THE MAGNETOROTATIONAL INSTABILITY IN CYLINDRICAL TAYLOR-COUETTE FLOW: FROM LINEAR EIGENVALUE TO NONLINEAR TIME-STEPPING CODES

Rainer Hollerbach

Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom

Key words: Magnetorotational instability, Taylor-Couette flow

Abstract. We consider the flow of an electrically conducting fluid between differentially rotating cylinders, in the presence of an externally imposed magnetic field. Several codes are presented to study this problem, consisting of linear eigenvalue and nonlinear time-stepping programs. Depending on whether the magnetic Reynolds number is small or large, the induction equation may also be treated as either diagnostic or predictive. These codes are used to study the magnetorotational instability in the presence of combined axial and azimuthal magnetic fields.

1 INTRODUCTION

The magnetorotational instability is a mechanism whereby a differential rotation flow that is hydrodynamically stable may nevertheless be magnetohydrodynamically unstable. It was first discovered by Velikhov\textsuperscript{1} in precisely the Taylor-Couette problem considered here, and was subsequently rediscovered in an astrophysical context by Balbus and Hawley\textsuperscript{2}. Because of its astrophysical importance, there is considerable interest in returning to the Taylor-Couette problem and studying it further, both theoretically and ideally also experimentally\textsuperscript{3}. In this note we outline a series of codes to solve this problem numerically.

2 BASIC EQUATIONS

Consider two concentric cylinders of radii \( r_i \) and \( r_o \), rotating at rates \( \Omega_i \) and \( \Omega_o \). Let \( B_0 = B_0 [\hat{e}_z + \beta(r_i/r)\hat{e}_\phi] \) be an externally imposed magnetic field, where the axial field \( B_0\hat{e}_z \) would be established by azimuthal electric currents in the region \( r > r_o \), and the azimuthal field \( B_0\beta(r_i/r)\hat{e}_\phi \) by an axial current in the region \( r < r_i \). Scaling length by \( r_i \), time by \( \Omega_i^{-1} \), \( \mathbf{U} \) by \( r_i \Omega_i \), and \( \mathbf{B} \) by \( B_0 \), the governing equations become

\[
Re \left( \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) = -\nabla p + \nabla^2 \mathbf{U} + Ha^2 Rm^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{1}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = Rm^{-1} \nabla^2 \mathbf{B} + \nabla \times (\mathbf{U} \times \mathbf{B}), \tag{2}
\]
where
\begin{align*}
Re &= \frac{\Omega_i r_i^2}{\nu}, \\
Rm &= \frac{\Omega_o r_i^2}{\eta}, \\
Ha &= \frac{B_o r_i}{\sqrt{\mu \mu / \eta}},
\end{align*}
with \( \rho, \nu, \eta \) and \( \mu \) the density, viscosity, magnetic diffusivity and permeability. The ratio \( Rm/Re = \nu/\eta = Pm \), the magnetic Prandtl number. Typical liquid metals have \( Pm = O(10^{-5}) \) or even smaller, meaning that the magnetic diffusive timescale is orders of magnitude smaller than the viscous timescale.

Next decompose the flow and field as
\begin{align*}
U &= U_0 + u, \\
B &= B_0 + Rm \mathbf{b},
\end{align*}
where
\begin{align*}
U_0 &= r \Omega(r) \hat{e}_\phi, \\
\Omega(r) &= A + Br^{-2}, \\
A &= \frac{\hat{\mu} - \hat{\eta}^2}{1 - \hat{\eta}^2}, \\
B &= \frac{1 - \hat{\mu}}{1 - \hat{\eta}^2}, \\
\hat{\mu} &= \frac{\Omega_o}{\Omega_i}, \\
\hat{\eta} &= \frac{r_i}{r_o},
\end{align*}
is the now suitably nondimensionalized differential rotation profile set up by the imposed rotation rates of the inner and outer cylinders, and
\begin{align*}
B_0 &= \hat{e}_z + \beta r^{-1} \hat{e}_\phi
\end{align*}
is the similarly nondimensionalized externally imposed field.

Equations (1–2) then become
\begin{align*}
Re \frac{\partial u}{\partial t} &= -\nabla p + \nabla^2 u + Ha^2 (\nabla \times \mathbf{b}) \times (B_0 + Rm \mathbf{b}) \\
&\quad - Re (U_0 \cdot \nabla u + u \cdot \nabla U_0 + u \cdot \nabla u), \\
Rm \frac{\partial \mathbf{b}}{\partial t} &= \nabla^2 \mathbf{b} + \nabla \times (u \times B_0) + Rm \nabla \times ((U_0 + u) \times \mathbf{b}).
\end{align*}

At this point we can understand also why we chose to include this factor \( Rm \) in the induced field \( Rm \mathbf{b} \); while the original system (1–2) makes no sense in the limit \( Rm \to 0 \), this new system does. \( Rm \mathbf{b} \) is then infinitesimally small, but \( \mathbf{b} \) itself is not, and is dynamically important. In this \( Rm \to 0 \) limit the mathematical character of the problem also changes; the induction equation (8) becomes an elliptic equation to be inverted at each timestep of (7), rather than a parabolic equation to be time-stepped along with (7). The advantage of this is that the timestep is then no longer limited by the magnetic diffusive timescale, which as we saw is typically very small, certainly compared with the viscous timescale. Of course, for many problems this \( Rm \to 0 \) limit is not appropriate, because the phenomena of interest only occur for sufficiently large \( Rm \). In that case one has no choice but to time-step (7) and (8) together, and simply accept that as one decreases \( Pm \) toward realistic values, the problem will inevitably get harder and harder, as the disparity between the magnetic and viscous timescales increases. It turns out that for the magnetorotational instability, both limits may be applicable; if the azimuthal field \( \beta r^{-1} \hat{e}_\phi \) is absent one cannot set \( Rm \to 0 \), but if it is present one can.
3 LINEAR EIGENVALUES

Apply the poloidal-toroidal decomposition
\[ u = \nabla \times (\psi \hat{e}_\phi) + v \hat{e}_\phi, \quad b = \nabla \times (a \hat{e}_\phi) + b \hat{e}_\phi \] (9)
applicable to axisymmetric solutions. Linearizing (7) and (8), and taking \( \psi, v, a \) and \( b \) to have the \( t \) and \( z \) dependence \( \exp(\gamma t + i k z) \), we obtain
\[
\text{Re} \gamma D^2 \psi &= D^4 \psi - \text{Re} 2i k \Omega v + Ha^2 ik (D^2 a + 2 \beta r^{-2} b), \\
\text{Re} \gamma v &= D^2 v + \text{Re} ik r^{-1}(r^2 \Omega)' \psi + Ha^2 ik b, \\
Rm \gamma b &= D^2 b - Rm ik \Omega' r a + ik v - 2ik \beta r^{-2} \psi, \\
Rm \gamma a &= D^2 a + ik \psi,
\] (10)
where \( D^2 = \nabla^2 - 1/r^2 \), and the primes denote \( d/dr \).

The associated boundary conditions (no-slip for \( u \), insulating for \( b \)) are
\[
v = \psi = \frac{\partial}{\partial r} \psi = 0 \quad \text{at} \quad r = r_i, \ r_o, \\
b = 0, \quad \frac{\partial}{\partial r} a - \left[ \frac{kI_1(kr_i)}{I_1(kr_i)} \right] a = 0 \quad \text{at} \quad r = r_i, \\
b = 0, \quad \frac{\partial}{\partial r} a - \left[ \frac{kK_1'(kr_o)}{K_1(kr_o)} \right] a = 0 \quad \text{at} \quad r = r_o,
\] (11)
where \( I_1 \) and \( K_1 \) are the modified Bessel functions.

Expanding the radial structures of \( \psi, v, a \) and \( b \) in terms of Chebyshev polynomials then converts this differential eigenvalue problem to a matrix eigenvalue problem, which can easily be solved for any choice of the various input parameters. Detailed results are presented by Hollerbach and Rüdiger\textsuperscript{4}. Here we merely note that in order to understand the results, one must focus on the two terms \( Rm ik \Omega' r a \) and \( 2ik \beta r^{-2} \psi \), both in (10c). If \( \beta = 0 \), the first term is needed to regenerate \( b \), and the magnetorotational instability does not set in until \( Rm \geq O(10) \). In this case therefore one cannot consider the \( Rm \to 0 \) limit. However, if \( \beta \geq O(1) \), the first term is no longer needed, and the magnetorotational instability turns out to set in when \( Re \geq O(10^3) \), with \( Rm \) no longer relevant at all. In this case one can therefore apply this \( Rm \to 0 \) simplification.

4 NONLINEAR TIME-STEPPING

To study the nonlinear equilibration of these instabilities, we still use the decomposition (9), but no longer restrict attention to a single \( e^{ikz} \) mode, and also no longer linearize the problem. Equations (7) and (8) then yield
\[
\left( Re \frac{\partial}{\partial t} - D^2 \right) D^2 \psi = -\hat{e}_\phi \cdot \nabla \times F_1, \\
\left( Re \frac{\partial}{\partial t} - D^2 \right) v = \hat{e}_\phi \cdot F_1, \\
\left( Rm \frac{\partial}{\partial t} - D^2 \right) a = \hat{e}_\phi \cdot F_2, \\
\left( Rm \frac{\partial}{\partial t} - D^2 \right) b = \hat{e}_\phi \cdot \nabla \times F_2,
\]
(12)

where
\[
F_1 = Ha^2 (\nabla \times b) \times (B_0 + Rm b) - Re (U_0 \cdot \nabla u + u \cdot \nabla U_0 + u \cdot \nabla u),
\]
\[
F_2 = u \times B_0 + Rm (U_0 + u) \times b.
\]
(13)

Expanding in Fourier modes in $z$, and Chebyshev polynomials in $r$, these equations may then be time-stepped, as described for example in the spherical case by Hollerbach\textsuperscript{5}. And again, we note that the code can easily be adapted to the $Rm \rightarrow 0$ limit, in which case only $\psi$ and $v$ are time-stepped, but $a$ and $b$ directly inverted for at each timestep of $\psi$ and $v$.

REFERENCES


