Stress Calculations on the Window Section of an All-composite Aircraft Fuselage

July 1992

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Summary

In this report an approximate solution is presented for the stresses in an infinite orthotropic plate with a row of holes. At first an infinite row of holes is considered. These holes are all equally shaped and equally spaced. Secondly a row of three holes is considered. These holes are still equally shaped but need not be equally spaced. In both cases the holes can be circular or elliptical. The calculations are based on the analytical method of complex stress functions.

The complex stress functions are evaluated from the boundary conditions of the holes. The influence of each hole is represented by a Taylor expansion of the elementary stress functions belonging to that hole. The use of series implies a limited accuracy of the solution. The numerical evaluation however shows that the convergence of the series is good.

In the stress calculations, arbitrary loading conditions can be applied at infinity and the row of holes can be orientated under an arbitrary angle with the symmetry axes of the material. Here two angles have been chosen (θ = 0° and θ = 30°) and a loading condition of plane stress (with the loads \( p_y = 1 \) and \( p_x = p_{xy} = 0 \)). Numerical results are obtained for five carbon fibre reinforced laminates with three different pitch to hole diameter ratios (s/D = 1.5, s/D = 3.0 and s/D = 4.5). The isotropic case has been added for reference purposes.

Also a preliminary design of an all composite sandwich fuselage (150 seats) is investigated, being the main subject of this report. With the analytical method developed before, the required facing thickness for the window section is calculated using a modified Tsai-Hill criterion. The calculations reveal that so far, the window section is most efficiently reinforced with a 1.9 mm C.F.R.P.-laminate consisting for 60% of [±45°]-C.F.R.P. and for 40% of [0°-90°]-C.F.R.P..
LIST OF SYMBOLS

b \quad \text{semi-axis of the ellips in y-direction} \quad \text{mm}

d \quad \text{semi-axis of the ellips in x-direction} \quad \text{mm}

D \quad \text{hole diameter} \quad \text{mm}

F_n^{(k)} \quad \text{series coefficient of the first order} \quad \text{mm}

f_n^{(k)} \quad \text{series coefficient of the second order} \quad \text{mm}

G_n^{(k)} \quad \text{series coefficient of the first order} \quad \text{mm}

g_n^{(k)} \quad \text{series coefficient of the second order} \quad \text{mm}

H_n^{(k)} \quad \text{series coefficient of the first order} \quad \text{mm}

h_n^{(k)} \quad \text{series coefficient of the second order} \quad \text{mm}

K_i \quad \text{integration constant} \quad \text{MPa}

p_x \quad \text{normal stress at infinity in x-direction} \quad \text{MPa}

p_y \quad \text{normal stress at infinity in y-direction} \quad \text{MPa}

p_{xy} \quad \text{shear stress at infinity} \quad \text{MPa}

p \quad \text{hole index} \quad \text{mm}

s \quad \text{pitch} \quad \text{MPa}^{-1}

s_{ij} \quad \text{material compliance} \quad \text{mm}

U \quad \text{displacement in x-direction} \quad \text{mm}

V \quad \text{displacement in y-direction} \quad \text{mm}

X \quad \text{external force in x-direction} \quad \text{N/mm}

Y \quad \text{external force in y-direction} \quad \text{N/mm}

z_k \quad \text{generalized complex variable of the first order} \quad \text{degr.}

z \quad \text{generalized complex variable of the second order} \quad \text{degr.}

\theta \quad \text{angle, indicating a position on the unit circle} \quad \text{degr.}

\lambda_k \quad \text{complex parameter of the second order} \quad \text{MPa}

\mu_k \quad \text{complex parameter of the first order} \quad \text{MPa}

\sigma \quad \text{unit circle} \quad \text{MPa}

\sigma_x \quad \text{normal stress in x-direction} \quad \text{MPa}

\sigma_y \quad \text{normal stress in y-direction} \quad \text{MPa}

\tau_{xy} \quad \text{shear stress} \quad \text{MPa}

\varphi \quad \text{angle between the material axes and the coordinate axes} \quad \text{degr.}

\phi_k \quad \text{complex stress function}
CHAPTER 1

INTRODUCTION

The principles of the theory of elasticity of anisotropic materials have been known for over a hundred years, and they were used in the very beginning of aircraft industry for calculations on structures made out of wood. Later on, aluminium took over as primary structural material, and interest in the theory decreased. Since the introduction of composites in aircraft industry it has become of interest again and, in combination with the modern computing facilities, it has shown to be very useful in the analysis of many structural problems. In the present work the theory will be used extensively for the analysis of stress distributions in composite fuselages.

In the first section of this introduction a short historical overview is given of the utilization of composites in aircraft industry over the last fifty years. It is followed by a brief discussion on the application of composites in a fuselage, being the main subject of this report. The introduction concludes with an outline of this report.

1.1 A general overview

The everlasting incentive behind developments in aerospace industry is the reduction of aircraft operating costs. Application of composites for instance, can result in less maintenance, a lower structural weight and, in case of thermoplastics, easier production methods. Therefore this could be effective in terms of operating costs, despite the high material costs. With composites a combination of qualities can be achieved, which cannot be realized in any other way. Unique qualities therefore, among which the high specific strength and stiffness in combination with the low specific weight are of decisive importance to the aerospace industry. Composites however, need adapted or new design practices and fabrication techniques. Their introduction therefore influences the development of new aircraft significantly. As a result the introduction of composites is a rather slow process.

Composites entered the aircraft structure during World War II, by means of glass fibre reinforced polyester of which radar domes were produced. With this material the complicated form of the radar dome could easily be produced while the structure would meet the strength requirements at the lowest weight possible. Since that time
engineers have always been fascinated by the high specific strength of glass fibres (theoretical values of 6000 MPa were found!). Despite the high strength of glass fibres, their application in fibre reinforced plastics for primary structures stayed behind because of two main reasons. First the combination of glass fibres and matrix material showed a specific stiffness which was far too low compared to conventional materials, and secondly the poor strength perpendicular to the fibre direction made it necessary to orientate the fibres in more directions which reduced the advantage of a high specific strength. Application of glass fibre reinforced plastics was therefore generally restricted to secondary parts with complicated doubly curved shapes, where the fabricational simplicity of the material was a dominant factor in the production process.

In the sixties the Apollo Project gave a new boost to the study of composites, since weight savings in the space industry are of great value (about $2,000 per ounce). New high strength fibres were developed like borium and carbon fibres, and in the seventies these fibres were applied in several test projects in the space industry. When prices dropped, application of composites in civil aircraft industry became attractive. The borium fibre, which remained expensive, had to hand over its place to the carbon fibre. At the same time the aramide fibre, developed by Du Pont, obtained a position next to the glass and carbon fibre. All of these fibres can be combined with different types of resin. Uptil now most often epoxy is being used although there is a tendency to increase the use of thermoplastic materials like polyetherimide.

Application of composite materials in aircraft structures is still increasing. Nowadays they are not only used for secondary structures, but for primary structures as well. Some of the latest examples of composite structures in modern civil aircraft are given below (Reference 1).
- The Boeing 737 composite horizontal stabilizer. This was the first certified primary carbon epoxy structure (1982), produced in small numbers.
- The composite vertical fin of the Airbus A320. This is an all composite structure, including the attachments to the fuselage, and is now in full production.
- The inner and outer flaps of the Fokker 100. These components are of a mixed composite-metallic type.
- The Beechcraft Starship, an all composite tandem wing twin pusher turboprop bussiness aircraft. Approximately 70% of the structural weight is composite structure. All the major components such as main and forward wing, pressure cabin, control surfaces and tipsails consist of a sandwich construction with graphite-epoxy facings and a Nomex® honeycomb core.
The great effort that is put in the research and application of composites in modern aircraft structures is not surprising. Today the expected life of a civil aircraft is about 20 years, and the safety precautions demand that after 20 years of flying the aircraft must be still as safe as it was when it was first delivered. This requires continuous maintenance which brings along tremendous costs. The maintenance costs after 20 years of a Boeing 747 for instance, add up to sixty million dollar which is half the amount of the purchase costs. The main reason for these high maintenance costs is the fatigue sensitivity of the conventional aluminium aircraft structure which must be checked on fatigue cracks on a regular basis. Therefore the application of composites, which are much less sensitive to fatigue then aluminium, could largely reduce the maintenance costs compensating for the high material costs. With this in mind, now the possibility of extending the Beechcraft Starship fuselage to the fuselage of a large composite civil aircraft is being studied.

1.2 An all composite sandwich fuselage

A conventional aluminium fuselage consists of a shell reinforced with stringers and frames. This is probably not a very efficient structure in case of the application of composites. The main reason for this is the sensitivity of composites for local, discrete changes in the geometry of the structure. Instead the use of sandwich panels seems to be more appropriate. Composite sandwiches are smooth, they can simplify the manufacturing of the fuselage and more over, they will enlarge the inside space in comparison to the stiffened shell structure since stringers and frames are no longer necessary.

Since the concept of a composite fuselage is in a preliminary phase, many problems still have to be solved. One of those problems is how to deal with the stress concentrations in the window section. In a conventional fuselage, this area is reinforced by window frames and a large doubling sheet (Figure 1). In a composite sandwich structure, the stress concentrations can be dealt with by a proper selection of facing material and thickness (Figure 2). Since a sandwich shell structure has a high bending stiffness of its own, in contrast with the skin used in the stiffened skin structure, the window frames do not need to be as stiff in bending as the conventional design and can therefore be much more simple. Their stress reduction function can probably be 'spread evenly' over the doubling material, in which case the window section can be treated as a smooth area without discrete reinforcements. The stress
problem of the window section then reduces to the problem of the determination of stresses in an anisotropic plate with a row of holes. Since the diameter of the fuselage is much larger than the facing thickness, and the sandwich shells are very stiff in bending, reducing the out of plane displacements of the structure around the windows, the curvature of the sandwich panels is assumed to be of no influence on the stresses. This design philosophy for a composite fuselage forms the base of the calculations presented in this report.

1.3 The outline of this report

In the next chapter an introduction to the plate problem in anisotropic materials is given, followed by chapter three concerning the theoretical background of the complex stress functions for an infinite row of holes in an infinite orthotropic plate loaded at infinity. After that, the same is done in chapter 4 for a row of three holes. Next, in chapter 5, an outline is given of the calculations that will be done with the theoretical background developed so far. Then, in chapter 6, results are obtained for five laminates of carbon fibre reinforced plastics, three pitch to hole diameter ratios (s/D = 1.5, s/D = 3.0 and s/D = 4.5) and two angles between the material axes and coordinate axes (ψ = 0° and ψ = 30°). The calculations are done on the infinite row of holes as well as on the row of three holes, for the situation of generalized plane stress (with the loads p_y = 1 and p_x = p_xy = 0) with the centre line of the holes equal to the x-axis. Also some practical measurements done on ARALL and GLARE are checked with analytical calculations done here.

After that, still in chapter 6, a strength prediction is given by combining stress calculations with a modified Tsai-Hill failure criterion (the Tsai-Hill failure criterion is modified to handle the difference between allowable tensile and allowable compression stresses). This is done for four different laminates containing an infinite row of holes with two different pitch to hole diameter ratios (s/D = 1.5 and s/D = 3.0) for the situation of generalized plane stress (with the loads p_y = 100 MPa and p_x = p_xy = 0 MPa). Chapter 6 concludes with the calculation of the required facing thickness of the sandwich panels in the window section, checked with the modified Tsai-Hill criterium. Finally, chapter 7 reveals the conclusions and recommendations.
CHAPTER 2

GENERAL EQUATIONS

The stress problem in anisotropic materials has been solved by Lekhnitskii by means of complex functions (Reference 2). His general solution is the point of departure for the two problems that are treated in this report. First the problem of generalized plane stress in an infinite orthotropic plate with an infinite row of holes is treated (Figure 3). These holes are equally shaped (circular or elliptical) and equally spaced. The purpose is to investigate the interaction between the holes by varying the pitch to hole diameter. Second the stress problem in an infinite orthotropic plate with a row of three holes is treated (Figure 4). These holes are also equally shaped (circular or elliptical) but need not be equally spaced. With this row the influence of the endholes on the stress distribution can be investigated.

In this chapter the approach to these two problems is given, which has been adopted from De Jong (Reference 3). In the neighbourhood of the central hole the elementary complex stress functions of the other holes are expanded into Taylor series. Since an elementary stress function is a series with unknown coefficients, the expansion results in an infinite double sum with one set of unknown coefficients. For an infinite row of holes the complex stress functions in the neighbourhood of the central hole become a triple sum. In the case of a row of three holes, the stress functions are summations of three double sums. By satisfying the load boundary conditions of the holes, an infinite set of linear equations for the coefficients of the stress functions is obtained. If the solution converges, an approximate solution for the coefficients is obtained by solving a truncated set of equations. A suitable way to do this is to truncate the previously mentioned Taylor expansions. Solving the truncated set of equations results in an approximation of the stress functions, with which an approximation of the stress distribution can be calculated.

2.1 Stresses and displacements

The general solution of the Airy differential equation for anisotropic materials contains two functions $\phi_k(z_k)$ satisfying the boundary conditions of the specific problem. The general expressions for the in-plane stresses in terms of $\phi_k(z_k)$ are:
\[
\sigma_x = 2Re \sum_{k=1}^{2} \mu_{k,\phi}^2 \phi_k'(z_k)
\]
\[
\sigma_y = 2Re \sum_{k=1}^{2} \phi_k'(z_k)
\]
\[
\tau_{xy} = -2Re \sum_{k=1}^{2} \mu_{k,\phi} \phi_k'(z_k)
\]

(2.1)

where the ' refers to differentiation of the function \(\phi_k(z_k)\) with respect to \(z_x\).

The boundary conditions for the external loads are

\[
2Re \sum_{k=1}^{2} \phi_k(z_k) = \int_{0}^{s} Y ds + K_1
\]
\[
2Re \sum_{k=1}^{2} \mu_{k,\phi} \phi_k(z_k) = -\int_{0}^{s} X ds + K_2
\]

(2.2)

in which \(X\) and \(Y\) are tractions on the plate contours and \(s\) is an arbitrary point along the path of integration.

In (2.1) and (2.2), \(z_k\) are complex variables

\[
z_k = x + \mu_{k,\phi} y
\]

(2.3)

\(\mu_{k,\phi}\) are the so-called complex material parameters of the first order. They are defined by

\[
\mu_{k,\phi} = \frac{\mu_k \cos \phi - \sin \phi}{\mu_k \sin \phi + \cos \phi}
\]

\(k = 1,2\)

(2.4)

for which \(\mu_k\) can be solved from

\[
\mu_1^2 + \mu_2^2 = \frac{s_{22}}{s_{11}}
\]
\[
\mu_1^2 \mu_2^2 = -\frac{2s_{12} + s_{66}}{s_{11}}
\]

(2.5)

\(s_{ij}\) are the material compliances in the principal material directions. \(\mu_{k,\phi}\) is either complex or imaginary.
The displacement terms of $\phi_k(z_k)$ are given by boundary conditions in

\[ U = 2Re \sum_{k=1}^{2} U_{k*} \phi_k(z_k) + K_3 y + K_4 \]

\[ V = 2Re \sum_{k=1}^{2} V_{k*} \phi_k(z_k) + K_3 x + K_5 \]

where

\[ U_{k*} = s_{11*} \mu_{k*}^2 + s_{12*} - s_{16*} \mu_{k*} \]

\[ V_{k*} = s_{12*} \mu_{k*} + \frac{s_{22*}}{\mu_{k*}} - s_{26*} \]

in which $s_{ij*}$ are the material compliances in the coordinate directions. The constants $K_1$ until $K_5$ are integration constants. $K_3$ is zero when no rotation of the plate as a rigid body is allowed. $K_4$ and $K_5$ represent the translation of the plate as a rigid body. The index $\varphi$ denotes the angle between the material axes and the coordinate axes and has a positive value when the material axes are rotated in clockwise direction relative to the coordinate axes.

2.2 The elementary stress function

Since the stress functions are analytical, they can be expressed as power series. Therefore the general expression for the stress functions $\phi_k(z_k)$, which describe the stress field in an infinite orthotropic plate with only one unloaded hole and loaded at infinity, is

\[ \phi_k(z_k) = g^{(k)}_{-2} z_k^2 + g^{(k)}_{-1} z_k + g^{(k)}_0 + \sum_{n=1}^{\infty} g^{(k)}_n z_k^{-n} \]

where the square term and the linear term represent respectively in-plane bending and a uniform stress field. The series represent the local influence of the hole. The powers of $z_k$ greater than 2 are not present in the formula for $\phi_k(z_k)$ because they would induce infinite stresses at infinity which is not realistic. That is why the corresponding coefficients must be zero. Since the function $\phi_k(z_k)$ is related to a single hole, it is called an elementary (complex) stress function.
In the general case of non-coinciding material and coordinate axes, the expressions for the coefficients $g_{-2}(k)$ and $g_{-1}(k)$ are very complicated. However, in the load boundary condition equations (2.2) they result in very simple terms, as will be shown in section 3.1 for the uniform loading case. For that reason the explicit expressions for $g_{-2}(k)$ and $g_{-1}(k)$ will be omitted here. The constant $g_0(k)$ in (2.8) has been added for formal reasons. It has no influence on the stress calculations. The unknown coefficients in the series must be found by solving the boundary conditions at the edge of the hole. In the next chapter this will be done for an infinite row of equally spaced and equally shaped holes in an infinite plate.
CHAPTER 3

STRESS CALCULATIONS FOR AN INFINITE ROW OF HOLES

In this chapter an analytical method is presented for the calculation of stresses in an infinite orthotropic plate with an infinite row of elliptical holes (Figure 3). First the complex stress functions which represent the problem must be composed. An important step in the solution is the determination of the complex coefficients in these functions from the boundary conditions. Finally the stresses are calculated from the differentiated functions, using equation (2.1).

3.1 The complex stress function for an infinite row of holes

Since for the present problem in-plane bending moments are not considered and the complex constants \( g_k^{(k)} \) have no influence on the stress calculations, the complex stress functions can be written as

\[
\phi_k(z_k) = g_{-1}^{(k)} z_k + \phi_k^0(z_k) \quad k=1,2
\]  

(3.1)

where \( g_{-1}^{(k)} z_k \) represents the undisturbed uniform stress as discussed in section 2.2 and \( \phi_k^0(z_k) \) represents the influence on the uniform stress field of the infinite row of holes with equal spacing \( s \). The functions \( \phi_k^0(z_k) \) can be considered as the sum of the elementary stress functions of the individual holes. If the centre points of the holes are on the x-axis and the centre point of the central hole coincides with the origin of the coordinate system, then \( \phi_k^0(z_k) \) can be written as

\[
\phi_k^0(z_k) = \sum_{p=1}^\infty \phi_k^p(z_k + ps) + \sum_{n=1}^\infty G_n^{(k)} z_k^{-n} + \sum_{p=1}^\infty \phi_k^p(z_k - ps)
\]  

(3.2)

In (3.2)

\[
\sum_{p=1}^\infty \phi_k^p(z_k + ps)
\]

are the elementary complex stress functions \((k = 1,2)\) belonging to hole \( p \), left of the central hole

\[
\sum_{n=1}^\infty G_n^{(k)} z_k^{-n}
\]

are the elementary complex stress functions belonging to the central hole

\[
\sum_{p=1}^\infty \phi_k^p(z_k - ps)
\]

are the elementary complex stress functions belonging to hole \( p \), right of the central hole

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Since for the present problem the row of holes is supposed to be infinitely long and all holes are of equal shape, the individual holes must have identical influence on the stress field. Therefore the elementary stress functions must be similar, having the same function coefficients. So \( \phi_k^0(z_k) \) can be represented as follows

\[
\phi_k^0(z_k) = \sum_{p=1}^{\infty} \phi_k^{(p)}(z_k - ps) = \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} G_n^{(k)}(z_k - ps)^{-n}
\]  

(3.3)

Note that for \( p = 0 \) or for \( s = \infty \), the elementary complex stress functions belonging to the central hole are achieved. The coefficients \( G_n^{(k)} \) in (3.3) must be obtained by satisfying the load boundary conditions of the holes. Since the stress distribution in the present case is periodic, it is sufficient to consider only one hole, for which the central hole \( (p = 0) \) is chosen.

The power series (3.3) substituted in the load boundary conditions (2.2) for the central hole produces very complicated expressions in \( z_k \) from which it is impossible to solve \( G_n^{(k)} \). For a plate with a single hole that problem can be circumvented by transforming the complex stress functions into functions of the new variables \( \zeta_k \)

\[
\sum_{n=1}^{\infty} G_n^{(k)} z_k^{-n} = \sum_{n=1}^{\infty} g_n^{(k)} \zeta_k^{-n}
\]  

(3.4) 

where

\[
\zeta_k = \frac{z_k \pm \sqrt{z_k^2 - \mu_k^2 b^2 - d^2}}{d - i \mu_k b}
\]  

(3.5)

The new variables have the interesting feature that on the contour of the elliptical hole

\[
\zeta_1 = \zeta_2 = e^{i\theta} = \cos \theta + i \sin \theta = \sigma \quad \text{(unit circle)}
\]

(3.6)

Therefore the boundary conditions of that hole can completely be defined in terms of the very easy to handle unit circle \( \sigma \) instead of the difficult complex variables \( z_k \). This simplifies the evaluation of the (new) set of coefficients \( g_n^{(k)} \) from these boundary conditions enormously.

With the intention of exploiting the same advantage in the case of a row of holes, the elementary stress functions of the individual holes are also transformed into functions of the variable
\[ r_k^{(p)} = \frac{(z_k - ps) \pm \sqrt{(z_k - ps)^2 - \mu_k b^2 - d^2}}{d - i\mu_k b} \]  

(yielding)

\[ \phi_k^0(z_k) = \sum_{p=-\infty}^{\infty} \phi_k^{(p)}(\zeta_k^{(p)}) = \sum_{p=-\infty}^{\infty} \sum_{n=1}^{\infty} g_n^{(k)}(\zeta_k^{(p)})^{-n} \]  

(3.8)

For \( p = 0 \) expression (3.7) obviously reduces to (3.5) which implies that the elementary stress functions of the central hole can be expressed in the unit circle on the contour of the central hole, in an identical way as for the single-hole problem. It is noted that it is not sufficient to transform only the elementary stress functions of the central hole and not the elementary stress functions of the other holes. In that case namely, a new set of coefficients \( g_n^{(k)} \) is introduced, due to the transformation of the functions of the central hole, while the old set of coefficients \( G_n^{(k)} \) remains in the non-transformed functions of the other holes. Then two sets of coefficients have to be solved for which a relation between \( g_n^{(k)} \) and \( G_n^{(k)} \) is needed.

Since only the boundary conditions of the central hole are considered and since the advantage of expressing the boundary conditions in the unit circle must be exploited, it is necessary to express the elementary stress functions of the holes other than the central hole (so for \( p \neq 0 \)) in the unit circle as well. For that reason these functions are expanded into Taylor series around the origin of the coordinate system, yielding

\[ (\zeta_k^{(p)})^{-n} = (\zeta_k^{(p)})^{-n} + \frac{z_k}{1!} \frac{d}{dz_k} (\zeta_k^{(p)})^{-n} |_{z_k=0} + \frac{z_k^2}{2!} \frac{d^2}{dz_k^2} (\zeta_k^{(p)})^{-n} |_{z_k=0} + \ldots \]  

(3.9)

In (3.9) the derivatives of \( (\zeta_k^{(p)})^{-n} \) for \( z_k = 0 \) are complex numbers, depending on the pitch \( s \) and the material parameter \( \mu_{k,\varphi} \). They are presented in Appendix C. A more suitable notation for these derivatives is

\[ \left( \frac{d_{\text{der}}}{dz_k} (\zeta_k^{(p)})^{-n} \right)_{z_k=0} = \zeta_k^{(\text{der})}(p, s) \]  

(3.10)

In (3.9) the complex coordinates \( z_k \) are still present. On the hole boundary they can be transformed into functions of the unit circle \( \sigma \) using the binomial of Newton; see appendix D. Here only the relevant expression is given.
\[ z_k^m = \left( \frac{d - i \mu_k b}{2} \right)^m \sum_{r=0}^{m} \lambda_k^r \binom{m}{r} \sigma^{m-2r} \]  

(3.11)

where

\[ \lambda_k = \frac{d + i \mu_k b}{d - i \mu_k b} \]  

(3.12)

With (3.8), (3.9), (3.11) and the elementary functions of the central hole \( \Sigma g_n^{(k)} \zeta_k^{-n} \), the boundary conditions of the central hole are now completely defined in terms of the unit circle. It is now possible to evaluate the unknown coefficients \( g_n^{(k)} \) from these conditions.

Note: The variables \( \zeta_k^{(p)} \) in (3.7) need extra attention in numerical calculations since they are double-valued. A choice has to be made between the + and - sign. For this, simple criteria are derived in Appendix A, based on the continuity of \( \zeta_k^{(p)} \) as a function of \( z_k \).

3.2 Evaluation of the series coefficients from the boundary conditions

The edge of the central hole is unloaded, so the boundary conditions (2.2) for the loads become

\[ 2 \text{Re} \sum_{k=1}^{2} \phi_k(z_k) = 0 \]  

(3.13)

\[ 2 \text{Re} \sum_{k=1}^{2} \mu_k \phi_k(z_k) = 0 \]

yielding the equations for \( \phi_k^0(z_k) \)

\[ 2 \text{Re} \sum_{k=1}^{2} \phi_k^0(z_k) = -p_y \cdot d \cos \theta + p_{xy} \cdot b \sin \theta \]  

(3.14)

\[ 2 \text{Re} \sum_{k=1}^{2} \mu_k \phi_k^0(z_k) = p_{xy} \cdot d \cos \theta - p_x \cdot b \sin \theta \]
Note: In section (2.2) the complicated expressions for $g_{-1}^{(k)}$ were mentioned. The right-hand side of equation (3.14) shows the simple terms in which they result on the hole boundary, in case of uniform loading $p_x$, $p_y$ and $p_{xy}$ at infinity.

Equations (3.14) yield an infinite set of linear equations for an infinite number of coefficients $g_n^{(k)}$. As stated before, a suitable way of truncating the set is to truncate the Taylor expressions (3.9). In the present work this is done after 8 terms. This truncation limits the number of coefficients $g_n^{(k)}$ which can be solved to 8 as well. The number of holes must also be limited. It is expected that 8 holes, both to the left and to the right of the central hole, yield sufficiently accurate stresses around the central hole. So

$$
\phi_k^0(z_k) = \sum_{n=1}^{8} g_n^{(k)} (\zeta_k^{(p=0)})^{-n} + \sum_{p=-8}^{8} \sum_{n=1}^{8} g_n^{(k)} (\zeta_k^{(p)})^{-n}
$$  \hspace{1cm} (3.15)

in which the ' in the summation over p denotes that $p=0$ has to be excluded. The term $(\zeta_k^{(p)})^{-n}$ has been worked out for $p=0$ as well as for $p\neq0$ in Appendix E.

$$
(\zeta_k^{(p)})^{-n} = \cos n\theta - isin n\theta \hspace{1cm} for \ p = 0
$$

$$
(\zeta_k^{(p)})^{-n} = \sum_{m=0}^{8} c_{k,n}^{(m)}(p,s)(\sigma^m + \lambda_k^m \sigma^{-m}) \hspace{1cm} for \ p \neq 0
$$  \hspace{1cm} (3.16)

In (3.16) $m=0$ produces a complex constant and can be omitted since it has no influence on the stress calculations. Using the property of the unit circle

$$
\sigma^{-n} = \cos n\theta - isin n\theta
$$  \hspace{1cm} (3.17)

$\sigma^m + \lambda_k^m \sigma^{-m}$ can be transformed to

$$
\sigma^m + \lambda_k^m \sigma^{-m} = (1 + \lambda_k^m)\cos m\theta + i(1 - \lambda_k^m)\sin m\theta
$$  \hspace{1cm} (3.18)

The characters $c_{k,n}^{(m)}(p,s)$ in (3.16) represent complex numbers and have different values for different indices $m$. With (3.14), (3.15) and (3.16), $\phi_k^0(z_k)$ can now be written as

$$
\phi_k^0(z_k) = \sum_{m=1}^{8} \sum_{n=1}^{8} \sum_{p=-8}^{8} (1 + \lambda_k^m) c_{k,n}^{(m)}(p,s) g_n^{(k)} \cos m\theta + \sum_{n=1}^{8} g_n^{(k)} \cos n\theta
$$

$$
+ \sum_{m=1}^{8} \sum_{n=1}^{8} \sum_{p=-8}^{8} i(1 - \lambda_k^m) c_{k,n}^{(m)}(p,s) g_n^{(k)} \sin m\theta - \sum_{n=1}^{8} ig_n^{(k)} \sin n\theta
$$  \hspace{1cm} (3.19)
Substitution of (3.19) into (3.14) and comparing the coefficients of sines and cosines with equal coefficients of the angle $\theta$ yields a set of 32 equations with 32 unknown coefficients, namely $\text{Re}\{g_n^{(k)}\}$ and $\text{Im}\{g_n^{(k)}\}$ for $k=1,2$ and $n=1...8$, yielding 8 coefficients $g_1^{(k)}$ and 8 coefficients $g_2^{(k)}$. The unknown coefficients are solved from the equations with the help of a computer program in HP Pascal. With known coefficients $g_n^{(k)}$, the complex stress functions are completely known and the stresses $\sigma_x, \sigma_y$ and $\tau_{xy}$ can be calculated. For this the same program is used. Since the edge of the hole is unloaded, the tangential stresses can simply be calculated with

$$\sigma_t = \sigma_x + \sigma_y$$  (3.20)

It is obvious that in (3.14) any combination of $p_x, p_y$ and $p_{xy}$ may be introduced. The numerical evaluation will be presented in the chapters 5 and 6.
CHAPTER 4

STRESS CALCULATIONS FOR A ROW OF THREE HOLES

In this chapter the complex stress function is derived for an orthotropic plate with three elliptical holes. Three sets of boundary conditions have to be solved, one for each hole. Solving the function coefficients from these boundary conditions yields the elementary stress functions of the holes.

4.1 The complex stress function for a row of three holes

As in the previous case of the infinite row of holes, the general expression for the complex stress function for the plate with three holes is

\[
\phi_k(z_k) = \phi_{k}^{\text{hom}}(z_k) + \phi_{k}^{0}(z_k)
\] (4.1)

\[\sum \phi_{k}^{0}(z_k)\] now represents the influence of the row of three holes on the undisturbed stress field and is considered to be the sum of the elementary stress functions of the single holes, so

\[
\phi_{k}^{0}(z_k) = \sum_{p=1}^{1} \phi_{k}^{(p)}(z_k - ps)
\] (4.2)

In the variable \((z_k - ps)\) the spacings \(s\) between the holes have arbitrary values and need an index number. In the present text however, these numbers will not be added to \(s\) since in the following expressions \(s\) is always connected directly to its related hole index \(p\).

The influence on the stress field of the individual holes is not identical, even in the case of identical shapes. The reason is that different types of holes can be distinguished: the central hole and the left and right end hole. In the case of unequal hole shapes the difference in influence on the stress field is obvious. So, in the case of a plate with three holes, the elementary stress functions are always different in that sense that they have different function coefficients. The elementary stress functions are now written as follows

\[
\phi_{k}^{0}(z_k) = \sum_{n=1}^{\tilde{r}} F_n^{(k)}(z_k - ps)_p^{-n} + \sum_{n=1}^{\tilde{n}} G_n^{(k)}(z_k - ps)_p^{-n} + \sum_{n=1}^{\tilde{r}+1} H_n^{(k)}(z_k - ps)_p^{-n+1}
\] (4.3)
In order to determine the function coefficients, three sets of boundary conditions have to be solved. First, the boundary conditions of the central hole are considered. The origin of the coordinate axes is placed in the centre of the central hole, and on the edge of the central hole all elementary stress functions will be written in terms of cosines and sines of the angle \( \theta \). To do so, the functions of \((z_k - ps)\) are transformed into functions of

\[
\zeta_k^{(p)} = \frac{(z_k - ps) \pm \sqrt{(z_k - ps)^2 - \mu_k^2 b^2 - d^2}}{d - i \mu_k b} \quad (4.4)
\]

Since the three holes in the general case have different shapes, the semi-axis of the ellips \( b \) in y-direction and the semi-axis of the ellips \( d \) in x-direction need index numbers as well. Again these are not added in order to keep the notation surveyable. With the variable \( \zeta_k^{(p)} \), the stress functions become

\[
\phi_k^0(z_k) = \sum_{n=1}^{\infty} f_n^{(k)}(\zeta_k^{(p-1)})^{-n} + \sum_{n=1}^{\infty} g_n^{(k)}(\zeta_k^{(p=0)})^{-n} + \sum_{n=1}^{\infty} h_n^{(k)}(\zeta_k^{(p+1)})^{-n} \quad (4.5)
\]

On the contour of the central hole, for \( p=0 \) again \( \zeta_k^{(p)} \) reduces to \( \sigma \),

\[
\zeta_1^{(p=0)} = \zeta_2^{(p=0)} = \sigma = \cos \theta + i \sin \theta \quad (4.6)
\]

and

\[
(\zeta_k^{(p=0)})^{-n} = \cos n \theta - i \sin n \theta \quad (4.7)
\]

The other two power series in (4.5) are expanded into Taylor series around the center of the coordinate system:

\[
(\zeta_k^{(p)})^{-n} = (\zeta_k^{(p)})^{-n} + \frac{z_k}{1!} \frac{d}{dz_k} (\zeta_k^{(p)})^{-n} |_{z_k=0} + \frac{z_k^2}{2!} \frac{d^2}{dz_k^2} (\zeta_k^{(p)})^{-n} |_{z_k=0} + \ldots \quad (4.8)
\]

The derivatives of \((\zeta_k^{(p)})^{-n}\) for \( z_k=0 \) are complex numbers, depending on \( s \) and \( \mu_k, \nu \). They are presented in Appendix c. In order to simplify the notation, they are written in the same way as the derivatives in the case of the infinite row of holes,

\[
\left( \frac{d}{dz_k} (\zeta_k^{(p)})^{-n} \right)_{z_k=0} = \zeta_{k,n}^{(der)}(p,s) \quad (4.9)
\]
The coordinates $z_k$ still present in (4.8) are transformed into functions of $\sigma$, see (3.11)

$$z_k^m = \left(\frac{d - i\mu_k b}{2}\right)^m \sum_{r=0}^{m} \lambda_k^r \binom{m}{r} \sigma^{m-2r}$$  \hspace{1cm} (4.10)

where

$$\lambda_k = \frac{d + i\mu_k b}{d - i\mu_k b}$$  \hspace{1cm} (4.11)

On the edge of the central hole the elementary stress functions are now expressed in the unit circle, and it is possible to work out the boundary conditions of that central hole.

4.2 Evaluation of the series coefficients from the boundary conditions

With only normal and shear loads on the plate at infinity and no tractions on the edge of the central hole, the boundary conditions for $\phi_k^0(z_k)$ can be written conform (3.13):

$$2\text{Re} \sum_{k=1}^{2} \phi_k^0(z_k) = -p_y d \cos \theta + p_{xy} b \sin \theta$$  \hspace{1cm} (4.12)

$$2\text{Re} \sum_{k=1}^{2} \mu_k \phi_k^0(z_k) = p_{xy} d \cos \theta - p_x b \sin \theta$$

Again eight terms of the Taylor expansions are taken into account, resulting in eight terms of the power series as well:

$$\phi_k^0(z_k) = \sum_{n=1}^{8} f_n^{(k)} (\zeta_k^{(p=-1)})^{-n} + \sum_{n=1}^{8} g_n^{(k)} (\zeta_k^{(p=0)})^{-n} + \sum_{n=1}^{8} h_n^{(k)} (\zeta_k^{(p=1)})^{-n}$$  \hspace{1cm} (4.13)

where (see (3.16) and Appendix E)

$$(\zeta_k^{(p)})^{-n} = \sum_{m=0}^{8} c_{k,n}^{(m)} (p,\zeta) (\sigma^m + \lambda_k^m \sigma^{-m})$$

for $p \neq 0$  \hspace{1cm} (4.14)

and

$$\sigma^m + \lambda_k^m \sigma^{-m} = (1 + \lambda_k^m) \cos m \theta + i (1 - \lambda_k^m) \sin m \theta$$  \hspace{1cm} (4.15)
With expressions (4.8), (4.10) and (4.13), \( \phi_k^0(z_k) \) in (4.12) is now completely defined in terms of sines and cosines:

\[
\phi_k^0(z_k) = \sum_{m=1}^{8} \sum_{n=1}^{8} \left(1 + \lambda_k^m\right) c_{k,n}^{(m)}(p,s) p_n f_n^{(k)} \cos m \theta \\
+ \sum_{n=1}^{8} g_n^{(k)} \cos n \theta \\
+ \sum_{m=1}^{8} \sum_{n=1}^{8} \left(1 + \lambda_k^m\right) c_{k,n}^{(m)}(p,s) p_{n-1} h_n^{(k)} \cos m \theta \\
+ \sum_{m=1}^{8} \sum_{n=1}^{8} i(1 - \lambda_k^m) c_{k,n}^{(m)}(p,s) p_{n-1} f_n^{(k)} \sin m \theta \\
- \sum_{n=1}^{8} i g_n^{(k)} \sin n \theta \\
+ \sum_{m=1}^{8} \sum_{n=1}^{8} i(1 - \lambda_k^m) c_{k,n}^{(m)}(p,s) p_{n-1} h_n^{(k)} \sin m \theta
\] (4.16)

Substitution of (4.15) in (4.11) and comparing the cosines and sines with the same coefficient of the angle \( \theta \) yields a set of 32 equations with 96 unknown coefficients, namely 8 real and 8 imaginary parts of \( f_n^{(k)} \), \( g_n^{(k)} \) and \( h_n^{(k)} \) for \( n=1...8 \) and \( k=1,2. \) So there are still 64 equations missing. These equations must be obtained from the boundary conditions of the two end holes. The system of constructing the boundary condition equations for the outer holes is identical to that of the central hole. The only difference is the shifting of the hole index \( p \) (Reference 4) as it appears in expression (4.5):

- For the boundary condition equations of the left hole, \( p \) has to run from zero (corresponding with the left hole) to 2 (corresponding with the right hole).
- For the boundary condition equations of the right hole, \( p \) has to run from -2 (corresponding with the left hole) to 0 (corresponding with the right hole).

The shifting of the index \( p \) is equivalent to translating the origin of the coordinate system to the centre point of the left hole and that of the right hole respectively. In constructing their respective boundary condition equations these holes take over the role of central hole. The mathematical treatment of the elementary stress functions of the other holes stays the same.
Thus for the left hole

\[ \phi_k^0(z_k) = \sum_{n=1}^{8} f_n^{(k)} (\zeta_k^{(p=0)})^{-n} + \sum_{n=1}^{8} g_n^{(k)} (\zeta_k^{(p=1)})^{-n} + \sum_{n=1}^{8} h_n^{(k)} (\zeta_k^{(p=2)})^{-n} \quad (4.17) \]

and for the right hole

\[ \phi_k^0(z_k) = \sum_{n=1}^{8} f_n^{(k)} (\zeta_k^{(p=-2)})^{-n} + \sum_{n=1}^{8} g_n^{(k)} (\zeta_k^{(p=-1)})^{-n} + \sum_{n=1}^{8} h_n^{(k)} (\zeta_k^{(p=0)})^{-n} \quad (4.18) \]

These equations are substituted in the boundary conditions (4.11) as was done before. \((\zeta_k^{(p)})^{-n}\) is transformed into cosines and sines and then terms with the same coefficient of the angle theta are compared. This yields another two sets of 32 equations, completing the total set of 96 equations. After solving the 96 function coefficients with the previously mentioned computer program, the complex stress functions for the plate with three holes of different spacing are known and the stresses can be calculated.
CHAPTER 5

NUMERICAL EVALUATION

In the previous two chapters the complex stress functions have been derived for an infinite orthotropic plate containing respectively an infinite row of equally spaced holes, and a group of three differently spaced holes. The stress functions allow for the calculation of stresses in any point of the orthotropic plate. However, for the present report only the tangential stress $\sigma_t$ around the holes and the stress $\sigma_y$ in the net area between the holes have been calculated for a selected group of materials. For the numerical evaluation five laminates of C.F.R.P. and one isotropic material (aluminium) have been used. The laminates were chosen because of their practical relevance.

The mechanical properties are listed in Table 1. Most of these properties are measured values. Note in Table 1 that the quasi-isotropic $(90^\circ/0^\circ/\pm 45^\circ)_s$-laminate does not fulfil the requirement for isotropic elasticity $\mu_1 = \mu_2 = 1$. Generally the $0^\circ$-direction has the highest Young's modulus. In the calculations of the present work this direction coincides with the y-axis or makes an angle $\varphi$ with that axis. In Table 1 the properties of a uni-directional material are given as well. This material is of interest because of its extremely anisotropic character. It will be used in the strength calculations of section 5.3.

5.1 The infinite row of holes

The calculation of the stress $\sigma_t$ will be referred to as an edge calculation. Because of the symmetry of the problem with respect to the x-axis, only the stresses along the upper half of a hole contour have been calculated. The first series of calculations have been done for a row of circular holes with the following parameters:

$p_y = 1$ MPa and $p_x = p_{xy} = 0$ MPa

$\varphi = 0^\circ$ and $\varphi = 30^\circ$

and

$s/D = 1.5$, $s/D = 3.0$ and $s/D = 4.5$
The results are presented in the Figures 5 - 15 where the graphs for three different pitch to hole diameter ratios have been combined in one figure.

Next, the stress ratio \( \sigma_y/p_y \) along the x-axis is calculated between two holes. These are called line calculations. For this the same materials, applied loads, laminate orientations and pitch to hole diameter ratios are used. The results are presented in the Figures 16 - 26.

Line calculations are also done on ARALL and GLARE (Figures 27 - 36), since practical measurements of these materials are available. With these, the adequacy of the analyses can be evaluated by comparing the practical and theoretical results.

5.2 The row of three holes

The second series include again edge and line calculations, only now on a row of three circular holes. The calculations have been done on three C.F.R.P. laminates and the isotropic material with the parameters used before:

\[ \varphi = 0^\circ \text{ and } \varphi = 30^\circ \]

\[ s/D = 1.5, \ s/D = 3.0 \text{ and } s/D = 4.5 \]

\[ p_y = 1 \text{ MPa and } p_x = p_{xy} = 0 \text{ MPa} \]

Here the tangential stress distribution along the edge of the central hole and along the edge of the right hole have been put in one figure, together with the tangential stress distribution on the edge of a hole from the infinite row (Figures 36 - 77).

5.3 Stress calculations as a design tool

The third series includes edge calculations on four different C.F.R.P. laminates completed with a strength prediction by applying the modified Tsai-Hill failure criterion. The strength values of the laminates used in this criterion, are listed in Table 2. The parameters used in the calculations are (see next page):
\[ \varphi = 0^\circ \]

\[ p_y = 100 \text{ MPa} \quad \text{and} \quad p_x = p_{xy} = 0 \text{ MPa} \]

\[ s/D = 1.5 \quad \text{and} \quad s/D = 3.0 \]

With these calculations done, it is possible to see the influence on the tangential stress distribution and the laminate strength when the fibre orientation of several layers in the laminate is changed. Clearly can be seen which laminate is able to carry the load \( p_y \) and which isn’t (Figures 78 - 85).

5.4 The all composite sandwich fuselage

Here the required laminate thickness is calculated for the facings in the window section of an all composite sandwich fuselage (in size comparable to an Airbus A320 fuselage). From the preliminary design process (Reference 5) it appears that for the undisturbed composite fuselage a facing thickness of \( t = 0.9 \) mm is just able to carry the ultimate load when a \([0^\circ-90^\circ]/\pm 45^\circ/0^\circ-90^\circ]\)-laminate of C.F.R.P. is used (the CD 282 prepreg from Ten Cate, see Table 1). The ultimate load is represented by the following two stress combinations:

\[ \sigma_x = 212 \text{ MPa}, \quad \sigma_y = 99 \text{ MPa} \quad \text{and} \quad \tau_{xy} = 160 \text{ MPa} \]

\[ \sigma_x = -113 \text{ MPa}, \quad \sigma_y = 99 \text{ MPa} \quad \text{and} \quad \tau_{xy} = 160 \text{ MPa} \]

However, this is only the case for the undisturbed fuselage without windows. Due to the presence of windows, the stresses in the window section will rise. Therefore the laminate thickness must be increased. Here the necessary thickness of the laminate facings is calculated. For these calculations the window dimensions of a Fokker F-100 aircraft are chosen:

\[ d = 130 \text{ mm} \quad \text{and} \quad b = 175 \text{ mm} \]

When the facing thickness in the window section is increased, the same laminate can be used, with exactly the same orientation and thickness ratio of the separate fibre layers. But then again it may be more efficient to apply a different orientation and/or thickness ratio of the fibre layers. Maybe it is even more efficient to use another material, with different mechanical properties. These situations have been investigated and are worked out in the Figures 86 - 92.
CHAPTER 6

DISCUSSION OF THE RESULTS

In this chapter the results of the calculations are discussed. In order to arrange this conveniently, the same sequence of series will be kept.

6.1 The infinite row of holes

In the isotropic case the elastic properties of the material result in $\mu_{k,\varphi} = i$, making $\lambda_k$ according to (3.12) equal to zero. For these materials it is not possible to obtain the coefficients $h_n^{(k)}$ from the equations that fulfil the boundary conditions. Therefore the calculations are made with values of the Young's modulus in the x- and y-direction slightly different from each other, resulting in values of $\mu_{k,\varphi}$ slightly different from $i$ (Table 1). For the isotropic case the stress ratio $\sigma_t/p_y$ is well known for an infinite plate with a single circular hole. On the edge of the hole $\sigma_t/p_y$ is equal to 3 for $\theta = 0^\circ$ and equal to -1 for $\theta = 90^\circ$. When an infinite row of holes is analysed, these values should be reached for an increasing pitch to hole diameter ratio $s/D$. This can be seen in Figure 5. Also the tendency of lower $\sigma_y/p_y$ values for larger $s/D$ ratio's in Figure 16 is correct.

For the orthotropic laminates the edge calculations show only a modest interaction between the holes for $s/D \geq 3$ (Figures 5 - 15); the numerical results indicate that peak stresses for $s/D = 3$ are only about 2% larger then the corresponding values of $s/D = 4.5$, both for the approximately quasi-isotropic [0°/90°/±45°]s-laminate and the laminates dominated by 90°-layers. The same can be said for the line calculations, only this can not be seen that easily, because along the x-axis the variable x/s has been plotted and not the distance between a point x on the x-axes and the edge of the hole.

The influence of $\varphi$ on the stress distribution is rather strong. As was expected the effect is stronger for the laminates dominated by the 90°-layers then for the laminates dominated by ±45°-layers. Although the [0°/90°/±45°]s-laminate is approximately quasi-isotropic, its elastic constants differ from the value for isotropy $\mu_k = i$. Therefore, for this laminate there is some influence of $\varphi$ on the stress distribution (Figures 14 and 15).
Figures 17 and 18 show a property of the $[\pm 45^\circ]_s$-laminate which is also known from the single hole solution (Reference 7). For $\varphi = 0^\circ$ the maximum stress ratio $\sigma_y/p_y$ on the x-axis does not occur on the edge of the hole. For $s/D = 1.5$ the stress ratio $\sigma_y/p_y$ on the edge of the hole is even the smallest of the whole section.

The orthotropic laminates chosen here, are the same as chosen in Reference 3. Because the same pitch to hole diameter ratios $s/D$ are taken, it is possible to compare the results of the stress calculations done here and those of Reference 3. Since in that report numerical results are only given in figures and not in tables, results can only be compared qualitatively. Although in Reference 3 calculations are done more long-winded, the results should still be the same. When results are compared, it can be seen that this is the case for all stress calculations except for one line: the edge calculation for the $[\pm 45^\circ]_s$-laminate with $\varphi = 30^\circ$ and $s/D = 1.5$. In Reference 3 this line doesn’t have the same tendency as the lines for $s/D = 3.0$ and $s/D = 4.5$, while in this report (Figure 7) it does. So most likely a mistake has been made there.

In Reference 3 numerical results in tables are only given for edge calculations on the isotropic case, with values for $s/D$ found in Reference 6. To be able to check the adequacy of the analysis of this report quantitatively, these calculations have been made here also (Table 3). Except for the very small pitch $s = 1.11D$, Table 3 shows good agreement with the calculated stresses for the isotropic case and the corresponding values of Reference 6. When the results, obtained with the present analysis, are compared with those of Reference 3, almost no differences can be found. This is not surprising; basically both analyses are the same. They both make use of power series and Taylor expansions. Only in the analysis presented here, a short cut to the solution has been made by directly transforming the variable $\xi_k^{(p)}$ into cosines and sines terms while Reference 3 uses more approximations.

Until now stress calculations have only been compared with theoretical values found in literature. However measurements from practical research are also available (Reference 8). In Figures 27 - 35 strains are given, measured on a finite plate (ARALL, GLARE) with only one central hole. These strains are compared with calculations done on an infinite plate (of the same material) with an infinite row of holes, with the pitch of the holes equal to width of the finite plate.
The first three figures show that the measured strains do not exactly match the strains calculated with the theory of anisotropic plates. Apparently it is not allowed to represent a finite plate with one hole by an infinite plate plate with an infinite row of holes. The differences between the measured and calculated strains get larger as the pitch to hole diameter ratio increases. Differences also get larger at higher stress levels. There the measured strain succeeds the yield strain of the aluminium whereas the strains, calculated with help of the computer program, are fully elastic.

Next, strain measurements done on a plate with slots are checked with calculated strains (Figures 30 - 31). Since the program doesn't allow calculations around slots, the slot must be approached by an ellipse with the following semi-axes

\[ d = \frac{W}{2} \]
\[ b = \sqrt{\rho \cdot d} \]

in which \( W \) represents the slot length and \( \rho \) the root radius of the slot (Reference 8). This naturally leads to differences between the measured and calculated strains, since the calculations are not in conformity with reality.

6.2 The row of three holes

The isotropic material (Figure 36) shows a first indication of the differences between the stress distribution around the holes of the infinite row and the row of three holes. As expected for larger \( s/D \) ratios the tangential stress ratio \( \sigma_t/p_y \) tends to go the value of 3 in the neighbourhood of \( \theta = 0^\circ \) and to the value of -1 around \( \theta = 90^\circ \). The differences in the stress distribution in the plate with the infinite row and the plate with the row of three holes are the largest for \( s/D = 1.5 \).

For the load case considered here (\( p_y = 1 \) and \( p_x = p_{xy} = 0 \)) it appears that for all laminates between \( \theta = 0^\circ \) and \( \theta = 180^\circ \) the tangential stresses in a plate with an infinite row are larger than the tangential stresses in a plate with a finite row (for \( \varphi = 0^\circ \)). This of course is due to the fact that the infinite row of holes reduces the area of the cross-section along the x-axis of the infinite plate, while the influence of the finite row of holes on the cross-section of the infinite plate is practically zero. For the finite row of holes, the tangential stresses around the central hole are larger than the tangential
stresses around the right hole. The reason for this is that the stress concentration around a hole decreases when there is less influence of the holes next to it. In the row of three holes, this is the case for the two end holes in comparison to the central hole.

Attention must be paid when looking at the figures of the edge calculations. The angle \( \theta \) runs both for the right hole and the central hole from 0° until 180° counterclockwise. So for instance, for the line of the right hole, \( \theta = 0° \) is the outer point on the right side of the finite row. When a point on the edge of the right hole, say \( \theta = 150° \), is to be compared with the opposite point on the edge of the central hole, then \( \theta = 30° \) must be taken. When looking at those figures for which \( \phi = 0° \), it can now be seen that for all laminates the tangential stress distribution on the edge of the central hole between \( \theta = 30° \) and \( \theta = 70° \) is practically equal to the tangential stress distribution of the right hole between \( \theta = 110° \) and \( \theta = 150° \).

The figures of the line calculations show clearly that for the finite row the stress ratio \( \sigma_y/p_y \) drops to 1 when moving away from the edge of the right hole. This is also the case for the \([\pm 45°]_s\)-laminate with \( s/D = 1.5 \), in contrast to the stress ratio of the infinite row of this laminate. There, the smallest stress ratio is found on the edge of the hole (Figure 60). As expected the stress ratio \( \sigma_y/p_y \) of the finite row does not increase between the holes when the \([\pm 45°]_s\)-laminate is orientated under an angle of \( \phi = 30° \). This is due to the large influence of the material orientation of this laminate.

6.3 Stress calculations as a design tool

The computer program does not only calculate stresses, it is also capable of strength calculations, using the modified Tsai-Hill failure criterion. To keep it simple, in this series only edge calculations have been made for two \( s/D \) ratios (\( s/D = 1.5 \) and \( s/D = 3 \), because they differ the most) and four different laminates under an angle \( \phi = 0° \). It should be noted that the failure criterion neglects important aspects such as three-dimensional stresses at the edge of the holes and non-linear elasticity of the laminates. Nevertheless the calculated stresses in combination with the corresponding strength values provide a good impression whether damage will occur or not, and where this happens first if it does.
On the edge of the hole, the peak in tangential stresses is always generated near the point where the direction of the highest Young’s modulus is parallel to the hole boundary. Since this direction coincides with the direction of the highest laminate strength, the maximum tangential stress does not generally produce first damage. First damage occurs on places with an unfavourable combination of $\sigma_t$-values and laminate strength in the direction of $\sigma_t$. Mostly this is the case near points with a high shear stress in the direction of the material axes. This is clearly shown in Figure 78, concerning the [$90^\circ$]$_s$-laminate. The highest stresses occur at $\theta = 0^\circ$ and $\theta = 180^\circ$. Yet the laminate strength there is sufficient. Damage will occur between $\theta = 2^\circ$ and $\theta = 30^\circ$. Above $\theta = 30^\circ$ the laminate can carry the stresses again.

When designing, a first reaction may be to increase the thickness of the [$90^\circ$]$_s$-laminate a little, so that tangential stresses along the edge of the hole are lowered. The stress line then keeps the same shape, and is lowered until it lies completely underneath the strength line. Since both the stress line and the strength line are more or less of the same shape, the material is used efficiently along the whole edge. Nevertheless attention must be paid here. When another material is chosen, not only the shape of the stress line but also the shape of the strength line changes and this might bring about a more favourable solution. When looking at the other laminates (Figures 80 - 85), the [$90^\circ/\pm 45^\circ$]$_s$-laminate is the most efficient (Figure 82). Not only can this laminate carry all stresses along the edge of the hole, but the laminate can also be lowered in thickness and still be able to carry the current stresses.

So as has been shown, changing the laminate changes both the strength line and the stress line. This makes the design process complicated. But that is not the only reason for this complexity. Another reason is that the maximum stresses do not always occur on places where they are expected. In a plate with holes you would for all laminates expect the maximum stress to occur somewhere on the edge of a hole, however this is not correct. An example is the [$\pm 45^\circ$]$_s$-laminate where the stresses rise when moving away from the edge of the hole at $\theta = 0^\circ$. 

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6.4 The all composite sandwich fuselage

As was stated in chapter five, the ultimate load on the undisturbed composite fuselage is represented by the following load cases:

\[
\begin{align*}
\sigma_x &= 212 \text{ MPa} \quad \sigma_y = 99 \text{ MPa} \quad \tau_{xy} = 160 \text{ MPa} \\
\sigma_x &= -113 \text{ MPa} \quad \sigma_y = 99 \text{ MPa} \quad \tau_{xy} = 160 \text{ MPa}
\end{align*}
\]

The stresses are carried by a sandwich shell consisting of a Nomex® core and two \([0^\circ-90^\circ/\pm 45^\circ/0^\circ-90^\circ]\)-C.F.R.P. facings with a thickness of 0.9 mm each (Reference 5). When windows are placed in the undisturbed fuselage, stress concentrations will occur on the edge of the holes. As can be seen in Figure 86 and 87, in both cases the facings are not able to carry the stresses and the sandwich shell will fail.

A simple solution to lower the stresses in the window section is to increase the laminate thickness. Figure 88 and Figure 89 show that if the total thickness of the two facings is increased with 4.2 mm to 6.0 mm, the laminate is just able to carry both load cases. The stress line stays between the upper and lower strength lines over the whole area. However the question arises whether it is possible to create a strength line more shaped like the stress line by using another laminate as reinforcement. Perhaps then the laminate doesn't need to be increased that much in thickness to be able to carry the stresses.

In order to create a more favourable strength line, a reinforcement is placed consisting of a \([\pm 45^\circ]\)-laminate. This it will increase the tangential strength around \(\theta = 45^\circ\) and \(\theta = 135^\circ\), just where it is needed. Figure 90 and Figure 91 show what happens when the window section is reinforced with a 4.2 mm \([\pm 45^\circ]\)-laminate. The strength around \(\theta = 45^\circ\) and \(\theta = 135^\circ\) has indeed risen, but not enough. It seems that the \([\pm 45^\circ]\)-laminate of C.F.R.P. does not only increase the strength, but also attracts load there. In fact, further calculations reveal that if only the \([\pm 45^\circ]\)-laminate is used as reinforcement, a total laminate thickness of 40 mm is required! This of course is no longer a realistic solution. However it indicates that there must be an optimum in the composition of the reinforcement sheet in such a way, that the thickness needed to carry the loadcases mentioned before, is reduced to a minimum. To determine this optimum, the minimal required thickness of the reinforcement sheet is calculated for different compositions (Figure 92).
The calculations are done for an increasing percentage of the [±45°]-laminate. From these calculations the conclusion can be drawn that the most efficient reinforcement sheet consists of (Figure 92):

60 % [±45°]-C.F.R.P. and 40 % [0°-90°]-C.F.R.P.

with a laminate thickness of

3.8 mm.

This means that both facings of the sandwich fuselage with an original thickness of 0.9 mm each, have to be reinforced with a laminate thickness of 1.9 mm.

It must be noted that the above discribed calculations reveal the most efficient configuration of the reinforcement sheet when it is composed of [±45°]- and [0°-90°]- laminates of C.F.R.P.. Of course other solutions are possible. For instance different orientations of the laminates can be used or a laminate with other mechanical properties. This has not been worked out here any further. More work however on the above mentioned will be published.
CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

The approximate solution for the calculation of stresses in an infinite orthotropic plate with an infinite row of circular holes has been evaluated by comparing the tangential stresses calculated for the isotropic case with values found in literature. Except for the very small pitch of $s=1.11D$, the agreement is good.

The influence of the pitch to hole diameter ratio $s/D$ on the stress distribution in the laminates shows a trend quite similar to that of the isotropic material, especially for $s/D \geq 3$.

The influence of the angle $\varphi$ on the stress distribution is stronger for the laminates dominated by the $90^\circ$-layers then for the laminates dominated by the $\pm 45^\circ$-layers.

Reasonable agreement was found between theoretical strain predictions and experimentally determined strains. Nevertheless deviations occurred for ARALL 1, GLARE 2 and GLARE 3, for specimens with large D/W values.

The maximum tangential stress concentration at the edge of the hole is not an indication for first significant damage of the laminate. First damage occurs mostly in the vicinity of points with a high shear stress in the direction of the material axes.

Designing in composites is a difficult process, because changing the laminate changes both the appearing stresses and strength values. Next to that, the maximum stresses do not always occur at places where they are expected.

So far, in the all composite sandwich fuselage the window section is most efficiently reinforced with a 1.9 mm C.F.R.P.-laminate consisting for 60% of $[\pm 45^\circ]$-C.F.R.P. and for 40% of $[0^\circ-90^\circ]$-C.F.R.P.. Nevertheless the application of other materials as reinforcement, like GLARE, must be investigated.

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REFERENCES

1. J. Laméris, *The use of load andhancement factors in the certification of composite aircraft structures*. Report TP 90068, National Aerospace Laboratory, 1990, Amsterdam, the Netherlands


7. Th. de Jong, *Versterkte materialen in de vliegtuigbouw* (Dutch). Report LR-84, Delft University of Technology, Faculty of Aerospace Engineering, August 1989, Delft, the Netherlands

Table 1: The mechanical properties and complex parameters of the materials

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<tr>
<th>material</th>
<th>$E_1$ (GPa)</th>
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<th>$G_{1,2}$ (GPa)</th>
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<td>0.976i</td>
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<td>$(90^\circ)_s$</td>
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<td>7.17</td>
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<td>0.897i</td>
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<td>$(\pm 45^\circ)_s$</td>
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<td>20.43</td>
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Table 2: The strength values of the laminates.

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Table 3: The tangential stress distribution $\sigma_r/p_y$ at the hole boundaries for an infinite row of circular holes with different values of $s/D$. Material: isotropic, load: $p_y=1$, $p_x=p_{xy}=0$. 
Fig. 1 A fuselage shell, reinforced with stringers and frames.

Fig. 2 An all composite sandwich fuselage.
Fig. 3 An infinite row of holes, equally shaped and equally spaced.

Fig. 4 A row of three holes, differently shaped and differently spaced.
Fig. 5 The tangential stress distribution at the hole boundaries in an isotropic material for an infinite row of holes. Load: $p_y=1$, $p_x=p_{xy}=0$. 
Material: $[\pm 45^\circ]_s$

![Graph](image)

**Fig. 6** The tangential stress distribution at the hole boundaries in a $[\pm 45^\circ]_s$-laminate for an infinite row of holes.
Laminate angle: $\phi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Material: $[\pm 45^\circ]_s$

![Graph](image)

**Fig. 7** The tangential stress distribution at the hole boundaries in a $[\pm 45^\circ]_s$-laminate for an infinite row of holes.
Laminate angle: $\phi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-41-
Fig. 8  The tangential stress distribution at the hole boundaries in a $[90^\circ/\pm 45^\circ]$-laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Fig. 9  The tangential stress distribution at the hole boundaries in a $[90^\circ/\pm 45^\circ]$-laminate for an infinite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-42-
**Fig. 10** The tangential stress distribution at the hole boundaries in a $[90^\circ_2/\pm45^\circ]_s$-laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

**Fig. 11** The tangential stress distribution at the hole boundaries in a $[90^\circ_2/\pm45^\circ]_s$-laminate for an infinite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-43-
Fig. 12 The tangential stress distribution at the hole boundaries in a $[90^\circ_4/\pm 45^\circ]_s$-laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Fig. 13 The tangential stress distribution at the hole boundaries in a $[90^\circ_4/\pm 45^\circ]_s$-laminate for an infinite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.  

-44-
Fig. 14 The tangential stress distribution at the hole boundaries in a $[90^\circ/0^\circ/\pm 45^\circ]_s$-laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Fig. 15 The tangential stress distribution at the hole boundaries in a $[90^\circ/0^\circ/\pm 45^\circ]_s$-laminate for an infinite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-45-
Fig. 16 The normal stress distribution along the x-axis in an isotropic material for an infinite row of holes. Load: $p_y = 1$, $p_x = p_{xy} = 0$. 
Material: $[\pm 45^\circ]_s$

**Fig. 17** The normal stress distribution along the x-axis in a $[\pm 45^\circ]_s$ laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Material: $[\pm 45^\circ]_s$

**Fig. 18** The normal stress distribution along the x-axis in a $[\pm 45^\circ]_s$ laminate for an infinite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-47-
**Fig. 19** The normal stress distribution along the x-axis in a $[90^\circ/\pm 45^\circ]_s$-laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

**Fig. 20** The normal stress distribution along the x-axis in a $[90^\circ/\pm 45^\circ]_s$-laminate for an infinite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-48-
**Fig. 21** The normal stress distribution along the x-axis
in a $[90^\circ_2/\pm 45^\circ]$-lamine for an infinite row of holes.
Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

**Fig. 22** The normal stress distribution along the x-axis
in a $[90^\circ_2/\pm 45^\circ]$-lamine for an infinite row of holes.
Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 
Fig. 23 The normal stress distribution along the x-axis in a $[90^\circ, \pm 45^\circ]$$_s$-laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Fig. 24 The normal stress distribution along the x-axis in a $[90^\circ, \pm 45^\circ]$$_s$-laminate for an infinite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-50-
Fig. 25 The normal stress distribution along the x-axis in a $[90^\circ/0^\circ/\pm 45^\circ]_s$-laminate for an infinite row of holes. Laminate angle: $\varphi = 0^\circ$, load: $p_y = 1$, $p_x = p_{xy} = 0$.

Fig. 26 The normal stress distribution along the x-axis in a $[90^\circ/0^\circ/\pm 45^\circ]_s$-laminate for an infinite row of holes. Laminate angle: $\varphi = 30^\circ$, load: $p_y = 1$, $p_x = p_{xy} = 0$. 

-51-
**Fig. 27** The comparison of experimental and theoretical values of the normal strain distribution along the x-axis for ARALL 1. Round hole: D = 50mm, finite width: s = 160mm.

**Fig. 28** The comparison of experimental and theoretical values of the normal strain distribution along the x-axis for ARALL 1. Round hole: D = 25mm, finite width: s = 160mm.
Fig. 29 The comparison of experimental and theoretical values of the normal strain distribution along the x-axis for ARALL 1. Round hole: D = 10mm, finite width: s = 160mm.
**Fig. 30** The comparison of experimental and theoretical values of the normal strain distribution along the x-axis for ARALL 1. Elliptical hole: W=25mm, ρ=5mm, finite width: s=160mm.

**Fig. 31** The comparison of experimental and theoretical values of the normal strain distribution along the x-axis for ARALL 1. Elliptical hole: W=25mm, ρ=2mm, finite width: s=160mm.
Fig. 32 The comparison of experimental and theoretical values of the normal strain distribution along the x-axis for GLARE 2.
Round hole: D = 25mm, finite width: s = 100mm.

Fig. 33 The comparison of experimental and theoretical values of the normal strain distribution along the x-axis for GLARE 2.
Round hole: D = 25mm, finite width: s = 100mm.
Fig. 34  The comparison of experimental and theoretical values of the normal strain distribution along the x-axis for GLARE 3. Round hole: $D = 25\text{mm}$, finite width: $s = 100\text{mm}$.

Fig. 35  The comparison of experimental and theoretical values of the normal strain distribution along the x-axis for GLARE 3. Round hole: $D = 25\text{mm}$, finite width: $s = 100\text{mm}$.
**Fig. 36** The tangential stress distribution at the hole boundaries in an isotropic material for an infinite and a finite row of holes. Load: $p_y = 1$, $p_x = p_{xy} = 0$.

**Fig. 37** The tangential stress distribution at the hole boundaries in an isotropic material for an infinite and a finite row of holes. Load: $p_y = 1$, $p_x = p_{xy} = 0$. 

-57-
Material: isotropic
s/D=4.5

Fig. 38  The tangential stress distribution at the hole boundaries
in an isotropic material for an infinite and a finite row of holes.
Load: p_y = 1, p_x = p_{xy} = 0.
Fig. 39  The tangential stress distribution at the hole boundaries in a \([\pm 45^\circ]_s\)-laminate for an infinite and finite row of holes. Laminate angle: \(\varphi=0^\circ\), load: \(p_y=1\), \(p_x=p_{xy}=0\).

Fig. 40  The tangential stress distribution at the hole boundaries in a \([\pm 45^\circ]_s\)-laminate for an infinite and finite row of holes. Laminate angle: \(\varphi=30^\circ\), load: \(p_y=1\), \(p_x=p_{xy}=0\).
Material: $[\pm 45^\circ]_s$

ts/D=3.0  $\phi=0^\circ$

---

**Fig. 41** The tangential stress distribution at the hole boundaries in a $[\pm 45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\phi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

---

Material: $[\pm 45^\circ]_s$

ts/D=3.0  $\phi=30^\circ$

---

**Fig. 42** The tangential stress distribution at the hole boundaries in a $[\pm 45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\phi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-60-
**Material: \([\pm 45^\circ]_s\)**

**s/D=4.5 \(\varphi=0^\circ\)**

Fig. 43 The tangential stress distribution at the hole boundaries in a \([\pm 45^\circ]_s\)-lamine for an infinite and finite row of holes.
Laminate angle: \(\varphi=0^\circ\), load: \(p_y=1\), \(p_x=p_{xy}=0\).

**Material: \([\pm 45^\circ]_s\)**

**s/D=4.5 \(\varphi=30^\circ\)**

Fig. 44 The tangential stress distribution at the hole boundaries in a \([\pm 45^\circ]_s\)-lamine for an infinite and finite row of holes.
Laminate angle: \(\varphi=30^\circ\), load: \(p_y=1\), \(p_x=p_{xy}=0\).
Fig. 45 The tangential stress distribution at the hole boundaries in a $[90_2/\pm 45^\circ]$$_L$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Fig. 46 The tangential stress distribution at the hole boundaries in a $[90_2/\pm 45^\circ]$$_L$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-62-
Fig. 47 The tangential stress distribution at the hole boundaries in a $[90^\circ, \pm 45^\circ]$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Fig. 48 The tangential stress distribution at the hole boundaries in a $[90^\circ, \pm 45^\circ]$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-63-
Material: $[90^\circ_2/\pm 45^\circ]_s$

$s/D=4.5 \; \varphi=0^\circ$

![Graph showing stress distribution](image)

**Fig. 49** The tangential stress distribution at the hole boundaries in a $[90^\circ_2/\pm 45^\circ]_s$-lamine for an infinite and finite row of holes.
Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Material: $[90^\circ_2/\pm 45^\circ]_s$

$s/D=4.5 \; \varphi=30^\circ$

![Graph showing stress distribution](image)

**Fig. 50** The tangential stress distribution at the hole boundaries in a $[90^\circ_2/\pm 45^\circ]_s$-lamine for an infinite and finite row of holes.
Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-64-
Material: $[90^\circ/0^\circ/\pm45^\circ]_s$

\[ s/D=1.5 \quad \phi=0^\circ \]

\[
\begin{array}{c}
\sigma_t/p_y \\
\theta \ (^\circ)
\end{array}
\]

---

**Fig. 51** The tangential stress distribution at the hole boundaries in a $[90^\circ/0^\circ/\pm45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\phi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

---

Material: $[90^\circ/0^\circ/\pm45^\circ]_s$

\[ s/D=1.5 \quad \phi=30^\circ \]

\[
\begin{array}{c}
\sigma_t/p_y \\
\theta \ (^\circ)
\end{array}
\]

---

**Fig. 52** The tangential stress distribution at the hole boundaries in a $[90^\circ/0^\circ/\pm45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\phi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

---

-65-
**Fig. 5.3** The tangential stress distribution at the hole boundaries in a \([90^\circ/0^\circ/\pm 45^\circ]_s\) laminate for an infinite and finite row of holes. Laminate angle: \(\varphi=0^\circ\), load: \(p_y=1\), \(p_x=p_{xy}=0\).

**Fig. 5.4** The tangential stress distribution at the hole boundaries in a \([90^\circ/0^\circ/\pm 45^\circ]_s\) laminate for an infinite and finite row of holes. Laminate angle: \(\varphi=30^\circ\), load: \(p_y=1\), \(p_x=p_{xy}=0\).
Fig. 55  The tangential stress distribution at the hole boundaries in a $[90^\circ/0^\circ/\pm 45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Fig. 56  The tangential stress distribution at the hole boundaries in a $[90^\circ/0^\circ/\pm 45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 
**Fig. 57** The normal stress distribution along the x-axis in an isotropic material for an infinite and a finite row of holes. Load: $p_y = 1$, $p_x = p_{xy} = 0$.

**Material: isotropic**

\[ s/D = 1.5 \]

\[ \sigma_y / p_y \]

\[ x/s \]

---

**Fig. 58** The normal stress distribution along the x-axis in an isotropic material for an infinite and a finite row of holes. Load: $p_y = 1$, $p_x = p_{xy} = 0$.
Material: isotropic
s/D=4.5

---

Fig. 59 The normal stress distribution along the x-axis in an isotropic material for an infinite and a finite row of holes. Load: \( p_y = 1, p_x = p_{xy} = 0 \).
**Fig. 60** The normal stress distribution along the x-axis in a $[\pm 45^\circ]$s-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

**Fig. 61** The normal stress distribution along the x-axis in a $[\pm 45^\circ]$s-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-70-
Material: $[\pm 45^\circ]_s$

$s/D=3.0 \ \ \phi=0^\circ$

Fig. 62 The normal stress distribution along the x-axis
in a $[\pm 45^\circ]_s$-lamine for an infinite and finite row of holes.
Laminate angle: $\phi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Material: $[\pm 45^\circ]_s$

$s/D=3.0 \ \ \phi=30^\circ$

Fig. 63 The normal stress distribution along the x-axis
in a $[\pm 45^\circ]_s$-lamine for an infinite and finite row of holes.
Laminate angle: $\phi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-71-
Fig. 64 The normal stress distribution along the x-axis in a $[\pm 45^\circ]^s$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_y=0$.

Fig. 65 The normal stress distribution along the x-axis in a $[\pm 45^\circ]^s$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_y=0$. 
Material: \([90^\circ_2/\pm45^\circ]_s\)

s/D=1.5  \(\varphi=0^\circ\)

![Graph](image)

**Fig. 66** The normal stress distribution along the x-axis in a \([90^\circ_2/\pm45^\circ]_s\)-laminate for an infinite and finite row of holes. Laminate angle: \(\varphi=0^\circ\), load: \(p_y=1\), \(p_x=p_{xy}=0\).

Material: \([90^\circ_2/\pm45^\circ]_s\)

s/D=1.5  \(\varphi=30^\circ\)

![Graph](image)

**Fig. 67** The normal stress distribution along the x-axis in a \([90^\circ_2/\pm45^\circ]_s\)-laminate for an infinite and finite row of holes. Laminate angle: \(\varphi=30^\circ\), load: \(p_y=1\), \(p_x=p_{xy}=0\).
Fig. 68 The normal stress distribution along the x-axis in a $[90^\circ_2/\pm 45^\circ]$\textsubscript{s}-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_z=p_{xy}=0$.

Fig. 69 The normal stress distribution along the x-axis in a $[90^\circ_2/\pm 45^\circ]$\textsubscript{s}-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_z=p_{xy}=0$. 
Material: $[90^\circ_2 / \pm 45^\circ]_s$
\[s/D=4.5 \ \varphi=0^\circ\]

![Graph](image1)

**Fig. 70** The normal stress distribution along the $x$-axis in a $[90^\circ_2 / \pm 45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Material: $[90^\circ_2 / \pm 45^\circ]_s$
\[s/D=4.5 \ \varphi=30^\circ\]

![Graph](image2)

**Fig. 71** The normal stress distribution along the $x$-axes in a $[90^\circ_2 / \pm 45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-75-
Fig. 72  The normal stress distribution along the x-axis in a $[90^\circ/0^\circ/\pm 45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

Fig. 73  The normal stress distribution along the x-axis in a $[90^\circ/0^\circ/\pm 45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-76-
Material: $[90^\circ/0^\circ/\pm 45^\circ]_s$
$s/D=3.0 \ \varphi=0^\circ$

\[
\begin{array}{c}
\sigma_y / P_y \\
0 \quad 0.50 \quad 1.00 \quad 1.50 \quad 2.00
\end{array}
\]

\[
\begin{array}{c}
\text{infinite row} \\
\text{finite row}
\end{array}
\]

**Fig. 74** The normal stress distribution along the x-axis in a $[90^\circ/0^\circ/\pm 45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Material: $[90^\circ/0^\circ/\pm 45^\circ]_s$
$s/D=3.0 \ \varphi=30^\circ$

\[
\begin{array}{c}
\sigma_y / P_y \\
0 \quad 0.50 \quad 1.00 \quad 1.50 \quad 2.00
\end{array}
\]

\[
\begin{array}{c}
\text{infinite row} \\
\text{finite row}
\end{array}
\]

**Fig. 75** The normal stress distribution along the x-axis in a $[90^\circ/0^\circ/\pm 45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-77-
Fig. 76 The normal stress distribution along the x-axis in a $[90^\circ/0^\circ/\pm 45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$.

Fig. 77 The normal stress distribution along the x-axis in a $[90^\circ/0^\circ/\pm 45^\circ]_s$-laminate for an infinite and finite row of holes. Laminate angle: $\varphi=30^\circ$, load: $p_y=1$, $p_x=p_{xy}=0$. 

-78-
Fig. 78  The tangential stress and strength distribution at the hole boundaries in a [90°]s-laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=100$ MPa, $p_x=p_{xy}=0$ MPa.

Fig. 79  The tangential stress and strength distribution at the hole boundaries in a [90°]s-laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=100$ MPa, $p_x=p_{xy}=0$ MPa.
Fig. 80  The tangential stress and strength distribution at the hole boundaries in a \([\pm 45^\circ]\)-laminate for an infinite row of holes. Laminate angle: \(\varphi=0^\circ\), load: \(p_y=100\) MPa, \(p_x=p_{xy}=0\) MPa.

Fig. 81  The tangential stress and strength distribution at the hole boundaries in a \([\pm 45^\circ]\)-laminate for an infinite row of holes. Laminate angle: \(\varphi=0^\circ\), load: \(p_y=100\) MPa, \(p_x=p_{xy}=0\) MPa.
Fig. 82 The tangential stress and strength distribution at the hole boundaries in a $[90^\circ / \pm 45^\circ]_s$-laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=100$ MPa, $p_x=p_{xy}=0$ MPa.

Fig. 83 The tangential stress and strength distribution at the hole boundaries in a $[90^\circ / \pm 45^\circ]_s$-laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=100$ MPa, $p_x=p_{xy}=0$ MPa.
Fig. 84 The tangential stress and strength distribution at the hole boundaries in a $[90^\circ/0^\circ/\pm 45^\circ]$-laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=100$ MPa, $p_z=p_{xy}=0$ MPa.

Fig. 85 The tangential stress and strength distribution at the hole boundaries in a $[90^\circ/0^\circ/\pm 45^\circ]$-laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_y=100$ MPa, $p_z=p_{xy}=0$ MPa.
Material: 1.8 mm $[0^\circ-90^\circ/\pm45^\circ/0^\circ-90^\circ]$  

---

**Fig. 86** The tangential stress and strength distribution at the hole boundaries in a 1.8 mm $[0^\circ-90^\circ/\pm45^\circ/0^\circ-90^\circ]$ -laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_x=212$ MPa, $p_y=100$ MPa, $p_{xy}=160$ MPa.

---

Material: 1.8 mm $[0^\circ-90^\circ/\pm45^\circ/0^\circ-90^\circ]$  

---

**Fig. 87** The tangential stress and strength distribution at the hole boundaries in a 1.8 mm $[0^\circ-90^\circ/\pm45^\circ/0^\circ-90^\circ]$ -laminate for an infinite row of holes. Laminate angle: $\varphi=0^\circ$, load: $p_x=-113$ MPa, $p_y=100$ MPa, $p_{xy}=160$ MPa.

-83-
Fig. 88  The tangential stress and strength distribution at the hole boundaries in a 6.0 mm [0°-90°/±45°/0°-90°]-laminate for an infinite row of holes. Laminate angle: φ=0°, load: p_x=63.6 MPa, p_y=29.7 MPa, p_xy=48.0 MPa.

Fig. 89  The tangential stress and strength distribution at the hole boundaries in a 6.0 mm [0°-90°/±45°/0°-90°]-laminate for an infinite row of holes. Laminate angle: φ=0°, load: p_x=-33.9 MPa, p_y=29.7 MPa, p_xy=48.0 MPa.
Material: 1.8 mm \([0^\circ-90^\circ/\pm 45^\circ/0^\circ-90^\circ]\) 
+ 4.2 mm \([\pm 45^\circ]\)

\[\sigma_t\text{ (MPa)}\]

\[\theta\text{ (°)}\]

**Fig. 90** The tangential stress and strength distribution at the hole boundaries in a 
1.8 mm \([0^\circ-90^\circ/\pm 45^\circ/0^\circ-90^\circ]\) + 4.2 mm \([\pm 45^\circ]\)-laminate for an infinite row of holes. 
Laminate angle: \(\varphi=0^\circ\), load: \(p_x=63.6\text{ MPa}, p_y=29.7\text{ MPa}, p_{xy}=48.0\text{ MPa}\).

Material: 1.8 mm \([0^\circ-90^\circ/\pm 45^\circ/0^\circ-90^\circ]\) 
+ 4.2 mm \([\pm 45^\circ]\)

\[\sigma_t\text{ (MPa)}\]

\[\theta\text{ (°)}\]

**Fig. 91** The tangential stress and strength distribution at the hole boundaries in a 
1.8 mm \([0^\circ-90^\circ/\pm 45^\circ/0^\circ-90^\circ]\) + 4.2 mm \([\pm 45^\circ]\)-laminate for an infinite row of holes. 
Laminate angle: \(\varphi=0^\circ\), load: \(p_x=-33.9\text{ MPa}, p_y=29.7\text{ MPa}, p_{xy}=48.0\text{ MPa}\).
Material: [0°-90°/±45°/0°-90°]

Fig. 92 The minimal thickness of the reinforcement sheet for different compositions of the [0°-90°/±45°/0°-90°] -laminate.
Appendix A

With the calculations on a stress field in an orthotropic plate, the complex variable $z_k$ is introduced (Reference 1)

$$z_k = x + \mu_k y$$  \hspace{1cm} (A.1)

$\mu_k$ is a complex parameter of the first order resulting from material properties, and can be represented by

$$\mu_k = \alpha_k + i \beta_k$$  \hspace{1cm} (A.2)

In this definition, $\alpha_k$ and $\beta_k$ are both real numbers. $\alpha_k$ can be smaller, equal or bigger than zero, unlike $\beta_k$ which can only be bigger than zero (Reference 3).

To simplify the analytical solution for an orthotropic plate with a row of elliptical holes, $z_k$ is being transformed to

$$\zeta_k^{(p)} = \frac{(z_k - ps) \pm \sqrt{(z_k - ps)^2 - \mu_k^2 b^2 - d^2}}{d - i \mu_k b}$$

$$- \frac{(z_k - ps) \pm z_k^*}{d - i \mu_k b}$$  \hspace{1cm} (A.3)

in which $d$ represents the semi-axis of the ellips in $x$-direction and $b$ the semi-axis of the ellips in $y$-direction. Equation (A.3) shows that $\zeta_k^{(p)}$ is double valued, but only one value of $\zeta_k^{(p)}$ belongs to the corresponding coordinate $z_k$. So a choice has to be made between the plus and minus sign in such a way that the next two conditions are satisfied

1) On the edge of hole $p$, the next equation must be fulfilled

$$\zeta_k = \zeta_1 = \zeta_2 = \sigma = \cos \theta + i \sin \theta$$

2) Outside the edge of hole $p$, $\zeta_k^{(p)}$ must be a continuous function of $z_k$

Here, criteria will be derived in such a way that these two conditions are satisfied.
The first condition

On the edge hole \( p \), the coordinates \( x \) and \( y \) are equal to

\[
\begin{align*}
    x &= ps + d\cos\theta \\
    y &= b\sin\theta
\end{align*}
\]  
(A.4)

With these coordinates, (A.4) can be worked out as follows

\[
\pm z'_k \quad = 
\begin{array}{l}
(d - i\mu_k b)\sigma - (z_k - ps) \\
- (d - i\mu_k b)(\cos\theta + is\sin\theta) - (d\cos\theta + \mu_k b\sin\theta) \\
- \beta_k b\cos\theta + i(d\sin\theta - \alpha_k b\cos\theta) \\
- \beta_k b x' + i\left(\frac{d}{b}y' - \alpha_k \frac{b}{d}x'\right)
\end{array}
\]  
(A.5)

in which \( x' \) and \( y' \) are the coordinates related to the origin of the coordinate axes of hole \( p \). The first condition now, demands that the complex numbers left and right in (A.5) are equal. Since \( \beta_k \) is always positive, the choice between the + and the - sign must be made in such a way that for \( x' \neq 0 \) it yields

\[
x'.Re\{z'_k\} > 0
\]  
(A.6)

In other words the real part of \( z'_k \) must have the same sign as \( x' \). When \( x' = 0 \), the upper criterion is no longer useful. Then the imaginary parts of the complex numbers left and right in (A.5) must be equal. Then the choice between the + and the - sign must be made in such a way that for \( x' = 0 \) yields

\[
y'.Im\{z'_k\} > 0
\]  
(A.7)

With equations (A.6) and (A.7) the choice between the + and - sign is fixed for \( z'_k \) on the edge of hole \( p \). This is graphically represented in the figure below (\( \theta = \tan\alpha_k \)).
The second condition

The continuity of the function \( \zeta_k^{(p)} \) requires that for a point B outside the hole p the same choice of sign is made as for a point A on the edge of the hole, unless, when going from A to B, boundaries are crossed on which either the imaginary part or the real part of \( z_k' \) is equal to zero. On such a boundary, where either the imaginary part or the real part of \( z_k' \) is equal to zero, it yields

\[
\text{Im}(z_k - ps)^2 - \mu_k^2b^2 - d^2 = 0
\]  
(A.8)

So a boundary is crossed when

\[
2\beta_k(\alpha_k(y^2 - b^2) + x'y') = 0
\]  
(A.9)

Because \( \beta_k \neq 0 \) this means for \( \alpha_k = 0 \)

\[
x' = 0 \lor y' = 0
\]  
(A.10)

and for \( \alpha_k \neq 0 \)

\[
y' = \frac{-x' \pm \sqrt{x'^2 + 4\alpha_k^2b^2}}{2\alpha_k}
\]  
(A.11)

These boundaries can be graphically represented as follows:

---

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Conclusion

From the two conditions in every quadrant the choice for $z_k'$ between a $+$ and a $-$ sign can now be made. For this, the following resulting criteria can be used ($\tan \theta = y/x$):

\[
\alpha_k > 0 \\
\begin{align*}
0 & \leq \theta < 1/2 \pi \\
1/2 \pi & \leq \theta < \pi \\
\pi & \leq \theta < 3/2 \pi \\
3/2 \pi & \leq \theta < 2\pi
\end{align*}
\begin{align*}
\Re z' & > 0 \\
\Im z' & > 0 \\
\Re z' & < 0 \\
\Im z' & < 0
\end{align*}
\]

(A.12-a)

\[
\alpha_k = 0 \\
\begin{align*}
-1/2 \pi & < \theta < 1/2 \pi \\
1/2 \pi & < \theta < 3/2 \pi \\
\theta & = 1/2 \pi \\
\theta & < -1/2 \pi
\end{align*}
\begin{align*}
\Re z' & > 0 \\
\Re z' & < 0 \\
\Im z' & > 0 \\
\Im z' & < 0
\end{align*}
\]

(A.12-b)

\[
\alpha_k < 0 \\
\begin{align*}
0 & < \theta \leq 1/2 \pi \\
1/2 \pi & < \theta \leq \pi \\
\pi & < \theta \leq 3/2 \pi \\
3/2 \pi & < \theta \leq 2\pi
\end{align*}
\begin{align*}
\Im z' & > 0 \\
\Re z' & < 0 \\
\Im z' & < 0 \\
\Re z' & > 0
\end{align*}
\]

(A.12-c)

With these criteria, every complex coordinate $z = x + iy$ corresponds with only one value of $z_k$. 

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Appendix B

To solve the boundary conditions, it is necessary that on the edge of the hole considered, the complex stress function is expressed into cosines and sines terms. What this comes down to, is that on the edge of that hole, $z_k$ is expressed into cosines and sines terms. With this purpose, $z_k$ will now be rewritten.

In the complex plane, the unity-circle is defined as

\[ \sigma = \cos \theta + i \sin \theta \]
\[ \bar{\sigma} = \cos \theta - i \sin \theta \]  \hspace{1cm} (B.1)

So,

\[ \cos \theta = \frac{\sigma + \bar{\sigma}}{2} \]  \hspace{1cm} (B.2)
\[ \sin \theta = \frac{\sigma - \bar{\sigma}}{2i} \]

In the real plane, the equation for an ellipse with semi-axis $d$ on the $x$-axis and semi-axis $b$ on the $y$-axis is

\[ \left( \frac{x}{d} \right)^2 + \left( \frac{y}{b} \right)^2 = 1 \]  \hspace{1cm} (B.3)

So, on the edge of the elliptical hole, $x$ and $y$ become

\[ x = d \cos \theta = d \frac{\sigma + \bar{\sigma}}{2} \]  \hspace{1cm} (B.4)
\[ y = b \sin \theta = b \frac{\sigma - \bar{\sigma}}{2i} \]

Furthermore $z_k = x + \mu_k y$, so on the edge of the elliptical hole $z_k$ is equal to

\[ z_k = d \frac{\sigma + \bar{\sigma}}{2} + \mu_k b \frac{\sigma - \bar{\sigma}}{2} \]  \hspace{1cm} (B.5)
or, after rewriting

$$z_k = \frac{d - i\mu_k b}{2} \sigma + \frac{d + i\mu_k b}{2} \frac{\sigma}{\bar{\sigma}}$$  \hspace{1cm} \text{(B.6)}$$

After multiplying (B.6) with $\sigma$, an equation of the second grade arises, from which $\sigma$ can be solved

$$\sigma = \frac{z_k \pm \sqrt{z_k^2 - \mu_k^2 b^2 - d^2}}{d - i\mu_k b}$$  \hspace{1cm} \text{(B.7)}$$

The right side of this equation is defined as the variable $\zeta_k$ and is corrected for the hole index

$$\zeta_k = \frac{(z_k - ps) \pm \sqrt{(z_k - ps)^2 - \mu_k^2 b^2 - d^2}}{d - i\mu_k b}$$  \hspace{1cm} \text{(B.8)}$$

On the edge of hole $p$, $\zeta_k$ then becomes

$$\zeta_1 - \zeta_2 - \sigma = \cos\theta + i\sin\theta$$  \hspace{1cm} \text{(B.9)}$$

which was the goal of this translation.
APPENDIX C

To solve the boundary conditions, the variable $\zeta_k^{(p)}$ has been introduced. Here eight derivatives of the variable in $z_k = 0$ are given, which are used in the Taylor expansion. The general expression for $\zeta_k^{(p)}$ is

$$
\zeta_k^{(p)} = \frac{(z_k - ps) \pm \sqrt{(z_k - ps)^2 - \mu_k^2 b^2 - d^2}}{d - i \mu_k b}
$$

(C.1)

Instead of writing the derivatives of $\zeta_k^{(p)}$ in $z_k = 0$ long-winded, a more comfortable notation is introduced.

$$
\left( \frac{d^{\text{der}}}{dz_k^{\text{der}}} (\zeta_k^{(p)})^{-n} \right)_{z_k = 0} = \zeta_k^{(p)}(p, s)
$$

(C.2)

Now the eight derivatives are given. The first derivative of $\zeta_k^{(p)}$ in $z_k = 0$ is

$$
(\zeta_k^{(1)}(p, s))^{-n} = -n \cdot \left( \frac{(-ps) \pm \sqrt{(-ps)^2 - \mu_k^2 b^2 - d^2}}{d - i \mu_k b} \right)^{-n}.
$$

(C.3)

The second derivative of $\zeta_k^{(p)}$ in $z_k = 0$ is

$$
(\zeta_k^{(2)}(p, s))^{-n} = -n \cdot \left( \frac{(-ps) \pm \sqrt{(-ps)^2 - \mu_k^2 b^2 - d^2}}{d - i \mu_k b} \right)^{-n}.
$$

(C.4)

The third derivative of $\zeta_k^{(p)}$ in $z_k = 0$ is

$$
(\zeta_k^{(3)}(p, s))^{-n} = -n \cdot \left( \frac{(-ps) \pm \sqrt{(-ps)^2 - \mu_k^2 b^2 - d^2}}{d - i \mu_k b} \right)^{-n}.
$$

(C.5)
The fourth derivative of $\zeta_k^{(p)}$ in $z_k = 0$ is

\[
\{\zeta_k^{(4)}(p,s)\}^{-n} = -n \cdot \frac{(-ps) \pm \sqrt{(-ps)^2 - \mu_k^2 b^2 - d^2}}{d - i\mu_k b} \cdot \left[ \frac{n^3 - 4n}{((-ps)^2 - \mu_k^2 b^2 - d^2)^2} - \frac{(6n^2 - 9)(ps)}{\pm ((-ps)^2 - \mu_k^2 b^2 - d^2)^{3/2}} \right. \\
+ \frac{15n(ps)^2}{((-ps)^2 - \mu_k^2 b^2 - d^2)^3} - \frac{15(ps)^3}{\pm ((-ps)^2 - \mu_k^2 b^2 - d^2)^{7/2}} \left. \right]
\]

(C.6)

The fifth derivative of $\zeta_k^{(p)}$ in $z_k = 0$ is

\[
\{\zeta_k^{(5)}(p,s)\}^{-n} = -n \cdot \frac{(-ps) \pm \sqrt{(-ps)^2 - \mu_k^2 b^2 - d^2}}{d - i\mu_k b} \cdot \left[ \frac{n^4 - 10n^2 + 9}{\pm ((-ps)^2 - \mu_k^2 b^2 - d^2)^{5/2}} - \frac{(10n^3 - 55n)(ps)}{((-ps)^2 - \mu_k^2 b^2 - d^2)^3} \right. \\
+ \frac{(45n^2 - 90)(ps)^2}{\pm ((-ps)^2 - \mu_k^2 b^2 - d^2)^{7/2}} - \frac{105n(ps)^3}{((-ps)^2 - \mu_k^2 b^2 - d^2)^4} \\
+ \frac{105(ps)^4}{\pm ((-ps)^2 - \mu_k^2 b^2 - d^2)^{9/2}} \left. \right]
\]

(C.7)

The sixth derivative of $\zeta_k^{(p)}$ in $z_k = 0$ is

\[
\{\zeta_k^{(6)}(p,s)\}^{-n} = +n \cdot \frac{(-ps) \pm \sqrt{(-ps)^2 - \mu_k^2 b^2 - d^2}}{d - i\mu_k b} \cdot \left[ \frac{n^5 - 20n^3 + 64n}{((-ps)^2 - \mu_k^2 b^2 - d^2)^3} - \frac{(15n^4 - 195n^2 + 225)(ps)}{\pm ((-ps)^2 - \mu_k^2 b^2 - d^2)^{7/2}} \right. \\
+ \frac{(15n^3 - 735n)(ps)^2}{((-ps)^2 - \mu_k^2 b^2 - d^2)^4} - \frac{(420n^2 - 1050)(ps)^3}{\pm ((-ps)^2 - \mu_k^2 b^2 - d^2)^{9/2}} \\
+ \frac{945n(ps)^4}{((-ps)^2 - \mu_k^2 b^2 - d^2)^5} - \frac{945(ps)^5}{\pm ((-ps)^2 - \mu_k^2 b^2 - d^2)^{11/2}} \left. \right]
\]

(C.8)
The seventh derivative of $\zeta_k^{(p)}$ in $z_k = 0$ is

\[
(\zeta_{k,n}^{(7)}(p,s))^{-n} = -n \cdot \frac{(-ps) \pm \sqrt{(-ps)^2 - \mu_k^2 b^2 - d^2}}{d - i\mu_k b} \\
\pm \frac{n^6 - 35n^4 + 259n^2 - 225}{((-ps)^2 - \mu_k^2 b^2 - d^2)^{7/2}} - \frac{(21n^5 - 525n^3 + 2079n)(ps)}{((-ps)^2 - \mu_k^2 b^2 - d^2)^4} \\
\pm \frac{(210n^4 - 3360n^2 + 4725)(ps)^2}{((-ps)^2 - \mu_k^2 b^2 - d^2)^{9/2}} - \frac{(1260n^3 - 10710n)(ps)^3}{((-ps)^2 - \mu_k^2 b^2 - d^2)^5} \\
\pm \frac{4725n^2 - 14175)(ps)^4}{((-ps)^2 - \mu_k^2 b^2 - d^2)^{11/2}} - \frac{(10395n)(ps)^5}{((-ps)^2 - \mu_k^2 b^2 - d^2)^6} \\
\pm \frac{10395(ps)^6}{((-ps)^2 - \mu_k^2 b^2 - d^2)^{13/2}}
\]

(C.9)

The eighth derivative of $\zeta_k^{(p)}$ in $z_k = 0$ is

\[
(\zeta_{k,n}^{(8)}(p,s))^{-n} = -n \cdot \frac{(-ps) \pm \sqrt{(-ps)^2 - \mu_k^2 b^2 - d^2}}{d - i\mu_k b} \\
\pm \frac{n^7 - 56n^5 + 784n^3 - 2304n}{((-ps)^2 - \mu_k^2 b^2 - d^2)^4} - \frac{(28n^6 - 1190n^4 + 10612n^2 - 11025)(ps)}{((-ps)^2 - \mu_k^2 b^2 - d^2)^{9/2}} \\
\pm \frac{(378n^5 - 11340n^3 + 53487n)(ps)^2}{((-ps)^2 - \mu_k^2 b^2 - d^2)^5} - \frac{(3150n^4 - 59850n^2 + 99225)(ps)^3}{((-ps)^2 - \mu_k^2 b^2 - d^2)^{11/2}} \\
\pm \frac{17325n^3 - 173250n)(ps)^4}{((-ps)^2 - \mu_k^2 b^2 - d^2)^6} - \frac{(62370n^2 - 218295)(ps)^5}{((-ps)^2 - \mu_k^2 b^2 - d^2)^{13/2}} \\
\pm \frac{(135135n)(ps)^6}{((-ps)^2 - \mu_k^2 b^2 - d^2)^7} - \frac{135135(ps)^7}{((-ps)^2 - \mu_k^2 b^2 - d^2)^{15/2}}
\]

(C.10)

which is the final derivative given here.
APPENDIX D

On the edge of the hole in which the coordinate system is placed, \((\zeta_k^{(p)})^{-n}\) has to be expressed into cosines and sines terms, to be able to calculate the stress functions coefficients. When \(p=0\), this is easy

\[
(\zeta_k^{(p)})^{-n} = \cos n\theta - isin n\theta \tag{D.1}
\]

When \(p\neq 0\) however, \((\zeta_k^{(p)})^{-n}\) is expanded into a Taylor series in an area around the center of the coordinate system.

\[
(\zeta_k^{(p)})^{-n} = \frac{z_k^0}{0!} \zeta_{k,n}^{(0)}(p,s) + \frac{z_k^1}{1!} \zeta_{k,n}^{(1)}(p,s) + \frac{z_k^2}{2!} \zeta_{k,n}^{(2)}(p,s) + \frac{z_k^3}{3!} \zeta_{k,n}^{(3)}(p,s) \tag{D.2}
\]

\[+ \frac{z_k^4}{4!} \zeta_{k,n}^{(4)}(p,s) + \frac{z_k^5}{5!} \zeta_{k,n}^{(5)}(p,s) + \frac{z_k^6}{6!} \zeta_{k,n}^{(6)}(p,s) + \frac{z_k^7}{7!} \zeta_{k,n}^{(7)}(p,s) + \frac{z_k^8}{8!} \zeta_{k,n}^{(8)}(p,s)\]

In these series, the complex powers \(z_k^m\) are converted into cosines and sines terms with help of the binomial of Newton.

As stated in Appendix B, on the edge of the hole in which the coordinate axes are placed, \(z_k\) can be written as

\[
z_k = \frac{d - i\mu_k b}{2}\sigma + \frac{d + i\mu_k b}{2}\bar{\sigma} \tag{D.3}
\]

or, in a more convenient way

\[
z_k = \left(\frac{d - i\mu_k b}{2}\right)(\sigma + \lambda_k\bar{\sigma}) \tag{D.4}
\]

with

\[
\lambda_k = \frac{d + i\mu_k b}{d - i\mu_k b} \tag{D.5}
\]

The binomial of Newton yields

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{D.6}
\]

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So, $z_k^m$ now becomes

$$z_k^m = \left( \frac{d - i\mu_k b}{2} \right) \sum_{r=0}^{m} \lambda_k^r \binom{m}{r} \sigma^{m-r} \bar{\sigma}^r$$

(D.7)

Since $\sigma \bar{\sigma} = 1$, this can be written as

$$z_k^m = \left( \frac{d - i\mu_k b}{2} \right) \sum_{r=0}^{m} \lambda_k^r \binom{m}{r} \sigma^{m-2r}$$

(D.8)
APPENDIX E

The variable $\zeta_k^{(p)}$ has been defined as

$$\zeta_k^{(p)} = \frac{(z_k - ps) \pm \sqrt{(z_k - ps)^2 - \mu_k^2 b^2 - d^2}}{d - i\mu_k b} \quad (E.1)$$

On the edge of the hole in which the coordinate system is placed, $(\zeta_k^{(p)})^{-n}$ has to be expressed into cosines and sines terms, to be able to calculate the coefficients of the complex stress function. When $p = 0$, this is easy

$$(\zeta_k^{(p)})^{-n} = \cos n\theta - i\sin n\theta \quad (E.2)$$

When $p \neq 0$ however, $(\zeta_k^{(p)})^{-n}$ has to be expanded into a Taylor series in an area around the center of the coordinate system. In this series, eight terms are accounted for.

$$(\zeta_k^{(p)})^{-n} = \frac{z_k^0}{0!} \zeta_{k,n}^{(0)}(p,s) + \frac{z_k^1}{1!} \zeta_{k,n}^{(1)}(p,s) + \frac{z_k^2}{2!} \zeta_{k,n}^{(2)}(p,s) + \frac{z_k^3}{3!} \zeta_{k,n}^{(3)}(p,s)$$
$$+ \frac{z_k^4}{4!} \zeta_{k,n}^{(4)}(p,s) + \frac{z_k^5}{5!} \zeta_{k,n}^{(5)}(p,s) + \frac{z_k^6}{6!} \zeta_{k,n}^{(6)}(p,s) + \frac{z_k^7}{7!} \zeta_{k,n}^{(7)}(p,s) + \frac{z_k^8}{8!} \zeta_{k,n}^{(8)}(p,s) \quad (E.3)$$

The values of the derivatives $\zeta_{k,n}^{(der)}(p,s)$ are given in Appendix C. The variable $z_k^m$ can be written as (Appendix D)

$$z_k^m = \left(-\frac{d - i\mu_k b}{2}\right)^m \sum_{r=0}^{m} \lambda_k^{(r)} \binom{m}{r} \sigma^{m-2r} \quad (E.4)$$

With the help of (E.4) $z_k^m$ is worked out for $m = 0$ until $m = 8$ (see next page).
\[ z^0_k = \left( \frac{d - i \mu_k b^0}{2} \right) \left[ \lambda_k^0 (0) \sigma^0 \right] \]

\[ z^1_k = \left( \frac{d - i \mu_k b^1}{2} \right) \left[ \lambda_k^1 (1) \sigma^{-1} \right] \]

\[ z^2_k = \left( \frac{d - i \mu_k b^2}{2} \right) \left[ \lambda_k^2 (2) \sigma^0 + \lambda_k^2 (2) \sigma^{-2} \right] \]

\[ z^3_k = \left( \frac{d - i \mu_k b^3}{2} \right) \left[ \lambda_k^3 (3) \sigma^1 + \lambda_k^3 (3) \sigma^{-1} + \lambda_k^3 (3) \sigma^{-3} \right] \]

\[ z^4_k = \left( \frac{d - i \mu_k b^4}{2} \right) \left[ \lambda_k^4 (4) \sigma^2 + \lambda_k^4 (4) \sigma^{-2} + \lambda_k^4 (4) \sigma^{-4} \right] \]

\[ z^5_k = \left( \frac{d - i \mu_k b^5}{2} \right) \left[ \lambda_k^5 (5) \sigma^3 + \lambda_k^5 (5) \sigma^{-1} + \lambda_k^5 (5) \sigma^{-3} + \lambda_k^5 (5) \sigma^{-5} \right] \]

\[ z^6_k = \left( \frac{d - i \mu_k b^6}{2} \right) \left[ \lambda_k^6 (6) \sigma^4 + \lambda_k^6 (6) \sigma^{-2} + \lambda_k^6 (6) \sigma^{-4} \right] \]

\[ z^7_k = \left( \frac{d - i \mu_k b^7}{2} \right) \left[ \lambda_k^7 (7) \sigma^5 + \lambda_k^7 (7) \sigma^3 + \lambda_k^7 (7) \sigma^{-1} + \lambda_k^7 (7) \sigma^{-3} \right] \]

\[ z^8_k = \left( \frac{d - i \mu_k b^8}{2} \right) \left[ \lambda_k^8 (8) \sigma^6 + \lambda_k^8 (8) \sigma^4 + \lambda_k^8 (8) \sigma^2 \right] \]

(E.5)
Equation (E.5) can be arranged more conveniently

\[ \zeta_k^0 = \left( \frac{d - i \mu_k b}{2} \right) \left[ \lambda_k^0 (\sigma^0) \right] \]

\[ \zeta_k^1 = \left( \frac{d - i \mu_k b}{2} \right) \left[ \sigma + \lambda_k \sigma^{-1} \right] \]

\[ \zeta_k^2 = \left( \frac{d - i \mu_k b}{2} \right) \left[ \binom{2}{0} (\sigma^2 + \lambda_k^2 \sigma^{-2}) + \lambda_k \binom{2}{1} \sigma^0 \right] \]

\[ \zeta_k^3 = \left( \frac{d - i \mu_k b}{2} \right) \left[ \binom{3}{0} (\sigma^3 + \lambda_k^3 \sigma^{-3}) + \lambda_k \binom{3}{1} (\sigma + \lambda_k \sigma^{-1}) \right] \]

\[ \zeta_k^4 = \left( \frac{d - i \mu_k b}{2} \right) \left[ \binom{4}{0} (\sigma^4 + \lambda_k^4 \sigma^{-4}) + \lambda_k \binom{4}{1} (\sigma^2 + \lambda_k^2 \sigma^{-2}) + \lambda_k^2 \binom{4}{2} \sigma^0 \right] \]

\[ \zeta_k^5 = \left( \frac{d - i \mu_k b}{2} \right) \left[ \binom{5}{0} (\sigma^5 + \lambda_k^5 \sigma^{-5}) + \lambda_k \binom{5}{1} (\sigma^3 + \lambda_k^3 \sigma^{-3}) + \lambda_k^3 \binom{5}{2} (\sigma + \lambda_k \sigma^{-1}) \right] \]

\[ \zeta_k^6 = \left( \frac{d - i \mu_k b}{2} \right) \left[ \binom{6}{0} (\sigma^6 + \lambda_k^6 \sigma^{-6}) + \lambda_k \binom{6}{1} (\sigma^4 + \lambda_k^4 \sigma^{-4}) + \lambda_k^2 \binom{6}{2} (\sigma^2 + \lambda_k^2 \sigma^{-2}) + \lambda_k^3 \binom{6}{3} \sigma^0 \right] \]

\[ \zeta_k^7 = \left( \frac{d - i \mu_k b}{2} \right) \left[ \binom{7}{0} (\sigma^7 + \lambda_k^7 \sigma^{-7}) + \lambda_k \binom{7}{1} (\sigma^5 + \lambda_k^5 \sigma^{-5}) + \lambda_k^2 \binom{7}{2} (\sigma^3 + \lambda_k^3 \sigma^{-3}) + \lambda_k^3 \binom{7}{3} (\sigma + \lambda_k \sigma^{-1}) \right] \]

\[ \zeta_k^8 = \left( \frac{d - i \mu_k b}{2} \right) \left[ \binom{8}{0} (\sigma^8 + \lambda_k^8 \sigma^{-8}) + \lambda_k \binom{8}{1} (\sigma^6 + \lambda_k^6 \sigma^{-6}) + \lambda_k^2 \binom{8}{2} (\sigma^4 + \lambda_k^4 \sigma^{-4}) + \lambda_k^3 \binom{8}{3} (\sigma^2 + \lambda_k^2 \sigma^{-2}) + \lambda_k^4 \binom{8}{4} \sigma^0 \right] \]

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These expressions for $z_k^n$ are substituted in (E.3). When equal terms of $\sigma$ are linked up, this yields

\[
(c_k^{(p)})^{-n} - \zeta_{k,n}(p,s) + \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^2 \zeta_{k,n}^{(2)}(p,s) + \frac{\lambda_k^4}{4!} \left( \frac{d - i\mu_k b}{2} \right)^4 \zeta_{k,n}^{(4)}(p,s) \\
+ \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^6 \frac{\zeta_{k,n}^{(6)}(p,s)}{6!} + \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^8 \frac{\zeta_{k,n}^{(8)}(p,s)}{8!} \\
+ (\sigma + \lambda_k \sigma^{-1}) \left[ \binom{1}{0} \left( \frac{d - i\mu_k b}{2} \right)^{\binom{1}{0}} \frac{\zeta_{k,n}^{(1)}(p,s)}{1!} + \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^{\binom{3}{1}} \frac{\zeta_{k,n}^{(3)}(p,s)}{3!} \\
+ \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^{\binom{5}{2}} \frac{\zeta_{k,n}^{(5)}(p,s)}{5!} + \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^{\binom{7}{3}} \frac{\zeta_{k,n}^{(7)}(p,s)}{7!} \right] \\
+ (\sigma^2 + \lambda_k^2 \sigma^{-2}) \left[ \binom{2}{0} \left( \frac{d - i\mu_k b}{2} \right)^2 \frac{\zeta_{k,n}^{(2)}(p,s)}{2!} + \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^4 \frac{\zeta_{k,n}^{(4)}(p,s)}{4!} \\
+ \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^6 \frac{\zeta_{k,n}^{(6)}(p,s)}{6!} + \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^8 \frac{\zeta_{k,n}^{(8)}(p,s)}{8!} \right] \\
+ (\sigma^3 + \lambda_k^3 \sigma^{-3}) \left[ \binom{3}{0} \left( \frac{d - i\mu_k b}{2} \right)^3 \frac{\zeta_{k,n}^{(3)}(p,s)}{3!} + \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^5 \frac{\zeta_{k,n}^{(5)}(p,s)}{5!} \\
+ \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^7 \frac{\zeta_{k,n}^{(7)}(p,s)}{7!} \right] \\
+ (\sigma^4 + \lambda_k^4 \sigma^{-4}) \left[ \binom{4}{0} \left( \frac{d - i\mu_k b}{2} \right)^4 \frac{\zeta_{k,n}^{(4)}(p,s)}{4!} + \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^6 \frac{\zeta_{k,n}^{(6)}(p,s)}{6!} \\
+ \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^8 \frac{\zeta_{k,n}^{(8)}(p,s)}{8!} \right] \\
+ (\sigma^5 + \lambda_k^5 \sigma^{-5}) \left[ \binom{5}{0} \left( \frac{d - i\mu_k b}{2} \right)^5 \frac{\zeta_{k,n}^{(5)}(p,s)}{5!} + \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^7 \frac{\zeta_{k,n}^{(7)}(p,s)}{7!} \right] \\
+ (\sigma^6 + \lambda_k^6 \sigma^{-6}) \left[ \binom{6}{0} \left( \frac{d - i\mu_k b}{2} \right)^6 \frac{\zeta_{k,n}^{(6)}(p,s)}{6!} + \lambda_k \left( \frac{d - i\mu_k b}{2} \right)^8 \frac{\zeta_{k,n}^{(8)}(p,s)}{8!} \right] \\
+ (\sigma^7 + \lambda_k^7 \sigma^{-7}) \left[ \binom{7}{0} \left( \frac{d - i\mu_k b}{2} \right)^7 \frac{\zeta_{k,n}^{(7)}(p,s)}{7!} \right] \\
+ (\sigma^8 + \lambda_k^8 \sigma^{-8}) \left[ \binom{8}{0} \left( \frac{d - i\mu_k b}{2} \right)^8 \frac{\zeta_{k,n}^{(8)}(p,s)}{8!} \right] \right]
\]
Then a more convenient notation is introduced

\[
(\zeta_k^{(p)})^{-n} = c_{k,n}^{(0)}(p,s) \cdot (\sigma^0 + \lambda_k^0 \sigma^{-0}) \\
+ c_{k,n}^{(1)}(p,s) \cdot (\sigma^1 + \lambda_k^1 \sigma^{-1}) \\
+ c_{k,n}^{(2)}(p,s) \cdot (\sigma^2 + \lambda_k^2 \sigma^{-2}) \\
+ c_{k,n}^{(3)}(p,s) \cdot (\sigma^3 + \lambda_k^3 \sigma^{-3}) \\
+ c_{k,n}^{(4)}(p,s) \cdot (\sigma^4 + \lambda_k^4 \sigma^{-4}) \\
+ c_{k,n}^{(5)}(p,s) \cdot (\sigma^5 + \lambda_k^5 \sigma^{-5}) \\
+ c_{k,n}^{(6)}(p,s) \cdot (\sigma^6 + \lambda_k^6 \sigma^{-6}) \\
+ c_{k,n}^{(7)}(p,s) \cdot (\sigma^7 + \lambda_k^7 \sigma^{-7}) \\
+ c_{k,n}^{(8)}(p,s) \cdot (\sigma^8 + \lambda_k^8 \sigma^{-8}) \\
- \sum_{m=0}^{8} c_{k,n}^{(m)}(p,s) \cdot (\sigma^m + \lambda_k^m \sigma^{-m})
\] (E.8)

and this is the final expression for \((\zeta_k^{(p)})^{-n}\) for \(p \neq 0\).