THE USE OF MATRIX ALGEBRA
IN
GEOMETRICAL OPTICS

PROEFSCHRIFT

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WILLEM BROUWER,
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geboren te Semarang.
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PROMOTOR PROF. DR. A. C. S. VAN HEEL.
Introduction.

The propagation of light is completely described by Maxwell's equations for an electromagnetic wave, but these equations are very difficult to solve in many practical problems. If we assume the wavelength to be zero in the wave equation, we have the geometrical description of optics. It is well known that in this case we can describe the behaviour of light in terms of light rays. For many purposes this assumption gives considerable simplification of mathematical difficulties involved in optical calculations without undue loss of precision. In many cases it is possible to arrive at a better description by taking the finite wavelength into account after the geometrical solution is found.

However, in the following pages we will restrict the discussion entirely to the geometrical approximation.

A light ray is completely defined by a point on it and its direction, i.e. in general by five coordinates. After passing through an optical system the light ray leaving the system is again defined by five coordinates. A light ray is given a transformation and the optical system is completely described if we know the transformation of the coordinates of a light ray passing the system.

If this transformation is a linear one, the matrix algebra is perfectly suitable to describe the transformation. All reflections of plane mirrors are linear transformations and are, for this reason, completely described by matrices.

Many optical devices, however, are not linear in this sense. This is easily seen. Refraction is governed by Snell's law which reads

\[ \mu \sin \varphi = \mu' \sin \varphi' \]

in which \( \mu \) and \( \mu' \) are the refractive indices of the medium before and after
refraction, $\varphi$ and $\varphi'$ the angles of incidence and refraction. Furthermore, the direction of the ray is given by its direction cosines and we have to transform to the sines to apply Snell's law. This is a quadratic relation. Many refracting surfaces used in optics are quadratic surfaces. To calculate the point of incidence we have to use these quadratic relations. There are, however, ways to describe geometrical optics, in spite of these difficulties, in matrix form. It is our aim to do so in the following as it results in a considerable simplification of the formulae and their derivation.

Systems consisting of plane mirrors are considered in chapter I. The theory of such systems was developed in matrix form by T. Smith in 1928 (see references at the end of chapter I). It seemed desirable to give here the results and their derivation since they are of utmost importance for practical purposes.
Chapter I.

Plane Reflecting Surfaces.

Rays passing through an optical system, consisting only of plane reflecting surfaces, undergo a linear transformation. The matrix which describes this transformation should give the transformation of the five coordinates of the incoming ray. In this case, however, we can treat the direction of the ray and the position of a point on the ray separately by asking where a point in space is imaged after passing the reflecting system.

It is clear that we have to solve the problem for one reflection only. If we have a system with more reflections, the matrix of each reflecting surface has to be found and then these matrices are multiplied in the proper sequence to get the matrix of the whole system.

The mirror is given by the direction cosines \( l, m \) and \( n \) of a normal to the mirror and a point \( P (f, g, h) \) on its surface with respect to a Cartesian coordinate system. See figure 1.1.

We will first determine the matrix describing the transformation of a point \( A (x, y, z) \) by this mirror into point \( A' (x', y', z') \). An easy way to find the matrix is to translate the coordinate system in such a way that its origin is in the point \( P \), then rotate the coordinate system until the \( z \)-axis coincides with the normal to the mirror. The image of point \( A \) in this new coordinate system is easily written down as we have only to change the sign of the \( z \)-coordinate. We now rotate the coordinate system back into its original direction and then translate in such a way as to bring its origin into the original position. The reason for this procedure is that each step can easily be represented by a matrix, and the final result can be found by multiplying the matrices of each step in the right sequence.

The translation to point \( P (f, g, h) \) is given by the matrix:
Figure 1.1
The rotation of the coordinate system till the z-axis coincides with the normal on the mirror is undetermined. In our case the position of the x-axis and y-axis is unimportant and can be chosen in a convenient way. Suppose that the new x-axis remains in the old x-z plane. This is given by:

\[
R_2 = \begin{bmatrix}
\frac{n}{w} & 0 & -\frac{\xi}{w} & 0 \\
-\frac{\xi m}{w} & w & -\frac{m n}{w} & 0 \\
\xi & m & n & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

in which \( w = \sqrt{\xi^2 + n^2} \).

It is easily seen that the reflection, derotation and translation back to the original origin are given by:

\[
R_3 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix};
\quad
R_4 = \begin{bmatrix}
\frac{n}{w} & -\frac{\xi m}{w} & \xi & 0 \\
0 & w & m & 0 \\
-\frac{\xi}{w} & -\frac{m n}{w} & n & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

and:

\[
R_5 = \begin{bmatrix}
1 & 0 & 0 & -f \\
0 & 1 & 0 & -g \\
0 & 0 & 1 & -h \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]
The reflection as a whole can now be described by the matrix \( R \) where

\[
R = R_2^f R_4^f R_2^x R_1^x .
\]

In doing so we find:

\[
H = \begin{bmatrix}
1-2\ell^2 & -2\ell m & -2\ell n & 2\ell (lf+mg+nh) \\
-2\ell m & 1-2m^2 & -2mn & 2m(lf+mg+nh) \\
-2\ell n & -2mn & 1-2n^2 & 2n(lf+mg+nh) \\
0 & 0 & 0 & 1
\end{bmatrix} . \tag{1.1}
\]

The elements of the principal diagonal are equal to the cosines of two times the angles between the coordinate axes and the normal to the mirror. The geometrical meaning of the term \( lf+mg+nh \) is easily shown. The direction cosines of the line \( OP \) are:

\[
\frac{f}{OP} ; \quad \frac{g}{OP} ; \quad \frac{h}{OP} .
\]

The angle \( \alpha \) between this line \( OP \) and the normal on the mirror through 0 is given by:

\[
\cos \alpha = \frac{lf+mg+nh}{OP} ,
\]

and the length \( p \) of this normal is:

\[
p = OP \cos \alpha = lf+mg+nh . \tag{1.2}
\]

This quantity thus appears to be equal to the distance between the mirror and the origin of the coordinate system. However, in practice it is often more advantageous to use an intersection of the mirror and one of the coordinate axes as point \( P \).

It is easily verified that the matrix (1.1) read as a determinant has
a value \(-1\). This makes it possible to determine directly whether a matrix represents an odd or an even number of reflections in that a determinant of \(-1\) indicates an odd number of reflections and \(+1\) an even number of reflections.

The matrix \(r\) formed by omitting the fourth column and row from the matrix \(R\) represents a mirror containing the origin of the coordinate system. We notice that terms with quantities of the form \(lf+mg+nh\) are appearing only in the fourth column and thus the matrix \(r\) for any given reflecting system consists only of elements formed with the direction cosines of the normals to the mirrors. Thus the matrix \(r\) formed in the above manner from the matrix \(R\) represents a reflecting system whereby all the surfaces are parallel to the given ones with, however, all surfaces passing through the origin of the coordinate system. This matrix represents a rotation of the coordinate system and is for this reason orthogonal and \(rr' = 1\) (1). For this reason we find that for each reflecting system the following equations hold:

\[
\begin{align*}
    r_{11}^2 + r_{12}^2 + r_{13}^2 &= 1 ; \\
    r_{11}r_{21} + r_{12}r_{22} + r_{13}r_{23} &= 0 ; \\
    r_{21}^2 + r_{22}^2 + r_{23}^2 &= 1 ; \\
    r_{11}r_{31} + r_{12}r_{32} + r_{13}r_{33} &= 0 ; \\
    r_{31}^2 + r_{32}^2 + r_{33}^2 &= 1 ; \\
    r_{21}r_{31} + r_{22}r_{32} + r_{23}r_{33} &= 0 .
\end{align*}
\]

The transformation \(r\) will image a point \(P\) at \(P'\) in such a way that the distance \(OP\) equals the distance \(OP'\). If the direction cosines of the line \(OP\) are \(L,M,N\) and the direction cosines for \(OP'\) are \(L',M',N'\), we have the following relations:

(1) See any textbook on matrices, for instance: Determinants and Matrices by A.C. Aitken, Oliver and Boid, 1951, pages 12, 15 and 57.
This shows that the matrix $r$ gives also the transformation of the direction cosines, thus:

\[
\begin{bmatrix}
L' \\
M' \\
N'
\end{bmatrix} = \begin{bmatrix}
L \\
M \\
N
\end{bmatrix} r \begin{bmatrix}
L \\
M \\
N
\end{bmatrix}.
\]

It often occurs in reflecting prisms, that light rays pass through them under the same angles of incidence and refraction as with a plane parallel plate of glass of appropriate thickness \(^{(2)}\). Let $L_\Phi$, $M_\Phi$ and $N_\Phi$ represent the normal to the entrance surface and $L_\Phi^*$, $M_\Phi^*$ and $N_\Phi^*$ the normal to the exit surface. Then the prism is equivalent to a plane parallel plate if the image of $L_\Phi$, $M_\Phi$ and $N_\Phi$, indicated by $L_\Phi^*$, $M_\Phi^*$ and $N_\Phi^*$, is parallel to $L_\Phi^*$, $M_\Phi^*$ and $N_\Phi^*$, or:

\[L_\Phi L_\Phi^* + M_\Phi M_\Phi^* + N_\Phi N_\Phi^* = 1.\]

In matrix notation:

\[
\begin{bmatrix}
L^* & M^* & N^*
\end{bmatrix} r \begin{bmatrix}
L_\Phi \\
M_\Phi \\
N_\Phi
\end{bmatrix} = 1.
\]

If the reflecting prism is made of glass, it is often advantageous to use total reflection. In order to investigate if a ray with direction

\(^{(2)}\) See e.g. J.C. Gardner;  

cosines $L, M$ and $N$ when entering the prism will be reflected totally at
the $j^{th}$ surface we have to find the angle between the normal on the $j^{th}$
surface given by $l_j, m_j$ and $n_j$ and the ray given by $L^j_{j-1}, M^j_{j-1}$ and $N^j_{j-1}$. This angle should be smaller than the critical angle. The cosine of the
angle between the normal and the ray should be smaller then a number $k$, being the cosine of the critical angle. This relation is given by:

$$
\frac{L^j_{j-1} l_j + M^j_{j-1} m_j + N^j_{j-1} n_j}{E_j} \leq k ,
$$
or in matrix notation:

$$
\begin{bmatrix}
  l_j & m_j & n_j
\end{bmatrix}
\begin{bmatrix}
  L^j_{j-1} \\
  M^j_{j-1} \\
  N^j_{j-1}
\end{bmatrix}
\leq

(1.6)

The coordinates of a point $V$ at a distance $\gamma$ from a point $P (x,y,z)$
along a ray with direction cosines $L, M$ and $N$ are given by:

$$
( x + \gamma L , y + \gamma M , z + \gamma N ) .
$$

The equation for the reflecting plane is:

$$
lx + my + nz - p = 0 ,
$$
in which $p$ is the distance between the origin and the plane as in equation
(1.2). So if we want the intersection point of the ray with the reflecting
surface, we combine the above equations and find, if we solve for the $j^{th}$
surface:

$$
\gamma_j \left( \begin{bmatrix}
  l_j & m_j & n_j
\end{bmatrix}
\begin{bmatrix}
  L^j_{j-1} \\
  M^j_{j-1} \\
  N^j_{j-1}
\end{bmatrix}
+ \left( \begin{bmatrix}
  l_j & m_j & n_j
\end{bmatrix}
\begin{bmatrix}
  r_{j-1}^l \\
  r_{j-1}^m \\
  r_{j-1}^n
\end{bmatrix}
\right) \cdot \begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}
\right) = 0 .
\quad (1.7)
$$
This equation is linear in $\gamma$. This quantity can thus easily be found. It gives us a direct way to see if the reflections take place in the right order. The condition in this case reads $\gamma_j + 1 > \gamma_j$.

It also gives us an easy way to calculate the optical thickness of a prism which is equivalent to a plane parallel plate. By choosing our coordinate system in such a way that the $z$-axis is perpendicular to the entrance surface, the $z$-axis has the direction cosines 0,0,1 and can be handled as a light ray. If the exit surface has the direction cosines $l_n$, $m_n$, $n_n$, then equation (1.7) gives us for the thickness, $\gamma_n$, of the prism:

$$
\gamma_n \begin{bmatrix} R_{13} \\ R_{23} \\ R_{33} \end{bmatrix} + \begin{bmatrix} R_{14} \\ R_{24} \\ R_{34} \end{bmatrix} = 0 . \quad (1.8)
$$

In which $R_{ij}$ are the appropriate elements of the matrix for the prism. $\gamma_n$ can be determined from this equation and multiplied by the refractive index of the prism material to give the optical thickness.

Bibliography:

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Uber Folgen ebener Spiegel.

Chapter II.

Refraction in Optical Systems with Rotational Symmetry.

Before analysing these systems we have to make some sign conventions. Unless otherwise stated, the light is always coming from the left. A distance is positive if measured in the direction of the light. A distance is always measured from a refracting surface.

The vertex of a refracting surface is its intersection point with the axis of symmetry of the system. A radius of curvature is positive if the direction from the vertex to the center of curvature is in the direction of the light.

A ray coordinate before and after refraction will be denoted with the same letter; the one after the refraction will have a prime.

Indices will be used to indicate at what surface the refraction is taking place. The numbering of surfaces will be in the order in which the light is passing through them.

The transformation of the coordinates of a light ray is in general not linear in this case. It is, however, possible to write the transformation for a particular ray in matrix form. We will show this for a spherical surface. Let AP be the incident ray (figure 2.1) and P the point of incidence on the spherical surface with radius \( r \) and center of curvature \( C \). We take a coordinate system with the \( z \)-axis coinciding with the axis of symmetry of the system. The \( z \)-axis is thus going through \( C \). According to Snell's law the refracted ray \( PV \) will be in the plane determined by the points \( A, P \) and \( C \). Now we take a distance \( PU \) on the incident ray with a length \( \mu \) equal to the refractive index to the left of the surface and a distance \( PV \) on the refracted ray with a length \( \mu' \) equal to the refractive index to the right of the surface. Let \( US \) and \( VT \) be perpendicular to \( PC \). According to Snell's law:
\[ US = \mu \sin \varphi = \mu' \sin \varphi' = VT, \]

and thus \( UV \) is parallel to \( PC \).

The direction cosines of the incident ray are \( \ell, m \) and \( n \), the direction cosines of the refracted ray \( \ell', m' \) and \( n' \). The coordinates of \( P \) are \((x, y, z)\). The direction cosines of \( PC \) and \( UV \) with respect to the \( x \)-axis, \( y \)-axis and \( z \)-axis are \(-xR, -yR \) and \((1-zR)\); where \( R = 1/r \). On projecting the triangle \( PUV \) on the coordinate axes we find:

\[ \mu' \ell' = UV (\ell R) - \mu \ell = 0, \]
\[ \mu' m' = UV (\ell R) - \mu m = 0, \]
\[ \mu' n' = UV (1-zR) - \mu n = 0, \]

and:

\[ UV = ST = \mu' \cos \varphi' - \mu \cos \varphi. \]

These equations describe the refraction completely if the point of incidence and the quantity

\[ A'_1 = (\mu' \cos \varphi' - \mu \cos \varphi)R, \quad (2.1) \]

are known, since

\[ x' = x, \]
\[ y' = y, \]
\[ z' = z, \]

the primed quantities being the coordinates of \( P \) after refraction.

It is convenient to introduce the so called optical direction cosines, which are direction cosines multiplied by the refractive index of the medium in which the ray is travelling. Thus:

\[ L = \mu \ell; \quad M = \mu m; \quad N = \mu n; \]
\[ L' = \mu' \ell'; \quad M' = \mu' m'; \quad N' = \mu' n'. \quad (2.2) \]
We can now describe the refraction with the following matrices (1):

\[
\begin{bmatrix}
L' \\
x'
\end{bmatrix} = \begin{bmatrix} 1 & -A_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} L \\ x \end{bmatrix} \quad \text{and} \quad
\begin{bmatrix}
M' \\
y'
\end{bmatrix} = \begin{bmatrix} 1 & -A_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M \\ y \end{bmatrix}.
\]

(2.3)

In order to follow a ray through an optical system we also need to know the transformation of the ray coordinates from one surface to another. Let us assume that the point of incidence \( Q \) on the next surface is known and the distance \( PQ \) equals \( t' \). We now define:

\[
T' = t' / \mu'.
\]

(2.4)

The transformation is now given by:

\[
\begin{align*}
L_Q &= L' \\
x_Q &= x' + T'L' \\
M_Q &= M' \\
y_Q &= y' + T'M'.
\end{align*}
\]

in which the subscripted quantities are the coordinates for point \( Q \). Or in matrix form:

\[
\begin{bmatrix}
L_Q \\
x_Q
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ T' & 1 \end{bmatrix} \begin{bmatrix} L' \\ x' \end{bmatrix} \quad \text{and} \quad
\begin{bmatrix}
M_Q \\
y_Q
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ T' & 1 \end{bmatrix} \begin{bmatrix} M' \\ y' \end{bmatrix}.
\]

(2.5)

In the following pages we will omit the matrices for \( M \) and \( y \) because they are analogous to those of \( L \) and \( x \).

(1) In some cases the transformation of \( N \) is useful. This transformation is given by:

\[
N' = N + A_1(r - z).
\]
If in a given optical system for a particular ray all the values of $A_i$ and $T'$ are known, we can form all the necessary matrices. By multiplying these in the right sequence we find the matrix describing the transformation of the ray coordinates by the whole system. In general this matrix can be written in the following form:

$$\begin{bmatrix}
B & -A \\
-D & C
\end{bmatrix} \quad (2.6)$$

This matrix is the product of matrices of the form (2.3) and (2.5), each of which has a value $+1$ if computed as a determinant. For this reason the following relation holds for (2.6):

$$BC = AD = +1 \quad (2.7)$$

To find formulae to calculate the quantities $A$, $B$, $C$ and $D$ we assume that these are known for the first $j$ surfaces of the system, denoted by $A_{1,j}$, $B_{1,j}$, $C_{1,j}$ and $D_{1,j}$. (A quantity with two indices like $A_{i,j}$ indicates that this quantity belongs to that part of the system consisting of the surfaces $i$, $i+1$, $\ldots$, $j-1$, $j$. A quantity with one index like $A_j$ belongs only to the $j^{th}$ surface). If we now add the $(j+1)^{st}$ surface, supposing $A_{j+1}$ and $T_j$ are known, we find for the system consisting of the first $j+1$ surfaces the following matrix:

$$\begin{bmatrix}
1 & -A_{j+1} \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
T_j & 1
\end{bmatrix} \begin{bmatrix}
B_{1,j} & -A_{1,j} \\
-D_{1,j} & C_{1,j}
\end{bmatrix} =$$

$$\begin{bmatrix}
B_{1,j} + A_{j+1}(D_{1,j} - T_jB_{1,j}) & -A_{1,j} + A_{j+1}(C_{1,j} - T_jA_{1,j}) \\
-(D_{1,j} - T_jB_{1,j}) & C_{1,j} + T_jA_{1,j}
\end{bmatrix} =$$
Thus we find the following relations:

\[ \begin{align*}
C_{1,j+1} &= C_{1,j} - T_j A_{1,j} \\
A_{1,j+1} &= A_{1,j} + A_{j+1}^* C_{1,j+1} \\
D_{1,j+1} &= D_{1,j} - T_j B_{1,j} \\
B_{1,j+1} &= B_{1,j} + A_{j+1}^* D_{1,j+1}
\end{align*} \]

Equation (2.3) shows that the quantities for the first surface are:

\[ B_1 = C_1 = +1 \quad ; \quad D_1 = 0 \] (2.9)

The relations (2.8) and (2.9) are sufficient to calculate \( A, B, C \) and \( D \) for the whole system if the individual \( A_j \) and \( T_j \) are known. T. Smith (2) gives algebraic formulae to calculate these. However they are easily calculated from any ray tracing data.

Suppose we know the quantities \( A, B, C \) and \( D \) for a system and the ray with direction cosines \( L_1 \) and \( M_1 \) and incidence point \( x_1, y_1 \) on the first surface. The direction cosines \( L'_e, M'_e \) and the coordinates of the emerging point \( x'_e, y'_e \) on the last surface can now be calculated. Take a point \( P(x,y) \) on the incident ray, a distance \( u \) from the point of incidence on the first surface. If we call \( U = u/\mu \), we find the ray coordinates with respect to \( P \) \((L,M,x,y)\) from the matrix:

with the same equation for M and y.

Taking also a point P' on the emerging ray a distance $U' = u'/u$ from the emerging point on the last surface, we find the following relation for the ray coordinates $L'$, $M'$, $x'$ and $y'$ with respect to $P'$:

\[
\begin{bmatrix}
L' \\
x'
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -U' & 1 \end{bmatrix} \begin{bmatrix} L \\ x \end{bmatrix},
\]

and the same matrix for $M'$ and $y'$.

For the transformation from $L$ and $x$ into $L'$ and $x'$ and $M$ and $y$ into $M'$ and $y'$ we get:

\[
\begin{bmatrix} 1 & 0 \\ U' & 1 \end{bmatrix} \begin{bmatrix} B & -A \\ -D & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -U & 1 \end{bmatrix} =
\begin{bmatrix} B + UA & -A \\ U'UA + U'B - UC - D & C - U'A \end{bmatrix}.
\]

The length of $U$ and $U'$ up to now can have any value. If we now assume $P'$ to be in the same axial plane as $P$ (Herzberger's diapoints), the following relation should hold:

\[
x'/x = y'/y = \beta',
\]

where $\beta'$ is a quantity corresponding to the transverse magnification in paraxial imagery (cf. Chapter III).

Combining (2.12) and (2.13) we find:

\[
U'UA + U'B - UC - D = 0
\]
Using the equations (2.7), (2.12) and (2.13) we find:

\[ \beta' = C - U'A, \quad (2.15) \]
\[ 1/\beta' = B + UA, \]

and the matrix (2.12) reduces to:

\[
\begin{bmatrix}
1/\beta' & -A \\
0 & \beta'
\end{bmatrix}
\quad (2.16)
\]

For a lens without aberrations it is clear that \( \beta' \) should be a constant for all rays emerging from a given point \( P \). When this is the case for any given point, but \( \beta' \) has different values for different object points, the system is afflicted with distortion. The fact that \( \beta' \) is a constant however is not sufficient for the system to be free of aberrations. All image points should also have the same \( z \) coordinate.

From the condition that \( \beta' \) is a constant Abbe's sine condition follows directly. If we take a point on axis as an object point, equation (2.16) reads:

\[ L'/L = 1/\beta', \]

which reduces to the familiar form if we remember that every ray containing a point on the axis is a meridional ray and \( L \) and \( L' \) can thus be replaced by the sines of the angles between the rays and the \( z \)-axis multiplied by the refractive indices, if the \( x \)-axis is chosen in this meridional plane (i.e. \( M \) and \( M' \) equal to 0).

It is interesting to analyze the formation of images by one surface. Equation (2.14) reduces in this case to:

\[ U'UA' + U' - U = 0, \]
or:

\[ \frac{1}{U'} = \frac{1}{U} + A_1 \]  \hspace{1cm} (2.17)

By taking a ray in a meridional plane and one just outside this plane, the same matrix is valid for both rays if we neglect second order quantities and (2.17) then gives the sagittal focal point on this ray.

To find the meridional focal point we have to alter \( L \) by an amount \( dL \) and find the intersection of the two rays. Starting from equation (2.3) we find, after replacing \( L \) and \( L' \) again by the sines of the angles between the ray and the axis:

\[ \mu' \sin \alpha' = \mu \sin \alpha - A_1 x \]

After differentiation we have:

\[ \mu' \cos \alpha' \, d\alpha' = \mu \cos \alpha \, d\alpha - A_1 \, dx - x \, dA_1 \]

If we now introduce the quantities \( \delta \) and \( \delta' \) (see figure 2.2) which are the lengths of the perpendiculars from the incidence point of the ray with angle \( \alpha \) on the rays with angles \( \alpha + d\alpha \) and \( \alpha' + d\alpha' \), a simple calculation shows that we have the following transformation:

\[
\begin{bmatrix}
-\mu' \, d\alpha' \\
\delta'
\end{bmatrix} = \begin{bmatrix}
\cos \varphi & -\frac{A_1}{\cos \varphi' \cos \varphi} \\
0 & \frac{\cos \varphi'}{\cos \varphi}
\end{bmatrix} \begin{bmatrix}
-\mu \, d\alpha \\
\delta
\end{bmatrix} \hspace{1cm} (2.18)
\]

The transformation of these quantities to the next surface is governed by the same matrix (2.5) as in our general ray tracing case.

The two rays considered intersect at a distance \( V \) from the first surface in the object space and \( V' \) from the last surface in the image space. The following relation holds between these quantities:

\[ d\alpha = \delta/V \hspace{1cm} ; \hspace{1cm} d\alpha' = \delta'/V' \hspace{1cm} (2.19) \]
For a single surface we find now by introducing (2.19) in the first equation of (2.18):

\[ \frac{\mu' \cos^2 \varphi'}{V'} = \frac{\mu \cos^2 \varphi}{V} + \Lambda_1. \quad (2.20) \]

(2.17) and (2.20) are the well known equations given by Young for the sagital and meridional focal lines of a narrow pencil of rays.

\[ \text{Figure 2.2} \]
Chapter III.

Paraxial Optics.

The formulae derived in the last chapter completely describe the behaviour of a ray. To gain insight into the optical system as a whole, we have to derive a set of equations describing the transformation of any ray passing through the system. Because these equations are very cumbersome, we develop them into a power series. If only the first term of this series is taken into account, we arrive at the, so called, paraxial laws of optics.

In this approximation we can replace \( \cos \varphi \) and \( \cos \varphi' \) by 1. The distances measured along rays reduce to the distances measured along the axis:

\[
T' \approx \frac{d'}{\mu'} = d \tag{3.1}
\]

From the equations (2.17) and (2.20) we see that both sagital and meridional image points coincide. For this reason in paraxial optics we have only to consider meridional rays. Now we can replace \( L \) and \( L' \) by \( \mu \sin \alpha \) and \( \mu' \sin \alpha' \), where \( \alpha \) and \( \alpha' \) are again the angles the rays make with the axis. In our approximation this reduces further to:

\[
L \approx \mu \alpha = \alpha \tag{3.2}
\]

\[
L' \approx \mu' \alpha' = \alpha' \tag{3.2}
\]

The \( x \) and \( x' \) coordinates in this approximation are equal to the distances between the axis and the intersection points of the ray with a plane through the vertex of the refracting surfaces:

\[
x \approx h \tag{3.3}
\]

\[
x' \approx h' \tag{3.3}
\]

The power \( A_i \) of a surface reduces to:
We can use all the equations given in Chapter II if we make the substitutions given above. We will write the constants A, B, C and D with small letters a, b, c and d to distinguish between the real and approximate cases. Thus equation (2.6) becomes:

\[
\begin{bmatrix} a' \\ b' \\ c' \\ d' \end{bmatrix} = \begin{bmatrix} b & -a \\ -d & c \end{bmatrix} \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}.
\]

The quantities a, b, c and d are called the Gaussian constants of the system and are calculated the same way as shown by (2.8), replacing the capital letters with small ones.

The equations for the position of object and image are derived from (2.14) by replacing \(U\) and \(U'\) by \(L\) and \(L'\), which are the distances from the object plane and image plane (the planes perpendicular to the axis going through object and image point) to the first and last refracting surface respectively and measured along the axis:

\[L'L'a + L'b - ac - d = 0.\]

The principal planes are the planes with magnification +1. We find from (2.15):

\[
\begin{align*}
\frac{L'}{H'} &= \frac{c - 1}{a}, \\
\frac{L}{H} &= \frac{1 - b}{a}.
\end{align*}
\]

If we measure the distances \(s\) and \(s'\) along the axis from object and image plane to the principal points (these are the intersection points of the principal planes with the axis) we find a very convenient image equation. We write for \(L\) and \(L'\):
Introducing these in the corresponding equation (2.12) we find:

\[
\begin{bmatrix}
    b + \xi a \\
    0 \\
    c - \xi' a
\end{bmatrix}
= \begin{bmatrix}
    1 + s a & -a \\
    0 & 1 - s' a
\end{bmatrix}.
\]

Using the condition (2.7) we find:

\[
\frac{1}{s'} = \frac{1}{s} + a .
\] (3.8)

The focal points are found by either making \( \beta^i = 0 \) or \( \beta^i = \infty \) in equation (2.15). Now we find for the distance between last vertex and image focal point and for the distance between first vertex and object focal point:

\[
\xi' = \frac{c}{a} ,
\]

\[
\xi = \frac{-b}{a} .
\] (3.9)

The focal lengths of a system are defined as the distance between a principal point and the corresponding focus. By making either \( s \) or \( s' \) in (3.8) infinitely large we find the two reduced focal distances:

\[
\xi'' = \frac{1}{a} ;
\]

\[
\xi = -\frac{1}{a} .
\] (3.10)

The paraxial equations derived above are the ones normally used. In the third order aberration theory it is convenient to use some additional less well known relations, which we will derive now.

In order to keep the form of the transformations as clear as possible, "reduced" coordinates were used. Up till now we multiplied the angles by the appropriate refractive index, and the distances along the axis were divided
by the appropriate refractive index, the distances perpendicular to the axis remained unchanged and were measured from the vertex of the refracting surfaces.

There is however another very useful set of variables which has the following significance. Here angles remain unchanged; distances along the axis are multiplied by the appropriate refractive index and measured from the center of curvature of the refracting surface; and the distances perpendicular to the axis are multiplied by the refractive index. We will distinguish reduced quantities in this system by overstripping them, e.g. \( \bar{h} \).

The transformation to our new variables is given by:

\[
\begin{bmatrix}
\alpha \\
h
\end{bmatrix} =
\begin{bmatrix}
\mu & 0 \\
0 & 1/\mu
\end{bmatrix}
\begin{bmatrix}
\alpha' \\
h'
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\alpha' \\
h'
\end{bmatrix} =
\begin{bmatrix}
1/\mu' & 0 \\
0 & \mu'
\end{bmatrix}
\begin{bmatrix}
\alpha \\
h
\end{bmatrix}.
\]

The matrix for the refraction at one surface becomes now:

\[
\begin{bmatrix}
1/\mu' & 0 \\
0 & \mu'
\end{bmatrix}
\begin{bmatrix}
1 & -a_i \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\mu & 0 \\
0 & 1/\mu
\end{bmatrix}
= \begin{bmatrix}
\mu/\mu' & -a_i/\mu \\
0 & \mu'/\mu
\end{bmatrix}.
\]

The transformation from one surface to another reads:

\[
\begin{bmatrix}
1/\mu' & 0 \\
0 & \mu'
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
\mu'' & 0
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
\mu'\bar{d}'' & 1
\end{bmatrix},
\]

in which, according to the new rule of reduction, we have:

\[
\bar{d}'' = \mu d'' = \mu^2 \bar{d}''.
\]

To convert completely to the new system, however, we should use the centers of curvature as reference points. By using (3.13), in which \( \bar{d}' = \bar{r} \), (3.12)
The elements of the principal diagonal reduce to 1 and:

\[ J_1 = \frac{a_1}{\mu_1}. \]  

(3.16)

If \( q' \) is the distance between two successive centers of curvature, equation (3.13) becomes:

\[
\begin{bmatrix}
1 & 0 \\
q' & 1 \\
\end{bmatrix}
\]  

(3.17)

In exactly the same manner as before, we find for the transformation of a complete optical system:

\[
\begin{bmatrix}
\alpha' \\
\beta' \\
\end{bmatrix} = \begin{bmatrix}
\bar{b} & -J \\
-d & c \\
\end{bmatrix} \begin{bmatrix}
\alpha \\
\bar{h} \\
\end{bmatrix}.
\]  

(3.18)

Again we take two points at distances \( \bar{z} \) and \( \bar{z}' \) from the first and last centers of curvature. Applying the condition that the two points are conjugate, we find the following transformation:

\[
\begin{bmatrix}
\bar{b} + \bar{z}J \\
\bar{z}' \bar{b} + \bar{z}' \bar{b} - \bar{z}c - \bar{a} \\
\end{bmatrix} = \begin{bmatrix}
1/G & -J \\
0 & G \\
\end{bmatrix}.
\]  

(3.19)

In order to investigate the relations between \( \beta' \) and \( G \), and between \( A \) and \( J \) we transform the original matrix into the one for the new variables:
Thus we find for a whole system:

\[ G = \beta' \mu' / \mu \]

\[ J = A / \mu \mu' \]  

(3.21)

Suppose we have an optical system with an object and image plane given by equation (3.20). We can now select a new set of conjugate planes, the object plane being at a distance \( \delta \) from the original object plane, the image plane being at a distance \( \delta' \) from the original image plane. If the new planes have a magnification \( S \) and \( \bar{p} \) is used to indicate \( \bar{h} \) for the planes with the new magnification, we find:

\[
\begin{bmatrix}
\alpha' \\
\bar{p}'
\end{bmatrix} = \begin{bmatrix}
\delta J + 1/G & -J \\
0 & G - \delta' J
\end{bmatrix} \begin{bmatrix}
\alpha \\
\bar{p}
\end{bmatrix},
\]

which is also equal to:

\[
\begin{bmatrix}
1/S & -J \\
0 & S
\end{bmatrix} \begin{bmatrix}
\alpha \\
\bar{p}
\end{bmatrix}.
\]

Thus:

\[
\delta = -\frac{S - G}{SGJ}, \]

\[
\delta' = -\frac{S - G}{J}.
\]

(3.22)

For two optical systems A and B we find:
If we have two optical systems A and B and two object planes imaged through these two systems, we find according to equation (3.22):

\[
\bar{\delta}_A = -\frac{S_A - G_A}{S_A G_A J_A} ; \quad \bar{\delta}'_A = -\frac{S_A - G_A}{J_A} ;
\]

\[
\bar{\delta}_B = -\frac{S_B - G_B}{S_B G_B J_B} ; \quad \bar{\delta}'_B = -\frac{S_B - G_B}{J_B} ;
\]

\[
\bar{\delta}_{AB} = -\frac{S_{AB} - G_{AB}}{S_{AB} G_{AB} J_{AB}} ; \quad \bar{\delta}'_{AB} = -\frac{S_{AB} - G_{AB}}{J_{AB}} ,
\]

and:

\[
\bar{\delta}_B = S_B G_B \bar{\delta}_B = S_B G_B \bar{\delta}_A = -S_B G_B \frac{S_A - G_A}{J_A} = \bar{\delta}_A = -\frac{S_{AB} - G_{AB}}{J_{AB}} ,
\]

or:

\[
S_B G_B \frac{S_A - G_A}{J_A} = \frac{S_{AB} - G_{AB}}{J_{AB}} . \quad (3.24)
\]
<table>
<thead>
<tr>
<th>Quantity</th>
<th>$A$ - system.</th>
<th>$J$ - system.</th>
<th>Relation between the two systems.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$, angle.</td>
<td>$\alpha = \mu \alpha$.</td>
<td>$\alpha$.</td>
<td>$\alpha = \mu \alpha$.</td>
</tr>
<tr>
<td>$h$, distance from axis.</td>
<td>$h$.</td>
<td>$h = \mu h$.</td>
<td>$h = \frac{h}{\mu}$.</td>
</tr>
<tr>
<td>$l$, distance along axis.</td>
<td>$l = \hat{l}/\mu$.</td>
<td>$l = \mu \hat{l}$.</td>
<td>$l = \frac{l}{\mu^2}$.</td>
</tr>
<tr>
<td>Magnification.</td>
<td>$\beta^* = h^*/h$.</td>
<td>$G = \bar{E}^*/\bar{E}$.</td>
<td>$\beta^* = \mu G/\mu^*$.</td>
</tr>
</tbody>
</table>
Chapter IV.
Small Decentrations in Optical Systems.

In many actual instruments the centers of curvature of the refracting elements may be displaced from the common axis. These displacements are called decenterations. Deformation of the instrument when in use will give rise to similar effects. Decentrations can cause unwanted effects, for instance, bending of a rangefinder will lead to inaccurate readings. For these reasons we will investigate the influence of these decenterations on the image formation, as long as they are small enough to be treated paraxially.

We will use an extra column and row in the paraxial matrix to describe the decenterations. For the centered system we have now:

\[
\begin{bmatrix}
\alpha' \\ h' \\ 1
\end{bmatrix} = \begin{bmatrix}
1/\beta' & -a & 0 \\
0 & \beta' & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\alpha \\ h \\ 1
\end{bmatrix}.
\]

(4.1)

From figure 4.1 we see that the following relations hold if the surface is decentered:

\[
\begin{bmatrix}
\alpha^* \\ h^* \\ 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & +e \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\alpha' \\ h' \\ 1
\end{bmatrix}.
\]

\[
\begin{bmatrix}
\alpha \\ h \\ 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -e \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\alpha^* \\ h^* \\ 1
\end{bmatrix}.
\]

(4.2)

So for a decentered surface:
Figure 4.1
The matrix for a system consisting of \( n \) refracting surfaces can now be written in the form:

\[
\begin{bmatrix}
1/eta^\prime_1, n & -a_{1, n} & k_{1, n} \\
0 & \beta^\prime_{1, n} & -f_{1, n} \\
0 & 0 & 1
\end{bmatrix}
\]

\[= \begin{bmatrix}
1/eta^\prime & -a & ea \\
0 & \beta^\prime & -e(\beta^\prime - 1) \\
0 & 0 & 1
\end{bmatrix} \left[\begin{array}{c}
a^* \\
h^*
\end{array}\right]. \tag{4.3}
\]

in which:

\[
\beta^\prime_{1, n} = \beta^\prime_1 \cdot \beta^\prime_2 \cdots \beta^\prime_1 \cdots \beta^\prime_n,
\]

\[
a_{1, n} = \sum_{i=1}^{n} \left( \beta^\prime_{1, n-i} a_{n+1-i} / \beta^\prime_{n+2-i, n} \right),
\]

\[
k_{1, n} = \sum_{i=1}^{n} a_{i+1, n} (e_i - e_{i+1}),
\]

\[
f_{1, n} = \sum_{i=1}^{n} \beta^\prime_{i+1, n} (e_i - e_{i+1}),
\]

where:

\[
\beta^\prime_{\ell, p} = 1 \quad \text{if} \quad \ell > p,
\]

\[
a_{\ell, p} = 0 \quad \text{if} \quad \ell > p,
\]

\[
e_0 = 0,
\]

\[
e_p = 0 \quad \text{if} \quad p > n.
\]

If the decentration is due only to construction errors, and for this
reason a constant condition of the instrument, we can define new z-axes in object and image space, parallel to the original one, in such a way that the transformation of the rays is described by the same matrix as for the perfectly centered instrument. If we take a z-axis in object space, translated a distance $k_{1,n}/a_{1,n}$ and in the image space a $z'$-axis translated over a distance $(\beta'_{1,n} k_{1,n}/a_{1,n} - a_{1,n} f_{1,n})/a_{1,n}$ we find:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -\frac{\beta'_{1,n} k_{1,n}/a_{1,n} - a_{1,n} f_{1,n}}{a_{1,n}} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1/eta'_{1,n} & -a_{1,n} & k_{1,n} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Now we can define principal points on these axes, in the usual way, getting exactly the same results as for the centered system.

It is clear that we could have used the Gaussian constants or the $J$ formulae to describe the decentered systems by starting from the appropriate matrix. The reason for using the magnification is the fact that these equations are the easiest to use when we want to design a system without image motion if the instrument is bending when in use. In this case it is easier to use the decenteration of a lens instead of a surface. It is easily seen that the elements of matrix (4.4) can be regarded as surface constants or as lens constants.

If one of the lenses is a field lens with magnification +1, we can
show that the decentration of this lens does not affect the position of the image. Suppose the \( i^{th} \) lens has a magnification +1. Now we find:

\[
\begin{bmatrix}
\frac{1}{\beta_{i+1,n}} & -a_{i+1,n} & k_{i+1,n} \\
0 & \beta_{i+1,n} & -f_{i+1,n}
\end{bmatrix}
\begin{bmatrix}
1 & -e_1 & e_1 a_1 \\
0 & 1 & 0
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{\beta_{i,n}} & -a_{i,n} & k_{i,n} \\
0 & \beta_{i,n} & \left(-\frac{1}{f_{i+1,n}} f_{i,n} + f_{i+1,n}\right)
\end{bmatrix} . \quad (4.7)
\]

The equation for \( h' \) is thus independent of \( e_1 \).

If the object is at infinity we have \( \beta_{1,n} = \beta_{i,n} = 0 \) and we cannot use the above formulae. However, if we start with the paraxial form of equation (2.3) we find:

\[
\begin{bmatrix}
a' \\
h'
\end{bmatrix} = \begin{bmatrix}
1 & -a_1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
a \\
h
\end{bmatrix} ,
\]

or for the decentered system:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & e_1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & -a_1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{a_1} & e_1 a_1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} . \quad (4.8)
\]
The values for $a_{1}^{*}$ and $h^{*}$ in the focal plane of the first lens are given by:

\[
\begin{bmatrix}
1 & 0 & 0 \\
1/a_{1} & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & -a_{1} & e_{1}a_{1} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
1 & -a_{1} & e_{1}a_{1} \\
1/a_{1} & 0 & e_{1} \\
0 & 0 & 1
\end{bmatrix}.
\]

By using (4.9) as the matrix for the first surface we find the image coordinates in terms of the reduced angle of the incident ray and the height of the incident ray on the first surface.

Bibliography:

Compensation of Flexure in Range Finders and Sighting Instruments.


Compensation of Flexure in Instruments with any Number of Lenses.

Chapter V.

Third Order Aberrations.

In order to obtain insight into the transformation of the ray coordinates, when a ray is refracted by a spherical surface, the equations were developed into a power series. By taking the linear terms only, we arrived at the paraxial optics. If more terms of this series are taken into account, provided that the series is convergent, we will find a better approximation to the real transformation. In the paraxial form all rays emerging from a given object point were going through the image point. The higher order terms will now describe the deviations of the ray from the path described by the paraxial terms.

In order to write the higher order terms in matrix form we have to use matrices with more rows and columns than in the paraxial case. Later on it will be shown that this power series does not have quadratic terms. In general a power series of two variables $x$ and $y$ without quadratic terms, of the form we will use to describe the third order aberrations, is given by:

$$\begin{align*}
    x' &= a_1 x + a_2 y + a_3 x^2 + a_4 x y + a_5 y^2 + a_6 x^3, \\
    y' &= b_1 x + b_2 y + b_3 x^2 + b_4 x y + b_5 y^2 + b_6 x^3.
\end{align*}$$

Using the matrix notation, this can be written in the following form:

$$\begin{bmatrix}
    x' \\
    y' \\
    x'^2 \\
    x'y' \\
    x'^3 \\
    y'^3
\end{bmatrix} =
\begin{bmatrix}
    a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\
    b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\
    0 & 0 & a_1^2 & 3a_1a_2 & 3a_1a_2^2 & a_1^2a_2 \\
    0 & 0 & a_1a_2 & a_1(2a_2b_1+a_1b_2) & a_2(a_2b_1+2a_1b_2) & a_2^2b_1 \\
    0 & 0 & a_1b_1 & b_1(2a_1b_2+a_2b_1) & b_2(a_1b_2+2a_2b_1) & a_1b_2^2 \\
    0 & 0 & b_1^2 & 3b_1b_2 & 3b_1b_2^2 & b_1^2b_2 \\
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    x^2 \\
    x'y \\
    x'^3 \\
    y'^3
\end{bmatrix}$$ (5.1)
A matrix of the form (5.1) might represent the transformation for one surface. To find the transformation for an optical system consisting of more surfaces, we have to multiply a number of matrices of the form (5.1) together. This will give a large number of sums of cross-products, and the calculation of the elements of the matrix representing the transformation for an optical system will be complicated.

Examination of the matrix (5.1), however, shows that it would be greatly simplified if we could find variables in which the coefficients $a_2$ and $b_1$ were equal to zero. This would give a matrix of the form:

\[
\begin{bmatrix}
a_1 & 0 & a_3 & a_4 & a_5 & a_6 \\
0 & b_2 & b_3 & b_4 & b_5 & b_6 \\
0 & 0 & a_1^2 & 0 & 0 & 0 \\
0 & 0 & 0 & a_1^2 b_2 & 0 & 0 \\
0 & 0 & 0 & 0 & a_1 b_2^2 & 0 \\
0 & 0 & 0 & 0 & 0 & b_2^3
\end{bmatrix}.
\] (5.2)

The paraxial part of the transformation is given by:

\[
\begin{bmatrix}
a_1 & 0 \\
0 & b_2
\end{bmatrix}.
\] (5.3)

The paraxial form, closest to (5.3), derived up till this point, is given by (3.19):

\[
\begin{bmatrix}
a' \\
\bar{x}'
\end{bmatrix} = \begin{bmatrix}
1/G & -J \\
0 & G
\end{bmatrix} \begin{bmatrix}
a \\
\bar{x}
\end{bmatrix},
\] (5.4)

where $\bar{x}$ and $\bar{x}'$ are used instead of $\bar{h}$ and $\bar{h}'$.

The equation describing the transformation of the $x$ coordinate is already...
in accordance with (5.3). However, the angular coordinate deviates from this form. This suggests a way to come to an equation of the form (5.3). If we take a second pair of conjugate planes perpendicular to the axis with magnification $S$ and coordinates $\bar{x}_1$ and $\bar{y}_1$, we can completely describe a ray by giving $\bar{x}$, $\bar{y}$ and $\bar{x}_1$ and $\bar{y}_1$. Paraxially, we now have the equations:

$$
\begin{bmatrix}
\bar{x}' \\
\bar{y}'
\end{bmatrix} =
\begin{bmatrix}
G & 0 \\
0 & S
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{x}_1
\end{bmatrix} ,
$$

and:

$$
\begin{bmatrix}
\bar{y}' \\
\bar{y}_1'
\end{bmatrix} =
\begin{bmatrix}
G & 0 \\
0 & S
\end{bmatrix}
\begin{bmatrix}
\bar{y} \\
\bar{y}_1
\end{bmatrix} .
$$

The equations, taking the third order coefficients into account, will, with these variables, have the form:

$$
\begin{align*}
\bar{x}' &= f(x, y, x_1, y_1) , \\
\bar{y}' &= f(x, y, x_1, y_1) , \\
\bar{x}_1' &= f_1(x, y, x_1, y_1) , \\
\bar{y}_1' &= f_1(x, y, x_1, y_1) .
\end{align*}
$$

We will restrict our further discussion to optical systems with an axis of symmetry. Now a change of sign of $\bar{x}$, $\bar{y}$, $\bar{x}_1$ and $\bar{y}_1$, should change the sign of $\bar{x}'$, $\bar{y}'$, $\bar{x}_1'$ and $\bar{y}_1'$. For this reason no quadratic terms are possible in equation (5.6). By replacing the variables by those in polar coordinates, it is easily shown that the equations (5.6) have the following form (compare also chapter VII):
in which:

\[
2\bar{u}_1 = \frac{x^2}{x} + \frac{y^2}{y},
\]

\[
\bar{u}_2 = \frac{xx}{x} + \frac{yy}{y},
\]

\[
2\bar{u}_3 = \frac{x^2}{x} + \frac{y^2}{y}.
\]

The elements \(c_{ij}\) in (5.7) are called the third order aberration coefficients.

For the transformation of \(\bar{y}\) and \(\bar{y}_1\) we have the same equations (5.7) where \(x\) is replaced by \(\bar{y}\) and \(x_1\) by \(\bar{y}_1\). Equation (5.8) is left unchanged.

For a meridional ray (5.7) reduces to:

\[
\begin{bmatrix}
\bar{x}' \\
\bar{x}'_1 \\
\bar{x}'_2 \\
\bar{x}'_3
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & c_{10} & c_{20} & c_{11} & c_{21} & c_{12} & c_{22} \\
0 & S & d_{10} & d_{20} & d_{11} & d_{21} & d_{12} & d_{22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{x}_1 \\
\bar{x}_2 \\
\bar{x}_3
\end{bmatrix} 
\]

\( (5.7) \)

Direct comparison of (5.7) with (5.9) in the case \(\bar{y} = \bar{y}_1 = 0\) gives:
For reasons of simplicity we will use the form (5.9) in the discussions about the third order aberrations and only give the general results at the end of this chapter.

Let us write equation (5.9) in the following symbolic form for the 1st surface:

\[ \overrightarrow{x_1^i} = \overrightarrow{M_1} \cdot \overrightarrow{x_1} \quad (5.11) \]

For the whole system consisting of n surfaces we find:

\[ \overrightarrow{x_n} = \overrightarrow{M_n \cdots M_1} \cdot \overrightarrow{x_1} = \overrightarrow{M_{1,n}} \cdot \overrightarrow{x_1} \quad (5.12) \]

To calculate the elements in \( \overrightarrow{M_{1,n}} \) we have to multiply all the elements of the matrices \( \overrightarrow{M_1} \) with powers of \( G_{1,i} \) and \( S_{1,i} \). If we could reduce the principal diagonal in \( \overrightarrow{M_1} \) to ones, then all the aberrational terms could simply be added. In order to do so we introduce a matrix \( A \) of the same order of \( \overrightarrow{M} \) and let \( A^{-1} \) be the reciprocal matrix of \( A \). We can now write (5.12) in the following form:

\[ \overrightarrow{x_n^i} = \overrightarrow{M_n \cdots M_1} \cdot A_{n-1} \cdot A_{n-2} \cdots A_1 \cdot A_1^{-1} \cdot A_1 \cdot \overrightarrow{x_1} \quad (5.13) \]

To make (5.13) more symmetrical we will define:

\[ \overrightarrow{x_n^{*i}} = A_{n+1} \cdot \overrightarrow{x_n^i} ; \quad \overrightarrow{x_1} = A_1^{-1} \cdot \overrightarrow{x_1^*} \quad (5.14) \]

and (5.13) becomes thus:
\[ \vec{x}_{n}^* = A_{n+1} \cdots M_{n} A_{n-1} \cdots A_{1+1} \cdots M_{1} A_{1-1} \cdots A_{1} \vec{x}_{1} \]
or:
\[ \vec{x}_{n}^* = M_{n}^{*} \cdots M_{1}^{*} \vec{x}_{1} \]
in which:

\[ M_{1}^{*} = A_{1+1} \cdots M_{1} A_{1-1} \]

The problem now is to determine \( A_{1} \) in such a way that the principal diagonal of \( M_{1}^{*} \) consists of only ones. This is easily done. A solution is given by:

\[
A_{1} = \begin{bmatrix}
G_{i,n} & 0 & 0 & 0 & 0 & 0 \\
0 & S_{i,n} & 0 & 0 & 0 & 0 \\
0 & 0 & G_{i,n}^{2} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{i,n} S_{i,n} & 0 & 0 \\
0 & 0 & 0 & 0 & G_{i,n} S_{i,n}^{2} & 0 \\
0 & 0 & 0 & 0 & 0 & S_{i,n}^{3}
\end{bmatrix} .
\]

Introducing this in equation (5.16) we find:

\[
M_{1}^{*} = \begin{bmatrix}
1 & 0 & m_{11}^{*}/2 & m_{21}^{*}/2 & m_{31}^{*}/2 & m_{41}^{*}/2 \\
0 & 1 & n_{11}^{*}/2 & n_{21}^{*}/2 & n_{31}^{*}/2 & n_{41}^{*}/2 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} ,
\]

with:
\[ \overrightarrow{m^*_{1_1}} = (g_{1+1,n_{1_1}}/\alpha_1^3, n_{1_1}), \quad \overrightarrow{n^*_{1_1}} = (s_{1+1,n_{1_1}}/\alpha_1^3, n_{1_1}) , \]
\[ \overrightarrow{m^*_{2_1}} = (g_{1+1,n_{2_1}}/\alpha_1^3, n_{2_1}, s_{1_1}, n_{1_1}) , \quad \overrightarrow{n^*_{2_1}} = (s_{1+1,n_{2_1}}/\alpha_1^3, g_{1_1}^2, s_{1_1}, n_{1_1}) , \]
\[ \overrightarrow{m^*_{3_1}} = (g_{1+1,n_{3_1}}/g_{1_1}, s_{1_1}^2, n_{1_1}) , \quad \overrightarrow{n^*_{3_1}} = (s_{1+1,n_{3_1}}/g_{1_1}, s_{1_1}^2, n_{1_1}) , \]
\[ \overrightarrow{m^*_{4_1}} = (g_{1+1,n_{4_1}}/s_{1_1}^3, n_{1_1}) , \quad \overrightarrow{n^*_{4_1}} = (s_{1+1,n_{4_1}}/s_{1_1}^3, n_{1_1}) . \]

The matrix \( A_{n+1} \) is, with this choice of \( A_1 \), reduced to the unit matrix and thus \( \overrightarrow{x^*_n} = \overrightarrow{x^*_1} \). Thus we find for the whole system:

\[
\overrightarrow{x^*_n} = \overrightarrow{x^*_1} = \begin{bmatrix}
1 & 0 & \sum_{i=1}^n \overrightarrow{m^*_{1_1}}/2 & \sum_{i=1}^n \overrightarrow{m^*_{2_1}}/2 & \sum_{i=1}^n \overrightarrow{m^*_{3_1}}/2 & \sum_{i=1}^n \overrightarrow{m^*_{4_1}}/2 \\
0 & 1 & \sum_{i=1}^n \overrightarrow{n^*_{1_1}}/2 & \sum_{i=1}^n \overrightarrow{n^*_{2_1}}/2 & \sum_{i=1}^n \overrightarrow{n^*_{3_1}}/2 & \sum_{i=1}^n \overrightarrow{n^*_{4_1}}/2 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \]

(5.20)

We now have coefficients \( \overrightarrow{m^*_{j_1}} \) which include magnifications of parts of the system. On first sight the situation does not appear to be much better than the original matrix, because in principal the same magnification factors which we tried to get rid of still exist. This form, however, has the advantage that the aberration coefficients for the whole system are now written down in a simple addition. We can try to find a factor by which we can multiply each term in this summation in such a way, that the calculation is simplest for each of these terms.
In order to arrive at the simplest form we will measure $x_1^*$ in length units equal to the distance between the two reference planes with magnification $G$ and $S$ in the image space. According to (3.22) the absolute value of this length is $(S_{1,n} - G_{1,n})/J_{1,n}$.

If we now define:

$$q = \frac{J_{1,n}}{(S_{1,n} - G_{1,n})},$$

(5.21)

and:

$$\begin{bmatrix}
q & 0 & 0 & 0 & 0 & 0 \\
0 & q & 0 & 0 & 0 & 0 \\
0 & 0 & q^3 & 0 & 0 & 0 \\
0 & 0 & 0 & q^3 & 0 & 0 \\
0 & 0 & 0 & 0 & q^3 & 0 \\
0 & 0 & 0 & 0 & 0 & q^3 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
q \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
$$

we have, using also (5.14) for the variables in object space $x$ and $x_1$:

$$\overrightarrow{x} = B \cdot A_1 \cdot x_1,$$

or:

$$x = \frac{G_{1,n} \cdot J_{1,n}}{(S_{1,n} - G_{1,n})} = \frac{G_{1,n} \cdot J_{1,n} \cdot n \cdot x}{(S_{1,n} - G_{1,n})},$$

$$x_1 = \frac{S_{1,n} \cdot J_{1,n} \cdot n}{(S_{1,n} - G_{1,n})} = \frac{S_{1,n} \cdot J_{1,n} \cdot n \cdot x_1}{(S_{1,n} - G_{1,n})}. \quad (5.22)$$

For the whole system we find with the help of (5.20) and (5.21):

$$\overrightarrow{x_1} = \overrightarrow{M_{1,n}^*} \cdot B^{-1} \cdot \overrightarrow{x} = M_{1,n} \cdot \overrightarrow{x}. \quad (5.23)$$

By defining:
we find by working out equation (5.23):

\[
\begin{align*}
m_1 &= q^{-3} \sum_{i=1}^{n} \bar{m}_{i1}^* \quad n_1 &= q^{-3} \sum_{i=1}^{n} \bar{n}_{i1}^* , \\
m_2 &= q^{-3} \sum_{i=1}^{n} \bar{m}_{21}^* \quad n_2 &= q^{-3} \sum_{i=1}^{n} \bar{n}_{21}^* , \\
m_3 &= q^{-3} \sum_{i=1}^{n} \bar{m}_{31}^* \quad n_3 &= q^{-3} \sum_{i=1}^{n} \bar{n}_{31}^* , \\
m_4 &= q^{-3} \sum_{i=1}^{n} \bar{m}_{41}^* \quad n_4 &= q^{-3} \sum_{i=1}^{n} \bar{n}_{41}^* .
\end{align*}
\]  

(5.25)

To simplify these equations we will make use of the relation (3.24), which reads in our case:

\[
\begin{align*}
(S_{1,n} - G_{1,n})/J_{1,n} &= S_{i+1,n}G_{i+1,n}(S_{1,i} - G_{1,i})/J_{1,i} .
\end{align*}
\]  

According to (3.22) we have:

\[
\begin{align*}
(S_{1,i} - G_{1,i})/J_{1,i} &= -\delta_{1,i} = -\delta_{1} = (S_{1} - G_{1})/J_{1} ,
\end{align*}
\]  

Thus:

\[
\begin{align*}
(S_{1,n} - G_{1,n})/J_{1,n} &= S_{i+1,n}G_{i+1,n}(S_{1} - G_{1})/J_{1} .
\end{align*}
\]  

(5.26)

For (5.25) we find, using (5.19) and (5.26):
\[ m_1 = \sum_1^n m_{11} = \sum_1^n q_1^{-3} g_{1,1} s_{1,1} \bar{m}_{11} / \sigma_1^2, \]
\[ m_2 = \sum_1^n m_{21} = \sum_1^n q_1^{-3} g_{2,1} s_{2,1} \bar{m}_{21} / \sigma_1^2 s_1, \]
\[ m_3 = \sum_1^n m_{31} = \sum_1^n q_1^{-3} g_{3,1} s_{3,1} \bar{m}_{31} / \sigma_1^2 s_1, \]
\[ m_4 = \sum_1^n m_{41} = \sum_1^n q_1^{-3} g_{4,1} \bar{m}_{41} / s_1^3, \]
\[ n_1 = \sum_1^n n_{11} = \sum_1^n q_1^{-3} g_{4,1} \bar{m}_{11} / \sigma_1^2, \]
\[ n_2 = \sum_1^n n_{21} = \sum_1^n q_1^{-3} g_{1,1} s_{1,1} \bar{m}_{21} / \sigma_1^2 s_1, \]
\[ n_3 = \sum_1^n n_{31} = \sum_1^n q_1^{-3} g_{2,1} s_{2,1} \bar{m}_{31} / \sigma_1^2 s_1, \]
\[ n_4 = \sum_1^n n_{41} = \sum_1^n q_1^{-3} g_{3,1} \bar{m}_{41} / s_1^3, \]
\[ (5.27) \]

in which:
\[ q_1^{-1} = (S_1 - G_1)/J_1. \]

We will show later that the factors \( g^{k}_{i+1,n} \) and \( s^{j}_{i+1,n} \), in which \( k \) and \( j \) represent the powers, are easily included in the calculations of the coefficients \( m_{j_1} \).

The coefficients \( n_{j_1} \) were carried along because they represent the aberrations of the aperture stop. When object and stop are moved, these coefficients are needed in the formulae describing the new aberration coefficients in terms of the old ones.

Our final result expresses \( \bar{x}' \) and \( \bar{x}'_1 \) in terms of \( x \) and \( x_1 \). \( x \) and
\(x_1\) are given by the equations (5.22) and the coefficients in these equations by (5.27). The final form of these equations is given by:

\[
\begin{bmatrix}
\bar{x}' \\
\bar{x}_1' \\
\bar{x}_2' \\
\bar{x}_3'
\end{bmatrix}
= \begin{bmatrix}
q^{-1} & 0 & m_1/2 & m_2/2 & m_3/2 & m_4/2 \\
0 & q^{-1} & n_1/2 & n_2/2 & n_3/2 & n_4/2 \\
0 & 0 & q^{-3} & 0 & 0 & 0 \\
0 & 0 & 0 & q^{-3} & 0 & 0 \\
0 & 0 & 0 & 0 & q^{-3} & 0
\end{bmatrix}
\begin{bmatrix}
x \\
x_1 \\
x_2
\end{bmatrix}
\]  
(5.28)

In the general case of a skew ray, we have to go through the same procedures, using matrices with two more rows and columns. It can be shown that if we introduce in (5.7) and (5.8) the variables given by (5.22) we get the following formulae:

\[
c_{10} = \sum_{i=1}^{n} q_{i+1}^{-3} g_{i+1, n}^{3} s_{i+1, n}^{3} \bar{c}_{01i}/g_{i}^{3},
\]

\[
c_{20} = \sum_{i=1}^{n} q_{i+1}^{-3} g_{i+1, n}^{2} s_{i+1, n}^{2} \bar{c}_{20i}/g_{i}^{2} s_{i},
\]

\[
c_{11} = \sum_{i=1}^{n} q_{i+1}^{-3} g_{i+1, n}^{2} s_{i+1, n}^{2} \bar{c}_{11i}/g_{i}^{2} s_{i},
\]

\[
c_{21} = \sum_{i=1}^{n} q_{i+1}^{-3} g_{i+1, n}^{3} s_{i+1, n}^{3} \bar{c}_{21i}/g_{i}^{3} s_{i},
\]

\[
c_{12} = \sum_{i=1}^{n} q_{i+1}^{-3} g_{i+1, n}^{3} s_{i+1, n}^{3} \bar{c}_{12i}/g_{i}^{3} s_{i},
\]

\[
c_{22} = \sum_{i=1}^{n} q_{i+1}^{-4} g_{i+1, n}^{4} \bar{c}_{22i}/s_{i}^{3},
\]

in which again:
\[ q_{i}^{-1} = (s_{i} - g_{i})/J_{i} \]

Now, if we have a combination of optical systems, each represented by its own matrix, it is clear that the same treatment, as given above for the combination of single surfaces, is valid for the combination of optical systems. All the relationships between the matrices, specifying surfaces in a system can be carried over to the matrices specifying the individual systems, forming a composite system.

Of special interest is the combination of the third order aberration coefficients of two optical systems A and B. If the third order aberration coefficients of A are given by \( m_{i}^{A} \) and those of system B by \( m_{i}^{B} \) and the magnifications are also indicated by the subscripts A and B, equation (5.27) reads:

\[ m_{1} = q_{A}^{-3}a_{B}^{3}s_{B}^{3}m_{1}^{A}/s_{A}^{3} + q_{B}^{-3}m_{1}^{B}/s_{B}^{3} \],

\[ m_{2} = q_{A}^{-3}a_{B}^{2}s_{B}^{2}m_{2}^{A}/s_{A}^{2} + q_{B}^{-3}m_{2}^{B}/s_{B}^{2} \],

\[ m_{3} = q_{A}^{-3}a_{B}^{1}s_{B}m_{3}^{A}/s_{A}^{1} + q_{B}^{-3}m_{3}^{B}/s_{B}^{1} \],

\[ m_{4} = q_{A}^{-3}a_{B}^{0}s_{B}m_{4}^{A}/s_{A}^{0} + q_{B}^{-3}m_{4}^{B}/s_{B}^{0} \]

in which \( m_{1} \) is the aberration coefficient for the whole system. With the help of (5.22) this reduces to:

\[ m_{1} = s_{B}^{3}a_{B}^{3}m_{1}^{A} + m_{1}^{B} \],

\[ m_{2} = s_{B}^{2}a_{B}^{2}m_{2}^{A} + m_{2}^{B} \],

\[ m_{3} = s_{B}a_{B}^{1}m_{3}^{A} + m_{3}^{B} \],

\[ m_{4} = a_{B}m_{4}^{A} + m_{4}^{B} \].
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The Changes in Aberrations when object and Stop are moved.

Discussion of the Mathematical Expressions for the Third Order Aberration Coefficients.

The mathematical expressions for the third order coefficients can be brought into many different forms. The most useful form is determined by the problem with which we want to use them. Different problems may require different expressions for the third order coefficients.

In optical design work in general there are two different problems:

a) What forms do we have to give the components of an optical system in order that the system has a minimum of certain aberrations?

b) What are the values of the aberrations for a given optical system?

For the third order aberrations problem a) is solved by T. Smith for thin lens systems by the introduction of the curved unit surfaces. The third order aberration coefficients to solve b) are given in many forms, for example by Seidel, Stewart, Smith and Stephan (See bibliography at the end of chapter V).

We will confine ourselves here to the second problem and then only in so far as third order aberrations are concerned. By doing so, we want the third order aberration coefficients in such a form that the calculations can be performed as rapidly and easily as possible. Unfortunately there are no absolute criteria by which we can know if a given form is the easiest possible. We can only determine if a given form is easier and faster in use than any other given form up to this date. However, there are some general approaches possible, one of which is to investigate the possibilities of factorizing the third order coefficients. It is well known that most aberrations are absent in the aplanatic points. To make use of this fact, we have to find variables, in such a way, that for each
aplanatic point one of these become zero.

The three aplanatic points are:

1) The vertex of the surface. Here:
\[ a - a' = 0 \]  \hspace{1cm} (6.1)

2) The center of curvature of the surface. Here:
\[ a - a' = 0 \]  \hspace{1cm} (6.2)

3) The third set of aplanatic points are at distances \( \mu' r/\mu \) and \( \mu r/\mu' \) from the center of curvature. Here:
\[ (a/\mu) - (a'/\mu') = 0 \]  \hspace{1cm} (6.3)

For this reason it seems profitable to introduce three quantities given by:
\[ a - a' \]  \hspace{1cm} (6.4)
\[ a - a' \]  \hspace{1cm} (6.5)
\[ (a/\mu) - (a'/\mu') \]  \hspace{1cm} (6.6)

It turns out that these quantities are closely related to the quantities \( h, d, \) and \( e \) used by T. Smith in his ray tracing equations (1).

For paraxial rays these quantities reduce to:
\( h \) as defined in equation (3.3),
\[ d = \varphi - \varphi' = a - a' \]  \hspace{1cm} (6.7)
\[ e = a + \varphi' \]  \hspace{1cm} (6.8)

If we trace a ray going through the axial point of the object we find according to (3.20):
\[ a' = a/g \]  \hspace{1cm} (6.9)

For one surface we have according to (3.12):

\[
\begin{bmatrix}
\alpha' \\
-\frac{\alpha'}{h'}
\end{bmatrix} = \begin{bmatrix}
\frac{\mu}{\mu'} & -J \\
0 & \frac{\mu'}{\mu}
\end{bmatrix} \begin{bmatrix}
\alpha \\
-\frac{\alpha}{h}
\end{bmatrix}.
\]

(6.10)

So for the ray going through the axial point of the object we find:

\[
\alpha - \alpha' = \mu \alpha - \mu' \left(\frac{\mu}{\mu'} \alpha - \frac{\mu}{\mu'} \alpha h\right) = \mu \mu' \alpha h,
\]

or:

\[
h = \frac{\alpha - \alpha'}{\mu \mu' J} = - \frac{\mu'}{\mu} \alpha' + \frac{\mu}{\mu} \alpha' \quad \text{(6.11)}
\]

For the same ray we find:

\[
d = \alpha - \alpha' = (\alpha - 1) \alpha', \quad \text{(6.12)}
\]

From figure 6.1 we see that:

\[
\varphi' = \alpha' + \gamma' \quad \text{(6.13)}
\]

Figure 6.1
In the paraxial approximation we have:

$$\gamma' = h R = \mu \mu' J h / (\mu' - \mu)$$  \hspace{1cm} (6.14)

So we find with the help of (6.8), (6.9), (6.11), (6.13) and (6.14):

$$e = \alpha + \alpha' + \mu \mu' J h / (\mu' - \mu) =$$

$$= \alpha + \alpha' - (\mu' - \mu \alpha') / (\mu' - \mu) =$$

$$= \frac{\mu \mu'}{\mu' - \mu} \left( \frac{\alpha}{\mu} - \frac{\alpha'}{\mu'} \right)$$ \hspace{1cm} (6.15)

This quantity shows the relation between $e$ and our third set of aplanatic points. For future calculations we rewrite (6.15) in the following way:

$$e = \frac{\mu \mu'}{\mu' - \mu} \left( \frac{G}{\mu} - \frac{1}{\mu'} \right) \alpha' = \frac{\mu' G - \mu}{\mu' - \mu} \alpha'$$ \hspace{1cm} (6.16)

If we also trace a ray paraxially through the axial point of the $x_1, y_1$ plane, which makes an angle $\chi$ with the axis, we find accordingly:

$$h' = -\frac{\mu' S - \mu S}{\mu \mu' J} \chi'$$ \hspace{1cm} (6.17)

$$d' = (S - 1) \chi'$$ \hspace{1cm} (6.18)

$$e' = \frac{\mu' S - \mu}{\mu' - \mu} \chi'$$ \hspace{1cm} (6.19)

We will now try to express the third order aberration coefficients in terms of $h$, $d$, $e$, $h'$, $d'$ and $e'$.

Only the aberrations, which are completely absent in the aplanatic points can be described completely in terms of the above variables.

For this reason we can expect some difficulties in the terms describing distortion and astigmatism. As shown in the next chapter, these difficulties are easily overcome.
Chapter VII.

The Calculation of the Third Order Aberration Coefficients.

To calculate the third order aberration coefficients for one surface we make use of equation (2.12), part of which reads:

\[ x' = ( U'^A + U' - U ) L + ( 1 - U'^A ) x \]
\[ y' = ( U'^A + U' - U ) M + ( 1 - U'^A ) y \]  \hspace{1cm} (7.1)

This equation uses the variables \( L \) and \( x \). We want to express \( x' \) in the variables \( x \) and \( x_1 \). In order to do this we have to express \( L \) and \( M \) in \( x \) and \( x_1 \).

If \( v \) is the distance, measured along the ray, between the two points \( (x, y, z) \) and \( (x_1, y_1, z_1) \), we have the following relations:

\[ L = \frac{\mu(x_1 - x)}{v} \] : \[ M = \frac{\mu(y_1 - y)}{v} \]  \hspace{1cm} (7.2)

If \( \delta \) is the distance between the \( x,y \) plane and the \( x_1,y_1 \) plane, measured along the \( z \)-axis, we find geometrically the following relationship:

\[ v = \sqrt{\delta^2 + (x_1 - x)^2 + (y_1 - y)^2} = \]
\[ = \delta \left[ 1 + \frac{(x_1 - x)^2 + (y_1 - y)^2}{2\delta^2} \right] + ..... \]

Introducing the variables:

\[ 2 u_1 = x^2 + y^2 \]
\[ u_2 = xx_1 + yy_1 \]
\[ 2 u_2 = x_1^2 + y_1^2 \]  \hspace{1cm} (7.3)

we find:
\[ L = (x_1 - x) \frac{\mu}{\delta} \left[ 1 - \frac{u_1 - u_2 + u_3}{\delta^2} \right], \]  
\[ M = (y_1 - y) \frac{\mu}{\delta} \left[ 1 - \frac{u_1 - u_2 + u_3}{\delta^2} \right]. \]  

To find \( U, U', \) and \( A \) as functions of \( x, y, x_1 \) and \( y_1 \) we will first express \( N, N' \) and \( z_0 \) (the \( z \) coordinate of the point of incidence of the ray on the surface) in these variables.

Now:
\[ L^2 + M^2 + N^2 = \mu^2; \quad L'^2 + M'^2 + N'^2 = \mu'^2, \]

or:
\[ N \approx \mu \left[ 1 - \frac{L^2 + M^2}{2\mu^2} \right], \]  
\[ N' \approx \mu' \left[ 1 - \frac{L'^2 + M'^2}{2\mu'^2} \right]. \]

With the help of (7.4) we find for (7.5):
\[ N = \mu \left[ 1 - \frac{L^2 + M^2}{2\mu^2} \right] \left( u_1 - u_2 + u_3 \right), \]  
\[ (7.5) \]

By introducing equation (3.22) this takes the form:
\[ N = \mu \left[ 1 - \frac{L^2 + M^2}{2\mu^2} \right] \left( u_1 - u_2 + u_3 \right), \]  
\[ (7.6) \]

\[ N = \mu \left[ 1 - \frac{L^2 + M^2}{2\mu^2} \right] \left( u_1 - u_2 + u_3 \right), \]  

In order to calculate \( N' \) we have to find \( L' \) and \( M' \) first. According to equation (2.16) we find for one surface:
\[ L' = (1/\beta') L - A x, \]  
\[ M' = (1/\beta') M - A y, \]  
\[ (7.9) \]
\[ L'1^2 + M'1^2 = (1/\beta')^2(L^2 + M^2) - 2 (A/\beta') \ (Ix + My) + A^2(x^2 + y^2) , \]

or, introducing \( u_1, u_2 \) and \( u_3 \) with the help of (7,4):

\[ L'1^2 + M'1^2 = 2 A^2 u_1 - 2 \frac{\mu}{\delta} A (1/\beta') \ (2u_1 - u_2) + \\
2 (\frac{\mu}{\delta})^2 (1/\beta')^2 (u_1 - u_2 + u_3) \quad (7.10) \]

This shows that we can use the paraxial approximation for \( A \) and \( \beta' \) in this expression and we find:

\[ \frac{L'1^2 + M'1^2}{2} = \frac{\mu^2 u^2_j}{(s - c)^2} \left[ u'^2 u_1 - S u'^2 u_2 + S^2 u_3 \right] \quad (7.11) \]

Using this expression in (7.6) we have:

\[ N' = \mu' \left[ 1 - \frac{\mu^2 j^2}{(s - c)^2} \left( u'^2 u_1 - S u'^2 u_2 + S^2 u_3 \right) \right] \quad (7.12) \]

Now, for a spherical surface, we have the following relation between the coordinates of the point of incidence of the ray with the surface:

\[ z_0 = \frac{x_0^2 + y_0^2}{2 r} \quad (7.13) \]

in which:

\[ x_0 = -UL + x \quad ; \quad y_0 = -UM + y \]

Thus, introducing \( u_1, u_2 \) and \( u_3 \):

\[ x_0^2 + y_0^2 = 2 u_1 + 2 U \frac{\mu}{\delta} (2u_1 - u_2) + 2 U^2 \left( \frac{\mu}{\delta} \right)^2 (u_1 - u_2 + u_3) \quad (7.14) \]

It is again clear that we can use the paraxial approximation for \( U \).
Using equation (2.15) for one surface:

\[ U \approx \frac{1}{A} = \frac{\mu^i - \mu G}{\mu^2 \mu^i G J} \]  \hspace{1cm} (7.15)

With this equation (7.14) becomes:

\[ z_0 = \frac{\mu J}{\mu^i (\mu^i - \mu)(s - g)^2} \left[ a^2 (\mu^i - \mu G)^2 u_1 - \right. \]
\[ \left. - SG (\mu^i - \mu G)(\mu^i - \mu S) u_2 + s^2 (\mu^i - \mu G)^2 u_3 \right] \]  \hspace{1cm} (7.16)

Now \( A \) can be calculated. According to (2.3) we have:

\[ A = \frac{N^i - N}{r - z_0} \]  \hspace{1cm} (7.17)

Combining (7.7) and (7.12) we find:

\[ N^i - N = (\mu^i - \mu) \left( \mu^3 \left( \frac{J^2}{(s - g)^2} - \right) \right) \left[ a^2 (\mu^i - \mu G)^2 u_1 - \right. \]
\[ \left. - SG (\mu^i - \mu G)(\mu^i - \mu S) u_2 + s^2 (\mu^i - \mu G)^2 u_3 \right] \]  \hspace{1cm} (7.18)

Thus with (7.17) and (7.16):

\[ A = \frac{N^i - N}{r - z_0} \approx (N^i - N) \left( \frac{1}{r} + \frac{z_0}{r^2} \right) = \]
\[ \frac{\mu^i - \mu}{r} \left( \frac{\mu^3 \mu^i J^3}{(\mu^i - \mu)(s - g)^2} \left[ a^2 (\mu^i - \mu G)^2 u_1 - \right. \right. \]
\[ \left. - SG (\mu^i - \mu G)(\mu^i - \mu S) u_2 + s^2 (\mu^i - \mu G)^2 u_3 \right] + \]
\[ + \frac{\mu^3 \mu^i J^3}{(\mu^i - \mu)^2 (s - g)^2} \left[ a^2 (\mu^i - \mu G)^2 u_1 - \right. \]
\[ \left. - SG (\mu^i - \mu G)(\mu^i - \mu S) u_2 + s^2 (\mu^i - \mu G)^2 u_3 \right] \]
or:

\[ A = a + \frac{\mu_2^4 \mu_2^2 \mu_3^3}{(\mu^2 - \mu)^2(S - G)^2} \left[ g^2(S - 1)^2 u_1 - \right. \]
\[ \left. - s g (G - 1)(S - 1) u_2 + s^2(G - 1)^2 u_3 \right] \]  \hspace{1cm} (7.19)

or:

\[ A = a + R_A \]  \hspace{1cm} (7.20)

where:

\[ R_A = \frac{\mu_2^4 \mu_2^2 \mu_3^3}{(\mu^2 - \mu)^2(S - G)^2} \left[ g^2(S - 1)^2 u_1 - \right. \]
\[ \left. - s g (G - 1)(S - 1) u_2 + s^2(G - 1)^2 u_3 \right] \]  \hspace{1cm} (7.21)

Now we calculate \( U \). From geometrical considerations it is easily seen that:

\[ U = \frac{t + z_0}{N} \]

Using the equations (7.16), (7.7) and (7.15) we find:

\[ U = \frac{t + J}{(\mu^2 - \mu)(S - G)^2} \left[ g \left\{ s^2(\mu^2 - \mu) - \mu s G (S - 1) - G (\mu^2 - \mu s) \right\} u_1 - \right. \]
\[ \left. \left\{ s g (S - 1)(\mu^2 - \mu G) \right\} u_2 + \left\{ s^2(\mu^2 - \mu G)(G - 1) \right\} u_3 \right] \]  \hspace{1cm} (7.22)

or:

\[ U = \frac{t}{N} + R_U \]  \hspace{1cm} (7.23)

in which:
\[ R_U = \frac{J}{(\mu' - \mu)(S - G)^2} \left[ g \left\{ s^2(\mu' - \mu) - sG(S - 1) - g(\mu' - \mu S) \right\} u_1 - \left\{ sG(S - 1)(\mu' - \mu G) \right\} u_2 + \left\{ s^2(\mu' - \mu G)(G - 1) \right\} u_3 \right]. \] (7.24)

In the same way we find:

\[ U' = \frac{\lambda' - s_o}{N'} = \]

\[ = \lambda' + \frac{\mu^2 J}{\mu^2(\mu' - \mu)(S - G)^2} \left[ g^2 \left\{ \mu'(S - 1) - (\mu' - \mu)G + s(\mu' - \mu s) \right\} u_1 - \left\{ sG(\mu' - \mu G)(S - 1) \right\} u_2 + \left\{ s^2(\mu' - \mu G)(G - 1) \right\} u_3 \right], \] (7.25)

or:

\[ U' = \lambda' + R_U \] (7.26)

in which:

\[ R_U = \frac{\mu^2 J}{\mu^2(\mu' - \mu)(S - G)^2} \left[ g^2 \left\{ \mu'(S - 1) - (\mu' - \mu)G + s(\mu' - \mu s) \right\} u_1 - \left\{ sG(\mu' - \mu G)(S - 1) \right\} u_2 + \left\{ s^2(\mu' - \mu G)(G - 1) \right\} u_3 \right]. \] (7.27)

We can now introduce the equations (7.20), (7.23) and (7.26) into (7.1) and find:

\[ x' = \left[ (\lambda' + R_U)(\lambda + R_A)(a + R_A) + \lambda' + R_U = \lambda - R_U \right] L + \]

\[ + \left[ 1 - (\lambda' + R_U)(a + R_A) \right] x. \]

Using the equations (2.14) and (2.15) reduced to the paraxial forms for one surface we find:

\[ x' = \left[ \lambda' R_A + a\lambda' R_U + a\lambda' R_U = R_U \right] L + \]

\[ + \left[ \beta' - \lambda' R_A - aR_U \right] x. \] (7.28)

Using (7.4) to replace \( L \) with the variables \( x \) and \( x_1 \) we have:
By using (2.15) in its paraxial form for one surface, it is easily seen that the following relations hold:

\[ x' = \beta'x - \left[ \ell'(1 + \frac{\mu}{\delta})R_A + \left\{ a + (a\ell + 1)\frac{\mu}{\delta} \right\} R_{U'} + (a\ell' - 1)\frac{\mu}{\delta}R_U \right] x + \]

\[ + \left[ \ell'^2 \frac{\mu}{\delta} R_A + (a\ell + 1)\frac{\mu}{\delta} R_{U'} + (a\ell' - 1)\frac{\mu}{\delta} R_U \right] x_1. \]  

(7.29)

With these relations (7.29) becomes:

\[ x' = \beta'x + \left[ \frac{(\mu' - \mu \ell)(\mu' - \mu s)G}{\mu' \mu^3 s (s - g) J} R_A + \frac{\mu' \mu' G J}{(s - g)} R_{U'} + \frac{\mu^3 s g^2 J}{\mu' (s - g)} R_U \right] x - \]

\[ - \left[ \frac{(\mu' - \mu \ell)^2 S}{\mu' \mu^3 s (s - g) J} R_A + \frac{\mu' s J}{(s - g)} R_{U'} - \frac{\mu^3 s g^2 J}{\mu' (s - g)} R_U \right] x_1 \]  

(7.30)

Introducing the relations:

\[ R_A = \frac{\mu^4 \mu' \mu^2 J^3}{(\mu' - \mu)^2 (s - g)^2} \]  

(7.31)
\[
R_U = \frac{J}{(\mu' - \mu)(s - g)^2} \cdot R_U, \tag{7.32}
\]
\[
R_{U_1} = \frac{\mu^2 J}{\mu'^2(\mu' - \mu)(s - g)^2} \cdot R_{U_1}, \tag{7.33}
\]

into equation (7.30) we find:

\[
\bar{x}' = G\bar{x} + \frac{\mu^3 J^2 \sigma}{(\mu' - \mu)^2(s - g)^3} \left[ (\mu' - \mu) (\mu' - \mu) x_A + \right.
\]
\[
+ (\mu' - \mu) x_{U_1} - (\mu' - \mu) S \sigma x_U \bigg] \cdot \bar{x} + 
\]
\[
+ \frac{\mu^3 J^2 S}{(\mu' - \mu)^2(s - g)^3} \left[ - (\mu' - \mu) x_A - \right.
\]
\[
- (\mu' - \mu) x_{U_1} + (\mu' - \mu) \sigma^2 x_U \bigg] x_1. \tag{7.34}
\]

If we now compare equation (7.34) with equation (5.7), using the relations (5.8), (7.3), (7.31), (7.32), (7.33), (7.21), (7.24) and (7.29) we find:

\[
\bar{c}_{10} = \frac{G J^2}{(\mu' - \mu)^2(s - g)^3} \left[ (\mu' - \mu) (\mu' - \mu) g^2 (s - 1)^2 + \right.
\]
\[
+ (\mu' - \mu) g^2 \left\{ \mu' (s - 1) - (\mu' - \mu) g + (\mu' - \mu) \right\} - 
\]
\[
- (\mu' - \mu) \sigma G^2 \left\{ s^2 (\mu' - \mu) + \sigma g (s - 1) - G (\mu' - \mu) \right\} \bigg].
\]

\[
\bar{c}_{20} = \frac{S J^2}{(\mu' - \mu)^2(s - g)^3} \left[ - (\mu' - \mu) g^2 (s - 1)^2 - \right.
\]
\[
- (\mu' - \mu) g^2 \left\{ \mu' (s - 1) - (\mu' - \mu) g + s (\mu' - \mu) \right\} + 
\]
\[
+ (\mu' - \mu) g^3 \left\{ s^2 (\mu' - \mu) - \sigma g (s - 1) - g (\mu' - \mu) \right\} \bigg].
\]

\[
\bar{c}_{11} = \frac{G J^2}{(\mu' - \mu)^2(s - g)^3} \left[ - (\mu' - \mu) (\mu' - \mu) \sigma g (s - 1) (s - 1) - \right.
\]
\[
- (\mu' - \mu) \sigma g (s - 1) + (\mu' - \mu) \sigma^2 g^2 (s - 1) \right\} \bigg].
\]
\[ \overline{\sigma}_{21} = \frac{S J^2}{(\mu^i - \mu)^2(s - a)^3} \left[ (\mu^i - \mu)^2s_2(\sigma - 1)(s - 1) + (\mu^i - \mu)s_2(\mu^i - \mu)(s - 1) - (\mu^i - \mu)s_2^2(\mu^i - \mu)(s - 1) \right] \]

\[ \overline{\sigma}_{12} = \frac{S J^2}{(\mu^i - \mu)^2(s - a)^3} \left[ (\mu^i - \mu)^2s_2(\sigma - 1)^2 + (\mu^i - \mu)s_2^2(\mu^i - \mu)(\sigma - 1) - (\mu^i - \mu)s_2^2(\mu^i - \mu)(\sigma - 1) \right] \]

\[ \overline{\sigma}_{22} = \frac{S J^2}{(\mu^i - \mu)^2(s - a)^3} \left[ - (\mu^i - \mu)^2s_2(\sigma - 1)^2 - (\mu^i - \mu)s_2^2(\mu^i - \mu)(\sigma - 1) + (\mu^i - \mu)s_2^2(\mu^i - \mu)(\sigma - 1) \right] \]

Reworking the above equations and introducing the relations (6.11), (6.12), (6.15), (6.17), (6.18), (6.19) and:
\[ P = \frac{\mu \mu^i}{\mu^i - \mu} \quad (7.44) \]

we find:
\[ \overline{\sigma}_{10} = \frac{J^2g_3}{(s - a)^3} \left[ - \frac{\mu^i - \mu s}{\mu \mu^i} \frac{(s - 1)(s - 1)}{J} \frac{\mu^i - s}{\mu^i - \mu} P + \frac{(s - g)}{J} (1 - s^2) \right] = \]
\[ = \frac{g^2J^3}{(s - a)^3} \left[ \frac{h^i}{\chi^i} \frac{d}{\alpha^i} \frac{d^i}{\chi^i} \frac{e^i}{\chi^i} P + \frac{S - g}{J} (1 - s^2) \right] \quad (7.45) \]

\[ \overline{\sigma}_{20} = \frac{g^2S^2j^3}{(s - a)^3} \left[ \frac{\mu^i - \mu g}{\mu \mu^i} (s - 1) \frac{2\mu^i g - \mu}{\mu^i - \mu} P - \frac{(s - g)^2}{J^2} \frac{R}{P} \right] = \]
\[ = \frac{g^2S^2j^3}{(s - a)^3} \left[ - \frac{h^i}{\alpha^i} \frac{d^i}{\chi_{12}^i} \frac{e^i}{\alpha^i} P - \frac{(s - g)^2}{J^2} \frac{R}{P} \right] \quad (7.46) \]
\[ \overline{c}\overline{11} = \frac{g^2 s_j^3}{(s - G)^3} \left[ \frac{\mu^i - \mu G}{\mu^i} \left( s - 1 \right)^2 \frac{\mu^i G - \mu}{\mu^i - \mu} P \right] = \]

\[ = \frac{g^2 s_j^3}{(s - G)^3} \left[ - \frac{h}{\alpha^i} \frac{d^i}{\chi^i} \frac{e}{\alpha^i} P \right] . \quad (7.47) \]

\[ \overline{c}\overline{21} = \frac{g s^2 j^3}{(s - G)^3} \left[ - \frac{\mu^i - \mu G}{\mu^i} \left( G - 1 \right) \left( s - 1 \right) \frac{\mu^i G - \mu}{\mu^i - \mu} P \right] = \]

\[ = \frac{g s^2 j^3}{(s - G)^3} \left[ \frac{h}{\alpha^i} \frac{d}{\alpha^i} \frac{d^i}{\chi^i} \frac{e}{\alpha^i} P \right] . \quad (7.48) \]

\[ \overline{c}\overline{12} = \frac{g s^2 j^3}{(s - G)^3} \left[ - \frac{\mu^i - \mu G}{\mu^i} \left( G - 1 \right) \left( s - 1 \right) \frac{\mu^i G - \mu}{\mu^i - \mu} P \right] = \]

\[ = \frac{g s^2 j^3}{(s - G)^3} \left[ \frac{h}{\alpha^i} \frac{d}{\alpha^i} \frac{d^i}{\chi^i} \frac{e}{\alpha^i} P \right] . \quad (7.49) \]

\[ \overline{c}\overline{22} = \frac{s^3 j^3}{(s - G)^3} \left[ \frac{\mu^i - \mu G}{\mu^i} \left( G - 1 \right)^2 \frac{\mu^i G - \mu}{\mu^i - \mu} P \right] = \]

\[ = \frac{s^3 j^3}{(s - G)^3} \left[ - \frac{h}{\alpha^i} \frac{d^2}{\alpha^i} \frac{e}{\alpha^i} P \right] . \quad (7.50) \]

In the equations above, the coefficients \( c_{1m} \) are given for one surface and should all have an index \( i \) added before they are introduced into (5.29). This gives:

\[ c_{10} = \sum_{l=1}^{n} \left[ h_{1l} e_{1l} P_{1} \frac{g_{1l+1,n} s_{1l+1,n}^3}{\alpha_{1l}^i \chi_{1l}^3} + \right. \]

\[ + \frac{s_{1l} - g_{1l}}{j_{1l}} g_{1l+1,n} s_{1l+1,n}^3 \right] . \quad (7.51) \]

\[ c_{20} = \sum_{l=1}^{n} \left[ - h_{1l} d_{1l} e_{1l} P_{1} \frac{g_{1l+1,n} s_{1l+1,n}^2}{\alpha_{1l}^i \chi_{1l}^2} - \right. \]

\[ - g_{1l+1,n} s_{1l+1,n}^2 \frac{(s_{1l} - g_{1l})^2}{j_{1l}^2} \frac{r_{1l}}{P_{1}} \right] . \quad (7.52) \]
Examination of these results show the frequent occurrence of the terms \( G_{i+1,n} / \alpha'_1 \) and \( S_{i+1,n} / \chi'_1 \). The only restriction so far on the two rays we used to calculate the \( h_1, d_1, e_1, h'_{1}, d'_{1}, \) and \( e'_{1} \) was that they went through the axial points of the \( x,y \) plane and \( x_1,y_1 \) plane. This raises the question whether the \( \alpha_1 \) and \( \chi_1 \) could be chosen in such a way, that the above terms assume a simple form. This can be done by taking:

\[
\alpha_1 = G_{1,n} \quad \text{and} \quad \chi_1 = S_{1,n} \quad . \tag{7.57}
\]

In this case the angles \( \alpha'_1 \) and \( \chi'_1 \), according to (3.20), have the values:

\[
\alpha'_1 = \alpha_1 / G_{1,i} = G_{1,n} / G_{1,i} = G_{i+1,n} \quad ,
\]

and:

\[
\chi'_1 = \chi_1 / S_{1,i} = S_{1,n} / S_{1,i} = S_{i+1,n} \quad ,
\]

and thus:

\[
G_{i+1,n} / \alpha'_1 = S_{i+1,n} / \chi'_1 = +1 \quad . \tag{7.58}
\]
This gives also a good check on the paraxial ray tracing because then, according to (3.20):

\[ \alpha_n' = +1 \quad \text{and} \quad \chi_n' = +1 \]  

(7.59)

The term in S and G in (7.51) can be written:

\[ \sum_{l=1}^{n} \frac{S_l - G_l}{J_l} G_{i+1,n} S_{i+1,n}^3 (1 - S_i^2) = \]

\[ = \sum_{l=1}^{n} \frac{S_l - G_l}{J_l} G_{i+1,n} S_{i+1,n} (S_{i+1,n}^2 - S_{i,n}^2) \]

With the help of (5.26) this is:

\[ \sum_{l=1}^{n} \frac{S_l - G_l}{J_l} G_{i+1,n} S_{i+1,n}^3 (1 - S_i^2) = \]

\[ = \frac{S_l,n - G_{l,n}}{J_l,n} \sum_{l=1}^{n} (S_{i+1,n}^2 - S_{i,n}^2) = \]

\[ = \frac{S_l,n - G_{l,n}}{J_l,n} (1 - S_{i,n}^2) \]  

(7.60)

With the same formula (5.26) we have for the term in (7.52):

\[ G_{i+1,n}^2 \frac{S_{i+1,n}^2}{J_l,n} \left( \frac{S_l - G_l}{J_l} \right)^2 = \left( \frac{S_l,n - G_{l,n}}{J_l,n} \right)^2 \]  

(7.61)

Combining these results, we have the following procedure:

Trace a ray paraxially through the axial point of the object with an angle \( \alpha_1 = G_{l,n} \) with the axis. Trace another ray paraxially through the axial point of the \( x_1,y_1 \) plane with an angle \( \chi_1 = S_{l,n} \) with the
axis. The angles $\alpha_n'$ and $\chi_n'$ should both be $+1$. The aberration coefficients are then calculated with:

$$c_{10} = \sum_{i=1}^{n} h_i d_i d_i' e_i P_i + \frac{S_{1,n} - G_{1,n}}{J_{1,n}} (1 - s_i^2) . \quad (7.62)$$

$$c_{20} = \sum_{i=1}^{n} - h_i d_i^2 e_i P_i - \left( \frac{S_{1,n} - G_{1,n}}{J_{1,n}} \right)^2 \sum_{i=1}^{n} R_i / P_i . \quad (7.63)$$

$$c_{11} = \sum_{i=1}^{n} - h_i d_i^2 e_i P_i . \quad (7.64)$$

$$c_{21} = \sum_{i=1}^{n} h_i d_i d_i' e_i P_i . \quad (7.65)$$

$$c_{12} = \sum_{i=1}^{n} h_i d_i d_i' e_i P_i . \quad (7.66)$$

$$c_{22} = \sum_{i=1}^{n} - h_i d_i^2 e_i P_i . \quad (7.67)$$

The coefficients $c_{lm}$ are the coefficients in the equations:

$$\bar{x}' = g \bar{x} + \frac{c_{10}}{2} (x^2 + y^2)x + \frac{c_{20}}{2} (x^2 + y^2)x_1 +$$

$$+ c_{11}(xx_1 + yy_1)x + c_{21}(xx_1 + yy_1)x_1 +$$

$$+ \frac{c_{12}}{2} (x_1^2 + y_1^2)x + \frac{c_{22}}{2} (x_1^2 + y_1^2)x_1 . \quad (7.68)$$

$$\bar{y}' = g \bar{y} + \frac{c_{10}}{2} (x^2 + y^2)y + \frac{c_{20}}{2} (x^2 + y^2)y_1 +$$

$$+ c_{11}(xx_1 + yy_1)y + c_{21}(xx_1 + yy_1)y_1 +$$

$$+ \frac{c_{12}}{2} (x_1^2 + y_1^2)y + \frac{c_{22}}{2} (x_1^2 + y_1^2)y_1 . \quad (7.69)$$
in which:
\[
\bar{x}' = \mu' x', \quad \bar{y}' = \mu' y', \\
\bar{x} = \mu x, \quad \bar{y} = \mu y, \\
x = \left( G_{1,n} J_{1,n} \mu x \right) / (s_{1,n} - G_{1,n}), \\
x_1 = \left( s_{1,n} J_{1,n} \mu x_1 \right) / (s_{1,n} - G_{1,n}), \\
y = \left( G_{1,n} J_{1,n} \mu y \right) / (s_{1,n} - G_{1,n}), \\
y_1 = \left( s_{1,n} J_{1,n} \mu y_1 \right) / (s_{1,n} - G_{1,n}).
\]

For meridional rays this reduces according to (5.10) and (5.9) to:

\[
m_1 = \sum_{i=1}^{n} h_i d_i d'_i e_i P_i + \frac{s_{1,n} - G_{1,n}}{J_{1,n}} \left( 1 - s_{1,n}^2 \right), (7.80)
\]

\[
m_2 = -3 \sum_{i=1}^{n} h_i d_i^2 d'_i e_i P_i - \left( \frac{s_{1,n} - G_{1,n}}{J_{1,n}} \right)^2 \sum_{i=1}^{n} R_i / P_i, (7.81)
\]

\[
m_3 = 3 \sum_{i=1}^{n} h_i d_i d'_i e_i P_i, (7.82)
\]

\[
m_4 = -\sum_{i=1}^{n} h_i d_i^2 d'_i e_i P_i. (7.83)
\]

The \( m_1 \) are the coefficients in the equation:

\[
\bar{x}' = G \bar{x} + m_1 x^3 / 2 + m_2 x^2 x_1 / 2 + m_3 x x_1^2 / 2 + m_4 x_1^3 / 2, (7.84)
\]

in which the same symbols are used as in equation (7.68).

A special case arises when \( G = 0 \). In this case we still take \( \alpha_1 = 0 \), but \( h_1 \) is not determined. However, \( \alpha_n' \) ought to be \( +1 \). From equation (3.8) follows that in this case:
\( \mu_n s' = a_{1,n} = \mu_{1} J_{1,n} \).

Now \( s' \) is measured from the image unit point. If \( a_{1,n} = +1 \) the intersection height of the emerging ray with this image unit surface is \(-s'\).

From the definition of the unit surfaces it follows that the intersection height of the incident ray with the object unit surface is also \(-s'\). The ray is parallel to the axis and thus is also:

\[
 h_1 = -s' = -1/\mu_1 J_{1,n} \quad (7.90)
\]

By a similar reasoning we find that for \( S = 0 \) the first incident height of the second ray (through the axial point of the \( x_1, y_1 \) plane) ought to be:

\[
 h'_1 = -1/\mu_1 J_{1,n} \quad (7.91)
\]

Since the calculation of the meridional or tangential aberration coefficients takes little time, the above set of equations has proved to be of the utmost practical value.

The meridional or tangential field curvature \( R_m \) and the sagittal curvature \( R_s \) are calculated from these aberration coefficients using the following equations:

\[
 R_m = \frac{\mu J_{1,n}^2}{(S_{1,n} - G_{1,n})^2} m_2 \quad , \quad (7.92)
\]

\[
 R_s = R_m/3 - 2 \mu_1/3 \sum R_i/P_i \quad , \quad (7.93)
\]

where \( R_i \) is the curvature of the \( i \)th refracting surface.
Appendix.

When object and stop are moved, the aberration coefficients change their value. With the help of the eikonal function $T$, Smith derives the relations between the coefficients before and after movement of object and stop (see the last reference at the end of chapter V). These relations are also easily derived with matrix algebra. For completeness we will outline the procedure in matrix theory and derive some specialized cases which are of practical importance.

If the reference surfaces, given by their magnifications $G$ and $S$ (we will omit the subscripts $l,n$ from here on), are moved to a new position, given by their new magnifications $\overline{G}$ and $\overline{S}$, we can write the transformations according to (5.24). These take the forms, for the old position:

$$X'_i = M_{i,n} X$$

and for the new position:

$$\overline{X}'_i = N_{i,n} \overline{X}$$

In order to find $N_{i,n}$ from $M_{i,n}$ we have to know the following transformations:

$$\overline{X}'_i = C X'_i \quad \text{and} \quad X = D \overline{X}$$

It is clear that we then have:

$$\overline{X}'_i = C X'_i = C M_{i,n} X = C M_{i,n} D \overline{X} = N_{i,n} \overline{X}$$

or:

$$N_{i,n} = C M_{i,n} D$$
The matrices C and D are easily found. From figure A.1 we see that:

\[ \tan \alpha = \frac{x_1 - x}{\delta} = \frac{x - \bar{x}}{p} = \frac{x_1 - \bar{x}_1}{p_1} \]

With the help of equations (3.22) and (5.22), using the appropriate values for the magnifications we find:

\[
sg \left[ \frac{x_1}{s} - \frac{x}{g} \right] = \frac{G(s - g) x - g(\bar{g} - g) \bar{x}}{g - \bar{g}} = \frac{\bar{G}(s - g) x_1 - g(\bar{g} - g) \bar{x}_1}{s - \bar{s}} ,
\]

or:

\[
(s - g)x = (\bar{s} - g)\bar{x} - (\bar{g} - g)\bar{x}_1 ,
\]

\[
(s - g)x_1 = (\bar{s} - s)\bar{x} - (\bar{g} - s)\bar{x}_1 . \quad \text{(A.5)}
\]

---

**Figure A.1**
In the same way we find for the image space:

\[
(S-G)x' = (S-G)x' + (G-G)x_1
\]

\[
(S-G)x'_1 = (S-S)x'_1 + (S-G)x_1
\]

Introducing the following quantities:

\[
\epsilon = \frac{G-S}{S-G}\quad \text{and} \quad s = \frac{S-S}{S-G}
\]

we can rewrite (A.5) and (A.6) into:

\[
x = (1+s)x - \epsilon x_1
\]

\[
x'_1 = sx + (1-\epsilon)x_1
\]

\[
\bar{x}' = (1-\epsilon)x'_1 + \epsilon x'_1
\]

\[
\bar{x}'_1 = -sx' + (1+s)x'_1
\]

The paraxial forms of the matrices (A.3) are thus:

\[
C_{\text{par}} = \begin{bmatrix}
1 - \epsilon & \epsilon \\
- s & 1 + s
\end{bmatrix}
\]

\[
D_{\text{par}} = \begin{bmatrix}
1 + s & - \epsilon \\
s & 1 - \epsilon
\end{bmatrix}
\]

These can easily be extended to the full matrices C and D in the way described in chapter V by the formula (5.1), realizing that in this matrix a_2, a_4, a_5, a_6 and b_3, b_4, b_5 and b_6 are equal to zero.

If we now perform the matrix multiplication C.M^D we find that the third order aberration coefficients are given as functions of the
\( m_1, n_1, s \) and \( g \) (see the form of matrix \( M_{1,n} \) given by equation 5.24).

However, up to this point, the coefficients \( n_1 \) are not open to numerical calculation. In order to arrive at numerical expressions for \( n_1 \) we examine the aberrations for a plane with magnification \( s \) and coordinates \( x^* \). If we use as a second reference plane the plane with magnification \( G \) and coordinates \( x_1^* \), we find:

\[
x_1^* = \frac{SJ(G - s)}{J(G - s)} x^* + \frac{m_1^*}{2} \left[ \frac{SJ x^*}{G - s} \right]^3 + \frac{m_2^*}{2} \left[ \frac{SJ x^*}{G - s} \right]^2 \left[ \frac{GJ x_1^*}{G - s} \right] + \frac{m_3^*}{2} \left[ \frac{SJ x^*}{G - s} \right] \left[ \frac{GJ x_1^*}{G - s} \right]^2 + \frac{m_4^*}{2} \left[ \frac{GJ x_1^*}{G - s} \right]^3.
\]

However, this equation describes the same transformation as the equation:

\[
x_1^* = \frac{SJ(S - G)}{J(S - G)} x_1 + \frac{n_1}{2} \left[ \frac{SJ x_1}{S - G} \right]^3 + \frac{n_2}{2} \left[ \frac{SJ x_1}{S - G} \right]^2 \left[ \frac{GJ x}{S - G} \right] + \frac{n_3}{2} \left[ \frac{SJ x_1}{S - G} \right] \left[ \frac{GJ x}{S - G} \right]^2 + \frac{n_4}{2} \left[ \frac{GJ x}{S - G} \right]^3.
\]

But since:

\[
x^* = x_1, \quad x_1^* = x, \quad x_1^{**} = x_1, \quad x_1^{*''} = x_1', \quad (A.9)
\]

we find from the equations above:

\[
m_1^* = -n_4, \quad (A.10)
m_2^* = -n_3, \quad \text{,}
m_3^* = -n_2, \quad \text{,}
m_4^* = -n_1.
\]
The calculation of the coefficients \( m_1^* \) are given by (7.80), (7.81), (7.82) and (7.83) if we interchange \( h_1^, d_1, e_1 \) and \( S \) respectively with \( h_1^, d_1^, e_1^ \) and \( G \).

Thus:

\[
\begin{align*}
    m_1^* &= \sum_{i=1}^{n} h_i^* d_i^* e_i^* P_i + \frac{(G - S)}{J} (1 - G^2), \\
    m_2^* &= -3 \sum_{i=1}^{n} h_i^* d_i^* e_i^* P_i - \frac{(S - G)^2}{J^2} \sum_{i=1}^{n} R_i/P_i, \\
    m_3^* &= 3 \sum_{i=1}^{n} h_i^* d_i^* e_i^* P_i, \\
    m_4^* &= -\sum_{i=1}^{n} h_i^* d_i^2 e_i^* P_i.
\end{align*}
(A.11)
\]

Now combining (A.10), (7.80), (7.81), (7.82), (7.83) and (A.11) we find:

\[
\begin{align*}
    n_1 &= \sum_{i=1}^{n} h_i^* d_i^2 e_i^* P_i, \\
    n_2 &= -3 \sum_{i=1}^{n} h_i^* d_i^* e_i^* P_i = -3 m_1 + 3 \frac{S - G}{J} (1 - S^2), \\
    n_3 &= 3 \sum_{i=1}^{n} h_i^* d_i^* e_i^* P_i + \frac{(S - G)^2}{J^2} \sum_{i=1}^{n} R_i/P_i, \\
    n_4 &= -\sum_{i=1}^{n} h_i^* d_i^* e_i^* P_i - \frac{S - G}{J} (G^2 - 1) = \\
    &= -\frac{m_3}{3} - \frac{S - G}{J} (G^2 - 1).
\end{align*}
(A.12)
\]

The form of \( n_3 \) can be rewritten with the help of equations (6.11), (6.12) and (6.16) as:
\[
\begin{align*}
    n_3 &= -m_2 - 3 \frac{S - G}{J} (SG - 1) \\
    n_1 &= m_o \\
    n_2 &= -3 m_1 + 3 \frac{S - G}{J} (1 - S^2) \\
    n_3 &= -m_2 - 3 \frac{S - G}{J} (SG - 1) \\
    n_4 &= -\frac{1}{3} m_3 - \frac{S - G}{J} (G^2 - 1)
\end{align*}
\]

In order to conform with the notation of T. Smith we will introduce:

Recapitulating we have:

\[
\begin{align*}
    m_o &= n_1 = \sum_{i=1}^{n} h_i d_i d_i^2 e_i^p_i \\
    n_2 &= -3 m_1 + 3 \frac{S - G}{J} (1 - S^2) \\
    n_3 &= -m_2 - 3 \frac{S - G}{J} (SG - 1) \\
    n_4 &= -\frac{1}{3} m_3 - \frac{S - G}{J} (G^2 - 1)
\end{align*}
\]

With the help of (A.4), (A.7) and (A.15) we can calculate the relations between the aberration coefficients before and after the two reference planes are moved.

Of special interest are the cases where either object or stop are moved.

If the object is moved without moving the stop, (A.7) written in matrix form, reduces to:

\[
C = \begin{bmatrix}
1 - g & g & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & (1 - g)^3 & 3(1 - g)^2 g & 3(1 - g) g^2 & g^3 \\
0 & 0 & 0 & (1 - g)^2 & 2(1 - g) g & g^2 \\
0 & 0 & 0 & 0 & (1 - g) & g \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Performing the multiplication $C.M_1.n.D$ and using the relations (A.15)

we find:

\[
\begin{align*}
\mathbf{s} &= 0 \\
\bar{m}_1 &= g m_0 + (1 - g) m_1 \\
\bar{m}_2 &= -3g^2 m_0 - 6g(1 - g) m_1 + (1 - g)^2 m_2 + \frac{3}{J} g (1 - s^2) (s - \tilde{g}) \\
\bar{m}_3 &= 3g^3 m_0 + 9g^2 (1 - g) m_1 - 3g (1 - g)^2 m_2 + (1 - g)^3 m_3 \\
&\quad - 3g \frac{s - \tilde{g}}{J} \left[ (2s^2 - s \tilde{g} - 1) g^2 - 2s (s - \tilde{g}) g + 1 - s \tilde{g} \right], \\
\bar{m}_4 &= -g^4 m_0 + 4g^3 (1 - g) m_1 + 2g^2 (1 - g)^2 m_2 - \frac{4}{J} g (1 - g)^3 m_3 \\
&\quad + (1 - g)^4 m_4 + g^4 \frac{s - \tilde{g}}{J} \left( \frac{s^3}{\tilde{g}} - 1 \right) - \frac{s - \tilde{g}}{J} \left( \frac{s^3}{\tilde{g}} - 1 \right).
\end{align*}
\]

If only the stop is moved ($g = 0$), we find in an analogous manner:

\[
\begin{align*}
\bar{m}_1 &= (1 + s)^3 m_1 + s (1 + s)^2 m_2 + s^2 (1 + s) m_3 + s^3 m_4 \\
\bar{m}_2 &= (1 + s)^2 m_2 + 2s (1 + s) m_3 + 3s^2 m_4 \\
\bar{m}_3 &= (1 + s) m_3 + 3s m_4 \\
\bar{m}_4 &= m_4.
\end{align*}
\]

In the case where $G = 0$ and $S = +1$, which often occurs, the

\textit{equations (A.18) and (A.19) reduce to:}
s = 0 ; G = 0 ; S = + 1.

\[
\begin{align*}
\overline{m}_1 & = \bar{G} m_0 + (1 - \bar{G}) m_1 , \\
\overline{m}_2 & = -3 \bar{G}^2 m_0 - 6\bar{G}(1 - \bar{G}) m_1 + (1 - \bar{G})^2 m_2 , \\
\overline{m}_3 & = 3\bar{G}^3 m_0 + 9\bar{G}^2 (1 - \bar{G}) m_1 - 3\bar{G}(1 - \bar{G})^2 m_2 + \\
& + (1 - \bar{G})^3 m_3 - 3 \frac{\bar{G}(1 - \bar{G})^2}{j} , \\
\overline{m}_4 & = -\bar{G}^4 m_0 - 4 \bar{G}^3 (1 - \bar{G}) m_1 + 2 \bar{G}^2 (1 - \bar{G})^2 m_2 - \\
& - \frac{4}{3} \bar{G}(1 - \bar{G})^3 m_3 + (1 - \bar{G})^4 m_4 + \frac{\bar{G}}{j} (-3 \bar{G}^2 + 2\bar{G}^3 + 1) ,
\end{align*}
\]  
(A.20)

and:

\[
\begin{align*}
g = 0 ; G = 0 ; S = + 1.
\end{align*}
\]

\[
\begin{align*}
\overline{m}_1 & = \bar{S}^2 m_1 + (\bar{S} - 1) \bar{S}^2 m_2 + (\bar{S} - 1)^2 \bar{S} m_3 + (\bar{S} - 1)^3 m_4 , \\
\overline{m}_2 & = \bar{S}^2 m_2 + 2(\bar{S} - 1) \bar{S} m_3 + 3(\bar{S} - 1)^2 m_4 , \\
\overline{m}_3 & = \bar{S} m_3 + 3(\bar{S} - 1) m_4 , \\
\overline{m}_4 & = m_4 ,
\end{align*}
\]  
(A.21)
SAMENVATTING.

In de geometrische optica wordt gebruik gemaakt van z.g. lichtstralen. Een lichtstraal is volkomen bepaald indien de richtingscosinussen en een punt op de straal gegeven zijn t.o.v. drie coördinatenassen. Indien een lichtstraal het scheidingsvlak tussen twee media met verschillende brekkingsindices passeert, ondergaan de coördinaten van de straal een transformatie.

Ondanks het feit, dat in de optiek deze transformatie in het algemeen niet lineair is, blijft de matrixalgebra grote voordelen bieden in het beschrijven van deze transformaties.

In hoofdstuk I worden systemen, bestaande uit vlakke spiegelende oppervlakken, behandeld. Deze systemen werden reeds behandeld met behulp van matrixalgebra door T. Smith. Volledigheidshalve zijn deze lineaire systemen in dit proefschrift opgenomen.

In hoofdstuk II worden systemen behandeld bestaande uit brekende oppervlakken die omwentelingssymmetrie hebben. De transformaties, die een lichtstraal in dergelijke stelsels ondergaat, zijn niet lineair. Het blijkt nu mogelijk te zijn ook hier de transformatie met een matrix te beschrijven, en wel door het invoeren van de scheve sterkte voor de beschouwde straal. Verder moeten afstanden gemeten worden langs de straal.

Indien als beeldpunt op de straal gedefinieerd wordt het doornijdingspunt van de straal met het vlak gaande door het voorwerpss punt en de symmetrieas van het systeem, kan een grootheid \( \beta^1 \) gedefinieerd worden corresponderende met de paraxiale dwarsvergroting.

In een aberratie-vrij stelsel moeten de grootheden \( \beta^1 \) en de scheve sterkte A voor de straal gelijk zijn aan de overeenkomstige paraxiale grootheden. De scheve sterkte hangt op eenvoudige wijze samen met de posities van de meridionale en sagittale brandlijntjes gegeven door de formules van Young.
Indien de grootheden voorkomende in bovengenoemde matrix in een machtreeks ontwikkeld worden en alleen de eerste term van deze reeks in aanmerking genomen wordt, hebben wij de paraxiale benadering. Dit wordt behandeld in hoofdstuk III.

Indien als referentiepunten de kromtemiddelpunten van de brekende oppervlakken genomen worden krijgt men de z.g. J-formules. Deze zijn van belang in de derde-orde aberratieetheorie.

In vele optische instrumenten treden ongewenste effecten op indien door een of andere oorzaak de optische elementen niet langer een gemeenschappelijke symmetrieas hebben. In hoofdstuk IV wordt de positie van het beeld als functie van de posities van de optische elementen beschreven, indien de afwijkingen klein genoeg zijn om met de paraxiale afbeeldingstheorie beschreven te worden.

Wanneer in bovengenoemde machtreeksen behalve de eerste-machtstermen ook de derde-machtstermen gebruikt worden, hebben wij te doen met een quasi-lineaire transformatie die weer m.b.v. een matrix beschreven kan worden. De coefficients van de derde-machtstermen zijn de derde-orde aberratiecoefficiënten. In hoofdstuk V wordt onderzocht door welke keuze van veranderlijken de berekening van deze coefficients kan worden vereenvoudigd.

In de aplanatische punten zijn de meeste aberraties afwezig en de aberratiecoefficiënten hebben daar de waarde nul. Voor andere posities van het voorwerp en het tweede referentievlak moet het dus mogelijk zijn de coefficients te ontbinden in factoren die nul worden in de aplanatische punten. Deze factoren worden ingevoerd in hoofdstuk VI.

Hoofdstuk VII behandelt de toepassing van de theorieën op het berekenen van de derde-orde coefficients voor een sferisch brekend oppervlak. De gevonden formules betekenen een grote tijdsbesparing in de berekening
van de derde-orde aberratiecoefficienten.

Indien de positie van voorwerp of intreepupil veranderd wordt, nemen de derde-orde aberratiecoefficienten nieuwe waarden aan. Door T. Smith zijn m.b.v. de eikonale functie, formules afgeleid die het verband aangeven tussen de aberratiecoefficienten in de nieuwe posities van voorwerp en intreepupil als functies van de coefficienten in de oude posities. In de appendix zijn deze formules afgeleid met behulp van matrixalgebra, voor een aantal speciale gevallen die van practisch belang zijn.
STELLINGEN.

I.

Een doorrekeningsmethode gebaseerd op de optische sterkte voor scheve stralen, geeft een beter inzicht in de correctiemogelijkheden van een optisch stelsel dan doorrekeningsmethoden, die de nadruk leggen op de snelheid van de berekeningen.

Hoofdstuk II van dit proefschrift.

II.

Het berekenen van een aplanatisch doublet of een triplet is eenvoudig, wanneer de oplossing voor een stelsel met dikte nul en zonder derde orde astigmatie en coma gevonden is, daar de correctie, benodigd voor het dikte geven, eenvoudig aan te brengen is.


III.

Het principe van de omgekeerde vlam biedt mogelijkheden tot het nauwkeurig bepalen van het zuurstofgehalte in gasmengsels.

IV.

Het beperken van de vliegtijd voor piloten in de burgerluchtvaart gebaseerd op het aantal vlieguren per tijdseenheid is principieel onjuist.

V.

De door Rank en Cronemeyer gevonden hoge waarde voor de infrarood-transmissie van germanium heeft een eenvoudige fysische verklaring.


VI.

De noodzakelijkheid om de electronische rekenmachine rendabel te maken heeft in vele gevallen een remmende invloed op het creëren van nieuwe optische ontwerpen.

VII.

De matrix schrijfwijze geeft een hulpmiddel om eenvoudig aan te tonen, dat het onmogelijk is duizenden variabelen te vinden voor de coördinaten van een lichtstraal, dat de derde- zowel als de vijfde-orde aberratie-coëfficiënten voor de verschillende oppervlakken van een stelsel additief worden.

Hoofdstuk V van dit proefschrift.

VIII.

De grootste moeilijkheden in de fabricatie van reflectie-tralies liggen niet in het verkrijgen van de precisie in de groefafstanden, doch in het vervaardigen van de juiste groefvorm.

IX.

Bij het ontwerpen van een mechanische trallesnijmachine dient men zeer langzaam op elkaar glijdende delen te vermijden, daar de wrijving bij zeer lage snelheden oorzaak is van de voornaamste moeilijkheden in de bestaande machines.

X.

Het is mogelijk reflectie-tralies te vervaardigen die geen tweede orde spectra hebben.

XI.

Uit het feit, dat slechts een klein gedeelte van de autobestuurders verantwoordelijk is voor het meerendeel der auto ongelukken, kan niet geconcludeerd worden, dat het aantal ongelukken noemenswaardig terughbracht kan worden door hun het rijbewijs te ontnemen.

XII.

Het aangeven van de aberraties van een golffront uitgedrukt in golflengten ten opzichte van een ideaal golffront is onvolledig en verwarrend. 