} \]

(5.50)

For finite depth the gradient term can be computed as follows. First we have:
\[
\frac{\partial W}{\partial x} = \frac{\partial}{\partial x} \Omega(\bar{k}_2, d) - \frac{\partial}{\partial x} \Omega(\bar{k}_z + \bar{P}, d) \\
\frac{\partial W}{\partial y} = \frac{\partial}{\partial y} \Omega(\bar{k}_2, d) - \frac{\partial}{\partial y} \Omega(\bar{k}_z + \bar{P}, d)
\] (5.51)

(5.52)

For deep water these derivatives are computed by writing:

\[
\omega_2 = \Omega(\bar{k}_2) = \sqrt{gk_2} \\
= \sqrt{g \left( k_{2,x}^2 + k_{2,y}^2 \right)^{1/2}} \\
= \sqrt{g \left( k_{2,x}^2 + k_{2,y}^2 \right)^{1/4}}
\] (5.53)

then

\[
\frac{\partial \omega_2}{\partial x} = \sqrt{g} \frac{\partial}{\partial x} \left| k_{2,x}^2 + k_{2,y}^2 \right|^{1/4} \\
= \sqrt{g} \left\{ \frac{1}{4} 2k_{2,x} \left| k_{2,x}^2 + k_{2,y}^2 \right|^{3/4} \\
= \sqrt{g} \frac{k_{2,x}}{2k_2^{3/2}}
\] (5.54)

and similarly for \( \omega_4 \) and the derivatives with respect to \( y \). For finite depth the derivation is as follows. To that end the radian frequency \( \omega_2 \) is written as:

\[
\omega_2 = \Omega(\bar{k}_2, d) = \sqrt{gk_2 \tanh(k_2d)} \\
= \sqrt{g \left( k_{2,x}^2 + k_{2,y}^2 \right)^{1/2} \tanh\left( \left( k_{2,x}^2 + k_{2,y}^2 \right)^{1/2} d \right)} \\
= \sqrt{g \left( k_{2,x}^2 + k_{2,y}^2 \right)^{1/4} \tanh^{1/2} \left( \left( k_{2,x}^2 + k_{2,y}^2 \right)^{1/2} d \right)}
\] (5.55)

then applying the chain for differentiation gives:
\[
\frac{\partial \omega_2}{\partial x} = \sqrt{g} \frac{\partial}{\partial x} \left[ \left( k_{2,x}^2 + k_{2,y}^2 \right)^{1/4} \tanh^{1/2} \left( \left( k_{2,x}^2 + k_{2,y}^2 \right)^{1/2} d \right) \right] \\
= \sqrt{g} \frac{1}{4} 2 k_{2,x} k_{2,y} \left( k_{2,x}^2 + k_{2,y}^2 \right)^{-3/4} \tanh^{1/2} \left( k_2 d \right) \\
+ \sqrt{g} \left( k_{2,x}^2 + k_{2,y}^2 \right)^{1/4} \frac{1}{2} k_{2,x} \frac{1}{2} k_{2,y} \frac{d}{\cosh^2 \left( k_2 d \right) \tanh^{1/2} \left( k_2 d \right)} \\
= \frac{\sqrt{g} k_{2,x}}{2 k_2} \frac{\sqrt{g} \tanh \left( k_2 d \right)}{\sqrt{g} k_2^{1/2} \tanh^{1/2} \left( k_2 d \right)} + \frac{\sqrt{g} k_{2,x}}{2 k_2} \frac{d}{\cosh^2 \left( k_2 d \right) \sqrt{g} k_2^{1/2} \tanh^{1/2} \left( k_2 d \right)} \\
= \frac{g}{2} \frac{k_{2,x}}{k_2 \omega_2} \left[ \tanh \left( k_2 d \right) + \frac{k_2 d}{\cosh^2 \left( k_2 d \right)} \right] \\
\]

(5.56)

and similarly for the derivatives with respect to \( y \) and for the radian frequency \( \omega_4 \).

\[ \omega_4 = \Omega \left( \vec{k}_2 + \vec{P}, d \right) \]

(5.57)

For finite depth this leads to:

\[
\frac{\partial \Omega \left( \vec{k}_2, d \right)}{\partial x} = \frac{g}{2 \omega_2} \frac{k_{2,x}}{k_2} \left[ \tanh \left( k_2 d \right) + \frac{k_2 d}{\cosh^2 \left( k_2 d \right)} \right] \\
\]

(5.58)

\[
\frac{\partial \Omega \left( \vec{k}_2, d \right)}{\partial y} = \frac{g}{2 \omega_2} \frac{k_{2,y}}{k_2} \left[ \tanh \left( k_2 d \right) + \frac{k_2 d}{\cosh^2 \left( k_2 d \right)} \right] \\
\]

(5.59)

\[
\frac{\partial \Omega \left( \vec{k}_2 + \vec{P}, d \right)}{\partial x} = \frac{g}{2 \omega_4} \frac{k_{2,x} + P_x}{\vec{k}_2 + \vec{P}} \\
\times \left[ \tanh \left( \vec{k}_2 + \vec{P} \right) d + \frac{\left| \vec{k}_2 + \vec{P} \right| d}{\cosh^2 \left( \left| \vec{k}_2 + \vec{P} \right| d \right)} \right] \\
\]

(5.60)

\[
\frac{\partial \Omega \left( \vec{k}_2 + \vec{P}, d \right)}{\partial y} = \frac{g}{2 \omega_4} \frac{k_{2,y} + P_y}{\vec{k}_2 + \vec{P}} \\
\times \left[ \tanh \left( \vec{k}_2 + \vec{P} \right) d + \frac{\left| \vec{k}_2 + \vec{P} \right| d}{\cosh^2 \left( \left| \vec{k}_2 + \vec{P} \right| d \right)} \right] \\
\]

(5.61)
For deep water \( \tanh \rightarrow 1 \) and \( \cosh \rightarrow \infty \) the above equations reduce to:

\[
\frac{\partial \Omega (|k_2|)}{\partial x} = \frac{\sqrt{g} k_{2,x}}{2 k_2^{3/2}} \tag{5.62}
\]

\[
\frac{\partial \Omega (|k_2|)}{\partial y} = \frac{\sqrt{g} k_{2,y}}{2 k_2^{3/2}} \tag{5.63}
\]

\[
\frac{\partial \Omega (|k_2 + \bar{p}|)}{\partial x} = \frac{\sqrt{g} (k_{2,x} + p_x)}{2 |k_2 + \bar{p}|^{3/2}} = \frac{\sqrt{g} k_{4,x}}{2 k_4^{3/2}} \tag{5.64}
\]

\[
\frac{\partial \Omega (|k_2 + \bar{p}|)}{\partial y} = \frac{\sqrt{g} (k_{2,y} + p_y)}{2 |k_2 + \bar{p}|^{3/2}} = \frac{\sqrt{g} k_{4,y}}{2 k_4^{3/2}} \tag{5.65}
\]

Using \( \omega_i = \sqrt{g k_i} \), the above expression can also be written as:

\[
\frac{\partial \Omega (|k_2|)}{\partial x} = \frac{g k_{2,x}}{2 k_2 \omega_2} \tag{5.66}
\]

\[
\frac{\partial \Omega (|k_2|)}{\partial y} = \frac{g k_{2,y}}{2 k_2 \omega_2} \tag{5.67}
\]

\[
\frac{\partial \Omega (|k_2 + \bar{p}|)}{\partial x} = \frac{g k_{4,x}}{2 k_4 \omega_4} \tag{5.68}
\]

\[
\frac{\partial \Omega (|k_2 + \bar{p}|)}{\partial y} = \frac{g k_{4,y}}{2 k_4 \omega_4} \tag{5.69}
\]

The same result was also obtained by Tracy and Resio (1982) by application of the chain rule for differentiation, except for the factor \( \sqrt{g} \), which they left out in their equation. However, this factor was included elsewhere in their system of equations for solving the Boltzmann equation.
5.3.4 Simplification of the gradient term

The gradient term is derived in the previous section can be simplified considerably by using simple geometric considerations and by inclusion of the group velocity. The gradient term is given by:

\[ R = \left( \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right)^{1/2} \]  

(5.70)

The derivatives can be simplified by replacing the ratio's of a wave number component and the magnitude of this wave numbers as sines and cosines. Thus \( k_{x,y} / k_i = \cos(\theta) \) and \( k_{x,y} / k_i = \sin(\theta) \).

\[
\frac{\partial W}{\partial k_x} = \frac{\partial}{\partial k_x} \Omega(|\bar{k}_2|) - \frac{\partial}{\partial k_x} \Omega(|\bar{k}_x + \bar{P}|)
\]
\[
= \frac{\sqrt{g} k_{2,x}}{2 k_2^{3/2}} - \frac{\sqrt{g} k_{4,x}}{2 k_4^{3/2}}
\]
\[
= \frac{g}{2 \sqrt{g k_2}} \frac{k_{2,x}}{k_2} - \frac{g}{2 \sqrt{g k_4}} \frac{k_{4,x}}{k_4}
\]
\[
= \frac{g}{2 \omega_2} \cos(\theta_2) - \frac{g}{2 \omega_4} \cos(\theta_4)
\]

(5.71)

and similarly for the derivative with respect to \( y \).

\[
\frac{\partial W}{\partial k_y} = \frac{\partial}{\partial k_y} \Omega(|\bar{k}_y|) - \frac{\partial}{\partial k_y} \Omega(|\bar{k}_y + \bar{P}|)
\]
\[
= \frac{\sqrt{g} k_{2,y}}{2 k_2^{3/2}} - \frac{\sqrt{g} k_{4,y}}{2 k_4^{3/2}}
\]
\[
= \frac{g}{2 \sqrt{g k_2}} \frac{k_{2,y}}{k_2} - \frac{g}{2 \sqrt{g k_4}} \frac{k_{4,y}}{k_4}
\]
\[
= \frac{g}{2 \omega_2} \sin(\theta_2) - \frac{g}{2 \omega_4} \sin(\theta_4)
\]

(5.72)

Substitution of (5.71) and (5.72) in (5.70) then gives:
\[ R = \frac{g}{2} \left( \left[ \frac{\cos(\theta_2) - \cos(\theta_4)}{\omega_2} \right]^2 + \left[ \frac{\sin(\theta_2) - \sin(\theta_4)}{\omega_4} \right]^2 \right)^{1/2} \]

\[ = \frac{g}{2} \left( \frac{1}{\omega_2^2} + \frac{1}{\omega_4^2} - \frac{2}{\omega_2 \omega_4} \left( \cos(\theta_2) \cos(\theta_4) + \sin(\theta_2) \sin(\theta_4) \right) \right)^{1/2} \]  

(5.73)

\[ = \frac{g}{2} \left( \frac{1}{\omega_2^2} + \frac{1}{\omega_4^2} - \frac{2}{\omega_2 \omega_4} \cos(\theta_4 - \theta_2) \right)^{1/2} \]

This equation has been checked numerically and graphically. The term \( R \) vanishes when both \( \omega_2 = \omega_4 \) and \( \theta_2 = \theta_4 \).

For finite depth the group velocity is given by:

\[ c_g = \frac{g}{2\omega} \left[ \tanh(kd) + \frac{kd}{\cosh^2(kd)} \right] \]

\[ = \frac{g}{2\omega} \tanh(kd) \left[ 1 + \frac{kd}{\sinh(kd) \cosh(kd)} \right] \]  

(5.74)

\[ = \frac{g}{2\omega} \tanh(kd) \left[ 1 + \frac{2kd}{\sinh(2kd)} \right] \]

Combining the Eq (5.74) with the Eqs. (5.58) – (5.61) leads to:

\[ R = \left( \left[ c_{g2} \cos(\theta_2) - c_{g4} \cos(\theta_4) \right]^2 + \left[ c_{g2} \sin(\theta_2) - c_{g4} \sin(\theta_4) \right]^2 \right)^{1/2} \]

\[ = \left( c_{g2}^2 + c_{g4}^2 - 2c_{g2} c_{g4} \cos(\theta_4 - \theta_2) \right)^{1/2} \]  

(5.75)

Rasmussen (1998) gives a direct and more elegant derivation of the gradient term. In his derivation Cartesian coordinates are not used. The definition of the gradient term can also be given by:

\[ R = \left| \frac{\partial \left( \omega_1 + \omega_2 - \omega_3 - \omega_{k_1+k_2-k_3} \right)}{\partial k_{2,n}} \right| \]

\[ = \left| \frac{\partial \Delta \omega_{1,2,3,k_1+k_2-k_3}}{\partial k_{2,n}} \right| \]  

(5.76)

In the first step of this derivation, the following gradient is computed:

\[ \nabla_{k_2} \Delta \omega_{1,2,3,k_1+k_2-k_3} = \nabla_{k_2} \left( \omega_1 + \omega_2 - \omega_3 - \omega_{k_1+k_2-k_3} \right) \]  

(5.77)

Since \( \vec{k}_1 \) and \( \vec{k}_3 \) are fixed, we find
\[
\n\nabla_{\tilde{k}_2} \Delta \omega_{1,2,3,\tilde{k}_1+\tilde{k}_2-\tilde{k}_3} = \nabla_{\tilde{k}_2} \omega_2 - \nabla_{\tilde{k}_2} \omega_3,
\]

(5.78)

Then, using the chain rule for differentiation this yields

\[
\nabla_{\tilde{k}_2} \Delta \omega_{1,2,3,\tilde{k}_1+\tilde{k}_2-\tilde{k}_3} = \nabla_{\tilde{k}_2} \omega_2 - \nabla_{\tilde{k}_2} \left( \tilde{k}_1 + \tilde{k}_2 - \tilde{k}_3 \right) \nabla_{\tilde{k}_1+\tilde{k}_2-\tilde{k}_3} \omega_{\tilde{k}_1+\tilde{k}_2-\tilde{k}_3}
\]

(5.79)

Which simplifies to:

\[
\nabla_{\tilde{k}_2} \Delta \omega_{1,2,3,\tilde{k}_1+\tilde{k}_2-\tilde{k}_3} = \tilde{c}_{g,2} - \tilde{c}_{g,4}
\]

(5.80)

It is easy to realise that on the locus the following criterion holds:

\[
\frac{\partial \Delta \omega_{1,2,3,\tilde{k}_1+\tilde{k}_2-\tilde{k}_3}}{\partial \tilde{k}_{2,t}} = 0
\]

(5.81)

Therefore, as

\[
\left| \nabla_{\tilde{k}_1+\tilde{k}_2} \Delta \omega \right| = \left| \nabla_{\tilde{k}_2} \Delta \omega \right|
\]

(5.82)

and

\[
\frac{\partial \Delta \omega}{\partial \tilde{k}_{2,t}} = 0
\]

(5.83)

it is easily found that the gradient is given by:

\[
\left| \frac{\partial \Delta \omega_{1,2,3,\tilde{k}_1+\tilde{k}_2-\tilde{k}_3}}{\partial \tilde{k}_{2,t}} \right| = \left| \tilde{c}_{g,2} - \tilde{c}_{g,4} \right|.
\]

(5.84)

Since

\[
\tilde{c}_{g,4} = \tilde{c}_{g,4}
\]

(5.85)

Eq. (5.84) can be written as:

\[
R = \left| \tilde{c}_{g,2} - \tilde{c}_{g,4} \right|
\]

(5.86)

where

\[
\left( c_{g,2}^2 + c_{g,4}^2 - 2c_{g,2}c_{g,4} \cos(\theta_2 - \theta_4) \right)^{1/2}
\]

This simple form of the gradient term is equal to (5.75).
5.3.5 Computational technique for the WRT-method

To compute the non-linear transfer rate for a given spectrum, it is assumed that the wave spectrum is given in terms of a discrete action density spectrum as a function of the discrete wave numbers $k_j$ and directions $\theta_j$. Using this assumption the basic equation of the WRT can be discretized as follows:

$$\frac{\partial N(\vec{k})}{\partial t} = \int d\vec{k}_5 T(\vec{k}_1, \vec{k}_5)$$

$$\approx \sum_{i=1}^{N_x} \sum_{j=1}^{N_x} k_{3,i,j} T(\vec{k}_1, \vec{k}_5) \Delta k_{3,i,j} \Delta \theta$$

(5.87)

Note the Jacobian factor $k_{3,i,j}$ when transforming the wave number vector $\vec{k}_5$ to its components. The outer loops (in fact summations) run over all discrete grid points of the spectrum. The term $T(\vec{k}_1, \vec{k}_5)$ is the integral over a closed locus in wave number space. This term can be discretized as follows:

$$T(\vec{k}_1, \vec{k}_5) = \oint G(s) R^{-1}(s) P(s) ds$$

$$\approx \sum_{i=1}^{N_x} G(s_i) R^{-1}(s_i) P(s_i) \Delta s_i$$

(5.88)

In (5.88) the term $G$ is the coupling coefficient, $R$ is the gradient term and $P$ is the action density product term. Expressions for these are given in previous sections.

The position of a $k_3$- and $k_r$- locus in wave number space and its position in a discretized polar spectral grid are visualised in Figure 5.17 in Appendix A.

In evaluating expression (5.88) the locus is given at discrete points. These points do not coincide with the discrete spectral grid points. The values for the coupling coefficient and the gradient term can be pre-computed in a pre-processing phase. The action densities at the points on the locus need to be obtained by bi-linear interpolation from the surrounding discrete spectral points. This procedure is visualised in Figure 5.18 in Appendix A. As shown in Figure 5.18 bi-linear interpolation is used to obtain the values for the action densities for the wave numbers $\vec{k}_2$ and $\vec{k}_4$. Note that no interpolation is necessary to obtain the action densities at the wave numbers $\vec{k}_1$ and $\vec{k}_5$. The action densities at the four wave numbers can then be used to compute the action density product term $P$.

In practice the action density spectrum is given for a limited range in wave number space. Above a certain wave number, a parametric decay of the spectrum is assumed. In the case the locus crosses part of the spectral tail the interpolation procedure can be simplified because the number of independent variables reduces from 4 to 2. This simplified procedure is illustrated in Figure 5.19 in Appendix A.

To reach optimal accuracy the number of points on the locus must be large enough that all cells in a discrete spectrum have a few points along a locus. If a cell does not contain a
discrete grid point, the integration procedure might miss relevant spectral information. On the other hand, too much locus points in cell results in an inefficient integration procedure. Therefore, the distribution of discrete grid points along a locus must reflect the local resolution of the spectral grid, such that each cell contains at least one or two discrete points of the \( k_2 \)-locus. Since the \( k_2 \) and \( k_3 \)-locus are coupled, an optimal distribution of grid points on the \( k_2 \)-locus, is not necessarily optimal for the \( k_3 \)-locus. Therefore additional grid points need to be added along both loci to satisfy the requirement of at least two grid points in each cell.

The integration along a closed locus is illustrated in Figure 5.20 in Appendix A. In the upper left panel the position of the loci for the \( k_2 \) and \( k_4 \) wave number are shown. In the other panels the variation of various parameters is shown as a function of a local coordinate \( s \) along the locus. Shown are the reciprocal value of the gradient term, the coupling coefficient, the action density product, the product of these terms (referred to as the total function) and the cumulative integral along the locus. This last term can mathematically be expressed as:

\[
T\left(\vec{k}_1, \vec{k}_3, s\right) = \int_{0}^{s} G(s) R^{-1}(s) P(s) ds
\]

(5.89)

The result of a full locus integration is equal to \( T(\vec{k}_1, \vec{k}_3) \).

The integration along the loci for a singular case is illustrated in Figure 5.21 in Appendix A. As can be seen in Figure 5.21. The total function approaches zero with increasing distance along the locus-line. This implies that the cumulative total function converges to a constant value.

The above procedure can be simplified by replacing the bi-linear interpolation by a method in which the action density at the nearest spectral bin is taken. This corresponds to representing the action density spectrum as a piecewise constant function, similar to the work carried out by Snyder et al. (1993).

The \textsc{wrt} method can be speeded up in various ways. With reference to the Eqs. (5.87) and (5.88) one can take less points on a locus, or one may consider only a few discrete wave number \( \vec{k}_3 \) which lie close to the wave number \( \vec{k}_1 \). Further test need to be carried out to assess acceptable limits of such reduction of the integration procedure.

## 5.4 The Generalized Discrete Interaction Approximation

### 5.4.1 Introduction

The computation of the Boltzmann integral is rather complicated and very time consuming since it requires the solution of a 3-fold integral over three 2-dimensional wave numbers. The delta functions in Eq.(5.4) reduce the integration space to a 3-dimensional surface. Because of its computational requirements it is not (yet) feasible to include a full and accurate solution of the Boltzmann integral in operational wave models. To overcome this practical difficulty Hasselmann et al. (1985) developed the Discrete Interaction
Approximation (DIA). In the DIA two wave numbers \( \vec{k}_1 \) and \( \vec{k}_2 \) are equal and the other two wave numbers \( \vec{k}_3 \) and \( \vec{k}_4 \) make an angle with the first two wave numbers.

To improve the DIA two types of extensions are given. Firstly, a generalisation to the DIA is derived. This generalisation has two characteristics. First, the wave numbers \( \vec{k}_1 \) and \( \vec{k}_2 \) may have different magnitudes and different directions, such that any possible wave number configuration can be selected. Further, the Generalized DIA (GDIA) is derived for finite depths, which makes the WAM depth scaling obsolete. The deep water DIA then is easily derived from the finite depth case. The extension consists of selecting a set of different wave number configurations, the so-called Multiple DIA.

### 5.4.2 The Discrete Interaction Approximation

The Discrete Interaction Approximation considers only one wave number configuration. In this configuration the wave numbers \( \vec{k}_1 \) and \( \vec{k}_2 \) are equal and the other two wave numbers have different magnitude and direction. Their frequencies are related via the parameter \( \lambda \) such that their configuration is uniquely determined:

\[
\begin{align*}
    f_1 &= f_2 = f \\
    f_3 &= (1+\lambda) f = f^+ \\
    f_4 &= (1-\lambda) f = f^-
\end{align*}
\]  
\[(5.90)\]

The parameter \( \lambda \) is limited to the range (0-0.5) to yield valid solutions. In Eq. (5.90) the superscripts + and – are used to emphasise the links with previously reported notations of the DIA. The directions of the wave numbers \( \vec{k}_3 \) and \( \vec{k}_4 \), relative to the direction of the wave numbers \( \vec{k}_1 \) and \( \vec{k}_2 \), follow from Eq. (5.90) and the resonance conditions:

\[
\begin{align*}
    \theta_3 &= \theta^+ = \arccos \left( \frac{4 + (1+\lambda)^4 - (1-\lambda)^4}{4(1+\lambda)^2} \right) \\
    \theta_4 &= \theta^- = -\arccos \left( \frac{4 + (1-\lambda)^4 - (1+\lambda)^4}{4(1-\lambda)^2} \right)
\end{align*}
\]  
\[(5.91, 5.92)\]

In the DIA proposed by Hasselmann et al. (1985) \( \lambda = 0.25 \), leading to the angles \( \theta_1 = \theta_2 = 0^\circ \), \( \theta_3 = \theta^+ = 11.48^\circ \) and \( \theta_4 = \theta^- = -33.56^\circ \). An example of a typical wave number configuration and its mirror image with \( \lambda = 0.25 \) is shown in Figure 5.12 in Appendix A. Figure 5.22 also shows the mirror image of a basic wave number configuration. Ueno (1997) and Hashimoto and Kawaguchi (2001) prefer \( \lambda = 0.19 \) because this value gives a better agreement with the exact transfer for JONSWAP type spectra, see also Figure 5.2. Tolman and Chalikov (1996) also use \( \lambda = 0.25 \).

The DIA source term describes the rate of change of energy density in all four (in fact three) wave numbers involved in an interaction. The corresponding energy densities are denoted
by \( E = E(f, \theta) \), \( E^+ = E(f^+, \theta^+) \) and \( E^- = E(f^-, \theta^-) \), and the corresponding contributions to the nonlinear transfer rates are denoted by \( S_{nl} \), \( S_{nl}^+ \) and \( S_{nl}^- \). The functional form of the DIA source term is given by:

\[
\begin{align*}
\left( \frac{\delta S_{nl}}{\delta S_{nl}^+} \right) &= \left( \begin{array}{c}
-2 \\
1 \\
1 \\
\end{array} \right) C_{nl4} g^{-4} f_1 \left( E^+ \left( \frac{E^+}{(1+\lambda)^2} + \frac{E^-}{(1-\lambda)^4} \right) - 2E \frac{E^+ E^-}{(1-\lambda^2)^4} \right)
\end{align*}
\]

(5.93)

in which \( C_{nl4} \) is a scale parameter. In the DIA proposed by Hasselmann et al. (1985), \( C_{nl4} = 3 \times 10^7 \) based on numerical growth curve experiments. With these values they obtained good agreement with results from computation in which the non-linear transfer was computed using an exact method. Tolman and Chalikov (1996) use \( C_{nl4} = 1 \times 10^7 \) in combination with a reduced dissipation function and wind input, such that the total source term is balanced.

To compute the non-linear quadruplet source term for a discrete wave spectrum, equation (5.93) is applied to all spectral bins, taking \( E \) as the energy density at this bin. The positions of the other two interacting bins with energy densities \( E^+ \) and \( E^- \) are determined relative to the central bin \( E \). In general, the locations \((f_3, \theta_3)\) and \((f_4, \theta_4)\) of the wave numbers \( k_3 \) and \( k_4 \) in the spectral grid do not coincide with discrete spectral grid points. To obtain the energy density at these points one can apply bi-linear interpolation between the four surrounding grid points or take the energy density at the nearest grid point. In the computational procedure Eq. (5.93) is also applied for its mirror image, obtained by reversing the signs of the angles \( \theta_3 \) and \( \theta_4 \).

In the computational procedure of the DIA special attention is given to wave number configurations that cross the boundaries of the spectral grid. Periodicity is assumed to take care of the directional boundaries. Further, in the case that the frequency \( f^- \) is lower than the lowest discrete frequency \( f_{\text{min}} \) the corresponding energy density \( E^- \) is set to zero, and in the case the frequency \( f^+ \) is higher than the highest discrete frequency a parametric decay of energy densities in the spectral tail is assumed, usually given by:

\[ E(f) = E(f_{\max}) \left( \frac{f}{f_{\max}} \right)^p \quad \text{for} \quad f \geq f_{\max} \]

(5.94)

in which \( p \) is the power of the spectral tail. In the WAM model (WAMDI, 1988) \( p = -5 \), whereas in the SWAN model (Booij et al., 1999) \( p = -4 \). To include all possible interactions between spectral bins in the prognostic range of the spectral grid and in the spectral tail, additional quadruplets need to be accounted for such that the bin \( E^- \) is located just at or just above the highest model frequency;

\[ f^- \geq f_{\max} \]

(5.95)
This is achieved by extending the spectral grid towards higher frequencies, e.g. by applying a geometric sequence of frequencies. Application of Eq. (5.93) produces the contributions to the rate of change of energy density at the interacting wave numbers. These rates are distributed among the four surrounding spectral bins with the same bilinear interpolation weights as used for the determination of the energy density at these wave numbers.

Extensions and modifications of the DIA have been proposed by various authors. Van Vledder et al. (2000) propose a DIA consisting of 2 different wave number configurations. Hashimoto and Kawaguchi, (2001) presented a multiple DIA with up to 6 wave number configurations (with 6 different \( \lambda \)-values). They show that the modelling error with respect to an exact solution can be reduced considerably. However, as can be seen in their Figure 1, the decrease in modelling error seems to have a lower limit. This figure is reproduced in Figure 5.23 in Appendix A. As will be shown later in this report, this is due to the fact that the DIA is not able to include arbitrarily shaped wave number configurations.

### 5.4.3 Derivation of the Generalized Discrete Interaction Approximation

The starting point for the derivation of the GDIA is the principle of detailed balance. This principle was formulated by Hasselmann (1966) and states that the change of wave action per unit time of each wave number involved in a resonant interaction is equal. To take advantage of this principle the Boltzmann integral is written in a symmetrical form:

\[
\begin{pmatrix}
\Delta n_1 \\
\Delta n_2 \\
\Delta n_3 \\
\Delta n_4 \\
\end{pmatrix} = \begin{pmatrix}
-1 \\
-1/4 \\
1 \\
1 \\
\end{pmatrix} \frac{G}{4} \delta (k_1 + k_2 - k_3 - k_4) \delta (\omega_1 + \omega_2 - \omega_3 - \omega_4) P d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\vec{k}_4 \Delta t
\]

(5.96)

The original integral follows by integrating Eq. (5.96) under the side conditions that one of the wave numbers is fixed. The factor \( \frac{1}{4} \) is due to the fact that each of the four wave numbers can be chosen in turn as the fixed wave number. In Eq. (5.96) we have the wave action product \( P \) given by:

\[
P = n_1 n_2 (n_3 + n_4) - n_3 n_4 (n_1 + n_2)
\]

(5.97)

Equation (5.96) has not the most convenient form to integrate with respect to the original wave numbers \( \vec{k}_1, \vec{k}_2, \vec{k}_3 \) and \( \vec{k}_4 \). Following Hasselmann and Hasselmann (1981) and Rasmussen (1995) the following integration variables are introduced:
$$\lambda_1 = \omega_1$$
$$\lambda_2 = \omega_2$$
$$\lambda_3 = \omega_2 = \omega_1 + \omega_2$$
$$\lambda_4 = \omega' = \omega_1 + \omega_2 - \omega_3 - \omega_4$$
$$\lambda_5 = k_5 = |\vec{k}_1| = |\vec{k}_1 + \vec{k}_2|$$
$$\lambda_6 = \theta_2 = \text{acos} \left( \frac{k_1 \cos \theta_1 + k_2 \cos \theta_2}{k_3} \right)$$
$$(\lambda_7, \lambda_8) = (\vec{k}'_1 = \vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4)$$

in which \((k_i, \theta_i)\) represent the polar co-ordinates of wave number vector \(\vec{k}_i\). After some algebra Eq. (5.96) can be written as (details are given in Rasmussen, 1995):

$$\begin{pmatrix}
\Delta n_1 \\
\Delta n_2 \\
\Delta n_3 \\
\Delta n_4
\end{pmatrix} =
\begin{pmatrix}
-1 \\
-1 \\
1 \\
1
\end{pmatrix}
\frac{G}{4} |J| P \Delta k_5 \Delta \theta_2 \Delta \omega_2 \Delta \omega_1 \Delta \omega_3 \Delta \tau$$

(5.99)

in which \(J\) the Jacobian of the transformation:

$$J = \frac{k_3}{c_1 c_2 c_3 c_4} \sin \left( \theta_2 \theta_1 \right) \sin \left( \theta_4 - \theta_3 \right)$$

(5.100)

However, due to the structure of the Jacobian additional singularities are introduced for \(\theta_2 = \theta_1\) and/or \(\theta_4 = \theta_3\) which have to be dealt with.

However, introducing the variable transformations from the WRT method, we first find by eliminating the first \(\delta\)-function with the integration variable \(d\vec{k}_4\) and rotating the integration variable \(d\vec{k}_2\) into components tangential and normal to the locus \(dk_{2,s}\) and \(dk_{2,n}\) (Note that the subscript \(s\) is used to refer to the tangential component for consistency with the notation used in the description of the WRT method).

$$\begin{pmatrix}
\Delta n_1 \\
\Delta n_2 \\
\Delta n_3 \\
\Delta n_4
\end{pmatrix} =
\begin{pmatrix}
-1 \\
-1 \\
1 \\
1
\end{pmatrix}
\frac{G}{4} \delta \left( \omega_1 + \omega_2 - \omega_3 - \omega_4 \right) P d\vec{k}_1 d\vec{k}_{2,s} d\vec{k}_{2,n} d\vec{k}_3 \Delta \tau$$

(5.101)

Then the integration variable \(dk_{2,n}\) is changed to the integration variable \((\omega_1 + \omega_2 - \omega_3 - \omega_4)\) to find
\[
\begin{pmatrix}
\Delta n_1 \\
\Delta n_2 \\
\Delta n_3 \\
\Delta n_4
\end{pmatrix} = \begin{pmatrix}
-1 \\
-1 \\
1 \\
1
\end{pmatrix} \frac{\left|J \right|}{4} Pd\tilde{k}_1dk_{2,s}d\tilde{k}_3 \Delta t
\]  

(5.102)

in which \(J\) is the Jacobian of the transformation:

\[
J = \frac{1}{c_{g1}^2 - c_{g1+2-3}^2}
\]  

(5.103)

See also the Eqs (5.77) through (5.86) and Rasmussen (1998). In fact this Jacobian, which also appears in the WRT method, is seldom given explicitly in the literature and never in this simple form. The complicated form was given in Van Vledder (2000). It is seen that the Jacobian is singular for \(c_{g2} = c_{g1+2-3}\) only.

Now \(d\tilde{k}_1\) and \(d\tilde{k}_3\) are changed into polar co-ordinates \(dk_1, d\theta_1, dk_3\) and \(d\theta_3\) and then \(dk_1\) and \(dk_3\) are transformed into \(d\omega_1\) and \(d\omega_3\). \(dk_{2,s}\) is transformed into the unphysical angular frequency \(d\omega_{2,s}\) by using the dispersion relation. Hence

\[
\begin{pmatrix}
\Delta n_1 \\
\Delta n_2 \\
\Delta n_3 \\
\Delta n_4
\end{pmatrix} = \begin{pmatrix}
-1 \\
-1 \\
1 \\
1
\end{pmatrix} \frac{\left|J'\right|}{4} Pd\omega_1d\theta_1d\omega_2_s d\omega_3 d\theta_3 \Delta t
\]  

(5.104)

where the Jacobian of the latter transformations is given by:

\[
J' = \frac{k_1 k_3}{c_{g1} c_{g3} c_{g2,s}}
\]  

(5.105)

This has removable singularities for \(k_1 = 0\) and \(k_3 = 0\) since

\[
\lim_{k \to 0} \frac{k}{c_g} = \lim_{k \to 0} \frac{k}{\sqrt{gh}} = 0
\]  

(5.106)

and no singularity for \(k_{2,s} = 0\) since

\[
\lim_{k_{2,s} \to 0} \frac{1}{c_{g2,s}^2} = \frac{1}{\sqrt{gh}}
\]  

(5.107)

The latter limit follows directly from the fact that for finite depth, the wave length \(L\) increases to infinity as the wave number \(k\) approaches zero. As a consequence the shallow water limit is applicable to compute the group velocity.
Note that the wave action \( n = \frac{E}{\omega} \) is undefined for \( k = 0 \) and the above limits can be avoided.

Next, the above equations are rewritten to obtain the nonlinear transfer rate in terms of energy densities as a function of frequency \( f \) and direction \( \theta \). First the action density \( n(\hat{k}) \) is converted to an energy density \( E(f, \theta) \) according to:

\[
n(\hat{k}) \propto \frac{c_k}{f_k} E(f, \theta)
\]  

(5.108)

The change of action \( \Delta n_i \) is replaced by the change of energy \( \Delta E_i \) using:

\[
\Delta n_i \propto \frac{1}{f_i} \Delta E_i
\]  

(5.109)

and the various increments scale as:

\[
\Delta \omega_1 \Delta \theta_1 \Delta \omega_2 \Delta \omega_3 \Delta \omega_4 \Delta \theta_3 \propto f^2 \Delta f \Delta \theta
\]  

(5.110)

Hence

\[
\begin{pmatrix}
\Delta E_1 \\
f_1 \\
\Delta E_2 \\
f_2 \\
\Delta E \\
f_3 \\
\Delta E_4 \\
f_4 \\
\end{pmatrix} = \begin{pmatrix}
-1 \\
-1 \\
1 \\
1 \\
\end{pmatrix} C'_{n4} G |JJ'| f^2 \times
\]

\[
\left[ \left( \frac{c_{g1}E_1}{f_1k_1} \right) + \left( \frac{c_{g2}E_2}{f_2k_2} \right) + \left( \frac{c_{g3}E_3}{f_3k_3} \right) + \left( \frac{c_{g4}E_4}{f_4k_4} \right) \right] +
\]

\[-\left[ \left( \frac{c_{g1}E_1}{f_1k_1} \right) + \left( \frac{c_{g2}E_2}{f_2k_2} \right) + \left( \frac{c_{g3}E_3}{f_3k_3} \right) + \left( \frac{c_{g4}E_4}{f_4k_4} \right) \right] \Delta f \Delta \theta \Delta \Delta t
\]  

(5.111)

The coefficient \( C'_{n4} \) is a scale factor in which all terms involving constants are assimilated. The source term in each interacting bin is given by:

\[
S_{n,(i,j)} \Delta f \Delta \theta_j = \frac{\Delta E_{(i,j)}}{\Delta t}
\]  

(5.112)
Finally, the basic expression of the Generalized DIA is obtained as:

\[
\begin{pmatrix}
\frac{\partial}{\partial f_1} & \frac{\partial}{\partial f_2} & \frac{\partial}{\partial f_3} & \frac{\partial}{\partial f_4} \\
\frac{\partial}{\partial \theta_1} & \frac{\partial}{\partial \theta_2} & \frac{\partial}{\partial \theta_3} & \frac{\partial}{\partial \theta_4}
\end{pmatrix} \begin{pmatrix}
S_{n,1} \\
S_{n,2} \\
S_{n,3} \\
S_{n,4}
\end{pmatrix} = C_{n4} G |JJ'| f^2 \times

\left[ \begin{pmatrix}
\frac{c_{g1} E_1}{f_1 k_1} \\
\frac{c_{g2} E_2}{f_2 k_2} \\
\frac{c_{g3} E_3}{f_3 k_3} \\
\frac{c_{g4} E_4}{f_4 k_4}
\end{pmatrix} + \begin{pmatrix}
\frac{c_{g1} E_1}{f_1 k_1} \\
\frac{c_{g2} E_2}{f_2 k_2} \\
\frac{c_{g3} E_3}{f_3 k_3} \\
\frac{c_{g4} E_4}{f_4 k_4}
\end{pmatrix} \right] - \left[ \begin{pmatrix}
\frac{c_{g1} E_1}{f_1 k_1} \\
\frac{c_{g2} E_2}{f_2 k_2} \\
\frac{c_{g3} E_3}{f_3 k_3} \\
\frac{c_{g4} E_4}{f_4 k_4}
\end{pmatrix} \right] \left[ \begin{pmatrix}
\frac{c_{g1} E_1}{f_1 k_1} \\
\frac{c_{g2} E_2}{f_2 k_2} \\
\frac{c_{g3} E_3}{f_3 k_3} \\
\frac{c_{g4} E_4}{f_4 k_4}
\end{pmatrix} \right]
\right]
\]

(5.113)

The Jacobian product can be written as:

\[
|JJ'| = \frac{1}{c_{g2} - c_{g2} c_{g2,\tau}} k_1 k_3
\]

(5.114)

in which \(c_{g2,\tau} \propto c_{g2}\). The ratio between these two group velocities follows from the particular location of the corresponding locus.

Expression (5.113) can be simplified further by assuming a geometric spacing of frequencies, i.e. succeeding frequencies have a constant ratio. Then,

\[
\Delta f_i \propto f_i
\]

(5.115)

and

\[
f_i = \frac{\Delta f_i}{\Delta f_1} f_1
\]

(5.116)

Further, assuming a constant directional step \(\Delta \theta = \Delta \theta_i\), the first term on the right-hand side of Eq. (5.113) reduces to \([-1 -1 1 1]^T f\).

For deep water the above expression can be simplified further. Since the coupling coefficient \(G\) scales as (Webb, 1978; Dungey and Hui, 1979):

\[
G \propto g^{-4} f^{12}
\]

(5.117)

the transformation Jacobians scale as

\[
JJ' \propto g^{-6} f^8
\]

(5.118)
and
\[
\frac{c_g}{f k} \propto \frac{g^2}{f^4}
\]
(5.119)

Substitution of these scaled quantities results in the following expression of the generalized deep water DIA.

\[
\begin{pmatrix}
\delta S_{nl,1} \\
\delta S_{nl,2} \\
\delta S_{nl,3} \\
\delta S_{nl,4}
\end{pmatrix} =
\begin{pmatrix}
-1 \\
-1 \\
1 \\
1
\end{pmatrix} C_{nl4} g^{-4} f^{23} \left[ \left( \frac{E_1 E_2}{f_1^4 f_2^4} \right) \left( \frac{E_3}{f_3^4} \right) + \left( \frac{E_4}{f_4^4} \right) - \left( \frac{E_1}{f_1^4} \right) + \left( \frac{E_2}{f_2^4} \right) \right] \left( \frac{E_3 E_4}{f_3^4 f_4^4} \right)
\]
(5.120)

Next, a general relationship between the wave numbers of an arbitrary wave number configuration is introduced. The radian frequencies of the four wave numbers in a quadruplet are related according to:

\[
\begin{align*}
\omega_1 &= \omega \\
\omega_2 &= (1 + \mu) \omega = \omega^+ \\
\omega_3 &= (1 + \lambda) \omega = \omega^* \\
\omega_4 &= (1 - \lambda + \mu) \omega = \omega^-
\end{align*}
\]
(5.121)

and the angles of the first two wave numbers are related as:

\[
\theta_2 = \theta_1 + \Delta \theta
\]
(5.122)

The choice of parameters is such to retain the notation of the DIA as presented by Hasselmann et al. (1985). An example of such a general wave number configuration is shown in Figure 5.24 in Appendix A.

The parameters \( \lambda, \mu \) and \( \Delta \theta \) are bounded since they must give valid solutions of the resonance conditions. Valid combinations of these parameters can be obtained by using the WRT method to select points on the locus. By definition these points satisfy the resonance conditions.

\[
\begin{pmatrix}
\delta S_{nl} \\
\delta S_{nl}^* \\
\delta S_{nl}^{**} \\
\delta S_{nl}^{***}
\end{pmatrix} =
\begin{pmatrix}
-1 \\
-1 \\
1 \\
1
\end{pmatrix} C_{nl4} g^{-4} f^{11} \times
\left[
\begin{pmatrix}
E \\
E^* \\
E^+ \\
E^-
\end{pmatrix}
\begin{pmatrix}
(1 + \mu)^4 & (1 + \lambda)^4 & (1 - \lambda + \mu)^4 & \\
(1 + \lambda)^4 & (1 - \lambda + \mu)^4 & \\
(1 + \mu)^4 & (1 - \lambda)^4 & (1 - \lambda + \mu)^4 & \\
(1 - \lambda^4) & (1 - \lambda + \mu)^4 & \\
\end{pmatrix}
\right]
\]
(5.123)
Putting $\mu = 0$ and $\Delta \theta = 0$ expression (5.93) reduces to the original DLA formulation (5.93). This analysis shows the origin of the factor 11 in the power of the frequency $f$, and the power 4 of the terms $(1 + \lambda)$, $(1 + \mu)$ and $(1 - \lambda + \mu)$.

For a given combination of the parameters $\lambda$, $\mu$ and $\Delta \theta$, the angles of the wave numbers $\vec{k}_3$ and $\vec{k}_4$ can be obtained from the resonance conditions by using a geometric method (Kahtri and Young, 1999 and Van Vledder, 2000). The principle of the geometric solution technique is explained as follows. The problem of finding the angles of the wave numbers $\vec{k}_3$ and $\vec{k}_4$ is mathematically speaking reduced in finding the crossing point of 2 circles. To compute the crossing points of the two circles the following equations the wave numbers $\vec{k}_3$ and $\vec{k}_4$ are written in polar co-ordinates:

\begin{align}
i_{3x} &= k_3 \cos(\theta_3) \\
i_{3y} &= k_3 \sin(\theta_3) \\ \text{(5.124)}
\end{align}

\begin{align}
i_{4x} &= k_4 \cos(\theta_4) \\
i_{4y} &= k_4 \sin(\theta_4) \\ \text{(5.125)}
\end{align}

For simplicity we define the vector $\vec{S}$:

\begin{align}
i_{x} &= S \cos(\theta_4) = k_{1x} + k_{2x} \\
i_{y} &= S \sin(\theta_4) = k_{1y} + k_{2y} \\ \text{(5.126)}
\end{align}

Then, the crossing point can be found by solving:

\begin{align}
i_{3} \cos(\theta_3) &= i_{x} - k_4 \cos(\theta_4) \\
i_{3} \sin(\theta_3) &= i_{y} - k_4 \sin(\theta_4) \\ \text{(5.127)}
\end{align}

\begin{align}
i_{1} \sin(\theta_3) &= i_{x} - k_4 \sin(\theta_4) \\
i_{2} \cos(\theta_3) &= i_{y} - k_4 \cos(\theta_4) \\ \text{(5.128)}
\end{align}

Squaring both equations and summing leads to:

\begin{align}
i_{3}^2 &= S^2 + k_4^2 - 2k_4 S \cos(\theta_4 - \theta_3) \\ \text{(5.129)}
\end{align}

From equation (5.129) we find the two solutions for the crossing point in terms of the angle $\theta_4$.

\begin{align}
i_{4} &= \theta_3 \pm \arccos \left( \frac{k_3^2 + S^2 - k_4^2}{2k_3 S} \right) \\ \text{(5.130)}
\end{align}

Similarly, the angle $\theta_4$ for the wave number $\vec{k}_3$ is given by:
\[ \theta_3 = \theta_1 \pm \cos \left( \frac{k_1^2 + S^2 - k_2^2}{2k_3S} \right) \] (5.131)

The Eqs. (5.130) and (5.131) are linked to the Eqs. (5.91) and (5.92) by writing \( k_3 = k(1 + \lambda)^2 \), \( k_4 = k(1 - \lambda)^2 \) and \( S = 2k \). Eqs. (5.130) and (5.131) can also be derived by applying the cos-rule on the configuration shown in Figure 5.24.

The resonance conditions for the GDIA for finite depth can be computed as follows:

Using the general dispersion relation \( \omega = \Omega(k, d) \) we find:

\[ \begin{align*}
\omega_1 &= \omega & \rightarrow & & k_1 = \Omega^{-1}(\omega_1, d) \\
\omega_2 &= (1 + \mu)\omega & \rightarrow & & k_2 = \Omega^{-1}(\omega_2, d) \\
\omega_3 &= (1 + \lambda)\omega & \rightarrow & & k_3 = \Omega^{-1}(\omega_3, d) \\
\omega_4 &= (1 - \lambda + \mu)\omega & \rightarrow & & k_4 = \Omega^{-1}(\omega_4, d)
\end{align*} \] (5.132)

Using \( \vec{S} = \vec{k}_1 + \vec{k}_2 \) we find the following expressions for the sum of the x- and y-components of the vectors:

\[ \begin{align*}
k_3 \cos(\theta_3) &= S \cos(\theta_1) - k_4 \cos(\theta_4) & (5.133) \\
k_3 \sin(\theta_3) &= S \sin(\theta_1) - k_4 \sin(\theta_4) & (5.134)
\end{align*} \]

yielding the same solution as for the deep water case, cf. The Eqs. (5.130) and (5.131). Note the agreement in functional form with the method described in Van Vledder (2000), including the symmetry axis, which has the same orientation as the vector \( \vec{S} \).

**5.4.4 The finite depth effects**

In present day third-generation wave prediction models, a simple method is applied to estimate the non-linear transfer in shallow water. Firstly, the nonlinear transfer rate is computed assuming deep water. Secondly, the resulting transfer rate is multiplied with a constant factor \( R \). This factor is a function of the dimensionless water depth \( \tilde{h} \), and constant for all spectral components of the spectrum. To enhance model robustness in the case of arbitrarily shaped spectra, the mean wave number is computed in a special way as (cf. Tolman, 1991):

\[ \tilde{k} = \frac{1}{E_{tot}} \int_0^{2\pi} \int_0^\infty \frac{1}{\sqrt{\tilde{k}}} E(f, \theta) df d\theta \] (5.134b)

with \( E_{tot} \) the total wave variance. The nonlinear finite depth transfer rate is computed as:

\[ S_{nl}(f, \theta) = S_{nl}^\infty(f, \theta) \times R(x) \] (5.135)
in which \( x = 0.75\tilde{k}h \) and the function \( R(x) \) is given by:

\[
R(x) = 1 + \frac{5.5}{x} \left( 1 - \frac{6x}{7} \right) \exp\left(-\frac{5x}{4}\right)
\]  

(5.136)

The depth scaling is based on a parameterisation of results of computations with the wave model EXACT-NL (Hasselmann and Hasselmann, 1981) of the non-linear transfer for a set of JONSWAP spectra and various water depths.

This scaling does not change the shape of the non-linear transfer. A careful analysis of depth effects on the non-linear interaction terms, however, shows that for \( \tilde{k}h < 1.0 \) the shape of the transfer function changes. This is illustrated in Figure 5.25 in Appendix A, but it is also already evident from Fig. 1 of Hasselmann and Hasselmann (1985a). This Figure is reproduced in Figure 5.26 in Appendix A.

The most striking feature is that the first positive lobe shifts towards lower frequencies and becomes wider and larger than the scaled deep water transfer. The parameterisation of the depth scaling has some weaknesses. It is based on computations with (deep water) JONSWAP spectra and not on shallow water spectra (cf Fig. 5.7). Only the minima and maxima of the transfer function were considered and no attention has been given to a dependence on frequency and direction. This deviation from the scaling law is important for the estimation of the rate of frequency down-shifting of swell waves in shallow water. As such, the WAM depth scaling may contribute to the underprediction of swell wave periods in, e.g. the Dutch, coastal waters.

The WAM depth scaling has the following disadvantages:

- It uses a scale factor constant for all bins of the spectrum, in other words, the shape of the non-linear transfer does not change, whereas the same computations on which the scaling is based show that the shape does change (cf. Figure 1 of Hasselmann and Hasselmann, 1985) especially for low values of the dimensionless depth \( \tilde{k}h \).
- The scaling method is based on computation for deep water JONSWAP spectra. Since these spectra are not representative for shallow water conditions (cf. Bouws et al., 1985 and Resio et al., 2001) it can be questioned if the WAM scaling was derived properly.
- The scaling method is based on an analysis of the minima and maxima of the non-linear transfer function. No distinction is made on dependence with frequency and/or direction.

In principle one may attempt to re-derive the scaling law for non-linear transfer. Such an attempt, however, might easily fail because the non-linear transfer is very sensitive to small changes in spectral shape and a scaling law might only work for smooth wave spectra. To avoid these problems, it is better to compute the non-linear transfer directly for finite depth conditions.

The finite depth DIA does not have this disadvantage since finite depth effects are automatically taken into account. The capabilities of the finite depth DIA are illustrated in Figure 5.27 in Appendix A. As can be seen in Figure 5.27 the finite depth DIA is able to reproduce some of the basic features of the changes of the nonlinear transfer rate in shallow water as inferred from exact computations. The overall magnitude increases, the first positive lobe shifts towards lower frequencies, and the first zero-crossing is closer to the peak frequency of the spectrum. Despite these improvements, the position of the negative
lobe along the frequency axis is still much too high and the second positive lobe is also still too high.

A detailed study of the finite depth DIA is described in Appendix C, "Improved modelling of nonlinear four-wave interactions in shallow water". This Appendix will be incorporated in the proceedings of the 28th Int. Conf. on Coastal Engineering (Van Vledder and Bottema, 2002).

### 5.4.5 The multiple Generalized Discrete Interaction Approximation

The Generalized DIA is able to compute the interaction between any possible wave number configuration. By using a finite set of GDA configurations it is possible to construct a Multiple GDIA, which contains the following parameters:

\[
\lambda_i, \mu_i, \Delta \theta_i, C_{nl4,i} \]  \quad \text{for}  \quad i = 1, N \tag{5.137}
\]

The idea of a multiple DIA was already considered by Hasselmann (1985), who already considered a second configuration with \( \lambda = 0.15 \). For efficiency reasons this double DIA was not included in the original WAM model.

For deep water the concept of the general multiple DIA can be written as:

\[
\begin{pmatrix}
S_{nl}^+ \\
S_{nl}^- \\
S_{nl}^* \\
S_{nl}^{* -1}
\end{pmatrix}
= 
\begin{pmatrix}
-1 \\
-1 \\
\sum_{i=1}^{N} C_{nl4,i} g^{-4} f_{11} \\
1 \\
1
\end{pmatrix}
\times
\begin{pmatrix}
E^+ \\
E^- \\
(1+\mu_i)^4 \\
(1+\lambda_i)^4
\end{pmatrix}
\left[
\frac{E^+}{(1+\mu_i)^4} + \frac{E^-}{(1-\lambda_i + \mu_i)^4}
\right]
\left[
\frac{E^+}{(1+\lambda_i)^4} + \frac{E^-}{(1-\lambda_i + \mu_i)^4}
\right]
\tag{5.138}
\]

### 5.4.6 Parameter estimation

The key problem in defining a Generalized DIA is the determination of the coefficients \( \lambda_i, \mu_i, \Delta \theta_i \) and \( C_{nl4,i} \). Various approaches can be followed to obtain these coefficients for deep and shallow water. Essential elements are the use of a large set of directional test spectra with a variety of characteristics, and the use of a benchmark model to compute the non-linear interactions with an exact method. Then, one dataset will be used to determine the coefficients and another (independent) dataset will be used to verify an extended DIA. A crucial step in this process is the choice of the test spectra. The range of spectral characteristics must be wide enough to ensure a wide range of applicability of the approximate methods.

One of the techniques for the determination of the coefficients is to treat it as a linear problem and to determine only the coefficients of proportionality by solving a least-squares (LSQ) problem. This can be achieved using Householder transformations of matrices, which are filled, with results of exact and approximate calculations of the non-linear transfer for a
set of test spectra (Hashimoto, 1999; Hashimoto and Kawaguchi, 2001). However, this approach the parameters (λ_i) defining the wave number configuration were pre-set and were not direct part of the calibration process. This method resulted in positive and negative coefficients of proportionality. These negative coefficients, however, are unphysical and limit the applicability of the Multiple DIA (MDIA) for spectra that are not similar in shape compared to the spectra used for determination of the coefficients of the MDIA. Still, the LSQ method provides a good starting point for the determination of the coefficients of a multiple DIA. Improvements can be made by, for instance, using unequal weights in defining the LSQ sum and/or by imposing constraints on the coefficients.

A disadvantage of computing the coefficients of proportionality with the previously described rigorous numerical method is that the inherent relative contribution of each individual wave number configuration is not accounted for. If these inter-relationships can be preserved, the optimisation procedure reduces to the estimation of only one coefficient. This has major benefits, in the sense that the numerical requirements for the LSQ problem reduce significantly.

The inherent relative contribution can be preserved by downscaling exact methods. Treating the WRT method (Tracy and Resio, 1982, Resio and Perrie, 1991) or Masuda’s method (Masuda, 1980; Komatsu and Masuda, 1996; Hashimoto et al., 1998) as a mathematical integration problem, down-scaling is similar to increasing the step size in the evaluation of integrals. In this way not all possible interactions are taken into account since a number of discrete spectral bins are skipped in the computation of the total non-linear transfer. In this way the computational requirements are reduced at the expense of accuracy. The down-scaling method provides an attractive alternative approach to extending the DIA with additional wave number configurations.

The WRT method can be used to generate a set of wave number configurations that satisfy the resonance conditions. For a given combination of the wave numbers \( \vec{k}_1 \) and \( \vec{k}_3 \), two loci can be generated, one for the wave number \( \vec{k}_2 \) and one for the wave number \( \vec{k}_4 \). Then, a set of wave number configurations can be generated by selecting a finite number of points on these loci. By definition, these quadruplets satisfy the resonance conditions. This procedure can be followed for various combinations of the wave numbers \( \vec{k}_1 \) and \( \vec{k}_3 \). An advantage of this procedure is that an regularly distributed set of quadruplets is generated. The corresponding weights of each quadruplet can be computed using a modified version of Eq. (5.88), such that the integral along the locus is splitted into the same number as there as selected points on the locus. To obtain a spectrum independent set of weights, the action density product P must be set to a constant value of, say, one. In this way, the relative contribution of each quadruplet is preserved, and a simple rescaling involving one parameter is needed.

### 5.5 An efficient computational technique: the triplet method

Based on an idea developed by Snyder et al. (1993) it is possible to simplify the computation of the non-linear transfer by considering its calculation as a finite sum of products of three energy densities at discrete spectral grid points. To that end, all unnecessary interpolations and computations need to be removed from the computational procedure. This concept will be illustrated on the basis of the DIA and the WRT method.
In the DIA the energy densities $E^-$, $E^0$ and $E^+$ are computed from the discrete energy density bins by means of linear interpolation from neighbouring bins:

$$E^-=\sum_{i=1}^{4} v_i E_i^-$$  \hspace{1cm} (5.139)$$

and

$$E^+=\sum_{j=1}^{4} w_j E_j^+$$ \hspace{1cm} (5.140)$$

in which $v_i$ and $w_j$ are the weights to obtain the energy density from the 4 surrounding points $E_i^-$ and $E_j^+$. By definition the weights satisfy:

$$\sum_{i=1}^{4} v_i = 1$$ \hspace{1cm} (5.141)$$

$$\sum_{j=1}^{4} w_j = 1$$ \hspace{1cm} (5.142)$$

Substitution of the expressions (5.139) and (5.140) into the general expression for the contributions to the nonlinear transfer rate the DIA (5.93) yields:

$$\begin{bmatrix}
\delta S_{nd} \\
\delta S_{nd}^+ \\
\delta S_{nd}^-
\end{bmatrix} = \begin{bmatrix}
-2 \\
1 \\
1
\end{bmatrix} C_{nd} \rho_4 g^2 f^{11} E^2 \left[ \sum_{j=1}^{4} w_j E_j^+ \left( \frac{\sum_{j=1}^{4} w_j E_j^+}{(1+\lambda)^4} + \frac{\sum_{i=1}^{4} v_i E_i^-}{(1-\lambda)^4} \right) - \frac{2E \left( \sum_{j=1}^{4} w_j E_j^+ \right) \left( \sum_{i=1}^{4} v_i E_i^- \right)}{(1-\lambda^2)^4} \right]$$ \hspace{1cm} (5.143)$$

Elaboration of the term between the square brackets gives an expression in the form of a finite sum of triplets (i.e. products of three energy densities at discrete spectral grid points):

$$\sum_{j=1}^{N} \alpha_j E_{p(j)} E_{q(j)} E_{r(j)}$$ \hspace{1cm} (5.144)$$

The terms $p(j)$, $q(j)$ and $r(j)$ refer to all interacting bins in a triplet, and $j$ refers to a spectral bin in a 2-d spectrum. Then, the change in energy density by the non-linear interactions can be written as:

$$\begin{bmatrix}
\delta S_{nd} \\
\delta S_{nd}^+ \\
\delta S_{nd}^-
\end{bmatrix} = \begin{bmatrix}
-2 \\
1 \\
1
\end{bmatrix} C_{nd} \rho_4 g^2 f^{11} \sum_{j=1}^{N} \alpha_j E_{p(j)} E_{q(j)} E_{r(j)}$$ \hspace{1cm} (5.145)$$
The resulting change in energy density within an interacting quadruplet is redistributed among the surrounding bins using the same weights as used for the determination of the energy densities:

\[
\delta S_{nl,j}^- = v_i \times \delta S_{nl}^-
\]

(5.146)

\[
\delta S_{nl,j}^+ = w_j \times \delta S_{nl}^+
\]

(5.147)

Finally, combination of all expressions results in a transfer for an arbitrary spectral bin with index \(s(k)\) which can be written as a finite number of weighted triple products of energy densities in discrete spectral grid points:

\[
S_{nl4,s(k)} = \sum_{k=1}^{N_4} \delta_k E_{p(k)} E_{q(k)} E_{r(k)}
\]

(5.148)

The problem is now reduced to the determination of all indices and weights. This is achieved by means of a dedicated program, which resembles an algebraic manipulator. In this process all contributions with equal indices for \(p(k), q(k), r(k)\) and \(s(k)\) are combined by adding their weights \(\delta_k\). In this way all possible symmetries are automatically accounted for. The weights in expression (5.148) need only to be computed for one direction sector, since the indices for all other directions can be obtained by a simple rotation. The triplet method also offers the possibility to filter out small contributions to the total transfer on the basis of the weights \(\delta_k\). All remaining weights and indices of the triplets can be stored in a database that can be stored in memory or on disk. The content of this database depends on the spectral resolution, the power of the spectral tail and the water depth. As such, it resembles the database file of the pre-computed integration space resulting from the WRT method.

The number of triplets can be greatly reduced by assuming a piece-wise constant representation of the spectrum in the same way as Snyder et al. (1993). Using this assumption the bi-linear interpolation of the Eqs. (5.139) and (5.140) is replaced by using a single value corresponding to the nearest bin. The effect on a discrete spectrum is shown in Figure 5.28 in Appendix A. It is to be discussed and analysed if this approach gives valid transfer rates for the higher frequencies, which are believed to follow a power law and not a staircase decrease.

Since there are much more possible triplets than bins in a discrete spectrum it is more efficient to evaluate expression (5.148) using a nested loop over the triplets. To that end the indices of all triplets are sorted such that \(p(k) \leq q(k) \leq r(k)\), followed by a factorisation. In this way the outer loop of the algorithm is over all \(p(k)\), the next loop is over all \(q(k)\), the next loop over \(r(k)\) and the most inner loop is over all \(\delta_k\) and \(s(k)\). The main advantage of the triplet method is that all weights and indices, including the combining, filtering, sorting and factorisation can be computed and stored outside the wave model. The number of lines of code for this algorithm is about 20 (twenty), except for I/O related operations to retrieve all weights and indices of the triplets.

The weights and indices of the triplet method can be derived for all types of computational methods for solving the Boltzmann integral. This includes the Generalised DIA, the Multiple DIA, the finite depth DIA, but also exact methods like the Webb/Resio/Tracy method (Tracy and Resio, 1982; Resio et al., 2001), and the Japanese methods (Masuda 1980; Hashimoto et al., 1998).
The triplet method can be considered as a unified method for the computation of the non-linear transfer. The size of the sum determines the accuracy and the corresponding computational requirement. The extremes of this range are the fast but inaccurate DIA and the time consuming full solution of the non-linear transfer. For application in an operational wave model an optimum needs to be found between the required accuracy and possible computational effort.

The triplet method has been applied to the MDIA presented by Hashimoto and Kawaguchi (2001) to quantify the number of unique triplets determining an MDIA and the relative error as a function of the number of wave number configurations. The results are shown in Figure 5.29 in Appendix A. The upper panel shows the number of basic and combined triplets as a function of number of wave number configurations. The number of basic triplets increases linearly with the number of configurations. However, with increasing number of configurations, more and more triplets can be combined. In this example, this number reaches an upper limit for eight wave number configurations. The lower panel shows the relative error with respect to an exact solution as a function of number of configurations. It can be seen that this error reaches a lower limit. This limiting behaviour indicates that a Multiple DIA, in which only the shape parameter $\lambda$ is used, has a limited capability to approximate the exact solution. The only way to further reduce the relative error is to include generally shaped configurations.

The triplet method is similar to the method derived by Snyder et al. (1993). It is noted here that the basic ideas behind the triplet method can also be applied to compute the triads interactions in a discrete wave spectrum yielding a duplet method.

## 5.6 Conclusions

In this study four major results have been obtained that form the basis of replacing the DIA with better techniques for computing the non-linear quadruplet wave-wave interactions in spectral wave models.

The first result is the establishment of a proper exact method for the evaluation of the Boltzmann integral describing the non-linear interactions. The basis for this method is the technique originally derived by Webb (1978) and Tracy and Resio (1982). Their method has been used by many researchers for deep water applications (e.g. Perrie and Resio, 1992; Young and Van Vledder, 1993), and for finite depth applications (e.g. Resio et al., 2001). Despite the fact this their 'code' exists for many years, this study and previous work carried out by Van Vledder (2000, 2001) and Van Vledder et al. (2000), yielded many algorithmic and mathematical improvements. In addition, a number of bugs where discovered and fixed in the original code of Tracy and Resio.

The WRT method developed in this study has been implemented in a test version of the SWAN model. Some first results of this method in the SWAN model are reported in Chapter 8.

With respect to the establishment of a proper benchmark model, it is worth noting that there is an ongoing discussion with N. Hashimoto about a difference with a certain factor ($3, \pi, \text{or}\ \nu g$) between computational results obtained with the WRT method and the Masuda-Hashimoto method. Preliminary tests indicate that the 2-dimensional shape of the computed transfer rate are more or less similar, but that the results only differ with previously
mentioned factor. At WISE 2002, Hashimoto and Van Vledder have discussed the origin of this factor, since then Hashimoto already found a factor 2 in his code, this would imply that the difference has reduced to a factor of about 1.5. Since a version of the Masuda/Hashimoto code is implemented in a test version of SWAN it is essential that the origin of this difference is found, fixed and published. In the framework of the Advanced Wave Prediction Program of the US Office of Naval Research, comparisons have been carried out with various codes for solving the Boltzmann integral. These include, the EXACT-NL model of Hasselmann and Hasselmann (1985), the original WRT code, the WRT version developed in this study, and the version of the EXACT-NL model developed by Rasmussen (1995). All these code gave comparable results. However, an independent check might be to use the results of narrow peak approximations developed by Fox (1976) and Dungey and Hui (1978).

The second result is the establishment of the Generalized Discrete Interaction Approximation (GSD). This work, carried out together with J.H. Rasmussen from Denmark, provides a method for the extension of the original DIA with additional wave number configurations, which in the limit has the ability to approach the exact solution to any desired degree of accuracy. The main advantages of the Generalized DIA with respect to the original DIA are:

- it includes finite depth effects;
- it uses a general wave number configuration;
- it can include multiple interacting wave number configurations;
- the WAM depth-scaling becomes obsolete.

The Generalized DIA has been implemented in a test version of the SWAN model, version 40.11. Some applications of this method are presented in Chapter 8. A more detailed description of the finite depth DIA and some applications in shallow water are presented in Appendix C, which was made as a contribution to the 28th Int. Conf. on Coastal Engineering.

The third result is the understanding that the DIA is a very small sub-set of any exact method for computing the non-linear interaction. In addition, the mathematical link between the DIA and exact methods has been explored. The crux of any exact method is the evaluation of many possible wave number configurations. Using discretisation techniques of the full solution, single wave number configurations can be identified. In this way the gap between accurate and costly exact methods and crude but fast DIA can be bridged, either by reducing the accuracy of exact methods using courser and courser resolutions, or by extending the DIA with additional generally shaped, finite depth interacting wave number configurations. This understanding needs some further work to assess the optimal setting for inclusion in a spectral model like SWAN.

The fourth result is the derivation of the triplet method. This method provides a simple way of computing all possible interactions between discrete spectral bins. The triplet method has a number of advantages:

- The method can be derived both for DIA-like methods as well as exact methods like the WRT method.
- The main work of computing all the weights can be evaluated outside the model run. These weights depend on the spectral grid, power of spectral tail and the water depth.
- The size of the sum for evaluating the non-linear transfer rate gives a direct measures of the degrees of freedom.
• It provides a simple way of performing filtering operations to reduce the computational burden.

Replacing the DIA with a more accurate method implies that the source term balance will change. Since shortcomings of the DIA are now masked by various fudge factors, replacing the DIA with something better will initially lead to a lower performance of the wave model. To improve the performance again, a re-calibration of the wave model is necessary.

The results of a first calibration and its effect on the predicted wave period are reported in Chapter 8 of this report. Regarding the role of the quadruplets, attention will be given to finite depth effects and the interaction between sea and swell waves on the predicted wave period. Since the role of the quadruplet interactions is important, further works needs to be carried out to speed up the evaluation of the Boltzmann integral.

5.7 Recommendations

The following recommendations are made with respect to the work carried on the quadruplets:

• Find and fix the origin of the ghost factor between computational results obtained with the WRT and Masuda/Hashimoto methods. This involves a detailed analysis of the Masuda/Hashimoto method to understand the integration procedure. Part of this analysis has already been done, but is reported elsewhere. In addition, comparisons should be made with results obtained from narrow peak approximations.

• Perform a detailed verification study of all available codes for computing the non-linear transfer rate (e.g. Benoit, Hashimoto, Lavrenov, Polnikov, EXACT-NL, Lin, Rasmussen, Pushkarev). In such a study attention should be paid to e.g. the treatment of symmetries, numerical integration techniques, limitations of integration techniques.

• Optimize the WRT method regarding the optimal number and distribution of points on a locus, optimal settings for filtering out unimportant contributions to the Boltzmann integral, the effect of using the nearest bin instead of bi-linear interpolation, and the handling of pre-computed databases in operational wave models.

• Develop a procedure to generate a set of wave number configurations for the GDIA. Here attention should be paid to the relative weight of each configuration, the computational requirements for a required accuracy and to finite depth effects.

• Determine the optimal spectral resolution for computing the Boltzmann integral accurately.

• Operationalisation and optimization of the triplet method. Here attention should be paid to procedures to generate sets of basic triplets from the DIA’s and exact methods, sorting, combining and factorisation of the triplets, and the handling of pre-computed datasets of triplets coefficients in operational wave models.

• Recalibrate spectral wave models.

• Perform detailed studies of wave evolution with various types of methods to compute the nonlinear four-wave interactions. Here, attention should be paid to situations with mixed seas, slowly turning wind situations and situations were also triads are active.
6 Triads

6.1 Introduction

In nearshore regions the short evolution distance and moderate dispersion suggest that second-order nonlinearities involving triad of waves are important. Triad interactions occur among waves with frequencies and wave numbers such that

\[ f_1 \pm f_2 = f_3 \]  \hspace{1cm} (6.1)

and

\[ k_1 \pm k_2 = k_3 \]  \hspace{1cm} (6.2)

where \( f \) and \( k \) are the scalar frequency and vector wavenumber, respectively. The wave components each satisfy the linear dispersion relation. The sum and difference interaction forces motions with the sum and difference frequency and wavenumber. If component 3 satisfies the linear dispersion relation, the interaction is resonant and a one-way transfer of energy to component 3 is caused. For gravity surface waves this is only possible in very shallow water. Here the waves are nondispersive, i.e. all wave components propagate with the same velocity. If component 3 does not satisfy the linear dispersion relation, then the interaction is non-resonant (e.g. in intermediate depths where waves are weakly dispersive) and the transfer is back-and-forth because of the mismatch in the phase speed.

Abreu et al. (1992) were the first who derived a phase-averaged formulation for the triad interactions, being suitable as a source term in a spectral energy balance. The formulation can only be applied to nondispersive shallow-water waves, causing a one-way transfer to higher harmonics. Therefore it is not suitable in many practical applications for wind waves. A step forward was made by Eldeberky and Battjes (1995). A discrete triad approximation (DTA) for co-linear waves was obtained by considering only the dominant self-self interactions. The model appeared to be fairly successful in describing the essential features of the energy transfer from the primary peak of the spectrum to the super harmonics. A slightly different version, the Lumped Triad Approximation (LTA) was later derived by Eldeberky (1996). This LTA is used in SWAN.

The aim of this part of the study is to find under which circumstances triad wave interactions play a role and to give a qualitative indication of the effect of triads on the spectrum. A phenomenological study will be carried out, using physical model results and model results obtained with the phase-resolving Boussinesq-type wave model TRITON and the phase-averaging model SWAN. We want to identify and quantify the deficiency of the LTA formulation in SWAN. TRITON will be used as a benchmark model, under the assumption that nonlinear interactions are determined correctly by this model. Note that the computational effort is significantly larger for the phase-resolving model TRITON.

In Section 6.2 the LTA formulation in SWAN is described. Comparisons between measurements in the laboratory flume of WL | Delft Hydraulics and the model results of SWAN and TRITON for the Petten case are described in Section 6.3. In Section 6.4 the results
of TRITON and SWAN computations are given for a number of schematized one-dimensional situations, which are representative for the bank system in front of the Dutch coast.

6.2 LTA in SWAN

In SWAN the triad wave-wave interactions are modeled by applying the Lumped Triad Approximation (LTA) of Eldeberky (1996) in each spectral direction. Eldeberky (1996) originally derived the net source term due to triad interactions for unidirectional waves. Formulated as a source term to be used in SWAN, the final result of these derivations reads

\[ S_{nl3}(\sigma, \theta) = S_{nl3}^{+}(\sigma, \theta) + S_{nl3}^{-}(\sigma, \theta), \]  

(6.3)

with

\[ S_{nl3}^{-}(\sigma, \theta) = -2S_{nl3}^{+}(2\sigma, \theta), \]  

(6.4)

and

\[ S_{nl3}^{+}(\sigma, \theta) = \alpha c_c R^2 \sin(\beta) \left[ E^2(\sigma/2, \theta) - 2E(\sigma/2, \theta)E(\sigma, \theta) \right]. \]  

(6.5)

where \( \alpha \) is an adjustable constant (a tuning) parameter. Dingemans (1998, p.48) remarked that it is not clear how this tuning parameter should be chosen. Experiments with one situation showed that \( \alpha = 1 \) led to acceptable results. Furthermore in (6.3)-(6.5) \( \sigma \) denotes relative frequency, \( \theta \) is wave direction, \( c \) propagation velocity, \( c_g \) the group velocity and \( R \) is an interaction coefficient taken from Madsen and Sørensen (1993),

\[ R = \frac{k_{\sigma/2}^2 (gd + 2c_{\sigma/2}^2)}{k_{\sigma}d \left( gd + \frac{2}{15}gd^3 k_{\sigma}^2 - \frac{2}{5} \sigma^2 d^2 \right)}, \]  

(6.6)

with \( k_{\sigma} \) and \( c_{\sigma} \) the wave number and propagation velocity corresponding to frequency \( \sigma \), respectively. Furthermore, \( d \) is the local water depth.

The bi-phase is approximated with

\[ \beta = -\frac{\pi}{2} + \frac{\pi}{2} \tanh \left( \frac{0.2}{Ur} \right), \]  

(6.7)

an expression that is obtained by fitting experimental results. The Ursell number is defined by

\[ Ur = \frac{g}{8 \sqrt{2} \pi^2} \frac{H_s \bar{T}^2}{d^2} \]  

(6.8)

with \( H_s = H_{m0} \) the significant wave height and \( \bar{T} \) the mean wave period. An important conclusion from the review of Dingemans (1998, p.48) is that the LTA approach seems to be
a rather robust method. The understanding of how to choose the tuning parameter $\alpha$ is seen as a major drawback.

The LTA method is a very approximate one. Part of the bi-spectrum, in casu the bi-phase, is not predicted at all. From a limited set of measurements a simple prescription is found. Consequently, the formulation may not be general enough to use for all kinds of conditions.

The simplified one-equation stochastic model represents an average effect of triad wave interaction, transferring energy from lower to higher frequencies through self-self interaction. The model does neither take into account the energy transfer to subharmonics nor the non-resonant wave interaction. Consequently, the approach is in general unsuitable for frequency spectra that are not uni-modal (and unidirectional). In cases of sea states involving swells and wind waves, the LTA model is not expected to model accurately the nonlinear energy exchange between the two frequency regimes.

Results of simulations on a horizontal bottom, performed by Rasmussen (1998, sect. 11), indicate that the LTA model and other one-equation models are not applicable for prediction of long-term evolution on a (nearly) horizontal bottom in shallow water. As a consequence of the introduced simplifications, the LTA model is mainly appropriate for relatively short evolution distances on sloping bathymetries, where the generation of bound superharmonics is substantial. However, Dingemans (1998) noticed that the LTA method is only valid for horizontal bottom, because, in course of the derivation, the shoaling term in the amplitude equations was temporarily neglected (Eldeberky, 1996, p.164). However, the term was never put back in the equations. This contradiction in the model setup might give rise to serious problems.

The triad wave interactions considered by Eldeberky (1996) are only valid in 1D and can be applied to two-dimensional problem accounting only for collinear interactions. In SWAN the LTA approach is used in each spectral direction. Interactions between waves with different directions have not been considered.

### 6.3 SWAN and TRITON computations of Petten case

#### 6.3.1 Description of numerical model computations

Within the framework of the European MAST-OPTICREST project physical model investigations and numerical model investigations have been performed by Van Gent and Doorn (2000). The numerical models SWAN (spectral wave model) and TRITON (time-domain model) have been applied to model wave propagation over the shallow foreshore of the Petten Sea-Defence and compared to the results of the physical model investigations. For the SWAN computations in that study SWAN version 30.75 was used. On the shallow foreshore depth-induced wave breaking and triad wave interaction are the major mechanisms involved. The conclusions drawn form the study by Van Gent and Doorn (2000) are still valid for SWAN, version 40.11. To confirm this statement we have added Figure 6.0 in Appendix A in which for test 1.01 the wave energy spectra are given, that have been computed with version 30.75 and version 40.11. Clearly, there is no difference at the output locations.

The physical model tests performed for studying the Petten Sea-defence were performed in a wave flume where about 1 km of the foreshore was schematized. Seaward of this part of
the foreshore an offshore bar is present on which wave breaking occurs under storm conditions ("Pettemer Polder"). For those tests in the physical model investigations that were aimed at representing measured storm conditions the effects of this bar were accounted for by generating the wave energy spectra that were measured (far) behind this bar in prototype. A second bar, closer to the dike, was modeled in the flume. Figure 6.1 shows the foreshore as constructed in the flume. This figure also shows at which locations wave conditions were measured during the tests. Figure 6.2 shows the dike as modeled in the flume.

Conditions that occurred in 6 storm periods for which prototype measurements were available were studied. Also conditions to study the influence of several parameters such as wave height, wave steepness, spectral shape, water level were studied. In addition, tests with regular waves were performed for validation of numerical models, especially for analysis of surface elevations on the slope.

![Figure 6-1](image1)
Figure 6-1  Model set-up in physical model tests (foreshore)

![Figure 6-2](image2)
Figure 6-2  Model set-up in physical model tests (dike)
The modelling of wave reflection near dikes has not been studied properly in both SWAN and TRITON. Therefore, in order to validate the model for the present situation with a complex foreshore, only tests where no structure was present in the flume are used (20 tests). Wave energy spectra measured in the flume at the start of the foreshore, corresponding to a location 1300m seaward of the crest of the dike on prototype-scale, were prescribed in the computations as incident wave energy spectra. The measured wave energy spectra were obtained from measured recordings of the surface elevations. Although the structure was not present in the flume, wave reflection by the foreshore itself still affects these surface elevations. Alternatively, the surface elevations could be corrected for by methods to eliminate the reflected waves from the recordings of the surface elevations. This has not been done because these methods involve loss in resolution of the spectrum. For the validation of the applied model the effects of reflected waves in the surface elevations at deep water were considered as less important than the loss of spectral resolution. Thus, the measured surface elevations that were used for comparison are those from individual wave gauges in tests without the structure in position. Because the measurements involve a relatively large amount of energy in long waves which is not modeled by SWAN, the parameters from the measurements and the numerical models SWAN and TRITON used for comparisons are based on the energy in the short waves only: between the frequencies 0.04 Hz and 0.3 Hz, using exactly the same software to obtain these parameters.

Apart from long wave reflection at the shoreline, another possible mechanism for the existence of long wave energy exists. The low frequency energy in the observations may be generated in the breaking process due to time variation of the breaking point or due to the presence of coupled modes at the group frequency of the short wave motion. The low frequency waves tend to shoal at a greater rate and energy may be transferred from the main spectral lobe towards lower frequencies in the breaking process and/or nonlinear coupling of wave components.

Some essential parameters used in the computations with the spectral wave model SWAN (Version 30.75) are as follows: for the physical process the default settings were used for wave breaking (depth-induced wave breaking and whitecapping), wave set-up, bottom friction and modeling triad wave-wave interaction. Quadruplet wave-wave interactions were not modeled. The constant space step was 10 m. The spectral resolution was 73 within the frequency range between 0.04 Hz and 0.35 Hz ($\Delta f/f_0=0.03$) and 60 in the directional sector between -7.5 and +7.5 degrees. The above mentioned settings were used without calibration to the specific application studied here.

In the wave model TRITON measured time signals of the incident waves at the start of the foreshore, corresponding to a location 1300 m seaward of the crest of the dike in prototype, were prescribed in the computations as incident waves. At this open seaward wave boundary the surface elevations were prescribed while at the landward boundary (toe of the structure) a weakly reflecting boundary was used. At this open boundary the waves can leave the computational domain, where use is made of the long-wave assumption at this boundary to assess the phase velocity of the outgoing waves. Some essential parameters used in the computations are the space-step $\Delta x=1.0$ m and time-step $\Delta t=0.06$ s (based on the Courant criterion).

For comparisons between the measured and computed wave parameters, again the measured recordings of the surface elevations of individual wave gauges are used, thus including incident and reflected waves. The wave parameters $H_{m0}$, $T_{mL0}$, and $T_p$ are used for comparison. The same software for analysis of time-signals was used for the measured and the computed time-signals. The parameters from the measurements and the computations
are based on the energy between the frequencies 0.04 Hz and 0.3 Hz. In the computations approximately 500 waves were computed which is shorter than the actual measurements, which had a duration of approximately 1000 waves. The measured and computed wave parameters were obtained from the time-series of these 500 waves.

### 6.3.2 Comparison with physical model tests

For each of the 20 tests used for comparisons, the measured and computed wave energy spectra at five positions on the foreshore are studied in Van Gent and Doorn (2000): The positions are denoted by DEEP, MP3, MP5, MP6 and TOE (see Figure 6.1). Figure 6.1 in Appendix A, taken from their Appendix ‘Figures’, show for a selection of the 20 tests the wave energy spectra. The wave energy spectra at the position DEEP are the measured wave energy spectra that were used as incident wave energy spectra for the numerical models.

The general impression from examining these figures is that SWAN provides realistic energy levels at all locations; the area underneath the wave energy spectra corresponding to the short waves is in general reproduced properly, see Figure 6.2 in Appendix A, in which the computed and measured significant wave height $H_{m0}$ for all 20 tests are compared. Some tests show that at the first location for comparison (MP3, 570m seaward from the toe of the construction), the total wave energy is somewhat overestimated by SWAN (e.g., tests 1.03 and 1.04). The modeling between deep water and this location also yields for some tests a too high shift of wave energy to higher frequencies; for some tests this results in a rather large amount of wave energy in a peak with a frequency twice the main peak (e.g., tests 2.11, 2.13 and 2.61). These differences are relatively large for tests with low water levels. Although the energy in these peaks is clearly overestimated, in locations further landward the dissipation of energy clearly reduces the amount of energy in these peaks again. Nevertheless, at the toe of the structure SWAN provides wave energy spectra where the wave energy is still distributed in peaks at the original deep-water peak frequency and a peak at twice this frequency peak, while the measurements show more flat wave energy spectra. Consequently the mean wave period $T_{m-1.0}$ computed by SWAN underestimates the measured wave period (see Figure 6.2 in Appendix A).

The wave energy spectra of the same selection of the 20 tests that were performed by the time-domain model TRITON are also presented in Figure 6.1 in Appendix A, taken from Figures F3.1a-d in the Appendix ‘Figures’ of Van Gent and Doorn (2000). In Figure 6.3 in Appendix A the significant wave height $H_{m0}$ and mean wave period $T_{m-1.0}$ for the 20 measurements are compared with their equivalent TRITON results. As mentioned above, both the computed and the measured spectral parameters were obtained from the time-series of approximately 500 waves. However, the measured wave energy spectra as shown in the Appendix ‘Figures’ are based on the complete measured time signals of 1000 waves, which explains the small differences between the measured and computed wave energy spectra at the location DEEP (see Figure 6.1, 1235m seaward from the toe of the structure).

The general impression from examining these figures is that TRITON simulates both the spectral shapes and the energy levels rather accurately. Also the energy shift to the lower frequencies is modeled surprisingly well (e.g., Tests 1.01, 1.02 and 2.51). Some tests in the measurements show clear peaks at low frequencies that are not as sharp in the computational results. This energy in low frequencies does not fully dissipate at the wave absorber at the rear-side of the flume and so this energy reflects for a relatively large part back into the flume; in the computations the rear-side of the flume was modeled as open which means that this reflection did not occur in the numerical model simulations, causing
differences which cannot be attributed to inaccurate modeling of the wave propagation in the numerical model.

Although only one geometry has been considered in this section, an entire spectrum of wave climates has been taken into account. TRITON seems to handle the variety of boundary conditions surprisingly well. Therefore, TRITON will be used as a benchmark model in a number of schematized situations in the next subsection.

6.4 Importance of triads in SWAN

For a number of schematized situations the importance of triads in SWAN is investigated. These tests are representative for the bank systems along the Dutch coasts and have been chosen such that triad wave interactions are expected to play an important role. In the SWAN computations a post-processing step is carried out to obtain the spatial distributions of the magnitudes of the source terms. The triad source term is compared with the source terms due to depth-induced wave breaking, quadruplets and bottom friction. In the tests wind input is not modeled. The tool to determine the magnitudes of the source terms is called HOTSOURCE and has been developed recently by Alkyon.

6.4.1 Description of tests

Three one-dimensional bathymetries have been considered, as well as three initial spectra at the deep-water boundary.

The following bathymetries are considered (see also upper plots in Figure 6.4 in Appendix A):

a. Constant slope: the off-shore boundary is at 24m and from the offshore boundary the bed constantly increases with a slope of 1:20, 1:50 or 1:150.
b. Foreshore of Petten Sea-defence with a still-water depth of 24m and a water level depending on the initial spectrum that is imposed.
c. Trapezoidal bathymetry: Initial water depth is 15m. After 150m the toe of a 1:20 slope is located. At a depth of 2m a horizontal plane is situated. Behind the horizontal plane the depth increases with a constant slope of 1:10, until a depth of 15m is reached. Two tests are considered: the horizontal plane is either 100m or 500m wide.

The schematization is explained in the Figures 6.4 in Appendix A, where the results of the computations are given as well as the bathymetry of the specific test case.

Three initial spectral density distributions are used in the tests. These spectra are the initial spectra that are used in the tests 2.11, 2.13 and 2.51 in the Petten study (Van Gent and Doorn, 2000) and are also plotted in the Figures 6-4a,b,c at the deep-water boundary location (location 1 in Figure 6.4). The integral wave parameters for these spectra are given in Table 6.1.

<table>
<thead>
<tr>
<th>Test</th>
<th>$H_m$ [m]</th>
<th>$T_m$ [s]</th>
<th>$T_p$ [s]</th>
<th>Water level [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.11</td>
<td>2.06</td>
<td>7.15</td>
<td>8.36</td>
<td>2.1</td>
</tr>
<tr>
<td>2.13</td>
<td>3.81</td>
<td>10.11</td>
<td>11.53</td>
<td>2.1</td>
</tr>
<tr>
<td>2.51</td>
<td>5.64</td>
<td>7.88</td>
<td>8.79</td>
<td>4.7</td>
</tr>
</tbody>
</table>

Table 6.1 Integral wave parameters for imposed spectra
We have performed 18 tests in total. The tests are indicated as xyz, with x equal to a,b,c, referring to a sloping bed, Petten bed and trapezoidal bed, respectively. Secondly, y is introduced to distinguish between slope or width of the horizontal plane. For the constant slope (test a) y equals 1, 2 or 3, for Petten-case only one bathymetry is considered, and for test c two widths for the horizontal plane are considered (y equals 1 or 2). Finally, z indicates which spectrum is imposed at the offshore boundary (2.11, 2.13 or 2.51). In Table 6.2 an overview of the tests is given. For all tests the three spectra mentioned above are imposed, which explains the z (either 1, 2 or 3) in the name of the tests.

<table>
<thead>
<tr>
<th>Test</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a1z</td>
<td>constant slope 1:20</td>
</tr>
<tr>
<td>a2z</td>
<td>constant slope 1:50</td>
</tr>
<tr>
<td>a3z</td>
<td>constant slope 1:150</td>
</tr>
<tr>
<td>b1z</td>
<td>Petten Sea-defence</td>
</tr>
<tr>
<td>c1z</td>
<td>trapezoidal bathymetry, hor. plane 100m</td>
</tr>
<tr>
<td>c2z</td>
<td>trapezoidal bathymetry, hor. plane 500m</td>
</tr>
</tbody>
</table>

Table 6.2 Name of tests

### 6.4.2 SWAN computations

SWAN computations have been carried out with the experimental version 40.16, supplied by RIZA. For the physical process the default settings were used for wave breaking (depth-induced wave breaking and whitecapping), wave set-up, bottom friction and modeling triad wave interaction. Quadruplet wave interactions were not modeled. The constant space step was 10 m. The spectral resolution was 73 within the frequency range between 0.04 Hz and 0.35 Hz ($\Delta f/f = 0.03$) and 15 in the directional sector between -7.5 and +7.5 degrees. In all computational points a numerical accuracy of 1E-5 was required for each of the three accuracy criteria for the iteration procedure, with a maximum of 50 iterations.

In SWAN version 40.16 the Ursell number can be specified above which the limiter will be switched off. Furthermore switching the limiter on or off is also a function of the breaker fraction. In the present computations the limiter is not switched off due to either breaking or triad wave-wave interaction.

The post-processing tool HOTSOURCE is used to obtain the spatial distribution of the magnitudes of the source terms from the SWAN hotfile of each test. This means that for each position the source terms are integrated over each frequency and each direction. In Appendix B a description of HOTSOURCE is given. With this tool we obtain more insight in the locations SWAN considers the triad wave-wave interactions to be important with respect to other source terms.

For all tests the energy density spectra at several locations have been plotted in the lower plot of Figure 6.4 in Appendix A. In the upper plot the bathymetry together with the output locations is given. In the middle plots the spatial distribution of the magnitudes of the breaking, bottom friction, triad and quadruplet source terms are given. Although the quadruplets are not taken into account in the SWAN computations, HOTSOURCE is able to reproduce its contribution from the SWAN hotfile.

In all tests a similar pattern can be found. Energy at the peak frequency is transferred towards the first harmonic, when entering shallow water. The triads are expected to be
responsible for this. In nature also long wave energy would be generated in this region due to nonlinear wave interactions. However, SWAN does not predict energy transfer towards lower frequencies.

In sloping regions the magnitude of the triads reaches its maximum in regions where wave breaking takes place. Obviously, these mechanisms are related. In the shallow areas where waves break, the magnitude of the triad source term is dominant over the magnitude of the other source terms. This behavior is to be expected in shallow water. Clearly, no energy at lower frequencies are generated.

The effect of the bottom slope on the quadruplet source term is negligible relative to the size of the triad term. For weakly sloping bottoms the bottom friction becomes relatively more important. In general, the magnitude of the triad source term is significant where the breaker source term is significant, the magnitude of the triad source term being larger than the breaker term. Clearly, a second peak at two times the peak frequency appears near the locations 3 and 4 (test a, Figures 6.4 a-i in Appendix A). By comparing Figures 6.2 a-c, with Figures 6.4 d-f and Figures 6.4 g-i we observe that over longer distances relatively more energy is dissipated. Less energy is left as the slope becomes milder and accordingly, the distance from the offshore boundary to the shoreline is larger. The trend in the wave energy spectra is nevertheless the same. Only the height of the first and second peak is different.

The magnitude of the source terms are sensitive to the wave steepness. The magnitude of the breaker term increases stronger than the triad source term. Both processes are still dominant over the other processes, except the quadruplets. However, the latter process is not modelled.

From the tests b and c we learn that SWAN only assumes that triads to play an important role in shallow uphill regions. In downhill regions the magnitude of the triad source term is negligible. Also the width of the bank system does not alter the magnitude of the source terms significantly.

In 3 of the 18 tests (a13, a23, a33) the spatial distribution of the magnitude of the quadruplet source term shows a wiggling behavior, while the other source terms are smooth. It is not yet clear what is the reason causes this behavior. However, in the modelling the quadruplets have not been taken into account and thus the conclusions drawn from the plots are not affected.

6.4.3 TRITON computations

Results of TRITON computations of the tests mentioned in section 6.4.1 will give insight in the locations where nonlinear wave-wave interactions are really important, assuming that TRITON represents nature. The nonlinear interactions are not only triad interactions, but also quadruplet interactions and even interactions between more than four waves. Nevertheless, we will assume that in shallow water the triad wave interactions will dominate, although SWAN predicts a strong effect of the quadruplets (see previous subsection). The nonlinear interactions predicted by TRITON will be attributed to the triads. Within the present version of TRITON it is impossible to obtain spectral distributions of source terms, due to wave breaking or nonlinear wave interactions. In TRITON only a mass and momentum equations are solved. Implicitly, also an energy equation is solved. However, it is yet uncertain what the form of this energy equation is. In order to gain insight in the transport, generation and
dissipation of wave energy, the energy equation underlying TRITON, should be determined. However, the determination of the energy equation is beyond the scope of this study.

Here the spectra in the same output locations as in the previous section are generated from the timeseries that have been computed with TRITON. The spatial evolution of both these energy spectra and the significant wave height $H_m$ and mean wave period $T_{m0}$ are studied.

TRITON computations have been carried out for the six geometries described in subsection 6.4.1. Here only the initial spectral density from test 2.11 (ref. Table 6.1) is used. Computations with the other two timeseries (test 2.13 and 2.51) have been performed as well, but led to unforeseen instabilities. It is not yet clear what causes these instabilities. More analysis is required to find a proper solution. As long as instabilities do not occur (e.g. test 2.11), realistic results are obtained.

In Figure 6.5 in Appendix A the results of the TRITON computations have been given. On a constant slope (test a, Figure 6.5a-c) energy at the peak frequency is shifted towards lower and higher frequencies if the water depth decreases, although marginally. According to the spatial distribution of the significant wave height (indicated with x), depth-induced wave breaking mainly occurs in the shallowest area (around locations 5 and 6). In that area the mean wave period $T_{m0}$ (indicated with o) decreases as well. First, nonlinear wave interactions shift the energy towards lower frequencies. In TRITON only the steepest waves break, which means that the generated long waves will not break. Only for frequencies higher than the peak frequency, wave energy is dissipated.

Also in test b some wave energy is shifted towards higher frequencies (around twice the peak frequency) in the shallowest area, i.e. near the toe of the construction. The mean wave period drops in this region, see Figure 6.5d. The amount of wave breaking is marginal. The variation in significant wave height is due to shoaling (uphill region) and de-shoaling (downhill region).

For the wave propagation over a submerged bar, the results are given in Figure 6.5e-f. The waves are strongly breaking on the bar (location 3 and 4). At location 3 and 4 the amount of wave energy at twice the peak frequency increases. However, relatively more low frequency wave energy is generated, causing an increase in mean wave period. On the lee side of the bar the waves are de-shoaling and the significant wave height decreases. The mean wave period does not change. The long waves that are generated are still in the system. It is uncertain whether these long waves are bound or free waves.

### 6.4.4 Comparison of SWAN and TRITON results

In general, the triad source term in SWAN causes a transport of wave energy towards higher frequencies that is strongly exaggerated. Furthermore, triads are active in SWAN in a larger region than where nonlinear interactions play a role in TRITON (visualized by considering the shift of wave energy towards higher harmonics in a number of locations), which also contributes to the overprediction.

SWAN is not able to generate low-frequency wave energy due to triad interaction or wave breaking. From the TRITON computations we have seen that in shallow regions nonlinear interaction become very important (though less important as predicted by SWAN), not only causing high-frequency waves (sum interaction) but also low-frequency (difference
interaction). Due to the existence of long waves the mean wave period will be larger, compared to situations in which long waves are neglected.

SWAN predicts the area where wave breaking occurs to be larger than TRITON does. In SWAN wave energy is dissipated at all frequencies, whereas in TRITON only the steep short waves are breaking. These waves become steeper due to shoaling and break only if they are steep enough. In SWAN the breaker model reacts instantaneously on a steeping slope. There is not some kind of delay mechanism.

6.5 Conclusions and recommendations

Based on the phenomenological study described in this chapter the following conclusions and recommendations can be given with respect to the modeling of triad wave interactions.

Conclusions

- Triad wave interactions are important in very shallow area. SWAN overestimates the area of influence of triads.
- Due to triad wave interactions wave energy from the peak frequency is shifted towards higher frequencies. SWAN strongly overpredicts the amount of wave energy that is transported from the peak frequency to the first harmonic, resulting in an underestimation of the mean wave period.
- Nonlinear wave interactions (also triad wave interactions) are responsible for the generation of long waves. In very shallow regions the shift of wave energy towards low frequencies is significant, causing the mean wave period to increase. In SWAN the LTA formulation neglects the triads for the difference frequencies and is therefore not able to shift energy to low frequencies, resulting in an underestimation of the mean wave period.

Recommendations

- Improve the modeling of triads in SWAN. Not only self-self interaction should be considered, but attention should be paid to the interaction of one wave with the entire spectrum of waves. For multi-peaked wave spectra the present LTA formulation leads to unrealistic wave-wave interactions. Furthermore, long wave generation should be made possible through the triad formulation to be developed and/or implemented.
- Add HOTSOURC* as a research tool for SWAN. The tool is very useful to gain insight in the relative importance of the mechanisms involved.
- Investigate the interaction between wave breaking and triad interaction. Both mechanisms generate long waves. Furthermore, triad wave interactions generate high frequency waves, which are more sensitive to breaking. Also for the implementation in SWAN it is necessary to have more insight in whether the two processes work sequentially or whether there is a continuous interplay between them in nature. Field experiments would be helpful.
- Extend TRITON or any other phase-resolving wave model with its underlying energy equation. As such, these models can be used to gain more insight in the effect of physical processes on the spectral distribution of wave energy. Furthermore, they might be useful as verification tool for newly developed functionalities in e.g. SWAN.
- Due to the limitations of the triad formulation the SWAN predictions are poor in very shallow water, e.g. in coastal areas, harbours or near constructions. An alternative for computations in these areas would be the time-dependent Boussinesq-type model
TRITON. However, the computational effort of TRITON is large. The odds and evens of SWAN and TRITON should be combined in a natural way. The wave motion in the entire area can be computed with SWAN, resulting in spectral information at the boundaries of the small region of interest. The wave motion in this area is successively be computed with TRITON, which uses the spectral information from SWAN as input.
7 Whitecapping

7.1 Introduction

Three physical processes contribute to wave growth and decay on open sea in deep water: energy input by the wind, whitecapping and quadruplet interactions. In first and second generation models, the combined effect of these processes is parameterised. Third generation models model each of the effects explicitly. This has the large advantage that they are applicable in a much wider range of circumstances. However, the disadvantage is that they are sensitive to factors that disturb the subtle balance between these three processes, which interact as follows:

- Energy is added by wind to the spectrum over various frequencies.
- Non-linear wave interactions redistribute energy to higher and to lower frequencies.
- Whitecapping dissipates some of the energy depending on the wave steepness.

These source terms have been developed and calibrated for open ocean conditions and have successfully been applied there. In recent years such models have also been applied in coastal regions. In that area processes like bottom friction, depth-limited wave breaking and triad wave-wave interactions start to play a role as well. This presents greater demands on the wave prediction models. Application in the coastal zone can lead to the occurrence of spectral forms that would not be encountered on the ocean. Formulations that work adequately for application on the ocean do not necessarily work adequately with these other spectral forms. This holds especially for multi-peaked spectra containing wind-sea and swell waves.

One such formulation is that for dissipation by whitecapping, as it is implemented in the WAM model (WAMDI, 1998) and the SWAN wave model (Booij et al. 1999). This formulation was proposed by Komen et al. (1984). In the following it is referred to as the KHH formulation. Hurdle (1998) discovered that when SWAN is applied in partly sheltered areas, the prediction of the local wave growth due to wind is significantly overestimated. A detailed analysis showed that the presence of small amounts of long period wave energy in the sheltered areas suppresses the dissipation by whitecapping in the model for the wind sea part. Field and laboratory observations of the effect of swell on the growth of wind sea are too few to draw firm conclusions. However, some of the few observations that are available (cf. Phillips and Banner, 1974; Donelan, 1987; Mitsuyasu and Kusaba, 1993) suggest that if swell has any effect, it is to suppress the growth due to wind rather than to enhance it. On the other hand, experiments of Mitsuyasu and Yoshida (1989) show the opposite effect: the presence of an opposing swell enhances the growth of the wind sea.

Another effect noted by Rogers et al. (2000) is that the KHH whitecapping formulation gives too much dissipation at low frequencies. This is an important problem for global wave models because it will lead to the prediction of swell wave heights lower than will occur in nature. Rogers et al. (2000) also noted that this may be a problem in shelf seas like the Southern North Sea and the North American continental shelf. Observations on the northeast American continental shelf near Duck suggest that the dissipation of swell energy is over-predicted thereby contributing to the under-prediction of period measures.
The whitecapping problem in situations with multi-peaked spectra may also be important for the prediction of the spectral wave period, since the overestimation of the wave growth is partly caused by too little dissipation at high frequencies and too much dissipation at lower frequencies. Both of these effects can be contributed to the use of a mean wave period to compute a mean wave steepness, which in turn is used to scale the dissipation by whitecapping.

Several alternative methods proposed in the literature to compute the dissipation by whitecapping are considered in the following pages. These include a formulation suggested by Donelan (1999) and methods suggested by Tolman and Chalikov (1996) and Holthuijzen and Booij (2000).

To be applicable, the formulations must satisfy at least the following conditions:

- The effect of adding small amounts of low frequency energy to the wave spectrum should not suppress the dissipation of waves at higher frequencies (as this would artificially enhance the growth of waves).
- The presence of a wind-sea should not lead to enhanced dissipation of swell waves.
- It should be possible to reproduce parametric formulations for wave growth in idealised situations equally well as with the current formulation applied in the WAM model and in the SWAN model.

Another relevant point of attention is the steepness spectrum. Although, the description of the steepness spectrum is trivial, we feel that its physical meaning is often neglected when the spectral form, in general, and whitecapping, in particular, are being considered. Little attention seems to be paid to the fact that the integral of the steepness spectrum is unbounded for the standard form applied for the spectral tail. This is obviously unrealistic since the integral of the steepness spectrum is a measure of the average surface slope and this is clearly not infinite. It is of course reasonable to consider that for those parts of the spectral tail dominated by quadruplet interactions the spectral density decreases according to the \(-4^{th}\) power of the frequency. However, this behaviour will change with increasing frequency as whitecapping and surface tension play an increasingly important role (Kitaigorodskii, 1983).

We therefore argue that the root cause of the current problems with the formulation for whitecapping is that the assumed form of the spectral tail is unrealistic. This makes it necessary to compute the wave steepness based on the wave height and an average wave period with all the associated disadvantages. The application of a realistic spectral tail is therefore preferred and, if wave steepness is required in the formulation for whitecapping, it should be computed based on the steepness spectrum.

### 7.2 The whitecapping problem

The formulation for dissipation by whitecapping as implemented in most numerical models is based on pulse-based description of Hasselmann (1974), as adapted by Komen et al. (1984). It is given by:

\[
S_{we}(\omega, \theta) = -C_{we} \omega \left( \frac{k}{k} \right) \left( \frac{\bar{s}}{s_{pt}} \right)^2 E(\omega, \theta)
\]  
(7.1)
where $\tilde{\phi}$ and $\tilde{K}$ are a mean frequency and a mean wave number, and where $\tilde{s}$ is a mean wave steepness, and $C_w$ a scaling parameter. The parameter $\tilde{s}_{pm}$ is the mean wave steepness for a Pierson-Moskowitz spectrum ($\tilde{s}_{pm} = 3.02 \times 10^{-3}$). In the SWAN model the coefficient $C_w$ is taken as $C_w = 2.36 \times 10^5$.

In many third generation models, the average wave steepness is computed from the energy based significant wave height and some average wave frequency $\tilde{s} \propto \tilde{\phi}^2 H_{mo}$. In finite depth the mean steepness is defined as:

$$\tilde{s} \propto \tilde{K} \sqrt{E_{tot}}$$

with $E_{tot}$ the total wave variance.

An undesired characteristic of this formulation is that the addition of small amounts of low frequency energy to the wave spectrum can have a large influence on the average wave period and thus significantly decrease the computed wave steepness. In this way, the low frequency energy has a large influence on a process that is occurring in the high frequency part of the spectrum. This behaviour is illustrated in Figure 7.1 in Appendix A, which shows the source term for whitecapping for a uni-modal wave spectrum and one for a double peaked spectrum with a small swell peak. It is clear that when a small amount of swell energy is added to a wind sea, the computed average wave number decreases and the computed steepness decreases. As a result the dissipation by whitecapping is reduced.

This reduction in wave steepness is sometimes so significant that dissipation by whitecapping becomes an order of magnitude smaller and leads to excessive growth in the wave energy at high frequencies. This behaviour with the SWAN model is reported by various authors (see e.g. Hurdle, 1998; Holthuijsen et al., 2001). The case reported by Hurdle is illustrated in Figure 7.2 in Appendix A in which the wind speed was 20 m/s and the incoming swell had a peak period of 6 s and a significant wave height of 0.1 m. As can be seen, wave growth is enhanced when some swell is present. This experiment shows that the simplified definition of steepness applied in the pulse-based model for whitecapping allows the presence of swell propagating in the wind direction to accelerate the wave growth due to the wind - the opposite of the effect expected from field observations (see section 7.1).

### 7.3 Alternative formulations

Three alternative formulations for dissipation by whitecapping are being considered:

- The cumulative steepness method. This method was first implemented by Cecchi (Ris et al., 1999) following suggestions by Donelan (1999) and Booij (private communication), that the dissipation by whitecapping at a particular frequency should depend on the steepness of the wave spectrum at and below that frequency. This means that the relative dissipation increases with frequency because the cumulative steepness also increases with frequency. This is physically plausible. It seems unlikely that the dissipation of wave energy at a certain frequency depends on what happens at higher frequencies. We expect the opposite: the slope in the water surface associated with lower frequencies should have a significant influence. (This is analogous to a car in a hilly area. The acceleration of a car rolling down a hill (analogous to whitecapping) is
influenced only by surface fluctuations on a length scale longer than the car, analogous to the cumulative steepness). One of the physical mechanisms behind the cumulative steepness method is that of surface straining, in which shorter waves riding on top of longer waves are modulated causing enhanced dissipation where the shorter waves are steepened. This process is illustrated in Figure 7.3.

The cumulative steepness method also has the considerable practical advantage that the shape of the tail of the spectrum need not be considered - it will result from the use of the formulation without being required by it. This behaviour follows directly from the basic concept of this method, only energy at lower frequencies is accounted for. Further, if too much energy accumulates at a certain frequency, the dissipation at that frequency will increase. As a consequence, the energy level will decrease until a new equilibrium is obtained.

- The extended Komen method. Ris et al. (1999) developed this method and Holthuijsen and Booij (2000) extended and tested it. In this method, the dissipation by whitecapping for a particular frequency depends only on parameters computed from the energy spectrum at frequencies above that frequency. The formulation behaves as expected and desired in the presence of swell (whitecapping in the wind sea part of the spectrum) and therefore may be considered a considerable improvement. An example of an application of this method to a field case is given below. Furthermore, it is unlikely to require extensive re-calibration of the model, since it should give very similar results to the previous formulation for situations with single peaked spectra. In fact, Holthuijsen and Booij (2000) require that their extended method reduces to the basic KHC formulation for single peaked spectra. It is therefore expected that their method still suffers from the problems associated using mean period measures to scale the whitecapping dissipation.

We do not favour the extended Komen method since the dissipation at a particular frequency depends on the steepness at frequencies higher than that being considered. Furthermore, since the questions discussed in the previous section over the form of the tail of the spectrum are unresolved, the steepness still has to be computed based on an average wave number and the corresponding spectral energy.

The Tolman and Chalikov (1996) method. In this method it is considered that the dissipation mechanism above the spectral peak frequency are different from those below the peak frequency. Below the peak frequency the dissipation is described using an analogy with dissipation of wave energy due to oceanic turbulence. They obtained a diagnostic parameterisation for the high frequency dissipation by assuming a quasi-steady balance of source terms in the corresponding regime. An intrinsic problem with the Tolman-Chalikov method is the fact that for many spectra the position of the peak frequency is not unique, especially in mixed seas this poses a problem. This is one of the reasons that we do not favour the Tolman-Chalikov model for whitecapping dissipation.

### 7.4 The steepness spectrum

The steepness spectrum for a frequency spectrum is derived by considering a realization of the energy spectrum. The surface elevation \( \eta \), is given by:

\[
\eta = \sum Y(\omega) e^{i(\omega - \omega_0)}
\]  

(7.3)
Where the expected value of $Y(\omega)$ is proportional to $\sqrt{E(\omega)\Delta\omega}$ and $E(\omega)$ is the spectral density at frequency $\omega$. The surface slope is then given by:

$$s = \frac{d\eta}{dx} = \frac{d}{dx}\left\{\sum Y(\omega) \cdot e^{i(kx-\omega t)}\right\} \quad (7.4)$$

Then

$$s = \sum ikY(\omega) \cdot e^{i(kx-\omega t)} \quad (7.5)$$

Analogously it can be shown that the expected value of the steepness spectrum is given by:

$$S(\omega) = k^2 E(\omega) \quad (7.6)$$

In deep water, the wave number, $k$, is proportional to $\omega^2/g$. This means that the expected value of the steepness is only defined for wave spectra in which the tail reduces faster than with $\omega^5$. This follows directly from Eq. (7.6). The integral $\int k^2 E(\omega) d\omega$ is unbounded when integrating to infinity for powers of the spectral decay larger than or equal to $-5$.

Kitaigorodskii (1983) analysed the expected form of the spectral tail for an inertial subrange of frequencies, i.e. the range of the spectrum where gravity plays a dominant role in determining the shape of the spectral tail. He considered an idealised situation in which energy is input at the low frequency end of the spectrum and, in the equilibrium situation, is transferred at a constant (independent of frequency) rate to higher frequencies. Because the transfer rate due to quadruplet interactions for a given spectral density is higher at higher frequencies, the spectral density has to decrease with frequency correspondingly. On this basis, the spectral tail must decrease according to $\omega^{-4}$ in the inertial subrange of frequencies. This is known as Kolmogoroff-type equilibrium. Kitaigorodskii goes on to argue that this sub-range must be roughly:

$$\omega_m < \omega < \frac{30g}{U_d} \quad (7.7)$$

in which $\omega_m$ is the peak frequency and $U_d$ the effective wind speed.

Kitaigorodskii gives a further restriction on the upper limit of the inertial sub-range based on consideration of whitecapping. This upper limit is given in terms of gravity and the rate of energy transfer through the spectrum. He goes on to consider the form of the spectral tail above this limit but below the range where surface tension and viscosity play a role. In this range, whitecapping determines the upper limit to the spectral density. He expresses this following Phillips (1958) as an instability, which occurs when the downward acceleration in the wave exceeds gravity. This leads to a form of the spectral tail that decreases according to $\omega^{-5}$. This is known as Phillips type equilibrium.

Subsequently, Kitaigorodskii, tried to find measured data to confirm the transition between the two forms for the spectral tail. He concluded, based on limited data, that the transition frequency is in the range given by:
\[ \omega_k = \frac{U_{\lambda_1}}{g} = 1.5 - 3.3 \quad (7.8) \]

This means that for higher wind speeds the transition occurs at a lower frequency as would be expected, since the spectral tail is more saturated. In private communication from Kahma (2001), he estimates the transition frequency to capillary waves is about 13 Hz. At this frequency the spectral tail will decrease more rapidly than with Phillips type equilibrium.

As an example, a spectrum is considered for the fully arisen sea corresponding to a wind speed, \( U_{\lambda_1} \), of 12.5 m/s (Bft. 6 - whitecapping begins to play a role at Bft. 4). According to the SPM (1984), the corresponding values for \( H_s \) and \( T_p \) are about 6 m and 13 s for a fully grown sea state. For these values \( \omega_k \) is 1.88 rad/s based on a value for the constant in Eq. (7.8) of 2.4 \((0.5 \times (1.5 - 3.3))\). The blue line in Figure 7.4 in Appendix A (upper panel) shows the form of the corresponding spectrum and the blue line in the lower panel of Figure 7.4 shows the related steepness spectrum with the adapted spectral form for the tail. The transition to the Phillips equilibrium form \((\omega^2)\) tail is at 0.3 Hz. It can be seen that the energy spectrum is well past its peak at this frequency. However, the steepness spectrum is just at its peak. When considering wave steepness it is therefore highly important to model the spectral tail properly. Also shown in Figure 7.4 is the spectrum (broken blue line) corresponding to young wind waves with the same wind speed and a small component of added swell. This spectrum has a tail with Kolmogorov type equilibrium \((\omega^4)\) tail. The steepness spectrum is roughly constant above 1 Hz, which intuitively seems rather unrealistic. The swell is just distinguishable in the energy spectrum but cannot be seen at all in the steepness spectrum. However, the effect on the steepness computed according to the pulse theory of Hasselmann is significant - a reduction from a steepness of 8.2% to 6.9%.

The explanation for this is the method to compute the mean wave number, which places more weight on the lower frequencies than on higher frequencies.

As shown above, theory of the equilibrium spectral form suggests that in the high frequency tail of the spectrum within some range of the peak frequency, the spectral density must decay with the fourth or fifth power of the frequency, \( \omega \). Traditionally, it has been assumed that this form for the tail of the spectrum continues beyond this range. However, this would mean that the spectral definition of wave steepness is unbounded. Therefore the spectral definition of wave steepness has been considered not to be usable. We, however, prefer to contend that this implies that the assumed form of the high frequency spectral tail is unrealistic.

### 7.5 The cumulative steepness method

Using the above concept the cumulative steepness wave method is defined as:

\[ S_{\omega} (\omega) = \int_{0}^{\omega} k^2 E(\omega') d\omega' \quad (7.9) \]

and the new whitecapping source term is given by:

\[ S_{\omega}^{\prime} (\omega, \theta) = -C_{uc}^{\prime} S_{\omega} (\omega) E(\omega, \theta) \quad (7.10) \]

with \( C_{uc}^{\prime} \) a tunable coefficient.
Directional effects are included in this method by considering the physical process of the straining mechanism (one of the ways in which wave energy at lower frequencies can influence the whitecapping). This states that short waves propagating on top of larger waves are compressed on the forward face of the longer waves and stretched at the backward face (cf. Figure 7.3). The compressed waves become steeper, resulting in enhanced dissipation. It is assumed that this process does not act when the waves propagate at an angle of 90°, but it does again when the waves have opposing directions of propagation. A simple way to account for this dependence is to introduce a cosine-term over the directional difference in combination with an absolute operation. This leads to the following form of the directionally dependent cumulative steepness spectrum.

\[ S_m(\omega, \theta) = \int \int k^2 |\cos(\theta - \theta')|^m E(\omega, \theta') d\omega d\theta' \quad (7.11) \]

In this expression the coefficients \( m \) controls the directional dependence. It is expected that this coefficient will be order 1 if the straining mechanism is dominant, \( m \) is less than 1 if other mechanisms play a role (e.g., instability that occurs when vertical acceleration in the waves becomes greater than gravity). For \( m = 0 \) expression (7.11) reduces to expression (7.9). The inclusion of directional effects does not include a normalisation over directions, because these effect directly affect the amount of dissipation.

The potential capabilities of the cumulative steepness method are illustrated in Figure 7.5 in Appendix A for a uni-modal and bi-modal spectrum. The upper part of the figure shows the variation of the density of a uni-modal and a bi-modal energy spectrum. The lower part of the figure shows the corresponding cumulative wave steepness as a function of frequency.

The whitecapping source function based on the cumulative steepness method was computed for a uni-modal and a bi-modal spectrum and is shown in Figure 7.6 in Appendix A. The upper panel shows the wave spectra. The middle panel shows the computed source term for the spectra using the Komen formulation. Although small, the dissipation of the swell peak decreases considerably. The lower panel shows the source terms based on the cumulative steepness method. As can be seen the addition of swell hardly affects the amount of dissipation when computed with the cumulative steepness method. One of the main differences is regarding the dissipation of the swell peak. It can also be seen that the addition of a swell peak slightly increases the dissipation based on the cumulative wave steepness. These results clearly show that the cumulative steepness method behaves according to physical expectations.

It is noted that the KHH formulation did not include any directional dependence. The whitecapping was simply proportional with the energy density. This is not logical in view of the physical mechanism considered in the cumulative steepness method. The effect of taking directional effects (using \( m=1 \)) into account is illustrated in the Figure 7.7 in Appendix A. This figure shows a comparison of the cumulative steepness whitecapping source term for the situation without and with a directional dependence. The situation considered is for a uni-modal spectrum. In the case no directional dependence is taken into account, the directional width of the dissipation source term is larger. The effect is more dissipation farther from the mean wave direction. Inclusion of a directional dependence reduces the directional width of the dissipation function.

The wave growth has also been simulated using both formulations for an idealized situation with restricted fetch. The formulations were implemented in a spectral wave model that
applies the same integration method applied in WAM. The results of this test are shown in Figure 7.8 in Appendix A. This figure shows the wave growth predicted using both formulations (black lines for the Komen model and red lines for the cumulative steepness method) for a situation beginning with a low frequency wave spectrum with a significant wave height of 0.1 m and a peak period of 6 s and for a situation initial wave growth starting with no waves. Also shown is the theoretical growth in the wave height according to Kahma and Calkoen (1992). The results indicate that the cumulative steepness method agrees reasonably well with growth curve of Kahma and Calkoen (1992) but that the Komen method tends to overestimate wave growth at this short fetch. However, these differences are in part related to the tuning of the model. Of more interest is that accelerated wave growth due to the presence of the swell can be seen for the Komen method but not for the cumulative steepness method.

Numerical fetch limited growth curve experiments with the SWAN model and the cumulative steepness methods were hampered by the occurrence of instabilities at the beginning of the fetch. It was found that along the fetch wave energy is transferred to directions opposite to the wind direction and at low frequencies by nonlinear four-wave interactions. Since the dissipation rate of these frequencies is very small in the cumulative steepness method, these wave components grow much faster than expected and accumulated at the beginning of the fetch. It is unclear if this accumulation of wave energy has a physical basis or if they are a result of the numerical propagation and integration scheme of SWAN. The use of a wind input source function with negative wind input for wind opposing direction might stabilise the results.

## 7.6 Discussion

Various studies have shown that the whitecapping formulation of Komen et al. (1984), as applied in many wave prediction models, underestimates the dissipation of the wind sea part in the case a swell (or any low frequency wave system) is present. This study also showed that the presence of a wind sea causes enhanced dissipation of the swell waves. Both of these effects can be attributed to the use of mean steepness based on a mean wave period.

In literature a number of solutions to this problem have been proposed, all of which aim to neutralise these unwanted effects of swell on the dissipation at the higher frequencies. The modified Komen method as proposed by Holthuijsen and Booij (2000) is probably able to resolve the overestimation of the wind sea peak in mixed seas for particular cases, but may fail as a generally applicable solution.

From a physical point of view the cumulative steepness method is more elegant because it has a better physical basis than the modified Komen method. Still, there are a number of problems to be resolved. One is to find the proper decay of the spectral tail, the dissipation of the swell and precise directional effects. A point of further investigation is if the straining mechanism has the same strength for short waves on a following swell or on an opposing swell. Numerical growth curve experiments and comparisons with field measurements are recommended to resolve these problems.

The whitecapping problem is part of the general problem of swell-sea interaction under wind-driven conditions. Sea and swell waves can interact in many other ways. If the sea and swell peak are close to one another they may exchange energy by non-linear quadruplet wave-wave interactions (e.g. Masson, 1993). Swells may also influence the wind-profile and surface drag above the sea-surface and affect indirectly the growth of the wind-sea. In
this respect one should be very careful when using results obtained in wind-flumes where
the wind profile might be affected by rigid-lid effects and the omission of the 2-D effects
that occur in nature. Therefore, field measurements of wave growth in the presence of swell
are needed to calibrate and validate the new formulations for whitecapping dissipation.

The cumulative steepness whitecapping dissipation has been build in a test version of the
SWAN model (version 40.11) and tests have been carried out to assess its capabilities of
resolving the so-called ‘whitecapping’ problem and the underprediction of period measures
in mixed sea conditions. These tests are reported in Chapter 8.

7.7 Conclusions and recommendations

In this study an analysis has been made of the source term for dissipation by whitecapping.
Special attention was paid to the behaviour of the widely used Komen formulation in mixed
seas. Based on the findings described in this chapter the following conclusions and
recommendations can be given with respect to the modelling of dissipation by
whitecapping.

Conclusions

- The widely used Komen formulation for whitecapping dissipation produces unrealistic
  results for multi-peaked wave spectra since it depends on a mean wave steepness.
- The Komen formulation under-predicts the dissipation of the wind-sea when a small
  amount of low frequency energy is added.
- The Komen formulation over-predicts the dissipation of swell waves when a wind sea is
  present.
- Both of the previous effects in the Komen method lead to an under-prediction of wave
  period measures.
- Any integral quantity of the spectrum depends on the shape of the spectral tail.
- Since wave steepness is not unbounded, the spectral tail must decreases faster than an
  $f^{-3}$ decay for frequencies above some (high) frequency.
- The cumulative steepness method does not depend on mean quantities of the spectrum,
  and is therefore independent of the shape of the spectral tail.
- In the cumulative steepness method the effect of adding low frequency wave energy
  enhances the dissipation at higher frequencies, probably leading to an increase in wave
  period measures.
- The cumulative steepness method predicts much lower dissipation rates at lower
  frequencies than the (modified) Komen method. This may solve some of the reported
  problems with the over-prediction of swell dissipation.
- The inclusion of directional effects in the cumulative steepness method reduces the
  amount of dissipation by whitecapping. However, one of the parameters in the
  formulation is a constant of proportionality that has to be obtained by calibration, so
  this has little practical effect.
- The Komen and extended Komen formulations do not include directional effects on the
  dissipation rate.
- The extended Komen formulation depends on integral quantities of the spectrum and
  relates the dissipation at a certain frequency with the occurrence of wave energy at
  higher frequencies.
Recommendations

- The reason for the occurrence of instabilities at low frequencies in directions opposite to the wind direction should be determined.
- The use of wind input source function, with negative input for opposing winds, should be considered. This may have a positive effect on the numerical stability of growth curve experiments performed with the SWAN model. However, the under-prediction of swell by the global models should be remembered.
- The directional dependence of the cumulative steepness method should be investigated, not only with numerical experiments but also by analysing field data. A particular point of interest is whether the straining mechanism has the same effect for following waves or for opposing waves.
- Further 1D and 2D tests are needed to assess the implications for growth curves of wave height, wave period and directional spreading.
- Detailed field data should be collected and analysed to determine the effect of lower frequency energy on the dissipation of wave energy at higher frequencies.
8 Calibration

8.1 Introduction

In this study attention has been paid to the development of new source terms for non-linear quadruplet wave-wave interactions, whitecapping dissipation and depth-induced wave breaking. New formulations for these source terms have been implemented in a test version of SWAN 40.11 and various tests and calibrations have been performed with this modified SWAN model. To isolate the effects of each of these improvements or modifications to the source terms, separate calibration and verification calculations have been performed, one for each source term.

The implementation and testing of the various new source terms was hampered by various numerical problems, which could not be solved in the framework of this project. Most of these problems are related to the convergence behaviour of SWAN and to the idiosyncrasies of the implementation of the various limiters, especially in shallow water. This situation forced us to limit the number and types of tests for this study. In consultation with the Client, it was therefore decided that extensive tests are of limited value until the numerics of the SWAN model are improved considerably.

The structure of this chapter is as follows. In Section 8.2 some effects of replacing the DIA with the exact solution of the non-linear quadruplet wave-wave interactions are described. Some effects of replacing the Komen whitecapping formulation with the cumulative steepness method are described in Section 8.3. Results of the modified Battjes-Janssen breaker formulation are already described in Chapter 2. Conclusions are given in Section 8.4, and recommendations are proposed in Section 8.5.

8.2 Quadruplets

To test the effect of the new source functions for nonlinear four-wave interactions, the SWAN model was extended with routines for the computation of the exact nonlinear interactions using the accurate WRT method. In addition the SWAN model was extended with routines for the computation of the nonlinear interactions with the shallow water version of the DIA. These routines can be activated in SWAN using the IQUAD command. In 1999 and 2000, J. Haagsma, then at the SWAN development group, enabled the use of these additional settings of the IQUAD parameter. Three values of IQUAD can be input to activate various implementations of the exact WRT method. When IQUAD=5, the non-linear source term is computed as if it were deep water. Setting IQUAD=6 also computes the non-linear source term as if it were deep water, but now it is followed by multiplication with the WAM depth scaling, see Eq. (5.136). When IQUAD=7, the non-linear source is computed using the local water depth (rounded off to decimetres). As shown in Chapter 5, finite depth effects not only increase the overall magnitude of the non-linear transfer, but they also affect the shape of the transfer rate. In a similar way the finite depth DIA is activated by setting the IQUAD parameter to the values 11, 12 or 13. Setting IQUAD =11 activates the deep water DIA, IQUAD=12 activates the WAM depth scaling, and IQUAD =13 activates the finite depth DIA.
The first step was to recalculate the SWAN model when the DIA is called with the exact WRt method. Following the work of Hasselmann and Hasselmann (1985) with the EXACT-NL model, the tuning parameter was the coefficient of proportionality of the Komen whitecapping formulation. In the SWAN and WAM model this coefficient has the value of \(3.33 \times 10^{-5}\). For the re-calibration the standard case f27grwd from the SWAN test bank was used. This case covers fetch-limited wave growth in deep water over a distance of 25 km. For the present calculations the wind speed was set to 20 m/s. Figure 8.1 shows a comparison of the growth curves obtained with the standard SWAN model (\(iQUAD=1\)) and the modified SWAN model (\(iQUAD=5\), deep water). As can be seen the SWAN model version with the accurate non-linear transfer results in a slower growth of the significant wave height \(H_s\) and mean wave period \(T_{m0}\). This implies that the source term balance is such that whitecapping dissipation is overpredicted. Following the calibration of the EXACT-NL model, the coefficient for whitecapping was set to \(2.22 \times 10^{-5}\). The result of this modification is seen in Figure 8.2. As can be seen, the growth curves for the significant wave height and the mean wave period are now close to one another. The main difference, however, is that the spectra obtained with the ‘exact’ SWAN are narrower. This follows directly from the smaller values of the directional spreading \(\sigma\) and higher values of the spectral narrowness parameter \(\kappa\). These results are as expected.

The second step was to determine the effect of replacing the WAM depth scaling with a direct depth dependent calculation of the non-linear transfer rate. To that end the SWAN version with exact method for computing the non-linear transfer rate was used with \(iQUAD=6\) and with \(iQUAD=7\). This SWAN version was used to compute fetch-limited wave growth in water with a finite depth and a wind speed of 20 m/s. Two situations were distinguished, a constant water depth of 10 m and a constant water depth of 5 m. In total 4 computations were performed. Figure 8.3 shows the results for a water depth of 10 m, and taking \(iQUAD=6\) (case S10XNL, in which S10 refers to a depth of 10 m and XNL to the exact transfer rate) and \(iQUAD=7\) (case S10XNL). Source terms for dissipation by finite depth effects (viz. bottom friction and depth induced wave breaking) were deactivated to isolate the effect of replacing the WAM depth scaling. These results indicate that almost no differences occur. The results of the computation for a water depth of 5 m are shown in Figure 8.4. For this situation some differences are visible. The mean wave period increases and the spectrum becomes slightly broader, as follows from the behaviour of the directional spreading and spectral narrowness.

The effect of the finite depth DIA in shallow water was tested for two situations with a flat bottom with a depth of 5 m and 2 m respectively and a wind speed of 25 m/s. The results of these tests are described in Van Vledder and Bottema (2002), which is included as Appendix C of this report.

The finite depth DIA has also been tested for a 2-dimensional field case, i.e. the Westerschelde estuary in the South-West of the Netherlands. To that end two runs were made with the SWAN model, one run with the DIA as implemented in SWAN using the WAM depth scaling (\(iQUAD=12\)) and one run with the finite depth DIA (\(iQUAD=13\)). The situation is taken from a storm situation on 28 Jan.1994 at high water such that current speeds were minimal. The wind speed was 21.5 m/s and the wind direction 237°N. The offshore wave boundary comprised a significant wave height of 5.20 m and a peak period of 9.12 s. The results of these tests computations are illustrated in the Figures 8.5 to 8.8 in Appendix A. Figure 8.5 shows the spatial distribution of the significant wave height \(H_s\) and Figure 8.6 shows the spatial variation of the mean wave period \(T_{m1.0}\). These results are computed using the DIA with WAM depth scaling. The effect of using the finite depth DIA instead of the DIA
with WAM depth scaling is illustrated by means of the spatial distribution of the differences in the significant wave height $H_s$ and mean wave period $T_{m-10}$. Figure 8.7 shows the difference in the significant wave height, obtained by subtracting the results for the DIA with WAM depth scaling from the results obtained with the finite depth DIA activated. Figure 8.8 shows the corresponding results for the difference in mean wave period $T_{m-10}$. These preliminary results indicate that the finite depth DIA has considerable effects on the prediction of the significant wave height and mean wave period. A striking result is that the areas where the significant wave height increases are coupled to the areas where the mean wave period decreases and vice versa. This behaviour suggests that a tuning of these source terms might be used to correct to observed underprediction of period measures and the observed over-prediction of wave heights. Another feature is the hypothetical line parallel to the coast where the sign of the difference changes. This line roughly corresponds to the area where finite depth effects start to play a role.

In view of the limitations of this study due to numerical problems with this version of SWAN, no conclusions can (yet) be drawn if these differences improve the predictive capabilities of the SWAN model. This can only be realised when the numerics of SWAN have improved and when also other new source terms are included in such tests, e.g. by including the cumulative steepness method for computing the dissipation by whitecapping.

### 8.3 White-capping

The Komen whitecapping formulation has been replaced with the cumulative steepness method described in Chapter 7. To activate this new whitecapping formulation a special configuration file must be present in the directory of the SWAN executable. This file, named `steepness.cfg`, contains the scale factor $C_{st}$ of the cumulative steepness, and two parameters of the DIA, viz. $C_{sid}$ and $\lambda$. Initial tests using only the cumulative whitecapping formulation and using the default DIA settings did not result in realistic growth curves. Since the total source balance changes with this new whitecapping formulation other parameters were needed which affect the source term balance. The obvious choice was to choose some parameters of the DIA source term. As known from other studies, e.g. Alkyon (1999), the parameters $C_{sid}$ and $\lambda$ are able to modify the source term balance.

A mathematical procedure was used to optimise these 3 parameters. The criterion was to minimise the least-square differences between the significant wave height at 3 locations along the fetch, 1, 10 and 100 km. To that end the fetch limited growth curve of Wilson (1955) was used as a benchmark. Two sets of optimisations were made for fetch limited conditions, one for a wind speed of 10 m/s and one for a wind speed of 20 m/s. The result of the calibration are summarised in Table 8.1.

The resulting values of the calibration parameters are: $C_{st}=1.8$, $\lambda=0.27$ and $C_{sid}=1.2 \times 10^8$. 
<table>
<thead>
<tr>
<th>Wilson 55</th>
<th>SWAN with cumulative steepness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hs (m)</td>
<td>Tp</td>
</tr>
<tr>
<td>U=10 m/s</td>
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</tr>
<tr>
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<td>10.02</td>
</tr>
</tbody>
</table>

Table 8.1 Computed significant wave height and peak period based on a calibration against the Wilson (1955 growth curves)

The results in Table 8.1 indicate that the computed significant wave heights are close to the values of Wilson, but that the computed peak periods are always higher than those given by Wilson.

### 8.4 Conclusions

Based on the results of the limited calibration of the SWAN model with new source terms for quadruplet wave-wave interactions and whitecapping dissipation, the following conclusions can be drawn.

- The results of the implementation and calibration of the newly developed source term for non-linear quadruplet wave-wave interactions indicate that replacing the DIA with an exact method improves the direction and frequency spreading.
- It is also found that replacing the WAM depth scaling with a direct computation of the non-linear transfer rate affects the prediction of the mean wave period (it increases) and other spreading measures.
- The results of the computations for the academic test spectrum indicate that the finite depth DIA leads to a better representation of shallow water effects on the nonlinear transfer rate compared to the present depth scaling. Still, a mismatch in the nonlinear transfer rate exists in comparison with results of exact computations. This is probably due to the fact that only one wave number configuration has been used in these computations.
- The results of the academic growth curves indicate that the 5 m depth shallow lake is not shallow in terms of $\tilde{k}h$ values, although the growth is depth-limited in terms of $H/h$ values. This is one of the reasons why the wave measurements described in Bottema et al., (2002) are not only interesting from an operational point of view, but also from a physical point of view. Only for the 2 m case depth effects become noticeable. The fact that the mean wave period decreases is surprising, but this may be due to the fact that only one wave number configuration was used. The decrease in directional spreading is probably due to the fact that the resonant wave number vectors become more aligned in shallow water, thus limiting the amount of energy transferred to off-wind directions.
- The cumulative steepness method can replace the Komen whitecapping formulation, but only in combination with other settings for an (extended) DIA.
- Initial experiments with the cumulative steepness in SWAN indicate that the dissipation of low frequency waves is much smaller than with the Komen formulation and that there is an accumulation of energy at those frequencies. This accumulation
increases with iteration number. Any (small) amount of energy transferred (by quadruplet interactions) is therefore not dissipated fast enough to avoid this accumulation. It is noted that this accumulation was not observed with the WAM type model used to access the (in)sensitivity to small amounts of low-frequency swell energy. The particular numerical scheme of SWAN might be responsible for this effect and this needs further study.

8.5 Recommendations

Based on the results obtained with this limited calibration and in view of other reported findings in this study, the following recommendations are proposed:

- The SWAN model should be re-calibrated when major improvements in the numerics of the SWAN model are realised.
- This re-calibration should be performed with all new source terms activated, and not per individual source term.
- During this re-calibration attention should be given to the interaction between the various source terms, the integration scheme and the role of limiters.
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A Figures

(Appendix A is bounded separately)
B Description of Hotsource

HOT_SOURCE is a post-processing program to obtain the magnitude of the SWAN source terms in all points of the computational grid. Knowledge about the spatial distribution of the magnitude of these source terms is an important tool to study the source term balance in coastal areas and to identify dominant physical processes. The program HOT_SOURCE is able to reconstruct the source terms in all grid points, whereas the TEST option of SWAN is only able to do that for a limited set of points.

The program HOT_SOURCE uses the SWAN restart file as input, together with output files with the wind and water depth in all grid points. The SWAN restart file (using the command HOTFILE) contains the wave spectra in all points of the computational grid. HOT_SOURCE reads all spectra, wind speed and direction and water depth and computes for all grid points the source terms for wind input, white-capping dissipation, surf breaking, bottom friction, triad and quadruplet interactions.

For the wind input source term the total magnitude is computed as:

$$\Sigma_{inp} = \iint S_{inp} (f, \theta) df d\theta$$

For the dissipation source terms the total (negative) magnitude is defined as:

$$\Sigma_{dis} = \iint S_{dis} (f, \theta) df d\theta$$

For the non-linear source terms 3 measures are defined, the positive contribution, the negative contribution and the total contribution.

$$\Sigma_{nl}^+ = \iint \frac{1}{2} \left( S_{nl} (f, \theta) + |S_{nl} (f, \theta)| \right) df d\theta$$

$$\Sigma_{nl}^- = \iint \frac{1}{2} \left( S_{nl} (f, \theta) - |S_{nl} (f, \theta)| \right) df d\theta$$

$$\Sigma_{nl} = \iint S_{nl} (f, \theta) df d\theta$$

$$= \Sigma_{nl}^+ + \Sigma_{nl}^-$$

Similar measures are computed for the total source function. The magnitude of the triad and quadruplet source terms is defined as the sum of the absolute values of the negative and positive sums:

$$\Sigma_{nl}^q = \iint |S_{nl} (f, \theta)| df d\theta$$

$$= \Sigma_{nl}^+ + |\Sigma_{nl}^-|$$

The tool HOT_SOURCE has been used to study the variation of the magnitude of the TRIAD source term along a 1-dimensional transect.
C Improved modelling of nonlinear four-wave interactions in shallow water
IMPROVED MODELLING OF NONLINEAR FOUR-WAVE INTERACTIONS IN SHALLOW WATER

Gerbrant Ph. van Vledder¹, and Marcel Bottema²

Abstract: A finite depth version of the Discrete Interaction Approximation (DIA) has been developed and implemented in the SWAN model. This modification of the DIA makes the presently used depth-scaling obsolete. The capabilities of the finite depth DIA have been compared with results from an exact technique for the calculation of the nonlinear transfer rate. Firstly, the nonlinear transfer rate was computed for a JONSWAP spectrum in deep and shallow water. Secondly, two growth curves have been computed for a shallow lake with a constant depth of 5 m and 2 m. The results of the computations indicate that for mean $kh$-values larger than 1.3 no effects are noticeable. Only when $kh<1.3$ the finite depth DIA yields different results. This leads to small changes in wave period and spreading measures.

INTRODUCTION
The present generation of full spectral discrete wave prediction models is based on the concept that each physical process can be modelled with a separate source term. In deep water source terms for wind input, whitecapping dissipation and nonlinear four-wave interactions are active. As the waves enter shallow water the source terms for bottom friction, depth-induced waves breaking as well as nonlinear three-wave interactions become important. Formulations of these source terms are often based on a combination of theoretical studies and analysis of field and laboratory measurements. Despite these efforts many different formulations for most of these source terms exist and no generally accepted formulation for each of these source terms exists.

The only source term for which a closed theoretical framework exists is the one describing the nonlinear four-wave interactions, which was proposed by Hasselmann (1962). The computation of the theoretical expression of the nonlinear four-wave

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interactions is very time consuming because its formulation contains a 6-dimensional integral. Due to these computational requirements it is not used in operational wave prediction models. To overcome this obstacle, Hasselmann et al. (1985) derived an approximation of the full expression. This approximation is known as the Discrete Interaction Approximation (DIA) and it initiated the development of the present day third generation wave prediction models like WAM (WAMDI, 1988), WAVESWATCH (Tolman, 1991), SWAN (Booij et al., 1999) and TOMAWAC (Benoit et al., 1996).

Depth effects on the nonlinear transfer rate can be incorporated in the full theoretical expression by using the finite depth dispersion relation and the finite depth interaction coefficient. Theoretical and numerical studies show that finite depth affects the transfer rate in various ways. Firstly, the overall magnitude increases as the water becomes shallower. Secondly, the frequency and directional distribution of the transfer rate change. In the DIA, however, finite depth effects are crudely schematised using a simple scaling law in which only the magnitude changes while the shape remains unchanged.

The last years it has become evident that the DIA is not able to properly represent the nonlinear transfer function for deep and shallow water (cf. Van Vleder et al., 2000). Consequently, it distorts the source term balance in a wind wave spectrum. To overcome the shortcomings of the DIA, coefficients in the source terms for wind input and whitecapping dissipation in WAM-type models are heavily tuned to compensate for the mismatch in the DIA. This situation hampers the further development of other source terms as long as the DIA is used to investigate source terms with such a numerical wave model.

The need to replace the DIA by better approximations has been widely accepted in the wave modelling community and various authors have proposed extensions or modifications to the original DIA (cf. Hashimoto and Kawaguchi, 2001, and Van Vleder 2001). However, until now these efforts are only aimed at improving the deep water transfer rate. In this paper attention is given to improve the modelling of the nonlinear four-wave interactions in shallow water. To that end a finite depth version of the DIA has been derived. This modification makes the currently used depth scaling obsolete. Its basic features will be illustrated by comparisons with exact solution techniques for a JONSWAP spectrum and some shallow water growth curves experiment.

**NUMERICAL MODELLING OF WIND WAVES**

The spatial and temporal evolution of the wave field is conveniently described by the wave action balance equation. In flux form this equation is given by:

\[
\frac{\partial}{\partial t}N + \frac{\partial}{\partial x}(c_{g,x}N) + \frac{\partial}{\partial y}(c_{g,y}N) + \frac{\partial}{\partial \theta}(c_{g,\theta}N) + \frac{\partial}{\partial k}(c_{g,k}N) = S
\]

in which \(N(N(\sigma, \theta, x, y, t))\) is the wave action density spectrum, \(c_{g,x}\) and \(c_{g,y}\) are the group velocities in x- and y-direction, and \(c_{g,\theta}\) and \(c_{g,k}\) are the spectral propagation velocities. The growth, decay and redistribution of wave action is given by the source term \(S\).
source term $S$ is considered to be the sum of individual source terms, each representing a specific physical process:

$$S = S_{\text{wind}} + S_{\text{wcap}} + S_{\text{nl4}} + S_{\text{fric}} + S_{\text{brk}} + S_{\text{nl3}}$$  \hspace{1cm} (2)

In this expression $S_{\text{wind}}$ is the wind input term, $S_{\text{wcap}}$ whitecapping dissipation, $S_{\text{nl4}}$ nonlinear four-wave interactions. In shallow water additional source terms are active; $S_{\text{fric}}$ energy decay by bottom friction, $S_{\text{brk}}$ energy decay by breaking waves as well as $S_{\text{nl3}}$ nonlinear three-wave interactions. The nonlinear interaction terms only redistribute wave action within the spectrum. Descriptions of these source terms can be found in papers describing particular wave models like WAVEWATCH and SWAN. For the purposes of this paper, the basic equations of the methods for computing the nonlinear four-wave interactions are repeated.

**NONLINEAR FOUR-WAVE INTERACTIONS**

Nonlinear wave-wave interactions between pairs of four wave components play an important role in the evolution of wind-generated waves (cf. Young and Van Vledder, 1993). Hasselmann (1962) developed the theoretical framework for these interactions and he formulated an integral expression for the computation of these interactions, which is known as the Boltzmann integral for surface gravity waves. Hasselmann (1962) found that a set of four waves, called a quadruplet, could exchange energy when the following resonance conditions are satisfied:

$$k_1 + k_2 = k_3 + k_4$$ \hspace{1cm} (3)

$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$ \hspace{1cm} (4)

in which $\omega_j$ the radian frequency and $k_j$ the wave number ($j=1,...,4$). The linear dispersion relation relates the frequency and the wave number:

$$\omega^2 = gk \tanh(kh)$$ \hspace{1cm} (5)

Here, $g$ is the gravitational acceleration and $h$ the water depth. Hasselmann (1962) describes the nonlinear interactions between wave components in a quadruplet in terms of their action density $n_i$, where $n(k_i) = E(k_i)/\omega_i$. The rate of change of action density at a wave number $k_i$ due to all quadruplet interactions involving $k_i$ is given by:

$$\frac{\partial n_i}{\partial t} = \iiint G(k_1, k_2, k_3, k_4) \times \delta(k_1 + k_2 - k_3 - k_4) \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \times [n_i n_2 (n_3 + n_4) - (n_i + n_2) n_3 n_4] dk_1 dk_2 dk_3 dk_4$$ \hspace{1cm} (6)

where $n_i = n(k_i)$ is the action density at wave number $k_i$ and $G$ is the coupling coefficient, which is a complicated function of the four wave numbers involved in an interaction. Deep and finite depth expressions for this coefficient have been given by Hasselmann (1962) and Herterich and Hasselmann (1980). The delta functions in (6) ensure that
contributions to the integral only occur for quadruplets that satisfy the resonance conditions. They also ensure conservation of energy, action and momentum.

The computation of the Boltzmann integral is rather complicated and very time consuming since it requires the solution of a 6-dimensional integral. In the numerical evaluation of Eq. (6) thousands of possible wave number configurations need to be evaluated to determine the nonlinear transfer rate for a particular wave spectrum. Numerical integration techniques for the Boltzmann integral have been developed by Hasselmann and Hasselmann (1981), Masuda (1980) and Resio et al. (2001). The latter use a technique based on methods developed by Webb (1978) and Tracy and Resio (1982). These techniques are referred to as exact methods because they are able to solve the Boltzmann integral to any prescribed degree of accuracy. Because of the computational requirements of these methods it is (still) not feasible to include a full solution of the Boltzmann integral in operational wave models.

THE DISCRETE INTERACTION APPROXIMATION

In contrast to exact methods, the Discrete Interaction Approximation considers only one wave number configuration, and its mirror image. In this configuration the wave numbers \( k_1 \) and \( k_2 \) are equal and the other two wave numbers have different magnitude and direction. Their frequencies are related via the parameter \( \lambda \) such that their configuration is uniquely determined:

\[
\begin{align*}
    f_1 &= f_2 = f \\
    f_3 &= (1 + \lambda) f = f^+ \\
    f_4 &= (1 - \lambda) f = f^-
\end{align*}
\]  

(7)

Here, the superscripts \( + \) and \( - \) are used to emphasise the link with previously reported notations of the DIA. The directions of the wave numbers \( k_1 \) and \( k_2 \), relative to the direction of the wave numbers \( k_1 \) and \( k_2 \), follow from Eq. (7) and the resonance conditions (3) and (4). In the DIA proposed by Hasselmann et al. (1985) \( \lambda = 0.25 \), leading to the angles \( \theta_1 = \theta_2 = 0^\circ \), \( \theta_3 = \theta' = +11.48^\circ \) and \( \theta_4 = \theta'' = 33.56^\circ \).

The DIA source term describes the rate of change of energy density in all four (in fact three independent) wave numbers involved in an interaction. The corresponding energy densities are denoted by \( E = E(f, \theta) \), \( E^+ = E(f^+, \theta') \) and \( E^- = E(f^-, \theta'') \), and the contributions to the corresponding transfer rates are denoted by \( \delta S_{nl} \), \( \delta S''_{nl} \) and \( \delta S^{'''}_{nl} \). The functional form of the DIA source term is given by:

\[
\begin{pmatrix}
    \delta S_{nl} \\
    \delta S'^{nl} \\
    \delta S''^{nl}
\end{pmatrix} =
\begin{pmatrix}
    -2 \\
    1 \\
    1
\end{pmatrix}
C_{nl4} S^{-4} f^{11}
E^2 \left( \frac{E^+}{(1 + \lambda)^4} + \frac{E^-}{(1 - \lambda)^4} \right) - 2 E \frac{E^+ E^-}{(1 - \lambda^2)^4}
\]  

(8)

in which \( C_{nl4} \) is a scale parameter. In the WAM and SWAN models \( C_{nl4} = 3 \times 10^7 \).
To compute the nonlinear quadruplet source term for a discrete wave spectrum, equation (8) is applied to all spectral bins, taking $E$ as the energy density at this bin. The positions of the other two interacting bins with energy densities $E^*$ and $E^-$ are determined relative to the central bin $E$. In general, the locations $(f^*, \theta^*)$ and $(f, \theta^-)$ of the wave numbers $k_j$ and $k_s$ in the spectral grid do not coincide with discrete spectral grid points. To obtain the energy density at these points one can apply bi-linear interpolation between the four surrounding grid points or one may take the energy density at the nearest grid point. In the computational procedure Eq. (8) is also applied to its mirror image, obtained by reversing the signs of the angles $\theta_j$ and $\theta_k$.

In the computational procedure of the DIA special attention is given to wave number configurations that cross the boundaries of the spectral grid. Periodicity is assumed to take care of the directional boundaries. Further, in the case that the frequency $f$ is lower than the lowest discrete frequency $f_{\text{min}}$, the corresponding energy density $E^*$ is set to zero, and in the case that the frequency $f^*$ is higher than the highest discrete frequency a parametric decay of energy densities in the spectral tail is assumed, usually given by:

$$ E(f) = E(f_{\text{max}}) \left( \frac{f}{f_{\text{max}}} \right)^p \quad \text{for} \quad f \geq f_{\text{max}} \quad (9) $$

in which $p$ is the power of the spectral tail. Additional quadruplets in the spectral tail need to be accounted for to ensure that all possible interactions between wave numbers in the prognostic range of the spectral grid and in the spectral tail are included. This is achieved by extending the spectral grid towards higher frequencies, such that the bin $E$ with frequency $f$ is located just at or just above the highest model frequency $f_{\text{max}}$. Application of Eq. (8) produces the rate of change of energy density at the interacting wave numbers. These rates are distributed among the four surrounding spectral bins using the same (interpolation) weights as used for the determination of the energy density at these wave numbers.

In WAMDI (1988) a simple method was proposed to include finite depth effects on the nonlinear transfer rate. Firstly, the nonlinear transfer rate is computed assuming deep water. Secondly, the resulting transfer rate is multiplied with a constant factor $R$. This factor is a function of the dimensionless water depth $\bar{k}h$, and constant for all spectral components of the spectrum. To enhance model robustness in the case of arbitrarily shaped spectra, the mean wave number is computed in a special way as (cf. Tolman, 1991):

$$ \bar{k} = \left( \frac{1}{E_{\text{tot}}} \int_0^{2\pi} \int_0^\infty \frac{1}{\sqrt{k}} E(f, \theta) df d\theta \right)^2 \quad (10) $$

with $E_{\text{tot}}$ the total wave variance. The nonlinear finite depth transfer rate is computed as:

$$ S_{nl4}^h(f, \theta) = S_{nl4}^\infty(f, \theta) \times R(x) \quad (11) $$

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in which \( x = 0.75 \tilde{k} h \) and where the function \( R(x) \) is given by:

\[
R(x) = 1 + \frac{5.5}{x} \left( 1 - \frac{6x}{7} \right) \exp \left( -\frac{5x}{4} \right)
\]  

(12)

To avoid numerical instabilities, the value of \( R \) is limited to at most 5. This parameterisation of the depth scaling is based on an analysis of results of computations for JONSWAP spectra on deep and finite depth with the wave model EXACT-NL (Hasselmann and Hasselmann, 1981). The functional behaviour of Eq. (12) is shown in Figure 1. As can be seen finite depth effects are only relevant for \( \tilde{k} h < 3 \), and an increase in magnitude of the nonlinear transfer rate only occurs when \( \tilde{k} h < 1 \).

![Figure 1. Parameterisation of depth scaling in the DIA](image)

**COMPARISONS OF NONLINEAR TRANSFER RATE FOR DEEP AND SHALLOW WATER**

To illustrate some of the shortcomings of the DIA a comparison was made of the nonlinear transfer rate computed with an exact method and with the DIA. The exact nonlinear transfer rates were computed with WRT method, developed by Webb (1978), Tracy and Resio (1982), and Resio et al. (2001), and rewritten by the first author. To that end the same deep water JONSWAP spectrum was used as in Hasselmann et al. (1985), viz. a JONSWAP spectrum with \( f_p = 0.1 \) Hz, \( \alpha = 0.0175 \), \( \gamma = 3.3 \) and a \( \cos^2(\theta) \)-directional spreading. The result is shown in Figure 2.

From Figure 2 it is evident that the DIA has the following deficiencies. The negative lobe is over-predicted. A relatively large positive lobe exists at about twice the peak frequency, which does not result from the exact method. Moreover, the frequency of the first zero-crossing of the transfer rate is much higher than the peak frequency, whereas the exact computation predicts its position to be located at the peak frequency.
Fig. 2. Comparison of nonlinear transfer rate computed with an exact method (solid line) and with the DIA (solid line with crosses) for a deep water JONSWAP spectrum with $f_p=0.1$ Hz, $\alpha=0.0175$, $\gamma=3.3$ and a $\cos^2(\theta)$-directional spreading.

In shallow water finite depth effects change the magnitude and shape of the nonlinear transfer rate. This is illustrated in Figure 3 on the basis of exact computations for a mean JONSWAP spectrum with a peak frequency of 0.1 Hz in deep and shallow water with a depth of 10 m. This figure also shows the scaled nonlinear transfer rate for which $\tilde{k}h = 0.78$, resulting in a multiplication factor of $R = 3.2$.

Fig. 3. Nonlinear transfer rates for a deep water JONSWAP spectrum with $f_p=0.1$ Hz, $\alpha=0.0175$, $\gamma=3.3$ and a $\cos^2(\theta)$-directional spreading. Results for deep water, a depth of 10 m and the scaled nonlinear transfer rate.

An important feature of finite depth effects on the nonlinear transfer rate is that the first positive lobe shifts towards lower frequencies and becomes wider than the scaled deep water transfer. This may lead to a faster shift of wave energy towards lower frequencies in shallow water in comparison with the scaled nonlinear transfer rate and to higher wave period measures.

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THE FINITE DEPTH DISCRETE INTERACTION APPROXIMATION

To improve the depth behaviour of the DIA, the DIA was re-derived while keeping all finite depth terms in the equations. A detailed derivation of the finite depth DIA can be found in Van Vledder and Rasmussen (2002) and its main result is repeated here. The starting point for the derivation of the finite depth DIA is the principle of detailed balance formulated by Hasselmann (1966). This principle states that the change of wave action per unit time of each wave number involved in a resonant interaction is equal. To take advantage of this principle the Boltzmann integral is written in a symmetrical form:

\[
\begin{pmatrix}
\Delta n_1 \\
\Delta n_2 \\
\Delta n_3 \\
\Delta n_4
\end{pmatrix} = \begin{pmatrix}
-1 \\
-1 \\
1 \\
1
\end{pmatrix} \frac{G}{4} \delta (k_1 + k_2 - k_3 - k_4) \delta (\omega_1 + \omega_2 - \omega_3 - \omega_4) P \, dk_1 \, dk_2 \, dk_3 \, dk_4 \Delta t
\]  
(13)

in which \( P \) is the wave action product term \( P = n_1 n_2 (n_3 + n_4) - n_3 n_4 (n_1 + n_2) \). Eq. (13) has not the most convenient form to integrate with respect to the original wave numbers \( k_1, k_2, k_3 \) and \( k_4 \). Following Hasselmann and Hasselmann (1981), and Rasmussen (1995, 2002). Eq. (13) can be written as:

\[
\begin{pmatrix}
\Delta n_1 \\
\Delta n_2 \\
\Delta n_3 \\
\Delta n_4
\end{pmatrix} = \begin{pmatrix}
-1 \\
-1 \\
1 \\
1
\end{pmatrix} \frac{G}{4} \left| J \right| P \, dk_1 \, dk_2 \, dk_3 \, dk_4 \Delta t
\]  
(14)

in which \( k_{2t} \) is a tangential component of the wave number vector \( k_2 \). \( J \) is the Jacobian of the transformation from Eq. (13) to Eq. (14). It is given by:

\[
J = \left| c_{g_2} - c_{g_4} \right|^{-1}
\]  
(15)

in which \( c_{g,i} \) is the group velocity vector of wave number vector \( k_i \). Next, a number of additional transformations are made to replace the change of wave action for a given wave number into the rate of change of energy density for a given frequency and direction. In addition the DIA assumption \( k_1 = k_3 \) is used. Details of this derivation will be given in Van Vledder and Rasmussen (2002). The result is the basic expression of the finite depth DIA:

\[
\begin{pmatrix}
\Delta S_{nl} \\
\Delta S_{nl}' \\
\Delta S_{nl}''
\end{pmatrix} = \begin{pmatrix}
-2 \\
1 \\
1
\end{pmatrix} C_{g4} G \left| J \right| J J' f^3 \times
\]

\[
\times \left[ \frac{c_g E}{f k} \right] \left[ \frac{c_{g1} E^*}{f' k^*} + \frac{c_{g2} E}{f k} \right] - 2 \left[ \frac{c_g E}{f k} \right] \left[ \frac{c_{g1} E^*}{f' k^*} \right] \left[ \frac{c_{g2} E}{f k} \right] \left[ \frac{c_{g1} E}{f' k^*} \right]
\]  
(16)

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in which $E=E(f, \theta)$ and $J'$ an additional Jacobian:

$$J' = \frac{k_1 k_3}{C_g^1 C_g^2 C_g^2 r}$$  \hspace{1cm} (17)

The main differences with the original DIA are the inclusion of the group velocities, frequencies and wave numbers in the product term, the explicit expression of the interaction coefficient $G$ and the scaling term $J'$ arising from the Jacobian's. Another difference with the DIA is that the shape of the interacting wave number configuration is depth dependent. This latter effect is illustrated in Figure 4. The upper left panel of this figure shows the standard DIA configuration with $\lambda=0.25$ for a central frequency of 0.0705 Hz, resulting in the wave numbers $k_1=k_3=0.02$ (1/m). The other panels show the modification of the DIA configuration with decreasing depth. The wave number magnitudes increases and their directions become more aligned.

![Fig. 4. Modification of the DIA wave number configuration with decreasing depth, for a central frequency of 0.0705 Hz and the water depths of 100 m, 20 m, 10 m and 5 m.](image)

For deep water, all terms in Eq. (16) can be written in terms of frequencies and the gravitational acceleration $g$, such that the original DIA formulation (8) is obtained. The performance of the finite depth DIA is illustrated in Figure 5, in which the nonlinear transfer rate was computed for a mean JONSWAP spectrum with a peak frequency of 0.1 Hz, in deep water and in shallow water with a depth of 10 m.
Figure 5 shows that the finite depth DIA is able to reproduce some of the basic features of the changes of the nonlinear transfer rate in shallow water as inferred from exact computations. The overall magnitude increases, the first positive lobe shifts towards lower frequencies, and the first zero-crossing is closer to the peak frequency of the spectrum. Despite these improvements, the position of the negative lobe along the frequency axis is still much too high and the second positive lobe is also still too high.

![Graph](image)

**Fig. 5.** Nonlinear transfer rate computed with the finite depth DIA for a deep water JONSWAP spectrum with $f_c=0.1$ Hz, $\alpha=0.0175$, $\gamma=3.3$ and a $\cos^2(\theta)$-directional spreading. Results for deep water, a depth of 10 m and the scaled nonlinear transfer rate.

**IMPLICATIONS FOR WAVE MODELLING**

To assess the implications of an improved modelling of the nonlinear four-wave interactions in shallow water the finite depth DIA was implemented in a test version of the SWAN model (Booij et al., 1999), version 40.11. Two test cases were defined, representing a shallow lake with a constant water depths of 5 and 2 m. These situations refer to typical RIZA problems (cf. Bottema et al., 2002). Wave model computations were made with the modified SWAN model in one-dimensional mode, a wind speed of 25 m/s and the source terms for bottom friction and wave breaking activated. The triad source term was deactivated to avoid numerical problems with this version of the SWAN model. The growth curves for the significant wave height $H_s$, mean wave period $T_{\text{m01}}$, directional spreading $\sigma$ and spectral narrowness $\kappa$ for the 5 m case are shown in Figure 6. The results in this figure show that the explicit inclusion of shallow water effects in the DIA hardly affects the results. Only the frequency spectra become more peaked. These results are not surprising since the $\tilde{k}h$ value at the end of the fetch is about 1.72 resulting in a scale factor of 0.91. Although the waves are depth limited, the water is not deep in terms of $kh$-values. The results for the 2 m case are shown in Figure 7, indicating that the finite depth DIA gives slightly lower wave periods, narrower directional distributions and somewhat wider frequency spectra. Inspection of the numerical results showed that at the end of the fetch the $\tilde{k}h$ value at the end of the fetch is about 1.21 resulting in a scale factor of 1.45.
Fig. 6. Growth curves computed with the standard DIA (solid line) and finite depth DIA (fDIA, dashed line) for a constant water depth of 5 m and a wind speed of 25 m/s using the modified SWAN model.

Fig. 7. Growth curves computed with the standard DIA (solid line) and finite depth DIA (fDIA, dashed line) for a constant water depth of 2 m and a wind speed of 25 m/s using the modified SWAN model.

DISCUSSION
The results of the computations for the academic test spectrum indicate that the finite depth DIA leads to a better representation of shallow water effects on the nonlinear transfer rate compared to the present depth scaling. Still, a mismatch in the nonlinear transfer rate exists in comparison with results of exact computations. This is probably due to the fact that only one wave number configuration has been used in these computations.

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The results of the academic growth curves indicate that the 5 m depth shallow lake is not shallow in terms of \( \tilde{k}h \) values, although the growth is depth-limited in terms of \( H/h \) values. This is one of the reasons why the wave measurements described in the companion paper (Bottema et al., 2002) are not only interesting from an operational point of view, but also from a physical point of view. Only for the 2 m case depth effects become noticeable. The fact that the mean wave period decreases is surprising, but this may be due to the fact that only one wave number configuration was used. The decrease in directional spreading is probably due to the fact that the resonant wave number vectors become more aligned in shallow water, thus limiting the amount of energy transferred to off-wind directions. Investigation of the spectral shapes, related source terms and more detailed comparison with the results of exact computation are needed to fully understand these results.

The finite depth version of the DIA is only a first step in bridging the gap between the fast but inaccurate DIA and the accurate but time consuming exact methods for computing the nonlinear transfer rate in deep and shallow water. Two other methods are suggested to further improve the DIA. The first method is to include more wave number configurations. Such multiple DIA's have been presented by Hashimoto and Kawagushi (2001) and Van Vledder et al. (2000). A basic shortcoming of these methods is that only a limited set of wave number configurations is considered. Therefore, a more general extension of the DIA with generally shaped wave number configurations is needed (Van Vledder, 2001).

CONCLUSIONS

The finite depth version of the DIA makes the presently used depth scaling obsolete. The finite depth DIA is able to account for some shallow water effects on the nonlinear transfer rate. Results of the computations indicate that for typical shallow lake situations the inclusion of finite depth effects in the DIA yields slightly different results only when \( \tilde{k}h < 1.3 \). Additional and more complicated tests are needed to assess the implications of the finite depth DIA. The potential benefits of the finite depth DIA are probably obscured by the mismatch in shape of the nonlinear transfer rate. This is due to the fact that only one wave number configuration has been used. Therefore, additional and arbitrarily shaped wavenumber configurations are needed to further improve the DIA.

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