COMPUTATIONAL GRADIENT PLASTICITY

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Abstract

The inclusion of higher-order deformation gradients is of paramount importance for properly describing strain localization within the framework of continuum mechanics. In this contribution a novel strategy for incorporating such gradients in numerical of failure processes in frictional-dilatant elasto-plastic materials is presented.

1. Introduction

Continuum models can only properly describe mechanical processes if the governing equations do not admit discontinuous solutions. Although classical continuum models are capable of predicting the onset of localization, the evolution of localization phenomena cannot be studied within the framework of these models. This is because in classical continua localization is synonymous with the emergence of discontinuities. Ellipticity of the governing set of equations is lost, so that the boundary value problem is no longer well-posed.

To study evolution of localization instabilities, for example within the context of the finite element method, enhanced continuum models are called for. Several different strategies have been pursued in the past, such as non-local approaches (e.g., Pijaudier-Cabot and Bazant 1987), gradient models (e.g., Muhlhaus and Alfantis 1991), addition of rate dependence (Needleman 1988, Sluys and de Borst 1991) and the use of Cosserat continua (Muhlhaus et al. 1991, de Borst and Sluys 1991). Currently, it is not clear which approach will be most successful. Probably, it will turn out that for each class of problems another model will be most appropriate. In this contribution we focus on gradient elasto-plasticity. In particular, we shall discuss some of the numerical implications of using such a model.

2. Mathematical formulation of a gradient plasticity theory

We consider the following set of equations:

\begin{align}
L^2 \sigma &= 0, \\
\dot{\sigma} &= D(\dot{\varepsilon} - \dot{\lambda} n), \quad \dot{\lambda} \geq 0, \\
f(\sigma, \kappa, V^\kappa) &= 0, \\
\dot{\varepsilon} &= Lu.
\end{align}

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which define the elasto-plastic rate problem during continued yielding. In the
equilibrium equation (1) \( \mathbf{\sigma} \) is a vector that contains the stress\ncomponents (Voigt's notation), \( \mathbf{L} \) is a differential operator matrix and the superscript \( T \) is the transpose symbol. As in standard plasticity stress and elastic strain, \( \bar{\mathbf{e}} = \mathbf{e} - \mathbf{A} \), are related through the elastic stiffness matrix \( \mathbf{D} \). The superimposed bars denote differentiation with respect to time. \( \lambda \) is a non-negative scalar-valued quantity which is a measure of
the plastic flow intensity and \( n \) is the gradient to the yield surface \( f \), that is, \( n = \partial f / \partial \mathbf{\sigma} \).

The salient feature of the plasticity theory adopted here is the functional
dependence of the yield function on the Laplacian of the hardening parameter \( \kappa \), in
addition to the usual dependence of \( f \) on \( \kappa \). In a continuum approach the gradient
terms reflect the fact that below a certain size scale the interaction between the
microstructural carriers of the deformation is nonlocal. For a detailed discussion of
nonlocal effects the reader is referred to the review paper by Kubin and Lépineaux
(1988). A particularly convincing example of nonlocal hardening models, based on
the interaction of a Frank Read source with dipolar dislocation arrangements, can be
found in the paper by Kraus (1988). For cementitious materials experimental results
which show evolving micro-structures during failure have been given by Van

The dependence of the yield function on spatial gradients of an invariant measure
of the plastic strain has a major impact when elaborating the condition that, during
plastic flow, the stress point must remain on the yield surface: \( f = 0 \) (consistency
condition). Taking into account eq. (3) the consistency condition becomes:

\[
\mathbf{n}^T \mathbf{\sigma} - h \lambda + c \nabla^2 \kappa = 0,
\]

with \( h = - \frac{\partial f}{\partial \kappa} \bar{\mathbf{\epsilon}} \), and \( \kappa = \nabla \lambda / \nabla^2 \lambda \). In the examples we shall adopt a Drucker-
Prager yield function

\[
f = \sqrt{3} J_2 + \alpha p - \kappa (\nabla^2 \kappa)
\]

with \( J_2 \) the second invariant of the deviatoric stress tensor, \( p \) the hydrostatic pressure,
\( \alpha \) a friction coefficient and \( \kappa \) reflecting the cohesion of the material, and a strain-
hardening hypothesis for the evolution of the hardening parameter \( \kappa = 2 \lambda (\bar{\mathbf{e}}^p)^T \bar{\mathbf{e}}^p \).

Substituting the yield function (6) into the flow rule and this result subsequently into
the strain-hardening hypothesis gives \( \kappa = \lambda (1 + 2 \alpha / 9) \). With \( c = \bar{\epsilon} (1 + 2 \alpha / 9) \) we obtain in lieu of eq. (5):

\[
\mathbf{n}^T \mathbf{\sigma} - h \lambda + c \nabla^2 \lambda = 0.
\]

From eq. (7) we observe that, unlike conventional plasticity, the consistency condition
results in a partial differential equation. This has the far-reaching consequence that an
explicit expression for \( \lambda \) cannot be obtained at a local (integration point) level. This
means that, within each equilibrium iteration loop, an internal loop must be set up to
solve for the plastic multipliers \( \lambda \) in the integration points, cf. de Borst and Mühlhaus
(1991). In non-local plasticity the same argument holds true.

3. A numerical strategy for gradient plasticity theories

To obviate the need for an iteration loop to solve for the partial differential equation
(7) within each iteration loop it is proposed to satisfy the consistency condition only in
a weak sense:

\[
\int_{\Omega} \delta \lambda (\mathbf{n}^T \mathbf{\sigma} - h \lambda + c \nabla^2 \lambda) d\Omega = 0.
\]

Together with the weak form of the equilibrium equation,
\[ \int_{V} \delta u^T (L^T \delta) \, dV = 0, \quad (9) \]

the elastic stress-strain relation (2) and the kinematic relation (4), both of which are satisfied in a pointwise manner, the elasto-plastic boundary value problem is then defined completely. However, the fact that the consistency condition is no longer satisfied in a pointwise manner marks a major departure from return-mapping algorithms as used in conventional plasticity. Now, the plastic multiplier \( \lambda \) is considered as a fundamental unknown and has a role equal to that of the displacements. It is solved at global level simultaneously with the displacement degrees-of-freedom.

The displacement field \( u \) and the field of plastic multipliers \( \lambda \) can now be discretized to nodal variables \( \bar{u} \) and \( \bar{\lambda} \) in the usual way, leading to the following set of equations:

\[
\begin{bmatrix}
K_{uu} & K_{u\lambda} \\
K_{\lambda u} & K_{\lambda\lambda}
\end{bmatrix}
\begin{bmatrix}
\bar{u} \\
\bar{\lambda}
\end{bmatrix} =
\begin{bmatrix}
f_u \\
0
\end{bmatrix},
\] (10)

\[K_{uu} = \int_{V} B^T D b u \, dV, \quad (11)\]

\[K_{u\lambda} = -\int_{V} B^T D n h^T \, dV, \quad (12)\]

\[K_{\lambda u} = \int_{V} \left( (h + n^T D n) h^T - c p^T \right) \, dV \quad (13)\]

and \( f_u \) the external force vector. In eqs. (10)-(13) \( B \) is the strain-nodal displacement matrix, \( h \) is the vector that contains the interpolation polynomials for the plastic multiplier and \( p \) contains the Laplacians of the latter functions.

4. Shear banding in a gradient-dependent Drucker-Prager plasticity theory

To demonstrate that the proposed dependence of the yield function on the Laplacian of the invariant inelastic strain measure \( \chi \) indeed results in a well-posed boundary value problem and in a finite width of the localization zone, a plane biaxial test was run with three different discretizations. The coarsest mesh consisted of 6x18 four-noded quadrilateral elements. The medium fine and the fine mesh consisted of 12x36 and 24x72 elements, respectively. The displacements were interpolated linearly with a \( B \)-enhancement (Hughes, 1980) to accommodate for the internal kinematic constraint caused by the plastic dilatancy. The plastic multiplier was interpolated by cubic Hermite interpolation polynomials in both directions.

The specimen had a width of 60 mm and height of 180 mm. Smooth boundary conditions (\( \mu = 0 \)) were assumed at the upper and lower boundaries. The following material properties were assumed: Young’s modulus \( E = 11920 \) MPa, Poisson’s ratio \( v = 0.49 \), \( \alpha = 2/3 \), \( k = 100(1-4\alpha) \) and \( c = 2500 \) MPa. Figure 1 shows that rather fine meshes are needed to capture the load-displacement curve accurately, since the medium mesh still results in a too stiff response. The deformation patterns converge much quicker, cf. Figure 2.

Initially two shear bands arise in the loading process. These shear bands grow from the weak element at the left boundary near the horizontal line of symmetry. But further down the descending branch of the load-displacement diagram only one shear band persists. This observation shows that the addition of higher-order gradients does not remove the existence of alternative equilibrium branches. Indeed, some form of geometric non-symmetry was necessary in the specimen to trigger the symmetry-breaking solution, where we have only one shear zone.
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Figure 1. Load-displacement curve for a biaxial test (gradient plasticity).

Figure 2. Incremental displacements at peak load (left) and beyond peak load (right).

Appendix. References