IMPACT ANALYSIS OF COMPOSITE STRUCTURES

Proefschrift

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To my loving parents Mina and Ali Talagani,
my lovely wife Sam Talagani and little Jami
Summary

Composite materials have been introduced in aerospace structures to increase their performance over weight ratio with the main goal to increase the economical efficiency of aircraft. In practice the use of composite materials has not led to considerable reduction of the weight. One important reason for this is the damage tolerant behaviour of composite structures. Especially due to foreign object impact loading composite structures show significant sub-surface delaminations, which in some cases are hard or even impossible to detect by visual inspections. These categories of hardly or undetectable damages are categorized as *Barely Visible Impact Damage* (BVID). The probability of existence of these category of damage needs to be accounted for during the design process, leading to an increase of knock-down factors on the material allowable values.

During the initial design phase of aerospace structures, laminates are selected by design engineers to meet the structural design requirements. Engineers are often left with several laminates which satisfy the structural requirements. Enabling the engineer to select the laminate that are less sensitive to impact damage can help designing more reliable composite structures. Numerical analysis like advanced finite element analyses can be used to model impact response on composite structures. However, these methods are usually extremely time inefficient due to the large number of degrees of freedom necessary for accurate results. Also considerable expertise is required for creating such a numerical model and the evaluation and validation of the results can be rather difficult.
In this thesis analytical methods are introduced for analysing composite structures subjected to low-velocity impact. Four aspects of impact are discussed in this work being:

- contact behaviour
- elastic dynamic impact behaviour
- stress analysis due to indentation
- fracture mechanics based delamination behaviour.

Contact behaviour during impact has a major effect on the damage mode created by the impact event. An impact event by a small rigid impactor will most likely result in local damages, while an impact event due to a large soft impactor will generate more global damages and delaminations. Obtaining an accurate contact behaviour is therefore considered important for damage analysis of impacted composites. The Hertzian contact formulation has been used extensively in analytical contact analyses, however aspects like damage and plasticity are not accounted for in this formulation. Two existing contact formulations proposed by Yang and Sun, and Yigit and Christoforuforou are investigated and validated by comparison with experimental test results. An improved contact model is obtained by combining the two contact models, which shows good comparison with experimental test results.

Dynamic behaviour of impact of composite plates is discussed by using plate theories and contact definitions. Classical plate theory is used to define the behaviour of the plate, which together with the Hertzian Contact law results in a differential equation representing the dynamic behaviour during the impact event. The methodology enables the analysis of elastic dynamic impact response of composite plates, impacted by spherical impactors. The method is validated by comparison with published numerical and experimental results.

For understanding the damage behaviour of composite plates during impact, knowing the correct stress state of the indented composite plate is crucial. Especially the stress state of an indented panel within the contact region is rather complex. A new methodology for analysing three dimensional stresses due to indentation is proposed and validated by comparison with numerical results. The method is based on the axi-symmetric Boussinesq equations, which are solved using Hankel transformations. The method shows good comparison compared to numerical results and proves to be extremely time efficient when compared to the numerical finite element models used for validation.
To analyse the delamination behaviour of composite plates, a new fracture initiation criteria was obtained by using linear elastic fracture mechanics on a circular plate, loaded quasi-statically in its out-of-plane direction. The obtained delamination onset criteria is a function of the layup architecture (i.e. the D-matrix) and due to its analytical nature and simplicity, it is very time efficient. The criteria can be used to compare laminates in order to select better damage resistant composite laminates. The low computational costs allow the criteria to be used for optimization purposes. In this work a genetic algorithm optimization routine is used to optimize composite laminate architecture to increase their damage resistance.
Samenvatting

Composietmaterialen worden in de lucht- en ruimtevaart sector voornamelijk gebruikt door hun theoretisch superieure eigenschappen in termen van stijfheid over gewicht overhouding. In de praktijk leidt het gebruik van composietmaterialen echter niet altijd tot lichtere constructies. Een belangrijke reden hiervoor is het schadegedrag van composieten. Met name schokbelastingen kunnen leiden tot significante schades (delaminaties), terwijl de schade aan het oppervlak van deze laminaten onder een schokbelasting onzichtbaar of nauwelijks zichtbaar kan zijn. Dit soort schades wordt over het algemeen gecategoriseerd als "Barely Visible Impact Damage (BVID)". Tijdens het ontwerp proces moet met de waarschijnlijkheid van BVID’s rekening gehouden worden. Dit verhoogt de zogenaamde "knock-down" factoren van het composietmateriaal.

In dit proefschrift worden analytische methoden gebruikt en geïntroduceerd voor het analyseren van composietconstructies onder schokbelastingen die worden gecategoriseerd als "low-velocity impact". De volgende vier aspecten van deze categorie van schokbelastingen worden behandeld:

- contact gedrag
- elastisch-dynamisch gedrag
- spanningsanalyse door indeuking
- delaminatieanalyse door fracture mechanics.

Het contactgedrag tijdens schokbelasting heeft een significant invloed op het schadege-drag van composieten. Schokbelasting door een klein rigide voorwerp zal voornamelijk voor lokale schades zorgen, terwijl een schokbelasting door een zacht en zwaar voorwerp voor meer globale schades kan zorgen. Een nauwkeurige analyse van het contactgedrag is daarom van groot belang. Het Hertz contactmodel wordt veelvuldig gebruikt in literatuur. Dit model is echter afgeleid met de aanname dat de indeuking tijdens het contact klein is. Ook wordt geen rekening gehouden met eventuele schades die lokaal het materiaal kunnen degraderen en de daardoor de elasticiteit van het materiaal lokaal aantasten. Twee modellen worden in de literatuur veelvuldig gebruikt voor het analyseren van het contactgedrag van composieten, deze zijn:

- Yang and Sun contactmodel
- Yigit and Christoforou contactmodel.

Deze twee contactmodellen zijn geëvalueerd en gevalideerd met behulp van experimenten. De twee modellen zijn samengevoegd tot een nieuw contactmodel, die goede correlaties laat zien met de experimentele resultaten.

Het elastisch-dynamisch aspect van composietplaten onder schokbelastingen is behandeld. Klassieke platentheorie is hier toepast om het gedrag van de plaat te beschrijven. In combinatie met het Hertz contactmodel ontstaat een differentiaalvergelijking die is opgelost met als resultaat het dynamisch gedrag van composietplaten onder een schokbelasting. Het model is gevalideerd door middel van vergelijking met andere analytische modellen en experimentele resultaten.
Voor een goed beeld van het schadegedrag van composieten onder een eventuele schokbelasting is een nauwkeurige kennis van de spanningen door de indeuking cruciaal. Met name het spanningsgedrag in de contactregio is complex. Een nieuw methode wordt gepresenteerd voor het bepalen van de driedimensionale spanningen door een indeuking. De methode is gebaseerd op de axi-symmetrische Boussinesq vergelijkingen, die opgelost zijn met behulp van Hankel transformaties. Het verkregen model is succesvol gevalideerd door vergelijkingen met numerieke modellen en vereist zeer weinig rekentijd in vergelijking met de numerieke analyses die zijn gebruikt voor validatie.

Een nieuw delaminatie initiatie criterium is verkregen om het delaminatiegedrag van composieten platen te analyseren. Hiervoor is een cirkelvormige plaat aangenomen, die quasi-statisch loodrecht op het oppervlak wordt belast. Het delaminatiecriterium is een functie van de architectuur van de composietplaat (D-matrix) en vanwege de analytische aard van deze criterium benodigt het geen noemenswaardige rekentijd. Dit criterium kan door een ontwerper gebruikt worden om laminaten te toetsen met betrekking tot hun schaderesistentie tegen een eventuele schokbelasting. Door de hoge efficiëntie van dit model met betrekking tot de benodigde rekentijd, is het model uiterst geschikt voor optimalisatie doeleinden. In dit werk wijn genetische algoritmen gebruikt om een optimalisatie uit te voeren op composietlaminaten, om de schade resistentie ten gevolge van een schokbelasting te verhogen.
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Chapter 1

Introduction

The introduction of composite materials in aerospace has led to a paradigm shift when it comes to performance over weight efficiency of structures. The possibility of manipulating the material behaviour by altering the composite layup architecture expands the design space, which in theory should lead to more efficient designs. In practice however, this is often not the case. The complex behaviour of composite materials, especially in terms of damage behaviour, due to the combination of different constituents leads to larger uncertainties. These uncertainties need to be accounted for during design, which lead to larger reserve factors and therefore a reduction in the efficiency of composite structures. Especially impact loading is a big concern, because it leads to significant reduction of the strength of composite structures. On top of this, impact induced damages could result in subsurface delaminations which are not or barely visible to the naked eye making them hard to detect during inspections. These impacts are usually probabilistic in nature and can occur during the entire life-cycle of the structure, starting with tool drops during the production process up to impact by foreign objects during service like impact by runway debris or by hail.

Composite structures designed with damage tolerant philosophy need to account for these types of damages, often resulting in relatively high reserve factors. Impact induced delaminations are highly dependent on the layup configuration, fracture toughness of the matrix material, boundary conditions, shape and mass of the impactor and the impact velocity. During the initial design phase of aerospace structures, laminates are selected by design engineers to meet the structural design requirements. Load case, production process, economic requirements and available test data are just a few examples of criteria that drive the selection process. This selection process often results in several laminates, from which the designers can choose. Enabling engineers with tools
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to select laminates that are less sensitive to impact damage during the initial design phase can help increasing the reliability of composite structures. Numerical analysis techniques like finite element methods are commonly used for analysing damage and fracture behaviour of composite laminates and structures under impact loading [2, 7, 8]. However, these methodologies are in general time consuming and are not suitable for quick decision making.

In this work analytical methods are introduced, which can be used to understand the behaviour of composite laminates under impact loading. Especially low-velocity impact is of interest, because this type of impact loading can lead to indentations which are barely visible on the surface (barely visible impact damage BVID) but can lead to significant subsurface delaminations. Here low-velocity impact is defined as an impact velocity that results in a dynamic response of the structure by flexural waves and transient shear waves and does not result in through the thickness stress waves [9, 10]. This behaviour is related to the speed of sound in the matrix material and for carbon/epoxy composites usually involves velocities higher than 70m/s. Low-velocity impact is often divided in two categories [11]:

- wave controlled impact
- boundary controlled impact.

This division is based on the impactor/plate mass ratio, where impact events with low impactor/plate mass ratio are categorized as wave controlled and impact events with high impactor/mass ratio as boundary controlled. Wave controlled impact events result in dynamic excitation of the structure. During such an event, the impact time is usually smaller than time needed for the elastic waves to reach the boundaries and the event is therefore often assumed to be insensitive to the boundary condition. Boundary controlled impact events result in a quasi-static response and can be analysed as such. For these events the boundary conditions play an important role [11].

In this research we are interested barely visible impact damages caused by relative small impactors like runway debris and hail. Therefore the work focuses on wave controlled low-velocity impact events. The goal of this research is to contribute to the understanding of impact behaviour of composite structures for low-speed impact loads. During low-speed impact events bending and contact stresses can lead to intra-ply damages and inter-ply fracture. Finite element analyses are often utilized for modelling these behaviours, however these numerical solutions do not result in a generic understanding of the problem and are often time consuming and are therefore unsuitable for initial design phase of composite structures. The main objective of this work is obtaining time efficient analytical methods to describe the aspects of low-velocity impact identified as:
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- contact behaviour
- stress due to indentation
- delamination behaviour.

In what follows, the chapters in which the above mentioned aspects of low-velocity impact are discussed.

In chapter 2 the aspect of contact mechanics is discussed, including elastic and in-elastic contact behaviour of composite layups with respect to spherical impactors. Significant work has been done in this field by Yang and Sun [1] and Yigit and Christoforou [12]. Yang and Sun [1] took an experimental approach while Yigit and Christoforou [12] analysed the impact of a transversely isotropic half-space by a spherical impactor using an elastic-plastic contact law. Four loading conditions are considered in both of the works being; elastic loading, in-elastic loading, unloading and reloading of contact. In this chapter both of the proposed models are evaluated by comparison with experimental test results. The models are then combined into a new model which shows better comparison with test results. In chapter 3 the dynamic behaviour of impact of composite plates is discussed by using plate theories and contact definitions. Classical plate theory is used to define the behaviour of the plate, which together with the Hertzian Contact law results in a differential equation representing the dynamic behaviour during the impact event. Fourier series are used to solve this equation and the result is validated through comparison with published experimental test results and other analytical and numerical results. Chapter 4 includes a model for obtaining the three dimensional stress state in a laminate due to spherical indentations. Impact loading often results in concentrated stress distributions in the contact region. The resulting transverse stresses can no longer be analysed using shell theories and three dimensional stress analysis is therefore required [13]. Apart from finite element analysis methods, a common approach is the assumption of an indented semi-infinite body for obtaining these stresses. A. E. H. Love [14]. L.M. Keer [15] and R. Olsson [13] have contributed to this methodology. A.P.S. Selvadurai [16] used the axisymmetric Boussinesq equations to analyse the three dimensional stresses in an elastic halfspace indented by a flat circular indenter. This work is used as inspiration to obtain a new methodology for analysing the transverse stresses in a finite thickness laminate indented by a spherical indenter. In chapter 5 the delamination behaviour of composite laminates is discussed through a fracture mechanics approach leading to a model for predicting delamination onset and growth. Delaminations are serious threats for composite structures, because they can occur at low impact loads and can cause significant reduction in flexural stiffness and buckling failure [10]. Fracture mechanics based cohesive behaviour has been used in numerical simulations of delamination due to low-velocity impact with success [2, 7, 8]. Due to the necessary detail in the numerical
models, the high computational time does not allow this method to be used efficiently during the initial design phase. The selection of laminates is an important aspect of the initial design phase. Therefore an analytical approach is used in this work to obtain a time efficient method for assessing composite laminates for their delamination behaviour during low-velocity impact. Great work has been done by Davies and Robinson [9], who obtained a mode-II delamination threshold load for layered beams under single delamination assumption. Olsson [10] expanded this model for multiple delaminations. Although this model shows good comparison with numerical simulations, it needs the number of delaminations as an input. In this work a new methodology is proposed, assuming a circular composite plate under a quasi-static out-of-plane loading, while a single circular delamination is assumed. An analytical delamination threshold load is obtained. In chapter 6, the obtained delamination threshold load from chapter 5 is used to create an objective function for optimizing composite laminates using genetic algorithms. The objective function is shown to be a good comparative tool that allows ranking of different impacted laminates for better damage resistance and can be used during the initial design phase for laminate selection.
Chapter 2

Contact Mechanics

Contact behaviour during impact is an important aspect, since the nature of it will effect the damage mode created by the impact event. An impact event due to a small and rigid impactor will result in a short impact time and local damages in the contact region [1], while a soft impactor will cause a more distributed contact stress and a longer impact time and more global bending damages can occur like matrix cracking and delaminations [17]. Obtaining an accurate contact behaviour is therefore considered important for damage analysis of impacted composites [1]. The Hertzian contact formulation [18] has been used extensively in analytical contact analyses, however aspects like damage and plasticity are not accounted for in this formulation. Yang and Sun [1] and Yigit and Christoforou[12] have obtained analytical formulations for inelastic contact on composite laminates, based on empirical data. In this chapter, these two models will be discussed and evaluated by comparison with experimental test results. An additional formulation is then proposed, which combines two formulations.

2.1 Elastic Contact Behaviour for Composite Materials

The first satisfactory contact model, representing the behaviour of two non-conforming bodies in contact, was developed by Hertz [18]. The Hertzian contact model was obtained for curved bodies and can be rewritten, by increasing one of the radii to infinity, to represent the contact behaviour of a curved body and a semi-infinite half-space. Even though the model is derived to represent the contact behaviour between a curved indenter and a half-plane, the model gives a good representation of the contact behaviour between curved indenters and plates with finite thickness. This is mainly because the contact formulation is derived with the assumption of small indentations, which results in a local behaviour. Eventhough the Hertzian contact formulation was obtained for
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isotropic bodies, the formulation has been shown to be applicable for modelling the contact behaviour between spherical indenters and anisotropic plates under the small indentation assumptions [19]. The Hertzian contact model includes relations between contact load and the accompanied indentation, contact area and contact pressure. The formulation can be summarised as:

\begin{equation}
F = k_h \delta^{3/2},
\end{equation}
\begin{equation}
c = \left[ 3 \left( \frac{1-\nu^2}{E} + \frac{1-\nu'^2}{E'} \right) R \right]^{\frac{1}{3}},
\end{equation}
\begin{equation}
P_z(x,y) = \frac{3F}{2\pi c^3} \sqrt{c^2 - (x^2 + y^2)},
\end{equation}
\begin{equation}
F = \frac{4}{3} \frac{\sqrt{R}}{1-\nu^2 + 1-\nu'^2},
\end{equation}

where \( k_h = \frac{4}{3} \frac{\sqrt{R}}{1-\nu^2 + 1-\nu'^2} \) and \( F, c, \delta \) and \( P \) are the contact load, radius of the contact area, indentation and contact pressure respectively. \( R, E, E', \nu, \nu' \) are the indenter radius, out-of-plane modulus of the plate, stiffness of the indenter material, plate material Poisson’s ratio and indenter material Poisson’s ratio respectively. In case of composite laminates the out-of-plane stiffness is usually taken as \( E_{22} \) [1, 17]. A single value for the out-of-plane Poisson’s ratio is in a general case not easy to determine, since \( \nu_{13} \neq \nu_{23} \), where the indices indicate material coordinates with 1 being the fibre direction. Often the Poisson’s ratio is neglected for indentation analysis on anisotropic plates [1, 17]. In this work the in-plane Poisson’s ratio \( \nu_{12} \) is used. For uni-directional composites it can be stated that \( \nu_{12} = \nu_{13} \) [20], however, in general, \( \nu_{23} \) has a higher value and thus the assumption of using \( \nu_{12} \) for the out-of-plane behaviour will result in the underestimation of the out-of-plane Poisson’s effect. Nevertheless we assume that this assumption will lead to a smaller error compared to the case where the Poisson’s effect of the indented plate is neglected. Figure 2.1 shows a comparison between the Hertzian contact formulation and results from experimental testing. The test was performed on a \([0/45/90/-45]_{5S}\) carbon composite layup, indented by a steel indenter with a radius of \( 3\text{mm} \). The details about the test and the material data are described in section 2.2.3.
The Hertzian contact model shows good agreement with the experimental test results for small indentations (approximately less than 0.2 mm). This result is a limitation of the Hertzian contact model, which was necessary in order to derive the results [18]. Composite laminates under impact loading could however show relatively large indentations. Also the large local stresses introduced by the indentation cause local material damage resulting in inelastic material behaviour, which is not accounted for in the Hertzian contact model.

Modifications are needed to obtain a contact model which accounts for inelastic behaviour and large indentations, since the Hertzian contact formulation is applicable in the elastic regime of the material under small indentations only. In the following section two existing models are evaluated, which include inelastic contact behaviour. These models are discussed in Sections 2.2.1 and 2.2.2. Section 2.2.3 handles the experimental procedure and results, used for verification of the proposed inelastic contact models. The verification by comparison is discussed in Section 2.2.4.

2.2 Inelastic indentation

2.2.1 Yang and Sun Contact Model

The problem of inelastic contact in composite materials was addressed by, among others Yang and Sun [1], in which experimental data were used to describe the contact behaviour of composite panels indented by a spherical indenter. The tests were performed
to investigate the contact behaviour during elastic and inelastic loading, unloading and reloading.

The resulting model is a modification of the Hertzian contact formulation. The load versus indentation formulations for three different loading conditions are presented in Table 2.1.

<table>
<thead>
<tr>
<th>Indentation form</th>
<th>Formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loading</td>
<td>( F = k_n \delta^n )</td>
</tr>
<tr>
<td>Unloading</td>
<td>( F = F_m \left( \frac{(\delta - \delta_0)}{(\delta_m - \delta_0)} \right)^q )</td>
</tr>
<tr>
<td>Reloading</td>
<td>( F = k_l (\delta - \delta_0)^p )</td>
</tr>
</tbody>
</table>

where \( n \), \( q \) and \( p \), shown in Table 2.1, are determined by fitting the experimental data. \( F_m \) and \( \delta_m \) are the maximum load and indentation prior to unloading and \( \delta_0 \) represents the permanent indentation. \( k_l \) is the reloading stiffness, which can be determined by the unloading behaviour, resulting in [1]:

\[
k_l = \frac{F_m}{(\delta_m - \delta_0)^p}. \tag{2.5}
\]

It should be noted that \( \delta_0 \) is not by definition constant in time. The inelastic compressive stresses during the indentation will result in residual stresses after removal of the indentation pressure. These stresses will result in a reduction of \( \delta_0 \) over time due to creep. This effect is considered secondary and is not taken into account in this work.

The parameters \( n \), \( q \) and \( p \) were determined by Yang and Sun [1] by fitting experimental data. The best comparison with experimental data was achieved with the parameters given in Table 2.2. The experiments were performed on graphite/epoxy specimens with a \([0/45/0/ - 45/0]_2s\) layup and a total laminate thickness of 2.54mm.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Determined value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>1.5</td>
</tr>
<tr>
<td>( q )</td>
<td>2.5</td>
</tr>
<tr>
<td>( p )</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Note that the loading curve presented in Table 2.1 combined with the value for \( n \) given in Table 2.2 is identical to the Hertz contact model. This formulation is however, as explained in section 2.1, only applicable for small indentations in absence of material damage. Figure 2.2 shows the comparison found by Yang and Sun for the loading case.
Note that the indentations and the corresponding indentation loads are small compared to indentations and loads found in low speed impact tests. In Figure 2.3, load versus time for a low speed 5J impact test is shown, performed by Lopes [2]. Significantly higher loads than the loads presented by Yang and Sun [1] are measured. Where Yang and Sun obtain test results up to 1200N, even at low impact energy level of 5J Lopes measures a maximum load of about 6000N.

![Figure 2.2: Test results for two configurations including the Hertzian contact prediction (source:[1])](image1)

This indicates that the model presented by Yang and Sun may not be complete and that the model will not be sufficiently accurate for many loading cases encountered in practice.

![Figure 2.3: 5J impact test result performed on [±45/90/0/45/0_4/ − 45/0]s carbon layup with a steel impactor with R = 8mm (source:[2])](image2)
A graphical comparison for the unloading model is also provided by Yang and Sun [1], shown in Figure 2.4. The unloading model given in Table 2.1 is used with $q = 2.5$. The predictions show good agreement with the test results, however it should be noted that for larger indentations this model may not be sufficient. The applicability of this model will be investigated in Section 2.2.4 using experimental results discussed in section 2.2.3.

![Figure 2.4: Unloading model compared with experimental results (source:[1])](image)

### 2.2.2 Yigit and Christoforu Contact Model

Yigit and Christoforu[12] proposed a contact model which, like the Yang and Sun model, is a modification of the Hertzian contact formulation. The model includes elastic loading, inelastic loading and elastic unloading. The model is summarised in Table 2.3,

<table>
<thead>
<tr>
<th>Indentation form</th>
<th>Formulation</th>
<th>Applicability range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic loading</td>
<td>$F = k_h \delta^{5/2}$</td>
<td>$0 \leq \delta \leq \delta_y$</td>
</tr>
<tr>
<td>Elastic-plastic loading</td>
<td>$F = k_y (\delta - \delta_y) + k_h \delta_y^{3/2}$</td>
<td>$\delta_y \leq \delta \leq \delta_m$</td>
</tr>
<tr>
<td>Elastic unloading</td>
<td>$F = k_h \left(\delta^{3/2} - \delta_m^{3/2} + \delta_y^{3/2} \right) + k_y (\delta_m - \delta_y)$</td>
<td></td>
</tr>
</tbody>
</table>

where $\delta_y$ is the critical indentation for "local yield" and is given by[12]

$$\delta_y = \frac{0.68 S_y^2 \pi^2 R}{E_h^2},$$

(2.6)

where $E_h = \frac{\sqrt{R}}{1 - \mu^2 + \frac{\mu^2 R}{E_h^2}}$, of which the parameters are discussed in Section 2.1. For composite materials it is assumed that $S_y = 2S_u$, with $S_u$ being the shear strength of
the laminate. $k_y$ is the indentation stiffness at $\delta_y$ and is given by

$$k_y = 1.5k_h \sqrt{\delta_y}. \quad (2.7)$$

It should be noted that the contact model proposed by Yigit and Christoforu\cite{12} gives no information about the reloading behaviour of the indented material for indentations larger than $\delta_y$. The dynamic impact response of an impacted composite, depicted in Figure 2.3, shows the periodic increase and decrease of the impact load due to the eigen frequency of the plate. In the contact region this results in unloading and reloading sequences. To capture this contact behaviour the reloading behaviour is important and should be included in the model. The applicability of this model will be investigated using experimental results discussed in section 2.2.3.

### 2.2.3 Experimental Evaluation of Spherical Indentation Behaviour of Composite Laminates

Effort is made to obtain a correct model for the contact behaviour between spherical indenters and composite laminates. In Sections 2.2.1 and 2.2.2, two contact models were discussed for these types of materials however, these models are not complete and their applicability in different loading conditions needs to be verified. Experimental tests have been performed to get a good understanding of the contact behaviour between spherical indenters and composite laminates and to verify the correctness and the applicability of the contact models proposed by Yang and Sun, and Yigit and Christoforu as discussed in Sections 2.2.1 and 2.2.2, respectively. In order to analyse the dependency of the contact behaviour on the layup configuration, two different layups are tested:

- $[0/45/90/ -45]_{mS}$
- $[0/90]_{(2m)S}$

The value for $m$ was determined during specimen design, which is discussed in the following sub-section along with the obtained experimental setup and experimental results. The details about the materials used for these laminates and the test procedure and setup are discussed in Page 17. Note that the first layup is quasi-isotropic and the second is highly orthotropic. The difference in layup configuration is chosen to examine the effect of the layup architecture on the contact behaviour.
2.2.3.1 Specimen design

Effort is made to isolate the contact behaviour during the tests, since we are only interested in the contact behaviour of the laminates. The specimens, therefore, need to be designed so that the effects of the boundaries are minimized. In other words, damage and fracture should remain within the specimen boundaries and the effect of the stiffness of the boundary fixtures should be reduced to a minimum. To achieve this goal, Finite Element analyses were performed in order to size the specimens and to ensure that the boundaries were not affecting the behaviour. It was decided to use a thick laminate, with respect to the indenter radius, supported at the bottom to eliminate membrane effects induced by bending of the specimen. A schematic representation of the specimen and the corresponding boundary conditions are given in Figure 2.5. The values for $a$, $b$, $h$ and $R$ are the design variables and are determined from the results of the numerical analyses.

In order to examine to what extent damage and fracture remain within the specimen boundaries, damage and fracture mechanics were included in the numerical model. The LARC continuum damage model [21] was used to model intra-laminar damage, while the inter-laminar fracture was modeled by fracture mechanics. Fracture mechanics was included by using cohesive elements available in ABAQUS FE code. These elements are based on a traction separation law, where a relation between the nodal tractions and the nominal strains are used for representing the constituent response. In Equation 2.8 the traction-separation response is given.

\[
\begin{bmatrix}
  t_n \\
  t_s \\
  t_t
\end{bmatrix} =
\begin{bmatrix}
  K_{nn} & K_{ns} & K_{nt} \\
  K_{ns} & K_{ss} & K_{st} \\
  K_{nt} & K_{st} & K_{tt}
\end{bmatrix}
\begin{bmatrix}
  \epsilon_n \\
  \epsilon_s \\
  \epsilon_t
\end{bmatrix},
\]  

Equation 2.8

Figure 2.5: Test specimen representation
where the strains are defined as \( \varepsilon_n = \frac{\delta_n}{T_0} \), \( \varepsilon_s = \frac{\delta_s}{T_0} \), and \( \varepsilon_t = \frac{\delta_t}{T_0} \), and \( \delta_i \) and \( T_0 \) are the nodal separations and the initial thicknesses respectively and the indices \( n, s, t \) refer to mode-I, mode-II and mode-III fracture modes. The cohesive elements are assumed to behave linearly prior to damage initiation. After damage has occurred a damage evolution law is used to model material degradation. In this work a linear damage evolution law has been used, which is a valid assumption for epoxy based composites due to the brittle behaviour of the matrix material. Figure 2.6 gives a typical traction separation response.

![Figure 2.6: Typical traction separation response](image)

Figure 2.7 shows how the intra and inter-ply elements were arranged. Frictionless contact is modelled between the indenter and the laminate as well as between the adjacent continuum damage layers, to assure contact after delamination.

![Figure 2.7: FE model description](image)

The Hexply AS4/8552 thermoset prepreg material was used. The corresponding material properties were characterized in earlier work by NLR\(^1\). For the intra-laminar ply

\(^1\)NLR is the National Aerospace Laboratory in The Netherlands
properties are given in Table 2.4. No tests have been performed to obtain the out-of-plane properties, therefore no data about the out-of-plane properties were available. Damage initiation and fracture energy values for the inter-ply fracture model are given in Table 2.5. It must be noted that the values corresponding with Mode-III were not tested and therefore not characterized. For this mode similar values as for Mode-II were used. This should not pose any problems since the delamination behaviour is highly Mode-II dependent. The fracture energies given in Table 2.5 correspond to the area under the traction separation curve as is indicated by $G$ in Figure 2.6.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{11}$ [MPa]</td>
<td>137800</td>
</tr>
<tr>
<td>$E_{22}$ [MPa]</td>
<td>8580</td>
</tr>
<tr>
<td>$G_{12}$ [MPa]</td>
<td>4920</td>
</tr>
<tr>
<td>$v_{12}$ [-]</td>
<td>0.32</td>
</tr>
<tr>
<td>$S_{11, r}$ [MPa]</td>
<td>2042.1</td>
</tr>
<tr>
<td>$S_{11, c}$ [MPa]</td>
<td>1495</td>
</tr>
<tr>
<td>$S_{22, r}$ [MPa]</td>
<td>66.1</td>
</tr>
<tr>
<td>$S_{22, c}$ [MPa]</td>
<td>257.0</td>
</tr>
<tr>
<td>$S_{12}$ [MPa]</td>
<td>105.2</td>
</tr>
<tr>
<td>$E_{\text{steel}}$ [MPa]</td>
<td>200000</td>
</tr>
<tr>
<td>$\nu_{\text{steel}}$ [-]</td>
<td>0.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode-I</th>
<th>Mode-II</th>
<th>Mode-III</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damage initiation [MPa]</td>
<td>66.1</td>
<td>105.2</td>
</tr>
<tr>
<td>Fracture energy [N/mm]</td>
<td>0.28</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Numerical analyses have been performed on various specimen configurations, as well as for various indenter radii until a configuration was found for which the indentation damage and the intra-laminar fracture remain within the specimen boundaries. The obtained specimen configuration including the overall dimensions are given in Figure 2.8. The obtained layups for the two specimens were:

- Quasi-isotropic: $[0/45/90/-45]_{55}$
- Cross-ply: $[0/90]_{105}$
The analysis resulted in rather thick configurations (40 layers). A reason for this is that the bottom surface is assumed infinitely stiff in the analysis, which is a good assumption since high stiffness steel is used as support during the test. The high stiffness of the support will affect the contact stiffness for thin laminates. Laminates thicker than the proposed 40 layer layups will theoretically give better results, however the cost of production will increase while the effect of the support stiffness will decrease exponentially [14], reducing the efficiency of adding additional plies.

In Figures 2.9 and 2.10 the delaminations and the indentation damage, provided by the numerical model for the [0/90]_{108} laminate, are visualized. From these results it can be concluded that the delaminations and the indentation damages remain within the specimen boundaries. Note that in Figures 2.9 and 2.10 a quarter of the model is presented for better visualization.
In Figure 2.9 the delaminations under indentation are presented. The colour-bar displays the degradation of the cohesive elements and the deleted elements are fully degraded, indicating full delamination.

The color bar in Figure 2.10 indicates the damage index for the out-of-plane damage (i.e. it represents $\sigma_{33}/S_{33}$). For more detail about the LARC criteria the author refers to the work of Maimi and Camanho [21].
2.2.3.2 Specimen Fabrication and Test Setup

Test specimens were produced using an Automated Dynamics Corporation fibre placement machine at NLR, to accurately control the fibre angles. Hexply AS4/8552 thermoset prepreg material was used and cured according to the manufacturer’s prescribed curing cycle. For each layup one plate of $200 \times 300 \ mm$ was manufactured, which after curing was cut into specimens of $20 \times 20 \ mm$. A total of 40 layers results in a cured thickness of the specimens of $7.2 \ mm$. The cut specimens are depicted in Figure 2.11.

![Test specimens](image)

**Figure 2.11:** Test specimens

As mentioned in section 2.2.2, the loading, unloading and reloading behaviours, in the elastic and inelastic regime, need to be determined. Therefore, these loading conditions are included in the test. Figure 2.12 shows the indentation pattern used in the tests, including elastic loading, inelastic loading, inelastic unloading and inelastic reloading.
An Instron 5882 test machine was used for the quasi-static indentation test, with the specimen fully supported along the bottom side as discussed in Section 2.2.3.1. A hardened steel spherical indenter was used with a radius of 3 mm. To accurately measure the displacement of the machine head, a strain gauge was attached to a steel sliding mechanism between the fixture and the machine head. In Figure 2.13, the spherical indenter, the measuring device and the composite specimen are shown. The loading followed the scheme displayed in Figure 2.12.

Figure 2.12: Loading pattern used for the experimental indentation tests

Figure 2.13: Experimental test setup displacement including test specimen
2.2.3.3 Test Results

Three specimens were tested per layup configuration to give an idea of the scatter. The test results, in terms of load versus indentation, are displayed in Figure 2.14 for all of the tested specimens.

![Indentation test results](image)

*Figure 2.14: Indentation test results*

The obtained test results will be used to investigate the applicability of contact models proposed by Yang and Sun, and Yigit and Christoforou as discussed in Sections 2.2.1 and 2.2.2.

2.2.4 Analytical Evaluation of Experimental Testing

2.2.4.1 Evaluation of the Yang and Sun Contact Model

The experimental results presented in section 2.2.3 will be used as comparison to evaluate the contact models proposed by Yang and Sun [1]. The three different loading cases discussed in Table 2.1 will be analysed and presented separately.

**Loading** Yang and Sun proposed the Hertzian contact model for the loading case. This model is only applicable for small indentations in absence of material damage, as discussed in section 2.1. The obtained test results are used to evaluate the loading model. The material properties and the testing configuration are given in Table 2.6, where $E$ is the plate out-of-plane modulus determined from $E_{22}$, which was presented...
Table 2.6: Material and testing configuration

<table>
<thead>
<tr>
<th>$E$ [Gpa]</th>
<th>$E'$ [Gpa]</th>
<th>$\nu$ [-]</th>
<th>$\nu'$ [-]</th>
<th>$R$ [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.58</td>
<td>200.0</td>
<td>0.3</td>
<td>0.3</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 2.4, assuming a transversely isotropic behaviour. As discussed in Page 6, the in-plane Poisson’s ratio ($\nu_{12}$) has been used for the plate out-of-plane Poisson’s ratio ($\nu$). A prime in Table 2.4 refers to indenter properties.

Using the data in Table 2.6 the loading curve during indentation can be analysed for the Yang and Sun/Hertz model. The comparison of the proposed loading model and the experimental data is presented in Figure 2.15.

![Figure 2.15: Evaluation of the Yang and Sun Contact contact model in loading condition](image)

In Figure 2.15 a good comparison between the loading model and the test results for small indentations is shown, while a deviation from the test results for larger indentations is clear. It should be noted that a better characterization of $E_{33}$ could result in a better representation of the test. Nevertheless, from literature [1, 12] and test results presented in this chapter, it can be concluded that the load-indentation relation becomes linear at larger indentations. This behaviour will cause a deviation from the Hertzian contact model, which includes a load-indentation relation that goes to the power of 1.5 (i.e. $F(\delta) = f(\delta^{1.5})$).

**Unloading** For the evaluation of the unloading behaviour proposed by Yang and Sun, additional data are needed. The unloading model proposed by Yang and Sun, as
summarized in Table 2.6, is given by

\[ F = F_m \left[ \frac{\delta - \delta_0}{\delta_m - \delta_0} \right]^q. \]  \hspace{1cm} (2.9)

In this model \( \delta_0 \), which represents permanent indentation, needs to be determined. Yang and Sun propose the following expression for \( \delta_0 \)

\[ \delta_0 = \delta_m \left[ 1 - \left( \frac{\delta_{cr}}{\delta_m} \right)^{2/3} \right], \]  \hspace{1cm} (2.10)

where \( \delta_{cr} \) represents the indentation corresponding to "local yield" and is, according to Yang and Sun, a material constant that needs to be determined from test results. Using the Yang and Sun unloading model, analyses were performed and compared to the test results to obtain the value of \( \delta_{cr} \) by matching the test results for the unloading curve. As proposed by Yang and Sun, \( q = 2.5 \) was used in these analyses, resulting in:

\[ \delta_{cr} = 0.3048 \text{ mm}. \]  \hspace{1cm} (2.11)

A graphical representation of the comparison between the Yang and Sun model and the experimental results, using the value for \( \delta_{cr} \) as given in Equation 2.11, is depicted in Figure 2.16.

Note that \( \delta_m \) is the maximum indentation prior to unloading and can be represented as:

\[ \text{if } \delta_{i+1} < \delta_i \text{ then } \delta_m = \delta_i. \]  \hspace{1cm} (2.12)
From the comparison between test and the Yang and Sun unloading model, graphically displayed in Figure 2.16, can be concluded that the model is able to predict the unloading behaviour with satisfactory accuracy. It should however be noted that for this model some experimental data are needed in order to get a good prediction for $\delta_{cr}$.

**Reloading** As discussed in Table 2.1 the reloading model proposed by Yang and Sun is described by

$$F = k_l (\delta - \delta_0)^p,$$

with

$$k_l = \frac{F_m}{(\delta_m - \delta_0)^p}.$$  \hspace{2cm} (2.15)

Also here $\delta_{cr}$ is needed in order to obtain $\delta_0$ as described in Equation 2.10. The same value as found for the unloading case ($\delta_{cr} = 0.3048$ mm) is used. The graphical comparison between the test data and the Yang ans Sun reloading model is given in Figure 2.17, where a good comparison with test data is shown.

![Graph](image)

**Figure 2.17**: Evaluation of the Yang and Sun Contact contact model in reloading condition

The Yang and Sun contact model shows a good representation of the contact behaviour between spherical indenters and composite plates during unloading and reloading cycles. The model is less accurate for the loading cycle at large indentations, therefore additional measures are needed to obtain more accurate results.
2.2.4.2 Evaluation of the Yigit and Christoforu Contact Model

The experimental results presented in section 2.2.3 will be used as comparison to evaluate the contact models proposed by Yigit and Christoforu [12]. The three different loading cases discussed in Table 2.3 will be analyzed and presented separately.

The material data presented in Table 2.6 is used to evaluate the Yigit and Christoforu contact model. In the loading condition, Yigit and Christoforu propose Hertzian contact for elastic loading and the linear expression: 

\[ F = k_y (\delta - \delta_y) + k_h \delta_y^{3/2}, \]

for inelastic loading. The transition between elastic and inelastic loading is determined by \( \delta_y \), which represents the indentation at "local yield". \( \delta_y \) used for the analysis is therefore chosen to be equal to \( \delta_{cr} \) found for the Yang and Sun model in Section 2.2.4.1:

\[ \delta_y = 0.3048 \text{ mm} \]  \hspace{1cm} (2.16)

and

\[ k_y = 1.5 k_h \sqrt{\delta_y}. \]  \hspace{1cm} (2.17)

In Figure 2.18, a comparison between the Yigit and Christoforu loading model and experimental results is displayed.

From the comparison results, presented in Figure 2.18, it can be concluded that the loading model proposed by Yigit and Christoforu gives a satisfactory representation of the loading behaviour found by the experimental tests. During unloading, Yigit and Christoforu propose an elastic behaviour, as is summarised in Table 2.3. In Figure 2.19, a comparison of the unloading behaviour and the test results is shown.
The unloading behaviour found by experimental testing is modelled less accurately by Yigit and Christoforu[12] than by Yang and Sun[1] (see in Figure 2.16 for the Yang and Sun unloading comparison results). As can be seen in Figure 2.19, in the Yigit and Christoforu unloading model, more energy is lost in a loading-unloading cycle than found by experimental testing, indicating that the Yigit and Christoforu model overestimates the local indentation damage. The Yigit and Christoforu contact model includes no reloading behaviour, therefore no comparison is made here for the reloading case.

2.2.4.3 Sun-Chistoforu combined Contact Model

The contact model determines the transfer of energy from the impactor/indenter to the plate and vice versa. It is important to have a model, which gives a good representation of the contact behaviour. The individual models discussed in the previous sections are not complete (i.e. they don’t cover the full elastic/elastic loading, unloading and reloading behaviour) or they don’t show a good representation when compared to the experimental test results. The models are summarised in Table 2.7, where the regions of applicability for the models are included.
TABLE 2.7: Applicability summary of the contact model

<table>
<thead>
<tr>
<th>Indentation form</th>
<th>Yang and Sun</th>
<th>Yigit and Christoforu</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic Loading</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Elastic-plastic loading</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>Unloading</td>
<td>✓</td>
<td>×</td>
</tr>
<tr>
<td>Reloading</td>
<td>✓</td>
<td>×</td>
</tr>
</tbody>
</table>

In order to have a contact model which satisfies the entire indentation regime (i.e. elastic/inelastic loading, unloading and reloading), the two contact models are combined. Table 2.8 concludes the combined contact model.

TABLE 2.8: Sun-Christoforu combined contact model

<table>
<thead>
<tr>
<th>Indentation form</th>
<th>Formulation</th>
<th>Origin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loading</td>
<td>$F = k_h \delta^{3/2}$</td>
<td>Hertz</td>
</tr>
<tr>
<td>Elastic-plastic</td>
<td>$F = k_y (\delta - \delta_y) + k_h \delta_y^{3/2}$</td>
<td>Yigit &amp; Christoforu</td>
</tr>
<tr>
<td>Unloading</td>
<td>$F = F_m [(\delta - \delta_0) / (\delta_m - \delta_0)]^{5/2}$</td>
<td>Yang &amp; Sun</td>
</tr>
<tr>
<td>Reloading</td>
<td>$F = k_i (\delta - \delta_0)^{3/2}$</td>
<td>Yang &amp; Sun</td>
</tr>
</tbody>
</table>

Where $\delta_y = \delta_{cr} = 0.3048 [\text{mm}]$.

A graphical comparison between the Sun-Christoforu combined contact model and the experimental data is shown in Figure 2.20.

![Figure 2.20: Sun-Christoforu contact model](image_url)

The Sun-Christoforu combined contact model, shown in Figure 2.20, gives a better representation of the contact behaviour than the Yang and Sun and Yigit and Christoforu
models individually, covering the entire loading, unloading and reloading regime. Note that experimental data is necessary for these models in order to obtain $\delta_{cr}$. It is also important to note that the measured $\delta_{cr}$ only includes information about the indentation and is therefore dependent on the material and indenter radius. This means that, as also mentioned by Yang and Sun [1], $\delta_{cr}$ is a constant for a given material system and indenter radius.

2.2.5 Conclusions

Two models for elastic-inelastic contact for composite laminates, proposed by Yang and Sun [1], and Yigit and Christoforu[12], were evaluated by comparison with experimental results. The two models were evaluated for contact behaviour defined as load versus indentation during:

- elastic loading
- inelastic loading
- unloading
- reloading.

The model proposed by Yang and Sun [1] showed good comparison with the experimental results for the elastic loading, unloading and reloading cycle, while for the inelastic loading cycle it showed a significant deviation compared to the experimental results. The Yigit and Christoforu[12] model showed good comparison with the experimental results for the elastic loading and the inelastic loading. The model showed significant deviation from the experimental results for the unloading case and does not include the reloading behaviour. The two models were combined to obtain a complete model as is shown in Table 2.8. The combined model shows good agreement with the experimental results for the all of the load cases (i.e. elastic loading, inelastic loading, unloading and reloading). In the next chapter the dynamic aspects of impact are discussed.
Chapter 3

Dynamic Impact Response

In this chapter the dynamic response of composite laminates due to impact is discussed. Especially for wave dominated impact events, where the ratio of impactor mass over plate mass is low, the dynamic behaviour is important [22]. An analytical dynamic model is discussed in this chapter, enabling the analysis of impact load and the corresponding out-of-plane displacement of composite laminates due to impact.

3.1 Analytical Elastic Dynamic Impact Response

The interest in using composite materials is increasing due to their superior stiffness over weight ratio. However, in practice much of this superiority is lost due to the poor performance of the layered material after impact, primarily under compression. Impact can lead to different types of damage in the material, such as matrix cracks, delaminations, broken fibres, etc. which, in turn, cause significant strength and stiffness degradation while little to no damage is visible on its surface.

Even though substantial effort has been spent to develop analysis methods that address this phenomenon, and significant progress has been made in quantifying the type and extend of damage and its effect on the residual strength [6, 20, 23–36], much work is still needed to improve insight and understanding and to translate the analysis methods to efficient (high performance over weight) designs. As a result, in general, a significant weight penalty is carried by today’s designs in order to make sure that composite laminates meet the applied loads with barely visible impact damage present.
Sun and Chattopadhyay [23] and Dobyns [24] were among the first to model impact damage on a composite plate. In both cases the governing equations including transverse shear effects derived by Whitney and Pagano [37] were solved numerically. Dobyns assumed that the contact force as a function of time was known a priori while Sun and Chattopadhyay derived it as part of their numerical solution. A more systematic way of dealing with the problem was proposed by Cairns and Lagace [6, 20, 25]. They were among the first to differentiate between damage resistance, which addresses the amount and types of damage created during an impact event and damage tolerance which determines the residual strength of an already impacted laminate. In their work, an explicit scheme is used in a global analysis, to determine the loads exerted on the plate during impact. Subsequently, a local analysis at the impact site uses the loads from the global analysis to perform a ply-by-ply analysis of the laminate.

The complexity of the damage mechanisms was studied experimentally by Starnes et al [26]. They used ultrasonic inspection and sectioning of impacted laminates, to determine the extent and location of delaminations and matrix cracks. Other representative experimental assessments of the effect of impact on composite plates can be found in [27] where scanning electron microscopy was used and [28] where X-ray radiography methods were employed.

Modelling of impact damage has been performed extensively with the use of finite element methods [29–33]. Further modelling efforts accounting for the details of the delaminations created during impact and their tendency to buckle under compressive load combined with modelling the impact site as a region of reduced stiffness were done by Dost et al [34–36]. Of particular interest for the present work is reference [35] where an attempt was made to use analysis methods in a design framework that allows comparison of different laminates on the basis of their performance during impact in order to determine the best performer(s).

The above discussion is by no means exhaustive and only brings out some of the representative approaches that have been used in the past. For additional references, the work by Cantwell and Morton [38] is a good starting point.

In the following two sub-sections, the dynamic displacement field of impacted plates and impactor projectiles were obtained and combined by contact definitions to obtain the elastic dynamic impact response. The methodology is based on the methodology presented by Abrate [39] and is used to emphasize on the ease of use of the method and the accuracy it provides. Comparisons with advanced numerical results and experimental
test results show how this method could be of great value for obtaining fast and accurate results. In addition, the use of this method to validate advanced numerical results is shown.

3.1.1 Dynamic displacement field of the plate

As discussed in Chapter 2, large indentations could be the result of high contact loads, which could result in less accurate representation of the contact behaviour when using the Hertzian contact formulation. In this section, the dynamic elastic response of low-velocity impact events is investigated. For these analyses, as will be shown in this section by comparison to experimental data, the Hertzian contact formulation is accurate enough for predicting the dynamic behaviour. An elastic impact model is therefore obtained based on Hertz' contact law. This law, summarised in Equations 2.1 to 2.3, gives a relation between the indentation during impact and the impact load, where the indentation is defined as:

\[ \delta = w_2 - w_1, \]  

where \( w_2 \) and \( w_1 \) represent the displacements of the impactor and the plate respectively. In Figure 3.1, the definitions of the displacements of the impactor and the plate as well as the definition of \( \delta \) are illustrated.

![Figure 3.1: Schematic representation of absolute and relative displacements during impact](image)

Classical plate theory (CLPT) is used as representation of the plate displacement. A partial differential equation representing the equation of motion of the plate is obtained from the CLPT, including the assumption that \( D_{1,6} \approx D_{2,6} \approx 0 \), compared to the remaining entries in the D-matrix. The resulting partial differential equation, representing...
the plate's equation of motion, is given as:

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + I_1 \ddot{w} = q, \tag{3.2}
\]

where \( w \) is the out-of-plane displacement of the plate and \( I_1 = \int_{-h/2}^{h/2} \rho \, dz \), which represents the mass of the plate per unit area. In order to solve this equation, assumptions have to be made about the boundary conditions of the plate. In general two categories of boundary conditions are used, segregating impact problems in; boundary dominated impact and wave dominated impact, as described by Olsson[22]. A graphical representation of the two impact categories is shown in Figure 3.2 [3].

![Figure 3.2: a: boundary dominated impact, b: wave dominated impact (source: [3])](image)

The first model includes a high impactor-plate mass ratio, as described by Olsson[22]. In this case the boundary conditions affect the displacement field significantly and can therefore not be neglected. These forms of impact have a quasi-static character and can therefore be analysed as such. The second model includes a low impactor-plate mass ratio as described by Olsson[22], where the interaction between the waves and the boundaries are negligible during impact. In this research the second assumption is used, resulting in a simply supported plate with an assumed displacement field, satisfying the simply supported boundary conditions, given as:

\[
w = \sum_{j=1,3..}^{\infty} \sum_{k=1,3..}^{\infty} W_{jk} \alpha_{jk}, \tag{3.3}
\]

where \( W_{jk} \) gives the displacement variation in space, satisfying the simply supported boundary conditions, while \( \alpha_{jk} \) represents the variation in time. The displacement in space is assumed to have the following form:

\[
W_{jk} = \cos \left( \frac{j\pi x}{a} \right) \cos \left( \frac{k\pi y}{b} \right), \tag{3.4}
\]
satisfying the simply supported boundary conditions \( w = 0 \) at \( x = \pm a/2 \) and \( y = \pm b/2 \). The definition for the dimensions and the chosen coordinate system is shown in Figure 3.3.

\[ w = 0 \text{ at } x = \pm a/2 \text{ and } y = \pm b/2 \]

Substituting the total displacement representation into Equation 3.2 gives:

\[
\sum_{jk} \left\{ \left[ D_{11} \left( \frac{j\pi}{a} \right)^4 + 2 \left( D_{12} + 2D_{66} \right) \left( \frac{j\pi}{a} \right)^2 \left( \frac{k\pi}{b} \right)^2 + D_{22} \left( \frac{k\pi}{b} \right)^4 \right] \alpha_{jk} + I_1 \bar{\alpha}_{jk} \right\} \cos \left( \frac{j\pi x}{a} \right) \cos \left( \frac{k\pi y}{b} \right) = q. \quad (3.5)
\]

Assuming a concentrated load defined as \( q = F \cdot \delta(x = 0) \cdot \delta(y = 0) \) and multiplying both sides of Equation 3.5 with \( \cos \left( \frac{j\pi x}{a} \right) \cos \left( \frac{k\pi y}{b} \right) \) and integrating over the plate area, results in a non-homogeneous ordinary differential equation, giving a representation of \( \alpha \).

\[
\bar{\alpha}_{jk} + \omega_{jk}^2 \alpha_{jk} = \frac{4F(t)}{ab I_1}. \quad (3.6)
\]

In Equation 3.6 \( F(t) \) represents the contact load, which is a result of the integration of the pressure over the area, \( a \) and \( b \) are the length and width of the plate respectively and

\[
\omega_{jk}^2 = \left[ D_{11} \left( \frac{j\pi}{a} \right)^4 + 2 \left( D_{12} + 2D_{66} \right) \left( \frac{j\pi}{a} \right)^2 \left( \frac{k\pi}{b} \right)^2 + D_{22} \left( \frac{k\pi}{b} \right)^4 \right] / I_1.
\]

Equation 3.6 can be solved, resulting in a formulation for \( \alpha \). The resulting expression for \( \alpha \) becomes:

\[
\alpha_{jk}(t) = \frac{4}{ab I_1 \omega_{jk}} \int_0^t F(\tau) \sin(\omega_{jk}(t - \tau)) d\tau. \quad (3.7)
\]
Substituting Equation 3.7 and Equation 3.4 into the assumed displacement field given in Equation 3.3 results in the displacement field of the plate, which is described as:

\[ w(x, y, t) = \sum_{j=1,3..} \sum_{k=1,3..} \frac{4}{abI_1 \omega_{j,k}} \cos \left( \frac{j \pi x}{a} \right) \cos \left( \frac{k \pi y}{b} \right) \int_0^t F(\tau) \sin(\omega_{j,k}(t - \tau)) d\tau. \]  

(3.8)

With Equation 3.8, the displacement field of a rectangular plate under a time-dependent load \( F(t) \) is complete. In an impact event \( F(t) \) is the contact load, which is a result of energy transfer between the impactor and the plate. In order to get a representation for \( F(t) \), the dynamic behaviour of the impactor needs to be obtained, which is discussed in the following sub-section.

### 3.1.2 Dynamic displacement of the impactor

The displacement of the impactor can be found by Newton’s second law. Figure 3.4 shows the free body diagram of the impactor.

The equilibrium equation for the impactor is given as

\[ -M \ddot{\omega} = F, \]

(3.9)

with \( M \) being the mass of the impactor.

Integrating the equation twice with respect to time and using the initial condition \( \dot{\omega}(t = 0) = \ddot{\omega}(t = 0) = 0 \) results in

\[ w(x, y, t) = \sum_{j=1,3..} \sum_{k=1,3..} \frac{4}{abI_1 \omega_{j,k}} \cos \left( \frac{j \pi x}{a} \right) \cos \left( \frac{k \pi y}{b} \right) \int_0^t F(\tau) \sin(\omega_{j,k}(t - \tau)) d\tau. \]  

(3.8)
0) = \( V_0 \), results in the displacement of the impactor, as given in Equation 3.10.

\[
w = V_0 t - \frac{1}{M} \int_0^t \int_0^\tau F(\tau') d\tau' d\tau. \tag{3.10}
\]

With the dynamic behaviour of the plate and the impactor known, the impact response can be finalised by combining the two displacement equations given in Equations 3.8 and 3.10 by means of a contact formulation. The dynamic impact response of a rectangular plate impacted by a spherical impactor is discussed in the following section.

### 3.1.3 Contact Load Determination

The obtained displacements of the plate and the impactor allow the derivation of the contact load. Substituting Equations 3.8 and 3.10 into the definition for indentation given by Equation 3.1 results in the indentation of the composite plate by a spherical impactor. In Equation 3.11 the indentation is given.

\[
\delta = V_0 t - \frac{1}{M} \int_0^t \int_0^\tau F(\tau') d\tau' d\tau - \sum_{j=1,3,} \sum_{k=1,3,} \frac{4}{abI_1\omega_{j,k}} \cos \left( \frac{j\pi x}{a} \right) \cos \left( \frac{k\pi y}{b} \right) \int_0^\tau F(\tau) \sin(\omega_{j,k}(t - \tau)) d\tau. \tag{3.11}
\]

Substituting the indentation found in Equation 3.11 into the non-linear relation between indentation and the contact load from Hertz contact law given in Equation 2.1, the contact load can be represented as:

\[
F(t)^{2/3} = k_h^{2/3} V_0 t - \frac{k_h^{2/3}}{M} \int_0^t \int_0^\tau F(\tau') d\tau' d\tau - \sum_{j=1,3,} \sum_{k=1,3,} \frac{4k_h^{2/3}}{abI_1\omega_{j,k}} \cos \left( \frac{j\pi x}{a} \right) \cos \left( \frac{k\pi y}{b} \right) \int_0^\tau F(\tau) \sin(\omega_{j,k}(t - \tau)) d\tau. \tag{3.12}
\]

Olsson [40] found an elegant solution to this problem by replacing the summations in Equation 3.12 by integrals and including factors of 1/2 to account for the fact that only the odd modes are taken into account. This method works well when the eigen frequencies of the system are relatively low. We will make an effort to divert from this restriction and to use numerical methods to solve Equation 3.12. For this purpose, the
integrals are replaced by Riemann summations, which results in:

\[ F_{n}^{2/3} = k_{h}^{2/3} V_{0} \tau_{n} - \frac{k_{h}^{2/3}}{M} \sum_{i=1}^{n} \sum_{l=1}^{i} F(\tau_{l}) \Delta \tau^{2} - \]

\[ \sum_{j=1,3..}^{\infty} \sum_{k=1,3..}^{\infty} \frac{4k_{h}^{2/3}}{abI_{1} \omega_{j,k}} \cos \left( \frac{j \pi x}{a} \right) \cos \left( \frac{k \pi y}{b} \right) \]

\[ \sum_{i=1}^{n} F(\tau_{i}) \sin(\omega_{j,k}(\tau_{n} - \tau_{i})) \Delta \tau. \quad (3.13) \]

This expression can be rewritten by separating the summations, resulting in the following expression:

\[ F_{n}^{2/3} = k_{h}^{2/3} V_{0} \tau_{n} - \frac{k_{h}^{2/3}}{M} \left[ \sum_{i=1}^{n-1} \sum_{l=1}^{i} F_{l} + \sum_{l=1}^{n-1} F_{l} + F_{n} \right] - \]

\[ \sum_{j=1,3..}^{\infty} \sum_{k=1,3..}^{\infty} \sum_{i=1}^{n-1} \frac{4k_{h}^{2/3}}{abI_{1} \omega_{j,k}} \cos \left( \frac{j \pi x}{a} \right) \cos \left( \frac{k \pi y}{b} \right) \]

\[ F_{i} \sin(\omega_{j,k}(\tau_{n} - \tau_{i})) \Delta \tau. \quad (3.14) \]

Writing explicitly for \( F(t) \) gives:

\[ F_{n}^{2/3} + \frac{k_{h}^{2/3} \Delta \tau^{2}}{M} F_{n} = k_{h}^{2/3} V_{0} \tau_{n} - \frac{k_{h}^{2/3} \Delta \tau^{2}}{M} \left[ \sum_{i=1}^{n-1} \sum_{l=1}^{i} F_{l} + \sum_{l=1}^{n-1} F_{l} \right] - \]

\[ \sum_{j=1,3..}^{\infty} \sum_{k=1,3..}^{\infty} \sum_{i=1}^{n-1} \frac{4k_{h}^{2/3}}{abI_{1} \omega_{j,k}} \cos \left( \frac{j \pi x}{a} \right) \cos \left( \frac{k \pi y}{b} \right) \]

\[ F_{i} \sin(\omega_{j,k}(\tau_{n} - \tau_{i})) \Delta \tau. \quad (3.15) \]

A Newton-Raphson method is used to obtain \( F(t) \) from the non-linear Equation 3.15, resulting in an iterative expression for \( F_{n} \):

\[ F_{n+1} = -k_{h}^{2/3} V_{0} \tau_{n} - \frac{k_{h}^{2/3} \Delta \tau^{2}}{M} \left[ \sum_{i=1}^{n-1} \sum_{l=1}^{i} F_{l} + \sum_{l=1}^{n-1} F_{l} \right] - \]

\[ \sum_{j=1,3..}^{\infty} \sum_{k=1,3..}^{\infty} \sum_{i=1}^{n-1} \frac{4k_{h}^{2/3}}{abI_{1} \omega_{j,k}} \cos \left( \frac{j \pi x}{a} \right) \cos \left( \frac{k \pi y}{b} \right) \]

\[ F_{i} \sin(\omega_{j,k}(\tau_{n} - \tau_{i})) \Delta \tau. \quad (3.16) \]

with

\[ \beta = k_{h}^{2/3} V_{0} \tau_{n} - \frac{k_{h}^{2/3} \Delta \tau^{2}}{M} \left[ \sum_{i=1}^{n-1} \sum_{l=1}^{i} F_{l} + \sum_{l=1}^{n-1} F_{l} \right] - \]

\[ \sum_{j=1,3..}^{\infty} \sum_{k=1,3..}^{\infty} \sum_{i=1}^{n-1} \frac{4k_{h}^{2/3}}{abI_{1} \omega_{j,k}} \cos \left( \frac{j \pi x}{a} \right) \cos \left( \frac{k \pi y}{b} \right) \]

\[ F_{i} \sin(\omega_{j,k}(\tau_{n} - \tau_{i})) \Delta \tau. \quad (3.17) \]
The iterative routine given in Equation 3.16 and 3.17 enables the analysis of the impact load for a given impact configuration (i.e. impactor mass, plate mass, impactor and plate stiffness, impactor radius and impact energy). Substitution of the obtained impact load in Equation 3.8 leads to the dynamic displacement field of the impacted plate. In the next subsection this method will be validated, by comparison to published experimental data and other analytical methods.

3.1.4 Validation and Results

Validation of the the semi-analytical model, discussed in the previous section, is accomplished by comparison to published experimental test results and other analytical results. It is important at this point to emphasize that there is no damage model present in the analytical model therefore, low energy test results are used for validation, reducing the effects of damage and fracture. Two different comparisons will be made; comparison with other analytical studies and comparison with physical test results published by Olson [4] and Lopes [2].

3.1.4.1 Validation using other analytical models

Studies have been performed in the past by Wu and Sun, and Cairns and Lagace [6]. These authors have analysed impact on a composite plate with a $[0/90/0/90/0]^s$ layup and an impact configuration as given in Table 3.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>200mm</td>
</tr>
<tr>
<td>$b$</td>
<td>200mm</td>
</tr>
<tr>
<td>$D_{11}$</td>
<td>154.9Nm</td>
</tr>
<tr>
<td>$D_{12}$</td>
<td>4.760Nm</td>
</tr>
<tr>
<td>$D_{22}$</td>
<td>91.4Nm</td>
</tr>
<tr>
<td>$D_{66}$</td>
<td>8.970Nm</td>
</tr>
<tr>
<td>$I_1$</td>
<td>4.132kg/m²</td>
</tr>
<tr>
<td>$h$</td>
<td>2.69mm</td>
</tr>
<tr>
<td>$V_0$</td>
<td>3m/s</td>
</tr>
<tr>
<td>$M$</td>
<td>8.3g</td>
</tr>
<tr>
<td>$R$</td>
<td>6.35mm</td>
</tr>
<tr>
<td>$E_c$</td>
<td>9.72</td>
</tr>
</tbody>
</table>

The analytical model is used to obtain the response of the impacted structure mentioned in Table 3.1. The results are concluded in Figure 3.5 along with the results obtained by Wu and Sun, and Cairns and Lagace [6].
In the figure on the left the comparison for the dynamic impact load is presented and the comparison for the plate displacement at the point of impact is demonstrated in the figure on the right. Good agreement compared to the analytical models, obtained by Wu and Sun and Cairns and Lagace, is shown in Figure 3.5. Some small low frequency oscillations are visible in both of the figures as indicated in Figure 3.5. These oscillations are the result of the eigen frequency oscillations of the plate.

3.1.4.2 Validation by comparison with experimental test results

Results from the analytical model are compared with experimental test results in this section. For a good comparison throughout the entire impact process, it is important to have an impact test where negligible damage occurs during impact, since the model summarized in Equations 3.8 and 3.16, includes no damage behaviour. Damage and fracture will result in degradation of the material stiffness, which can lead to significant changes in the impact response. Olson [4] has published experimental test results for a low speed impact test on a [45/ − 45]_{0s} with impact configuration given in Table 3.2.
Table 3.2: Impact configuration used by Olson [4]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>500 mm</td>
</tr>
<tr>
<td>b</td>
<td>400 mm</td>
</tr>
<tr>
<td>$D_{11}$</td>
<td>632.2 Nm</td>
</tr>
<tr>
<td>$D_{12}$</td>
<td>198.0 Nm</td>
</tr>
<tr>
<td>$D_{22}$</td>
<td>273.5 Nm</td>
</tr>
<tr>
<td>$D_{66}$</td>
<td>214.6 Nm</td>
</tr>
<tr>
<td>$I_1$</td>
<td>7.36 kg/m²</td>
</tr>
<tr>
<td>$h$</td>
<td>4.572 mm</td>
</tr>
<tr>
<td>$V_0$</td>
<td>0.38 m/s</td>
</tr>
<tr>
<td>$M$</td>
<td>30 g</td>
</tr>
<tr>
<td>$R$</td>
<td>9.5 mm</td>
</tr>
</tbody>
</table>

The comparison between test and analysis is shown in Figure 3.6.

Figure 3.6 shows good agreement between test and analysis, where in the left figure the comparison for the impact load is depicted and the comparison for the plate displacement at the point of impact is presented in the figure on the right.

A second comparison is made between the analytical model and test results published by Lopes [2]. Lopes performed a 5J impact test on a [±45/90/0/45/0/−45/0]_s layup with the impact configuration given in Table 3.3.
TABLE 3.3: impact configuration used by Lopes [2]

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$</td>
<td>135 GPa</td>
</tr>
<tr>
<td>$E_2$</td>
<td>9.6 GPa</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>5.3 GPa</td>
</tr>
<tr>
<td>$t$</td>
<td>0.182 mm</td>
</tr>
<tr>
<td>$R$</td>
<td>8 mm</td>
</tr>
<tr>
<td>$a$</td>
<td>125 mm</td>
</tr>
<tr>
<td>$b$</td>
<td>75 mm</td>
</tr>
<tr>
<td>$M$</td>
<td>0.95 kg</td>
</tr>
</tbody>
</table>

A graphical representation of the comparison between the analysis (Equation 3.16) and experimental results is shown in Figure 3.7.

![Figure 3.7](image)

**Figure 3.7**: Comparison of the analysis with experimental 5J impact test result published by Lopes [2]

Good agreement with the 5-J impact test performed by Lopes [2] can be concluded from Figure 3.7. In Figure 3.7, the smoothening of the response on right half of the graph should be noted. This is the effect of energy consumption due to damaged material. The local peaks correspond to the eigen frequency of the system and because there is no damage behaviour present in the analytical model, no smoothing is observed in the analytical response.

Lopes simulated this impact event by using advanced finite element analysis in combination with a LARC continuum damage model [21] for the intra-ply damage and a cohesive fracture model for modelling inter-ply delaminations (for details see reference [2]). The simulation of low energy impact often results in increasing mesh density, because damage and fracture occur in a small region. High computation times is a direct result of the high mesh density. In Figure 3.8 the results presented by Lopes [2] are displayed for the 5-J impact event along with the experimental results and the analytical elastic response. From Figure 3.7 it can be concluded that the analytical solution can accurately predict the dynamic response of low-velocity impact events and due to its relative
simplicity and low computation times, compared to advanced numerical methods, this method is suitable for fast decision making during initial design. The method can also be of great value for validating advanced numerical solutions. In case of Figure 3.8, it can be concluded that the numerical model needs to be revisited.

![Figure 3.8: Dynamic impact response obtained by Lopes (source: [2])](image)

In Figure 3.9, the comparison between the analytical method and experimental results for the 5-J impact test is presented in terms of the dynamic load-displacement response.

![Figure 3.9: Comparison of the analysis with experimental 5J impact test result published by Lopes [2]](image)

Note the degradation of the oscillation behaviour in the experimental test results in Figure 3.7. This indicates the effects of damage and fracture in the material. This behaviour
is absent in the analytical result, since no damage model is included. Nevertheless, the agreement with the test result is satisfactory.

### 3.1.5 Conclusions

An analytical dynamic impact model was obtained using classical plate theory. The model was validated by comparison with test results published by Olsson [4] and Lopes [2]. With the obtained model the impact load and the laminate out-of-plane displacement can be analysed. The bending caused by the out-of-plane displacements can be used to analyse the stresses in the laminate due to bending. In addition to the bending stresses the indenter introduces significant local stresses. Obtaining these stresses is essential for analysing delamination initiation. In the next chapter an analytical model for analysing local stresses due to indentation is obtained.
For understanding the damage behaviour of composite plates during impact, knowing the correct stress state of the indented composite plate is crucial. Especially the stress state of an indented panel within the contact region is rather complex. Often the out-of-plane shear stress distribution is assumed to be piecewise parabolic, with respect to the out-of-plane coordinate, with the maximum stress in the mid-plane [10]. In general this assumption is not correct within the contact region, due to the large compressive stress gradients (i.e. $\partial \sigma_z / \partial z$). For more detailed and accurate damage analysis in the vicinity of the indentation, more accurate representation of the out-of-plane stresses is necessary. Good solutions can be obtained through detailed finite element analyses. These analyses are however very time consuming since high mesh-density is needed in order to obtain the correct stresses in the contact zone. Therefore, in this work, effort is made to obtain a semi-analytical representation of the stress state of a transversely isotropic plate under quasi-static indentation. The analysis is split into two sections, in which the first section considers the local indentation stress and the second section handles the global stresses due to bending. The two solutions are finally superposed to get the complete stress field.

4.1 Local Indentation Stress Analysis

In the literature, often the inter-laminar shear stress distribution through the thickness is assumed to be piecewise parabolic with maximum shear stress in the mid-plane [10, 41]. However, research done on stress analysis of beams indented by a cylindrical indenter[42], indicates that this is generally not the case.
Love\cite{4} obtained solutions for semi-infinite bodies under pressure loading on the surface. Keer\cite{15} and Olsson\cite{13} used the semi-infinite half-plane solutions and combined these results with flexural stresses obtained using Plate Theories to approximate the stress response of indented plates with finite thickness. These methods will result in good approximations for laminates with large thicknesses with respect to the impactor diameter. A general methodology is provided in this work, which enables the determination of out-of-plane stresses due to indentation of plates without restrictions to laminate thickness and impactor diameter. For obtaining the out-of-plane stresses during impact, the analysis is performed in two parts. In the first part local stresses due to indentation pressure are obtained. In this part the plate is assumed to be transversely isotropic and the Boussinesq equations\cite{43} are used for obtaining the out-of-plane stresses due to indentation. In the second part, the inter-laminar shear stresses due to bending are obtained using the equilibrium equations, giving a representation of the inter-laminar stresses in the contact region due to quasi-static bending. Finally the results of both parts are combined to obtain the correct out-of-plane stresses, combining local indentation and global bending effects.

4.1.1 Out-of-Plane Stresses due to Axi-Symmetric quasi-static Indentation Pressure

It is assumed that the displacements due to indentation by a spherical indenter are axi-symmetric and therefore cylindrical polar coordinates are used for representation. The axi-symmetric assumption of the indentation is valid only in this case, because of the axi-symmetric spherical indenter that has been used. Hence the limitation of the this model, which is only applicable for such an indenter. Because most impact tests on composite panels are performed using spherical indenters, the model still has a wide range of applicability and can be useful for such applications.

The Boussinesq equations \cite{43}, which include two Laplace equations as representation of the displacement potentials, are used to obtain the displacement as well as the stress fields. The Laplace equations and the corresponding displacements in cylindrical coordinates are defined as:

\begin{align}
\nabla^2 B_o &= 0, \quad (4.1) \\
\nabla^2 B_z &= 0, \quad (4.2) \\
u_r &= -\frac{1}{4(1-\nu)} \frac{\partial}{\partial r} (B_o + zB_z), \quad (4.3) \\
u_z &= B_z - \frac{1}{4(1-\nu)} \frac{\partial}{\partial z} (B_o + zB_z), \quad (4.4)
\end{align}
where $u_r$ and $u_z$ are the displacement fields in the radial and the out-of-plane direction respectively, while $B_o$ and $B_z$ are the potentials used for describing the displacement fields. The coordinates are displayed in Figure 4.1. At this point it is important to emphasize on some restrictions that the Boussinesq equations introduce. The axi-symmetric nature of the displacement fields given in Equations 4.3 and 4.4 does not allow a distinction between $\nu_{13}$ and $\nu_{23}$ for a composite laminate. Therefore, in this work the value for $\nu_{12}$ is used for $\nu$.

![Figure 4.1: Cylindrical coordinates](image)

The strain-displacement equations in cylindrical coordinates are given to be [44]:

$$
\begin{align*}
\varepsilon_r &= \frac{\partial u_r}{\partial r}, \\
\varepsilon_\theta &= \frac{u_r}{r}, \\
\varepsilon_z &= \frac{\partial u_z}{\partial z}, \\
\gamma_{rz} &= \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right).
\end{align*}
$$

Linear elastic material is assumed, therefore Hooke’s law is used. In cylindrical coordinates this includes [44]:

$$
\begin{align*}
\sigma_r &= \lambda (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2\mu \varepsilon_r, \\
\sigma_z &= \lambda (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2\mu \varepsilon_z, \\
\sigma_{rz} &= 2\mu \gamma_{rz},
\end{align*}
$$

where $\lambda$ and $\mu$ represent the first and second Lame constants respectively. Assuming a single transverse shear stiffness, represented here with $\mu$, has an implication with respect to composite laminates. Composite materials show different transverse shear stiffness values with respect to the fibre orientation. In other words $G_{23} \neq G_{13}$, in fact we can state that in general $G_{23} < G_{13}$, where these parameters define the out-of-plane shear moduli with 1, 2 and 3 referring to the material coordinate system with 1 being the fibre direction. The assumption of a single transverse shear stiffness will
result in an overestimation of the stiffness in the "23-direction", which will eventually result in overestimation of the stresses in the "23-direction" (i.e. \( \sigma_{23} \)). We have assumed this effect as being conservative and used \( \mu \) as the transverse modulus for the remaining work. This assumption has enabled us to obtain a semi-analytical solution for the rather complex problem posed in Equations 4.1 to 4.4.

Substituting the strain-displacement relations into Hooke’s law results in the stresses as function of the displacements. Since here the out-of-plane stresses are of interest, only these stresses are considered from here on. The out-of-plane stresses in terms of the displacement fields are defined as:

\[
\sigma_z = \lambda \left( \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) + 2\mu \frac{\partial u_z}{\partial z}, \quad (4.12)
\]

\[
\sigma_{rz} = \mu \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right). \quad (4.13)
\]

Substitution of these stresses into the Boussinesq equations results in the stress formulations as functions of the Boussinesq displacement potentials:

\[
\sigma_z = \lambda \left[ -\frac{1}{4(1-\nu)} \frac{\partial^2}{\partial r^2} (B_o + zB_z) - \frac{1}{4(1-\nu)} \frac{1}{r} \frac{\partial}{\partial r} (B_o + zB_z) + \frac{\partial B_z}{\partial z} - \frac{1}{4(1-\nu)} \frac{\partial^2}{\partial z^2} (B_o + zB_z) \right] + 2\mu \left[ \frac{\partial B_z}{\partial z} - \frac{1}{4(1-\nu)} \frac{\partial^2}{\partial z^2} (B_o + zB_z) \right], \quad (4.14)
\]

\[
\sigma_{rz} = \mu \left[ -\frac{1}{4(1-\nu)} \frac{\partial^2}{\partial r \partial z} (B_o + zB_z) + \frac{\partial B_z}{\partial r} - \frac{1}{4(1-\nu)} \frac{\partial^2}{\partial r \partial z} (B_o + zB_z) \right]. \quad (4.15)
\]

Rearranging and simplifying these equations, the following expressions can be obtained:

\[
\sigma_z = \lambda \left[ -\frac{1}{4(1-\nu)} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial^2}{\partial z^2} \right) (B_o + zB_z) \right] + 2\mu \left[ \frac{\partial B_z}{\partial z} - \frac{1}{4(1-\nu)} \frac{\partial^2}{\partial z^2} (B_o + zB_z) \right], \quad (4.16)
\]

\[
\sigma_{rz} = \mu \left[ -\frac{1}{2(1-\nu)} \frac{\partial^2}{\partial r \partial z} (B_o + zB_z) + \frac{\partial B_z}{\partial r} \right]. \quad (4.17)
\]

Substituting \( \nabla^2 B_o = 0 \) and \( \nabla^2 B_z = 0 \) and rearranging gives:

\[
\sigma_z = (\lambda + 2\mu) \frac{\partial B_z}{\partial z} - \frac{\mu}{2(1-\nu)} \left( \frac{\partial^2 B_o}{\partial z^2} + z \frac{\partial^2 B_z}{\partial z^2} \right), \quad (4.18)
\]
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\[ \sigma_{rz} = \mu \left[ \frac{\partial B_z}{\partial r} - \frac{1}{2(1-\nu)} \left( \frac{\partial^2 B_o}{\partial r \partial z} + z \frac{\partial^2 B_z}{\partial r \partial z} \right) \right]. \quad (4.19) \]

For solving the set of partial differential equations, the Hankel transform technique \cite{45} is used. This method is quite efficient for axi-symmetric problems, allowing for the partial differential equations to be rewritten as ordinary differential equations in the Hankel space. The Hankel transform and its inverse transform are defined as:

\[ F(\zeta, z) = \int_0^\infty f(r, z) r J_0(\zeta r) \, dr, \quad (4.20) \]
\[ f(r, z) = \int_0^\infty F(\zeta, z) \zeta J_0(\zeta r) \, d\zeta, \quad (4.21) \]

where \( J_0(\zeta r) \) is the zeroth order Bessel function of the first kind.

The Laplace equations can be transformed to a second order linear homogeneous ODE in the Hankel space using the Hankel transform. For the Laplace equation \( \nabla^2 B_o = 0 \) this results in:

\[ \int_0^\infty \nabla^2 B_o(r, z) r J_0(\zeta r) \, dr = 0, \quad (4.22) \]
\[ = -\zeta^2 \hat{B}_o(\zeta, z) + \frac{d^2 \hat{B}_o(\zeta, z)}{dz^2} = 0, \quad (4.23) \]

where \("^H\) indicates the Hankel transform. The homogeneous ODE can be solved, which can be presented as:

\[ \hat{B}_o(\zeta, z) = C_1(\zeta) e^{\zeta z} + C_2(\zeta) e^{-\zeta z}. \quad (4.24) \]

The same analysis can be performed for the Laplace equation \( \nabla^2 B_z = 0 \) resulting:

\[ \hat{B}_z(\zeta, z) = C_3(\zeta) e^{\zeta z} + C_4(\zeta) e^{-\zeta z}. \quad (4.25) \]

To obtain the displacement potentials in the \( r - z \) space the inverse transform is used

\[ B_o(r, z) = \int_0^\infty \left( C_1(\zeta) e^{\zeta z} + C_2(\zeta) e^{-\zeta z} \right) \zeta J_0(\zeta r) \, d\zeta, \quad (4.26) \]
\[ B_z(r, z) = \int_0^\infty \left( C_3(\zeta) e^{\zeta z} + C_4(\zeta) e^{-\zeta z} \right) \zeta J_0(\zeta r) \, d\zeta. \quad (4.27) \]

Substituting \( B_o \) and \( B_z \) in the stress formulations given in Equations 4.18 and 4.19 results in:

\[ \sigma_z = \int_0^\infty \left\{ (\lambda + 2\mu) \left( C_3 e^{\zeta z} - C_4 e^{-\zeta z} \right) \zeta - \frac{\mu\zeta^2}{2(1-\nu)} \left[ C_1 e^{\zeta z} + C_2 e^{-\zeta z} + z \left( C_3 e^{\zeta z} + C_4 e^{-\zeta z} \right) \right] \right\} \zeta J_0(\zeta r) \, d\zeta, \quad (4.28) \]
\[
\sigma_{rz} = \mu \int_{0}^{\infty} \left\{ - \left( C_3 e^{\zeta z} + C_4 e^{-\zeta z} \right) \zeta + \frac{\zeta^2}{2(1-\nu)} \left( C_1 e^{\zeta z} - C_2 e^{-\zeta z} + z \left( C_3 e^{\zeta z} - C_4 e^{-\zeta z} \right) \right) \right\} \zeta J_1(r\zeta) \, d\zeta, \quad (4.29)
\]

where \( J_1(r\zeta) \) is the first order Bessel function of the first kind and \( C_i(\zeta), \, i = 1..4, \) are unknown functions which need to be determined.

For obtaining the functions \( C_i(\zeta) \) with \( i = 1..4, \) four boundary conditions are used, assuming frictionless contact behaviour:

\[
\begin{align*}
\sigma_z(r, -h/2) &= P(r), \quad (4.30) \\
\sigma_z(r, h/2) &= 0, \quad (4.31) \\
\sigma_{rz}(r, -h/2) &= \sigma_{rz}(r, -h/2) = 0, \quad (4.32)
\end{align*}
\]

where \( P(r) \) represents the contact pressure on the top surface.

These equations lead to the following four equations:

\[
(\lambda + 2\mu) \left( C_3(\zeta) e^{\zeta h} - C_4(\zeta) e^{-\zeta h} \right) \zeta - \frac{\mu \zeta^2}{2(1-\nu)} \left[ C_1(\zeta) e^{\zeta h/2} + C_2(\zeta) e^{-\zeta h/2} + \frac{h}{2} \left( C_3(\zeta) e^{\zeta h/2} + C_4(\zeta) e^{-\zeta h/2} \right) \right] = P(\zeta), \quad (4.33)
\]

\[
(\lambda + 2\mu) \left( C_3(\zeta) e^{-\zeta h} - C_4(\zeta) e^{\zeta h} \right) \zeta - \frac{\mu \zeta^2}{2(1-\nu)} \left[ C_1(\zeta) e^{-\zeta h/2} + C_2(\zeta) e^{\zeta h/2} - \frac{h}{2} \left( C_3(\zeta) e^{-\zeta h/2} + C_4(\zeta) e^{\zeta h/2} \right) \right] = 0, \quad (4.34)
\]

\[
- \left( C_3 e^{\zeta h/2} + C_4 e^{-\zeta h/2} \right) \zeta + \frac{\zeta^2}{2(1-\nu)} \left( C_1 e^{\zeta h/2} - C_2 e^{-\zeta h/2} + \frac{h}{2} \left( C_3 e^{\zeta h/2} - C_4 e^{-\zeta h/2} \right) \right) = 0, \quad (4.35)
\]

\[
- \left( C_3 e^{-\zeta h/2} + C_4 e^{\zeta h/2} \right) \zeta + \frac{\zeta^2}{2(1-\nu)} \left( C_1 e^{-\zeta h/2} - C_2 e^{\zeta h/2} - \frac{h}{2} \left( C_3 e^{-\zeta h/2} - C_4 e^{\zeta h/2} \right) \right) = 0, \quad (4.36)
\]
where \( \hat{P}(\zeta) \) is the Hankel transform of the contact stress on the upper surface \( P(r) \).

Writing these equations in matrix format gives:

\[
\begin{bmatrix}
A & B & C & D \\
B & A & E & F \\
G & H & I & J \\
-H & -G & J & I
\end{bmatrix}
\begin{bmatrix}
C_1(\zeta) \\
C_2(\zeta) \\
C_3(\zeta) \\
C_4(\zeta)
\end{bmatrix}
= \begin{bmatrix}
\hat{P}(\zeta) \\
0 \\
0 \\
0
\end{bmatrix},
\]

with:

\[
A = -\frac{\mu\zeta^2}{2(1-\nu)}e^{\frac{\zeta h}{2}},
B = -\frac{\mu\zeta^2}{2(1-\nu)}e^{-\frac{\zeta h}{2}},
C = \left[ (\lambda + 2\mu)\zeta - \frac{\mu\zeta^2}{2(1-\nu)} \frac{h}{2} \right] e^{\frac{\zeta h}{2}},
D = -\left[ (\lambda + 2\mu)\zeta - \frac{\mu\zeta^2}{2(1-\nu)} \frac{h}{2} \right] e^{-\frac{\zeta h}{2}},
E = \left[ (\lambda + 2\mu)\zeta + \frac{\mu\zeta^2}{2(1-\nu)} \frac{h}{2} \right] e^{-\frac{\zeta h}{2}},
F = \left[ - (\lambda + 2\mu)\zeta + \frac{\mu\zeta^2}{2(1-\nu)} \frac{h}{2} \right] e^{\frac{\zeta h}{2}},
G = \frac{\zeta^2}{2(1-\nu)}e^{\frac{\zeta h}{2}},
H = -\frac{\zeta^2}{2(1-\nu)}e^{-\frac{\zeta h}{2}},
I = \left[ -\zeta + \frac{\zeta^2}{2(1-\nu)} \frac{h}{2} \right] e^{\frac{\zeta h}{2}},
J = \left[ -\zeta - \frac{\zeta^2}{2(1-\nu)} \frac{h}{2} \right] e^{-\frac{\zeta h}{2}}.
\]

Solving the \( 4 \times 4 \) matrix equation results in \( C_i(\zeta) \ i = 1..4 \).

### 4.1.2 Inter-Laminar Stress due to quasi-static Bending

In addition to the local indentation stress, more global bending stresses will also be present due to the bending of the plate. Especially for thin laminates impacted by a large radius impactor, the stresses due to bending can become dominant. We obtain the out-of-plane shear stresses due to bending by utilizing the classical plate theory, after which the total stress due to indentation of a plate will be the superposition of the indentation stress and the bending stress.

From the displacement of the plate, the stresses and strains due to bending can be obtained. Assuming that bending is the main driver for the in-plane stresses, the following
can be stated about the relation between the out-of-plane displacements and the in-plane displacements for the plate using the classical plate theory:

\[
\begin{align*}
    u &= -z \frac{\partial w}{\partial x}, \\
    v &= -z \frac{\partial w}{\partial y},
\end{align*}
\]

which results in the following strain field:

\[
\begin{align*}
    \epsilon_x &= \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2}, \\
    \epsilon_y &= \frac{\partial u}{\partial y} = -z \frac{\partial^2 w}{\partial y^2}, \\
    \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = - \left( \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y} \right) = -2z \left( \frac{\partial^2 w}{\partial x \partial y} \right).
\end{align*}
\]

The stresses in each layer due to the strains are obtained using Hooke’s law, in matrix notation this law can be represented as:

\[
\sigma^{(x,y)} = T^{-1} C^{(1,2)} T \epsilon^{(x,y)} = C^{(x,y)} \epsilon^{(x,y)},
\]

where the superscripts \((x, y)\) and \((1, 2)\) refer to the global Cartesian and local material coordinate systems respectively with:

\[
C = \begin{bmatrix}
    E_{11} & \frac{E_{12}E_{22}}{1-\nu_{12}\nu_{21}} & 0 \\
    \frac{E_{12}E_{22}}{1-\nu_{12}\nu_{21}} & E_{22} & 0 \\
    0 & 0 & G_{12}
\end{bmatrix}
\]

and

\[
T = \begin{bmatrix}
    \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\
    \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\
    -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta
\end{bmatrix}.
\]

Substitution of the stress and strain formulations results in the in-plane stresses as function of the displacement field, which can be written as:

\[
\begin{align*}
    \sigma_x &= -z \left( C_{11}^{(x,y)} \frac{\partial^2 w}{\partial x^2} + C_{12}^{(x,y)} \frac{\partial^2 w}{\partial y^2} + 2C_{13}^{(x,y)} \frac{\partial^2 w}{\partial x \partial y} \right), \\
    \sigma_y &= -z \left( C_{12}^{(x,y)} \frac{\partial^2 w}{\partial x^2} + C_{22}^{(x,y)} \frac{\partial^2 w}{\partial y^2} + 2C_{23}^{(x,y)} \frac{\partial^2 w}{\partial x \partial y} \right), \\
    \sigma_{xy} &= -z \left( C_{13}^{(x,y)} \frac{\partial^2 w}{\partial x^2} + C_{23}^{(x,y)} \frac{\partial^2 w}{\partial y^2} + 2C_{33}^{(x,y)} \frac{\partial^2 w}{\partial x \partial y} \right).
\end{align*}
\]
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From the in-plane stresses, the out-of-plane stresses can be obtained using the equilibrium equations in absence of body forces, given as:

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0, \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0, \\
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= 0.
\end{align*}
\] (4.49) (4.50) (4.51)

The integration of the first two equations over \( z \) will lead to the inter-laminar stresses \( \sigma_{xz} \) and \( \sigma_{yz} \) resulting in:

\[
\sigma_x^{(i)} = \left[ C_{11}^{(x,y)} \frac{\partial^3 w}{\partial x^3} + \left( C_{12}^{(x,y)} + 2C_{33}^{(x,y)} \right) \frac{\partial^3 w}{\partial x \partial y^2} + 3C_{13}^{(x,y)} \frac{\partial^3 w}{\partial x^2 \partial y} + C_{23}^{(x,y)} \frac{\partial^3 w}{\partial y^3} \right] \frac{1}{2} z^2 + C_1^{(i)},
\] (4.52)

\[
\sigma_y^{(i)} = \left[ C_{13}^{(x,y)} \frac{\partial^3 w}{\partial x^3} + 3C_{23}^{(x,y)} \frac{\partial^3 w}{\partial x \partial y^2} + \left( 2C_{33}^{(x,y)} + C_{12}^{(x,y)} \right) \frac{\partial^3 w}{\partial x^2 \partial y} + C_{22}^{(x,y)} \frac{\partial^3 w}{\partial y^3} \right] \frac{1}{2} z^2 + C_2^{(i)}.
\] (4.53)

\( C_1 \) and \( C_2 \) are integration constants in \( z \) and are functions of \((x, y)\), which are obtained using layer by layer compatibility and the index \( i \) refers to the \( i^{th} \) layer in the layup.

The compatibility equations for \( \sigma_{xz} \) are discussed in more detail.

For \( z_1 = -h/2 \), \( \sigma_{xz} = 0 \) resulting in:

\[
\begin{align*}
C_{11} &= - \left[ C_{11}^{(x,y)} \frac{\partial^3 w}{\partial x^3} + \left( C_{12}^{(x,y)} + 2C_{33}^{(x,y)} \right) \frac{\partial^3 w}{\partial x \partial y^2} + 3C_{13}^{(x,y)} \frac{\partial^3 w}{\partial x^2 \partial y} + C_{23}^{(x,y)} \frac{\partial^3 w}{\partial y^3} \right] \frac{1}{2} \left( \frac{-h}{2} \right)^2.
\end{align*}
\] (4.54)
For the remaining layers,

\[
C_{1i} = \left[ C_{11}^{(x,y)} \frac{\partial^3 w}{\partial x^3} + \left( C_{12}^{(x,y)} + 2C_{33}^{(x,y)} \right) \frac{\partial^3 w}{\partial x \partial y^2} + 
3C_{13}^{(x,y)} \frac{\partial^3 w}{\partial x^2 \partial y} + C_{23}^{(x,y)} \frac{\partial^3 w}{\partial y^3} \right] \frac{1}{2} z_i^2
\]

\[
- \left[ C_{11}^{(x,y)} \frac{\partial^3 w}{\partial x^3} + \left( C_{12}^{(x,y)} + 2C_{33}^{(x,y)} \right) \frac{\partial^3 w}{\partial x \partial y^2} + 
3C_{13}^{(x,y)} \frac{\partial^3 w}{\partial x^2 \partial y} + C_{23}^{(x,y)} \frac{\partial^3 w}{\partial y^3} \right] i_{i-1} \frac{1}{2} z_i^2 + C_{1_{i-1}}, \tag{4.55}
\]

where \(i\) indicating layer-\(i\) within the laminate. The constants for \(\sigma_{yz}\) are obtained in a similar manner. It should be noted at this point that Equation 4.51 will only result in the trivial solution \(\sigma_z = 0\) and is therefore not considered.

### 4.1.3 Plate Displacement

Classical Plate Theory (CPT) is used to obtain the plate displacement due to the indentation load. Assuming that \(D_{1,6} \approx D_{2,6} \approx 0\) with respect to the other entities in the D-matrix and assuming a quasi-static state, CPT provides a partial differential equation, representing the equation of motion for the plate. This equation is given to be [39]:

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 2 \left( D_{12} + 2D_{66} \right) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = q, \tag{4.56}
\]

where \(w\) is the out-of-plane displacement and \(q\) represents the out-of-plane indentation loading.

The solution for the differential equation is found in the form of Fourier series

\[
w = \sum_{j=1,3..}^{\infty} \sum_{k=1,3..}^{\infty} W_{jk} \alpha_{jk}, \tag{4.57}
\]

with

\[
W_{jk} = \cos \left( \frac{j \pi x}{a} \right) \cos \left( \frac{k \pi y}{b} \right). \tag{4.58}
\]

Substitution of Equation 4.58 into Equation 4.56 gives:

\[
\sum_{jk}^{\infty} \left\{ \left[ D_{11} \left( \frac{j \pi}{a} \right)^4 + 2 \left( D_{12} + 2D_{66} \right) \left( \frac{j \pi}{a} \right)^2 \left( \frac{k \pi}{b} \right)^2 + D_{22} \left( \frac{k \pi}{b} \right)^4 \right] \alpha_{jk} \right\} \cos \left( \frac{j \pi x}{a} \right) \cos \left( \frac{k \pi y}{b} \right) = q. \tag{4.59}
\]
Note that Equation 4.59 is the quasi-static version of the dynamic displacement field given in Equation 3.5.

Rewriting the loading $q$ as a concentrated load (i.e. $F \delta(x = 0) \delta(y = 0)$), multiplying both sides of Equation 4.59 with $\cos\left(\frac{j \pi a}{a}\right) \cos\left(\frac{k \pi b}{b}\right)$ and integrating over the plate area, the following can be obtained.

$$\alpha_{jk} = \frac{4F}{ab \omega_{jk}^2},$$  \hspace{1cm} (4.60)

with

$$\omega^2 = \left[D_{11} \left(\frac{j \pi a}{a}\right)^4 + 2(D_{12} + 2D_{66}) \left(\frac{j \pi a}{a}\right)^2 \left(\frac{k \pi b}{b}\right)^2 + D_{22} \left(\frac{k \pi b}{b}\right)^4\right],$$  \hspace{1cm} (4.61)

resulting in the displacement field

$$w(x, y) = \frac{4F}{ab} \sum_{j=1,3,\ldots}^{\infty} \sum_{k=1,3,\ldots}^{\infty} \frac{1}{\omega_{jk}^2} \cos\left(\frac{j \pi x}{a}\right) \cos\left(\frac{k \pi y}{b}\right).$$  \hspace{1cm} (4.62)

The Hertzian contact formulation [46], given in Equations 2.1 to 2.3 can be written in the form:

$$F = \frac{4}{3} \frac{\sqrt{R}}{1 - \nu_p^2 + \frac{1 - \nu_i^2}{E_i}} \delta^3,$$  \hspace{1cm} (4.63)

with $\nu_p$, $\nu_i$, $E_p$, $E_i$ being the Poisson’s ratios and the stiffness values for the plate and impactor respectively. $\delta$ represents the indentation and is defined in Equation 3.1.

Substituting Equations 4.63 and 3.1 into Equation 4.62 for the point of contact $(x,y) = (0,0)$ gives

$$w_p = \frac{16\sqrt{R} (w_i - w_p)^3}{3 \left(1 - \nu_p^2 + \frac{1 - \nu_i^2}{E_i}\right) ab} \sum_{j=1,3,\ldots}^{\infty} \sum_{k=1,3,\ldots}^{\infty} \frac{1}{\omega_{jk}^2}.$$  \hspace{1cm} (4.64)

For a certain impactor displacement $w_i$, Equation 4.64 gives the plate displacement at the point of impact, where in this displacement based quasi-static case, $w_i$ is chosen in a similar way as the FE analysis discussed in the following section. Equations 3.1 and 4.63 are substituted in Equation 4.62, resulting in the displacement field which is then substituted in Equations 4.52 and 4.53 to obtain the out-of-plane shear stress due to bending. Note that there is no contribution to the out-of-plane normal stress due to bending because, as already mentioned, Equation 4.51 gives $\sigma_n = 0$.

The out-of-plane shear stress due to bending and the local out-of-plane shear stress due to indentation are finally added to result the complete out-of-plane shear stress.
4.1.4 Validation

In order to validate the proposed model for obtaining the out-of-plane stresses due to spherical indentation, two numerical models using ABAQUS FE are created and the results are used for comparison. The reference models include two quasi-static indented plates, one with a 16-ply [0/45/90/-45]_{2S} layup and the second consisting of monolithic aluminium. Both of the models are simply supported along all sides. The dimensions of the models are given in Figure 4.2.

The material properties used for the analysis are given in Table 4.1 for the composite plate, which originate from Lopes [2]. In Table 4.2 the properties for the aluminium plate and the steel impactor are presented. Because no out-of-plane properties were provided, the value for \( G_{12} \) was used for the out-of-plane shear moduli \( G_{13} \) and \( G_{23} \), overestimating the value for \( G_{23} \).

<table>
<thead>
<tr>
<th>( E_{11} ) [GPa]</th>
<th>( E_{22} ) [GPa]</th>
<th>( \nu_{12} ) [-]</th>
<th>( G_{12} ) [GPa]</th>
<th>( G_{13} ) [GPa]</th>
<th>( G_{23} ) [GPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>135</td>
<td>9.6</td>
<td>0.3</td>
<td>5.3</td>
<td>5.3</td>
<td>5.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( E_i ) [GPa]</th>
<th>( \nu_i ) [-]</th>
<th>( E_{al} ) [GPa]</th>
<th>( \nu_{al} ) [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>0.3</td>
<td>70</td>
<td>0.3</td>
</tr>
</tbody>
</table>

where the index \( i \) represents the impactor properties.

A frictionless contact model is used for modelling the contact between the impactor and the plates. Figure 4.3 gives a detailed representation of the mesh used for the numerical analysis with a minimum element size at point of impact of \( 50 \cdot 10^{-3}mm \times 50 \cdot 10^{-3}mm \times 46.25 \cdot 10^{-3}mm \).
An indenter-displacement of 2mm is applied in small steps. At two different steps \( \sigma_{xz} \) and \( \sigma_z \) are obtained at the edge of the contact region and at the point of contact respectively. At each step the contact load from the FE result is used to obtain the analytical out-of-plane stresses for comparison.

In order to solve Equation 4.37, the Hankel transform of the load distribution on the upper surface \( \hat{P}(\zeta) \) is needed. In Hertzian contact formulation the contact pressure is given in Equation 2.3, which can be rewritten in polar coordinates as:

\[
P(r) = \frac{3F}{2\pi c^3} \sqrt{c^2 - r^2} \quad \text{For } r \leq c, \tag{4.65}
\]

\[
P(r) = 0 \quad r > c \quad \tag{4.66}
\]

where \( F \) and \( c \) represent the contact force and the contact area respectively, where the contact area is given by Equation 2.2.

The Hankel transform of the load distribution then becomes:

\[
\hat{P}(\zeta) = \int_0^\infty P(r)rJ_0(\zeta r) \, dr = \int_0^a P(r)rJ_0(\zeta r) \, dr + \int_a^\infty P(r)rJ_0(\zeta r) \, dr, \tag{4.67}
\]

which results in

\[
\hat{P}(\zeta) = \int_0^a P(r)rJ_0(\zeta r) \, dr = -\frac{3F}{2} \frac{\cos(\alpha\zeta)\zeta c - \sin(\alpha\zeta)}{\zeta^3\pi c^3}. \tag{4.68}
\]

For obtaining the out-of-plane stresses, the matrix Equation 4.37 gives the necessary functions \( G_i(\zeta) \) \( i = 1..4 \), after which the integrals defined in Equations 4.28 and 4.29 are solved numerically. The first and second Lame constants were determined assuming
transverse isotropy, resulting in:

\begin{align*}
\lambda & = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \\
\mu & = G_{12}
\end{align*}

(4.69) \hspace{1cm} (4.70)

Figures 4.4 and 4.5 show the comparison between the finite element results and the analytical results for \( \sigma_{rz} \) and \( \sigma_{zz} \) for two different indentation loads for the isotropic Aluminium plate, while Figures 4.6 and 4.7 show the comparison between finite element results and analysis results for the 16-ply composite plate. The stresses presented in these graphs are obtained at the point of contact for \( \sigma_{zz} \) and at the edge of the contact region for \( \sigma_{rz} \). In these figures, \( z = 0 \) corresponds to the mid-plane of the laminate and is defined through the thickness as is shown in Figure 4.1.

**Figure 4.4:** \( \sigma_{rz} \) and \( \sigma_{zz} \) as function of the z-coordinate for an indentation load of \( F = 357.98N \) for the isotropic case.
Figure 4.5: $\sigma_{yz}$ and $\sigma_{zz}$ as function of the z-coordinate for an indentation load of $F = 2.0 \cdot 10^3 N$ for the isotropic case

Figure 4.6: $\sigma_{yz}$ and $\sigma_{zz}$ as function of the z-coordinate for an indentation load of $116.9 N$ for the composite case
4.1.5 Result Discussion

The results show a difference between the comparison of the isotropic analysis and the composite analysis. This difference is mainly due to the occurrence of local in-plane effects. Equation 4.12 and 4.13 are used to obtain the out-of-plane stresses, however these stresses are being influenced by the in-plane stresses due to local bending introduced by the indentation. For similar plate dimensions and boundary conditions, composites show significant jumps in the in-plane stress gradients through the thickness because of the sudden changes of the stiffness compared to isotropic materials, resulting in the jumps shown in Figures 4.6 and 4.7.

The influence of the in-plane bending can be accounted for by solving the system of axi-symmetric equilibrium equations shown in Equations 4.71 and 4.72.

\[
\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r} (\sigma_r - \sigma_\theta) = 0, \tag{4.71}
\]

\[
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r} \sigma_{rz} = 0. \tag{4.72}
\]

It should be noted that the out-of-plane stresses are only valid for stresses below damage stress. In case of Epoxy based composites the inter-laminar shear strength lies in general between 90 MPa and 160 MPa. These stress levels are reached at moderate indentations allowing the proposed model to obtain the maximum stress with sufficient accuracy in order to capture damage initiation. Including the in-plane effects will significantly
increase the complexity of the method and therefore reduce its efficiency, without much contribution to the accuracy of analysing damage initiation.

Another aspect that needs to be discussed are the horizontal lines that are visible in the numerical results. In Figure 4.8, which shows the comparison between the analytical results and the numerical results for the composite laminate at \( F = 1.0294 \times 10^3 \) N, these horizontal lines are emphasized.

![Figure 4.8: \( \sigma_{zz} \) and \( \sigma_{zz} \) as function of the z-coordinate for an indentation load of \( 1.03 \times 10^3 \) N with the arrows emphasizing the horizontal lines due to post processing](image)

These points show a constant inter-laminar shear stress as function of the z-coordinate. Due to the large indentation load and the small out-of-plane stiffness of the composite material, significant distortion will occur in the elements. Especially at the edge of the contact region, the large shear stress gradient through the thickness in combination with the low stiffness of the 90 deg layers will result in the collapse of elements. This is schematically depicted in Figure 4.9, where node A will collapse towards node B due to large shear stresses at the edge of contact.
For comparison between the semi-analytical result and the FE results, it was chosen to plot the results as function of the thickness. For this reason, the obtained stresses from the FE results are projected onto the undeformed configuration for a clear comparison as function of the thickness. The horizontal lines correspond to the collapsed nodes. In the deformed configuration these nodes have z-coordinates that are almost identical due to collapse, which results in similar values for the shear stress. During projection of the stresses to the undeformed configuration, the stress values do not change while the coordinate values do change, hence the horizontal lines in Figure 4.9.

The presented methodology for predicting the out-of-plane stresses for quasi-static indentation on transverse isotropic plates with finite thickness was validated using detailed finite element analyses for two material systems. The out-of-plane stresses at the point of contact and at the edge of the contact region were used for comparison with the finite element results. Good agreement was achieved for both material systems. The methodology was shown to be very efficient in terms of necessary computation time, without any restrictions to plate thickness and indenter radius, accounting for local indentation effects as well as the more global bending effects.

### 4.2 Damage Mechanics and Failure Criteria Discussion

Delaminations in composite laminates have a significant effect on the residual strength of laminates after impact. Even though matrix cracks do not affect the residual strength of composite laminates significantly, they could introduce delaminations [39] and therefore they need to be addressed. Choi et al. [31] propose a failure criterion for matrix cracks
in composite materials, which is given as:

\[
\left( \frac{\sigma_{22}}{X_2} \right)^2 + \left( \frac{\tau_{23}}{S_i} \right)^2 = 1,
\]

(4.73)

where the subscripts 2 and 3 are the local material coordinates transverse to the fibre direction and in the out-of-plane direction respectively and \( X_2 \) and \( S_i \) are the material transverse and shear strength respectively.

This model indicates that matrix cracks occur due to transverse tensile and out-of-plane shear. It is however not easy to determine the matrix crack initiation strength \( S_i \), since the matrix material shows significant plasticity in shear as is shown in a typical in-plane-shear test result in Figure 4.10.\(^1\)

\[\text{FIGURE 4.10: In-plane shear experimental test result}\]

In the transverse direction, the material shows negligible plasticity, which makes it easier to determine the matrix crack initiation strength. In Figure 4.11, a typical transverse tensile test result is shown.

\(^{1}\)The tests are performed by the National Aerospace Laboratory in The Netherlands (NLR) on a Hexply AS4/8552 material system
It is therefore easier to understand the phenomenon of matrix crack initiation by using the transverse stress in the principal directions. In other words:

\[
\left( \frac{\tilde{\sigma}_{2+}}{X_{2+}} \right)^2 = 1,
\]

(4.74)

where \( \tilde{\sigma}_{2+} \) is the tensile principal stress and is defined as:

\[
\tilde{\sigma}_{2+} = \frac{\sigma_{22} + \sigma_{33}}{2} + \sqrt{\left( \frac{\sigma_{22} - \sigma_{33}}{2} \right)^2 + \tau_{23}^2}.
\]

(4.75)

For an impact event on a relatively thick laminate (with respect to the impactor radius), the stress state relevant for matrix cracks as described in Equation 4.73 can be segregated in an out-of-plane shear stress distribution due to local indentation and in-plane transverse stresses due to global bending. This segregation is schematically depicted in Figure 4.12.

**Figure 4.11:** Transverse tensile experimental test result

**Figure 4.12:** Contribution of transverse shear stress on matrix cracks in composite layups due to impact
As discussed in section 4.1, the maximum out-of-plane shear stress due to local indentation occurs in the vicinity of the indented (top) surface, while the maximum tensile transverse stress occurs at the opposite (bottom) surface. According to Choi et al. [31], these maximum stresses will introduce matrix cracks.

The segregation of the stresses responsible for matrix crack initiation and the use of the principal stresses, allows the analysis of the matrix cracks and the corresponding damage modes. In Figure 4.13 the most probable matrix crack initiation locations for an impacted composite layup are shown along with the corresponding contributors.

The matrix initiation damage mode can be analysed by obtaining the principal direction, which is given by:

\[ \alpha_p = \frac{1}{2} \arctan \left( \frac{2\tau_{23}}{\sigma_{22} - \sigma_{33}} \right). \]  

(4.76)

Assuming that the local indentation causes a shear dominated stress state in the vicinity of the top surface (i.e. \( \sigma_{i3} \gg \sigma_{ii} \ i = 1,2 \)) and the global bending causes a transverse tensile dominated stress state at the bottom surface (i.e. \( \sigma_{ii} \gg \sigma_{i3} \ i = 1,2 \)), the corresponding matrix crack initiation modes, according to Equation 4.76, become \( \alpha_p \approx \frac{1}{2} \arctan(\infty) = \pi/4 \) for the out-of-plane shear dominated region and \( \alpha_p \approx \frac{1}{2} \arctan(0) = 0 \) for the transverse tensile dominated region. In Figure 4.14 the matrix crack initiation modes for out-of-plane shear and the transverse tensile dominated regions are shown schematically.
The matrix crack modes can also be observed in experimental test results. In Figure 4.15 a section cut image of an impacted composite plate with a \([45/0^-/45/90]_{2S}\) layup configuration, impacted with an 40 J energy with an impactor mass and radius of 2.441 kg and 8 mm respectively, is shown.
Figure 4.15 shows the out-of-plane shear stress induced and the transverse tensile stress induced matrix cracks and the corresponding damage modes.

4.3 Conclusions

A mathematical model was developed to obtain the three dimensional stresses due to indentation in the contact region. The analysis is based on an axi-symmetric assumption, where the Boussinesq equations were used as representation of the displacement field due to indentation. The obtained partial differential equations which resulted from the Boussinesq equations, strain-displacement relations and the constitutive equations were solved using the Hankel transform. In addition to the local indentation stresses the bending stresses were obtained using classical plate theory, after which the results were combined by superposition in order to get the complete stress field.

The results were successfully validated through comparison with numerical results for an isotropic plate and a quasi-isotropic plate. The obtained model showed to be computationally efficient, especially when comparing to the numerical model used for comparison.
Chapter 5

Fracture Mechanics and Delamination Analysis

The existence of size effects on strength properties of solids have been shown by Leonardo da Vinci (1452-1519) [47] long before the existence of Fracture Mechanics. Da Vinci tested the strength of iron wires with constant cross section for different wire lengths. His experiments showed that the strength of the wire is inversely proportional to the wire length.

The hypothesis that explains this behaviour is the fact that in damage mechanics the material is often assumed to be in a pristine condition, while in reality all solids contain flaws which can affect the strength of the material. Assuming that the existence of flaws in materials is probabilistic, the larger the material volume the higher the probability of flaws of a given size being present. This hypothesis explains da Vinci’s findings not only by explaining the reduction of strength with increasing length, but also they explain the convergence behaviour which is observed as the tested specimen becomes larger. Many tests, similar to the tests performed by da Vinci, have been conducted confirming this hypothesis. Irwin [48] performed tests on glass fibres, showing a logarithmic decrease of the strength as function of the logarithm of the specimen length and he also showed the dependency of the strength behaviour on existing flaws by conducting the test on specimens with little handling damage and specimens with larger initial damage. Griffith [49, 50] extended the theorem of minimum potential energy by including energy dissipation due to crack growth. The energy balance according to the law of conservation of energy can be expresses as:

\[ W = \hat{E} + \hat{K} + \hat{\Gamma} \quad (5.1) \]
Where $W$ is the work performed per time unit, $E$ and $K$ are the internal end kinetic energy per time unit respectively and $\dot{\Gamma}$ is the energy per time unit needed to increase an existing crack. The internal energy can be subdivided into an elastic part $U^e$ and an in-elastic or plastic part $U^p$. Assuming that the behaviour is quasi-static, $K$ can be neglected. The differentials with respect to time can be rewritten as:

$$\frac{\partial}{\partial t} = \frac{\partial A}{\partial t} \frac{\partial}{\partial A}$$

(5.2)

where $A$ is area of the crack surface. Substituting this into the energy balance given in Equation 5.1 gives

$$\frac{\partial W}{\partial A} = \left( \frac{\partial U^e}{\partial A} + \frac{\partial U^p}{\partial A} \right) + \frac{\partial \Gamma}{\partial A}$$

(5.3)

Substituting $\Pi = U^e - W$, where $\Pi$ is the potential energy, results in

$$-\frac{\partial \Pi}{\partial A} = \frac{\partial U^p}{\partial A} + \frac{\partial \Gamma}{\partial A}$$

(5.4)

Equation 5.4 shows that the change in potential energy with respect to the crack area equals the change of the plastic energy with respect to the crack area plus the change of energy needed for crack growth with respect to the crack area.

Griffith simplified the relation given in Equation 5.4 by assuming brittle material, which implies that the energy dissipated due to plasticity can be neglected. In other words:

$$-\frac{\partial \Pi}{\partial A} = \frac{\partial W}{\partial A} - \frac{\partial U^e}{\partial A} = \frac{\partial \Gamma}{\partial A} \equiv G$$

(5.5)

This theorem, often written as:

$$G = \frac{\partial W}{\partial A} - \frac{\partial U^e}{\partial A}$$

(5.6)

is referred to as Griffith's theorem, where $G$ represents the available energy for crack growth. In general, when neither the load nor the displacement are predefined (e.g. in an impact event), both the displacement and the load will change during crack growth. However two limiting cases can be recognized, where in the first case the load is assumed to remain constant during crack growth known as the "dead-load" condition and a second case where the displacement is assumed to remain constant during crack growth known as "fixed-grips" condition.

According to Clapeyron's theorem [51], in the dead-load condition, linear elastic behaviour leads to:

$$\frac{\partial W}{\partial A} = 2 \frac{\partial U^e}{\partial A}$$

(5.7)
Substituting this in Equation 5.6 gives:

\[ G = \frac{\partial U^e}{\partial A} \]  

(5.8)

In the fixed-grips condition, since the displacement remains constant, the change of work is eliminated from Equation 5.6 resulting in:

\[ G = -\frac{\partial U^e}{\partial A} \]  

(5.9)

Griffith’s theory is very powerful for brittle materials, as is indicated by the assumption of the lack of plasticity. Therefore, the use of this theory for epoxy based composite materials, which show very little plasticity, has shown to be very useful [51]. In section 5.1, this theory is used to analyse the delamination behaviour of a composite beam. The results are compared to published bench mark solutions based on Finite Element analyses for validation. This will increase the confidence of using this method for more complex plate structures, which is discussed in section 5.2.

5.1 Fracture Mechanics of a Beam

Delamination is believed to be one of the dominant failure modes in most layered materials [52]. Due to the complex stress state around cracks, often finite element analyses are used for obtaining the delamination behaviour. Because these finite element based analyses often lead to high computation times, effort is made to use Griffith’s theory in order get an estimate of the delamination behaviour in an analytical manner. Krueger [5] obtained a benchmark model for analyzing delamination initiation and progression in an end-notched-flexure beam with an initial crack at the centre plane, using finite elements in combination with the Virtual Crack Closure Technique (VCCT) [53]. Krueger used a [0]_{24} layup made of IM7/8552 carbon/epoxy system. A schematic representation of the beam is shown in Figure 5.1.
The dimensions and material properties used by Krueger [5] are given in Figure 5.2.

\[ a = 25.4 \text{ mm} \quad L = 101.6 \text{ mm} \quad 2h = 4.5 \text{ mm} \]

The obtained results for delamination initiation and growth are used in the following section to validate the applicability and accuracy of Griffith's method for fracture initiation and growth discussed in the previous section.

### 5.1.1 Beam Single Delamination Analysis

In order to implement Griffith's theorem given in Equation 5.6, the displacement field of the beam under the loading and boundary conditions as shown in Figure 5.2 is obtained. The displacement field is subsequently used to analyse delamination initiation.
and delamination growth. These steps are discussed in the following sub-sections.

5.1.1.1 Displacement Field of a Delaminated Beam

As mentioned before, in this sub-section the displacement field of the beam shown in Figure 5.1 is obtained. The Euler-Bernoulli beam equation is used as representation, which is given as:

\[
\frac{\partial^2 w}{\partial x^2} = \frac{M}{EI}
\]  

(5.10)

where \( M \) is the bending moment in the beam, \( E \) is the material’s Young’s modulus and \( I \) is the moment of inertia of the beam.

As shown in Figure 5.3, the bending moment in the beam is given by \( M = \frac{F}{2}x \). Substituting in Equation 5.10 and integrating twice with respect to \( x \) results in the displacement fields of the delaminated and undelaminated section:

\[
w_d = \frac{Fx^3}{12EI_d} + C_1x + C_2 \quad 0 \leq x \leq a
\]  

(5.11)

\[
w_u = \frac{Fx^3}{12EI_u} + C_3x + C_4 \quad a \leq x \leq L
\]  

(5.12)

\[
w_{ur} = \frac{Fx^3}{12EI_u} + C_5x + C_6 \quad L \leq x \leq 2L
\]  

(5.13)

where \( C_1 \) to \( C_4 \) are integration constants and the subscripts \( d \) and \( u \) refer to the delaminated and undelaminated section respectively. The boundary conditions corresponding with the beam shown in Figure 5.1 are given to be:

\[
w_d(x = 0) = 0
\]  

(5.14)

\[
w_u(x = 2L) = 0
\]  

(5.15)
Displacement and rotation compatibility are enforced between the delaminated and undelaminated regions. In other words, the compatibility conditions between the delaminated and un-delaminated sections are:

\[
\frac{\partial w_d}{\partial x} \bigg|_{x=a} = \frac{\partial w_{ul}}{\partial x} \bigg|_{x=a}
\]

\(w_d(x = a) = w_{ul}(x = a)\) \hspace{1cm} (5.17)

\[
\frac{\partial w_{ul}}{\partial x} \bigg|_{x=L} = \frac{\partial w_{ur}}{\partial x} \bigg|_{x=L}
\]

\(w_{ul}(x = L) = w_{ur}(x = L)\) \hspace{1cm} (5.18)

Using the boundary conditions and the compatibility equations, the constants of Equations 5.11 to 5.13 are solved and the resulting integration constants are given as:

\[
C_1 = -\frac{F \left(3a^2L(J_u - I_d) + 3I_dL^3 + a^3(I_d - I_u)\right)}{12I_dLEI_u}
\]

\(C_2 = 0\) \hspace{1cm} (5.20)

\[
C_3 = -\frac{F \left(3I_dL^3 + a^3(I_d - I_u)\right)}{12I_dLEI_u}
\]

\(C_4 = \frac{Fa^3(I_d - I_u)}{6I_dI_u}\) \hspace{1cm} (5.21)

\[
C_5 = -\frac{F \left(a^3(I_d - I_u) + 9I_dL^3\right)}{12I_dLEI_u}
\]

\(C_6 = \frac{F \left(I_dL^3 + a^3(I_d - I_u)\right)}{6EI_dI_u}\) \hspace{1cm} (5.22)

\(C_7 = 0\) \hspace{1cm} (5.23)

\[
C_8 = \frac{F \left(a^3(I_d - I_u) + 9I_dL^3\right)}{12I_dLEI_u}
\]

\(C_9 = \frac{F \left(I_dL^3 + a^3(I_d - I_u)\right)}{6EI_dI_u}\) \hspace{1cm} (5.24)

\(C_{10} = 0\) \hspace{1cm} (5.25)

The constants result in the complete displacement field of the beam. In the following discussion the displacement of the beam is used in combination with Griffith’s theorem to analyse delamination initiation.

5.1.1.2 Delamination Initiation by Linear Elastic Fracture Mechanics

Impact-induced damage leads to degradation of strength and stiffness of composite structures, therefore information about damage size could be of great interest. Damage is initiated when a certain load level has been reached. In terms of delamination damage, this critical load level is often referred to as the delamination initiation load. Early works by Dobyns and Porter [54] proposed the use of damage mechanics, where the high transverse shear stresses induced by the impact event will lead to damage. However, Davies et al. [55] showed the existence of distinct and sudden increase of
damage area after a certain critical load. The delamination is assumed to extend once a critical energy has been reached. This observation is very much in line with Griffith’s theorem, where energy is dissipated by increasing the crack size. Assuming linear elastic behaviour, Griffith’s theorem of Linear Elastic Fracture Mechanics (LEFM) shown in Equation 5.6, gives the energy balance during crack growth.

The displacement field in Equations 5.11 to 5.13 are used to obtain the work performed on the beam by the external load. As discussed earlier in this chapter, two limiting cases are often used:

- fixed-grips
- dead-load

In a dead-load situation the applied load is assumed to be constant during crack growth, this method does not allow for analysing load-drops during crack growth. In this case it is chosen to use a displacement driven analysis (fixed-grips) since this method enables the analysis of load-drops during delamination growth. The load as function of the displacement is found by using Equations 5.11 to 5.13 which results in:

\[
F = \frac{12w_o EI_d I_u}{2 I_d L^3 + a^3 (I_u - I_d)}
\]  

(5.26)

where \(w_o\) is the displacement at the loading point and is assumed to be positive in the loading direction.

The elastic strain energy in terms of the deflection at the loading point then becomes:

\[
U^e = \frac{1}{2} F \cdot w_o = \frac{6w^2_o EI_d I_u}{2 I_d L^3 + a^3 (I_u - I_d)}
\]  

(5.27)

In the fixed-grips assumption, the displacement remains constant during crack growth. This means that \(\partial W/\partial A = 0\) for which Griffith’s theory reduces to Equation 5.9. Substituting this in Griffith’s energy balance under the assumption of fixed-grips given in Equation 5.9 gives:

\[
G = -\frac{\partial U^e}{\partial A} = -\frac{\partial}{\partial A} \left[ \frac{6w^2_o EI_d I_u}{2 I_d L^3 + a^3 (I_u - I_d)} \right]
\]  

(5.28)

As shown in Figure 5.4, it is assumed that the delamination grows in a self similar manner resulting in:

\[
\frac{\partial}{\partial A} = \frac{1}{B} \frac{\partial}{\partial a}
\]  

(5.29)
This allows for Equation 5.28 to be rewritten as:

\[
G = -\frac{1}{B} \frac{\partial}{\partial a} \left[ \frac{6w_o^2EI_dI_u}{2I_dL^3 + a^3 (I_u - I_d)} \right] = \frac{18w_o^2EI_dI_u a^2 (I_u - I_d)}{B (2I_dL^3 + a^3 (I_u - I_d))^2}
\]

Equation 5.30 represents the energy available for crack growth. Increasing the load will lead to a situation where the critical energy release rate \((G_c)\) is reached, after which the crack will grow rapidly. Assuming a critical material fracture toughness of \(G_c\), the critical displacement at which the delamination will start to grow is obtained to be:

\[
w_{oc} = \sqrt{\frac{G_cB}{18EI_dI_u (I_u - I_d)}} \frac{(2I_dL^3 + a^3 (I_u - I_d))}{a}
\]

Substitution into Equation 5.26 gives the critical load at which delamination growth is initiated which is given as:

\[
F_c = \frac{12w_{oc}EI_dI_u}{2I_dL^3 + a^3 (I_u - I_d)} = \frac{1}{a} \sqrt{\frac{8EI_dI_u G_cB}{(I_u - I_d)}}
\]

The benchmark problem discussed in [5] is used to validate the fracture initiation method discussed in the section above. The dimensions and properties used in the benchmark problem are shown in Figure 5.2. The comparison between the analysis given above and the benchmark results are summarized in Table 5.1.

<table>
<thead>
<tr>
<th></th>
<th>Benchmark</th>
<th>Analytical model</th>
<th>Error [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement [mm]</td>
<td>1.430</td>
<td>1.332</td>
<td>-6.8</td>
</tr>
<tr>
<td>Load [N]</td>
<td>1525.14</td>
<td>1594.68</td>
<td>4.6</td>
</tr>
</tbody>
</table>

The existing error between the analysis and the benchmark results is due to the use of the Euler-Bernoulli beam equation given in Equation 5.10, which does not take into account the effects of shear deformation. This effect can be demonstrated in Figure 5.5, where can be seen that the stiffness of the analytical method is higher than the
stiffness of the benchmark problem. This is due to the lack of shear deformation in the Euler-Bernoulli beam equation.

According to Gere and Timoshenko [56], the displacement due to shear deformation can be described by:

$$\frac{\partial w_s}{\partial x} = \frac{3}{2} \frac{V}{GA}$$  \hspace{1cm} (5.33)

where $V/A$ is the average shear stress and $G$ is the shear modulus. Solving this equation leads to:

$$w_s = \frac{3}{2} \frac{V}{GA} L$$  \hspace{1cm} (5.34)

The relative error due to the stiffening effect now becomes:

$$\Delta K = \frac{w_o + w_s}{w_o}$$  \hspace{1cm} (5.35)

As a correction we relax the stiffness with the reciprocal value resulting in:

$$E_{corr} = E \frac{w_o}{w_o + w_s}$$  \hspace{1cm} (5.36)

The comparison between the analysis and the benchmark results including correction for shear deformation effects are summarized in Table 5.2, showing good agreement.

<table>
<thead>
<tr>
<th></th>
<th>Benchmark</th>
<th>Analytical model</th>
<th>Error [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement [mm]</td>
<td>1.430</td>
<td>1.430</td>
<td>0</td>
</tr>
<tr>
<td>Load [N]</td>
<td>1525.14</td>
<td>1484.82</td>
<td>-2.64</td>
</tr>
</tbody>
</table>

Figure 5.5: Load displacement comparison between the analytical model and the benchmark results.
A graphical comparison between the improved analytical model and the benchmark model is shown in Figure 5.6, where it is shown that both the delamination initiation load as well as the stiffness of the analytical model match the benchmark results.

**Figure 5.6:** Load displacement comparison between the analytical model and the benchmark results including correction for shear deformation.
5.1.1.3 Delamination Growth

The delamination indicated in Figure 5.1, with an initial size of $a_i$, will grow if the fracture energy will exceed the critical material fracture toughness. As shown in Figure 5.7 [3], a drop in the load will be a result of delamination growth.

![Figure 5.7: Load-deflection curve](image)

Using Griffith’s theory, the energy balance for the beam after delamination has been initiated, can be obtained in a differential manner and is represented as [57]:

$$G\Delta A = \Delta U^e = \frac{1}{2} (F_i + F_{i+1}) (w_{i+1} - w_i)$$  \hspace{1cm} (5.37)

where subscript $i$ indicates the $i^{th}$ increment (in this case displacement increment).

Substituting Equation 5.26 for $F$ results in:

$$GB (a_{i+1} - a_i) = 6EI_dI_u \left( \frac{w_{o_i}}{2I_dL^3 + a_i^3 (I_u - I_d)} + \frac{w_{o_{i+1}}}{2I_dL^3 + a^3_{i+1} (I_u - I_d)} \right) (w_{o_{i+1}} - w_{o_i})$$  \hspace{1cm} (5.38)

Equation 5.38 is solved for $a_{i+1}$ by using the Newton-Raphson method, which results in the following routine:
Solving Equation 5.38 for \( a_{i+1} \) and using Equation 5.26 for rewriting the applied displacement to applied loads, the load vs. displacement behaviour can be obtained. In Figure 5.8 a graphical representation of the comparison between the analytical solution and the benchmark finite element solution is given for the load-displacement behaviour.

The delamination growth as function of the applied displacement is shown in Figure 5.9.
Figure 5.9: Delamination size as function of the displacement

Figure 5.8 shows good agreement between the analytical method and the benchmark finite element analysis, both in terms of delamination initiation as well as delamination growth. The methodology has been used extensively for analysing delamination behaviour of beams [9, 10]. In the next section, the methodology will be expanded to plate structures, resulting in a new delamination onset criteria for composite plates assuming circular delaminations.
5.2 Fracture Mechanics of Impacted Composite Plates

Properties of composites can degrade significantly due to the existence of delaminations. Especially delamination onset is very important in engineering because once this occurs, as is shown in Figure 5.8, the delamination could grow rapidly (even when stable) and detecting it is not an easy task. In this section a model is proposed for analysing delamination onset and growth for composite plates. In this case it is assumed that the delamination will have a circular shape and will grow in a self-similar manner. For this reason it is chosen to use a circular plate for the analysis with a concentrated load at its centre. In the case of an impact-event on a rectangular plate, it is assumed that the impact event is dominated by the plate dynamics and that the boundary conditions don’t play a crucial role, as discussed in Figure 3.2 and explained by Olsson [3].

In what follows, the model for delamination onset and growth is derived in a similar manner as performed for the beam model in the previous section, including the assumption that the delaminations have a circular shape and grow in a self similar manner.

5.2.1 Single Delamination Analysis For a Circular Composite Plate

In the following sections, fracture mechanics is used for analyzing delamination initiation in a circular composite plate. First the displacement field of a circular composite plate is obtained, after which this expression is used in combination with fracture mechanics to obtain a delamination initiation criteria.

5.2.1.1 Circular Plate Displacement Field

As a first approximation, it is assumed that the delaminations will have a circular shape and will grow in a self-similar manner, therefore a circular disk is used to analyse delamination onset and growth. Figure 5.10 gives the free body diagram of a circular disk, under concentrated load in the centre, where $M_\theta$ and $M_r$ are the moments per unit length and $Q$ represents the shear force per unit length.
The equilibrium equation in terms of moments are:

\[
\sum M^\uparrow + 0 = \left( M_r + \frac{dM_r}{dr} dr \right) (r + dr) \cdot d\theta
- M_r \cdot r \cdot d\theta - M_\theta \cdot dr \cdot d\theta + Q \cdot r \cdot d\theta \cdot dr = 0
\]  

(5.39)

Simplification gives:

\[
M_r + \frac{dM_r}{dr} r - M_\theta + Q \cdot r = 0
\]  

(5.40)

The material bending properties in Cartesian coordinates are given as:

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
D_{11} & D_{12} & 0 \\
D_{12} & D_{22} & 0 \\
0 & 0 & D_{66}
\end{bmatrix}
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]  

(5.41)

where

\[
\kappa_x = \frac{\partial^2 w}{\partial x^2} 
\]  

(5.42)

\[
\kappa_y = \frac{\partial^2 w}{\partial y^2} 
\]  

(5.43)

\[
\kappa_{xy} = 2 \frac{\partial^2 w}{\partial x \partial y} 
\]  

(5.44)
Assuming axi-symmetric displacement, the following transformation is used to rewrite the expressions from Cartesian coordinates to cylindrical coordinates:

\[
\frac{\partial^2 w}{\partial x^2} = \cos^2(\theta) \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \sin^2(\theta) \frac{\partial w}{\partial r} \tag{5.45}
\]

\[
\frac{\partial^2 w}{\partial y^2} = \sin^2(\theta) \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \cos^2(\theta) \frac{\partial w}{\partial r} \tag{5.46}
\]

\[
\frac{\partial^2 w}{\partial x \partial y} = \sin(\theta) \cos(\theta) \frac{\partial^2 w}{\partial r^2} - \frac{1}{r} \sin(\theta) \cos(\theta) \frac{\partial w}{\partial r} \tag{5.47}
\]

The stresses in cylindrical coordinates are given as [44]:

\[
\sigma_r = \sigma_x \cos^2(\theta) + \sigma_y \sin^2(\theta) + 2\tau_{xy} \sin(\theta) \cos(\theta) \tag{5.48}
\]

\[
\sigma_\theta = \sigma_x \sin^2(\theta) + \sigma_y \cos^2(\theta) - 2\tau_{xy} \sin(\theta) \cos(\theta) \tag{5.49}
\]

The moments corresponding to these stresses are:

\[
M_r = - \int_{-h/2}^{h/2} \sigma_r z dz \tag{5.50}
\]

\[
M_\theta = - \int_{-h/2}^{h/2} \sigma_\theta z dz \tag{5.51}
\]

Substituting Equation 5.48 into Equation 5.50 and Equation 5.49 into Equation 5.51 results:

\[
M_r = - \int_{-h/2}^{h/2} \sigma_x z dz \cos^2(\theta) - \int_{-h/2}^{h/2} \sigma_y z dz \sin^2(\theta) - 2 \int_{-h/2}^{h/2} \tau_{xy} z dz \sin(\theta) \cos(\theta) \tag{5.52}
\]

\[
M_\theta = - \int_{-h/2}^{h/2} \sigma_x z dz \sin^2(\theta) - \int_{-h/2}^{h/2} \sigma_y z dz \cos^2(\theta) + 2 \int_{-h/2}^{h/2} \tau_{xy} z dz \sin(\theta) \cos(\theta) \tag{5.53}
\]

In Cartesian coordinates, the relation between moments and stresses are defined as:

\[
M_x = - \int_{-h/2}^{h/2} \sigma_x z dz \tag{5.54}
\]

\[
M_y = - \int_{-h/2}^{h/2} \sigma_y z dz \tag{5.55}
\]

\[
M_{xy} = - \int_{-h/2}^{h/2} \tau_{xy} z dz \tag{5.56}
\]
Substitution of these relations into the moment Equations 5.52 and 5.53 gives:

\[ M_r = M_x \cos^2(\theta) + M_y \sin^2(\theta) + 2M_{xy} \sin(\theta) \cos(\theta) \]  
(5.57)

\[ M_\theta = M_x \sin^2(\theta) + M_y \cos^2(\theta) - 2M_{xy} \sin(\theta) \cos(\theta) \]  
(5.58)

Substituting Equations 5.41 to 5.47 into Equations 5.57 and 5.58 gives:

\[ M_r = D_{11} \left[ \cos^2(\theta) \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \sin^2(\theta) \frac{\partial w}{\partial r} \right] \cos^2(\theta) + \]
\[ D_{12} \left[ \sin^2(\theta) \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \cos^2(\theta) \frac{\partial w}{\partial r} \right] \sin^2(\theta) + \]
\[ D_{12} \left[ \cos^2(\theta) \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \sin^2(\theta) \frac{\partial w}{\partial r} \right] \cos^2(\theta) + \]
\[ D_{22} \left[ \sin^2(\theta) \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \cos^2(\theta) \frac{\partial w}{\partial r} \right] \sin^2(\theta) + \]
\[ 4D_{66} \left[ \sin(\theta) \cos(\theta) \frac{\partial^2 w}{\partial r^2} - \frac{1}{r} \sin(\theta) \cos(\theta) \frac{\partial w}{\partial r} \right] \sin(\theta) \cos(\theta) \]  
(5.59)

\[ M_\theta = D_{11} \left[ \cos^2(\theta) \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \sin^2(\theta) \frac{\partial w}{\partial r} \right] \sin^2(\theta) + \]
\[ D_{12} \left[ \sin^2(\theta) \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \cos^2(\theta) \frac{\partial w}{\partial r} \right] \cos^2(\theta) + \]
\[ D_{12} \left[ \cos^2(\theta) \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \sin^2(\theta) \frac{\partial w}{\partial r} \right] \sin^2(\theta) + \]
\[ D_{22} \left[ \sin^2(\theta) \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \cos^2(\theta) \frac{\partial w}{\partial r} \right] \sin^2(\theta) + \]
\[ 4D_{66} \left[ \sin(\theta) \cos(\theta) \frac{\partial^2 w}{\partial r^2} - \frac{1}{r} \sin(\theta) \cos(\theta) \frac{\partial w}{\partial r} \right] \sin(\theta) \cos(\theta) \]  
(5.60)

Rewriting Equations 5.59 and 5.60 results:

\[ M_r = \frac{\partial^2 w}{\partial r^2} \left[ D_{11} \cos^4(\theta) + D_{22} \sin^4(\theta) + 2 \sin^2(\theta) \cos^2(\theta) (D_{12} + 2D_{66}) \right] + \]
\[ \frac{1}{r} \frac{\partial w}{\partial r} \left[ D_{12} (\cos^4(\theta) + \sin^4(\theta)) + \sin^2(\theta) \cos^2(\theta) (D_{11} + D_{22} - 4D_{66}) \right] \]  
(5.61)

\[ M_\theta = \frac{\partial^2 w}{\partial r^2} \left[ D_{12} (\cos^4(\theta) + \sin^4(\theta)) + \sin^2(\theta) \cos^2(\theta) (D_{11} + D_{22} - 4D_{66}) \right] + \]
\[ \frac{1}{r} \frac{\partial w}{\partial r} \left[ D_{11} \cos^4(\theta) + D_{22} \sin^4(\theta) + 2 \sin^2(\theta) \cos^2(\theta) (D_{12} + 2D_{66}) \right] \]  
(5.62)
Now let:

\[ M_r = \frac{\partial^2 w}{\partial r^2} \tilde{S}_1 + \frac{1}{r} \frac{\partial w}{\partial r} \tilde{S}_2 \]  
(5.63)

\[ M_\theta = \frac{\partial^2 w}{\partial r^2} \tilde{S}_2 + \frac{1}{r} \frac{\partial w}{\partial r} \tilde{S}_1 \]  
(5.64)

with

\[ \tilde{S}_1 \equiv [D_{11} \cos^4(\theta) + D_{22} \sin^4(\theta) + 2\sin^2(\theta) \cos^2(\theta) (D_{12} + 2D_{66})] \]  
(5.65)

\[ \tilde{S}_2 \equiv [D_{12} (\cos^4(\theta) + \sin^4(\theta)) + \sin^2(\theta) \cos^2(\theta) (D_{11} + D_{22} - 4D_{66})] \]  
(5.66)

Substitution of Equations 5.63 and 5.64 into Equation 5.40 and integrating over \( \theta \) (i.e. \( \int_0^{2\pi} d\theta \)) gives:

\[ \frac{\partial^3 w}{\partial r^3} + \frac{1}{r} \frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2} \frac{\partial w}{\partial r} + \frac{4F}{\pi r \tilde{D}_1} = 0 \]  
(5.67)

with

\[ \tilde{D}_1 = (3D_{11} + 3D_{22} + 2(D_{1,2} + 2D_{66})) \]  
(5.68)

The solution for the homogeneous equation is given in the form:

\[ w_h = C_1 + C_2 \ln(r) + C_3 r^2 \]  
(5.69)

The particular solution is of the type:

\[ w_p = Ar^2 \ln(r) \]  
(5.70)

resulting in the solution for Equation 5.67 to be:

\[ w = C_1 + C_2 \ln(r) + C_3 r^2 - \frac{Fr^2 \ln(r)}{\pi \tilde{D}_1} \]  
(5.71)

The clamped boundary conditions for Equation 5.71 are:

\[ \frac{\partial w}{\partial r} = 0 \quad \text{For} \quad r = 0 \wedge r = R \]  
(5.72)

\[ w(r = R) = 0 \]  
(5.73)
The derivative of the displacement solution given in Equation 5.71 shows a singularity at \( r = 0 \), which results in \( C_2 = 0 \). The remaining two boundary conditions result in:

\[
2C_3R - \frac{F(2R\ln(R) + R)}{\pi D_1} = 0 \tag{5.74}
\]

\[
C_1 + C_3R^2 - \frac{FR^2\ln(R)}{\pi D_1} = 0 \tag{5.75}
\]

For the clamped boundary conditions given in Equation 5.72 the constants become:

\[
C_3 = \frac{F}{\pi D_1} \left( \ln(R) + \frac{1}{2} \right) \tag{5.76}
\]

\[
C_1 = -\frac{FR^2}{2\pi D_1} \tag{5.77}
\]

Resulting in the displacement field for an anisotropic circular plate with clamped boundary conditions under the assumption of axi-symmetric displacement, which is given as:

\[
w(r) = -\frac{FR^2}{2\pi D_1} + \frac{F}{\pi D_1} \left( \ln(R) + \frac{1}{2} \right) r^2 - \frac{Fr^2\ln(r)}{\pi D_1} \tag{5.78}
\]

It should be noted that the axi-symmetric assumption enforces additional constraints to the model. This will artificially increase the stiffness and result in lower displacement than in reality, resulting in underestimating the work performed on the specimen and delamination size. Also, as discussed for the beam model in Section 5.1.1, the lack of shear deformation leads to an increase of the stiffness, which is not accounted for. To get a feel of the error, two finite element models are used as comparison, including a \([90/0]_S\) and a \([0/45/90]_S - 45\] layup. Details about the dimensions of the models and the used materials are given in Table 5.3.

**Table 5.3: Dimensions and elastic properties of the circular plate used in the FE analysis**

<table>
<thead>
<tr>
<th>Radius [m]</th>
<th>Thickness [m]</th>
<th>Load [N]</th>
<th>( E_{11} ) [Gpa]</th>
<th>( E_{22} ) [Gpa]</th>
<th>( \nu_{12} ) [-]</th>
<th>( G_{12} ) [Gpa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.2 ( \cdot 10^{-3} )</td>
<td>1</td>
<td>135</td>
<td>9.6</td>
<td>0.3</td>
<td>5.3</td>
</tr>
</tbody>
</table>

The corresponding D-matrices of the two models are obtained to be:

\[
D_{[90/0]_S} = \begin{bmatrix}
1.085 & 0.1237 & \approx 0 \\
0.1237 & 5.124 & \approx 0 \\
\approx 0 & \approx 0 & 0.2261
\end{bmatrix} \tag{5.79}
\]
In Figures 5.11 and 5.12, the displacement fields of the two finite element results are displayed. Note that the displacement field is not axi-symmetric.

\[
D_{\{0/45/90/-45\}_S} = \begin{bmatrix}
31.77 & 4.150 & \approx 0 \\
4.150 & 11.58 & \approx 0 \\
\approx 0 & \approx 0 & 4.97
\end{bmatrix}
\] (5.80)

Even though the error between the analytical method and the FE results is clearly visible in Figures 5.13 and 5.14, the model seems to capture the overall behaviour. It should also be noted that the model is developed under axi-symmetric assumptions and therefore is only applicable for laminates with \(D_{11}/D_{22} \approx 1\).
5.2.2 Linear Elastic Fracture Mechanics for Single Delamination

5.2.2.1 Displacement field of a circular plate with a circular delamination

In Figure 5.15 a schematic representation of the single delamination model is shown.

\[ \text{FIGURE 5.15: Schematic representation of the single delamination model} \]

The displacement field of the single delamination model represented in Figure 5.15 is obtained by using Equation 5.71 for the outer plate and the inner plate (represented in Figure 5.15 with (2) and (1) respectively), including displacement and rotational compatibility boundary conditions between the two plates at \( r = a \). The boundary conditions then are:

\[
\begin{align*}
\frac{\partial w^{(2)}}{\partial r} |_{r=R} &= 0 \\
w^{(2)}(r = a) &= w^{(2)}(r = a) \\
\frac{\partial w^{(2)}}{\partial r} |_{r=a} &= \frac{\partial w^{(1)}}{\partial r} |_{r=a}
\end{align*}
\]

where \( w^{(1)} \) and \( w^{(2)} \) are given by:

\[
\begin{align*}
w^{(1)} &= C_1 + C_2 \ln(r) + C_3 r^2 - \frac{F r^2 \ln(r)}{\pi D_1} & 0 \leq r \leq a \\
w^{(2)} &= C_4 + C_5 \ln(r) + C_6 r^2 - \frac{F r^2 \ln(r)}{\pi D_2} & a \leq r \leq R
\end{align*}
\]

Resulting in the following four boundary conditions for a single delamination model.

\[
\begin{align*}
C_4 + C_6 R^2 &= \frac{F R^2 \ln(R)}{\pi D_2} \\
2C_6 R &= \frac{F R}{\pi D_2} (2 \ln(R) + 1) \\
C_1 + C_3 a^2 - C_4 - C_6 a^2 &= \frac{F a^2 \ln(a)}{\pi} \left( \frac{1}{D_1} - \frac{1}{D_2} \right) \\
2C_3 a - 2C_6 a &= \frac{F a (2 \ln(a) + 1)}{\pi} \left( \frac{1}{D_1} - \frac{1}{D_2} \right)
\end{align*}
\]
Solving these equations results:

\[
C_1 = \frac{F \left( a^2 \bar{D}_2 + R^2 \bar{D}_1 - a^2 \bar{D}_1 \right)}{2\pi \bar{D}_1 \bar{D}_2} \quad (5.91)
\]

\[
C_3 = \frac{F \left( a^2 \bar{D}_1 \ln(R) + 2 \ln(R) \bar{D}_2 - 2 \ln(a) \bar{D}_1 + \bar{D}_2 \right)}{2\pi \bar{D}_1 \bar{D}_2} \quad (5.92)
\]

\[
C_4 = -\frac{FR^2}{2\pi \bar{D}_2} \quad (5.93)
\]

\[
C_6 = \frac{F(2 \ln(R) + 1)}{2\pi \bar{D}_2} \quad (5.94)
\]

In order to use Griffith's theory the load-displacement relation at the point of loading is needed, since this allows the analysis of the performed work during loading. The deflection at the loading point \( w_0 \) as function of the load \( F \) and delamination radius \( a \) is therefore given by:

\[
w_0 = \frac{F \left[ a^2 \left( \bar{D}_2 - \bar{D}_1 \right) + R^2 \bar{D}_1 \right]}{2\pi \bar{D}_1 \bar{D}_2} \quad (5.95)
\]

### 5.2.2.2 Delamination Onset

Assuming a linear elastic behaviour until delamination initiation, Equation 5.9 in combination with Equation 5.95 is used. Assuming a circular delamination it can be shown that:

\[
\frac{\partial}{\partial A} = \frac{\partial}{\partial a} \frac{\partial}{\partial A} = \frac{1}{2\pi a} \frac{\partial}{\partial a} \quad (5.96)
\]

Substituting Equation 5.96 in Griffith's theory under fixed-grips condition given in Equation 5.9 results in:

\[
w_c = R^2 \sqrt{\frac{G_{IIc} \bar{D}_1}{\bar{D}_2 \left( \bar{D}_2 - \bar{D}_1 \right)}} \quad (5.97)
\]

And

\[
F_c = \sqrt{\frac{4\pi^2 \bar{D}_1 \bar{D}_2 \bar{G}_{IIc}}{\bar{D}_2 - \bar{D}_1}} \quad (5.98)
\]

Equations 5.97 and 5.98 give the critical displacement and the corresponding critical load respectively, after which delamination will grow. The growth behaviour of the delamination is discussed in the following sub-section.
5.2.2.3 Delamination Growth for the Single Delamination Model

As discussed in section 5.1.1, the Griffith method under the "fixed-grips" condition, given in Equation 5.9, is implemented. Equation 5.95 can be rewritten as:

\[ F = \frac{2\pi\bar{D}_1\bar{D}_2w_0}{a^2 \left( \bar{D}_2 - \bar{D}_1 \right) + R^2\bar{D}_1} \]  
(5.99)

As for the beam case, Equation 5.37 is used to analyse the delamination growth, resulting in:

\[ G\Delta A = \pi\bar{D}_1\bar{D}_2 \left[ \frac{w_0}{a^2 \left( \bar{D}_2 - \bar{D}_1 \right) + R^2\bar{D}_1} + \frac{w_{0_{i+1}}}{a^2_{i+1} \left( \bar{D}_2 - \bar{D}_1 \right) + R^2\bar{D}_1} \right] (w_{0_{i+1}} - w_0) \]  
(5.100)

As mentioned earlier, it is assumed that the delamination has a circular shape and grows in a self-similar manner. From this it can be concluded that \( \Delta A = \pi (a^2_{i+1} - a^2_i) \). Substituting this into Equation 5.100 gives:

\[ G (a^2_{i+1} - a^2_i) = \bar{D}_1\bar{D}_2 \left[ \frac{w_0}{a^2_i \left( \bar{D}_2 - \bar{D}_1 \right) + R^2\bar{D}_1} + \frac{w_{0_{i+1}}}{a^2_{i+1} \left( \bar{D}_2 - \bar{D}_1 \right) + R^2\bar{D}_1} \right] (w_{0_{i+1}} - w_0) \]  
(5.101)

Similar to the beam case, the Newton-Raphson method is used to obtain the delamination size as function of the applied displacement.

A circular plate with dimensions and elastic properties as given in Table 5.3 was analysed, including a fracture toughness value of \( G_c = 0.78N/mm \). In Figure 5.16, the load-displacement curve is plotted showing the degradation due to delamination onset and growth.
A model was developed for analysing delamination initiation and growth of a circular plate under an out-of-plane load at its centre, representing the indentation load. To achieve this, it was assumed that the displacement due to indentation is axi-symmetric and that the model includes a single delamination on its mid-plane. Furthermore, it was assumed that the delamination has a circular shape and remains circular during growth. Linear elastic fracture mechanics was used to obtain an analytical representation of the delamination initiation load. The principle was validated by analysing the delamination behaviour of a beam, after which the results were successfully compared to published benchmark results. The delamination onset model was however not validated by test or analysis.

Due to its analytical nature, the model is very efficient in terms of computation times. In the next chapter this model is used to perform an optimization, where an objective function is obtained from the delamination initiation load and the layup architecture is used as design variable.
Chapter 6

Laminate Optimization for Delamination Onset during Low-Speed Impact Loading

In Chapter 5, a model for delamination initiation was derived for a circular plate. The analytical solutions presented in Equations 5.97 and 5.98 enable the analysis of delamination onset with low computational cost, which makes these expressions suitable for optimization purposes. In this chapter the optimization process and results are discussed.

6.1 Optimization Problem

The defined optimization problem, needed to find an optimum layup architecture for low-velocity impact loading, is set-up to maximize the delamination initiation load given in Equation 5.98. From this equation it can be concluded that the absolute value of the delamination threshold is dependent on the value of the energy release rate $G_{IC}$. Theoretically this value can vary through the layup thickness, when different ply materials or adhesives are used to create the layup. In such layups, the different $G_{IC}$ values can be used to control the delamination location, which can be used to increase the compression after impact strength of the material. In the current analysis it is however assumed that the energy release rate $G_{IC}$ is constant for the entire layup. Since for the optimization we are interested in capturing the relative response rather than the absolute delamination onset load, an objective function representing delamination initiation is obtained.
using Equation 5.98, which can be represented as:

\[
 f \left( \tilde{D}_1, \tilde{D}_2 \right) = \sqrt{\frac{\tilde{D}_1 \tilde{D}_2}{\tilde{D}_2 - \tilde{D}_1}},
\]

with \( \tilde{D}_i \ i = 1, 2 \) defined in Equation 5.68.

The optimization problem can therefore be represented as:

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{f \left( \tilde{D}_1, \tilde{D}_2 \right)} \\
\text{such that} & \quad |D_{11} - D_{22}| \leq \epsilon,
\end{align*}
\]

where \( \epsilon \) is taken as a measure for the axi-symmetric bending behaviour. It should be noted that the constraint \( |D_{11} - D_{22}| \leq \epsilon \) will not guarantee an axi-symmetric bending behaviour. For example a pure \( \pm 45 \) layup could show a low value for \( \epsilon \) with a bending behaviour that is not be axi-symmetric. Therefore all the solutions were checked their validity with respect to the axi-symmetric assumption.

Due to the nature of the objective function, which is based on the assumption of axi-symmetric bending of a circular plate, a constrain function is introduced to make sure that the optimized layups are consistent with this assumption, which was made in Chapter 5. It is noteworthy to mention that this restriction will reduce the design-space resulting in possibly less optimal results. However, a correct representation of the problem in the form of the objective function is essential for reliable outcome, therefore the decision of restricting the results to be consistent with the axi-symmetric assumption is considered a valid assumption.

In order to make an assessment of the objective function, experimental results published by Dost et al [35] are used as comparison. Dost et al [35] performed low-velocity impact tests on various composite layups to analyse the effect of layup architecture on the impact behaviour of composite layups. The delamination radius was measured by Dost et al for all the experimental tests. The delamination radii presented by Dost et al are used to make an assessment of the objective function. The objective function is based on the delamination onset load given in Equation 5.98, which is defined as the load after which the delamination will start growing. For the comparison between the objective function and the delamination radius published by Dost et al [35], the assumption is made that for laminates with similar geometries, a lower delamination onset load will result in a larger delamination radius. It should be noted that the goal of the assessment is not a quantitative comparison but to capture the trend that is found in the experimental research.
The tested specimens are fabricated from Hercules IM7/8551-7, having a fibre volume fraction of $V_f = 0.57$ and a ply thickness of $t = 0.19\,mm$. For an impact energy of 16.3 J, four different layups are used by Dost et al:

- Layup 1: $[45_3, 90_3, -45_3, 0_3]_t$
- Layup 2: $[45, (90, -45)_3, (0, 45)_2, 0]_s$
- Layup 3: $[45, (0, -45)_3, (90, 45)_2, 90]_s$
- Layup 4: $[45, 90, -45, 0]_3$

The impact tests were performed using a steel spherical tub with a 1.59 cm diameter with a mass of 5.44 kg. The weight was dropped in a drop tower, where it was caught after impact to prevent a second impact event.

Ultra sonic Non Destructive Inspection (NDI) was utilized to measure the damage diameter. In Figure 6.1 the measured delamination diameter obtained by Dost et al [35] is presented together with the values for objective function given in Equation 6.1.

![Figure 6.1: Objective function assessment by comparison with Dost et al experimental results](image.png)

Since the objective function is used to obtain the best layups for impact delamination initiation, it is important that the objective function shows a similar trend as the target. As is shown in Figure 6.1, laminate 1 shows the highest value of the objective function, indicating low damage resistance. Laminates 2 and 3 show a similar value for the
objective function and laminate 4 shows the lowest value for the objective function, which was defined in Equation 6.1. This trend is also visible for tests performed by Dost et al [35]. The objective function seems to be useful in selecting the layup, which is most (or least) resistant to low speed impact damage.

In the following section, this objective function is used to optimize an 8-ply layup.

6.2 Laminate optimization

Altering the stacking sequence of a layup in a design is relatively inexpensive, while it may have significant effects on the impact response. It is however difficult to find optimal layups for maximum damage resistance due to the complex relation between stiffness and strength. An optimization procedure can be of great help in an attempt to find optimal composite layups for maximum impact damage resistance.

Evolutionary algorithms are suitable methods for optimizing the layup architecture, because of the discrete nature of the optimization problem [58]. Genetic algorithms (GA), based on evolutionary theories [59], are used extensively to optimize composite structures. Especially in combination with finite element analyses, these methods have been used with success. Yong et al. [60] used GA in combination with finite element analyses for optimizing the architecture of composite laminates subjected to low-velocity impact loading, where the objective function included the peak deflection and penetration. The objective of the optimization procedure is to find layups that are more damage resistant for impact loading. In this work, damage resistance of a certain layup is defined as the resistance of the layup to the onset of damage, measured by the magnitude of the impact force required for delamination onset. The higher the magnitude of this force, the higher the impact resistance. Structures for which impact loading is not the primary loading case are in particular suitable for a damage resistance design [60].

In the following section, the use of GA is discussed along with the obtained optimization results for an 8-ply laminate.

6.2.1 Optimization process and results

GAs are evolutionary algorithms based on natural selection, where the selection criteria determines the "survival" of a certain arrangement of the "chromosomes". GAs were introduced by J. Holland [61], who proposed a method of moving from one population of "chromosomes" to another population based on natural selection and genetic modification [59]. The genetic modification results in alteration of the chromosomes from that of the parents by means of crossover and mutations. The genes which construct the
chromosomes are coded in a genetic code (e.g. binary coded) and during genetic modification the genes are altered either due to crossover (where genes are exchanged from two chromosomes (the parents)) or due to mutations (which occur in a probabilistic manner and result in altering some of the genes). The survival of chromosomes is based on the selection criteria, where chromosomes selected by the selection procedure are allowed to reproduce. In general the fitter the chromosome the higher the probability that it will reproduce resulting in an evolutionary approach towards an optimum chromosome with respect to the selection criteria.

In the current work the ply angles of a layup with fixed number of layers are chosen as the "chromosomes" that need to be optimized, with respect to the selection criteria which in this case is the objective function given in Equation 6.1. Layup symmetry is an important aspect in composite design and is applied in most cases to reduce deformation of the part during the curing cycle and to avoid unwanted coupling of in-plane loads with out-of-plane strains and curvatures in subsequent service. This production aspect is taken into account in the optimization routine ensuring symmetric optimized layups. The GA algorithm present in the open-source program Scilab\(^1\) has been used, of which the details can be found in [62] and [58].

As mentioned earlier, the objective function is obtained under axisymmetric assumptions and the optimized results need to conform to this assumption for the sake of validity of the results. Due to the architecture of the GA within Scilab, it is chosen not to include this aspect as a constraint, but rather to modify the objective function to enforce this condition. The modified objective function is defined as:

\[
    f = \sqrt{\frac{\bar{D}_2 - \bar{D}_1}{\bar{D}_1 \bar{D}_2}} + \lambda |D(1,1) - D(2,2)|, \tag{6.4}
\]

where \(\lambda\) is a penalty factor enforcing the constraint. The value of lambda determines the strength of the constraint. For the optimization routine parameters need to be selected for the population size, probabilities of mutation and crossover, number of couples to perform crossover and the maximum number of generations. The values of these parameters used for the optimization within Scilab are given in Table 6.1.

\(^1\)www.scilab.org
The optimization is performed for an 8-layer plate with a symmetric layup. The optimum layup resulting from the GA procedure then becomes \([35/-35/34/-34]_s\). The minimum and maximum value in the population converging towards the optimized solution is displayed in Figure 6.2.

Two finite element models are created and analysed to validate the optimization results. For this purpose the optimized result is compared to a quasi-isotropic layup. The models consist of continuum elements for modelling the individual plies and cohesive elements are used to model delamination. In Figure 6.3 a graphical representation of the model is shown.

<table>
<thead>
<tr>
<th>TABLE 6.1: Optimization settings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population size</td>
</tr>
<tr>
<td>Probability of crossover</td>
</tr>
<tr>
<td>Probability of mutation</td>
</tr>
<tr>
<td>Number of couples to perform crossover</td>
</tr>
<tr>
<td>Maximum number of generations</td>
</tr>
<tr>
<td>(\lambda)</td>
</tr>
</tbody>
</table>

\[\text{minimum value in the population}\]

\[\text{maximum value in the population}\]
As is shown in Figure 6.3 a quarter of the model is used to reduce the model size. The dimensions of the plate and the indenter are given in Figure 6.4.

The material properties used for the lamina and for the cohesive material are given in Table 2.4 and Table 2.5 respectively. The plate is simply supported on the sides and symmetry boundary conditions are used on the symmetry axes. A rigid spherical indenter with a radius of 4 mm was used and was given an out-of-plane displacement of 10 mm. The degraded cohesive elements due to fracture are shown in Figures 6.5 to 6.10 for both of the laminates, where the value of 1 indicates fully degraded (fractured) cohesive elements. The degradations of the cohesive elements follow the traction separation law discussed in section 2.2.3.
FIGURE 6.5: Delamination initiation criteria for the quasi-isotropic layup [0/45/90/ -45]s (top view)

FIGURE 6.6: Delamination initiation criteria for the optimized layup [35/ -35/34/ -34]s (top view)

FIGURE 6.7: Delamination initiation criteria for the quasi-isotropic layup [0/45/90/ -45]s (bottom view)

FIGURE 6.8: Delamination initiation criteria for the optimized layup [35/ -35/34/ -34]s (bottom view)

FIGURE 6.9: Delamination initiation criteria for the quasi-isotropic layup [0/45/90/ -45]s (3-D view)

FIGURE 6.10: Delamination initiation criteria for the optimized layup [35/ -35/34/ -34]s (3-D view)
The quasi-isotropic layup shows slightly more delaminations compared to the optimized layup. In Figure 6.11 the values for the objective function for the two layups are plotted, showing a higher value for the quasi-isotropic laminate compared to the optimized laminate.

![Figure 6.11: Comparison of the objective function for the optimized [35/-35/34/-34]₄ layup and the quasi-isotropic [0/45/90/-45]₄ layup](image)

The results of the finite element analyses are in line with the optimization results. It should however be noted that the objective function was developed under axi-symmetric assumptions. This means that more optimal layups are probable for layups with high $D_{11}/D_{22}$ ratios (i.e. $D_{11}/D_{22} >> 1$).

### 6.2.2 Conclusions

The critical delamination load, discussed in section 5.2.1, was modified to obtain an objective function which was used to optimize the delamination onset load using the ply angles as design variable. The objective function was compared to test results published by Dost et al [35] for validity, resulting in a satisfactory comparison in terms of capturing the trend shown by the experimental results. Eventually, a GA optimization routine present in the open-source program Scilab was utilized for performing the optimization. It is noteworthy to mention that the delamination load obtained in section 5.2.1, which was used as the basis for the objective function, was obtained under axi-symmetric assumptions. The objective function is modified in order to make sure the optimization results meet this assumption, which reduces the design-space and therefore will theoretically result in less optimal laminates.
Chapter 7

Conclusions and Recommendations

Analytical methods have been presented for analysing low velocity impact behaviour of composite laminates. The methods include the following aspects of low velocity impact behaviour on composite laminates:

- inelastic contact behaviour
- elastic dynamic impact response
- stress analysis of impacted composites
- delamination behaviour of impacted composites

Looking at the contact behaviour, it was important to understand this behaviour beyond the small indentation assumptions which are the restrictions of the Hertzian contact model. Impacted composites show indentation damage at small indentations, which results in degradation of the local material stiffness. Also the indentations at low velocity impact were shown to be larger than accepted for the Hertzian contact model to give accurate results. Therefore an inelastic contact model was considered to be important. Since the dynamic impact response of an impacted composite shows periodic increase and decrease of the contact load due to the eigen frequency of the system, it was important to understand the contact behaviour in the following loading regions:

- loading
- unloading
Two frequently used inelastic contact models were considered:

- Yang and Sun Contact Model [1]
- Yigit and Christoforu Contact Model [12]

Experiments were conducted to evaluate these two models. The experiments included indentation of 40-layer composite layups. The test specimens were analysed prior to the experiment to ensure that the contact behaviour was not influenced by the boundaries. Finite element analyses were performed, including the influence of intra-ply damage and inter-ply fracture to make sure damage and fracture do not exceed the boundaries of the specimen. During the test the specimens were indented with a steel tub in such a way that the complete loading, unloading and reloading behaviour was included.

The Yang and Sun Contact Model showed good comparison to the test results for the unloading and reloading case, while the Yigit and Christoforu Contact Model showed good comparison to the test results for the inelastic loading case. The two models were combined, resulting in a new model which showed very good agreement with the experimental results for loading, unloading and reloading. Both of the models that were considered include a yield-indentation parameter, which can only be obtained by experimental testing and is depending on the indented material and indenter radius. This is considered as a limitation of these models.

A dynamic impact model was obtained, using classical plate theory, based on the methodology proposed by Abrate [39]. The obtained partial differential equation describing the problem was solved using Fourier series and numerical results were obtained through a Newton-Raphson method. The model was used to obtain the dynamic behaviour of impacted composite laminates in terms of load versus displacement and was verified through comparison with published test results. The model proved to be efficient in terms of computation time. The methodology showed to be valuable for obtaining the load-time response for low-velocity impact events. Also it was shown that such analytical models can be of great value for validating advanced numerical simulations.

A new model was introduced to obtain the out-of-plane stresses due to indentation during impact. The mathematical model was based on the Boussinesq equations, assuming axi-symmetric behaviour within the contact region. The resulting set of partial differential equations were solved using the Hankel transform, transforming the set of partial differential equations to a set of ordinary differential equations in the Hankel space, after which the stresses were obtained using the inverse transform. In addition to the local indentation stresses, the interlaminar stresses due to bending were added to
get a complete solution. The model was validated through comparison with numerical analysis results for an isotropic material and a 16-ply quasi-isotropic carbon epoxy laminate. The comparison was made at different load levels, resulting in good comparison between the model and numerical finite elements results. The model is highly efficient in terms of computation time, especially compared to the numerical method, since the finite element mesh needed to be refined significantly around the indentation area (i.e. a minimum meshsize of $50 \times 10^{-3} \times 50 \times 10^{-3} \times 46.25 \times 10^{-3}$ was needed) for the result to converge.

The presented model shows a significant peak in the out-of-plane shear stress due to indentation, just below the indented surface. This implies that damage due to indentation is most likely going to occur just below the indented surface. The results also show a significant value of the maximum out-of-plane shear stress, even at low forces. This makes it rather hard to avoid damage initiation due to impact, even at low impact energies.

Fracture mechanics was used to analyse delamination onset and growth due to impact. For the sake of obtaining an analytical solution to reduce the necessary computation time, it was assumed that the delamination is circular and will remain circular during growth.

A new single-delamination model was created based on this assumption, where the delamination was assumed to be situated in the mid-plane of the laminate. Using these assumptions a critical fracture load was obtained, which is a function of the material fracture toughness and the bending stiffness of the subsequent delaminated sub-laminates. In this case the critical fracture load is defined as the load after which the delamination will start growing. The resulting model is very computationally efficient and therefore suitable for optimization purposes.

The obtained single delamination model was used to create an objective function, which then was used for optimizing a composite layup architecture. It was shown that the objective function is able to predict the trend of delamination size versus layup architecture through comparison with published experimental data. Since the objective function was obtained under axi-symmetric assumptions, it was modified to enforce the results to be in accordance to the axi-symmetric assumption.

A Genetic Algorithm was used to optimize an 8-layer laminate, subjected to an indentation by a 4mm radius steel tup, resulting in a [35/-35/34/-34]$_s$ layup as optimum. The fracture behaviour of this layup was compared to that of an 8-ply quasi-isotropic layup, using finite element analyses. The fracture behaviour was modelled using cohesive zones between the layers. The results show slightly less delaminations in the optimized layup, compared to the quasi-isotropic layup. The axi-symmetric constraint on the optimization, which was necessary to stay consistent with earlier made assumptions, reduces the theoretical design space significantly. However, the results show that analytical methods
have great potential in obtaining better and more efficient structures, while adding to the understanding of the underlying physics.

The presented work has introduced many questions that need to be answered, before the methodologies described can be used in practice. Additional research is recommended to investigate the following subjects:

- The inelastic contact definitions discussed in chapter 2 need a yield-indentation parameter, which has to be obtained through experimental testing. This is a significant limitation of these methods, since this value depends on the impactor radius and the impacted material strength. Additional work is recommended to obtain an analytical estimate of the yield-indentation, which will increase the usability of these methods significantly.

- The delamination onset criteria developed in Chapter 5 was not validated. Test validation is recommended for this methodology.

- The fracture analyses that were performed to analyse the delamination behaviour, were based on a single delamination assumption. It is recommended to improve this model by including multiple delaminations, analysing the distribution of the fracture energy through the layup thickness. This will result in less conservative and more accurate description of the delamination behaviour.

- The axi-symmetric assumption used to obtain the delamination initiation load, has proven to limit the applicability of the model. This limitation has resulted in significant reduction of the design space during the optimization process, resulting in suboptimal outcome. Removing the axi-symmetric assumption will lead to a better description of the delamination problem and will increase the applicability region and therefore the design space during optimization, resulting in more optimal laminates.

- An optimization routine was utilized to optimize the layup architecture, while maximizing the delamination initiation load. This objective does not include any information about the post delamination strength of the laminate like, for example, compression after impact (CAI) strength. To include post delamination behaviour, a multiple delamination model is needed as was recommended above. In addition to this the Mode-I delamination growth behaviour of the sub-laminates due to local buckling in compression needs to be addressed.
Bibliography


1. Schade initiatie door schokbelastingen is moeilijk te voorkomen, het is efficiënter om de focus op schadegroei te richten.

2. Bijna alles om ons heen is een product van wetenschap en engineering. In zekere mate geven wetenschappers en ingenieurs vorm aan de wereld. Het bewustzijn hiervan zou meer verantwoordelijkheid moeten creëren van de sociale effecten die ons werk heeft op de wereld.

3. De sleutel tot het verbeteren van het schadegedrag van composiet laminaten ten gevolge van schokbelastingen ligt in het ontwikkelen van taaiere matrixmaterialen.

4. Door de toename van gebruiksvriendelijkheid van commerciële eindige elementen programma’s kunnen complexe modellen, zelfs zonder goede kennis van de achterliggende mechanica, makkelijk en snel gecreëerd en geanalyseerd worden. Dit is niet per definitie een goede ontwikkeling.

5. De mens lijkt meer angst te tonen voor het idee van de dood dan de dood zelf.

6. Er is gat ontstaan tussen wat theoretisch mogelijk is met composiet materialen en wat er in de praktijk mogelijk wordt gemaakt. Betere methodieken om het schadegedrag van composieten te analyseren en te simuleren kan helpen om dat gat te dichten.


8. Ondanks het feit dat veel aanname en simplificaties nodig zijn om analytische of semianalytische resultaten te verkrijgen, kunnen deze resultaten het aard van het probleem beschrijven, wat van een grotere waarde kan zijn dan nauwkeurige resultaten verkregen door numerieke methoden.

9. Het beloningsysteem dat wordt toegepast in het huidige kapitalistisch systeem is niet gerelateerd aan de sociale effecten van de producten en diensten die door de industrie worden geleverd. Deze worden geïmplementeerd als wetten wat het systeem inefficiënt en onbetrouwbaar maakt.

10. Wiskunde en kunst horen bij de grootste prestaties van de mensheid, omdat ze een universele waarde bezitten.
1. Damage initiation due to impact loading is hard to avoid, it is more efficient to focus on damage growth.

2. Almost everything around us is a product of science and engineering. In a way scientists and engineers shape the world. Awareness of this fact should increase the responsibility for the social effects of our work.

3. The key to improve impact damage resistance of composite laminates lies in the developing tougher resin materials.

4. Due to the increase in user-friendliness of commercial finite element tools complex models can be created and analysed, even without good knowledge of the corresponding mechanics. This is not by definition a good thing.

5. People tend to have more fear for the idea of death, than for death itself.

6. There is a large gap between what theoretically can be achieved with composites and what actually is being achieved in practice. More accurate methodologies for analysing and simulating composite damage behaviour could help close that gap.

7. A good education will probably not make you happier in life. It can however make life more interesting.

8. Even though significant assumptions and simplifications are needed to obtain analytical or semi-analytical solutions to engineering problems, the result could reveal the nature of the problem, which can be of greater value than accurate numerical results.

9. The rewarding method of the capitalistic system as it is applied is not related to the social effects of the products or services provided by the industry, these aspects are enforced by laws which makes the system inefficient and less reliable.

10. Mathematics and art should be valued as two of the greatest achievements of the human kind, because their value is universal.
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Stellingen behorende by het proefschrift

Impact Analysis of Composite Structures
door: M.R. Talagani

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