Master of Science Thesis
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Stress Intensity Factors for Fatigue Crack Growth Analysis

November 2014
“Stress Intensity Factors for Fatigue Crack Growth Analysis”

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November 2014  Delft, the Netherlands
PREFACE

The report “Stress Intensity Factors for Fatigue Crack Growth Analysis” describes the research conducted by Dimitris Chrysafopoulos for the Master of Science degree in Structural Engineering at Delft University of Technology.

The research was executed at Delft University of Technology during the period March 2014 to November 2014.

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I would like to express my gratitude and my thanks to the persons who have supported me during the writing of my thesis. I would like to thank, Prof. Frans Bijlaard, Dr. Henk Kolstein, Dr. Ir. Max Hendriks from Delft University of Technology and Dr. Ir. Erwan Karjadi from Heerema Marine Contractors for their supervision and comments. Special thanks to Ir. Abdulkadir Akyel for his continuous support and comments. Last but not least I would like to thank my family and friends for their mental support and advice.

November 2014, Delft
Dimitris Chrysafopoulos
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<thead>
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<th>Symbol</th>
<th>Definition</th>
<th>Unit</th>
</tr>
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<tbody>
<tr>
<td>A</td>
<td>material constant in fatigue crack growth relationship</td>
<td></td>
</tr>
<tr>
<td>α</td>
<td>half length for through thickness flaw, height for surface flaw or half height for embedded flaw</td>
<td>mm</td>
</tr>
<tr>
<td>(a_0)</td>
<td>initial flaw size</td>
<td>mm</td>
</tr>
<tr>
<td>(a_f)</td>
<td>final flaw size</td>
<td>mm</td>
</tr>
<tr>
<td>(a_N)</td>
<td>flaw size after n cycles</td>
<td>mm</td>
</tr>
<tr>
<td>B</td>
<td>section thickness in plane of flaw</td>
<td>mm</td>
</tr>
<tr>
<td>c</td>
<td>half flaw length for surface or embedded flaws</td>
<td>mm</td>
</tr>
<tr>
<td>(c_f)</td>
<td>final half flaw length for surface or embedded flaws</td>
<td>mm</td>
</tr>
<tr>
<td>(da/dN)</td>
<td>crack growth rate</td>
<td>mm/cycle</td>
</tr>
<tr>
<td>F</td>
<td>function to account for the influence of various boundaries</td>
<td></td>
</tr>
<tr>
<td>(g, f_w, f_o)</td>
<td>correction terms in stress intensity factors for elliptical cracks</td>
<td>N/mm</td>
</tr>
<tr>
<td>(G_{\text{max}})</td>
<td>energy release rate at maximum load</td>
<td>N/mm</td>
</tr>
<tr>
<td>(G_{\text{pl}})</td>
<td>energy release rate upper limit</td>
<td>N/mm</td>
</tr>
<tr>
<td>(G_{\text{thresh}})</td>
<td>threshold energy release rate</td>
<td>N/mm</td>
</tr>
<tr>
<td>(G_{\text{equiv,c}})</td>
<td>critical equivalent energy release rate</td>
<td>N/mm</td>
</tr>
<tr>
<td>H</td>
<td>bending multiplier</td>
<td></td>
</tr>
<tr>
<td>K</td>
<td>stress intensity factor (SIF)</td>
<td>N/mm(^{3/2})</td>
</tr>
<tr>
<td>(K_I)</td>
<td>applied tensile (mode I) stress intensity factor</td>
<td>N/mm(^{3/2})</td>
</tr>
<tr>
<td>(K_{II})</td>
<td>mode II linear elastic stress intensity factor</td>
<td>N/mm(^{3/2})</td>
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<tr>
<td>(K_{III})</td>
<td>mode III linear elastic stress intensity factor</td>
<td>N/mm(^{3/2})</td>
</tr>
<tr>
<td>(K_t)</td>
<td>stress concentration factor (SCF)</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>plate length</td>
<td>mm</td>
</tr>
<tr>
<td>m</td>
<td>material constant exponent in crack, Paris growth law</td>
<td></td>
</tr>
<tr>
<td>(M_b)</td>
<td>stress intensity magnification factor for bending loading</td>
<td></td>
</tr>
<tr>
<td>(M_i)</td>
<td>curve fitting boundary correction factors</td>
<td></td>
</tr>
<tr>
<td>(M_k)</td>
<td>correction factor for welded joints</td>
<td></td>
</tr>
<tr>
<td>(M_m)</td>
<td>stress intensity magnification factor for membrane loading</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>fatigue life</td>
<td>cycles</td>
</tr>
<tr>
<td>(N_c)</td>
<td>number of cycles</td>
<td>cycles</td>
</tr>
<tr>
<td>(N_f)</td>
<td>final number of cycles</td>
<td>cycles</td>
</tr>
<tr>
<td>r</td>
<td>distance from the tip of the crack</td>
<td>mm</td>
</tr>
<tr>
<td>R</td>
<td>stress ratio</td>
<td></td>
</tr>
<tr>
<td>W</td>
<td>plate width in plane of flaw</td>
<td>mm</td>
</tr>
<tr>
<td>Y</td>
<td>stress intensity correction factor</td>
<td></td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
<td>Unit</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
<td>--------------------</td>
</tr>
<tr>
<td>(\beta)</td>
<td>shape factor</td>
<td></td>
</tr>
<tr>
<td>(\Delta c)</td>
<td>increment in (c)</td>
<td>(\text{mm})</td>
</tr>
<tr>
<td>(\Delta G)</td>
<td>relative fracture energy release rate</td>
<td>(\text{N/mm})</td>
</tr>
<tr>
<td>(\Delta K)</td>
<td>stress intensity factor range</td>
<td>(\text{N/mm}^{3/2})</td>
</tr>
<tr>
<td>(\Delta K_{\text{max}})</td>
<td>maximum stress intensity factor range</td>
<td>(\text{N/mm}^{3/2})</td>
</tr>
<tr>
<td>(\Delta K_{\text{th}})</td>
<td>threshold stress intensity factor range</td>
<td>(\text{N/mm}^{3/2})</td>
</tr>
<tr>
<td>(\Delta \alpha_i)</td>
<td>increment in (\alpha)</td>
<td>(\text{mm})</td>
</tr>
<tr>
<td>(\Delta \alpha_{\text{max}})</td>
<td>maximum crack growth increment</td>
<td>(\text{mm})</td>
</tr>
<tr>
<td>(\Delta N)</td>
<td>increment in the number of cycles</td>
<td>(\text{cycles})</td>
</tr>
<tr>
<td>(\Delta \sigma)</td>
<td>applied stress range</td>
<td>(\text{N/mm}^2 \text{ (Mpa)})</td>
</tr>
<tr>
<td>(\theta)</td>
<td>parametric angle to identify position along an elliptical flaw front</td>
<td>(\text{radians})</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>general symbol for applied stress</td>
<td>(\text{N/mm}^2 \text{ (Mpa)})</td>
</tr>
<tr>
<td>(\sigma_y)</td>
<td>tensile yield strength of the material</td>
<td>(\text{N/mm}^2 \text{ (Mpa)})</td>
</tr>
<tr>
<td>(\Phi)</td>
<td>complete integral of the second kind</td>
<td></td>
</tr>
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### ABBREVIATIONS

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Meaning</th>
</tr>
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<tbody>
<tr>
<td>API</td>
<td>American Petroleum Industry</td>
</tr>
<tr>
<td>BS</td>
<td>British Standard</td>
</tr>
<tr>
<td>CCT</td>
<td>Central Cracked Tension</td>
</tr>
<tr>
<td>DNV</td>
<td>Det Norske Veritas</td>
</tr>
<tr>
<td>ECA</td>
<td>Engineering Critical Assessment</td>
</tr>
<tr>
<td>EPFM</td>
<td>Elastic Plastic Fracture Mechanics</td>
</tr>
<tr>
<td>FE</td>
<td>Finite Element</td>
</tr>
<tr>
<td>FEM</td>
<td>Finite Element Method</td>
</tr>
<tr>
<td>ISO</td>
<td>International Organization for Standardization</td>
</tr>
<tr>
<td>LEFM</td>
<td>Linear Elastic Fracture Mechanics</td>
</tr>
<tr>
<td>NORSOK</td>
<td>Norsk Sokkels Konkuranseposisjon</td>
</tr>
<tr>
<td>PD</td>
<td>Published Document</td>
</tr>
<tr>
<td>SCF</td>
<td>Stress Concentration Factor</td>
</tr>
<tr>
<td>SIF</td>
<td>Stress Intensity Factor</td>
</tr>
<tr>
<td>XFEM</td>
<td>Extended Finite Element Method</td>
</tr>
</tbody>
</table>
SUMMARY

Fatigue loads and failures in metallic structures are a well-known technical problem. Commonly, the fatigue life of the steel elements is determined by an analytical calculation using S-N curves as design resistance to periodic action. However, the technological improvements in almost all structures continuously stretch the limits of acceptance levels of flaws in structures. Consequently, crack growth control might provide an assessment tool in lifetime predictions.

BS 7910 provides a general procedure to predict fatigue crack growth commonly addressed as fracture mechanics fatigue assessment. It is based on the well-known Paris Law relationship which relates the crack growth rates to the stress intensity factor range. BS 7910 provides a library of SIF solutions to support this procedure. However, several limitations are contained which trigger research for possible extensions. Furthermore, it is common that a more sophisticated method than the analytical is usually required to predict fatigue crack growth. FEM, as one of the most sophisticated numerical methods; it is widely used as a tool to predict fatigue life. From this, stems the need for research on the FEM to predict fatigue crack growth.

The purpose of the present thesis is to present these topics, evaluate the current status and examine possible extensions. At first, the terminology related to fracture mechanics and fatigue crack growth is explained and an evaluation of the current BS 7910 stress intensity factor solutions is conducted. Then, the results of a research on new SIF solutions outside the limitations of BS 7910 based on FEM analyses are presented. At the end of the thesis, a study on the various numerical and analytical techniques to perform a fatigue crack growth analysis is conducted.
1. INTRODUCTION

1.1 GENERAL

A component or a structure might resist the application of a single static load far below the static strength of the structure, but fail due to repeated application of the same load. This phenomenon is called fatigue failure [1]. When a structure is subject to fluctuated loading, it may lead to the development of fatigue cracks. Fatigue cracks extend slowly (depending on the loading conditions and the material properties), generally with a very small increment of crack growth occurring after each cycle. The crack continues to grow until it causes complete failure by fracture, plastic collapse or other mode which prevents service duties of the structure being performed [2].

Fatigue failures in metallic structures are a well-known technical problem. Numerous fatigue failures of welded structures, aircraft, machinery, offshore structures etc. have been reported. Currently fatigue design in structures is mainly based on S-N curves contained in codes and standards such as Eurocode 3, DNV, and API. However, the ever growing need for derivation of acceptance levels based on the principle of fitness for service (i.e. a structure is considered to be adequate for its purpose provided the conditions to cause failure are not reached [3]) rendering necessary the determination of the tolerance of structures to cracks and crack-like defects[4]. In order to achieve this, the reliability of an existing defect is considered based on fracture mechanics principles. It is based on the observed relationship between the rate of crack growth, \( \frac{dc}{dN} \), and the change in the stress intensity factor range \( \Delta K \) (\( \Delta K = K_{\text{max}} - K_{\text{min}} \)). This relationship suggests that the stress intensity factor range is characterizing the crack growth...
per cycle since many test data collapse in a single power law in the $d\alpha/dN-\Delta K$ diagram [5]. This fact reveals the importance of the parameter $K$ and widows it to the governing parameter for fatigue crack growth analysis.

British standard, BS 7910, and API 579 fitness for service (as the two most commonly used procedures [6]) based on fracture mechanics propose each, a general procedure to assess the acceptability of flaws found in service and to estimate the tolerable flaw sizes in relation to the fatigue life of the member containing flaws [7]. Of course, since as mentioned before, the stress intensity factor is a governing parameter for crack growth, BS 7910 and API 579 are also providing a library of $K$ solutions for various types and geometries of cracks. The recognition of BS 7910 by the offshore structure’s standards ISO 19902 Fixed steel offshore structures [8] and reference by DNV-RP-C203 Fatigue design of offshore steel structures [9] and NORSOK M-101 Structural steel fabrication [10] led to the selection as leading standard for the assessment of the library of stress intensity factor contained.

The general procedure proposed by BS 7910 is an important assessment tool for the prediction of the fatigue life of a specimen or structure. However, in several cases it is important to be able to use a more sophisticated method to predict fatigue crack growth. Numerical techniques have been successfully employed to predict crack shape evolution and fatigue life. One of the most widely quoted in literature for this purpose is the Finite Element Method (FEM). Subsequently, a research on the FEM capability to perform a fatigue crack growth analysis is justified and will be part of this report.
1.2 DEFINITION OF THE PROBLEM

The technological improvements in almost all structures continuously stretch the limits of acceptance levels of flaws in structures. As aforementioned, stress intensity factor solutions are very important for the fatigue crack growth analysis based on fracture mechanics. BS 7910 is adopting the fracture mechanics method for fatigue crack growth analysis and provides a library of K solutions. The existing library of BS 7910 contains several solutions for a wide range of flaw types and different member geometries. However, several geometrical limitations in terms of crack aspect ratio are imposed to those solutions. Furthermore, the growing need for a numerical approach on the problem of fatigue crack growth necessitates the research on the FEM capability to perform a reliable fatigue crack growth analysis. For these reasons, this project is focused on the compilation of K solutions enclosed in annex M of BS 7910 and on a study on the various numerical and analytical procedures used to conduct a fatigue crack growth analysis.

1.3 THE OBJECTIVES OF THE REPORT

The limitations on the library of stress intensity factor solutions contained in annex M of BS 7910, the importance of the stress intensity factor, K in fatigue life assessment and the need for a finite element tool to predict fatigue crack growth trigger the research for possible extensions. This thesis is focused on:

- Insight to the existing SIF solutions provided by Annex M of BS 7910 and investigation for possible extensions.
- Determination of Stress Intensity Factors for a number of solutions outside the limits defined by BS 7910 using FEM analysis.
• Fatigue crack growth analysis using analytical calculations based on Paris Law and the SIFs from the BS 7910 library of K solutions.

• Fatigue crack growth analysis using FEM to calculate the SIF and evaluation of the method by comparing the results with the respective from the analytical calculations.

• Research on the XFEM capabilities to reliably predict fatigue crack growth.
2. **LITERATURE OVERVIEW**

2.1 **GENERAL**

Cracks and other forms of defects might be introduced during manufacturing, especially if welding is used, or form during service under repeated load applied to the structure. Hence, if the structure is subjected to fatigue loading, virtually the whole life of the structure or component of the structure is occupied by fatigue crack growth [11]. Therefore, the study of cracks and their growth is of great importance. In order to examine and assess, if a crack under the present or likely to happen conditions will develop by fatigue crack growth to such an extent that fracture will occur or whether the crack will not cause fracture; a fracture mechanics assessment is needed [12]. It is a powerful tool in making fitness for purpose assessments for the effects of flaws, for decision of inspection periods for structures and for determination of acceptance levels for flaws found during inspection [2].

2.2 **FRACTURE MECHANICS CONCEPTS**

2.2.1 **Linear Elastic Fracture Mechanics**

When a member containing a crack is loaded; the stresses surrounding the crack tip region are elevated. Linear elastic analysis of the stress and displacement state of the cracked member shows that the stresses around the crack tip vary according to a singular term, \( r^{-1/2} \), where \( r \) is the distance from the tip of the crack. Of course as \( r \) approaches to zero, the stresses tend to infinity [13]. But an infinite stress could be a disaster for the material. Instead, the materials, even the high strength materials, have some ductility. As a result of the ductility, yielding of the material takes place. A small plastic zone is created and the
infinite peak stress is levelled off [1]; see Fig. 2-1.

![Fig. 2-1 Formation of plastic zone around the crack tip due to local yielding [1]](image)

This could invalidate the assumed elastic behaviour and cancel the validity of the equations which were based on linear elastic analysis. However, as long as the plastic zone remains small (usual check is that the nominal net section stress does not exceed $0.8 \cdot \sigma_y$ [11]) relative to the crack magnitude, the elastic solution is not seriously disturbed [12]. After all, the science of dealing with a cracked member as linear elastic while plasticity or other non-linear effects are assumed to be negligible and therefore ignored is called Linear Elastic Fracture Mechanics (LEFM) [13].

### 2.2.2 Elastic Plastic Fracture Mechanics

The LEFM based on stress intensity factor, $K$, is very successful in predicting fracture when the $K$-dominance zone (i.e. the zone where the near-tip stress field can be described by the singular stress field or equivalently by the stress intensity factor $K$ [13]) is larger than the crack tip plastic zone [13], see Fig. 2-2.
However, if the material has low yield strength or the load is relatively high, larger plastic zones will be formed. As a result, assessing the crack under such conditions using LEFM would be inadequate because the severity of the crack tip conditions would be underestimated [12]. Consequently, Elastic Plastic Fracture Mechanics (EPFM) should be employed in that case to assess the crack. Parameters that describe the overall plastic deformation such as the J-Integral and the Crack Tip Opening Displacement (CTOD) are needed [13]. Nevertheless, as this is beyond the scope of this document it will not be further explained.

Fig. 2-2 K-dominated region [11]
2.3 STRESS INTENSITY FACTOR

2.3.1 History of stress intensity factor

Application of repeated loads can start a fatigue mechanism in the material which leads to nucleation of a small crack, followed by crack growth and finally failure [1], see Fig. 2-3.

![Fig. 2-3 Different phases of the fatigue life and relevant factors][1]

Therefore, the fatigue life until failure can be divided into three distinct steps. Firstly, crack initiation where a small crack forms at a point of high stress concentration. Secondly, crack propagation whereby crack grows incrementally after each stress cycle; and finally failure.

When a member with a discontinuity (i.e. notch, hole or weld) is loaded by homogeneous stress distribution, the discontinuity will cause inhomogeneous stress distribution at the cross section of the member where it exists. This difference in the stress distribution is defined by the stress concentration factor, $K_c$, which is a dimensionless shape factor defined as the ratio of the peak stress due to a discontinuity, to the nominal stress, see Fig. 2-4.
Therefore, the crack initiation period is highly dependent on the value of the stress concentration factor, $K_t$, since as mentioned before a crack may initiate at a point of high stress concentration.

However, during the crack growth period due to the crack formation; $K_t$ value can no longer describe the severity of the stress distribution around the crack tip. This lies on the difference between a crack and a notch. The crack is a notch with a zero tip radius. Therefore, if $K_t$ would be used, according to the stress concentration formula the stresses close to the crack tip independently from the crack shape and length, would reach infinity. This of course is not true since as mentioned before, the stress field surrounding the crack tip is finite. Instead, a new concept is introduced to describe the severity of stress distribution surrounding the crack tip, the so-called stress intensity factor, $K$ [1].

This concept was originally developed by Irwin [14]. In the early 1950s, based on the method
of Westergaard [15] to solve elastic problems, it was made possible to obtain the singularity term in the elastic crack tip stress field series expansion and to lead to the nowadays general formula of stress intensity factor [1]:

\[ K = \beta \sigma \sqrt{a} \]  

\text{Eq. 2-1}

Where \( K \), is the stress intensity factor, \( \beta \), is a dimensionless factor depending on the geometry of the specimen or structure, \( \sigma \), is the remote loading stress on the member and \( a \), is the length of the crack [1].

The first known expansion of the crack tip stress field was done by Sneddon in 1946 for the “penny shaped crack”. Due to the relative simple boundary conditions; Sneddon, obtained a closed form solution of the stress distribution around a circular crack in an infinite solid in tension [1]. Later on, at the end of 1950s, Irwin [16] defined the three modes of fracture (Mode I, Mode II and Mode III, see Fig. 2-5) and the elastic analysis methods to determine their stress intensity factors (\( K_I \), \( K_{II} \), and \( K_{III} \)) [17]. Mode I (tensile mode) represent a crack whereby the opposing crack surfaces move directly apart. Mode II is related to in plane shear and Mode III is related to out of plane shear, see Fig. 2-5.

Fig. 2-5 Schematic of the basic fracture modes: (a) Mode I, (b) Mode II, (c) Mode III [13]
These results were put to use by various researchers and produced stress intensity factor solutions for several crack shapes such as elliptical shaped cracks, edge cracks, semi-elliptical surface cracks etc. Following those developments, at 1961 the first publication [5] on fatigue crack growth using K was revealed.

The following years, till today, plenty of research has been conducted on the field of stress intensity factor solutions. This research led to the development of various methods to determine SIF solutions; including analytical, numerical and experimental approaches while many results of calculations for various geometries and loading cases have been published. Books like *Stress Intensity Factors (1976)* by Rooke and Cartwright, *The Stress Analysis of Cracks Handbook (1985)* by Tada, Paris and Irwin and *Stress Intensity Factors Handbook (1987)* by Murakami are frequently cited and contain compilations of K solutions [1]. One such compilation of K solutions was published in 1999 as annex of the BS 7910 [18] and is going to be subject of research of this thesis.
2.3.2 Theoretical background of stress intensity factor

As mentioned before, the Stress Intensity Factor (SIF) is a fundamental importance fracture mechanics concept. It arises from linear elastic analysis and it is central to the theory of linear elastic fracture mechanics. A common reference often used, is that of an infinite plate with a central through thickness crack of length 2a under a perpendicular to the crack plane stress, $\sigma$, see Fig. 2-6 [12].

![Stress distribution at the crack tip of an infinite plate with a central crack of length 2a](image)

The stresses near the crack tip ($r \ll a$) for this case are given from the following forms [12]:

$$\sigma_x = \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

Eq. 2-2
\[
\sigma_y = \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \quad \text{Eq. 2.3}
\]

\[
\tau_{xy} = \frac{K}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \quad \text{Eq. 2.4}
\]

Where \( r \) and \( \theta \) define the polar coordinates of the system centered at the crack tip and \( K \), is the stress intensity factor. The above equations can also be applied to a finite dimensions element by taking into account the geometry correction factor \( \beta \) (see Eq. 2-1) which for the infinite sheet equals to 1 [12].

The stress intensity factor, \( K \), in the above equations is referring to a tensile mode (or Mode I) form of crack opening. Mode I, as mentioned before, represents a crack whereby the opposing crack surfaces move directly apart. However cracks may be subjected to different types of loads, leading to different crack opening modes (Mode II or Mode III, see Fig. 2-5). Nevertheless in practice most cracks tend to grow under Mode I, therefore attention is mainly focused on this crack opening mode.

In the previous reference example, the infinite sheet was under uniform loading conditions. However in many engineering applications, a cracked body is generally subjected to combined loading. In that case the principle of superposition is used and the total stress intensity factor is obtained by adding algebraically the fundamental stress intensity factor solutions for each load case [1]. Consider a cracked body subjected to a combined bending and tensile loading as shown in Fig. 2-7.
In this situation, the principle of superposition is used; the total stress intensity factor is calculated by summing up the separate stress intensity factor solutions for tension and bending.

### 2.3.3 Types of cracks

Cracks occur when highly localized stresses exceed the tensile strength of the material. A crack in a solid consists of disjoined upper and lower faces. The joint of the two faces form the crack front wherefrom the crack extends [13]. Fatigue cracks can be categorised in geometrical terms as [1]:

I. Through thickness cracks; whereby the crack extends through the full thickness of the member with the crack front perpendicular to the material surface, see Fig. 2-8

II. Part through cracks; where the crack front does not cover the whole thickness and
develops a curved shape, see Fig. 2-8

![Diagram of cracks](image)

Fig. 2-8 Different types of cracks emanating from a hole [1]

To the part through cracks are included the corner cracks, the embedded cracks and the surface cracks. As mentioned before the part through cracks propagate through a curved crack front which is usually approximated with an elliptic or near elliptic shape for modelling reasons [2]. Of course, this means that the K factor is no longer constant but it varies along the curved shape crack front [1]. In order to take this into account, Irwin [19] derived the following equation for an infinite plate:

$$K(\theta) = \frac{\sqrt{\pi}a}{\Phi} \left[ \sin^2 \theta + \left( \frac{a}{c} \right)^2 \cos^2 \theta \right]$$  \hspace{1cm} \text{Eq. 2-5}

Where c and a, are the crack half- length and crack depth of the elliptical crack respectively while \( \Phi \) is the shape factor and \( \theta \) is the parametric angle of the ellipse, see Fig. 2-9. The shape factor \( \Phi \) is the so-called complete integral of the second kind which depends on the aspect ratio of the ellipse. The difficulty to be solved analytically has lead to tabulated numerical values. However, a very good approximation was developed by Rawe and is given
in reference [20]. Both solutions are provided by BS 7910 [3]. Regarding the parametric angle $\theta$, it is used as shown in Fig. 2-9, to define the location of a point B at the crack front [1].

![Parametric angle $\theta$ to define location B at the crack front](image)

Fig. 2-9 Parametric angle $\theta$ to define location B at the crack front

Of course to the Eq. 2-5 correction to account for the finite specimen size effects mentioned before should be made. Therefore, a dimensionless function $F$ accounting for the width, $W$, and the thickness, $B$, is applied [13], see Eq. 2-6.

$$K = K_{Eq.2-5} F \left[ \frac{c}{B}, \frac{c}{W}, \frac{2c}{W}, \theta \right]$$  \hspace{1cm} Eq. 2-6

The corrections made, also depend on the aspect ratio of the crack, $c/a$, and the parametric angle of the ellipse, $\theta$. An example correction factor, is the finite specimen size effects correction factor, $f_w$, which is used in BS 7910 and will be explained in the next section.
2.4 OVERVIEW OF ANNEX M OF BS7910

2.4.1 Introduction

In 1980, the procedure for the assessment of flaws in metallic structure was published as PD 6493. To this document, solutions of the stress intensity factors were provided in the form of simple graphical methods. Nevertheless, the only geometry explicitly considered was that of the flat plate considering surface cracks and embedded flaws. Following research and new developments, the PD 6493 was revised in 1991 and extensive stress intensity factor solution formulas were added. In 1999 the document was upgraded to become a British Standard, BS 7910. To this document an expanded library of K-solutions was added as an annex; allowing for analysis of plates, cylinders, round bars and welded joints [21].

The last updated version of BS 7910 in 2013 [3] contains stress intensity factor solutions divided into nine parts (M3-M11). However, the solutions can be conveniently concentrated in five main areas. Namely, the flat plates, the curved shells and spheres, the welded joints the tubular joints and the round bars and bolts.

2.4.2 Flat plates

Due to its simple geometry, the crack in a sheet (or plate) was the first [14] to be developed. Since fatigue failures usually occur from the initiation and propagation of cracks from notches or defects in the material that are either embedded, on the surface, or at a corner, stress intensity factor solutions for plates containing through-thickness flaws, edge flaws, surface flaws, embedded flaws, corner flaws and corner flaws at a hole are enclosed [22]. BS 7910 is mainly adopting the stress intensity factor solutions proposed by Raju and
Newman which are based on parametric equations derived from curve fitting of an extensive set of finite element results [22],[23]. The basis of the stress intensity factor equations provided for plates is the Eq. 2-6. The finite specimen size effects correction function F is accounted by the term \( f_w \) and the angular function \( \theta \) for the curved shape crack front is taken into account by the term \( f_\theta \). Of course, since the results arise from curve fitting, extra terms are added. Particularly, the \( M_i \) (i=1, 2 or 3) terms which are curve fitting boundary correction factors accounting for the influence of the ratio of crack depth (\( \alpha \)) to plate thickness (\( B \)) and the term \( g \), which is a product of functions that are used to fine-tune the equations to the finite element results. The final outcome is termed \( M_m \) (m stands for membrane loading) and is called stress intensity magnification factor, see below:

\[
M_m = \left[ M_1 + M_2 \left( \frac{\alpha}{B} \right)^2 + M_3 \left( \frac{\alpha}{B} \right)^4 \right] \frac{g f \theta f w}{\phi} \quad \text{Eq. 2-7}
\]

Solutions are given for membrane loading. For bending loading, a bending multiplier \( H \) is applied to the membrane solution to account for the corresponding bending correction [3].

2.4.3 Curved Shells and spheres

Stress intensity factor solutions for flaws in pipes and vessels are of great technical importance. Therefore, \( K \) solutions for curved shells under internal pressure, curved shells under combination of internal pressure and mechanical loads, and spheres are contained in this part of Annex M. Due to the curved geometry of shells and spheres there is a stress increasing which is taken into account by introducing the curvature or “bulging” correction factor. This factor accounts for the difference in behaviour of a crack in a plate compared to that of a crack in a curved shell and it is defined as the ratio of the stress intensity factor for a flat cracked plate to
the stress intensity factor for a cracked shell [24]. Consequently, K solutions for curved shells under internal pressure are mainly derived from the K solutions for plates with the addition of the bulging factor.

For the case of curved shells under mechanical loads and internal pressure; solutions derived from finite element analysis [25], [26], [27], are used. These solutions are implemented as correction factors \( M_i \) \((i=m,b)\) for tension and bending and are mainly applicable to surface and through thickness flaws. They derive from direct finite element analysis results of cracked shell geometries which mean that there is no need to account for the bulging effect. For the rest of the flaws; again, flat plate solutions are used with the addition of the bulging factor. As far as the spheres are concerned, the same approach is used, after correction factors originating from finite element analysis results [28].

### 2.4.4 Welded joints

Probably the most common site for fatigue crack initiation in welded joints due to fatigue loading is the weld toe. Due to the high stress concentration and the crack-like flaws which are inherent feature of most welds, fatigue cracks initiate at the weld toe [29]. Nevertheless, flaws may also arise from the root of the weld. Annex M includes \( M_k \) solutions for weld joint geometries for flaws initiating either from the weld root or from the weld toe based on 2D and 3D finite element analysis. Essential to the methodology adopted by BS 7910 for the stress intensity factor solutions for welded joints is the correction factor \( M_k \) defined as [30]:

\[
M_k = \frac{K_{\text{in a plate with attachment}}}{K_{\text{in the same plate but with no attachment}}}
\]

Eq. 2-8

\( M_k \)
When the crack is located in a region of stress concentration as for example in the weld toe, there will be an increase of the stress intensity factor compared to the regions where the stress concentration is absent. The $M_k$ factor quantifies this change in stress intensity factor due to the presence of the weld and the attachment [30]. It is applied to plain plate solutions in order to derive solutions for a plate with an attachment.

Considering the surface cracks at the weld toe, the 2D FEM analysis is providing general solutions[31]; however conservatism is included due to the fact that the solutions derive from crack edge models which are unable to provide accurate solutions for the crack ends [3]. The solutions based on curve fitting of 3D FEM analysis [30] are more accurate but results are available only for the surface and the deepest points of the weld toe flaw [3].

### 2.4.5 Tubular joints

For stress intensity factor solutions of tubular joints, BS 7910 is addressing to Annex B whereby two options are given. Firstly, the use of FEM to analyze tubular joints which requires complex finite element modelling and stress analysis. For this case, limited tubular joint solutions are available, e.g. [32], [33]. Secondly, the complexity of the tubular joint can be surpassed by simplifying the joint to a plate or cylinder. Consequently flat plate solutions are employed by referring again to Raju and Newman solutions [23], [34]. Of course due to the geometry of the tubular joints and the existence of welding, correction factor $M_k$ is again used [3].

### 2.4.6 Round bars and bolts

Threaded fasteners in the form of bolts are widely employed in the assembly of many large
structural components. High strength bolts are very common in joints for aerospace, structural components and offshore structures. Although failures from bolted fasteners are not frequent, when they occur there might be considerable consequences [35]. Therefore, a number of K solutions for straight crack fronts, semi-circular and semi-elliptical surface flaws, and fully circumferential flaws for bolts and round bars are contained in annex M.

There are solutions for semi-elliptical crack fronts taking into account the crack aspect ratio and the effect of the threaded shape of the bolts [36]. However due to the fact that the influence of the thread upon the stress intensity factor is diminishing as the crack becomes larger [35]; the solutions are mainly based on round bars. Consequently, solutions obtained by polynomial equations (fitting to various previous FEM results) as functions of the crack depth and bolt radius (or round bar radius) are obtained for the various crack front shapes (e.g. straight fronted cracks, semi circular crack front etc.) [35] [36].
2.5 FATIGUE CRACK GROWTH

In principle, determination of fatigue crack growth data is straightforward. An example of a crack propagation test is the one that can be carried out on a simple CCT (Central Cracked Tension) specimen containing a crack. The specimen is provided with a central notch whereby crack initiates rapidly. The specimen is clamped at the ends between two steel plates which are fastened by a number of bolts. The clamping should ensure that the fatigue load from the testing machine is distributed homogeneously [1]. The crack growth records can be plotted as a simple graph of crack length (a) data versus the number of cycles (N), see Fig. 2-10. From the slope of this graph, the crack growth rate data can be extracted and plotted as a function of the crack length.

Fig. 2-10 Example of crack length versus number of cycle [1]
In 1961, Paris et al. [5] observed that under different crack length and different stress amplitude, similar crack growth rates appeared. They assumed that since the stress ratio and the magnitude of the stresses surrounding the crack tip are the governing parameters during a cycle of loading; the crack growth rate must be a function of these two parameters.

### 2.5.1 Crack growth laws

A fatigue load on a cracked specimen introduces a cyclic stress varying between $S_{\text{max}}$ and $S_{\text{min}}$. Therefore, taking into account Eq. 2-1, the corresponding stress intensity factor also varies between $K_{\text{max}}$ and $K_{\text{min}}$. Thus, since crack growth rate can be described as the crack opening under one cycle stress range, it appears to be fully correct to assume that the crack growth rate is a function of $K_{\text{max}}$ and $K_{\text{min}}$ or in other words a function of $\Delta K$ and $R$[1]:

\[
\frac{dn}{dN} = f(\Delta K, R)
\]

Eq. 2-9

Where $\Delta K$ is the range between the two extreme stress intensity factors ($\Delta K = K_{\text{max}} - K_{\text{min}}$) and $R$ is the stress ratio expressed in terms of stress intensity factor ($R = K_{\text{min}}/K_{\text{max}}$).

Fatigue crack growth results, cover a wide range of $\Delta K$ values and are plotted in Fig. 2-11 on a double log scale. Three regions are covered: the threshold region, the Paris region and the stable tearing crack growth region.
In the threshold region, it is assumed that at some critical value of $\Delta K$ which is the threshold value, $\Delta K_{th}$, the crack growth rate tends to zero.

For the Paris region, the power law relation first formulated by Paris and Erdogan [37] is adopted:

$$\frac{da}{dN} = A(\Delta K)^m$$  \hspace{1cm} \text{Eq. 2-10}

Where $A$ and $m$, are material constants, $da/dN$, is the crack growth rate and $\Delta K$, is the stress intensity range. At this point it should be mentioned that in this power function the effect of
the stress ratio is not taken into account. In literature, several alternative functions have been proposed by Forman, Priddle, Klesnil and Lukas to overcome this problem [1]. However, since BS 7910 is recommending this approach, this law will be used further.

Regarding the stable tearing crack growth region; the crack growth rates are increasing rapidly (in the order of 0.01 mm/cycle and above) as the conditions for static failure are approached. Final unstable failure will occur when \( K_{\text{max}} \) is equal to \( K_c \). Where, \( K_c \) is the stress intensity factor causing final failure. The crack growth life spent in this region is rather small, limiting its engineering significance [1].

2.5.2 Prediction of fatigue crack growth

The prediction of fatigue crack growth requires two types of information. Firstly, the stress intensity factor range as a function of crack length, \( \Delta K[\alpha] \), which account for the crack driving force and secondly the crack growth rate as a function of the stress intensity factor range [1]:

\[
\begin{align*}
\frac{da}{dN} &= A(\Delta K)^m \\
\Delta K &= Y(\Delta \sigma)\sqrt{\pi \alpha}
\end{align*}
\]

Eq. 2-11

By substituting the stress intensity factor range, \( \Delta K \), to the crack growth rate formula, and following integrate from initial crack, \( \alpha_0 \), to maximum allowable crack, \( \alpha_f \), the fatigue life (in cycles) of a known crack can be calculated:

\[
N = \frac{1}{A(\Delta \sigma)^m} \int_{\alpha_0}^{\alpha_f} \frac{da}{(Y(\Delta \sigma)\sqrt{\pi \alpha})^m}
\]

Eq. 2-12
The previous procedure is given for constant amplitude loading [1]. However, the structures are commonly loaded under variable amplitude loading. This type of loading causes variable amplitude stress ranges. For each stress range of the stress spectrum, the stress intensity factor range, $\Delta K$, has to be calculated. With the stress intensity factor range, the crack growth, $\Delta \alpha$, for one cycle can be calculated and added to the initial crack dimension, see Eq. 2-13. This procedure is repeated for each cycle until the maximum allowable crack size is reached [1].

$$\alpha_n = \alpha_0 + \sum_{i=1}^{N} \Delta \alpha_i \quad \text{Eq. 2-13}$$

Where $\alpha_n$, is the crack length after N cycles, $\alpha_0$, is the initial crack length and $\Delta \alpha_i$, is the extension $\Delta \alpha$ of the crack in cycle number i [1].

The preceding procedure, for a single crack would need a great number of iterations until initial crack size reaches maximum allowable crack dimension. This makes difficult to perform assessment procedure for a structure under variable amplitude loading. Therefore, an alternative can be used [3]. The stress spectrum of the structure during its life is converted to identifiable stress ranges. Following, it is divided into blocks creating a histogram, see Fig. 2-12. Each block is represented by constant stress amplitude versus the number of cycles. Consequently the number of repetitions is reduced.
The problem with this method is that the crack may reach the maximum allowable crack size after performing the first block of the spectrum because the block length (number of cycles) is defined for the life time of the structure. The problem can be overcome by dividing each block into increments. The calculations will be done for each block increment and the check will be done after each increment for all blocks is completed [3].

Finally, it should be noted that Eq. 2-13, assumes that the crack size is fully defined by a single size parameter, the crack length, $\alpha$. However, for part-through cracks, as aforementioned, a curved crack front is adopted [1]. Therefore, the shape of the crack must be taken into account. This can be done by assuming that the same crack growth relationship applies for crack growth in both $\alpha$ and $c$ directions. The growth of the crack after
each cycle is added to the initial dimensions. Therefore, the growth of crack, $\Delta \alpha$, should be added to the dimension, $\alpha$, and the same (growth of crack $\Delta c$) should be done for the other dimension, $c$, [3].
2.6 CONCLUDING REMARKS

To conclude this chapter, beyond the brief introduction to the theoretical background of the notions of fracture mechanics, stress intensity factor and fatigue crack growth; the importance of the flat plate solution should be pointed out. More specifically:

- The adoption of $M_k$ factor for welded plates or joints directly links the flat plate stress intensity factor solutions with the welded joint solutions.
- For the derivation and verification of the stress intensity factor solutions of round bars a comparison with the solution of the semi elliptical crack in a flat plate is carried out.
- Curved shells are basing their denouements to the ones given for flat plates with the respective flaw geometry.
- In offshore industry and particularly in tubular joints the most common option is to adopt flat plate solutions to obtain the stress intensity factor.

As mentioned before, the flat plate solutions are mainly based on the equations proposed by Raju and Newman which are based on curve fitting of finite element results. However, the solutions proposed are restricted in terms of crack aspect ratio ($\alpha/c$) even though there is reported literature (tubular joints [32], flat plates [38]) where a much larger variety of crack aspect ratio is considered.

Furthermore, since fatigue cracks usually grow from an initial part thickness crack (either surface or embedded) to a full thickness crack and due to the fact that BS 7910 [3] proposes (unless failure has occurred):
• For embedded flaws when the crack breaks through to one surface it should be treated subsequently as a surface crack of length 2c

• For surface flaws, when the crack breaks to the far surface, it should be treated further as a through thickness crack of length 2c

The research on the flat plate stress intensity factor solutions of surface flaws is justified and will be in the field of research of this thesis.
3. **SIF SOLUTIONS FROM FEM ANALYSIS**

The determination of the stress intensity factor is extremely difficult when it comes to analytical solutions. As a result, numerical modelling appears to be the only feasible solution. For problems in solid mechanics the most common numerical modelling method applied is the finite element [39], [40]. BS 7910 is adopting for the flat plate stress intensity factor solutions the parametric equations proposed by Raju and Newman. Those equations derived from curve fitting of an extensive set of stress intensity factor solutions which are the outcome of application of the finite element method. On this chapter, the finite element method is used again. Firstly, to validate the method used to extract the stress intensity factor and secondly, to determine the stress intensity factor for chosen crack aspect ratios that exceed the limits defined in annex M of BS 7910.

3.1 **NUMERICAL MODELLING OF PLAIN PLATE**

3.1.1 **Details of the analysis**

Chosen for the analysis is a plate of 10 mm length (L), 10 mm width (W) and 1 mm thickness (B), see Fig. 3-1. The material properties used for the analysis are that of steel. Therefore, the Young’s modulus and the Poisson’s ratio used are 210 KN/mm$^2$ and 0.3 respectively.
Each model is analysed under membrane and bending loading in accordance with BS 7910 solutions. The models are loaded by prescribed displacements. For membrane loading, one end of the plate is restrained whilst the other is given a uniform longitudinal displacement. A deformed model under membrane loading is shown in Fig. 3-2.

For bending loading, four-point bending is chosen. The ends of the plate are restrained while a vertical displacement is applied at 1/3 of the plate length from each end. A deformed model
under bending is shown in Fig. 3-3.

![Plate with a crack under bending loading, general view of deformed half model](image1)

**Fig. 3-3** Plate with a crack under bending loading, general view of deformed half model

In order to accelerate the analysis, the symmetry of the plate on the plane vertical to the crack plane, see Fig. 3-4, is exploited. Therefore, only half of the body is modelled. The finite element package used for the analysis is the commercial software ABAQUS [41].

![Symmetry plane](image2)

**Fig. 3-4** Symmetry plane used for the analysis
3.1.2 Mesh generation

The creation of the 3-D plain plate model is a standard finite element modelling procedure. The main difficulty lays on the set up of the mesh surrounding the crack-tip. For the crack-tip region, the most efficient mesh design has proven to be the “spider-web” configuration. It consists of concentric rings of elements that are focused towards the crack tip creating a regular and focused mesh [40], see Fig. 3-5.

Close to the crack-tip region, steep stress and strain gradients are contained. Therefore, a greater refinement is necessary. The spider-web design facilitates a smooth transition from the fine mesh in the vicinity of the crack tip to a relatively coarser mesh remote from the crack tip. Furthermore, this configuration results in a series of concentric integration domains (contour integrals) which are useful for the numerical evaluation of the stress intensity factor, $K$ [40].

3.1.3 Elements

The choice of the elements used in the analysis is also of great importance. In Fig. 3-6 the most common three-dimensional continuum element shapes are illustrated [40].
The elements depicted have mid-side nodes and quadratic shape functions. Typical crack analyses use brick elements for three dimensional problems [40], [42]. However, close to the crack tip, the crack is introducing a singular stress field. Therefore, in order to create this singularity at the crack tip, the problem should be approached differently. Consequently, at the crack front, the brick elements are collapsed to wedge elements. For elastic problems, the nodes at the crack tip are tied and the mid-side nodes are moved to ¼ points, see Fig. 3-7. Such modifications result in the required $1/\sqrt{r}$ singularity which dominates the solution close to the crack tip [40].
Finally, the element pattern around the crack front is following the set-up depicted in Fig. 3-8.

Fig. 3-8 Element pattern around crack front [20]

Taking all the aforementioned into account, the element type chosen for the analyses is the reduced integration 20-noded brick element (C3D20R from the ABAQUS element library), see Fig. 3-9.

Fig. 3-9 20-noded brick element [41]
3.2 VALIDATION OF THE FEM MODEL

3.2.1 Calculation of the Stress Intensity Factor

The virtual crack extension technique, implemented in ABAQUS [41], is used to provide estimates of the stress intensity factor. In the elastic regime the stress intensity factor can be calculated from the J-integral value (energy release rate), for plane stress, as

\[ K = \sqrt{J} \]  

Eq. 3-1

For plain strain, \( E \) is replaced with \( E/(1-\nu^2) \). For each crack front node three contours are requested, each representing a ring of crack front elements. Each crack front node SIF is then taken to be the average of those calculated from contours 2 and 3, with the first contour which is prone to numerical error, being ignored [41].

3.2.2 Calculation of the shape factor \( Y \)

Once the stress intensity factor is evaluated, it is necessary to non-dimensionalise it with respect to the crack length and loading in order to compare it with the BS 7910 solution. This non-dimensional stress intensity factor, is calculated according to

\[ Y = \frac{K}{\sigma_{\text{nom}}\sqrt{a}} \]  

Eq. 3-2

Where \( \sigma_{\text{nom}} \), is the nominal stress in the plate and \( a \), is the crack depth. The calculation of the nominal stress resulted from FEM analysis of the plain plate without the crack present.
3.2.3 Model validation

For the validation of the model, solutions for a plain plate with three different crack aspect ratios ($\alpha/2c$ = 0.2, 0.5, 0.8) are being compared with the respective obtained from BS 7910 stress intensity factor equation. The crack depth ratio ($\alpha/B$) is kept constant to 0.5 in all cases. Both membrane and bending loading are included in the analyses and the results can be seen in Fig. 3-10 for membrane loading, in Fig. 3-11 for bending loading while in Fig. 3-12 the stress intensity variation along the crack front is also presented.
Fig. 3-10 $K_I$ for a surface crack in a plate under membrane loading. (a) surface, (b) maximum depth
Fig. 3-11 $K_I$ for a surface crack in a plate under bending loading, (a) surface, (b) maximum depth.
The results compare well, with a maximum difference of around 10 percent. Best agreement is obtained for tension but larger differences occur for bending. The differences between the current results and BS 7910 can be attributed mainly to the modelling methods used by Raju and Newman (whose solutions are adopted by BS 7910). Raju and Newman used full integration, applied stress loading and a force based stress intensity factor evaluation method [43], whereas the current modelling method uses reduced integration, displacement loading and virtual crack extension technique to evaluate the stress intensity factor. These differences however, are not significant and the results can be said that are in good agreement with BS 7910 solutions[43].
3.3 SIF FOR LARGER ASPECT RATIOS

3.3.1 Parametric ranges for the study

The crack aspect ratios ($\alpha/2c$) studied are 1.5 and 2.0. These values double the existing range of crack aspect ratios for which the parametric equations of BS 7910 provide stress intensity factor solutions. Due to the limited capabilities of the workstation used for the analyses and considering the element distortion resulting from the large crack aspect ratios, the thickness of the plate, ($B$), changed to 2 mm, while the crack depth ratio ($\alpha/B$) kept constant to 0.5. The modified FE model is illustrated in Fig. 3-13.

![Finite element model after modifications](image)

Since, not only the thickness but also the mesh refinement had to be modified, additional validation of the FE model is carried out in order verify that the model still gives accurate results. Chosen, is a crack of aspect ratio equal to 1, while the crack depth ratio is again 0.5.
The results can be seen in Fig. 3-14 for both membrane and bending loading.

Fig. 3-14 $K_I$ for a surface crack in a plate subjected to (a) membrane loading, (b) bending loading.
The results are again in good agreement with the results from the parametric equations proposed by BS 7910. Subsequently, the model can be used for the evaluation of the stress intensity factors for the chosen aspect ratios.

3.3.2 Results

The chosen crack aspect ratios ($\alpha/2c=1.5, 2.0$) are analysed both in membrane and bending loading. The value of the crack depth, ($\alpha$), is 1 mm (half specimen depth) and the value of the crack length, ($2c$), is changing in order to meet the crack aspect ratios value. The results can be seen in Fig. 3-15 for both loading cases.
Fig. 3-15 $K_I$ for a surface crack in a plate subjected to (a) membrane loading, (b) bending loading

Since, there is no data available in the literature for comparison, the results cannot be verified. However, based on the proven validity of the modelling method, the stress intensity factors obtained herein should be valid and should be useful in correlating fatigue crack growth rates as well as fracture toughness calculations for the surface crack configurations considered.
4. **A STUDY ON FATIGUE CRACK GROWTH ANALYSIS**

4.1 **FATIGUE CRACK GROWTH USING ANALYTICAL FORMULAS**

4.1.1 **Basic procedure for performing fatigue crack growth calculation**

The basic knowledge and the theoretical background for fatigue crack growth analysis have already been presented in Chapter 2 where the reader is referred (see section 2.5). In this section, the basic procedure for performing fatigue crack growth analytical calculations is documented. In the procedure, which is implemented in Mathcad®, the following guidelines are followed based on the BS 7910 [3] general procedure for fracture mechanics fatigue assessment:

- The crack is grown in the width and depth directions using the Paris crack growth law. It is assumed that the same relationship applies for crack growth in both directions.
- The Paris law constants are taken to be $A=5.21\times10^{-13}$ and $m=3.0$ as recommended by BS 7910 [3] for steels operating in the air or other non-aggressive environments.
- A single crack is assumed to exist, unless otherwise stated.
- The number of cycles is incremented by one for each increment of crack growth.
- Failure is deemed to have occurred when the crack grows through the specimen thickness or width, or when the predetermined crack length is reached.
- For the flaw dimensions and position, the stress intensity factor range, $\Delta K$, corresponding to the applied stress range, $\Delta \sigma$, is estimated from the stress intensity factor solutions provided by Annex M of BS 7910 [3].
In Fig. 4-1, the basic procedure in a step by step flow chart is illustrated.

Fig. 4-1 Basic procedure for fatigue crack growth calculations
4.1.2 Fatigue crack growth of a surface cracked plate

With purpose a fatigue crack growth analysis, a steel specimen containing a surface crack is assumed. The steel specimen is a plate of 10 mm length (L), 10 mm width (W) and 1 mm thickness (B) while the crack is 1 mm wide (2c) and 0.3 mm deep (a). The rest of the input parameters required, see Fig. 4-1, are as follows:

- The stress range, $\Delta \sigma = 100 \text{ N/mm}^2$
- The threshold value for the stress intensity factor range, $\Delta K_{th} = 63 \text{ N/mm}^{3/2}$
- Crack growth increment, $\Delta a = 0.03 \text{ mm}$
- The final crack length, $a_f = 0.6 \text{ mm}$

For the calculations, as mentioned before, a Mathcad® sheet was created. The full Mathcad® sheet can be found in Appendix A, where a solved example is given. The results of the calculations are illustrated in Fig. 4-2, where the crack shape evolution is depicted and in Fig. 4-3, where a crack length versus number of cycles graph is presented.

Fig. 4-2 Crack shape evolution from Mathcad® sheet calculations
Fig. 4-3 Crack length versus number of cycles data from analytical solution
4.2 FATIGUE CRACK GROWTH USING FEM

4.2.1 Introduction

Numerical techniques have been successfully employed to predict crack shape evolution and fatigue life. One of the most powerful and widely used in literature for this purpose consists of an iterative procedure based on 3D FEM analysis [44], [45], [46]. Firstly, a 3D finite element model is developed considering problem specificities, such as the geometry of the cracked body, the initial crack shape and the respective dimensions, the boundary conditions and the properties of the material. Secondly, the stress intensity factors along the crack front are calculated. Finally, the crack advances using a crack growth law relationship creating this way the new crack front. The whole procedure is repeated until a predetermined failure criterion is reached [47].

Two main methodologies can be distinguished in relation to the aforementioned technique. The first, proposed by Raju and Newman [44],[45], considers the surface and deepest points at the crack front and assumes that a particular crack shape (semi-elliptical) is maintained during the whole crack propagation procedure. The second, proposed by Lin and Smith[46], considers several points along the crack front eliminating in this way the crack shape restriction. In this chapter, the methodology used by Raju and Newman is utilized again in order to predict fatigue crack growth in a surface cracked steel specimen.

4.2.2 Crack advance and re-meshing

The crack growth technique employed in this study is illustrated in Fig. 4-4. The procedure can be divided into three main parts: pre-processing, processing and post-processing. The
first one is dedicated to the definition of the problem. Therefore, material properties, geometry, boundary conditions, loading and crack shape must be defined. The second step consists of three successive cyclic steps which are repeated until the predetermined failure criterion is reached: the finite element model generation, the stress intensity factor calculation and the crack growth model. The last step, the post-processing, consists of the analysis of the results.

Fig. 4-4 Algorithm of the re-meshing crack growth technique

The effect of the processing part on the crack front is shown in Fig. 4-5. First, the FE mesh with the initial crack front is generated in order to extract the stress intensity factors. Then the displacements of the nodes are calculated based on Paris law and eventually the new crack front is generated.

Fig. 4-5 Crack advance: a) initial crack front b) K calculation c) node displacement d) new crack front[47]
4.2.3 Fatigue crack growth of surface cracked plates using FEM

For the purpose of comparison, the same surface cracked steel specimen used in section 4.1 is used again. Therefore a steel specimen 10 mm wide (W), 1 mm thick (B) and 10 mm long (L), containing a 0.3 mm deep (α) and 1 mm wide (2c) initial crack, is generated, see Fig. 4-6. The material properties used for the analysis are that of steel. Therefore, the Young’s modulus and the Poisson’s ratio used are 210 KN/mm² and 0.3, respectively. The FEM analyses are conducted in ABAQUS[41] finite element software package.

![Surface cracked steel specimen with dimensions](image_url)

For the generation of the FE mesh and the calculation of the stress intensity factor, K, the methodology described in chapter 3 is utilized. Once the FE calculation is successfully completed, the stress intensity factors are extracted and the stress intensity factor ranges are calculated. Fig. 4-7 illustrates the stress intensity factors range at the deepest and the surface points of the crack for each crack growth step.
Following, the increments of crack growth for the deepest and the surface points of the crack front and the corresponding increment of fatigue cycles are calculated based on Eq. 4-1 and Eq. 4-2 respectively [46].

\[ \Delta a_i = \left( \frac{\Delta K_i}{\Delta K_{max}} \right)^m \Delta a_{max}, \quad i = 1, 2, ..., \]  

Eq. 4-1

\[ \Delta N = \frac{\Delta a_{max}}{C(\Delta K_{max})^m} \]  

Eq. 4-2

Where, \( \Delta a_{max} \) is the maximum crack growth increment at the point where the maximum stress intensity factor range along the crack front occurs (in our case, the surface or the deepest point). The value of \( \Delta a_{max} \) is specified and it should usually be kept a small constant.
throughout the analysis in order to achieve good numerical accuracy [46]. For the purpose of the present thesis the value of $\Delta a_{\text{max}}$ is chosen as 0.03 mm.

Then, the new crack front is reconstructed based on the new position of the surface and deepest nodes, while the shape remains semi-elliptical. Repeating the calculations enable the step by step tracking of the development of the fatigue crack.

As in section 4.1.2, the crack growth until it reaches a crack depth ($a$) of 0.6 mm is investigated. Subsequently, as illustrated in Fig. 4-7, ten crack growth steps are performed. The results can be seen in Fig. 4-8, where the crack length versus number of cycles graph from the FEM solution alongside with the respective results from the analytical calculations (see section 4.1.2) is illustrated and in Fig. 4-9, where the crack shape evolution is depicted.

![Crack length versus number of cycles data from numerical and analytical solutions](image)
The results from the FEM analyses are in general in good agreement with the respective from the analytical solution. For the FEM solution the crack grows to a depth of 0.6 mm in 582740 cycles while for the analytical solution in 592600 cycles which yields a deviation of 1.66%. The FEM solution after 582740 cycles yields a crack length of 1.55 mm while for the analytical solution the respective crack length is 1.58 mm in 592600 which leads to a difference of 1.96%.

Fig. 4-9 Crack shape evolution from finite element calculations
4.3 FATIGUE CRACK GROWTH USING XFEM

4.3.1 XFEM framework

Modelling stationary or growing cracks with the conventional finite element method requires that the mesh conforms to the geometrical discontinuities. Therefore, a focused mesh with the highest level of mesh refinement at the crack tip is required (see section 3.1.2). In case of a growing crack as that is the case in fatigue cracks, this method is not appropriate unless the focused region moves with the crack tip (see section 4.2). However, this is quite cumbersome and time consuming since the mesh must be updated continuously in order to match the geometry of the discontinuity as the crack progresses [41].

The shortcomings confronted in conventional FEM associated with meshing crack surfaces can be alleviated by the Extended Finite Element Method (XFEM). XFEM is an extension of the conventional finite element method based on the partition of unity [48]. Using the partition of unity concept, XFEM adds a priori knowledge about the solution in the finite element space and makes it possible to model discontinuities independently of the mesh. This a priori knowledge consists of enrichment functions which are added to the finite element approximation to represent the discontinuity and the singularity around the crack tip. This makes it very attractive in simulation of crack propagation since it is not necessary to update the mesh continuously in order to match the geometry of the discontinuity. The crack can propagate in an arbitrary, solution-dependent path [41]. ABAQUS commercial software enables the use of XFEM to model crack propagation and will be the finite element package used for the purpose of the present thesis.
4.3.2 Modelling fatigue crack propagation based on the principles of LEFM

The XFEM based LEFM approach provided by ABAQUS [41] can be used to simulate discrete crack growth in the bulk material. Therefore, problems in which brittle fracture or small scale yielding occur can be confronted. The crack growth is characterized by using the Paris Law through which the fracture energy release rates can be related to the crack growth rates. Subsequently, the fracture energy release rate is calculated and the rate of the crack growth per cycle is given[41].

![Fatigue crack growth governed by the Paris law](image)

Fig. 4-10 Fatigue crack growth governed by the Paris law [41]

If $G_{\text{thresh}} < G_{\text{max}} < G_{\text{pl}}$, ($G_{\text{max}}$ corresponds to the cyclic energy release rate when the structure is loaded up to its maximum value [41]) see Fig. 4-10, then the crack length is extended by fracturing at least one element ahead of the crack front. Because the progressive damage in a material is quite slow and the computational costs are high, the damage extrapolation
technique is implemented [41] to accelerate the analysis. Therefore, the crack length, \( a_N \), at the end of a stabilized cycle \( N \) is extended forward over a number of cycles, \( \Delta N \), to \( a_{N+\Delta N} \), by releasing at least one element at the crack front, see Fig. 4-11. As the crack propagates, the load is redistributed and a new fracture energy release rate is calculated. In case \( G_{\text{thresh}} > G_{\text{max}} \), then there is no consideration of fatigue crack propagation. When \( G_{\text{max}} > G_{\text{pl}} \), the elements at the crack tips will be released by increasing the cycle number count, \( \Delta N \), by one only in order to represent the accelerated crack growth rate [41]. The whole fatigue crack growth procedure in ABAQUS is conveniently summarized in Fig. 4-11.

\[
\Delta G = G_{\text{max}}(P_{\text{max}}) - G_{\text{min}}(P_{\text{min}})
\]

\[
\alpha_{N+\Delta N} = \alpha_N + \Delta N \cdot \Delta a \\
N + \Delta N
\]

\[
\frac{da}{dN} = \frac{c_3 \Delta G^{\alpha_4}}{c_2}
\]

\[
N_0 = c_1\Delta G^{\alpha_2}
\]

\[
\text{Damage extrapolation: Calculate the incremental number of cycles, } \Delta N, \text{ for each crack tip and find minimum cycles to fail, } \Delta N_{\text{min}}
\]

\[
\text{Repeat the above process until the maximum number of cycles is reached or until the ultimate load carrying capability is reached.}
\]

Fig. 4-11 Fatigue crack growth in ABAQUS [42]

For the use of the XFEM based LEFM approach in ABAQUS, as can be seen in Fig. 4-11, several additional information are needed. All the details about the required input parameters are discussed in Appendix B.
4.3.3 Modelling difficulties in the use of the XFEM

Despite the advantages that XFEM provides as a numerical method, the several modelling difficulties aroused during this research, did not make it possible to acquire reliable fatigue crack growth results in the time frame of the present thesis. The results acquired were either no crack propagation or very fast failure (in 15-20 cycles) of the steel plate. The modelling difficulties which prevented an accurate fatigue crack growth analysis are summarized below:

- The crack has to propagate across an entire element at a time to avoid the need to model the stress singularity, see Fig. 4-12. Consequently, one of the key parameters to simulate crack growth is the mesh size. This mesh dependency prevents the acquisition of reliable fatigue crack growth results.

![Fig. 4-12 Crack propagation in XFEM](image)

- The crack shape and dimensions always adapt to the shape and the dimensions of the element selected for the analysis. Therefore, it is not possible to model precisely a crack with a semi-elliptical shape, see Fig. 4-13.

![Fig. 4-13 Crack shape transformation for FE calculations](image)
The damage extrapolation technique which is used to accelerate the analysis inserts uncertainty to the solution and does not allow for realistic prediction of the evolution of the crack.

The mixed mode law which is used for the calculation of the critical equivalent release rate, $G_{\text{equiv},C}$, is chosen empirically [41]. However, the choice of the mixed mode law proves to significantly affect the results of the analysis and does not allow for accurate crack growth predictions.

As a concluding remark, from the research on the capabilities of XFEM to reliably predict the fatigue crack growth of a steel plate, it stem that further research and clarifications on the XFEM theory and application are needed. As mentioned before, in appendix B of the present thesis all the required parameters to perform a fatigue crack growth analysis are described. Therefore, we welcome future researchers to use this appendix in order to improve the current results by eliminating the modelling difficulties listed before.
5. CONCLUSIONS AND RECOMMENDATIONS

5.1 CONCLUSIONS

The aim of this thesis was to provide an insight on the stress intensity factor and on the various numerical and analytical methods to perform a fatigue crack growth analysis. During this study, it aroused that further research on the limitations of the stress intensity factor solutions given by BS 7910 is required. Furthermore, from research on the various fatigue crack growth procedures, it has become evident that some can easily be applied while other require further clarification and research. Based on this study the following statements can be concluded:

- It has been seen from literature study that the flat plate SIF solutions contained in Annex M of BS 7910 are the most important since they are the basis for the derivation of the SIF of almost any other geometrical shape considered.

- If the SIF is not available in BS 7910, FE calculations should be considered.

- The comparison of the SIFs for semi-elliptical cracks resulted from FEM analysis with the respective BS 7910 solutions, shows excellent accuracy. Consequently, the FEM procedure employed in this report to acquire the SIFs is quite acceptable.

- The “spider-web” meshing technique has been proved through literature study and practical implementation in this report to be the most appropriate mesh configuration for the FE calculation of the SIF.
SIFs outside the limits of BS 7910 have been developed. However further research is required in order to validate the results and to obtain SIFs for the full range of crack aspect and crack depth ratios.

In Chapter 4 of this report, a fatigue crack growth example is solved using both analytical and FE procedures. By comparison, it is evident that both methods provide approximately the same results. Therefore the FEM procedure can be used in case the SIFs required are not available.

From research on the XFEM capabilities to perform fatigue crack growth analysis, it is evident that the crack growth is mesh-dependent. Therefore the CPU capabilities and the choice of the meshing strategy significantly affect the results.

The use of XFEM to predict fatigue crack growth needs further clarifications and research in order to provide realistic results.
5.2 RECOMMENDATIONS

Based on this study, the following recommendations are suggested for future research:

- The SIF solutions for tubular joints contained in BS 7910 need further research since the solutions provided are limited.

  The SIFs for embedded flaws in flat plates provided by BS 7910, have similar limitations to those of the surface flaws. A study might be done to determine the SIFs outside these limits.

- In the context of the re-meshing technique, a maximum crack growth increment, $\Delta a_{\text{max}}=0.03$ mm is used. A research on the magnitude of the crack growth increment, $\Delta a_{\text{max}}$, in relation to the crack depth, $a$, and comparison with the analytical solution is suggested to fully verify the method.

- In Appendix B of this report all the required information to conduct a three dimensional fatigue crack growth analysis in ABAQUS commercial software are given. Taking into account the modelling difficulties presented in section 4.3.3, further research might be done on the feasibility of a three dimensional fatigue crack growth analysis.

- In the time frame of the present thesis it was not possible to conduct a two dimensional fatigue crack growth analysis. For such situation, study has to be carried out to examine the additional parameters needed for the analysis. The results can be compared with the respective from the analytical solution and the FEM (re-meshing) technique.
REFERENCES


APPENDICES

A. MATHCAD SHEET ........................................................................................................................................ 2
B. REQUIRED INPUT FOR ABAQUS ........................................................................................................... 5
A. MATHCAD SHEET

A surface crack steel specimen in accordance with section 4.1.2 is assumed:

<table>
<thead>
<tr>
<th>Crack Dimensions</th>
<th>Specimen Dimensions</th>
<th>Input data</th>
</tr>
</thead>
<tbody>
<tr>
<td>a1 = 0.3</td>
<td>L1 = 10</td>
<td>( \Delta \sigma = 100 )</td>
</tr>
<tr>
<td>c1 = 0.5</td>
<td>W1 = 10</td>
<td>( \Delta K_{\text{threshold}} = 63 )</td>
</tr>
<tr>
<td></td>
<td>B1 = 1</td>
<td>( N_c = 0 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( A_p = 1.21 \cdot 10^{-13} )</td>
</tr>
</tbody>
</table>

\[
\text{Crack Growth} = \text{for } N_c = 1,2, 500000
\]

\[
\begin{align*}
\text{break if } & a_1 \geq 0.33 \\
\text{break if } & c_1 \geq \frac{W_1}{2} \\
M_1 & = 1 \\
M_2 & = 1.13 - 0.05 \left( \frac{a_1}{c_1} \right) \text{ if } \left[ \frac{a_1}{2c_1} > 0 \right] \land \left( \frac{a_1}{2c_1} \leq 0.5 \right) \\
& \left[ \left( \frac{c_1}{a_1} \right)^{0.5} \left( 1 + 0.64 \left( \frac{c_1}{a_1} \right) \right) \right] \text{ otherwise} \\
M_3 & = 0.5 - \frac{1}{0.65 + \frac{a_1}{c_1}} + 14 \left( 1 - \frac{a_1}{c_1} \right)^{24} \text{ if } \left[ \frac{a_1}{2c_1} > 0 \right] \land \left( \frac{a_1}{2c_1} \leq 0.5 \right) \\
& \left[ -0.11 \left( \frac{c_1}{a_1} \right)^{4} \right] \text{ otherwise} \\
g_1 & = \left[ 1.1 + 0.35 \left( \frac{a_1}{B_1} \right)^2 \right] \text{ if } \left[ \frac{a_1}{2c_1} > 0 \right] \land \left( \frac{a_1}{2c_1} \leq 0.5 \right) \\
& \left[ 1.1 + 0.35 \left( \frac{c_1}{a_1} \right) \left( \frac{a_1}{B_1} \right)^2 \right] \text{ otherwise} \\
g_4 & = 1 \\
f_{0,1} & = \left( \frac{a_1}{c_1} \right)^{0.5} \text{ if } \left[ \frac{a_1}{2c_1} > 0 \right] \land \left( \frac{a_1}{2c_1} \leq 0.5 \right)
\end{align*}
\]
The previous Mathcad® sheet demonstrates the first crack growth step of the fatigue crack growth analysis from, \( \alpha=0.3 \text{ mm} \), to \( \alpha=0.33 \text{ mm} \) (\( \Delta \alpha_{\text{max}}=0.03 \text{ mm} \)). The above procedure is
repeated until the crack grows to a depth of 0.6 mm.

For the mathcad calculations all the input data and results are given in N, mm and MPa. The basic procedure and the Mathcad® sheet were validated in accordance with reference [49].
B. REQUIRED INPUT FOR ABAQUS

The input required to perform fatigue crack growth analysis in ABAQUS is discussed in the following paragraphs. It is assumed that the reader is familiar with ABAQUS [41] and the syntax used in the input file (.inp). Therefore, the focus is on the specific input that relates to low-cycle fatigue and to direct cyclic approach in ABAQUS.

Input for cyclic loading

The direct cyclic approach in ABAQUS is used to obtain the stabilized response of a structure directly:

*Direct Cyclic, fatigue

<\i_0>, <\i_t>, <\n>, <\n_{max}>, <\Delta n>, <\i_{max}>,
<\Delta N_{min}>, <\Delta N_{max}>, <\i_T>,

Where, the parameter fatigue is used to perform a low-cycle fatigue analysis. In the first line the parameters defined are, the initial time increment, <\i_0>, the time of a single loading cycle (default used), <\i_t>, the minimum and maximum time increment (not used), the initial number of terms in the Fourier series, <\n>, the maximum number of terms in the Fourier series, <\n_{max}>, the increment in number of terms in the Fourier series, <\Delta n>, and the maximum number of iterations allowed in a step, <\i_{max}>. The parameters specified in the second line are the minimum and maximum increment in number of cycles over which the damage is extrapolated forward, <\Delta N_{min}>, <\Delta N_{max}>, the total number of cycles allowed in a step, <\i_T>, and the damage extrapolation tolerance (default used) [50], [41].

For the needs of the present thesis, a load cycle of 1 second (default value) is used with an initial time increment of 0.25 second. The initial and maximum numbers of terms in the Fourier series which partly control the solution accuracy, are set as 25 while for the
increment, $<\Delta n>$, the default value is used. Regarding the maximum number of iterations allowed in a step $<i_{\text{max}}>$, which also controls the accuracy of the solution, a value of 20 is set in accordance with ABAQUS example problems [51]. The number of cycles allowed in a step, as well as the minimum and the maximum number of cycles over which the damage is extrapolated forward, depend on the analysis.

**Input for low-cycle fatigue criterion**

In order to carry out a low-cycle fatigue analysis a fracture criterion must be defined:

```
*FRACTURE CRITERION, TYPE=fatigue, MIXED MODE BEHAVIOR=POWER
<\text{c}_1>, <\text{c}_2>, <\text{c}_3>, <\text{c}_4>, <G_{\text{thresh}}/G_{\text{equivC}}>, <G_p/G_{\text{equivC}}>, <G_{\text{IIC}}>, <G_{\text{IIIC}}>
<G_{\text{IIIC}}>, <a_m>, <a_n>, <a_o>
```

Where, **MIXED MODE BEHAVIOR** is the type of mode-mix formulae for computing the critical equivalent energy release rate, $G_{\text{equivC}}$. ABAQUS provides three mode-mix formulae for the calculation of the critical equivalent energy release rate: the BK law, the Power law and the Reeder law [41]. The parameters $<\text{c}_1>$, $<\text{c}_2>$ are obtained by solving the law for growth onset, see Eq. 0-1.

$$\frac{N}{c_1 \Delta \text{c}^2} \geq 1$$  \hspace{1cm} \text{Eq. 0-1}

In case immediate onset is desirable in the analysis as that is in the present thesis, parameters suggested in ABAQUS example problems [51] may be chosen ($c_1=0.5$, $c_2=-0.1$) [50]. The parameters $<\text{c}_3>$ and $<\text{c}_4>$ are obtained directly from the Paris Law [3], see Eq. 0-2.
Next, the ratio $<G_{\text{thresh}}/G_{\text{equivC}}>$, and the ratio $<G_{\text{pl}}/G_{\text{equivC}}>$, are calculated based on the energy release rate cut-off value, see BS 7910 [3], $G_{\text{th}}$, and the energy release rate upper limit, $G_{\text{pl}}$ (default used). The parameters $<G_{I}>$, $<G_{II}>$ and $<G_{III}>$, represent the fracture toughness of the material, while, the rest of the parameters, $<a_m>$, $<a_n>$ and $<a_o>$ are related to the mode-mix formulae which in this case is the Power law [41].