FLOW EQUATIONS FOR RIVER BENDS DERIVED BY TENSOR CALCULUS

and

FLOW ACCELERATION IN DEPTH-AVERAGED MODELS

by

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ABSTRACT

Flow equations for application in river bends are derived. Coordinate transformations are achieved by tensor calculus. The momentum equations are formulated in a cylindrical coordinate system and in a cylindrical bed-following coordinate system. The depth-averaged momentum equations in a cylindrical bed-following coordinate system are derived. The resulting equations agree with the formulations known from literature.

In depth-averaged flow models the flow is mostly modelled by similarity velocity profiles that are based on the assumption of local equilibrium. In regions of gently varying bed topography the effect of flow accelerations, or decelerations, on the velocity profiles and bed shear-stresses is studied. The conclusion is that the bed shear-stresses are the most affected. The effect on the flow velocities is small.

PREFACE

This document contains some theoretical analyses which were conducted during the writer’s PhD-study, entitled "Bed topography in river bends with suspended sediment transport". This study was supported by the Netherlands Technology Foundation (STW).

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</tr>
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<td>A non-orthogonality coefficient</td>
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<td>A averaged water depth</td>
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<td>b bed level</td>
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<td>B non-orthogonality coefficient</td>
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<td>fₘ similarity function main flow velocity</td>
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<td>fᵤ, fᵤ₁ main flow similarity functions</td>
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<td>g₁      gravitational acceleration</td>
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<tr>
<td>gᵢⱼ^{-¹} inverse of the metric tensor</td>
<td>[-]</td>
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<td>J Jacobian</td>
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<td>k secondary flow convection coefficient</td>
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<td>K coefficient</td>
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<td>L length scale of variations in main flow direction</td>
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<td>Mᵢ, Mᵢ₀ coefficients</td>
<td>[-]</td>
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<tr>
<td>p pressure</td>
<td>[N/m²]</td>
</tr>
<tr>
<td>Pᵣ₁ pressure at the rigid lid</td>
<td>[N/m²]</td>
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<tr>
<td>P atmospheric pressure</td>
<td>[N/m²]</td>
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<td>r radius</td>
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<td>s water surface</td>
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<td>sᵢⱼ strain rate component</td>
<td>[ls⁻¹]</td>
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<td>t time</td>
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<td>T ratio local and base state bed shear stress</td>
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<tr>
<td>u time-averaged velocity in x-direction</td>
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<td>uₙ bed shear-stress velocity</td>
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<td>uᵢ time averaged velocity component</td>
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<td>uᵢ⁽¹⁾ local i th order velocity</td>
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<tr>
<td>uᵢ, uᵢ², uᵢ³ covariant velocity components</td>
<td>[m²/s], [m/s], [m²/s]</td>
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</tbody>
</table>
\( u^1, u^2, u^3 \) contravariant velocity components [1/s], [m/s], [1/s]
\( u_1, u_2, u_3 \) velocity components in \( x_1, x_2, x_3 \) direction [m/s]
\( u_\theta, u_\tau, u_\zeta \) velocity components in \( \theta, \tau, \zeta \) direction [m/s]
\( u^\theta, u^\tau, u^\zeta \) depth-averaged velocity components in \( \theta, \tau, \zeta \) direction [m/s]
\( U \) depth-averaged velocity in \( x \)-direction [m/s]
\( x \) coordinate [m]
\( z \) vertical coordinate [-]
\( z_b \) bed level [m]

\( \alpha \) roughness coefficient; \( \alpha = g/(\kappa c) \) [-]
\( \beta \) direction coefficient of the secondary flow [-]
\( \beta_i \) coefficients in asymptotic approximation [-]
\( \Gamma^{jk}_{\ell} \) Christoffel symbol [-]
\( \delta \) small parameter; \( \delta = 1/\kappa A/L C/g \) [-]
\( \delta_{ij} \) Kronecker delta [-]
\( \epsilon \) small parameter; \( \epsilon = A/L C^2/g \) [-]
\( \xi \) stretched vertical coordinate [-]
\( \xi_0 \) zero-velocity level [-]
\( \theta \) angular coordinate [rad]
\( \kappa \) von Kármán constant, \( \kappa = 0.4 \) [-]
\( \nu \) kinematic viscosity [m^2/s]
\( \rho \) density water [kg/m^3]
\( \sigma_{ij} \) stress components [ ]
\( \tau \) bed shear-stress [N/m^2]
\( \tau_i \) local i\(^{th}\) order shear stress contribution [N/m^2]
\( \tau_i(b) \) local i\(^{th}\) order bed shear-stress contribution [N/m^2]
\( \tau_{\theta \tau}, \tau_{\tau \tau}, \tau_{\tau \zeta}, \tau_{\zeta \zeta} \) Reynolds stresses [N/m^2]
\( \tau_{\text{bed s}} \) bed shear-stress in s-direction [N/m^2]
\( \tau_{\text{bed n}} \) bed shear-stress in n-direction [N/m^2]

\( A \) second order tensor [m/s^2]
\( g \) gravitation vector \((0,0,-g)\) [m/s^2]
\( u \) velocity vector [m/s]
\( \alpha_\perp, \alpha_\parallel \) unit base vectors of rectangular cartesian coordinates [-]
\( \alpha_1, \alpha_2, \alpha_3 \) covariant base vectors \([m],[-],[m]\)
\( \eta_1, \eta_2, \eta_3 \) contravariant base vectors \([1/m],[-],[1/m]\]
\( \lambda_\perp, \lambda_\parallel \) unit vectors in \( \theta \) - and \( \tau \) -directions [-, m]
\( \sigma \) second order stress tensor
1 INTRODUCTION

Formal derivations of the equations for the flow in river bends are given in this report. In river bends it advantageous to use a coordinate system which is aligned with the bend. The flow is governed by the continuity and momentum equations. These have to be formulated in such an coordinate system. In the literature the equations are given on many occasion. Their formal derivation, however, is often lacking. In order to understand the origin of terms and to formally derive the basic equations, tensor calculus is used.

In depth-averaged flow models the flow is mostly modelled by similarity velocity profiles that are based on the assumption of local equilibrium. In regions of gently varying bed topography the effect of flow accelerations, or decelerations, on the velocity profiles and bed shear-stresses is studied. In morphological models this could be important because sediment transport rates are very sensitive to bed shear-stress variations.

In chapter 2 the momentum equations in a cylindrical coordinate system are derived. In chapter 3 the momentum equations in a cylindrical bed-following coordinate system are derived. In chapter 4 the depth-averaged momentum equations in a cylindrical bed-following coordinate system are derived. In chapter 5 the method of Kalkwijk and de Vriend (1980) of simplifying the set of momentum equations, by considering the characteristics of the set of equations, is documented. In chapter 6 the effect of flow accelerations (or decelerations), due to gently varying bed topography, on the flow velocity and the bed shear-stress are investigated. The conclusions are given in chapter 7.
2 MOMENTUM EQUATIONS IN CYLINDRICAL COORDINATES

2.1 Introduction

The momentum equations are derived in a cylindrical coordinate system. To transform the equations from an orthogonal Cartesian coordinate system to the cylindrical coordinate system tensor calculus is applied. The formulation of the equations in this coordinate system has been presented by many authors. The purpose of the derivations presented in this chapter is to get acquainted with the formal derivation by tensor calculus and to understand the origin of terms.

In section 2.2 the momentum equation in orthogonal Cartesian coordinates is discussed. In section 2.3 the appropriate tensor calculus theory is given. In section 2.4 the one to one relation between the coordinate systems is given. In section 2.5 the transformation of the convective term is given. In section 2.6 the pressure term is transformed. In section 2.7 the turbulent stress term is transformed. In section 2.8 finally the momentum equations in the cylindrical coordinate system are given.

2.2 The equations in rectangular Cartesian coordinates

The constitutive equations are most clearly formulated in a rectangular Cartesian coordinate system \((x_1,x_2,x_3)\) with right-hand orientation of the unit vectors: \(i_1 \times i_2 = i_3\). This coordinate system is used as a reference, stating basic definitions and for reason of comparison. The designation Cartesian refers to the property that the positions of the points are determined by their distance from intersecting planes of \(x_i = \text{const.}\) (the unit vectors need not to be orthogonal). If the unit vectors are orthogonal the system is referred to as a rectangular Cartesian coordinate system.

In a rectangular Cartesian coordinate system \((x_1,x_2,x_3)\), the Navier-Stokes equations have the following form (conservative form, constant \(\rho\)):

\[
\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} - \delta_{ij} \frac{\partial \mathbf{g}}{\partial t}, \quad \text{for } i=1,2,3. \tag{2.1}
\]

With: \(\sigma_{ij} = \) time averaged turbulent stress components;

\[
\sigma_{ij} = -\rho \overline{u_i' u_j'} = \rho \nu 2 s_{ij} \quad \text{(no summation)} \tag{2.2}
\]

\(s_{ij} = \) the strain rate components; \(s_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{2.3}\)

\(\nu = \) kinematic viscosity (isotropic)
In tensor notation the Navier-Stokes equations are formulated by:

\[
\frac{\partial u_i}{\partial t} + \nabla \cdot (u_i u) = -\frac{1}{\rho} \nabla p + \nabla \cdot (\tau_i) + g
\]

in which: \( A = \) second order momentum tensor, \( A_{ij} = u_i u_j \)

\[
(A)_i = i^{\text{th}} \text{ row or column of the second order tensor}
\]

\( \sigma = \) second order stress tensor

\( i_i = \) unit vector of the coordinate system

\( g = \) gravitation vector, \( g = (0,0,-g) \)

2.3 Tensor calculus

In an other coordinate system \((x_1,x_2,x_3)\) the equations have to be invariant, i.e. the physical meaning has to be the same. The new coordinate system can be specified by equations:

\[
x_1 = f_1(x_1,x_2,x_3), \quad x_1 = h_1(x_1,x_2,x_3)
\]

or in vector notation:

\[
\hat{x} = f(x), \quad x = h(x)
\]

The functions \( f \) and \( h \) relate one and only one value of \( x \) to \( \hat{x} \) and vise-versa. The functions are such that the three coordinates are independent of each other. Because of these properties of the functional relationship, the new coordinates are called curvilinear. If the surfaces \( x_i = \text{const.} \) are orthogonal, these are called orthogonal curvilinear coordinates.

Tensor calculus will be used to transform the equation to a cylindrical coordinate system.

In tensor calculus two sets of base vectors of a coordinate system are defined \((\hat{r}_1, \hat{r}_2, \hat{r}_3 \) and \( \hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3 \), fig.2.1.
The co- and contra-figure 2.1 Co- and contra-variant base vectors \( (\hat{r}_i, \hat{\eta}^i) \). The co- and contra-variant components of \( u \) are \( \hat{u}_i \) and \( \hat{u}^i \).

The covariant base-vectors are tangent to the coordinate axes;
\[
\hat{r}_i = \frac{\partial x}{\partial \hat{x}_i}
\tag{2.7}
\]
The contravariant base-vectors are normal to surfaces of \( \hat{x}^i \) = const.;
\[
\hat{\eta}^i = \frac{\partial \hat{x}^i}{\partial x_j}
\tag{2.8}
\]
The two sets of base vectors are perpendicular: \( \hat{r}_i \cdot \hat{\eta}^j = \delta^j_i \) \( \tag{2.9} \)
in which: \( \delta^j_i = 1 \) for \( i = j \), \( \delta^j_i = 0 \) for \( i \neq j \)
A vector, for instance, can be represented in both sets of base vectors. The representation of a vector \( u \) in respectively the covariant and contravariant base is given by, fig.(2.1):
\[
u = u^i \hat{r}_i = u^1 \hat{r}_1 + u^2 \hat{r}_2 + u^3 \hat{r}_3
\tag{2.10}
\]
\[
u = u_i \hat{\eta}^i = u_1 \hat{\eta}^1 + u_2 \hat{\eta}^2 + u_3 \hat{\eta}^3
\tag{2.11}
\]
in which: \( \hat{u}_1 \), \( \hat{u}_2 \) and \( \hat{u}_3 \) are the orthogonal components of the vector \( u \) defined in the covariant base \( \hat{r}_i \). These are called contravariant components of the vector \( u \).
\(\hat{u}_1, \hat{u}_2, \text{and } \hat{u}_3\) are the orthogonal components of the vector \(u\) defined in the contravariant base \(\hat{\eta}^i\). These are called covariant components of the vector \(u\).

To determine to co- and contravariant components of a vector the property given by eq. (2.9) is very useful. Multiplication of eq. (2.10) and eq. (2.11) by \(\hat{\eta}^i\) and \(\hat{\tau}^i\) respectively yields:

\[
\begin{align*}
\text{u. } \eta^i &= u_1 \hat{\tau}^1 \eta^i = u^j \\
\text{u. } \tau_j &= u_1 \hat{\eta}^1 \tau_j = u_j
\end{align*}
\]

(2.11a)

(2.11b)

A correct representation of the mathematical operators is important. The formulation of the gradient of a scalar \(A\) in the new coordinate system is, for example, given by:

\[
\nabla A = \frac{\partial A}{\partial x^j} \hat{\eta}^j = \frac{1}{\sqrt{g}} \frac{\partial A}{\partial \hat{\eta}^j}
\]

(2.11c)

This represents the variation of \(A\) along the transformed coordinate lines, nonuniformity and nonorthogonality of the coordinate system are accounted for by \(\hat{\eta}^j\). The gradient of \(A\), \((\nabla A)\), is a vector. With the representation of eq. (2.11c) the vector is represented in the contravariant base, with \(\partial A/\partial \hat{\eta}^j\) being called the contravariant component of \(\nabla A\).

The geometrical properties of a transformation are reflected in the metric tensor:

\[
\hat{g}_{ij} = \hat{\tau}_i \cdot \hat{\tau}_j
\]

(2.12)

The inverse of the metric tensor is defined by:

\[
\hat{g}^{ij} = \hat{\eta}^i \cdot \hat{\eta}^j
\]

(2.13)

The transformation of unit volume is reflected in the Jacobian of the transformation:

\[
J = \left| \frac{\partial (x_1, x_2, x_3)}{\partial (\hat{x}_1, \hat{x}_2, \hat{x}_3)} \right| = \sqrt{g}
\]

(2.14)

For convenience the position of the index on a coordinate will be changed from a subscript to a superscript.

In a nonorthogonal curvilinear coordinate system the differentiation of a vector or a tensor is more complicated than in an orthogonal Cartesian system. Because of this a different notation for differentiation is used: a
The differentiation of a scalar with respect to the transformed coordinates is given by:

\[ A_{,j} = \frac{\partial A}{\partial x^j}, \quad A = \text{scalar} \]  
(2.16)

The differentiation of the contravariant components of a first order tensor (\(A\) - vector) is more complicated, because it also involves nonuniformity and nonorthogonality of the coordinate system. The derivative is given by:

\[ A^{,i}_{,j} = \frac{\partial A^{,i}}{\partial x^j} + \Gamma^i_{jk} A^k, \quad A = \text{first order tensor} \]  
(2.17)

in which: \(\Gamma^i_{jk}\) = the Christoffel symbol;

\[ \Gamma^i_{jk} = \frac{\partial x^i}{\partial x^j} \frac{\partial x^j}{\partial x^k} = \eta^{ij} \frac{\partial x^i}{\partial x^k} \]  
(2.18)

This is proven by the following:

(application of the chain rule of differentiation and \(\delta^{p}_{k} = \delta_{k}^{p}\))

\[ \frac{\partial A}{\partial x^j} = \frac{\partial (A^{,i}_{,j})}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} = \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} = \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} + \Gamma^i_{jk} \frac{\partial A^{,p}}{\partial x^j} \frac{\partial A^{,p}}{\partial x^i} = \]  
(p=k)

\[ \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} = \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} + \Gamma^i_{jk} \frac{\partial A^{,p}_{,i}}{\partial x^j} \frac{\partial A^{,p}_{,i}}{\partial x^i} = \]

\[ \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} = \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} + \Gamma^i_{jk} \frac{\partial A^{,p}_{,i}}{\partial x^j} \frac{\partial A^{,p}_{,i}}{\partial x^i} = \]

\[ \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} = \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} + \Gamma^i_{jk} \frac{\partial A^{,p}_{,i}}{\partial x^j} \frac{\partial A^{,p}_{,i}}{\partial x^i} = \]

\[ \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} = \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} + \Gamma^i_{jk} \frac{\partial A^{,p}_{,i}}{\partial x^j} \frac{\partial A^{,p}_{,i}}{\partial x^i} = \]

\[ \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} = \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} + \Gamma^i_{jk} \frac{\partial A^{,p}_{,i}}{\partial x^j} \frac{\partial A^{,p}_{,i}}{\partial x^i} = \]

\[ \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} = \frac{\partial A^{,i}_{,j}}{\partial x^i} \frac{\partial A^{,i}_{,j}}{\partial x^i} + \Gamma^i_{jk} \frac{\partial A^{,p}_{,i}}{\partial x^j} \frac{\partial A^{,p}_{,i}}{\partial x^i} = \]

Terms incorporating a Christoffel symbol are generally referred to as curvature terms (non-orthogonality of a coordinate system also yields non-zero Christoffel symbols).

In regard of the transformation of the momentum equations some specific transformation rules will be applied (these will be given without proof).

The divergence of a first order tensor \(A\) (proof in Dutton 1976 p. 142):
\[ \nabla \cdot \mathbf{A} = \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} \hat{A}^i)}{\partial x^i} \]  

(2.20)

It represents the change of the contravariant component \( \hat{A}^i \) and the change of unit volume in the direction of the coordinate axes of the transformed coordinate system.

The divergence of a second order tensor \( \mathbf{A} \) with one of its indices constant, Cuvelier 1988 (the tensor elements in the Cartesian orthogonal system are; \( A_{kp} \)):

\[ \nabla \cdot (\mathbf{A})_k = \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} (\eta^i)_{p,kp})}{\partial x^i} \]  

(2.21)

in which: \( (\eta^i)_p \) = \( p \) th element of the \( \eta^i \) base-vector

2.4 Coordinate transformation

The equations have to be transformed from the orthogonal Cartesian coordinate system to a orthogonal cylindrical coordinate system. The one to one relationship between these coordinate systems is given by (the water depth \( a \) is constant):

\begin{align}
\hat{x}^1 &= r \cos \theta \\
\hat{x}^2 &= r \sin \theta \\
\hat{x}^3 &= a \zeta \\
\end{align}

(2.22)

\begin{align}
\hat{x}^1 &= \theta = \arctg (x^2 / x^1) \\
\hat{x}^2 &= r = ((x^1)^2 + (x^2)^2)^{1/2} \\
\hat{x}^3 &= \zeta = x^3 / a \\
\end{align}

The covariant base-vectors are (tangent to the coordinate axes):

\[ \hat{\mathbf{A}}_1 = \frac{\partial \mathbf{x}}{\partial x^i}, \quad \hat{\mathbf{A}}_1 = \begin{pmatrix} -r \sin \theta \\ r \cos \theta \\ 0 \end{pmatrix}, \quad \hat{\mathbf{A}}_2 = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \quad \hat{\mathbf{A}}_3 = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \]  

(2.23)
The contravariant base-vectors are (normal to surfaces of \( x^i = \text{const.} \));

\[
\hat{\eta}^i = \frac{\partial x^i}{\partial \xi^j}, \quad \hat{\eta}^1 = \begin{pmatrix} 1/r \sin \theta \\ 1/r \cos \theta \\ 0 \end{pmatrix}, \quad \hat{\eta}^2 = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \quad \hat{\eta}^3 = \begin{pmatrix} 0 \\ 0 \\ 1/a \end{pmatrix}
\] (2.24)

The coordinate system is orthogonal, the \( \hat{\eta}^i \) and \( \hat{\eta}^i \) base vectors are identical in direction.

The only non-zero Christoffel symbols are:

\[
\hat{\gamma}^{2}_{11} = -r, \quad \hat{\gamma}^{1}_{12} = \hat{\gamma}^{1}_{21} = \frac{1}{r}
\] (2.25)

The metric tensor and the inverse of the metric tensor are (\( g_{ij} = \hat{\eta}^i \cdot \hat{\eta}^j \), \( g^{ij} = \eta^i \cdot \eta^j \)):

\[
\hat{g}_{ij} = \begin{pmatrix} r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^2 \end{pmatrix}, \quad \hat{g}^{ij} = \begin{pmatrix} \frac{1}{r^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a^2} \end{pmatrix}
\] (2.26)

The determinant (Jacobian) of the transformation is:

\[
J = \left| \frac{\delta(x_1, x_2, x_3)}{\delta(x_1, x_2, x_3)} \right| = \sqrt{g} = ra
\] (2.27)

The contravariant velocity components (= velocity components in directions tangential to the coordinate lines) are:

\[
\hat{u}^i = \eta^i \cdot u
\]

\[
\hat{u}^1 = \frac{1}{r} \sin \theta \cdot u^1 + \frac{1}{r} \cos \theta \cdot u^2
\]

\[
\hat{u}^2 = \cos \theta \cdot u^1 + \sin \theta \cdot u^2
\]

\[
\hat{u}^3 = \frac{1}{a} \cdot u^3
\] (2.28)

The physical velocity components in \( \theta \), \( r \) and \( \xi \) direction are given by:
\[ u_\theta = \lambda_\theta \cdot u = \lambda_\theta \cdot \hat{r}_1 \hat{u}_1, \quad u_r = \hat{r} \cdot \hat{u}_1, \quad u_\xi = \hat{r} \cdot \hat{u}_2, \quad u_\zeta = \hat{r} \cdot \hat{u}_3 \]  

(2.29)

In which: use is made of the fundamental relation: \( u = \hat{r}_1 \hat{u}_1 \)

: the unit vectors in \( \theta \), \( r \) and \( \xi \) direction are:

\[
\lambda_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \quad \lambda_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \lambda_\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

(2.31)

2.5 Transformation of the convective term

The convective term of the momentum transport vector is:

\[
i_1 \cdot (u_1 u) = i_1 \frac{\partial A_{i j}}{\partial x_j}, \quad \text{with } A_{i j} = u_i u_j
\]

(2.32)

According to the transformation law for the divergence of a second order tensor, eq.(2.21):

\[
i_1 \cdot (u_1 u) = i_1 \frac{\partial (\hat{g} \cdot (\eta) \hat{u}_p \cdot u_p)}{\partial \hat{x}_k} = i_1 \frac{\partial (\hat{g} \cdot (\eta) u_i u_j)}{\partial \hat{x}_k} =
\]

\[
= i_1 \frac{\partial (\hat{g} \cdot u_i u_k)}{\partial \hat{x}_k} = i_1 \frac{\partial (\hat{g} \cdot u_i u_k)}{\partial \hat{x}_k} = i_1 \frac{\partial (\hat{g} \cdot u_i u_k \hat{r}_1)}{\partial \hat{x}_k} =
\]

\[
= \frac{1}{\hat{g}} \left( \frac{\partial (\hat{g} \cdot u_i u_k)}{\partial \hat{x}_k} \hat{r}_1 + \hat{g} \cdot u_i u_k \frac{\partial \hat{r}_1}{\partial \hat{x}_k} \right) =
\]

\[
= \frac{1}{\hat{g}} \left( \frac{\partial (\hat{g} \cdot u_i u_k)}{\partial \hat{x}_k} + \hat{g} \cdot u_i u_k \frac{\partial \hat{r}_1}{\partial \hat{x}_k} \right) \hat{r}_1
\]

(2.33)

Substitution of the Jacobian and the Christoffel symbols yields:

The \( \hat{r}_1 \)-direction component is (yielded by multiplication with \( \hat{\eta}^{\hat{1}} \)):

\[
\hat{\eta}^{\hat{1}} \cdot i_1 \cdot (u_1 u) = \frac{\partial (u_1 u_1^{\hat{1}})}{\partial \hat{x}_1} + \frac{\partial (u_1 u_2^{\hat{1}})}{\partial \hat{x}_2} + \frac{\partial (u_1 u_3^{\hat{1}})}{\partial \hat{x}_3} + 2 \frac{\hat{1}_{21}^{\hat{1}} (u_1 u_1^{\hat{2}})}{\hat{r}_1} =
\]

(2.34)
Substitution of the physical velocity components, eq. (2.29), and $\hat{\Gamma}_{21}$ yields:

$$
\begin{align*}
&\frac{\partial (u_\theta u_\theta)}{\partial \theta} + \frac{\partial (u_\theta u_r)}{\partial r} + \frac{\partial (u_\theta u_r)}{\partial \zeta} + 2 \frac{1}{r} u_\theta u_r \frac{1}{r} \\
&= \left( \frac{\partial (\hat{u}_\theta \hat{u}_\theta)}{\partial \theta} + \frac{\partial (\hat{u}_\theta \hat{u}_r)}{\partial r} + \frac{\partial (\hat{u}_\theta \hat{u}_r)}{\partial \zeta} + 2 \frac{1}{r} \hat{u}_\theta \hat{u}_r \frac{1}{r} \right) \\
&= 2.35
\end{align*}
$$

The $\hat{\tau}_2$-direction component is (yielded by multiplication with $\hat{\eta}^2$):

$$
\hat{\eta}^2 \cdot \hat{\tau}_2 \nabla \cdot (u_2 u) = \left( - \frac{\partial (\hat{u}_2 \hat{u}_1)}{\partial x^1} + \frac{\partial (\hat{u}_2 \hat{u}_2)}{\partial x^2} + \frac{\partial (\hat{u}_2 \hat{u}_3)}{\partial x^3} + \hat{\Gamma}_{11} \hat{u}_1 \hat{u}_1 \frac{1}{r} \right) = 2.36
$$

Substitution of the physical velocity components and $\hat{\Gamma}_{11}$ yields:

$$
\begin{align*}
&\frac{\partial (u_r u_\theta)}{\partial \theta} + \frac{\partial (u_r u_r)}{\partial r} + \frac{\partial (u_r u_\zeta)}{\partial \zeta} + \frac{1}{r} u_r u_r - \frac{1}{r} u_\theta u_\theta \\
&= \left( \frac{\partial (\hat{u}_r \hat{u}_\theta)}{\partial \theta} + \frac{\partial (\hat{u}_r \hat{u}_r)}{\partial r} + \frac{\partial (\hat{u}_r \hat{u}_\zeta)}{\partial \zeta} + \frac{1}{r} \hat{u}_r \hat{u}_r - \frac{1}{r} \hat{u}_\theta \hat{u}_\theta \right) \\
&= 2.37
\end{align*}
$$

These expressions for the convective terms of the $u_\theta$ and $u_r$ momentum equations are the same as the expressions given by Kalkwijk and de Vriend (1980).

The $\hat{\tau}_3$-direction component is (yielded by multiplication with $\hat{\eta}^3$):

$$
\hat{\eta}^3 \cdot \hat{\tau}_3 \nabla \cdot (u_3 u) = \left( - \frac{\partial (\hat{u}_3 \hat{u}_1)}{\partial x^1} + \frac{\partial (\hat{u}_3 \hat{u}_2)}{\partial x^2} + \frac{\partial (\hat{u}_3 \hat{u}_3)}{\partial x^3} \right) = 2.38
$$

Substitution of the physical velocity components yields:

$$
\begin{align*}
&\frac{\partial (u_\zeta u_\theta)}{\partial \theta} + \frac{\partial (u_\zeta u_r)}{\partial r} + \frac{\partial (u_\zeta u_\zeta)}{\partial \zeta} + \frac{1}{r} u_\zeta u_r \frac{1}{a} \\
&= \left( \frac{\partial (\hat{u}_\zeta \hat{u}_\theta)}{\partial \theta} + \frac{\partial (\hat{u}_\zeta \hat{u}_r)}{\partial r} + \frac{\partial (\hat{u}_\zeta \hat{u}_\zeta)}{\partial \zeta} + \frac{1}{r} \hat{u}_\zeta \hat{u}_r \frac{1}{a} \right) \\
&= 2.39
\end{align*}
$$

### 2.6 Transformation of the Pressure Term

The pressure term of the momentum transport vector is: $\nabla p$

According to the transformation rule for the gradient vector a scalar field, eq. (2.19):
\[ \nabla p = \frac{\partial p}{\partial x} \hat{\eta} = \frac{\partial p}{\partial x} \hat{\eta}_i \quad (2.40) \]

The \( \hat{\eta}_1 \)-direction component is:

\[ \hat{\eta}_1 \cdot \nabla p = g \frac{\partial p}{\partial x} \hat{\eta}_1 + g \frac{\partial p}{\partial y} \hat{\eta}_2 + g \frac{\partial p}{\partial z} \hat{\eta}_3 - \frac{\partial p}{\partial \theta} \frac{1}{r} \quad (2.41) \]

The \( \hat{\eta}_2 \)-direction component is:

\[ \hat{\eta}_2 \cdot \nabla p = g \frac{\partial p}{\partial x} \hat{\eta}_1 + g \frac{\partial p}{\partial y} \hat{\eta}_2 + g \frac{\partial p}{\partial z} \hat{\eta}_3 = \frac{\partial p}{\partial r} \quad (2.42) \]

The \( \hat{\eta}_3 \)-direction component is:

\[ \hat{\eta}_3 \cdot \nabla p = g \frac{\partial p}{\partial x} \hat{\eta}_1 + g \frac{\partial p}{\partial y} \hat{\eta}_2 + g \frac{\partial p}{\partial z} \hat{\eta}_3 = \frac{\partial p}{\partial \varphi} \frac{1}{r} \quad (2.43) \]

2.7 Transformation of the stress term

In an orthogonal coordinate system the components of the strain rate tensor are given by:

\[ s_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{2} \left( \frac{\partial n_i}{\partial x_j} + \frac{\partial n_j}{\partial x_i} \right) = \frac{1}{2} \left( (\hat{\eta})_j \frac{\partial u_i}{\partial x} + (\hat{\eta}^p)_i \frac{\partial u_j}{\partial x} \right) \quad (2.44) \]

The strain rate tensor is:

\[ s = \frac{1}{2} (i_j)^T s_{ij} \quad (2.45) \]

\[ s = \frac{1}{2} i_j (\hat{\eta})_j \left( \hat{\eta}^k \hat{u}^p_{:n} \right)_{;i} + (\hat{\eta}^p)_i \left( \hat{\eta}^k \hat{u}^p_{:n} \right)_{;j} - \frac{1}{2} \left( \hat{\eta}^k \hat{u}^p_{:n} + \hat{\eta}^p \hat{u}^k_{:n} \right) \quad (2.46) \]

The contravariant strain rate tensor components are calculated by:
\[ s_{ij} = (\eta^i_j)^T(s^i_j) = \frac{1}{2} \left( \delta^i_k \delta^j_l + \eta^i_k \eta^j_l \right) \]

Substitution of the contravariant derivatives yields:

\[ s_{ij} = \frac{1}{2} \left( g^{nj} \frac{\partial u^i}{\partial x^j} + g^{pj} \frac{\partial u^i}{\partial x^j} \right) \]  \hfill (2.48)

Substitution of the non-zero elements of the inverse of the metric tensor \((g^{11}, g^{22}, g^{33})\), the non-zero Christoffel symbols \((\Gamma^1_{12}, \Gamma^2_{11})\) and the physical velocity components yields the contravariant strain rate tensor components in the cylindrical coordinate system.

\[
\hat{s}^{ij} = \frac{1}{2} \begin{bmatrix}
\frac{2}{r^2} \left( \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} \right) & -\frac{1}{r} \left( \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) & \frac{1}{r} \left( \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} \right) \\
\frac{1}{r} \left( \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} \right) & -\frac{1}{r} \left( \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right) & \frac{1}{r} \left( \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} \right) \\
\frac{1}{\rho a^2} \frac{\partial u_r}{\partial \zeta} & \frac{1}{\rho a^2} \frac{\partial u_\phi}{\partial \zeta} & -\frac{1}{\rho a^2} \frac{\partial u_\phi}{\partial \zeta} & -\frac{1}{\rho a^2} \frac{\partial u_r}{\partial \zeta} & \frac{1}{\rho a^2} \frac{\partial u_\phi}{\partial \zeta}
\end{bmatrix}
\]  \hfill (2.49)

The physical strain rates \(\theta, r\) and \(\zeta\) direction are:

\[
s_{\theta\theta} = (\lambda_\theta)^T(s^\lambda_\theta) = (\lambda_\theta)_{i}^{\lambda} (\eta^i_j)^T \lambda_\theta s_{ij} = r^3 s_{\theta\theta}^{11}
\]

\[
s_{rr} = s^{22}, s_{\zeta\zeta} = a^2 s^{33}, s_{\theta r} = r s^{12}, s_{\theta\zeta} = ra s^{13}, s_{r\zeta} = a^2 s^{23}
\]  \hfill (2.50)

The turbulent stresses are modelled by an eddy viscosity model:

\[
\tau_{\theta\theta} = \rho \nu_{\theta\theta} 2 s_{\theta\theta}, \text{ etc... The six turbulent stress components are:}
\]

\[
\tau_{\theta\theta} = \rho \nu_{\theta\theta} 2 \left( \frac{\partial u_\theta}{\partial \theta} + \frac{u_\theta}{r} \right) \quad \tau_{r\theta} = \rho \nu_{r\theta} \left( \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \right)
\]
The turbulent stress term of the momentum transport vector is: \( \mathbf{i}_1(\nabla \sigma)_i \)

According to the transformation rule for the divergence of a second order tensor with one of its indices constant, eq.(2.21), and mathematical manipulations equivalent to that of convective terms, eq.(2.33), the following expression is obtained.

\[
\mathbf{i}_1(\nabla \sigma)_i = \frac{1}{\sqrt{g}} \left( \frac{\partial \hat{\sigma}^k}{\partial x^k} + \hat{\sigma}^{pk} \gamma^i_{pk} \right) \hat{r}_i
\]

in which: \( \hat{\sigma}^{pk} \) = contravariant stress tensor components

\[
\hat{\sigma}^{pk} = (\eta^p)^T (\sigma \eta^k)
\]

The physical turbulent stresses in \( \theta \), \( r \) and \( \zeta \) direction are related to \( \hat{\sigma}^{ij} \) by:

\[
\begin{align*}
\tau_{\theta\theta} &= (\lambda_\theta)^T (\sigma \lambda_\theta) = (\lambda_\theta)^T \tau^i_1 (\lambda_\theta^j) T_{ij} \hat{\sigma}^{ij} = r^2 \hat{\sigma}^{11} \\
\tau_{r\zeta} &= \hat{\sigma}^{22} = a^2 \hat{\sigma}^{33}, \quad \tau_{\theta r} = r \hat{\sigma}^{12}, \quad \tau_{\theta \zeta} = ra \hat{\sigma}^{13}, \quad \tau_{\zeta r} = a \hat{\sigma}^{23}
\end{align*}
\]

Substitution of the Jacobian and the Christoffel symbols in the turbulent stress term of the momentum transport equation yields:

The \( \hat{r}_1 \)-direction component:

\[
\eta \mathbf{i}_1 \nabla \cdot (\eta \sigma)_i = \left( \frac{\partial (\hat{\sigma}^{11})}{\partial x^1} + \frac{\partial (\hat{\sigma}^{12})}{\partial x^2} + \frac{\partial (\hat{\sigma}^{13})}{\partial x^3} + 2 a \hat{\sigma}^{12} \right) \frac{1}{ra} =
\]

\[
= \frac{\partial \tau_{\theta\theta}}{\partial \theta r} \frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{r\zeta}}{\partial \zeta} + 2 \frac{1}{r} \tau_{r\theta} \frac{1}{r}
\]

These normal and tangential turbulent stresses are identical to the stresses reported by Hinze (1975) and de Vriend (1981). (in case of isotropic viscosity)
The \( r_2 \)-direction component:

\[
\eta \cdot i_2 \nabla \cdot (\sigma) = \left( \frac{\partial (r \sigma^{12})}{\partial x^1} + \frac{\partial (r \sigma^{22})}{\partial x^2} + \frac{\partial (r \sigma^{32})}{\partial x^3} - r^2 a_{11} \right) \frac{1}{ra} = \\
\frac{\partial r}{\partial \theta} + \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial r}{\partial \tau_{rr}} - \frac{1}{r} \frac{\partial \tau_{\theta \theta}}{\partial \tau_{rr}}
\]

(2.56)

These expressions are equivalent to the expressions given by De Vriend 1981.

The \( r_3 \)-direction component:

\[
\eta \cdot i_3 \nabla \cdot (\sigma) = \left( \frac{\partial (r \sigma^{13})}{\partial x^1} + \frac{\partial (r \sigma^{23})}{\partial x^2} + \frac{\partial (r \sigma^{33})}{\partial x^3} \right) \frac{1}{ra} = \\
\frac{\partial \tau_{\theta \theta}}{\partial \theta} + \frac{\partial \tau_{r \theta}}{\partial \theta} + \frac{\partial \tau_{\theta \theta}}{\partial \theta} + \frac{1}{r} \frac{\partial r}{\partial \tau_{\theta \theta}} \frac{1}{a}
\]

(2.57)
2.8 The momentum equations in cylindrical coordinates

The momentum equations in the cylindrical coordinate system are obtained by multiplication of the momentum transport vector, eq.(2.4) by the contravariant base vectors \( \hat{\eta}_1 \), \( \hat{\eta}_2 \) and \( \hat{\eta}_3 \). Consequently the contravariant components of the momentum vector are yielded. Each of the three components represents one of the momentum equations in \( \hat{r}_1 \), \( \hat{r}_2 \) and \( \hat{r}_3 \) directions.

In the cylindrical, constant depth, coordinate system the mathematical formulation of the momentum equations is:

In \( \hat{r}_1 \) direction:

\[
\frac{\partial u_\theta}{\partial t} + \frac{\partial (u_\theta u_\theta)}{\partial r} + \frac{\partial (u_\theta u_r)}{\partial \theta} + \frac{\partial (u_\theta u_\phi)}{\partial \phi} + 2 \frac{u_\theta u_r}{r} = \frac{1}{\rho} \left( - \frac{\partial p}{\partial \theta} + \frac{\partial \rho \theta}{\partial \theta} + \frac{\partial \rho \theta}{\partial r} + \frac{\partial \rho \theta}{\partial \phi} + 2 \frac{1}{r} \frac{\partial \rho \phi}{\partial \phi} \right) + \frac{1}{\rho} \left( - \frac{\partial \rho \phi}{\partial \theta} + \frac{\partial \rho \phi}{\partial \theta} + \frac{\partial \rho \phi}{\partial r} + \frac{\partial \rho \phi}{\partial \phi} + 2 \frac{1}{r} \frac{\partial \rho \phi}{\partial \phi} \right)
\]

(2.58)

In \( \hat{r}_2 \) direction:

\[
\frac{\partial u_r}{\partial t} + \frac{\partial (u_r u_\theta)}{\partial r} + \frac{\partial (u_r u_r)}{\partial \theta} + \frac{\partial (u_r u_\phi)}{\partial \phi} + \frac{u_r u_r}{r} - \frac{1}{\rho} \frac{u_\theta u_\theta}{r} = \frac{1}{\rho} \left( - \frac{\partial p}{\partial \theta} + \frac{\partial \rho \theta}{\partial \theta} + \frac{\partial \rho \theta}{\partial r} + \frac{\partial \rho \theta}{\partial \phi} + \frac{1}{r} \frac{\partial \rho \phi}{\partial \phi} \right) + \frac{1}{\rho} \left( - \frac{\partial \rho \phi}{\partial \theta} + \frac{\partial \rho \phi}{\partial \theta} + \frac{\partial \rho \phi}{\partial r} + \frac{\partial \rho \phi}{\partial \phi} + \frac{1}{r} \frac{\partial \rho \phi}{\partial \phi} \right)
\]

(2.59)

In \( \hat{r}_3 \) direction:

\[
\frac{\partial u_\phi}{\partial t} + \frac{\partial (u_\phi u_\theta)}{\partial r} + \frac{\partial (u_\phi u_\phi)}{\partial \theta} + \frac{\partial (u_\phi u_r)}{\partial \phi} + \frac{1}{\rho} \frac{u_\phi u_\phi}{r} = \frac{1}{\rho} \left( - \frac{\partial p}{\partial \theta} + \frac{\partial \rho \theta}{\partial \theta} + \frac{\partial \rho \theta}{\partial r} + \frac{\partial \rho \theta}{\partial \phi} + \frac{1}{r} \frac{\partial \rho \phi}{\partial \phi} \right) + \frac{1}{\rho} \left( - \frac{\partial \rho \phi}{\partial \theta} + \frac{\partial \rho \phi}{\partial \theta} + \frac{\partial \rho \phi}{\partial r} + \frac{\partial \rho \phi}{\partial \phi} + \frac{1}{r} \frac{\partial \rho \phi}{\partial \phi} \right) - g
\]

(2.60)
3 MOMENTUM EQUATIONS IN CYLINDRICAL BED-FOLLOWING COORDINATES

3.1 Introduction

The momentum equations are derived in a cylindrical bed-following coordinate system. To yield a bed-following coordinate system a sigma coordinate transformation (Phillips, 1957) of the vertical is used. The coordinate transformation is achieved by one transformation step which incorporated the joint transformation to a cylindrical and a bed-following coordinate system. Obviously this could also been achieved by two subsequent transformations. The joint coordinate transformation is given in Talmon (1989) for application to the convection-diffusion equation. The theory of the coordinate transformation will be repeated in this chapter. The transformed coordinate system is non-orthogonal. The eddy viscosity is assumed isotropic.

In section 3.2 the momentum equations in orthogonal Cartesian coordinates are repeated. In section 3.3 the coordinate transformation is given. In section 3.4 the transformation of the convective term is given. In section 3.5 the pressure term is transformed. In section 3.6 the gravitation term is transformed. In section 3.7 the turbulent stress term is transformed. In section 3.8 the turbulent stresses are transformed. In section 3.9 finally the momentum equations in the cylindrical bed-following coordinate system are given.

3.2 The equations in orthogonal Cartesian coordinates

In an orthogonal Cartesian coordinate system \((x_1, x_2, x_3)\), the Navier-Stokes equations have the following form (conservative form, constant \(\rho\)):

\[
\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = \frac{1}{\rho} \left( \frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} \right) - \delta_{ij} g , \text{ for } i=1,2,3. \tag{3.1}
\]

With:

- \(u_i\) = time averaged velocity component
- \(\sigma_{ij}\) = Reynolds stress components;
- \(s_{ij}\) = the strain rate components;
- \(\nu\) = kinematic viscosity (isotropic)
- \(u_i'\) = mean velocity component
- \(u_i''\) = fluctuating velocity component

\[
\sigma_{ij} = -\rho u_i' u_j'' = \rho \nu_{ij} 2 s_{ij} \text{ (no summation)} \tag{3.2}
\]

\[
s_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{3.3}
\]
In tensor notation the Navier-Stokes equations are formulated by:

\[
\frac{\partial u_i}{\partial t} + \mathbf{V}.(u_i u) = \frac{\partial u_i}{\partial t} + i_i \mathbf{V} . (A)_i = \frac{1}{\rho} \left( - \mathbf{V} p + i_i \mathbf{V} . (\sigma)_i + g \right) 
\]  

(3.4)

in which: \( A \) = second order momentum tensor, \( A_{ij} = u_i u_j \)  

(3.5)

\( (A)_i \) = \( i^{th} \) row or column of the second order tensor  
\( \sigma \) = second order stress tensor  
\( i_i \) = unit vector of the coordinate system  
\( g \) = gravitation vector, \( g = (0,0,-g) \)  

(3.6)

### 3.3 Coordinate transformation

The equations have to be transformed from the orthogonal coordinate system to a nonorthogonal cylindrical bed-following coordinate system. The one to one relationship between these coordinate systems is given by, fig. (3.1):

![Bed following coordinate system](image)

figure 3.1 Bed following coordinate system

\[
x_1 = r \cos \theta \\
x_2 = r \sin \theta \\
x_3 = a \zeta + z_b, \quad a = s - z_b, \quad x^3 = a \zeta + s - a
\]

(3.7)

\[
\dot{x}_1 = \theta = \arctg(x^2/x^1) \\
\dot{x}_2 = r = ((x_1)^2 + (x_2)^2)^{1/2} \\
\dot{x}_3 = \zeta = (x^3 - z_b)/(s - z_b)
\]

In which: \( s \) = water surface, \( s \) = constant
\( x_b \) = bed level  \\
\( a \) = local water depth

The covariant base-vectors are (tangent to the coordinate axes);

\[
\hat{\mathbf{r}}_1 = \frac{\partial \mathbf{x}}{\partial \theta} = \begin{bmatrix} -r \sin \theta \\ r \cos \theta \\ (\xi-1) \frac{\partial \mathbf{a}}{\partial \theta} \end{bmatrix}, \quad \hat{\mathbf{r}}_2 = \frac{\partial \mathbf{x}}{\partial r} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ (\xi-1) \frac{\partial \mathbf{a}}{\partial r} \end{bmatrix}, \quad \hat{\mathbf{r}}_3 = \frac{\partial \mathbf{x}}{\partial \xi} = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \tag{3.8}
\]

The contravariant base-vectors are (normal to surfaces of \( x^i = \) const.);

\[
\hat{\mathbf{\eta}}^1 = \begin{bmatrix} -\frac{1}{r} \sin \theta \\ \frac{1}{r} \cos \theta \\ 0 \end{bmatrix}, \quad \hat{\mathbf{\eta}}^2 = \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix}, \quad \hat{\mathbf{\eta}}^3 = \begin{bmatrix} -\frac{\xi-1}{a} \frac{\partial \mathbf{a}}{\partial \mathbf{x}^1} \\ \frac{\xi-1}{a} \frac{\partial \mathbf{a}}{\partial \mathbf{x}^2} \\ \frac{1}{a} \end{bmatrix} \tag{3.9}
\]

The coordinate system is non-orthogonal, the \( \hat{\mathbf{r}}_i \) and \( \hat{\mathbf{\eta}}^i \) base vectors are not identical in direction.

The non-zero Christoffel symbols of the transformation are, eq.(2.17):

\[
\Gamma_{11}^2 = -r \tag{3.10a}
\]

\[
\Gamma_{12}^1 = \Gamma_{21}^1 = \frac{1}{r} \tag{3.10b}
\]

\[
\Gamma_{11}^3 = (\xi-1) \frac{\partial \mathbf{a}}{\partial r} + (\xi-1) \frac{1}{a} \frac{\partial^2 \mathbf{a}}{\partial \theta^2} \tag{3.10c}
\]

\[
\Gamma_{12}^3 = \Gamma_{21}^3 = -(\xi-1) \frac{\partial \mathbf{a}}{\partial r} + (\xi-1) \frac{1}{a} \frac{\partial^2 \mathbf{a}}{\partial r \partial \theta} \tag{3.10d}
\]

\[
\Gamma_{22}^3 = (\xi-1) \frac{\partial^2 \mathbf{a}}{\partial r^2} \tag{3.10e}
\]

\[
\Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{a} \frac{\partial \mathbf{a}}{\partial \theta} \tag{3.10f}
\]
\[
\hat{\gamma}_{23} = \hat{\gamma}_{32} = \frac{1}{a} \frac{\partial a}{\partial r}
\]  
(3.10g)

The \(\hat{\gamma}_{11}\) and \(\hat{\gamma}_{12}\) Christoffel symbols are due to the transformation to a cylindrical coordinate system. The other symbols are due to the transformation to a bed following coordinate system, compare eq.(2.25). The second order derivatives of the bed level in eq.(3.10c,d,e) and the first order derivatives in eq.(3.10f,g) are due to the bed following system. The first order derivatives in eq.(3.10c,d) are due to the combined cylindrical and bed following coordinate transformation.

The metric tensor and the inverse of the metric tensor are \((\hat{g}_{ij} = \hat{\gamma}_{i} \cdot \hat{\gamma}_{j}, \hat{g}^{-1} = \eta_{i} \cdot \eta_{j})\):

\[
\hat{g}_{ij} = \begin{pmatrix}
\frac{r^2}{a^2} & (\zeta-1)^2 \frac{\partial a}{\partial \theta} \\
(\zeta-1)^2 \frac{\partial a}{\partial r} & 1 + (\zeta-1)^2 \left(\frac{\partial a}{\partial r}\right)^2 \\
(\zeta-1)^2 a \frac{\partial a}{\partial \theta} & (\zeta-1)^2 a \frac{\partial a}{\partial r}
\end{pmatrix}
\]  
(3.11)

\[
\hat{g}^{ij} = \begin{pmatrix}
\frac{1}{r^2} & 0 & (\zeta-1) \frac{\partial a}{\partial r} a \\
0 & 1 & (\zeta-1) \frac{\partial a}{\partial \theta} \\
(\zeta-1) \frac{\partial a}{\partial r} a & (\zeta-1) \frac{\partial a}{\partial \theta} (\zeta-1)^2 \left(\frac{\partial a}{\partial r}\right)^2 - (\zeta-1)^2 \left(\frac{\partial a}{\partial r}\right)^2 + \frac{1}{a^2}
\end{pmatrix}
\]  
(3.12)

The determinant (=Jacobian) of the transformation is:

\[
J = \left| \frac{\partial (x_1, x_2, x_3)}{\partial (\hat{x}_1, \hat{x}_2, \hat{x}_3)} \right| = \sqrt{\hat{g}} = ra
\]  
(3.13)

The contravariant velocity components (=velocity components in directions tangential to the coordinate lines) are: \(\hat{u}^{i} = \eta^{i} \cdot u\)

\[
\hat{u}^{1} = \frac{1}{r} \sin \theta \ u^{1} + \frac{1}{r} \cos \theta \ u^{2}
\]
\[ \hat{u}^2 = \cos \theta u^1 + \sin \theta u^2 \]  
\[ \hat{u}^3 = - \frac{c-1}{a} \frac{\partial a}{\partial x^1} u^1 - \frac{c-1}{a} \frac{\partial a}{\partial x^2} u^2 + \frac{1}{a} u^3 \]

with: \( u^i \) = the velocity components in the rectangular Cartesian coordinate system [m/s]  
\( \hat{u}^i \) = the contravariant velocity components in the transformed coordinate system  
(with dimensions governed by \( \hat{\eta}^i . u \))

The dimensions of the base vectors and the contravariant velocity components of the transformed coordinate system are:

\( \hat{r}_1 = [m] \quad \hat{\eta}^1 = [l/m] \quad \hat{u}^1 = [l/s] \)
\( \hat{r}_2 = [-] \quad \hat{\eta}^2 = [\cdot] \quad \hat{u}^2 = [m/s] \)
\( \hat{r}_3 = [m] \quad \hat{\eta}^3 = [l/m] \quad \hat{u}^3 = [l/s] \)

The physical velocity components in \( \theta, r \) and \( \zeta \) direction are given by:

\[ u_\theta = \lambda_\theta . u = \lambda_\theta \cdot \hat{r}_1 \hat{u}^i, \quad u_\theta = r \hat{u}^1, \quad u_r = \hat{u}^2, \quad u_\zeta = \hat{u}^3 \]  
(3.15)

In which: use is made of the fundamental relation: \( u = \hat{r}_1 \hat{u}^i \)

: the unit vectors in \( \theta, r \) and \( \zeta \) direction are:

\[ \lambda_\theta = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}, \quad \lambda_r = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \quad \lambda_\zeta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]  
(3.16)

### 3.4 Transformation of the convective term

The convective term of the momentum transport vector is:

\[ i_i \nabla (u_i u) = \frac{\partial A_{ij}}{\partial x^j}, \quad \text{with} \quad A_{ij} = u_i u_j \]  
(3.17)

According to the transformation law for the divergence of a second order tensor with one if its indices constant, eq.(2.21):
Substitution of the Jacobian and the Christoffel symbols yields:

The \( r_1 \)-direction component is:

\[
\eta \cdot i_1 \nabla (u_1 u) = \frac{\partial (\hat{u}^1 u)}{\partial x^1} + \frac{\partial (\hat{u}^2 u)}{\partial x^2} + \frac{\partial (\hat{u}^3 u)}{\partial x^3} + 2 r_1^1 \hat{u}^2 \hat{u}^1 \frac{1}{ra} = (3.19)
\]

Substitution of the physical velocity components, eq.(3.15), and \( r_{11} \) yields:

\[
= \frac{\partial (au_r u_\theta)}{r \partial \theta} + \frac{\partial (au_r u_r)}{\partial r} + \frac{\partial (au_\theta u_\theta)}{\partial \theta} + 2 \frac{a}{r} u_\theta u_r + \frac{\partial a}{r \partial \theta} \frac{\partial (1-\zeta)u_\theta u_\theta}{\partial \zeta} + \frac{\partial a}{\partial r} \frac{\partial (1-\zeta)u_\theta u_\theta}{\partial \zeta} \frac{1}{ra} = (3.20)
\]

The \( r_2 \)-direction component is:

\[
\eta \cdot i_2 \nabla (u_2 u) = \frac{\partial (\hat{u}^2 u)}{\partial x^1} + \frac{\partial (\hat{u}^2 u)}{\partial x^2} + \frac{\partial (\hat{u}^2 u)}{\partial x^3} + r_1^2 \hat{u}^1 \hat{u}^1 \frac{1}{ra} = (3.21)
\]

Substitution of the physical velocity components and \( r_{11} \) yields:
\[- \left( \frac{\partial (au_1 u_1)}{r} \right) \frac{\partial}{\partial \theta} + \frac{\partial (au_1 u_1)}{r} \frac{\partial}{\partial r} + \frac{\partial (au_1 u_1)}{a} \frac{\partial}{\partial \xi} + \frac{a u_1 u_1}{r} - \frac{a u_1 u_1}{r} \right] + \frac{\partial a}{\partial \theta} \left( 1 - \xi \right) u_1 u_1 \frac{\partial}{\partial \xi} + \frac{\partial a}{\partial r} \left( 1 - \xi \right) u_1 u_1 \frac{\partial}{\partial r} + \frac{1}{a} \quad (3.22)\]

These expressions for the convective terms in \( \hat{r}_1 \) and \( \hat{r}_2 \)-direction are extendend with two non-orthogonality terms with respect to a cylindrical constant-depth coordinate system, eq.(2.34) and eq.(2.36).

The \( \hat{r}_3 \)-direction component is:

\[
\hat{\eta} \cdot \hat{r}_3 \tau (u_3 u) = \left( \frac{\partial (au_1 u_1)}{\hat{x}_1} + \frac{\partial (au_1 u_1)}{\hat{x}_2} + \frac{\partial (au_1 u_1)}{\hat{x}_3} \right) \frac{1}{r a} + \frac{\hat{r}_3 \cdot \hat{r}_1 \cdot \hat{r}_1 u_1}{r a} + \Gamma_{11} u_1 u_1 + \Gamma_{22} u_2 u_2 + \Gamma_{12} u_1 u_2 + \Gamma_{33} u_3 u_3 + \Gamma_{23} u_2 u_3 \quad (3.23)\]

Substitution of the velocities, eq.(3.15) and Christoffel symbols yields a very lengthy expression which will not be given here.

3.5 Transformation of the pressure term

The pressure term of the momentum transport vector is: \( \nabla p \)

According to the transformation rule for the gradient vector a scalar field, eq.(2.19):

\[
\nabla p = \frac{\partial p}{\partial x_k} \hat{\eta} k = \frac{\partial p}{\partial x_k} \hat{x} \quad (\nabla p) = \hat{\nabla} p \quad (3.25)\]

The formulation of \( \nabla (p) = (\nabla p) \) in the transformed coordinate system is thus given by the product of the contravariant components of \( \nabla (p) \) and the covariant base vectors \( \hat{r}_i \).

The \( \hat{r}_1 \)-direction component is:

\[
\hat{\eta} \cdot \nabla p = g \frac{\partial p}{\partial \hat{\eta}} + g \frac{\partial p}{\partial \hat{x}_1} + g \frac{\partial p}{\partial \hat{x}_2} + g \frac{\partial p}{\partial \hat{x}_3} = \left( \frac{\partial p}{r \partial \theta} + \frac{\partial a}{r \partial \theta} + \frac{\partial p}{r \partial \xi} \right) \hat{r}_1 \quad (3.26)\]
The \( r_2 \)-direction component is:

\[
\eta^2 \cdot \nabla p = \frac{\partial p}{\partial x_1} + \frac{\partial p}{\partial x_2^+} + \frac{\partial p}{\partial x_3} = \frac{\partial p}{\partial r} + (1-\xi) \frac{\partial a}{\partial a \delta \xi} \tag{3.27}
\]

The \( r_3 \)-direction component is:

\[
\eta^3 \cdot \nabla p = \frac{\partial p}{\partial x_1} + \frac{\partial p}{\partial x_2^+} + \frac{\partial p}{\partial x_3} = \\
\left( \frac{\partial p}{\partial r} \frac{\partial a}{\partial a \delta \xi} \right) + (1-\xi) \frac{\partial a}{\partial r} \frac{\partial p}{\partial r} \tag{3.28}
\]

The contravariant components of \( \nabla p \) involve non-orthogonality terms. The components \( \eta^1 \cdot \nabla p \) and \( \eta^2 \cdot \nabla p \) incorporate an additional term involving the pressure gradient in vertical direction times the local non-orthogonality of the coordinate system in vertical planes.

The component \( \eta^3 \cdot \nabla p \) incorporates additional terms involving pressure gradients in horizontal directions times local non-orthogonality.

### 3.6 Transformation of the gravitation term

The gravitation vector in an orthogonal Cartesian coordinate system is given by: \( g = (0,0,-g) \). In the transformed coordinate system the gravitation vector can be expressed in the set of covariant base vectors \( \hat{r}_i \) and the contravariant components \( \hat{g}^i \) of the gravitation vector: \( g = \hat{r}_i \hat{g}^i \) \( \tag{3.29} \)

The contravariant components \( \hat{g}^i \) of the gravitation vector are equated by (the contravariant component multiplied by the covariant base vector \( \hat{r}_i \)

yields a vector which points in the direction of \( \hat{r}_i \)):

\[
\hat{g}^i = \eta \cdot g = (i_j) \frac{\partial x}{\partial x_j} g = - \frac{\partial x}{\partial x_3} \hat{g}^i \tag{3.30}
\]

The gravitation term in the momentum equation in \( \hat{r}_1 \)-direction is: \( \eta^1 \cdot g = 0 \)

The gravitation term in the momentum equation in \( \hat{r}_2 \)-direction is: \( \eta^2 \cdot g = 0 \)

The gravitation term in the momentum equation in \( \hat{r}_3 \)-direction is:

\[
\eta \cdot g = -g/a \tag{3.31}
\]
3.7 Transformation of the stress term

The turbulent stress term of the momentum transport vector is: \( \mathbf{i}_i \cdot \nabla (\sigma) \).

According to the transformation rule of the divergence of a second order tensor with one of its indices constant, eq. (2.21), and mathematical manipulations equivalent to that of convective terms, the following expression is obtained.

\[
\mathbf{i}_i \cdot \nabla (\sigma) = \frac{1}{\sqrt{g}} \left( \frac{\partial (\hat{\sigma}^{ik})}{\partial x_k} + \sqrt{g} \hat{\sigma}^{pk} \Gamma^i_{pk} \right) \hat{\tau}^i
\]  

(3.32)

in which: \( \hat{\sigma}^{pk} \) = contravariant stress tensor components,

\( \hat{\sigma}^{pk} = (\eta^p)^T \sigma \eta^k \)  

(3.33)

The physical turbulent stresses in \( \theta \), \( r \) and \( \zeta \) direction are calculated by:

\[
\tau_{\theta \theta} = (\lambda_\theta)^T \sigma \lambda_\theta = (\lambda_\theta)^T \hat{\tau}_i (\hat{\tau}_j)^T \lambda_\theta \sigma^i j = r^2 \sigma^{11}
\]  

(3.34)

The remaining stresses are calculated the same way. The result is:

\[
\begin{pmatrix}
    r^2 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    A^2 & B^2 & a^2 & 2AB & 2Aa & 2Ba \\
    0 & 0 & 0 & r & 0 & 0 \\
    rA & 0 & 0 & rB & ar & 0 \\
    0 & B & 0 & A & 0 & a
\end{pmatrix}
\begin{pmatrix}
    \sigma^{11} \\
    \sigma^{22} \\
    \sigma^{33} \\
    \sigma^{12} \\
    \sigma^{13} \\
    \sigma^{23}
\end{pmatrix}
\begin{pmatrix}
    \tau_{\theta \theta} \\
    \tau_{rr} \\
    \tau_{\zeta \zeta} \\
    \tau_{\theta \zeta} \\
    \tau_{\theta r} \\
    \tau_{r \zeta}
\end{pmatrix}
\]  

(3.35)

in which: \( A = (\zeta - 1) \frac{\partial}{\partial \theta} \) and \( B = (\zeta - 1) \frac{\partial}{\partial r} \)

(3.36)

The inverse relation is calculated by matrix manipulation:
Substitution of the Jacobian and the Christoffel symbols in the turbulent stress term of the momentum transport equation yields:

The $\tau_1$-direction component is:

\[
\eta \cdot \mathbf{i}_1 \nabla \cdot (\sigma) = \left( \frac{\partial (\tau_{11})}{\partial x^1} + \frac{\partial (\tau_{12})}{\partial x^2} + \frac{\partial (\tau_{13})}{\partial x^3} + 2 a \tau_{12} \right) \frac{1}{ra} - \\
- \left( \frac{\partial a r_{\theta \theta}}{r \theta} + \frac{\partial a r_{\tau \tau}}{r \tau} \right) - \frac{\partial (a A / r r_{rr})}{r r_{rr}} - \frac{\partial (a B r_{rr})}{r r_{rr}} + 2 - \frac{1}{r} \frac{1}{ra} = \\
- \left( - \frac{\partial r_{\theta \theta}}{r \theta \theta} + \frac{\partial r_{\tau \tau}}{r \tau} \right) + 2 \frac{1}{r} \frac{1}{ra} = \\
- \frac{\partial a}{a \theta} + \frac{\partial a}{a \tau} + \frac{\partial a}{a \sigma} + 2 \frac{1}{r} \frac{1}{ra} = \\
\eta \cdot \mathbf{i}_2 \nabla \cdot (\sigma) = \left( \frac{\partial (\tau_{12})}{\partial x^1} + \frac{\partial (\tau_{22})}{\partial x^2} + \frac{\partial (\tau_{32})}{\partial x^3} + r^2 a \tau_{11} \right) \frac{1}{ra} - \\
- \left( \frac{\partial a r_{\theta \theta}}{r \theta} + \frac{\partial a r_{\tau \tau}}{r \tau} \right) - \frac{\partial (a A / r r_{rr})}{r r_{rr}} - \frac{\partial (a B r_{rr})}{r r_{rr}} + 2 - \frac{1}{r} \frac{1}{ra} = \\
- \left( - \frac{\partial r_{\theta \theta}}{r \theta \theta} + \frac{\partial r_{\tau \tau}}{r \tau} \right) + 2 \frac{1}{r} \frac{1}{ra} = \\
(\tau_{12}) = \tau_{13} = \tau_{23} = 0,
\[
\frac{\partial \tau_{\theta \theta}}{\partial \theta} + \frac{\partial \tau_{rr}}{\partial \theta} + \frac{\partial \tau_{\theta r}}{\partial \theta} = \frac{1}{r} \frac{\partial \tau_{rr}}{\partial r} - \frac{1}{r} \frac{\partial \tau_{\theta \theta}}{\partial r} + \frac{\partial a}{\partial \theta} \frac{\partial (1 - \gamma) \tau_{\theta \theta}}{\partial \theta} + \frac{\partial a}{\partial r} \frac{\partial (1 - \gamma) \tau_{rr}}{\partial r} = \frac{1}{r} \frac{\partial \tau_{rr}}{\partial r} - \frac{1}{r} \frac{\partial \tau_{\theta \theta}}{\partial r} + \frac{\partial a}{\partial \theta} \frac{\partial (1 - \gamma) \tau_{\theta \theta}}{\partial \theta} + \frac{\partial a}{\partial r} \frac{\partial (1 - \gamma) \tau_{rr}}{\partial r}
\tag{3.39}
\]

The 3-direction component is:

\[
\eta^3 \cdot I_3 \nabla \cdot \sigma_3 = -\left( \frac{\partial (\tau_{\theta \theta})}{\partial x_1} + \frac{\partial (\tau_{rr})}{\partial x_2} + \frac{\partial (\tau_{\theta r})}{\partial x_3} \right) \frac{1}{ra} + \tau^3 \gamma^1 \sigma^1 + \tau^3 \gamma^2 \sigma^2 + \tau^3 \gamma^3 \sigma^3
\tag{3.40}
\]

Substitution of \( ^{\gamma^i j} \), eq. (3.37), and substitution of the Christoffel symbols, eq. (3.10) yields a very lengthy expression which will not be given here.

### 3.8 Transformation of stresses

In an orthogonal coordinate system the components of the stress tensor are modelled by (in case of an isotropic eddy viscosity \( \nu \)):

\[
\sigma_{ij} = \frac{\tau_{ij}}{r} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}
\tag{3.42}
\]

The stress tensor is given by:

\[
\sigma = \sigma_{ij} (i_j)^T \sigma_{ij}
\tag{3.43}
\]

\[
\sigma = \frac{\tau_{ij}}{r} (i_j)^T \tau_{ik} \cdot \frac{\tau_{ij}}{r} \cdot \frac{\tau_{ik}}{r} = \frac{\tau_{ij}}{r} (i_j)^T \tau_{ik} \cdot \frac{\tau_{ij}}{r} \cdot \frac{\tau_{ik}}{r}
\tag{3.44}
\]

The contravariant stress tensor components are calculated by:

\[
\tau_{ij} = \frac{\tau_{ij}}{r} \tau_{ik} \cdot \frac{\tau_{ij}}{r} \cdot \frac{\tau_{ik}}{r} = \frac{\tau_{ij}}{r} (i_j)^T \tau_{ik} \cdot \frac{\tau_{ij}}{r} \cdot \frac{\tau_{ik}}{r}
\tag{3.45}
\]
\[ - \rho \nu \left( g^{nj} \frac{\partial u^i}{\partial x_n} + g^{pi} \frac{\partial u^j}{\partial x_p} \right) \quad (3.45) \]

Substitution of the contravariant derivatives yields:

\[ \frac{\partial}{\partial x} \left( g^{nj} \frac{\partial u^i}{\partial x_n} + g^{pi} \frac{\partial u^j}{\partial x_p} \right) \quad (3.46) \]

Substitution of the non-zero elements of the inverse of the metric tensor in eq.(3.12) and substitution of eq.(3.45) in the relations given by the matrix eq.(3.35) the relation between the covariant derivatives and the physical stresses is obtained:

\[
\begin{pmatrix}
\tau_{\theta\theta} \\
\tau_{rr} \\
\tau_{r\xi} \\
\tau_{\theta\xi} \\
\tau_{r\xi}
\end{pmatrix} = \rho \nu \begin{pmatrix}
2 & 0 & -2A & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -2B & 0 & 0 & 0 \\
0 & 0 & 2A & 0 & 2B & 0 & 0 & 2 \\
0 & r & -B & 1 & 0 & A & 0 & 0 \\
0 & A & 0 & (\frac{r}{a} - \frac{2}{a}) & B & 0 & A & 0 \\
0 & A & 0 & \frac{AB}{a} & 0 & (\frac{1}{a} - \frac{B^2}{a}) & 0 & 0 & 0 & -B
\end{pmatrix} \quad (3.47)
\]

The contravariant derivatives are related to the contravariant velocities by (substitution of non-zero Christoffel symbols in eq.(3.46)):  

\[
\begin{pmatrix}
\frac{\partial}{\partial x} u^1 \\
\frac{\partial}{\partial x} u^2 \\
\frac{\partial}{\partial x} u^3 \\
\frac{\partial}{\partial x} u^4 \\
\frac{\partial}{\partial x} u^5 \\
\frac{\partial}{\partial x} u^6 \\
\frac{\partial}{\partial x} u^7 \\
\frac{\partial}{\partial x} u^8 \\
\frac{\partial}{\partial x} u^9 \\
\frac{\partial}{\partial x} u^{10}
\end{pmatrix} = \rho \nu \begin{pmatrix}
\frac{\partial}{\partial x} u^{-1} \\
\frac{\partial}{\partial x} u^{-2} \\
\frac{\partial}{\partial x} u^{-3} \\
\frac{\partial}{\partial x} u^{-4} \\
\frac{\partial}{\partial x} u^{-5} \\
\frac{\partial}{\partial x} u^{-6} \\
\frac{\partial}{\partial x} u^{-7} \\
\frac{\partial}{\partial x} u^{-8} \\
\frac{\partial}{\partial x} u^{-9} \\
\frac{\partial}{\partial x} u^{-10}
\end{pmatrix}
\]
\[
\begin{pmatrix}
\hat{u}_1 \\
\hat{u}_2 \\
\hat{u}_3
\end{pmatrix}
- I
\begin{pmatrix}
\frac{\partial u_1}{\partial x_1} \\
\frac{\partial u_2}{\partial x_2} \\
\frac{\partial u_3}{\partial x_3}
\end{pmatrix}
+ \begin{pmatrix}
0 & \Gamma_{12} & 0 \\
\Gamma_{21} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\hat{u}_1 \\
\hat{u}_2 \\
\hat{u}_3
\end{pmatrix}
\]

Multiplication of matrix (3.48) and matrix (3.47) and substitution of the physical velocity components yields:

\[
\tau_{\theta\theta} = \rho \nu \left( \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + (1-\xi^2) \frac{\partial u_\phi}{\partial \phi} \right) \frac{a \partial u_\phi}{a \xi^2}
\]

(3.49)

\[
\tau_{rr} = \rho \nu \left( \frac{\partial u_r}{\partial r} + (1-\xi^2) \frac{\partial u_\phi}{\partial \phi} \right)
\]

(3.50)

\[
\tau_{\xi\xi} = \rho \nu \frac{\partial u_\xi}{\partial \xi}
\]

(3.51)

\[
\tau_{\theta r} = \tau_{r \theta} = \rho \nu \left( \frac{\partial u_\theta}{\partial \phi} + \frac{u_r}{\partial r} + (1-\xi^2) \frac{\partial u_\phi}{\partial \phi} + (1-\xi^2) \frac{\partial u_\phi}{\partial \phi} \right) \frac{a \partial u_\phi}{a \xi^2}
\]

(3.52)
The second term of $\tau_{\theta\theta}$ is due to radial velocities, which diverge, that causes a normal stress in the $\theta$-const. surface. The third term of $\tau_{\theta\theta}$ is due to a changing velocity along a line perpendicular to the $\theta$-const. surface because of a non-parallel bed a correction for the shape of the main flow velocity profile necessary. The second term of $\tau_{rr}$ is also due to the non-parallel bed, as are the last two terms of $\tau_{r\theta}$, the last term of $\tau_{\theta\zeta}$ and the last term of $\tau_{rr}$.

### 3.9 The equations in cylindrical bed-following coordinates

The momentum equations in the cylindrical bed-following coordinate system are obtained by multiplication of the momentum transport vector, eq.(3.4) by the contravariant base vectors $\hat{\eta}^1$, $\hat{\eta}^2$ and $\hat{\eta}^3$ (sect. 3.4..3.7). Consequently the contravariant components of the momentum vector are yielded. These components point in $\hat{r}_1$ and $\hat{r}_2$ directions, the $\hat{r}_3$ component is directed in vertical direction. The equation in $\hat{r}_3$-direction will be very lengthy and is consequently not further elaborated.

In the cylindrical bed-following coordinate system the mathematical formulations of the momentum equations in $\hat{r}_1$ and $\hat{r}_2$-directions, in conservative form, are:

In $\hat{r}_1$-direction:

$$
\frac{\partial u_\theta}{\partial t} + \frac{\partial (au_\theta u_\theta)}{\partial \theta} + \frac{\partial (au_\theta u_r)}{\partial r} + \frac{\partial (u_r u_r)}{\partial \zeta} + \frac{2}{r} \frac{a}{u_\theta u_r} = \\
\frac{\partial a}{\partial \theta} \frac{\partial (1-\zeta)u_\theta u_\theta}{\partial \zeta} + \frac{\partial a}{\partial r} \frac{\partial (1-\zeta)u_r u_r}{\partial \zeta} =
$$

$$
\tau_{\theta\zeta} = \rho \nu \left( \frac{\partial u_\theta}{\partial \zeta} + \frac{\partial u_\zeta}{\partial \theta} + (1-\zeta)\frac{\partial a}{\partial \theta} \frac{\partial u_\zeta}{\partial \zeta} \right)
$$

$$
\tau_{rr} = \rho \nu \left( \frac{\partial u_r}{\partial r} + \frac{\partial u_\zeta}{\partial \zeta} + (1-\zeta)\frac{\partial a}{\partial \zeta} \frac{\partial u_r}{\partial r} \right)
$$
In $\tau_2$-direction:

\[

\begin{align*}
- \frac{a}{\rho} & \left( - \frac{\partial p}{\partial \theta} + \frac{\partial r \theta}{\partial \theta} + \frac{\partial r}{\partial r} + \frac{\partial r \theta}{\partial \theta} + 2 \frac{1}{r} \tau \theta + \frac{1}{a \partial \theta} \tau \theta + \frac{1}{a \partial r} \tau \theta + \\
& \frac{\partial a}{\partial \theta} + \frac{\partial a}{\partial \theta} \frac{\partial (1-\xi)}{\partial \theta} \frac{\tau \theta}{\partial \theta} + \frac{\partial a}{\partial r} \frac{\partial (1-\xi)}{\partial r} \frac{\tau \theta}{\partial \theta} \right) \tag{3.55}
\end{align*}
\]

The mathematical expressions for the convective and stress terms are the same. This agrees with $\tau_{ij}$ being equal to: $\tau_{ij} = \rho \bar{u}_i \bar{u}_j$.
4 DEPTH-AVERAGED MOMENTUM EQUATIONS IN CYLINDRICAL BED-FOLLOWING COORDINATES

4.1 Introduction

The momentum equations, which are derived in chapter 3, are formulated in depth-averaged variables. To this purpose similarity profiles for the main and secondary flow components are applied.

In section 4.2 the momentum equations in the cylindrical bed-following coordinate system are given. In section 4.3 depth-averaging of the momentum equations is performed. In section 4.4 expressions for the (depth-averaged) shear stresses are given. In section 4.5 the resulting equations are given.

4.2 The 3-D equations

The constitutive equations in a cylindrical, bed-following, coordinate system are:

The continuity equation:

\[
\frac{\partial (\bar{u}_\theta)}{\partial \theta} + \frac{\partial (\bar{u}_r)}{\partial r} + \frac{\partial (\bar{u}_z)}{\partial z} + \frac{\bar{a}}{r} \bar{u}_r = 0
\]  

(4.1)

The momentumequations in \( \theta \) and \( r \)-directions eq. (3.55) and eq. (3.56) (\( r_1 \) and \( r_2 \) directions coincide with the \( \theta \) and \( r \) directions).

4.3 Depth-averaging of continuity and momentum equations

A depth-averaged variable, for instance the velocity, is defined by (overbar denotes depth-averaging):

\[
\bar{u}_\theta = \int_0^1 u_\theta \, d\zeta
\]

(4.2)

The depth-averaged continuity equation is:

\[
\frac{\partial (\bar{u}_\theta)}{\partial \theta} + \frac{\partial (\bar{u}_r)}{\partial r} + \frac{\bar{a}}{r} \bar{u}_r = 0
\]

(4.3)
The effect of vertical accelerations can be neglected. Consequently the pressure distribution is hydrostatic: \( \frac{\partial p}{\partial \zeta} = -\rho g \) \hspace{1cm} (4.4)

The pressure is given by: \( p = p_{rl} + \rho g (1 - \zeta) \) \hspace{1cm} (4.5)

in which: \( p_{rl} \) - the pressure at the rigid lid.

The pressure at the rigid lid is given by:

\[ p_{rl} = -i_\theta \rho g \theta - i_r \rho g r + \text{const.} \] \hspace{1cm} (4.6)

in which: \( i_\theta, i_r \) - water level slopes in \( \theta \)- and \( r \)-directions

The pressure terms of the momentum equations in horizontal directions are:

\[ \frac{\partial p}{\partial \theta} + (1 - \zeta) \frac{\partial a}{\partial \theta} = \frac{\partial p_{rl}}{\partial \theta} = -i_\theta \rho g \] \hspace{1cm} (4.7)

\[ \frac{\partial p}{\partial r} + (1 - \zeta) \frac{\partial a}{\partial r} = \frac{\partial p_{rl}}{\partial r} = -i_r \rho g \] \hspace{1cm} (4.8)

The depth-averaged, conservative, momentum equation in \( \theta \)-direction is:

\[ \frac{\partial u_\theta}{\partial t} + \frac{\partial (au_\theta u_\theta)}{\partial \theta} + \frac{\partial (au_\theta u_r)}{\partial r} + \frac{\partial (u_\zeta u_\zeta)}{\partial \zeta} + 2 \frac{a}{r} u_\theta u_r + \]

\[ \frac{\partial (1 - \zeta)u_\theta u_\zeta}{\partial \zeta} + \frac{\partial (1 - \zeta)u_\theta u_r}{\partial r} = \]

\[ \frac{\partial p_{rl}}{\partial \theta} + \frac{\partial r}{\partial \theta} + \frac{\partial r}{\partial r} + \frac{\partial r}{\partial \zeta} + 2 \frac{1}{r} \frac{\partial a}{\partial r} + \frac{1}{a} \frac{\partial a}{\partial \theta} + \frac{1}{\theta} \frac{\partial a}{\partial r} + \]

\[ \frac{\partial (1 - \zeta) \tau_{\theta \zeta}}{\partial \zeta} + \frac{\partial (1 - \zeta) \tau_{\theta r}}{\partial r} = \]

\[ \left( \frac{\partial p_{rl}}{\partial \theta} + \frac{\partial r}{\partial \theta} + \frac{\partial r}{\partial r} + \frac{\partial r}{\partial \zeta} + 2 \frac{1}{r} \frac{\partial a}{\partial r} + \frac{1}{a} \frac{\partial a}{\partial \theta} + \frac{1}{\theta} \frac{\partial a}{\partial r} + \right) \] \hspace{1cm} (4.9)

The depth-averaged, conservative, momentum equation in \( r \)-direction is:
\[ \frac{\partial u}{\partial t} + \frac{\partial (au \cdot u)}{\partial r} + \frac{\partial (au \cdot u)}{\partial \theta} + \frac{\partial (u \cdot u)}{\partial r} + \frac{\partial (u \cdot u)}{\partial \theta} + a \frac{u \cdot u}{r} - a \frac{u \cdot u}{r} + \]

\[ + \frac{\partial (1-\xi)u \cdot u}{\partial r} \frac{\partial a}{\partial r} + \frac{\partial (1-\xi)u \cdot u}{\partial \theta} \frac{\partial a}{\partial \theta} = \]

\[ = \frac{a}{\rho} \left( \frac{\partial p}{\partial r} + \frac{\partial \theta}{\partial r} + \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \theta} + \frac{1}{a} \frac{\partial a}{\partial \theta} + \frac{1}{a} \frac{\partial a}{\partial \theta} \right) \]

\[ + \frac{1}{a} \left( \frac{r}{\partial \theta} - \frac{r}{\partial \theta} \right) + \frac{\partial (1-\xi)r}{\partial r} \frac{\partial a}{\partial r} + \frac{\partial (1-\xi)r}{\partial \theta} \frac{\partial a}{\partial \theta} \]

\[ \text{(4.10)} \]

The last terms on both left and right hand sides of the equations represent momentum transport by convection and stresses through surfaces of \( \xi = \text{const.} \). By depth-averaging the convective part of the terms (left hand side) becomes zero due to the integration involved, because at the upper boundary of integration, \( \xi = 1 \) and at the lower boundary, the velocities are zero. The depth-averaged value of the vertical convection term, \( \frac{\partial u \cdot u}{\partial \xi} \), is also zero.

The last terms of eq.(4.9) and eq.(4.10) are due to non-orthogonality of the coordinate system and are part of the bed shear stresses directed in the \( \theta \) and \( r \)-directions. This can be shown by considering the force transmitted through a piece of surfaces of \( \xi = \text{const.}, \) Flugge (1972):

\[ dF = \sigma^3 \hat{j} dA \hat{r} \]

in which: \( dA \hat{3} = \text{area of } \xi = \text{constant surface} \)

\( dF = \text{force (vector)} \)

This area is given by: \( dA = dA_1 \eta \times r_1 \times r_2 = \epsilon_{123} \eta ^3 \)

Consequently: \( dA_1 = dA_2 = 0, dA_3 = \epsilon_{123} = \frac{\eta}{g} \). The physical dimension of the \( dA_3 \) area is: \( /g \mid \hat{\eta} ^3 \), its orientation is determined by the direction of \( \hat{\eta} ^3 \).

The transmitted force is: \( dF = \sigma^3 \hat{j} \hat{r} \). The transmitted stress \( r \) (a vector, because only the flux through one boundary is considered) is:

\[ r = \frac{\hat{\eta} j}{/g \hat{\eta} ^3} \]
The horizontal and vertical components of the stress are obtained by substitution of the contravariant stress components and the covariant base vectors.

\[
\tau_\theta = \frac{\sigma^{31} \theta + 0 + 0}{|\eta^3|} - \frac{(1-\zeta) \frac{\partial a}{\partial \theta} \tau_{\theta \theta} + (1-\zeta) \frac{\partial a}{\partial \tau} \tau_{\tau \theta} + \tau_{\theta \zeta}}{|\eta^3|a} \\
(4.12)
\]

\[
\tau_r = \frac{\sigma^{32} + 0}{|\eta^3|} - \frac{(1-\zeta) \frac{\partial a}{\partial r} \tau_{r r} + (1-\zeta) \frac{\partial a}{\partial \theta} \tau_{\theta r} + \tau_{r \zeta}}{|\eta^3|a} \\
(4.13)
\]

\[
\tau_\zeta = \frac{\sigma^{31} \zeta - (\sigma^{32} + \zeta \sigma^{33})}{|\eta^3|} = - \frac{(1-\zeta) \frac{\partial a}{\partial \theta} \tau_{\theta \zeta} + (1-\zeta) \frac{\partial a}{\partial \tau} \tau_{\tau \zeta}}{|\eta^3|a} \\
(4.14)
\]

By depth-averaging the shear stresses are modelled such that they apply to the projection of the bed surface on a virtual horizontal surface of unit area. The bed shear-stresses acting on this virtual surface are:

\[
\tau_{\text{bed} \theta} = \tau_{\theta} \left(\frac{\eta^3}{\eta a}\right) = (1-\zeta_0) \frac{\partial a}{\partial \theta} \tau_{\theta \theta} + (1-\zeta_0) \frac{\partial a}{\partial \tau} \tau_{\theta r} + \tau_{\theta \zeta} \\
(4.15)
\]

\[
\tau_{\text{bed} r} = \tau_{r} \left(\frac{\eta^3}{\eta a}\right) = (1-\zeta_0) \frac{\partial a}{\partial r} \tau_{r r} + (1-\zeta_0) \frac{\partial a}{\partial \theta} \tau_{\tau r} + \tau_{r \zeta} \\
(4.16)
\]

Upon depth-averaging the momentum equations, and substitution of these stresses the depth-averaged momentum equations read:

\[
\begin{align*}
\frac{\partial u_\theta}{\partial t} + \frac{\partial (au_\theta u_\theta)}{\partial \theta} + \frac{\partial (au_\theta u_r)}{\partial r} + 2 \frac{a}{r} u_\theta u_r &= 0 \\
- \rho \frac{\partial p}{\partial \theta} + \tau_{\text{bed} \theta} a + 1 \frac{\partial a}{\partial r} \tau_{\theta \theta} + \frac{1}{ar^2} \frac{\partial a}{\partial r} \tau_{\theta r} &= 0 \\
(4.17)
\end{align*}
\]

The depth-averaged, conservative, momentum equation in \(r_2\)-direction is:

\[
\begin{align*}
\frac{\partial u_r}{\partial t} + \frac{\partial (au_r u_r)}{\partial r} + \frac{\partial (au_r u_\theta)}{\partial \theta} + \frac{a}{r} u_r u_\theta - \frac{a}{r} u_\theta u_r &= \left(\frac{\partial a}{\partial r} \tau_{r r} + \frac{1}{ar^2} \frac{\partial a}{\partial r} \tau_{r \theta} \right)
\end{align*}
\]
The depth-averaged velocity products are further elaborated. The velocity profiles are modeled by similarity profiles:

\[ u_\theta = f_m u_\theta^m + f_s u_\theta^s \]

\[ u_r = f_s u_\theta^s + f_m u_\theta^m \]

in which: \( f_m = \frac{u}{u_m} \), \( f_s = \frac{u}{u_s} \), \( f_m^2 = 1 \), \( f_s^2 = 0 \), \( f_m^2 \approx 1 \), \( f_s^2 \approx 0.5 \), \( f_m^2 \approx 10 \), \( f_s^2 \approx 10 \).

The depth averaged velocity products are:

\[
\begin{pmatrix}
    u_\theta u_\theta \\
    u_r u_r \\
    u_\theta u_r
\end{pmatrix}
= \begin{pmatrix}
    \frac{u_\theta^2}{f_m^2} & \frac{u_\theta^2}{f_s^2} & 2 f_m f_s \\
    \frac{u_r^2}{f_s^2} & \frac{u_r^2}{f_m^2} & 2 f_m f_s \\
    \frac{u_\theta u_r}{f_m^2} & \frac{u_\theta u_r}{f_s^2} & \frac{u_\theta^2}{f_m^2} + \frac{u_r^2}{f_s^2}
\end{pmatrix}
\begin{pmatrix}
    u_\theta u_\theta \\
    u_r u_r \\
    u_\theta u_r
\end{pmatrix}
\]

The orders of magnitude of the terms are:

\( \frac{u_\theta^2}{f_m^2} \approx 1 \), \( \frac{u_\theta^2}{f_s^2} \approx 0.5 \), \( \frac{u_r^2}{f_s^2} \approx 10 \), \( \frac{u_r^2}{f_m^2} \approx 10 \).

Kalkwijk and de Vriend (1980) have simplified these relations by stating that \( \frac{u_\theta^2}{f_m^2} \approx 1 \), \( \frac{u_r^2}{f_s^2} \) is small and \( u_r^2 \) is small.

Substitution of these products in the depth-averaged equations yields:

\[ \frac{u_\theta u_\theta}{f_m^2} = \frac{f_m f_s}{f_m f_s} i_s r = k_s i_s r \]

in which: \( i_s = \) strength of secondary flow, full strength: \( i_s = 1 \).

The turbulent normal stresses are considered small in comparison with the turbulent shear stresses. Consequently these terms are neglected. (In absense of dominant shear stresses, however, the anisotropy of turbulent normal stresses generates flow vorticity)
The depth-averaged momentum equation in \( \theta \)-direction is:

\[
\frac{\partial u_\theta}{\partial t} + k_{uu} \left( \frac{\partial (au_\theta^2)}{\partial \theta} + \frac{\partial (au_\theta u_r)}{\partial r} \right) + 2a \frac{\partial u_\theta}{\partial r} + k \frac{\partial (i \frac{u_\theta^2 a^2}{r})}{\partial r} + \frac{2a^2}{r^2} =
\]

\[
- a \left( - \frac{\partial p_{rl}}{\partial \theta} + \frac{\partial r r_\theta}{\partial r} + \frac{\partial r r_\theta}{\partial \theta} + 2 \frac{\partial a}{\partial r \theta} + \frac{\partial (1-\xi) r_\theta}{\partial \theta} \right) \frac{\partial a}{\partial r} \quad (4.22)
\]

The depth-averaged momentum equation in \( r \)-direction is:

\[
\frac{\partial u_r}{\partial t} + k_{uu} \left( \frac{\partial (au_r^2)}{\partial \theta} + \frac{\partial (au_r u_\theta)}{\partial \theta} \right) - \frac{a^2}{r^2} + \frac{\partial (i u_r a^2/r)}{\partial \theta} + k \frac{\partial (i \frac{u_r^2 a^2}{r})}{\partial \theta} + \frac{2a^2}{r^2} =
\]

\[
- a \left( - \frac{\partial p_{rl}}{\partial r} + \frac{\partial r r_\theta}{\partial \theta} + \frac{\partial r r_\theta}{\partial \theta} + 2 \frac{\partial a}{\partial r \theta} + \frac{\partial (1-\xi) r_\theta}{\partial \theta} \right) \frac{\partial a}{\partial r} \quad (4.23)
\]

Subtraction of the depth-averaged continuity equation from the momentum equations yields the non-conservative form of the equations:

The depth-averaged momentum equation in \( \theta \)-direction:

\[
\frac{\partial u_\theta}{\partial t} + k_{uu} \left( \frac{\partial (au_\theta^2)}{\partial \theta} + \frac{\partial (au_\theta u_r)}{\partial r} \right) + 2a \frac{\partial u_\theta}{\partial r} + k \frac{\partial (i \frac{u_\theta^2 a^2}{r})}{\partial r} + \frac{2a^2}{r^2} =
\]

\[
- a \left( - \frac{\partial p_{rl}}{\partial \theta} + \frac{\partial r r_\theta}{\partial r} + \frac{\partial r r_\theta}{\partial \theta} + 2 \frac{\partial a}{\partial r \theta} + \frac{\partial (1-\xi) r_\theta}{\partial \theta} \right) \frac{\partial a}{\partial r} \quad (4.24)
\]

The depth-averaged momentum equation in \( r \)-direction:

\[
\frac{\partial u_r}{\partial t} + k_{uu} \left( \frac{\partial (au_r^2)}{\partial \theta} + \frac{\partial (au_r u_\theta)}{\partial \theta} \right) - \frac{a^2}{r^2} + \frac{\partial (i u_r a^2/r)}{\partial \theta} + k \frac{\partial (i \frac{u_r^2 a^2}{r})}{\partial \theta} + \frac{2a^2}{r^2} =
\]

\[
- a \left( - \frac{\partial p_{rl}}{\partial r} + \frac{\partial r r_\theta}{\partial \theta} + \frac{\partial r r_\theta}{\partial \theta} + 2 \frac{\partial a}{\partial r \theta} + \frac{\partial (1-\xi) r_\theta}{\partial \theta} \right) \frac{\partial a}{\partial r} \]
4.4 Shear stresses

Three shear stresses are present in the depth-averaged momentum equations are: \( \tau_{\xi \theta} \), \( \tau_{\zeta r} \) and \( \tau_{\theta r} \). The \( \tau_{\xi \theta} \) stress is the dominating stress term, at the bed level it is equal to the bed shear-stress. The \( \tau_{\zeta r} \) stress is small because the flow deviates little from the \( \theta \)-coordinate. The \( \tau_{\theta r} \) stress is considered because it is to be incorporated to model side wall shear stress, which could play a role in the side wall regions.

The shear stress in main flow direction \( \tau_{\theta \xi} \) is given by, eq.(3.53):

\[
\tau_{\theta \xi} = \rho \nu \left( \frac{\partial u_\theta}{\partial \zeta} + \frac{\partial u_\xi}{\partial \theta} + (1-\xi) \frac{\partial a}{\partial \zeta} \right) \tag{4.26}
\]

The most important contribution is given by the first term. The shear at the bed level functions in the \( \theta \)-momentum equation.

The transverse shear stress \( \tau_{\theta r} \) is modelled by:

\[
\tau_{\theta r} = \rho \nu \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + (1-\xi) \frac{\partial u_\xi}{\partial \zeta} \right) \tag{4.27}
\]

The first term is equivalent to the stress term in an orthogonal Cartesian coordinate system. The second term is due to curvature of the coordinate system. If the fluid velocity increases linearly with the radius, i.e. no shear stress, (the fluid rotates like a solid body) the sum of the first and second term is zero, \( \tau_{\theta r} = 0 \). The third term is due to non-orthogonality of the coordinate system; the surfaces of \( \zeta \)-const. are not perpendicular to the surfaces of \( r \)-const., in which the stress is evaluated.

The depth-averaged transverse shear stress is:
\[
\tau_{\theta r} = \rho (\nu \frac{\partial u_{\theta}}{\partial r} - \nu \frac{u_{\theta}}{r} + \nu (1-\zeta) \frac{\partial u_{\theta}}{a \partial \zeta} \frac{\partial a}{\partial r}) 
\]  
(4.28)

The bed shear-stress is modelled by (\(\beta\)-secondary flow direction coefficient): \(r_{\text{bed} \, \theta} = \rho u_{*}^{2} \), \(r_{\text{bed} \, r} = \rho u_{*}^{2} \frac{\tilde{u}_{r}}{u_{\theta}} + \beta \frac{a}{r} r_{\text{bed} \, \theta} \)

The friction velocity is related to the main flow velocity by: \(u_{*} = \frac{g}{c} u_{\theta}^{2} \)

The bed shear-stress terms are given by:

\[
r_{\text{bed} \, \theta} = \rho g \frac{1}{C^{2}} u_{\theta}^{2} 
\]  
(4.29)

\[
r_{\text{bed} \, r} = \rho g \frac{1}{C^{2}} u_{\theta} \frac{u_{r}}{u_{\theta}} + \beta \rho g \frac{1}{C^{2}} u_{\theta}^{2} 
\]  
(4.30)

The depth-averaged value of \(r_{\theta r}\) is calculated assuming an isotropic eddy viscosity profile. Anisotropy can be accounted for by raising the resulting \(r_{\theta r}\) value artificially. The eddy viscosity profile is assumed parabolic, the main flow velocity profile \((f)\) is assumed logarithmic:

\[
\nu = \kappa^{2} a_{\zeta} (1-\zeta) a u_{\theta} 
\]  
(4.31)

\[
f_{m} = \frac{u}{\bar{u}} = 1 + \alpha (1 + \ln \zeta) 
\]  
(4.32)

In which: \(\alpha = \sqrt{g}/(\kappa C)\)  
(4.33)

Substitution of the viscosity and velocity profiles, eq.(4.31) and eq.(4.32) yields:

\[
\nu \frac{\partial u_{\theta}}{\partial r} = \left[ \int_{r=0}^{1} \nu \frac{\partial u_{\theta}}{\partial r} d\zeta = \rho \kappa^{2} \frac{1}{6} \alpha (1+\frac{1}{6} \alpha) \frac{a}{2} \frac{\partial u_{r}}{\partial \zeta} \right]^{2} 
\]  
(4.34)

\[
\nu \frac{u_{\theta}}{r} = \left[ \int_{\zeta=0}^{1} \nu \frac{u_{\theta}}{r} d\zeta = \rho \kappa^{2} \frac{1}{3} \alpha (1+\frac{1}{6} \alpha) \frac{a}{r} \frac{1}{2} \frac{u_{\theta}^{2}}{u_{\theta}} \right]^{2} 
\]  
(4.35)
\[ \nu \underbrace{\frac{\partial u}{\partial \xi} \frac{\partial a}{\partial r}}_{\partial \xi \partial r} = \rho \kappa^2 \frac{1}{3} \alpha \frac{\partial a}{\partial r} \frac{\partial u}{\partial r} \]  
\quad \text{(4.36)}

Theoretically depth-averaging should be performed over the interval \( r_0 \ldots 1 \), because below \( \xi_0 \) no flow exists. In that case the result is the same when terms with \( r_0^2 \) or smaller are neglected.

Substitution of eq.(4.34)...(4.36) in eq.(4.28) yields the depth-averaged transverse stress component:

\[ r_{\theta r} = \rho \kappa^2 \alpha \left( \frac{1}{6} (1+\frac{1}{6} \alpha) \right) \left( \frac{1}{2} \frac{\partial u}{\partial r} - \frac{u}{r} \right) + \frac{1}{3} \alpha \frac{\partial a}{\partial r} \frac{u}{r} \]  
\quad \text{(4.37)}

which is equal to:

\[ r_{\theta r} = \rho A \alpha \left( \frac{\partial u}{\partial r} \right) + \rho \frac{\kappa^2 \alpha}{3} \frac{\partial a}{\partial r} \frac{u}{r} \]  
\quad \text{(4.38)}

in which: \( A = \kappa^2 \frac{1}{6} \alpha (1+\frac{1}{6} \alpha) \)

4.5 The depth-averaged momentum equations

Considering the arguments discussed in the previous chapters and neglect of the normal stresses, the depth-averaged momentum equations are given by (in non-conservative form):

The depth-averaged momentum equation in \( \theta \)-direction:

\[ \frac{\partial u}{\partial t} + k_{uu} \left( \frac{\partial u}{\partial \theta} \right) + \frac{\partial u}{\partial r} + \frac{a}{r} \frac{\partial u}{\partial r} + \frac{a}{u} \frac{\partial u}{u} \left( \frac{a}{u} \frac{\partial a}{\partial r} \right) + k_{sn} \left( \frac{\partial (i u_s a_r^2)}{\partial r} \right) + 2 i s \frac{a^2}{r^2} = \]  
\quad \text{(4.39)}

The depth-averaged momentum equation in \( r \)-direction:
\[
\begin{align*}
&\frac{\partial u}{\partial t} + k_u (au_\theta \frac{\partial u}{\partial r \theta} + au_r \frac{\partial u}{\partial r} - a u_\theta^2 ) + \\
&+ k \frac{\partial (i s u_\theta a^2 / r)}{r \partial \theta} + 2 \frac{\partial (i s u_r a^2 / r)}{\partial r} = \\
&= \frac{1}{\rho} \left( - \frac{\partial p r_1}{\partial r} - \tau_{b e d} r + \frac{\partial a r}{\partial r \theta} \right) \\
\end{align*}
\]
5 THE KALKWIJK AND DE VRIEND DEPTH-AVERAGED MODEL

5.1 Introduction

Kalkwijk and de Vriend (1980) have simplified the depth-averaged momentum equations by considering the characteristics of the problem. In their publication they do not give a derivation of the momentum equations in a cylindrical bed-following coordinate system, instead they cite an Russian author (who in turn cites a publication in Russian). The equations are, however, identical to the equations derived in chapter 4 (except for \( r \theta_r \) terms which were not considered). The model has been extensively tested since then. The model has also been used in a morphological model for the bed-topography in river bends (Olesen, 1987).

In this chapter the derivation of the model is repeated. The method of determining the characteristics is documented in detail. The model of Kalkwijk and de Vriend is also extended with transvers shear stress, which offers a possibility to incorporate side wall friction.

In section 5.2 the depth-averaged momentum equations are simplified. In section 5.3 the characteristics of the simplified equations are determined. In section 5.4 the construction of a flow model, as originally formulated by Kalkwijk and de Vriend (1980), is given. An extension of the model with transverse shear stress is given in section 5.5.

5.2 Simplification of the momentum equations

In the depth-averaged momentum equations the normal stresses are neglected. Non-orthogonality effects on the \( r \theta_r \) term are also neglected. The depth-averaged momentum equation in \( \theta \)-direction is (non-conservative form):

\[
\begin{align*}
&k \left( \frac{\partial u_{\theta}}{r \partial \theta} + \frac{u_{\theta}}{r} \frac{\partial u_{\theta}}{\partial r} + \frac{a \Delta u_{\theta}}{r} \right) + k \left( \frac{\partial (i u_{\theta} a^2/2r)}{\partial r} + 2i \frac{a u_{\theta}}{s} \right) = \\
&\quad a \left( \frac{\partial p_{\theta \theta}}{r \partial \theta} + \frac{\partial \delta_{\theta \theta}}{a \partial \zeta} + \frac{1}{a} \frac{\partial \delta_{\theta \theta}}{\partial r} + 2 \frac{1}{r} \frac{\partial \delta_{\theta r}}{\partial \theta} \right) 
\end{align*}
\]

The convective terms in the depth-averaged momentum equation in \( r \)-direction are neglected, all the stress terms are also neglected. The remaining equation reads:
If bed shear-stresses in transverse direction are incorporated (only due to the main flow) and the convective term is included (which is a way to account for streamline curvature), the momentum equation in transverse direction reads:

\[
-k_{uu} \frac{a}{r} u_\theta u_\theta - k_{uu} \frac{a}{r} u_\theta u_\theta - \frac{\partial p_{r1}}{\rho \partial r} - \frac{g}{c^2} u_\theta u_\theta u_r = 0 \quad (5.2a)
\]

5.3 The characteristics

The characteristics of the flow model follow from the set of equations eq.(5.1) and eq.(5.2). Discontinuities in the derivatives of the solution propagate in the direction of the characteristics. The unknown derivatives of the simplified flow model are (\(r_{\theta r}\) terms are not considered, the Kalkwijk and de Vriend model):

\[
\begin{align*}
\frac{\partial u_\theta}{\partial r} & \frac{\partial u_\theta}{\partial r} \frac{\partial p_{r1}}{\partial r} \frac{\partial p_{r1}}{\partial r} \\
\frac{\partial p_{r1}}{\partial r} & \frac{\partial p_{r1}}{\partial r} \frac{\partial p_{r1}}{\partial r} \frac{\partial p_{r1}}{\partial r}
\end{align*}
\]

The two momentum equations, eq.(5.1) and eq.(5.2), and two additional equations for the total derivatives of \(u_\theta\) and \(p_{r1}\) yield a system of 4 equations with 4 unknown derivatives.

The total derivatives of \(u_\theta\) and \(p_{r1}\) are:

\[
\begin{align*}
\frac{\partial u_\theta}{\partial r} = \frac{\partial u_\theta}{\partial r} & \frac{\partial u_\theta}{\partial r} \frac{\partial u_\theta}{\partial r} \frac{\partial u_\theta}{\partial r} \\
\frac{\partial p_{r1}}{\partial r} & \frac{\partial p_{r1}}{\partial r} \frac{\partial p_{r1}}{\partial r} \frac{\partial p_{r1}}{\partial r}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u_\theta}{\partial r} = \frac{\partial u_\theta}{\partial r} & \frac{\partial u_\theta}{\partial r} \frac{\partial u_\theta}{\partial r} \frac{\partial u_\theta}{\partial r} \\
\frac{\partial p_{r1}}{\partial r} & \frac{\partial p_{r1}}{\partial r} \frac{\partial p_{r1}}{\partial r} \frac{\partial p_{r1}}{\partial r}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u_\theta}{\partial r} = \frac{\partial u_\theta}{\partial r} & \frac{\partial u_\theta}{\partial r} \frac{\partial u_\theta}{\partial r} \frac{\partial u_\theta}{\partial r} \\
\frac{\partial p_{r1}}{\partial r} & \frac{\partial p_{r1}}{\partial r} \frac{\partial p_{r1}}{\partial r} \frac{\partial p_{r1}}{\partial r}
\end{align*}
\]

These relations give the increment of \(u_\theta\) and \(p_{r1}\) for an increment of coordinates \(r\partial \theta\) and \(\partial r\).

The following system of equations results; \(A \phi = g\) (the unknown derivatives \(\phi\) on the left hand side):
The values of the derivatives are undetermined in case the determinant of the matrix \( \mathbf{A} \) is zero. The solution is the characteristic:

\[
\frac{dr}{rd\theta} = \frac{u}{u} + 2k \frac{a}{sn} \frac{i}{s}
\]

The convectiveterm and the pressureterm of the \( \theta \)-momentum equation are joined. The increment of \( p_{rl}/\rho + 0.5u_{\theta} \) over an increment of coordinates \( rd\theta \) and \( dr \) is:

\[
d(p_{rl}/\rho + 0.5u_{\theta})_\theta = \frac{\partial (p_{rl}/\rho + 0.5u_{\theta})}{\partial r} dr + \frac{\partial (p_{rl}/\rho + 0.5u_{\theta})}{\partial \theta} rd\theta
\]

division by \( rd\theta \) yields:

\[
d(p_{rl}/\rho + 0.5u_{\theta})_\theta = \frac{\partial (p_{rl}/\rho + 0.5u_{\theta})}{\partial r} \frac{dr}{rd\theta} + \frac{\partial (p_{rl}/\rho + 0.5u_{\theta})}{\partial \theta} \frac{dr}{rd\theta}
\]

On the characteristic, eq.(5.5) there results by substitution of eq.(5.5) and the momentum equations, eq.(5.1) and eq.(5.2), in eq.(5.7):

\[
d(p_{rl}/\rho + 0.5u_{\theta})_\theta = - \frac{u_{\theta}}{a} \left( \frac{g}{c^2} + k \frac{\partial i a^2}{\partial r} \right)
\]

This is the flow model formulation along the characteristic.
5.4 The Kalkwijk and de Vriend flow model

The essence of the model of Kalkwijk and de Vriend is that the numerical approach is such that the characteristics nearly coincide with the \( \theta \)-coordinate. The terms incorporating \( u_r \) are assumed to be known (they are calculated from the preceding timestep).

The system of equations now becomes:

\[
\begin{align*}
\begin{bmatrix}
\frac{\partial u_\theta}{\partial \theta} & 2k & \frac{u_r}{r} & \frac{1}{\rho} & 0 \\
0 & 0 & 0 & \frac{1}{\rho} & 0 \\
rd\theta & dr & 0 & 0 & \frac{\partial p_{rl}}{\partial r} \\
0 & 0 & rd\theta & dr & \frac{\partial p_{rl}}{\partial r}
\end{bmatrix}
&= \begin{bmatrix}
\text{remaining terms} \\
\frac{u_\theta}{r} \\
\frac{\partial u_\theta}{\partial r} \\
\frac{\partial p_{rl}}{\partial r}
\end{bmatrix}
\end{align*}
\]

The characteristic of this set of equations is:

\[
\frac{dr}{rd\theta} = 2k \frac{a}{sn \frac{1}{r} s} \quad (5.10)
\]

The momentum equation along this characteristic, with does not deviate much from the \( \theta \)-coordinate, is:

\[
\frac{d(p_{rl}/\rho+0.5u_\theta^2)}{rd\theta} = -\frac{u_\theta^2}{a} \left( \frac{g}{c^2 + k sn \frac{1}{\rho} r} \right) - \left( \frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r} \right) \quad (5.11)
\]

5.5 Extension with transverse shear stress

In case of transverse friction the \( \theta \)- and \( r \)-momentum equations have the following additional terms:

\[
\begin{align*}
\theta \text{-mom. eq.: } & \frac{1}{a} \frac{\partial a \tau_{\theta \theta}}{\partial r} + 2 \frac{1}{r} \frac{\partial (1-\xi) \tau_{\theta r}}{\partial \theta} + \frac{\partial a}{a \partial \xi} \frac{\partial a}{\partial \xi} \\
r \text{-mom. eq.: } & \frac{1}{a} \frac{\partial a \tau_{\theta \theta}}{\partial r} + \frac{\partial (1-\xi) \tau_{\theta r}}{a \partial \xi} \frac{\partial a}{\partial \xi} \frac{\partial a}{\partial \xi} 
\end{align*}
\]

\[
\begin{align*}
(5.12) \\
(5.13)
\end{align*}
\]
The last shear stress term of the \( \theta \)-momentum equation, which is due to non-orthogonality, will be neglected. All shear stress terms of the \( r \)-momentum equation will be neglected.

The solution method is chosen the same as in the model of Kalkwijk and de Vriend. That is the transverse stress terms are taken from the preceding time step.

The system of equations now becomes:

\[
\begin{bmatrix}
\frac{u_\theta}{\rho} & 2k \frac{a}{\text{sn} r} & \frac{1}{\rho} & 0 \\
0 & 0 & 0 & \frac{1}{\rho} \\
r \frac{\partial u_\theta}{\partial \theta} & dr & 0 & 0 \\
0 & 0 & \frac{rd \theta}{dr} & dr
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u_\theta}{\partial \theta} \\
\frac{\partial u_\theta}{\partial r} \\
\frac{\partial p_r l}{\partial \theta} \\
\frac{\partial p_r l}{\partial r}
\end{bmatrix}
= \begin{bmatrix}
\text{remaining terms} \\
\frac{u_\theta}{r} \\
- \frac{du_\theta}{dr} \\
\frac{dp_r l}{dr}
\end{bmatrix}
\] (5.14)

The characteristic of this set of equations is the same as in the model of Kalkwijk and de Vriend: \( \frac{dr}{rd \theta} = 2k \frac{a}{\text{sn} r} \) (5.15)

The momentum equation along this characteristic, with does not deviate much from the \( \theta \)-coordinate, is:

\[
d\left(\frac{p_r l}{\rho+0.5u_\theta} \right)_{\frac{rd \theta}{dr}} = \frac{u_\theta}{a} \left( \frac{g}{c^2 + k \frac{\partial^2 a}{\partial r}} \right) - \left( \frac{\partial u_\theta}{\partial r} + \frac{u_r \frac{u_\theta}{r}}{r} \right) + \frac{1}{\rho} \frac{\partial a}{\partial r} \right) (5.16)

The depth averaged transverse shear stress is given by (A parabolicaleddy viscosity distribution is assumed, eq. 4.33):

\[
r_{\theta r} = \rho \alpha (\frac{1}{6} \left( 1, \frac{1}{2} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) + \frac{1}{3} \frac{\partial a}{\partial r} \frac{u_\theta}{r})^2
\] (5.17)

If \( \frac{\partial a}{\partial r} = 0 \) eq.(5.17) is similar to the expression given by de Vriend (1981).

The shear stress at the side walls is related to the velocity at a reference location nearby the side walls. The depth-averaged side wall shear stress is modelled by:
\[
\tau_w = \rho \frac{u_{*w}^2}{C_w^2} = \rho \frac{g}{C_w^2} u_{\theta \text{ref}}^2
\]  

(5.18)

In which: \(C_w [m^{-0.5}/s]\) a side wall roughness coefficient in analogy to Chezy's definition.

- \(u_{*w} [m/s]\) depth-averaged friction velocity at the side wall
- \(u_{\theta \text{ref}} [m/s]\) a depth-averaged velocity at a nearwall reference location

In a numerical model the momentum equations may be represented by discretized versions of the momentum equations eq.(5.16) and eq.(5.2(a)). The pressure at the rigid lid is than eliminated by subtraction of discretized equations. This method is given in Kalkwijk & de Vriend (1980) and Olesen (1987).
6 FLOW ACCELERATIONS IN DEPTH-AVERAGED MODELS

6.1 Introduction

In depth-averaged models of the flow in rivers and estuaries, similarity profile functions for the velocity are assumed. These functions are then on the assumption of equilibrium flow conditions. In situations in which the flow is accelerating or decelerating, but the local velocity remains constant, such profile functions are not appropriate from a theoretical point of view. The effect of non-equilibrium flow on the main-flow momentum equation and the bed shear-stresses is investigated. The sediment transport rate is very sensitive to small variations of the bed shear-stress. Consequently it could be important to account for non-equilibrium flow in morphological models.

In section 6.2 the main flow momentum equation is simplified and depth-integrated for the study of acceleration effects. In section 6.3 the solution is approximated by asymptotic expansion of the variables. In section 6.4 a solution is derived with the assumption of local equilibrium of the eddy viscosity. In section 6.5 the implementation of the results in a depth-averaged flow model is discussed. In section 6.6 the results are applied to a hypothetical situation of two-dimensional flow over sinuous bed topography. In section 6.7 the results of other work on this subject is summarized. The conclusions are given in section 6.8.

6.2 The main flow momentum equation

The simplified momentum equation in main flow direction is (steady flow, vertical convection term neglected, discard of transverse-direction):

$$\frac{\partial u}{\partial x}^2 = \frac{1}{\rho}(-\frac{\partial p}{\partial x} + \frac{\partial r}{\partial z})$$ (6.1)

The pressure is assumed hydrostatic:

$$p = P + \rho g (s - z)$$ (6.2)

in which: 
- $P$ = pressure at water surface
- $s$ = $z$-coordinate of water surface
- $b$ = $z$-coordinate of bed level

The momentum equation is to be integrated over the flow depth. The integrated pressure term is given by:

$$\int_b^s \rho \frac{\partial p}{\partial x} \, dz = \frac{1}{\rho} (s - b) (\frac{\partial P}{\partial x} + \rho g \frac{\partial s}{\partial x})$$ (6.3)

The integrated friction term is ($r(s) = 0$):
\[ \int_{s} s \frac{1}{\rho} \frac{\partial r}{\partial z} \, dz = - \frac{r(b)}{\rho} \quad (6.4) \]

Consequently the depth-integrated main flow momentum equation reads:

\[ \int_{s-b} s \frac{\partial u}{\partial x} \, dz = - \frac{1}{\rho(s-b)} \left( \frac{\partial p}{\partial x} + \rho \frac{\partial s}{\partial x} + r(b) \right) \quad (6.5) \]

Substitution of the hydrostatic pressure in eq. (6.1) and elimination of the pressure term of the local and the depth-integrated momentum equations eq. (6.1) & eq. (6.5) yields:

\[ \frac{1}{s-b} \int_{s} s \frac{\partial u}{\partial x} \, dz - \frac{\partial u}{\partial x} = - \frac{r(b)}{\rho(s-b)} \frac{1}{\rho} \frac{\partial r}{\partial z} \quad (6.6) \]

The primitive of eq. (6.6) is:

\[ \int_{z} \frac{1}{s-b} \left( \int_{s} s \frac{\partial u}{\partial x} \, dz \right) \, dz - \int_{z} \frac{\partial u}{\partial x} \, dz = - z \frac{r(b)}{\rho(s-b)} - \frac{r}{\rho} + A \quad (6.7) \]

The integration constant is determined by the condition of a vanishing shear stress at the water level. This yields:

\[ A = \int_{s} \frac{1}{s-b} \left( \int_{s} s \frac{\partial u}{\partial x} \, dz \right) \, dz - \int_{s} \frac{\partial u}{\partial x} \, dz + \frac{r(b)}{\rho(s-b)} \frac{s}{s-b} \quad (6.8) \]

Substitution in eq. (6.7) yields:

\[ \int_{s} \frac{1}{z} \left( \int_{s-b} s \frac{\partial u}{\partial x} \, dz \right) \, dz - \int_{z} \frac{\partial u}{\partial x} \, dz = \frac{r(b)}{\rho} \frac{s-z}{s-b} + \frac{r}{\rho} \quad (6.9) \]

This is equal to:

\[ \frac{s-z}{s-b} \int_{s-b} s \frac{\partial u}{\partial x} \, dz - \int_{z} \frac{\partial u}{\partial x} \, dz = \frac{r(b)}{\rho} \frac{s-z}{s-b} + \frac{r}{\rho} \quad (6.10) \]

The z-coordinate is made non-dimensional with a sigma-transformation:

\[ \zeta = \frac{z-b}{s-b} \quad (6.11) \]

Substitution in eq. (6.10) yields:

\[ (s-b) \left( (1-\zeta) \int_{0}^{1} \frac{\partial u}{\partial x} \, d\zeta - \int_{\zeta}^{1} \frac{\partial u}{\partial x} \, d\zeta \right) = -(1-\zeta) \frac{r(b)}{\rho} + \frac{r}{\rho} \quad (6.12) \]
6.3 Approximation by asymptotic expansions

The objective is to express the velocity field by similarity functions. The velocity and the shear-stress are approximated by asymptotic expansions:

\[ f_u = \frac{u}{U} = \beta_0 f_{u0} + \varepsilon \beta_1 f_{u1} + \varepsilon^2 \beta_2 f_{u2} + 6.6. \] (6.13)

\[ f_r = \frac{r}{r_0} = \beta_0 f_{r0} + \varepsilon \beta_1 f_{r1} + \varepsilon^2 \beta_2 f_{r2} + 6.6. \] (6.14)

In which \((f)\) are vertical profiles of velocity and shear stress. Similarity profiles are assumed (no dependence on \(x\)). This assumption has to be verified during the analysis. The \(f_u\) profile represents the local velocity profile normalized with the local mean velocity. The \(f_r\) profile represents the local shear-stress normalized with the local zeroth-order bed-shear stress. The parameter \((\varepsilon)\) is a small parameter independent of \(x\). Amplitudes of the profiles are determined by \(\beta_0, \beta_1\) and \(\beta_2\), these are assumed to have values of the order one: \(O(1)\). They are allowed to be a function of \(x\).

\[ f_{u0} = \frac{u_0}{U}, \quad f_{u1} = \frac{u_1}{U}, \quad f_{u2} = \frac{u_2}{U} \] (6.15)

\[ f_{r0} = \frac{r_0}{r_0(b)}, \quad f_{r1} = \frac{r_1}{r_0(b)}, \quad f_{r2} = \frac{r_2}{r_0(b)} \] (6.16)

in which: \(u_i\) = local \(i^{th}\) order velocity contribution

\(r_i\) = local \(i^{th}\) order shear stress contribution

\(r_i(b)\) = local \(i^{th}\) order bed shear-stress contribution

\(U\) = local depth averaged velocity

A base state is introduced. This is a situation in which no bed undulations are present. For these conditions the velocity profile is given by \(f_{u0}\). Equation (6.12) is used to calculate the variables \(r\) and \(U\) at surfaces of \(\xi=\text{constant}\). Substitution of the velocity profile in eq.(6.12) yields:

\[
\frac{A_u^{\text{bs}}}{L_u^{\text{xbs}}} \left[ (1-\xi) \int_0^1 f^2 u_0 d\xi - \int_0^1 f^2 u_0 d\xi \right] \frac{\partial^2 \beta_0^{2(s'-b')}U'^2}{\partial \xi'} + \\
\frac{A_r^{\text{bs}}}{L_r^{\text{xbs}}} \varepsilon \left[ (1-\xi) \int_0^1 f^2 u_0 u_1 d\xi - 2 \int_0^1 f^2 u_0 u_1 d\xi \right] \frac{1}{U'^2} \frac{\partial \beta_1 \beta_0^{2(s'-b')}U'^2}{\partial \xi'} + \\
+ \ldots \ H.O.T. = (1-\xi) f(b) + f_r) T
\] (6.17)
in which: $A$ = length scale variations in vertical direction
(average flow depth)
$L$ = length scale variations in horizontal direction
$U_{bs}$ = velocity scale (= mean velocity base state)

$(s'-b')A = s-b$  
$s' = 0(1)$,  
$b' = 0(1)$
$x'L = x$  
$x' = 0(1)$
$U'U_{bs} = U$  
$U' = 0(1)$

subscript $bs$: base state ($A/L = def. = 0$)

$T = \frac{U''}{U'_{bs}} = \text{ratio local and base state bed shear-stress}$  
(6.18)

The small parameter ($\epsilon$) is be chosen equal to: 
$$\epsilon = \frac{A U^2}{L u^2} = \frac{A}{L g}$$  
(6.19)

Substitution of the asymptotic expansion of $f_r$ in eq.(6.17) and collecting terms at $0^{th}$, $1^{st}$ and $2^{nd}$ order yields:

at $0^{th}$ order: 
$$-(1-\zeta) \tau_0 (b) + \tau_0 = 0$$  
(6.20)

at $1^{st}$ order: 
$$(1-\zeta) \int_0^1 f^2 u_0 d\zeta - \int_0^1 f^2 u_0^2 d\zeta = \frac{1}{U'^2} \frac{\partial \beta_0^2 (s'-b') U'^2}{\partial x'}$$
$$= \beta_1 T \left[ -(1-\zeta) (f_{r1} (b) + f_{r1}) \right]$$  
(6.21)

at $2^{nd}$ order: 
$$(1-\zeta) 2 \int_0^1 f u_0 f u_0 d\zeta - 2 \int_0^1 f u_0 f u_1 d\zeta = \frac{1}{U'^2} \frac{\partial \beta_1 \beta_0 (s'-b') U'^2}{\partial x'}$$
$$= \beta_2 T \left[ -(1-\zeta) (f_{r2} (b) + f_{r2}) \right]$$  
(6.22)

6.4 Solution in case of an equilibrium eddy viscosity

The shear stress is modelled by the eddy viscosity hypothesis: 
$$\tau = \nu \frac{\partial u}{\partial z}$$

The eddy viscosity is assumed not to be affected by the spatially varying flow field. Local equilibrium is assumed. Consequently: $\nu = \nu_0$. The shear stress is modelled by: 
$$\tau = \nu_0 U ( \beta_0 \frac{\partial f}{\partial z} + \epsilon \beta_1 \frac{\partial f}{\partial z} + \epsilon^2 \beta_2 \frac{\partial f}{\partial z} + ...)$$  
(6.23)

A parabolic eddy viscosity profile is applied:

$$\nu_0 = \kappa \zeta (1-\zeta) u_{x0}$$  
(6.24)
in which: $u_{*0}$ - the local zeroth-order bed shear-velocity

The solution of the 0th order problem, eq(6.20), are is the standard logarithmic velocity profile and the linear shear-stress profile:

$$f_{u0} = \frac{u_0}{U} = \frac{\alpha}{\beta_0} \ln(\xi/\xi_0) \quad (6.25)$$

$$f_{r0} = \frac{(1-\xi)}{\beta_0} \quad (6.26)$$

in which: $\alpha = \frac{u_{*0}}{kU} = \text{friction factor} \quad (6.27)$

$\xi_0 = \text{dimensionless zero velocity level}$

The shape and magnitude of the $f_{u0}$ and $f_{r0}$ profiles was assumed to be independent on the x-coordinate. Because the shape has to remain constant it is concluded that: $\alpha$ and $\xi_0$ are constant. The latter because of:

$$\xi_0 = e^{-1-1/\alpha} \quad (6.28)$$

Because the magnitude has been assumed constant it has to be concluded from eq.(6.25) and eq(6.26) that $\beta_0$ is constant. For ease $\beta_0$ is chosen equal to one: $\beta_0 = 1 \quad (6.29)$

The solution of the 1th order problem is somewhat more elaborate. The integration intervals of eq.(6.21) are reformulated.

$$\int_{\xi_0}^{\xi} \int \frac{2}{u_0} d\xi = \beta_1 T \left( \left( (1-\xi) f_{r1}(b) + f_{r1} \right) \right) \quad (6.30)$$

in which: $K = \frac{1}{U} \frac{\delta(s'-b')U}{U^2} \quad (6.31)$

The primitive of $f_{u0}$ is equal to:

$$P(f_{u0}) = \alpha^2 \xi \left( (\ln \frac{\xi}{\xi_0})^2 - 2\ln \frac{\xi}{\xi_0} + 2 \right) =$$

$$= \alpha^2 \xi \left( \ln^2(\xi) + 2(\ln(\xi)-1)(1+\frac{1}{\alpha}) + \ln^2(\xi_0) - 2\ln(\xi) + 2 \right) \quad (6.32)$$

In which use has been made of: $\ln(\xi_0) = -1-1/\alpha \quad (6.33)$

The integrals in eq.(6.30) are given by:

$$\int_{\xi_0}^{\xi} \int \frac{2}{u_0} d\xi = \alpha^2 \left( \xi \ln^2(\xi) + \frac{2}{\alpha}(\ln(\xi)-1) + 1n^2(\xi_0) - 2\xi_0 \right) \quad (6.34)$$
Substitution of these equations in eq.(6.30) yields:

\[ K \frac{\alpha^2}{\alpha^2} \left( \zeta \ln^2(\zeta) + \frac{2}{\alpha} \zeta \ln(\zeta) - 2\zeta_0 (1-\zeta) \right) = \beta_1 T \left( - (1-\zeta) f_r(b) + f_{r1} \right) \]  

(6.36)

The 1st order contribution to the shear stress is given by:

\[ \tau_1 = \nu_0 \frac{\partial u_1}{\partial z} \]  

(6.37)

this is equal to:

\[ f_{r1} = \frac{\zeta (1-C)}{\alpha} \frac{\partial f_{u1}}{\partial \zeta} \]  

(6.38)

Substitution of eq.(6.38) in eq.(6.36) and integration over \( \zeta \) yields:

\[ f_{ul} = \frac{K}{\beta_1 T} \alpha^2 \left[ (\alpha \ln^2(\zeta) - 2 \ln(\zeta)) \left( f_r(b) - 2 \frac{K}{\beta_1 T} \zeta_0^2 \zeta_0 \right) \right] \]  

(6.39)

The integration constant \( B \) is obtained by the condition of zero velocity at the zero velocity level; \( u_1(\zeta_0) = 0 \). Consequently:

\[ f_{ul} = \frac{K}{\beta_1 T} \alpha^2 \int_{\zeta_0}^{\zeta} \left( \alpha \ln^2(\zeta) - 2 \ln(\zeta) \right) d\zeta + \left( f_r(b) - 2 \frac{K}{\beta_1 T} \zeta_0^2 \right) a \ln(\zeta) - B \]  

(6.40)

The shear-stress at the bed, "the zero velocity level", is obtained by the condition of zero net flow by \( u_1 \) (\( \int u_1 = 0 \) the continuity equation can be based on \( u_0 \) only, conservation of mass in a 2D case): for small \( \zeta_0 \):

\[ \int_{\zeta_0}^{1} f_{ul} d\zeta = 0 = \frac{K}{\beta_1 T} (2\alpha - 2) \zeta_0^2 + f_{r1}(b) = 0 \]  

(6.41)

Consequently the first order perturbation of the bed shear-stress is given by:

\[ f_{r1}(b) = 2 \frac{K}{\beta_1 T} (1-\alpha) \zeta_0^2 \]  

(6.42)

Substitution of eq.(6.42) in eq.(6.40) yields the first order contribution to the velocity profile:

\[ f_{ul} = \frac{K}{\beta_1 T} \alpha^2 \left( \int_{\zeta_0}^{\zeta} \left( \alpha \ln^2(\zeta) - 2 \ln(\zeta) \right) d\zeta - 2(1-\alpha) a \ln(\zeta) \right) \]  

(6.43)

(\( f_{ul} \) positive in lower half of flow depth)
It was assumed that $f_{ul}$ is independent on $x$. Consequently: $\beta_1 = K/T$ (6.44)

Remark:
The derivation of the secondary flow velocity profile in fully developed river bend flow follows the same method. In that case the secondary flow profile has to be solved instead of $f_{ul}$. Shear stresses in transverse direction are then substituted instead of $f_r$. The small parameter is then:

$$\epsilon = -\frac{a}{r g}, \ T = 1$$ (6.45)

The same procedure applies for the computation of the second order velocity profile. No analytical solution is found. The profile is calculated numerically. In appendix A the base equations to determine the solutions for the 1st, 2nd and 3rd order approximations are given.

For the second order solution, the derivative with respect to $x$ in eq.(6.22), is with the aid of eq.(6.31) written as:

$$\frac{1}{u'^2} \frac{\partial \beta_1}{\partial x'} (s'-b') u'^2 = \beta_1 K + (s'-b') \frac{\partial \beta_1}{\partial x'} = \frac{K^2}{T} + (s'-b') \frac{\partial K/T}{\partial x'}$$ (6.46)

Like in case of the derivation of the first order profile, the second order velocity profile $f_{u2}$ will be linear with eq.(6.46) divided by $\beta_2 T$, compare eq.(6.21) and eq.(6.43). In order to express the velocity by similarity functions, $\beta_2$ is to be taken equal to:

$$\beta_2 = \frac{K^2}{T^2} + \frac{(s'-b')}{T} \frac{\partial K/T}{\partial x'}$$ (6.47)

The resulting second order velocity profile approximation will be given by:

$$f_u = f_{u0'} + \epsilon \frac{K}{T} f_{u1} + \epsilon^2 \left( \frac{K^2}{T^2} + \frac{(s'-b')}{T} \frac{\partial K/T}{\partial x'} \right) f_{u2} + \ldots \text{H.O.T.}$$ (6.48)

Results of numerical computation of $f_{u1}$, $f_{u2}$, $f_{u3}$, $f_{u3b}$ and $f_{u4a}$ are given in fig. 6.1 (the third order profile consists of two contributions, the most important is $f_{u3a}$, appendix A, the fourth order profile consist of three separate contributions). The profiles, which are $O(1)$, have been calculated with a small parameter $\delta$ given by eq.(6.49) (instead of $\epsilon$ given by eq.(6.19)). The largest contributions to the perturbation of the bed shear-stress occur for large bed roughness: at $C=20m^{0.5}/s$, $f_{ri}=0.5$. The $f_{ui}$ and $f_{ri}$ contributions are decreasing with the order (i). This indicates that a
Asymptotic approximation profiles
C=20 m0.5/s

Asymptotic approximation profiles
C=50 m0.5/s

Figure 6.1 Similarity functions of flow velocity perturbation. The results are based on the small parameter δ.
better choice for the small parameter satisfying the \( O(1) \) condition for \( f_{u_1} \) and \( f_{r_1} \) is:

\[
\delta = \frac{1}{\kappa L} \frac{A C}{\sqrt{g}}
\]  

(6.49)

The bed shear-stress ratio \( (T) \) variable is related to the local depth-averaged flow velocity, because of its definition eq.(6.19):

\[
T = U'^2
\]  

(6.50)

A solution in which the eddy-viscosity is also perturbated has been investigated, appendix C. It turns out that this is not possible.

6.5 Implementation in a depth-averaged flow model

The application of the theory will be in numerical models. In the theory non-dimensional variables were used. For practical application the variables will be expressed in metric units. The variables to be used in a numerical implementation are designated by: \( \text{(subscript}_n \)).

\[
M_{1n} = \left( \frac{A}{L} M_1 \right) = \left( \frac{A}{L} K \right) = \frac{1}{U^2} \frac{\delta (s-b) U^2}{\delta x}
\]  

(6.51)

\[
M_{2n} = \left( \left( \frac{A}{L} \right)^2 M_2 \right) = \frac{1}{U^2} \frac{\delta K / T (s-b) U^2}{\delta x} = \frac{\delta K (s-b)}{\delta x} \frac{U_{bs}^2}{U^2}
\]  

(6.52)

etc...

The \( M_i \) parameters are given in appendix A.

The depth-integrated squared main velocity component is modelled by:

\[
\int_0^1 u^2 \, d\xi = U^2 \, k_{uu}
\]  

(6.53)

The \( k_{uu} \) variable is calculated by substitution of \( K, T \) and \( \epsilon \) in eq.(6.48), with \( \epsilon \) given by eq.(6.19). \( (k_{uu} \) is approximated to third order):

\[
k_{uu} = k_{00} + [ 2k_{01} M_{1n} + 2k_{02} M_{2n} + k_{11} M_{1n}^2 + \\
+ 2k_{03a} M_{3an} + 2k_{03b} M_{3bn} + 2k_{12} M_{1n} M_{2n} ] \frac{U_{bs}^2}{U^2}
\]  

(6.54)
The \( k_{ij} \) coefficients are (numerical values for \( C=20m^{0.5}/s \)):

\[
\begin{align*}
    k_{00} &= \int_0^1 f_u^2 \, d\zeta = 1.14, \\
    k_{01} &= \frac{c^2}{g} \int_0^1 f_u f_u^1 d\zeta = -0.40, \\
    k_{02} &= \frac{C_4}{g} \int_0^1 f_u f_u^2 d\zeta = 4.3, \\
    k_{11} &= \frac{C_4^2}{g} \int_0^1 f_u^2 d\zeta = 4.6, \\
    k_{03a} &= \frac{C_6}{g} \int_0^1 f_u f_u^3a d\zeta = -47, \\
    k_{03b} &= \frac{C_6}{g} \int_0^1 f_u f_u^3b d\zeta = 1.0, \\
    k_{12a} &= \frac{C_6^2}{g} \int_0^1 f_u f_u^3d\zeta = -48.
\end{align*}
\]

The local bed friction coefficient is determined by: \( C = C_{bs}/f(b) \) (6.55)

The local bed friction ratio \( f(b) \) (=ratio of: local bed shear/zeroth-order bed shear) is given by:

\[
f(b) = 1 + \left[ t_1 M_{1n} + t_2 M_{2n} + t_3a M_{3an} + t_3b M_{3bn} \right] \frac{U_{bs}^2}{U^2} \tag{6.56}
\]

with: \( t_1 = 2(1-a)a^2 \)

The \( t_i \) coefficients are calculated from \( \tau = \nu_0 \left( \frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + 6 \right) \) at the zero-velocity level. The \( t_i \) coefficients are computed from the similarity profiles, \( f_{ui} \), multiplied by the eddy viscosity. The \( t_i \)-values at \( C=20m^{0.5}/s \) are (with \( \epsilon \) given by eq.(6.19)):

\[
\begin{align*}
    t_1 &= 7.9, \\
    t_2 &= -40, \\
    t_3a &= 412, \\
    t_3b &= -11.
\end{align*}
\]

The \( M_{in} \) terms will have to decrease for increasing (i) to achieve convergence of the solution. The scaled \( M_i \) variable was expected to be of the order \( O(1), \) eq.(6.51). Consequently convergence of the asymptotic expansion is only expected to occur for small \( A/L. \)

6.6 Approximation of the flow over sinusoidal bed topography

The asymptotic solution for flow over a 2-D sinusoidal bed topography is given in appendix B. The result for \( C=20 \, m^{0.5}/s, \, Fr=0.4, \, A/L=0.02 \) is given in fig. 6.2. The zeroth, first, second and the third order approximations are
Figure 6.2: The flow over sinuous bed topography. 0th, 1st, 2nd and 3rd order approximations of the velocity field, the bed shear stress ($f_r(b)$) and the convective term ($k_{uu}$).
figure 6.3
6th order approximation of the bed shear stress perturbations

base state: C=20, A/L=0.02, dA/A=0.35
6th order
5th order
4th order
3rd order
2nd order
1st order
0th order

base state: C=20, A/L=0.01, dA/A=0.2
6th order
5th order
4th order
3rd order
2nd order
1st order
0th order

base state: C=20, A/L=0.01, dA/A=0.35
6th order
5th order
4th order
3rd order
2nd order
1st order
0th order

base state: C=20, A/L=0.01, dA/A=0.1
6th order
5th order
4th order
3rd order
2nd order
1st order
0th order
displayed. Compared to the zeroth-order logarithmic profile skewing of the velocity profile occurs in the deepest part of the flow, here decelerations occur. Due to flow momentum the flattening lags behind the local maximal (down)slope bed topography. Accelerations occur uphill, the profile becomes flatter. Also indicated are $k_{uu}$ maximal in the zone of skewed velocity profile, minimal in the zone of flattened profile. The bed shear-stress (normalized by the local zeroth-order bed-shear stress is also indicated. Low values occur in the zone skewed profiles. High values in the flattening zone (acceleration).

The third order has a spurious peak in the bed shear-stress perturbation at the downhill region. The convergence of the solution has been investigated numerically up to the 6th order. The result shows that the parameter set used in fig. 2 does not yield convergence of the solution (fig. 3a, the bed shear-stress). Only in case of a smaller A/L ratio, A/L=0.01, and a smaller bed amplitude, 0.1A, the solution seems to converge. The bed shear-stress perturbation is maximal 5% of the zeroth order solution. If the bed is smooth, $C=50 \text{ m}^3/\text{s}$, a smaller A/L and bed amplitude are necessary to yield an apparent convergent solution. Bed shear-stress perturbations are then about 1%.

It will be very difficult to prove mathematically the convergence of the solution, because of the complicated analytical expressions for $M_1/T$, appendix B.
6.7 Discussion

A theory has been outlined to calculate the flow field over a non-horizontal bed by a depth-averaged flow model accounting for non-equilibrium conditions. The theory is based on asymptotic expansion of variables. The solution consists of similarity profiles for velocity and shear stress. It has been found that:

- It is not possible to find similarity profiles eddy viscosity is also approximated by asymptotic series. Consequently only when a zeroth order solution for the eddy viscosity is modeled a solution is found. This means that an equilibrium shape eddy viscosity profile has to be assumed.

- The velocity profiles remain of $O(1)$ and their amplitude decreases with increasing order if the small parameter is taken equal to:

$$\delta = \frac{1}{\kappa L} \frac{C}{g} \quad (\delta = \frac{1}{\kappa}(2\lambda A/L^2), \text{ with } \lambda = \frac{G^2}{2g A})$$

- The effects of acceleration (flattening of the velocity profile) and deceleration (skewing of the velocity profile) are reproduced.

- The bed shear-stress perturbations are larger than perturbations of the convective term in the flow model.

- The convergence limits have not been mathematically investigated, but a solution of the bed shear-stress up to the sixth order in a 2-D case indicates that only very modest bed undulations are allowed for:

$$C=20m^{-0.5}, A/L=0.01, \text{ bed amplitude } 0.1A. \text{ In that case the bed shear perturbations are only about } 5\% \text{ of the zeroth order solution. Still this may lead to a } 20 \% \text{ perturbation of the sediment transport rate. It may be expected that the ratio } A/L \text{ as well as the ratio } \Delta A/A \text{ determine the convergence of the solution.}$$

- Convergence in a 3-D case has not been investigated, but experience gained with the 2-D case indicates that convergence of the solution will also be poor in the 3-D case, because near river banks the flow becomes more 2-D like.

Some aspects of the present study have been identified before. De Vriend (1977) has formulated the first order approximation in a general coordinate system. The resulting first order profile formulation is the same as Eq.(6.39). De Vriend (1976) presents the approximation technique in more detail. But the analysis of the flow remains restricted to the first order solution. Perturbations of the surface level and the zero velocity level are considered. In the present method this is circumvented due to the normalization of vertical distances by the local water depth. De Vriend uses the traditional Prandlt mixing length theory:

$$\nu=(\kappa z) \left| \frac{\partial u}{\partial z} \right|^2$$

and derives higher order approximations of $\nu$, but does not check whether these satisfy the similarity conditions for the flow and shear stress profiles at higher orders (the problem associated with Eq.(C.7)).

De Vriend (1979, 1981) has included main flow accelerations in a flow model. The method he employed is summarized as follows:
1- Start with estimated u and v velocity profiles.
2- Solve depth-averaged flow equations.
3- Calculate in each vertical the u and v velocity profiles on basis of depth-averaged values. Calculate also the associated bed friction coefficient.
4- return to 2

The velocity profiles are determined numerically, no similarity profile formulations are used (de Vriend 1979 appendix B, 1981 appendix E). The flow is calculated for some curved-flume experiments with a horizontally smooth bed. Conclusions are:

1 Neglect of main flow accelerations is allowable for the prediction of the depth-averaged velocity field.
2 Neglect of main flow accelerations is not allowable when calculating the secondary flow and the magnitude and the direction of the bed shear-stress.

The first conclusion agrees with the small $k_{uu}$ perturbations found in the present study. The second conclusion confirms that the main effect is on the bed shear-stress.
The momentum equations for application in river bend flow are derived. The river bend is skematized. Its width is assumed constant and curvature is assumed constant. The fundamental 3-dimensional flow equations are derived by applying tensor transformations rules. This is done in two steps.

First the equations in a cylindrical coordinate system are derived. In this system the curvature is constant. The resulting equations are identical to those given in the literature and in textbooks.

Next the equations are derived for a cylindrical bed-following coordinate system, which is a coordinate system that is aligned with a non-horizontal river-bed in a bend of constant curvature. The coordinate system is non-orthogonal in vertical planes. The turbulent stresses are rather involved. By depth-averaging the equations, the well-known depth-averaged flow equations result.

The derivation of the Kalkwijk and de Vriend depth-averaged flow model and its extension with transverse shear-stresses is given.

Although it involves a lot of mathematics, it is concluded that tensor transformation rules are very useful to study the transformation of basic equations from formulation in an orthogonal Cartesian coordinate system to any other coordinate system.

It has been shown that it is possible, by asymptotic expansion of variables, to account for non-equilibrium flow over gently varying bed topography in a depth-averaged flow model. The momentum equation in main flow direction is considered, transverse flow and centripetal forces are not accounted for. The theory has identified the mathematical formulation of a small parameter. The hydraulic conditions have, at least, to satisfy the conditions imposed by the mathematical formulation of this parameter. It is concluded that bed shear-stresses are more affected by flow accelerations (or decelerations) than the convective term in the flow model.
Appendix A: Summary solution method for asymptotic approximations

The generalized version of the solution method for the velocity and shear stress perturbations is given. The $i$th order equation to be solved reads, if $v=v_0$, and parabolic (no summation).

$$F_i M_i = \beta_i T ( -(1-\zeta) f_{\tau i}^{(b)} - f_{\tau i} )$$  \hspace{1cm} (A.1)

at 0th order: $F_0 = 0, M_0 = 0, \beta_0 = 1, f_{\tau 1} = 1$

$$f_u^{(0)} = \alpha \ln(\zeta/\zeta_0)$$  \hspace{1cm} (A.2)
$$\alpha = \text{constant}$$  \hspace{1cm} (A.3)

at 1st order:

Asymptotic expansion: $f_u = f_u^{(0)} + \epsilon \beta_1 f_{u1} + \epsilon^2 \beta_2 f_{u2} + \ldots.$  \hspace{1cm} (A.4)
$$f_{\tau} = f_{\tau 0} + \epsilon \beta_1 f_{\tau 1} + \epsilon^2 \beta_2 f_{\tau 2} + \ldots.$$  \hspace{1cm} (A.5)

$$F_1 = (1-\zeta) \int_{\zeta_0}^{1} f_{u0}^2 d\zeta - \int_{\zeta}^{1} f_{u0}^2 d\zeta$$  \hspace{1cm} (A.6)
$$M_1 = K$$  \hspace{1cm} (A.7)

in which: $K = \frac{1}{u^2} \frac{\partial (z'-b')}{\partial x'}$  \hspace{1cm} (A.8)

Solution: $f_{u1} = \frac{K}{\beta_1 T} \left( \int_{y=\zeta_0}^{y=1} \frac{\alpha F_1}{(1-\zeta)} dy - \int_{y=\zeta}^{y=1} \int_{y=\zeta_0}^{y=1} \frac{\alpha F_1}{(1-\zeta)} dy d\zeta f_{u0} \right)$  \hspace{1cm} (A.9)
$$f_{\tau 1}^{(b)} = \frac{K}{\beta_1 T} \int_{y=\zeta_0}^{y=1} \int_{y=\zeta_0}^{y=1} \frac{\alpha F_1}{(1-\zeta)} dy d\zeta$$  \hspace{1cm} (A.10)

A similarity $f_{u1}$ profile (= independent on $x$) results if: $\beta_1 = K/T$  \hspace{1cm} (A.11)

at 2nd order:

Updated asymptotic expansion: $f_u = f_u^{(0)} + \frac{M_1}{T} f_{u1} + \epsilon^2 \beta_2 f_{u2} + \ldots.$  \hspace{1cm} (A.12)
$$f_{\tau} = f_{\tau 0} + \frac{M_1}{T} f_{\tau 1} + \epsilon^2 \beta_2 f_{\tau 2} + \ldots.$$  \hspace{1cm} (A.13)

$$F_2 = (1-\zeta) 2 \int_{\zeta_0}^{1} f_{u0} f_{u1} d\zeta - 2 \int_{\zeta}^{1} f_{u0} f_{u1} d\zeta$$  \hspace{1cm} (A.14)
\[ M_2 = \frac{K^2}{T} + (s'-b') \frac{\partial K/T}{\partial x'} \text{ (from: } \frac{1}{U',2} \frac{\partial}{\partial x'} \frac{M_1}{T}(s'-b')U',2) \]  
(A.15)

Solution: \[ f_{u2} = \frac{M_2}{2\beta_2 T} \left( \int_{y_0}^{y} \frac{\alpha F_2}{\xi(1-\xi)} \, dy - \int_{y_0}^{y} \frac{1}{\xi(1-\xi)} \, dy \, \frac{\alpha F_2}{\xi} \, d\xi \right) f_{u0} \]  
(A.16)

\[ f_{r2}(b) = -\frac{M_2}{2\beta_2 T} \int_{r_0}^{r} \frac{1}{\xi(1-\xi)} \, dy \, \frac{\alpha F_2}{\xi} \, d\xi \]  
(A.17)

Similarity profiles are found for: \[ \beta_2 = \frac{M_2}{T} \]  
(A.18)

\( \beta \)rd-order

Updated asymptotic expansion:  
\[ f_u = f_{u0} + \varepsilon \frac{M_1}{T} f_{u1} + \varepsilon^2 \frac{M_2}{T} f_{u2} + \ldots \]  
(A.21)

\[ f_r = f_{r0} + \varepsilon \frac{M_1}{T} f_{r1} + \varepsilon^2 \frac{M_2}{T} f_{r2} + \ldots \]  
(A.22)

The asymptotic expansion of the convective term reads:

\[ \frac{\partial (s-b)U^2}{\partial x} \int f^2_u \, d\xi = \frac{B}{L} \left[ \frac{1}{U',2} \frac{\partial}{\partial x'} \left( s'-b' \right) U',2 \right] \int f^2_u \, d\xi U'^2 - \]  

\[ - \frac{B}{L} U^2 \left[ \int f^2_{u0} \, d\xi \frac{1}{U',2} \frac{\partial}{\partial x'} \left( s'-b' \right) U'^2 + 2 \varepsilon \int f_{u0} f_{u1} \, d\xi \frac{1}{U',2} \frac{\partial K/T(s'-b')U'^2}{\partial x'} + \right] \]

\[ + \varepsilon^2 \left[ 2 \int f_{u0} f_{u2} \, d\xi \frac{1}{U',2} \frac{\partial}{\partial x'} \left( \frac{K^2}{T} + (s'-b') \frac{\partial K/T}{\partial x'} \right) (s'-b')U'^2 \right] + \]

\[ + \int f_{u1} \, d\xi \frac{1}{U',2} \frac{\partial K^2/T^2 (s'-b')U'^2}{\partial x'} + 6 \ldots \right] = \]

\[ - \frac{B}{L} U^2 \left[ \int f^2_{u0} \, d\xi \frac{K^2}{2} + 2 \varepsilon \int f_{u0} f_{u1} \, d\xi \left( K^2 + (s'-b') \frac{\partial K}{\partial x'} \right) + \right] \]
\[ + \epsilon^2 \left( 2 \int u_0 f u_2 d\xi - \left[ \frac{K^3}{T} + \frac{K}{T} (s'-b') \frac{\partial K/T}{\partial x'} \right] + \right. \]
\[ + \left. (s'-b') \frac{1}{U',2} \frac{\partial \left[ \frac{K^2}{T} + (s'-b') \frac{\partial K/T}{\partial x'} \right]}{\partial x'} \right) \]
\[ + \int f_u^2 d\xi \left[ \frac{K^3}{T} + (s'-b') \frac{\partial K/T}{\partial x'} \right] + \ldots \]  
(A.23)

In which:
\[ K = \frac{1}{U',2} \frac{\partial (s'-b')U'^2}{\partial x'} \]  
(A.24)

The equation to be solved is:
\[ F_{3a} M_{3a} + F_{3b} M_{3b} = \beta_{3a} T \left[ -(1-\xi) f_{r3a} - f_{r3a} \right] + \]
\[ + \beta_{3b} T \left[ -(1-\xi) f_{r3b} - f_{r3b} \right] \]  
(A.25)

With:
\[ F_{3a} = (1-\xi) \int_0^1 f u_0 f u_2 d\xi - \int_0^1 f u_0 f u_2 d\xi \]  
(A.26)

\[ M_{3a} = \frac{K^3}{T} + \frac{K}{T} (s'-b') \frac{\partial K/T}{\partial x'} \left[ s'-b' \right] \frac{1}{U',2} \frac{\partial \left[ \frac{K^2}{T} + (s'-b') \frac{\partial K/T}{\partial x'} \right]}{\partial x'} \]  
(from: \[ \frac{1}{U',2} \frac{\partial \left( \frac{M_2}{T} (s'-b')U'^2 \right)}{\partial x'} \] )

(A.27)

\[ F_{3b} = (1-\xi) \int_0^1 f u_1^2 d\xi - \int_0^1 f u_1^2 d\xi \]  
(A.28)

\[ M_{3b} = \frac{K^2}{T} + (s'-b') \frac{\partial K^2/T}{\partial x'} \]  
(from: \[ \frac{1}{U',2} \frac{\partial (s'-b')U'^2}{\partial x'} \] )

(A.29)

Solution:
\[ f u_{3a} = \beta_{3a} T \int \left( \int_0^\xi \frac{f F_{3a}}{\xi(1-\xi)} dy - \int_0^\xi \int_0^\xi \frac{x}{\xi(1-\xi)} dy \right) f u_0 \]  
(A.30)
\[ f_{r3a} = - \frac{M_{3a}}{\beta_{3a} T} \int_{y_T}^{\xi} \int_{\xi_0}^{\xi} \frac{\alpha F_{3a}}{\xi (1-\xi)} \ dy \ dx \]  
(A.31)

\[ f_{u3b} = \frac{M_{3b}}{\beta_{3b} T} \left( \int_{y_T}^{\xi} \int_{\xi_0}^{\xi} \frac{\alpha F_{3b}}{\xi (1-\xi)} \ dy \ dx + \int_{y_T}^{\xi} \int_{\xi_0}^{\xi} \frac{\alpha F_{3b}}{\xi (1-\xi)} \ dy \ dx \right) \]  
(A.32)

\[ f_{r3b} = - \frac{M_{3b}}{\beta_{3b} T} \int_{y_T}^{\xi} \int_{\xi_0}^{\xi} \frac{\alpha F_{3b}}{\xi (1-\xi)} \ dy \ dx \]  
(A.33)

Similarity functions are obtained for: \( \beta_{3a} = M_{3a}/T, \beta_{3b} = M_{3b}/T \)  
(A.34)

Updated asymptotic expansion:

\[ f_u = f_{u0} + \epsilon \frac{M_1}{T} f_{u1} + \epsilon^2 \frac{M_2}{T} f_{u2} + \epsilon^3 \left( \frac{M_{3a}}{T} f_{u3a} + \frac{M_{3b}}{T} f_{u3b} \right) + \ldots \]  
(A.39)

\[ f_r = f_{r0} + \epsilon \frac{M_1}{T} f_{r1} + \epsilon^2 \frac{M_2}{T} f_{r2} + \epsilon^3 \left( \frac{M_{3a}}{T} f_{r3a} + \frac{M_{3b}}{T} f_{r3b} \right) + \ldots \]  
(A.40)
Appendix B: 2D flow over sinuous bed undulations

The velocity profile is approximated by:

\[ f_u = f_{u0} + \epsilon \frac{M_1}{T} f_{u1} + \epsilon^2 \frac{M_2}{T} \frac{f_{u2}}{2} + \epsilon^3 \frac{M_3a}{T} \frac{f_{u3a}}{3} + \frac{M_3b}{T} \frac{f_{u3b}}{2} \ldots \text{H.O.T.} \quad (B.1) \]

in which: \[ \frac{M_1}{T} = \frac{K}{T} \quad (B.2) \]

\[ \frac{M_2}{T} = \frac{1}{T U^2} \frac{\partial M_1}{\partial x'} (s'-b') U'^2 = \frac{1}{U^4} \frac{\partial M_1 (s'-b')}{\partial x'} \quad (B.3) \]

\[ \frac{M_3a}{T} = \frac{1}{T U^2} \frac{\partial M_2}{\partial x'} (s'-b') U'^2 = \frac{1}{U^4} \frac{\partial M_2 (s'-b')}{\partial x'} \quad (B.4) \]

\[ \frac{M_3b}{T} = \frac{1}{T U^2} \frac{\partial M_3}{\partial x'} (s'-b') U'^2 = \frac{1}{U^4} \frac{\partial M_3 (s'-b')}{\partial x'} \quad (B.5) \]

For 2D flow the flow rate \( U'(s'-b') \) is constant. The \( M_1 \) term is written as:

\[ \frac{M_1}{T} = \frac{(s'-b')^2}{U'^2} \frac{\partial (1/(s'-b'))}{\partial x'} = \frac{1}{U'} \frac{\partial (s'-b')}{\partial x'} = (s'-b')^2 \frac{\partial (s'-b')}{\partial x'} \quad (B.6) \]

\[ \frac{M_2}{T} = \frac{1}{U^4} \frac{\partial (M_1 (s'-b'))}{\partial x'} = 0.5 (s'-b')^4 \frac{\partial^2 (s'-b')}{\partial x'^2} = \]

\[ = - (s'-b')^4 \left( \frac{\partial (s'-b')}{\partial x'} \right)^2 + (s'-b') \frac{\partial^2 (s'-b')}{\partial x'^2} \quad (B.7) \]

\[ \frac{M_3a}{T} = \frac{1}{U^4} \frac{\partial (M_2 (s'-b'))}{\partial x'} = \text{(unfinished)} \quad (B.8) \]

\[ \frac{M_3b}{T} = \frac{1}{U^4} \frac{\partial (M_3 (s'-b'))}{\partial x'} = \text{(unfinished)} \quad (B.9) \]

A sinuous bed level is assumed. The amplitude of the bed level is assumed larger than the surface level amplitude. Consequently \( \partial s'/\partial x' \) will be
neglected with respect to $\frac{\partial b'}{\partial x'}$ and $s' = 1$. The water depth and its derivatives as they occur in eq. (B.9) are given by:

$$s' - b' = 1 - \frac{\alpha}{A} \sin(\pi x'), \quad \alpha = \text{amplitude bed oscillation} \quad (B.10)$$

$$\frac{\partial (s' - b')}{\partial x'} = - \frac{\partial b'}{\partial x'} = - \frac{\alpha}{A} \pi \cos(\pi x') \quad (B.11)$$

$$\frac{\partial^2 (s' - b')}{\partial x'^2} = \frac{\alpha}{A} \pi^2 \sin(\pi x') \quad (B.12)$$

The $M_1$ terms are:

$$\frac{M_1}{T} = (1 - \frac{\alpha}{A} \sin(\pi x'))^2 \frac{\alpha}{A} \pi \cos(\pi x') \quad (B.13)$$

$$\frac{M_2}{T} = - \pi^2 (1 - \frac{\alpha}{A} \sin(\pi x'))^4 \left[ \frac{\alpha}{A} \sin(\pi x') + \left( \frac{\alpha}{A} \right)^2 \cos(2\pi x') \right] \quad (B.14)$$

$$\frac{M_3}{T} = \pi^3 (1 - \frac{\alpha}{A} \sin(\pi x'))^6 \left[ 3 \left( \frac{\alpha}{A} \sin(\pi x') + \left( \frac{\alpha}{A} \right)^2 \cos(2\pi x') \right) + 
\left( 1 - \frac{\alpha}{A} \sin(\pi x') \right) \left( \frac{\alpha}{A} \cos(\pi x') - 2 \left( \frac{\alpha}{A} \right)^2 \sin(2\pi x') \right) \right] \quad (B.15)$$

The $M_i/T$ terms increase with increasing order $i$. For the asymptotic series to converge the $\epsilon^i f_{ri}$ (no summation) terms have to decrease at a faster rate than $M_i/T$ for increasing $i$. Inspection of eq. (B.13) and eq. (B.15) reveals that the solutions show an alternating behaviour. The 1st, 3rd, 5th ... orders are mainly related to the local bed slope. The 2nd, 4th, 6th ... orders are mainly related to local bed curvature. Due to this alternating character the condition for convergence will be (the $f_{ri}$ contributions decrease at a slower rate than the $f_{ui}$ profiles):

$$|\epsilon^2 \frac{M_{i+2}}{T} f_{ri+2}| < \left| \frac{M_i}{T} f_{ri} \right| \quad (B.16)$$

The perturbation profiles and the bed shear-stress contributions have been shown to be of $O(1)$ and to decrease for increasing order $i$ if the small parameter is defined by: $\delta = \frac{1}{\kappa} \frac{\Delta}{L} \sqrt{g}$, instead of $\epsilon = \frac{\Delta}{L} \sqrt{g}$. 

In that case the F functions, appendix A, are to be multiplied with \( \kappa c/g \).

Due to the complicated formulation of the \( M_i/T \) terms it is practically impossible to investigate mathematically the convergence limits of the expansion. The system has been solved numerically up to the sixth order. Some results are shown in fig. 6.3. Convergence seems to be achieved for very small depth/length ratios and small bed amplitudes.
Appendix C: A solution including a perturbation of the eddy viscosity

The formulation of the eddy viscosity profile is traditionally achieved by considering mixing length theory near the bed and a curve fit of the velocity profile in the outer region (In this region mixing length theory cannot be applied because more than one length scale determines the problem, then mixing length theory can not predict anything substantial, Tennekes & Lumley 1973). In the wall region the length scale is: \( \kappa \), the velocity scale is: \( u_* \).

It is assumed that the shape of the eddy viscosity profile is maintained. Near the bed mixing length theory has to be satisfied, which means that close to the bed the eddy viscosity is linear with \( \zeta \) in the limit \( \zeta \to 0 \). The constraints to the eddy viscosity in the outer region are unclear, consequently the same shape is assumed as for the zeroth-order solution.

The asymptotic expansion for the bed friction velocity \( u_* \) (normalized with \( kU \)) and the bed shear-stress (normalized with the 0th order solution) reads:

\[
\frac{u_*}{kU} = a = a_0 + \epsilon a_1 + \epsilon^2 a_2 \tag{C.1}
\]

\[
\frac{r(b)}{r_0(b)} = f_r(b) = 1 + \epsilon \beta_1 f_{r_1}(b) + \epsilon^2 \beta_2 f_{r_2}(b) \tag{C.2}
\]

in which:
\[
\alpha_0 = \frac{u_{*0}}{kU}, \quad \alpha_1 = \frac{u_{*1}}{kU}, \quad \alpha_2 = \frac{u_{*2}}{kU} \tag{C.3}
\]

where: \( u_{*i} \) - local ith-order bed shear velocity contribution

The definition of \( u_* \) is: \( r(b) = \rho u_*^2 \) \( \tag{C.4} \)

This is equal to: \( f_r(b) = (\alpha/\alpha_0)^2 \) \( \tag{C.5} \)

Substitution of the asymptotic expansions eq.(C.1) and eq.(C.2) in eq.(C.5) and collecting 0th, 1st, 2nd order terms yields:

0th order: \( 1 = (\alpha_0/\alpha_0)^2 \) \( \tag{C.6} \)

1st order: \( \beta_1 f_{r_1}(b) = 2 (\alpha_1/\alpha_0) \) \( \tag{C.7} \)

2nd order: \( \beta_2 f_{r_2}(b) = (\alpha_1/\alpha_0)^2 + 2 (\alpha_2/\alpha_0) \) \( \tag{C.8} \)

3rd order: \( \beta_3 f_{r_3}(b) = 2\alpha_1\alpha_2/(\alpha_0)^2 + 2 (\alpha_3/\alpha_0) \) \( \tag{C.9} \)
From eq.(C.7) and eq.(C.8) relations between the perturbations of the bed shear-stress and the bed friction velocity are derived:

\[ \alpha_1/\alpha_0 - \beta_1 f_{r1}(b) \text{, } \quad \alpha_2/\alpha_0 = \frac{1}{2} \beta_2 f_{r2}(b) - \frac{1}{8} \beta_1^2 f_{r1}(b) \]  

(C.10ab)

The eddy viscosity relation, with preserved profile shape reads:

\[ \nu = \kappa_\xi (1-\xi) \alpha u_* \]  

(C.11)

The shear stress is given by: \( \tau = \nu \frac{\partial u}{\partial z} \). Substitution of (C.11) yields:

\[ f_r = \frac{r}{\tau(0)} = \kappa_\xi (1-\xi) \frac{u}{u_*} \frac{\partial u}{\partial z} = \alpha_2 \xi (1-\xi) \frac{\partial f}{\partial \xi} \]  

(C.12)

Substitution of the asymptotic expansions for \( f_r \) and \( f_u \) and collecting by order yields:

\[ 0^{\text{th}} \text{ order: } f_{r0} = \xi (1-\xi) \frac{\partial f}{\partial \xi} \]  

(C.13)

\[ 1^{\text{th}} \text{ order: } \beta_1 f_{r1} = \beta_1 \frac{\xi (1-\xi)}{\alpha_0} \frac{\partial f}{\partial \xi} + \frac{\alpha_1}{\alpha_0} \xi (1-\xi) \frac{\partial f}{\partial \xi} \]  

(C.14)

\[ 2^{\text{nd}} \text{ order: } \beta_2 f_{r2} = \beta_2 \frac{\xi (1-\xi)}{\alpha_0} \frac{\partial f}{\partial \xi} + 2 \beta_1 \frac{\alpha_2}{\alpha_0} \xi (1-\xi) \frac{\partial f}{\partial \xi} + \]  

\[ + \beta_0 \frac{\alpha_1}{\alpha_0} \xi (1-\xi) \frac{\partial f}{\partial \xi} \]  

(C.15)

The solution of the zeroth order problem is the logarithmic profile:

\[ f_{u0} = \frac{\alpha_0}{\beta_0} \ln \xi \quad \text{with: } \beta_0 = \alpha_0 \text{ to satisfy similarity} \]  

(C.16)

Substitution of \( \partial f_{u0}/\partial \xi = 1/\xi \) in eq.(C.14) and dividing by \( \beta_1 \) yields:

\[ f_{r1} = \frac{\xi (1-\xi)}{\alpha_0} \frac{\partial f}{\partial \xi} + \frac{\alpha_1}{\alpha_0} \frac{(1-\xi)}{\beta_1} \]  

(C.63)

Substitution of eq.(C.17) in the first order momentum equation, eq.(C.20) and application of the boundary condition \( u(\xi_0) = 0 \) yields the following perturbation profile, compare with eq.(C.37):
The first order contribution to the bed shear-stress is obtained by the boundary condition: \( \int u_1' = 0 \).

\[
\int_{\xi_0}^{1} u_1'\,d\xi = 0 = \frac{K}{\beta_1 T} (2\alpha_0 - 2)\alpha_0^2 - \frac{\alpha_1}{\alpha_0 \beta_1} + f_{\tau_1}(b) = 0 \tag{C.19}
\]

\( f_{\tau_1}(b) = \frac{K}{\beta_1 T} 2(1 - \alpha_0)\alpha_0^2 + \frac{\alpha_1}{\alpha_0 \beta_1} \tag{C.20} \)

Substitution of eq.(C.10a) yields:

\[ f_{\tau_1}(b) = \frac{K}{\beta_1 T} 4(1 - \alpha_0)\alpha_0^2 \tag{C.21} \]

The first order contribution to the velocity profile is obtained by substitution of eq.(C.21) in eq.(C.19). The result eq.(C.22) is the same as the profile in the case of an undisturbed eddy viscosity profile. The first order contribution to the bed shear-stress is however doubled eq.(C.21).

\[
f_{u_1} = -\frac{K}{\beta_1 T} \alpha_0^2 \int_{\xi_0}^{1}(\alpha_0 \ln \xi - 2(1 - \alpha_0)\alpha_0 \ln \xi) \,d\xi - 2(1 - \alpha_0)\alpha_0 \ln \xi \tag{C.22}
\]

To satisfy similarity: \( \beta_1 = K/T \) \tag{C.23}

The second order contribution to the velocity profile can only be computed numerically. The solution in case of the undisturbed eddy viscosity profile can be used because of the same first order velocity profile. Further the computed first order contribution to the bed shear-stress will have to be modified for the perturbered eddy viscosity, like in eq. (C.14). The second order contribution to the shear stress is equal to.

\[
f_{\tau_2} = \frac{\xi(1 - \xi)}{\alpha_0} \frac{\partial f_{u_2}}{\partial \xi} + 2 \frac{\beta_1}{\beta_2} \frac{\alpha_1}{\alpha_0} \xi(1 - \xi) \frac{\partial f_{u_1}}{\partial \xi} - \frac{1}{\beta_2} \frac{\alpha_2}{\alpha_0} (1 - \xi) \tag{C.24}
\]

Substitution of eq.(C.24) in eq.(C.21) yields:
\[(1-\zeta) \left( 2 \int_0^1 f u_0 f u_1 d\zeta - 2 \int_0^1 f u_0 f u_1 d\zeta \right) M_2 = \beta_2 T \left[ -(1-\zeta) f r_2 (b) + \right. \]
\[+ 2 \frac{\beta_1}{\beta_2} \frac{a_1}{a_0} \zeta (1-\zeta) \frac{\partial f}{\partial \zeta} + \frac{1}{\beta_2} \frac{a_2}{a_0} (1-\zeta) + \frac{1}{a_0} \frac{\partial f u_2}{\partial \zeta} \right] \quad (C.25)\]

in which: \[M_2 = \frac{1}{U,2} \frac{\partial \beta_1 (s'-b') U'}{\partial x'} \quad (C.26)\]

It is seen that the \( f r_2 (b) \) term in comparison to a solution without a perturbated eddy viscosity is extended to:
\[-(1-\zeta) f r_2 (b) := -(1-\zeta) f r_2 (b) + 2 \frac{\beta_1}{\beta_2} \frac{a_1}{a_0} \zeta (1-\zeta) \frac{\partial f u_1}{\partial \zeta} + \frac{1}{\beta_2} \frac{a_2}{a_0} (1-\zeta) \quad (C.27)\]

To determine the velocity profile this term is divided by the eddy viscosity. The velocity profile is then given by, compare eq.(C.36) for first order unperturbated profile.
\[f u_2 = \frac{M_2}{\beta_1^2} \int u_0 f u_1 \text{ integr. part} + (f r_2 (b) - \frac{\alpha_2}{\beta_2^2} \alpha_0 \ln \zeta) - \frac{\beta_1}{\beta_2} \frac{\alpha_1}{\alpha_0} f u_1 + B \quad (C.28)\]

Application of the boundary condition \( f u_2 (\zeta_0) = 0 \) to determine \( B \) yields:
\[f u_2 = \frac{M_2}{\beta_1^2} \int u_0 f u_1 \text{ integr. part} + (f r_2 (b) - \frac{\alpha_2}{\beta_1^2} \alpha_0 \ln \zeta) - \frac{\beta_1}{\beta_2} \frac{\alpha_1}{\alpha_0} f u_1 \quad (C.29)\]

Similarity is yielded for: \( \beta_2 = M_2 / T \) and \( \frac{\beta_1}{\beta_2} \frac{\alpha_1}{\alpha_0} = \text{const.} \). These are conflicting conditions, except if \( \alpha_1 = 0 \). Consequently no solution is obtained by this method of perturbation of the eddy viscosity.
REFERENCES

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