Stellingen behorende bij het proefschrift

TRANSIENT ACOUSTIC WAVES IN CONTINUOUSLY LAYERED MEDIA

van

Martin Daniël Verweij

Delft, 17 december 1992
1. De in hoofdstuk 3 van dit proefschrift beschreven methode voor de bepaling van het transiënte akoestische golveld in continu gelaagde media kan op eenvoudige wijze worden aangepast voor de bepaling van transiënte elastodynamische of elektromagnetische golvelden.

2. Hoewel het mogelijk is om een interpretatie te geven van de WKBJ-iteratieve oplossing die in hoofdstuk 3 van dit proefschrift wordt gebruikt, mag deze interpretatie niet worden verheven tot de feitelijke fysische situatie die optreedt bij golfpropagatie in continu gelaagde media.

3. Het is een inherent gegeven dat een term van de orde \( i \geq 1 \) van de WKBJ-iteratieve oplossing slechts afhangt van een term van de orde \( (i - 1) \). Het is dan ook niet zinvol om te spreken van eventuele "reflectieverliezen" van een term van de orde \( (i - 1) \) ten gevolge van het ontstaan van een term van de orde \( i \).


4. Door het gebruiken van dezelfde notatie voor een exponentiële functie met een scalair argument en een exponentiële functie met een matrix-argument, wordt ten onrechte gesuggereerd dat de eigenschappen van beide functies gelijk zijn. Zo is de incorrecte analyse die in het artikel van Nwoke is gemaakt, vermoedelijk terug te voeren tot het niet opmerken van de verschillen tussen beide typen exponentiële functies bij differentiatie.


5. Omdat de verbinding tussen de aanwijzer en het aangewezen ontbreekt, is een "laserpointer" in principe minder geschikt om iets aan te wijzen dan een aanwijsstok.
6. In het onderzoek kan veel moeite worden bespaard indien er ook artikelen zouden verschijnen, waarin wordt uitgelegd waarom een op het eerste gezicht succesvol lijkende wijze van aanpak toch niet tot het gewenste resultaat leidt.

7. Het belang van het analyseren van benaderende, eenvoudige modellen komt vooral voort uit het feit dat het soort resultaten waarover wij gemakkelijk kunnen praten in de praktijk feitelijk niet optreden.


8. Veel mensen die zich gaan toeleggen op een studie techniek beschikken nog niet over een goede studietechniek.

9. Het is belangrijk om aan jonge mensen duidelijk te maken dat een individualistische levensstijl zoals deze onder andere door de media vaak wordt ge-propageerd, hen hoogstwaarschijnlijk niet het gesuggereerde geluk zal opleveren. Juist vrijwilligers in het jeugdwerk kunnen in dit opzicht een voorbeeld voor een andere dan bovengenoemde levensstijl geven.

10. Het is wenselijk dat een nieuwe versie van het tekstverwerkingsprogramma \LaTeX\ zich verontschuldigt na het optreden van cynische soutmeldingen van het onderliggende programma \TeX\ ten gevolge van op zich redelijke opdrachten aan \LaTeX\.
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TRANSIENT ACOUSTIC WAVES IN CONTINUOUSLY LAYERED MEDIA

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"There is a mine for silver and a place where gold is refined.
Iron is taken from the earth, and copper is smelted from ore.
Man puts an end to the darkness; he searches the farthest recesses for ore in the blackest darkness.
Far from where people dwell he cuts a shaft, in places forgotten by the foot of man; far from men he dangles and sways.
The earth, from which food comes, is transformed below as by fire;
sapphires come from its rocks, and its dust contains nuggets of gold.
No bird of prey knows that hidden path, no falcon's eye has seen it.
Proud beasts do not set food on it, and no lion prowls there.
Man's hand assaults the flinty rock and lays bare the roots of the mountains.
He tunnels through the rock; his eyes see all its treasures.
He searches the sources of the rivers and brings hidden things to light.

"But where can wisdom be found? Where does understanding dwell?
Man does not comprehend its worth; it cannot be found in the land of the living.
The deep says, 'It is not in me'; the sea says, 'It is not with me.'
It cannot be bought with the finest gold, nor can its price be weighted in silver.
It cannot be bought with the gold of Ophir, with precious onyx or sapphires.
Neither gold nor crystal can compare with it, nor can it be had for jewels of gold.
Coral and jasper are not worthy of mention; the price of wisdom is beyond rubies.
The topaz of Cush cannot compare with it; it cannot be bought with pure gold.

"Where then does wisdom come from? Where does understanding dwell?
It is hidden from the eyes of every living thing, concealed even from the birds of the air.
Destruction and Death say, 'Only a rumour of it has reached our ears.'
God understands the way to it and he alone knows where it dwells,
for he views the ends of the earth and sees everything under the heavens.
When he established the force of the wind and measured out the waters,
when he made a decree for the rain and a path for the thunderstorm,
then he looked at wisdom and appraised it; he confirmed it and tested it.
And he said to man, 'The fear of the Lord—that is wisdom, and to shun evil is understanding.'"

— The Holy Bible, the book of Job, chapter 28
Chapter 1

Introduction

In this chapter it is first explained how physical fields of different nature and time behavior are used as a tool for the determination of the distribution of one or more medium parameters that characterize the materials inside the Earth. From these medium parameter distributions, a prediction can be made about the composition of the Earth and, as in the case of exploration geophysics, about the possible location of natural resources (oil, gas, minerals, ores, water). However, in order to deduce the distribution of medium parameters from a measured field, it is often necessary to be able to determine the field in a known configuration. The subject of this thesis is the development of methods for the determination of the space-time domain acoustic wavefield in an horizontally continuously layered and isotropic medium; although there are situations in which such methods can be useful, up till now these are rare. Since the configuration is shift invariant in the horizontal directions as well as in the time, integral transformation methods are considered suitable for this. Each integral transformation method consists of three basic steps: the transformation of the space-time domain problem using integral transformations, the solution of the resulting transform domain problem, and the transformation of this solution back to the space-time domain. For each of these steps several mathematical methods exist, from which we make specific choices in this thesis in order to derive two complete integral transformation methods. At the end of this chapter an outline of this thesis is presented.
1.1. Statement of the problem

One of the aims of geophysics is the determination of the distribution of the materials (rock, sediments) inside the Earth. Since it is very expensive to take samples from all but the most shallow parts of the Earth’s interior, the composition of the Earth is often determined using so-called imaging and inversion techniques. These techniques have been based on two facts. Firstly, physical fields of different nature (seismic, electric, magnetic, electromagnetic, or gravitational) and of different time behavior (static, harmonic, or transient) have the ability to penetrate into a wide class of materials. Secondly, the behavior of these physical fields at one point is influenced by the properties of the materials elsewhere else. Using physical fields as a tool, knowledge can be obtained about the distribution of one or more medium parameters characterizing the materials inside the Earth (GRANT & WEST, 1965; TELFORD et al., 1976; PARASNIS, 1986). In turn, with these data the distribution of the actual materials in the Earth’s interior is predicted. The application of physical fields for the determination of the composition of the Earth is used for various objectives, such as the location of oil, gas, minerals, ores, and water, for economic exploration in case of exploration geophysics. To indicate its importance, we note that in the year 1990 the total worldwide expenditures for exploration geophysics were approximately US$ 2.2 billion, of which no less than approximately US$ 2.1 billion was spent on hydrocarbon (oil and gas) exploration (GOODFELLOW, 1991).

In this thesis we focus our attention on the transient seismic (acoustic in fluids and gasses, or elastodynamic in solids) wavefield. This kind of wavefield is widely used with exploration geophysics. We define the ensemble of a certain number of seismic sources, a certain number of seismic receivers (geophones or hydrophones), and the distribution of the relevant medium parameters (acoustic or elastic constants) of the materials that occupy space, as the configuration in which the seismic wavefield is present. The problem of determining an unknown wavefield in a known configuration is called a direct problem. If, however, the wavefield at the receiver(s) is known and we want to obtain knowledge about the distribution of the medium parameters, we speak of an inverse parameter problem. An overview of the known and unknown aspects that play a part in the direct problem and in the inverse parameter problem is given in table 1.1. Clearly, the most important problem in
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<table>
<thead>
<tr>
<th>Aspect</th>
<th>Direct problem</th>
<th>Inverse parameter problem</th>
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<tbody>
<tr>
<td>Configuration</td>
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<td></td>
<td>Receiver(s)</td>
<td>known</td>
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<td>Medium parameters</td>
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<tr>
<td>Wavefield at receivers</td>
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Table 1.1. Classification of the direct problem and the inverse parameter problem.

exploration geophysics is the inverse parameter problem. For the solution of an inverse parameter problem, it is often important, however, that we are able to solve the direct problem in the expected class of configurations. This both follows from a theoretical point of view (since we must know which behavior of the wavefield we can expect under what circumstances), and from a numerical point of view (many computer programs used for solving the inverse parameter problem require the repeated evaluation of the wavefield).

After these introductory remarks, we will now formulate the subject of this thesis, being: the development of mathematical methods for solving the space-time domain direct problem for an acoustic wavefield in a horizontally continuously layered and isotropic fluid.

In order to motivate the development of methods for finding a wavefield in a continuously layered configuration, we note that there exist situations in which the medium parameters of the materials inside the Earth vary in a certain direction in a continuous fashion. Another possibility is that we only want to take into account the slow, continuous changes of the medium parameters, i.e., the trend of the medium parameters. In this thesis we develop mathematical methods that can deal with continuously layered configurations since up till now such methods are rare. To fully concentrate on the implications of the continuous behavior of the medium parameters, we have decided to keep the analysis as simple as possible. For this reason we have assumed that the medium is horizontally layered, i.e., the medium parameters only vary in the vertical direction. Moreover, we have assumed that the medium is isotropic. Another way of keeping the analysis as simple as possible
is achieved by considering the *acoustic wavefield* and not the elastodynamic wavefield. The reason for the development of space-time domain methods is two-fold. Firstly, all physical processes take place in the *space-time domain*, and in essence all wavefield quantities are space-time-domain quantities. Another point is that in the space-time domain all physical quantities are causal; in many imaging and inversion methods the causality of the applied quantities plays an important role.

As a possible application of the methods to be developed we mention the generation of synthetic seismograms for horizontally continuously layered configurations. These can be used as a test case for methods that can deal with more complex configurations for which the solution of the wave propagation problem is more difficult. Moreover, the methods can be employed with imaging and inversion methods for the determination of the wavefield in the background medium or macro model, which is described by the trend of the medium parameters.

1.2. Integral transformation methods

In view of the invariance of the medium with respect to the horizontal directions and the time, we restrict our research to the class of integral transformation methods. With these methods, we employ the specific invariances of the medium by applying integral transformations in order to reduce the complexity of the problem (reduction of dimension through parametrization). After solving the transformed version of the problem, the results are transformed back to the space-time domain. Thus, the three basic steps of our integral transformation methods are

1. the transformation of the space-time domain problem using integral transformations with respect to the spatial directions and the time;

2. the solution of the resulting transform domain problem;

3. the transformation of the transform domain solution back to the space-time domain.

Next, we discuss some frequently employed methods for the three parts of an integral transformation method.
1.2.1. Methods for the transformation of the space-time domain problem to the transform domain problem

For the transformation with respect to time we consider

The (two-sided) temporal Fourier transformation
This integral transformation (PAPoulis, 1962) can be applied to a large class of two-sided functions. The inverse transformation is of the same form as the forward transformation. Many numerical routines exist for a fast evaluation of the discretized version of both the forward and the inverse Fourier transformation (FFT, Fast Fourier Transform). From a theoretical point of view the Fourier transformation with a real transformation parameter $\omega$ is not well suited for the transformation of temporal functions, since causality (positive-sidedness) can not be conserved.

The (one-sided) temporal Laplace transformation
With this integral transformation (Widder, 1946, 1989; Van der Pol & Bremmer, 1959; Henrici, 1977) a large class of one-sided functions can be dealt with. From a theoretical point of view, the Laplace transformation is well suited for the transformation of temporal functions, since causality can be conserved by taking a transformation parameter $s$ with a positive real part. In fact, it is possible to keep the transformation parameter strictly real. There also exists an inverse Laplace transformation in which the transformation parameter is kept real as well, but it is of a different form as the forward Laplace transformation. Numerical routines for performing the inverse transformation with a real transformation parameter are rare.

For the transformation with respect to the horizontal directions we mention

The (two-sided) spatial Fourier transformation
For the transformation with respect to the horizontal directions, in virtually all cases the two-sided Fourier transformation (Papoulis, 1962) is used. As
already mentioned, the inverse transformation is of the same form as the forward transformation, and many numerical routines exist for a fast evaluation of the discretized version of both the forward and the inverse Fourier transformation. The Fourier transformation can deal well with spatial functions, which are essentially two-sided.

1.2.2. Methods for the solution of the transform domain problem

In case of a horizontally layered configuration, the transform domain problem consists of solving a coupled system of ordinary differential equations, or an equivalent system of integral equations. The general mathematical form of the system of differential equations is known to be (Chapman, 1974a, 1976; Kennett, 1983; De Hoop, 1988, 1990; Verweij & De Hoop, 1990)

\[
\frac{db(x)}{dx} + M(x)b(x) = u(x),
\]

(1.1)
in which \(b(x)\) is a vector representing the wavefield, \(M(x)\) is a matrix composed of medium parameters, and \(u(x)\) is a vector that accounts for the action of the source. In the literature the following four methods have been commonly employed for solving the transform domain problem (Chapman, 1981; Chapman & Orcutt, 1985; Kennett, 1983)

WKBJ asymptotics

With this method we first derive the WKBJ asymptotic expansion of the solution of the transform domain problem. This WKBJ asymptotic expansion consists of an asymptotic expansion (Erdélyi, 1956) times an exponential function, which represents wave propagation. Once the WKBJ asymptotic expansion is known, an WKBJ asymptotic representation of the transform domain solution is derived by omitting the terms higher than a certain order in the WKBJ asymptotic expansion. Two features limit the application of this method: firstly, the elements of the matrix \(M\) must satisfy requirements with respect to their differentiability in order to avoid the breakdown of the WKBJ asymptotic expansion after a number of terms, and secondly the method breaks down at a zero of the eigenvalues of \(M\).
Langer asymptotics
This method is similar to the previous method, except that it does not break
down at a point where the squares of the eigenvalues of $M$ have a single
zero. To achieve this, wave propagation must now be represented by the
more intricate Airy functions. The elements of the matrix $M$ must satisfy
differentiability requirements in order to avoid the breakdown of the Langer
asymptotic expansion after a number of terms, and the method breaks down
at a point where the squares of the eigenvalues of $M$ have a multiple zero.

The WKBJ iterative solution
This method is also known under several other names, such as the Bremmer
series and the Picard method of successive approximations. With this method
the solution of the transform domain problem is in fact the Neumann series
solution of an integral equation that is equivalent with eq. (1.1), in which
exponential functions occur that represent the propagation of the wavefield.
In principle the method can be applied in those cases where the elements of
$M$ are continuous; the derivatives of the elements of $M$ may be discontinu-
ous. However, the WKBJ iterative solution does not always converge, so it is
important to determine whether the class of elements of $M$ must be further
restricted. The method breaks down in a point where the eigenvalues of $M$
become zero.

The Langer iterative solution
This method resembles the WKBJ iterative solution, except that another in-
tegral equation is used as the equivalent of eq. (1.1). In this integral equation
the propagation of the wavefield is represented by the more intricate Airy func-
tions. The resulting Neumann series solution will not break down at a point
where the squares of the eigenvalues of $M$ have a single zero. In principle the
method can be applied in those cases where the elements of $M$ are contin-
uous; the derivatives of the elements of $M$ may be discontinuous. However,
the Langer iterative solution does not always converge, so it is important to
determine whether the class of elements of $M$ must be further restricted. The
method breaks down at a point where the squares of the eigenvalues of $M$
have a multiple zero.
1.2.3. Methods for the transformation of the transform domain solution to the space-time domain solution

In order to return to the space-time domain, the transform domain solution must be subjected to an inverse transformation process that forms the counterpart of the first step of an integral transformation method. The inverse transformation process is frequently performed using a straightforward numerical evaluation of the inverse integral transformations. Although this approach is simple, drawbacks are that the required numerical effort is in general large. Therefore, methods have been derived in which analytical techniques are applied in order to obtain expressions that can be evaluated with less numerical effort. The following four methods are regularly used (Chapman, 1978; Chapman & Orcutt, 1985).

The spectral method with a real slowness contour

With this method the inverse spatial transformation is performed first. As a part, this requires the evaluation of an integral in which the integration variable is called the horizontal slowness. In the present case the contour of integration of this integral is kept real; on this real contour, the integrand is in general highly oscillatory. Nevertheless, a fast evaluation of this integral is made possible by using approximate analytical methods. After performing the inverse spatial transformation, the intermediate result is in the time-transformed domain. Finally, the transformation from this spectral domain to the time domain is performed. Due to the nature of the result in this spectral domain, this transformation is carried out numerically.

The spectral method with a complex slowness contour

This method resembles the previous method, except that the oscillatory behavior of the integrand of the slowness integral is (partly) circumvented by deformation of the real contour of integration into the complex slowness plane. Although this leads to some numerical advantages, the analytical effort in choosing the correct complex contour can be considerable.

The slowness method with a real slowness contour

This method is also known as the Chapman method. With this method, first the inverse temporal transformation is performed in a purely analytical manner. The intermediate result lies in the space-transformed domain, which is
also called the slowness domain. The transformation from the slowness domain to the space-domain requires, among others, the evaluation of an integral in which the integration variable is called the horizontal slowness. In this case the contour of integration of this integral is kept real. It has turned out that this integral can be approximated by a discrete sum, the evaluation of which requires little numerical effort. As such, in its approximate form the method is elegant and fast. If, however, the latter integral has to be evaluated exactly, a more complicated situation arises and the method looses much of its attraction.

The slowness method with a complex slowness contour

Other names of this method are the generalized ray method, the modified Cagniard method, or the Cagniard-De Hoop method. Using this method, first the inverse spatial transformation is carried out formally. The resulting integral expression is manipulated in such a way that at the end it contains an integral that can be recognized as the integral that defines the forward temporal transformation. At this point the time-domain result is determined by inspection. By doing so, both the inverse temporal transformation is carried out, while at the same time we eliminate one of the integrals due to the inverse spatial transformation. Even if no approximations are made during the inversion process, the numerical gain of this method is considerable. The method can be speeded up even more by using an appropriate approximation.

1.3. Objectives and outline of this thesis

In this thesis two different integral transformation methods are presented for the determination of the space-time domain acoustic wavefield in a horizontally continuous layered and isotropic fluid.

In chapter 2, a precise description is given of the continuously layered configuration that is used in our investigations. Further, the space-time domain basic acoustic equations that describe the wavefield in this configuration are presented. This is a system of coupled partial differential equations. As the first step of both integral transformation methods to be developed, we subject the space-time domain basic acoustic equations to a Laplace transformation with respect to the time and a
Fourier transformation with respect to the horizontal directions. Through this, the dimension of the remaining problem is reduced to one by parametrization. The system of coupled ordinary differential equations that describes the transform domain problem is derived, as well as an equivalent system of coupled integral equations. The fact that in this thesis we keep the transformation parameter of the temporal Laplace transformation real and positive is of fundamental importance, since hereby we avoid many difficulties that are commonly encountered, such as the occurrence of turning points in the transform domain system of differential equations, are avoided.

In chapter 3, a first integral transformation method is completed. The aim of this chapter is to develop a method by which it is in principle possible to obtain sufficiently accurate results for the acoustic space-time domain acoustic wavefield at any time instant and in any continuously layered configuration. Taking the transform domain system of integral transformations as a starting point, the second step of this integral transformation method, which is the solution of the transform domain problem, is performed using the WKBJ iterative solution (Neumann series). Due to the transformation scheme that has been applied in chapter 2, we are able to prove the convergence of the transform domain solution for every horizontally continuously layered configuration. Moreover, due to the transformation scheme, we achieve convergence of the corresponding space-time domain solution for any time instant as well. This implies that with this integral transformation method it is in principle possible to get sufficiently accurate space-time domain results for any time instant and for every horizontally continuously layered configuration. Further, the transformation back to the space-time domain, which forms the third step of this integral transformation method, is performed with the aid of the slowness method with a complex slowness contour (Cagniard-De Hoop method). As an important item in this thesis, we explain in detail the application of this method in case of continuous layered media. Next, the numerical implementation of the theory is discussed and numerical results are presented. The chapter concludes with a discussion of the method.

In chapter 4, an alternative integral transformation method is described. In this chapter our goal is to derive a method by which the higher-order behavior of the space-time domain acoustic wavefield right after its arrival can be determined. We expect that the resulting higher-order early-time asymptotic representations will
lead to accurate approximations of the wavefield over a time interval of nonzero length beyond the arrival time. To achieve this, as the second step of this integral transformation method we solve the transform domain problem using higher-order WKBJ asymptotic representations around the point infinity of the Laplace transformation parameter. The coefficient functions that occur in these representations satisfy a recurrence scheme. We show how it is possible to obtain higher-order coefficient functions by evaluation of this recurrence scheme with the aid of symbolic manipulation. Due to the structure of the transform domain solution, we are also in a position to efficiently perform the transformation back to the space-time domain, which is the third step of this integral transformation method, by means of the Cagniard-De Hoop method. Further, numerical results are presented. These show that we indeed obtain an interval of nonzero length beyond the arrival time for which subsequent higher-order approximations to the space-time domain wavefield nearly coincide. In the final section of this chapter the method is discussed.

Finally, in chapter 5 the main conclusions with respect to the work described in this thesis are drawn.
Chapter 2

Basic relations for the acoustic wavefield

In this chapter the continuously layered configuration that will be used in our investigations is described in detail. Next, the space-time domain linearized equations for the acoustic wavefield quantities (the particle velocity and the acoustic pressure) are given. These basic acoustic equations form a system of coupled partial differential equations in which derivatives with respect to the three spatial coordinates and the time occur. As the first step of both integral transformation methods to be developed, we reduce the number of derivatives in this system by applying a one-sided Laplace transformation with respect to the time and a double-sided Fourier transformation with respect to both horizontal directions. As a result, the dimension of the problem is reduced to one by parametrization, and an ordinary differential equation for the acoustic state vector (with the transformed vertical particle velocity and the transformed acoustic pressure as its components) is obtained. By subjecting the acoustic state vector to a linear transformation, the acoustic wavevector is arrived at, whose components represent either downwardly or upwardly propagating waves. Subsequently, the wavevector differential equation and the equivalent wavevector integral equation are derived. It is of fundamental importance that in this thesis we apply a real and positive Laplace transform parameter, since in this way many difficulties are avoided that are commonly encountered during the further analysis (e.g., the occurrence of turning points in the transform domain differential equations).
2.1. Description of the configuration and conventions

A schematic picture of the continuously layered configuration in which the acoustic wave field of interest propagates, is shown in figure 2.1. To indicate the position of a point in this configuration, a right-handed Cartesian frame of reference with origin $O$ and three orthonormal base vectors $\{i_1, i_2, i_3\}$ has been defined. In accordance with geophysical practice, the axis in the $i_3$-direction points vertically downward. Within this reference frame, the three coordinates $\{x_1, x_2, x_3\}$ specify the position vector $x$ of a point as

$$x = x_1 i_1 + x_2 i_2 + x_3 i_3.$$  \hspace{1cm} (2.1)

In this equation two conventions are shown that will frequently be used. The first one is the subscript notation, by which a vector is denoted by a symbol with a single subscript and, more generally, a Cartesian tensor of rank $K$ is denoted by a symbol with $K$ subscripts. If we use lower-case Latin subscripts, these will be assigned the values 1 to 3 to indicate the individual elements of the vector or tensor. Secondly, we have employed the summation convention (GOODBODY, 1982), which prescribes summation over repeated subscripts, e.g., $x_k i_k = x_1 i_1 + x_2 i_2 + x_3 i_3$. The time coordinate is denoted by $t$. Differentiation with respect to $x_k$ and $t$ is denoted

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**Figure 2.1.** The horizontally continuously layered configuration with the applied coordinate system.
by $\partial_k$ and $\partial_t$, respectively. As an exception, the subscript $t$ in $\partial_t$ does not range from 1 to 3 but refers to the time coordinate, so $\partial_t$ is a reserved symbol and the subscript $t$ will never be used otherwise. All quantities will be measured in SI-units (Cohen & Giacomo, 1987).

The continuously layered configuration consists of a medium that extends to infinity in all directions, in which a source and a receiver are located at fixed positions. The medium is time-invariant, linear, instantaneously reacting (and thus without losses), locally reacting and isotropic. The medium parameters that are relevant to the propagation of acoustic waves are the volume density of mass $\rho(x_3)$ and the compressibility $\kappa(x_3)$. These parameters are independent of the $x_1$- and $x_2$-coordinates, are continuous and nonvanishing functions of the $x_3$-coordinate, and have piecewise continuous derivatives with respect to $x_3$. The associated acoustic wavespeed is

$$c(x_3) = [\rho(x_3) \kappa(x_3)]^{-1/2}.$$  \hfill (2.2)

In the configuration the acoustic wavefield is generated by an acoustic point source located at $\{x_1^S, x_2^S, x_3^S\}$, which starts to act at $t = t_0$. The quantities describing the acoustic wavefield are measured by a point receiver with position $\{x_1^R, x_2^R, x_3^R\}$. It is assumed that the receiver does not influence the acoustic wavefield in any respect. To the acoustic wave propagation the principle of causality applies, which states that the wavefield due to the action of the source can only manifest itself after the source has started to act. This implies that the acoustic wavefield quantities are zero for $t < t_0$ in the entire configuration.

### 2.2. Space-time domain acoustic equations

The linearized acoustic equations form the basis of our investigations. These are given by

$$\partial_k p + \rho(x_3) \partial_t v_k = f_k,$$  \hfill (2.3)

$$\partial_k v_k + \kappa(x_3) \partial_t p = q.$$  \hfill (2.4)
In these acoustic equations,

\[ p = \text{acoustic pressure [Pa]}, \quad (2.5) \]
\[ v_r = \text{particle velocity [m/s]}, \quad (2.6) \]
\[ f_k = \text{volume density of volume force [N/m}^3], \quad (2.7) \]
\[ q = \text{volume density of volume injection rate [s}^{-1}], \quad (2.8) \]
\[ \rho(x_3) = \text{volume density of mass [kg/m}^3], \quad (2.9) \]
\[ \kappa(x_3) = \text{compressibility [Pa}^{-1}. \quad (2.10) \]

The terms \( f_k \) and \( q \) describe the action of the acoustic source. A monopole source is a source of volume injection characterized by \( f_k = 0 \) and \( q \neq 0 \), while a dipole source is a source of volume force characterized by \( f_k \neq 0 \) and \( q = 0 \). Since the medium is time invariant, the origin of the time axis may be shifted to the instant at which the source starts to act, so we take \( t_0 = 0 \). Due to the invariance of the medium in both horizontal directions, the frame of reference may be chosen in such a way that its vertical axis runs through the source position. As a result, our fixed acoustic point source is represented by

\[
\begin{pmatrix} q \\ f_k \end{pmatrix} = \delta(x_1, x_2, x_3 - x_3^S) \begin{pmatrix} Q^S(t) \\ F_k^S(t) \end{pmatrix}, \quad (2.11)
\]

where \( \delta(x_k) \) denotes the three-dimensional impulse function, also known as the delta function or the Dirac distribution. The functions \( Q^S(t) \) and \( F_k^S(t) \) describe the temporal behavior of the source and are therefore denoted as the source signatures.

### 2.3. Transformation of the acoustic equations

Equations (2.3) and (2.4) describe a system of four coupled partial differential equations in which derivatives with respect to the three spatial coordinates and to time occur. The number of derivatives occurring in these acoustic equations will now be reduced by applying integral transformations with respect to those coordinates in which the medium is invariant. The acoustic pressure will be used as an example in the description of the transformations.
2.3.1. Temporal transformation

Firstly, a one-sided Laplace transformation with respect to the time coordinate is carried out. As an example, the space-temporal Laplace domain counterpart \( \hat{p}(x_k, s) \) of the space-time domain acoustic pressure \( p(x_k, t) \) is defined as

\[
\hat{p}(x_k, s) = \int_{t_0=0}^{\infty} p(x_k, t) \exp(-st) \, dt. \tag{2.12}
\]

Due to the boundedness of the physical space-time domain quantities, their Laplace transforms exist for \( \text{Re}\{s\} > 0 \). Since all relevant quantities are zero for \( t < 0 \), the operator \( \partial_t \) in the space-time domain corresponds to a multiplication by \( s \) in the space-temporal Laplace domain. The inverse Laplace transformation in fact consists of solving Carson's integral equation (Carson, 1926), which is obtained when in eq. (2.12) one considers \( \hat{p}(x_k, s) \) as a known function and \( p(x_k, t) \) as an unknown function. Lerch's theorem (Widder, 1946, 1989; Henrici, 1977) states that a unique and causal solution \( p(x_k, t) \) of Carson's integral equation exists if \( \hat{p}(x_k, s) \) is bounded for an infinite sequence of points \( \{s_n \in \mathbb{R} | s_n = s_0 + n\ell\} \), where \( s_0 \in \mathbb{R}^+ \) and sufficiently large, \( n \in \mathbb{N} \), and \( \ell \in \mathbb{R}^+ \). The fact that causality can be ensured is one reason for preferring the temporal Laplace transformation over the temporal Fourier transformation. It is of fundamental importance to this thesis that, as a consequence of Lerch's theorem, it is sufficient to restrict our analysis to real and positive values of the Laplace transform parameter \( s \) (cf. Cagniard, 1939, 1962). The fact that causality can be ensured is one reason for preferring the temporal Laplace transformation over the temporal Fourier transformation. Although they will not be used in our further analysis, we note that several versions of the inverse Laplace transformation are known (Widder, 1946; Van der Pol & Bremmer, 1959; Henrici, 1977). A frequently applied version is the Bromwich integral

\[
p(x_k, t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \hat{p}(x_k, s) \exp(st) \, ds. \tag{2.13}
\]

The value of \( \sigma \) must be chosen large enough to ensure that \( \hat{p}(x_k, s) \) is analytic in the halfplane \( \text{Re}\{s\} \geq \sigma > 0 \).

2.3.2. Spatial transformation

Secondly, transformations with respect to the horizontal coordinates are carried out. The spatial Fourier – temporal Laplace domain counterpart or, shortly, the
transform domain counterpart \( \tilde{p}(\alpha_1, \alpha_2, x_3, s) \) of the space-temporal Laplace domain quantity \( \tilde{p}(x_k, s) \) is defined by

\[
\tilde{p}(\alpha_1, \alpha_2, x_3, s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{p}(x_k, s) \exp[is(\alpha_1 x_1 + \alpha_2 x_2)] \, dx_1 \, dx_2.
\]  
\hfill (2.14)

A sufficient condition for the existence of \( \tilde{p}(\alpha_1, \alpha_2, x_3, s) \) for all values of \( \alpha_1 \) and \( \alpha_2 \) is that \( \tilde{p}(x_k, s) \) is absolutely integrable over the unbounded domain \( \{ (x_1, x_2) \in \mathbb{R}^2 \} \). This implies that \( \tilde{p}(x_k, s) \to 0 \) for \( \{|x_1|, |x_2|\} \to \infty \) or, in words, that \( \tilde{p}(x_k, s) \) will be a localized function in the \( (x_1, x_2) \)-plane. Assuming that this requirement is met, the operators \( \partial_1 \) and \( \partial_2 \) in the space-time domain correspond in the transform domain to a multiplication by \( -is\alpha_1 \) and \( -is\alpha_2 \), respectively. The corresponding inverse spatial Fourier transformation is

\[
\tilde{p}(x_k, s) = \left( \frac{s}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{p}(\alpha_1, \alpha_2, x_3, s) \exp[-is(\alpha_1 x_1 + \alpha_2 x_2)] \, d\alpha_1 \, d\alpha_2.
\]  
\hfill (2.15)

Note that the actual Fourier transformation parameters are \( s\alpha_1 \) and \( s\alpha_2 \); the factor \( s \) has been incorporated for later convenience.

2.3.3. Transform domain basic acoustic equations

Applying the respective transformations to the acoustic equations (2.3) and (2.4) we arrive at the transform domain system of coupled differential equations

\[
-is\alpha_1 \tilde{p} + s\rho(x_3) \tilde{u}_1 = \tilde{f}_1,
\]  
\hfill (2.16)

\[
-is\alpha_2 \tilde{p} + s\rho(x_3) \tilde{u}_2 = \tilde{f}_2,
\]  
\hfill (2.17)

\[
\partial_3 \tilde{p} + s\rho(x_3) \tilde{v}_3 = \tilde{f}_3,
\]  
\hfill (2.18)

\[
is\alpha_1 \tilde{v}_1 - is\alpha_2 \tilde{v}_2 + \partial_3 \tilde{v}_3 + s\kappa(x_3) \tilde{p} = \tilde{q}.
\]  
\hfill (2.19)

Elimination of \( \tilde{v}_1 \) and \( \tilde{v}_2 \) and application of eq. (2.2) gives

\[
\partial_3 \tilde{p} + s\rho(x_3) \tilde{v}_3 = \tilde{f}_3,
\]  
\hfill (2.20)

\[
\partial_3 \tilde{v}_3 + s\rho^{-1}(x_3) \left[ \epsilon^{-2}(x_3) + \alpha^2 + \alpha_2^2 \right] \tilde{p} = \tilde{q} + \rho^{-1}(x_3)[is\alpha_1 \tilde{f}_1 + is\alpha_2 \tilde{f}_2].
\]  
\hfill (2.21)

Using the subscript notation and the summation convention, where upper-case Latin subscripts can be assigned the values 1 and 2, this can be cast in the more compact matrix form

\[
\partial_3 \tilde{v}_I + sA_{IJ}(x_3) \tilde{b}_J = \tilde{u}_I.
\]  
\hfill (2.22)
In this differential equation we have introduced the transform domain state vector
\[ \tilde{b}_I = \begin{pmatrix} \tilde{v}_3 \\ \tilde{p} \end{pmatrix}, \]  
(2.23)
the transform domain notional source strength vector [cf. eq. (2.11)]
\[ \tilde{u}_I = \delta(z_3 - z_3^S) \begin{pmatrix} \tilde{Q}^N \\ \tilde{P}_N \end{pmatrix}, \]  
(2.24)
in which
\[ \begin{pmatrix} \tilde{Q}^N \\ \tilde{P}_N \end{pmatrix} = \begin{pmatrix} Q^S(s) + \rho^{-1}(z_3) [i\alpha_1 \tilde{P}_1^S(s) + i\alpha_2 \tilde{P}_2^S(s)] \\ \tilde{P}_3^S(s) \end{pmatrix}, \]  
(2.25)
and the system matrix
\[ A_{IJ}(z_3) = \begin{pmatrix} 0 & \gamma(z_3) Y(z_3) \\ \gamma(z_3) Y^{-1}(z_3) & 0 \end{pmatrix}. \]  
(2.26)
In the system matrix the vertical slowness
\[ \gamma(z_3) = [\epsilon^{-2}(z_3) + \alpha_1^2 + \alpha_2^2]^{1/2} \]  
(2.27)
and the vertical acoustic wave admittance
\[ Y(z_3) = \frac{\gamma(z_3)}{\rho(z_3)} \]  
(2.28)
occur. With our transformation scheme both \( \gamma(z_3) \) and \( Y(z_3) \) are real and positive, and the system matrix will never become singular.

--- Comparison with the frequency domain analysis ---

The application of the temporal Fourier transformation is less convenient, for in this case the equivalents of eqs. (2.12) and (2.14) are,
\[ \hat{p}(x_k, i\omega) = \int_{-\infty}^{\infty} p(x_k, t) \exp(-i\omega t) \, dt \]
and
\[ \hat{p}(\alpha_1, \alpha_2, x_3, i\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{p}(x_k, i\omega) \exp[i\omega(\alpha_1 x_1 + \alpha_2 x_2)] \, dx_1 \, dx_2, \]
respectively. With this transformation scheme a vertical slowness
\[ \gamma(z_3) = [c^{-2}(z_3) - \alpha_1^2 - \alpha_z^2]^{1/2} \]
is arrived at that for some values of the transformation parameters becomes zero, see figure 2.2. Under those circumstances the system matrix becomes singular, which will make the solution of the differential equation (2.22) more intricate (Chapman, 1974b, 1981) than in our case.

\[ \text{Im}(\alpha_2) \]
\[ \text{Re}(\alpha_1) \]
\[ \alpha_2 \]
\[ O \]
\[ c^{-2} - \alpha_1^2 > 0 \]
\[ c^{-2} - \alpha_1^2 < 0 \]

**Figure 2.2.** The position of the branch points of the vertical slowness \( \gamma(z_3) \) in the complex \( \alpha_2 \)-plane, for a fixed real value of \( \alpha_1 \): (a) with the temporal Laplace transformation; (b), (c) with the temporal Fourier transformation.

### 2.4. Differential equation for the wavevector

In order to show that the components of the acoustic state vector both consist of terms that represent waves traveling downwardly (in the direction of increasing \( z_3 \)) and terms that represent waves traveling upwardly (in the direction of decreasing \( z_3 \)), we apply a linear transformation of the state vector. As a result, we obtain the so-called acoustic wavevector, whose components represent either a downgoing or an upgoing wave.
As a first step, following the literature (Chapman, 1974a, 1976; Kennett, 1983; Van der Hilden, 1987), we decompose the system matrix $A_{IJ}(x_3)$ according to

$$A_{IJ}(x_3) = N_{IK}(x_3) \Lambda_{KL}(x_3) N_{LJ}^{-1}(x_3), \quad (2.29)$$

where

$$\Lambda_{KL}(x_3) = \begin{pmatrix} \gamma(x_3) & 0 \\ 0 & -\gamma(x_3) \end{pmatrix}, \quad (2.30)$$

is the eigenvalue matrix of $A_{IJ}(x_3)$, while

$$N_{IK}(x_3) = \frac{1}{2} \sqrt{2} \begin{pmatrix} Y^{1/2}(x_3) & -Y^{1/2}(x_3) \\ Y^{-1/2}(x_3) & Y^{1/2}(x_3) \end{pmatrix}, \quad (2.31)$$

is a matrix with normalized eigenvectors of $A_{IJ}(x_3)$ as its columns. The matrix $N_{IJ}(x_3)$ is called the composition matrix because it is used to relate the acoustic wavevector $\tilde{w}_J$ to the acoustic state vector $\tilde{b}_I$ through the so-called composition relation

$$\tilde{b}_I = N_{IJ}(x_3) \tilde{w}_J. \quad (2.32)$$

Analogously, the matrix

$$N_{LJ}^{-1}(x_3) = \frac{1}{2} \sqrt{2} \begin{pmatrix} Y^{-1/2}(x_3) & Y^{1/2}(x_3) \\ -Y^{-1/2}(x_3) & Y^{1/2}(x_3) \end{pmatrix} \quad (2.33)$$

is called the decomposition matrix since it occurs in the decomposition relation

$$\tilde{w}_I = N_{IJ}^{-1}(x_3) \tilde{b}_J. \quad (2.34)$$

Using eqs. (2.22) and (2.32), the wavevector differential equation is found as

$$\partial_3 \tilde{w}_I + s \Lambda_{IJ}(x_3) \tilde{w}_J = \Delta(x_3)_{IK} \tilde{w}_K + N_{IJ}^{-1}(x_3) \tilde{a}_J. \quad (2.35)$$

The first term on the right-hand side contains the coupling matrix

$$\Delta_{IK}(x_3) = -N_{IJ}^{-1}(x_3) \partial_3 N(x_3)_{JK} = \chi(x_3) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.36)$$

that establishes the coupling between $\tilde{w}_1$ and $\tilde{w}_2$ at those levels where the medium is inhomogeneous and thus the inhomogeneity function

$$\chi(x_3) = \frac{\partial_3 Y(x_3)}{2 Y(x_3)} = \frac{1}{2} \partial_3 \ln[Y(x_3)]$$

$$= -\frac{\partial_3 c(x_3)}{2 \gamma^2(x_3) c^2(x_3)} \frac{\partial_3 \rho(x_3)}{2 \rho(x_3)}$$

$$= \frac{\rho(x_3) \partial_3 \kappa(x_3) + \kappa(x_3) \partial_3 \rho(x_3)}{4 \gamma(x_3)} - \frac{\partial_3 \rho(x_3)}{2 \rho(x_3)} \quad (2.37)$$
differs from zero. In homogeneous regions of the configuration the inhomogeneity function is zero and no coupling takes place. Due to our specific choice of the normalization of $N_{iJ}(x_3)$, the diagonal components of $\Delta_{iK}(x_3)$ are zero and no coupling of a wavevector component with itself takes place, while both off-diagonal elements of $\Delta_{iK}(x_3)$ are equal and both wavevector components couple in equal amounts on the same level. This simplifies the analysis of eq. (2.35). In our case the inhomogeneity function is always bounded since $\partial_3 c(x_3)$ and $\partial_3 \rho(x_3)$ are bounded and $\gamma(x_3)$ is bounded away from zero. The second term on the right-hand side of eq. (2.35) is representative for the source and is given by

$$N_{ij}^{-1}(x_3) \tilde{u}_j = \frac{1}{2} \sqrt{2} \delta(x_3 - x_3^S) \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix},$$

(2.38)

with

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} = \begin{pmatrix} \tilde{F}^N Y^{1/2}(x_3^S) + \tilde{Q}^N Y^{-1/2}(x_3^S) \\ \tilde{F}^N Y^{1/2}(x_3^S) - \tilde{Q}^N Y^{-1/2}(x_3^S) \end{pmatrix}.$$  

(2.39)

In a homogeneous and sourcefree layer of the configuration the right-hand side of eq. (2.35) is equal to zero. Since $\Lambda(x_3)$ is a diagonal matrix, the components of the wavevector equation in this part of the configuration are

$$\tilde{w}_1 = K_1 \exp[-s \int_{x_3^{(1)}}^{x_3} \gamma \, d\zeta],$$

(2.40)

$$\tilde{w}_2 = K_2 \exp[-s \int_{x_3^{(2)}}^{x_3} \gamma \, d\zeta],$$

(2.41)

where $K_1$, $K_2$, $x_3^{(1)}$, and $x_3^{(2)}$ are constants. This shows that in a homogeneous layer the wavevector components $\tilde{w}_1$ and $\tilde{w}_2$ indeed represents independently propagating downgoing and upgoing waves, respectively.

2.5. Integral equation for the wavevector

In this thesis two different methods will be employed to find the solution of the transform domain wave propagation problem. Both methods do not directly use the wavevector differential equation (2.35), but are based on the corresponding integral equation. This integral equation will now be derived. First, we determine the causal Green's functions of the left-hand side of eq. (2.35) that satisfy

$$\partial_3 \tilde{g}_1 + s \gamma(x_3) \tilde{g}_1 = \delta(x_3 - x_3'),$$

(2.42)

$$\partial_3 \tilde{g}_2 - s \gamma(x_3) \tilde{g}_2 = \delta(x_3 - x_3').$$

(2.43)
The general solutions of these equations satisfy
\begin{align}
\tilde{g}_1(x_3, x'_3) &= [k_1 + H(x_3 - x'_3)] \exp[-s \int_{x'_3}^{x_3} \gamma(\zeta) \, d\zeta], \\
\tilde{g}_2(x_3, x'_3) &= [k_2 + H(x_3 - x'_3)] \exp[-s \int_{x'_3}^{x_3} \gamma(\zeta) \, d\zeta],
\end{align}
(2.44)
(2.45)
In these equations, $H$ denotes the Heaviside unit step function
\begin{equation}
H(x_3) = \begin{cases}
0 & \text{if } x_3 < 0 \\
\frac{1}{2} & \text{if } x_3 = 0 \\
1 & \text{if } x_3 > 0
\end{cases}.
\end{equation}
(2.46)
Causality requires that $\tilde{g}_1(x_3, x'_3)$ and $\tilde{g}_2(x_3, x'_3)$ are bounded for all $x_3$ and all $s \geq a_0 > 0$, so we must necessarily set $k_1 = 0$ to keep $\tilde{g}_1(x_3, x'_3)$ bounded for $x_3 \to -\infty$. Physically speaking this means that no waves come from $-\infty$. In the same way the boundedness of $\tilde{g}_2(x_3, x'_3)$ for $x_3 \to \infty$ necessarily leads to $k_2 = -1$. As a result, we obtain
\begin{align}
\tilde{g}_1(x_3, x'_3) &= H(x_3 - x'_3) \exp[-s \int_{x'_3}^{x_3} \gamma(\zeta) \, d\zeta], \\
\tilde{g}_2(x_3, x'_3) &= -H(x'_3 - x_3) \exp[-s \int_{x'_3}^{x_3} \gamma(\zeta) \, d\zeta].
\end{align}
(2.47)
(2.48)
Knowing $\tilde{g}_1$ and $\tilde{g}_2$, we can use the superposition principle and the right-hand sides of eq. (2.35) to obtain
\begin{align}
\tilde{\omega}_1(x_3) &= \int_{-\infty}^{\infty} \lambda(x'_3) \tilde{g}_1(x_3, x'_3) \, dx'_3 + \frac{1}{2} \sqrt{2} \tilde{a}_1 \tilde{g}_1(x_3, x'_3), \\
\tilde{\omega}_2(x_3) &= \int_{-\infty}^{\infty} \lambda(x'_3) \tilde{g}_2(x_3, x'_3) \, dx'_3 + \frac{1}{2} \sqrt{2} \tilde{a}_2 \tilde{g}_2(x_3, x'_3).
\end{align}
(2.49)
(2.50)
This integral equation for the wavevector can be rewritten as an operator equation
\begin{equation}
\tilde{\omega}_I = L_{IJ} \tilde{\omega}_J + \tilde{h}_I,
\end{equation}
(2.51)
in which the integral operator $L_{IJ}$ is
\begin{equation}
L_{IJ} \tilde{\omega}_J = \left( \int_{\gamma}^{x} \chi(x'_3) \exp[-s \int_{x'_3}^{x} \gamma(\zeta) \, d\zeta] \, dx'_3 \right),
\end{equation}
(2.52)
where the integration limits follow from the step functions in eqs. (2.47) and (2.48). The source vector $\tilde{h}_I$ is given by
\begin{equation}
\tilde{h}_I = \left( \begin{array}{c}
\frac{1}{2} \sqrt{2} \tilde{a}_1 H(x_3 - x'_3) \exp[-s \int_{x'_3}^{x_3} \gamma(\zeta) \, d\zeta] \\
\frac{1}{2} \sqrt{2} \tilde{a}_2 H(x'_3 - x_3) \exp[-s \int_{x'_3}^{x_3} \gamma(\zeta) \, d\zeta]
\end{array} \right).
\end{equation}
(2.53)
Equation (2.51) is the wavevector integral equation that is equivalent to the wavevector differential equation (2.35).
Chapter 3

Combination of the WKBJ iterative solution and the Cagniard-De Hoop method

In this chapter a first complete integral transformation method is presented. The first step, i.e., the transformation of the space-time domain problem, has already been performed in the previous chapter. As the second step, in this chapter the transform domain equivalent of the wave propagation problem in a horizontally continuously layered configuration is solved by means of the WKBJ iterative solution (Neumann series) of the wavevector integral equation. Due to the chosen transformation scheme, we can prove the convergence of the transform domain solution for every horizontally continuously layered configuration. Moreover, we show that this implies the convergence of the space-time domain solution for any time instant as well. Thus, it is in principle possible to obtain sufficiently accurate results for every horizontally continuously layered configuration and for any time instant. The transformation of the derived solution back to the space-time domain forms the third step of the present integral transformation method. This step is performed by applying the Cagniard-De Hoop method to the individual terms of the transform domain solution. It is shown how the Cagniard-De Hoop method is adapted in order to deal with continuously layered configurations. Further, the numerical implementation of the zeroth-order term and the first-order term of the space-time domain solution is discussed and for these terms various numerical results are presented.
3.1. The WKBJ iterative solution

As the first step of a possible integral transform method, in chapter 2 we have introduced the transform domain equivalent of the space-time domain wave propagation problem in a horizontally continuously layered configuration. In order to form a complete integral transformation method, in this chapter we will perform the other two steps. The aim of this chapter is to obtain an integral transformation method by which it is in principle possible to obtain sufficiently accurate results for the acoustic wavefield at any time instant and in every continuously layered configuration. The second step of an integral transformation method consists of solving the transform domain problem. In the introduction we have described four methods that are commonly employed to perform this task: WKBJ asymptotics, Langer asymptotics, the WKBJ iterative solution and the Langer iterative solution. Reviews of these methods can be found in Chapman (1981), Chapman & Orcutt (1985) and Kennett (1983). To proceed with our investigation, out of these methods we must select the one that seems most promising in view of the aim of this chapter. At this stage, however, this selection will only be based on the following general considerations. Firstly, we recall that the WKBJ-type methods are less intricate than the Langer-type methods. On the other hand the Langer-type methods can under circumstances deal with a vertical slowness that becomes zero, while the WKBJ-type methods break down in that case. However, due to the fact that in our transformation scheme the temporal Laplace transformation with a real and positive transform parameter has been used, the vertical slowness does never become zero. Thus the Langer-type methods do not benefit and the less complex WKBJ-type methods are preferred. Secondly, we have mentioned that the iterative methods allow the medium parameter profiles to possess discontinuous derivatives, while the asymptotic methods impose a more severe requirement on the differentiability of these profiles. Since the class of configurations that can be dealt with by using the asymptotic methods is more restricted than by using the iterative methods, the latter are preferred. Combining both considerations, for the moment the WKBJ iterative solution seems to be the most suitable method for solving the transform domain problem.
In the literature several different ways of deriving the WKBJ iterative solution have appeared. Originally, the WKBJ iterative solution was found by BREMMER (1939, 1951), who used physical arguments to derive the total solution of the one-dimensional Helmholtz equation for an inhomogeneous medium. BREKHOVSKIKH (1960), investigating the propagation of electromagnetic waves in a continuously layered medium, arrived at a system of two coupled ordinary differential equations containing a small parameter. He applied a perturbation-like approach and assumed that the solutions of the system of differential equations can be written as series expansions in the small parameter. After establishing the relations between the coefficients occurring in these series and eliminating the small parameter, he arrived at the WKBJ iterative solution. This approach has also been referred to by CHAPMAN (1974a). Quite another approach was followed by WING (1974), who used the theory of invariant embedding to find the solution of a general system of two coupled ordinary differential equations. The resulting solution was found to be equivalent with the solution derived by Bremmer. To conclude this discussion of the literature, we mention that CHAPMAN (1976), searching for a solution of a matrix form differential equation for the acoustic wave propagation in a continuously layered fluid, obtained the WKBJ iterative solution by simply using a vectorial extension of the scalar Picard method of successive approximations, for which he refers to CODDINGTON & LEVINSON (1955).

Despite of all the derivations available in the literature, we present our own way of deriving the WKBJ iterative solution for the transform domain wave propagation problem. Instead of taking the wavevector differential equation (2.35) as point of departure, we start with the wavevector integral equation defined in eqs. (2.51)-(2.53). This can be considered as a Volterra integral equation of the second kind for the wavevector \( \tilde{w}_f \). Therefore we assume that its solution can be obtained by a vectorial extension of the iterative solution of scalar Volterra integral equation of the second kind (POGORZELSKI, 1966; HOCHSTÄDT, 1973), more specifically known as the Neumann series solution (WHITTAKER & WATSON, 1952). Our approach has the benefit that the WKBJ iterative solution is obtained in a straightforward manner. In particular, the various integration bounds that show up in the WKBJ iterative solution, follow automatically and need not be determined ad-hoc.
To start with, we note that according to eq. (2.51) the wavevector satisfies the operator equation

$$\tilde{\omega}_I = L_{IJ} \tilde{\omega}_J + \tilde{h}_I. \quad (3.1)$$

This is equivalent to

$$(L_{IJ} - L_{IJ}) \tilde{\omega}_J = \tilde{h}_I. \quad (3.2)$$

where $I_{IJ}$ is the $2 \times 2$ unit matrix. For this moment we assume that the solution of this equation can be written as

$$\tilde{\omega}_I = (1 - L)_{IJ}^{-1} \tilde{h}_J. \quad (3.3)$$

Here $(1 - L)_{IJ}^{-1}$ indicates the inverse of the operator $(L_{IJ} - L_{IJ})$, i.e.,

$$(1 - L)_{IJ}^{-1} (L_{JK} - L_{JK}) \tilde{\omega}_K = \tilde{\omega}_I. \quad (3.4)$$

We further assume, as with the scalar Volterra integral equation (HOCHSTADT, 1973), that this inverse operator can be expanded in the series

$$(L_{IJ} - L_{IJ})^{-1} = \sum_{i=0}^{\infty} L_{IJ}^i. \quad (3.5)$$

This series shows a resemblance with the geometric series of $(1 - x)^{-1}$. In the equation above powers of the integral operator $L_{IJ}$ show up, which are defined in a recurrent way by

$$L_{IJ}^0 \tilde{\omega}_J = L_{IJ} \tilde{\omega}_J = \tilde{\omega}_I, \quad (3.6)$$

$$L_{IJ}^i \tilde{\omega}_J = L_{IK} (L_{KJ}^{-1} \tilde{\omega}_J), \quad (i \geq 1). \quad (3.7)$$

Substitution of eq. (3.5) in eq. (3.3) yields

$$\tilde{\omega}_I = \sum_{i=0}^{\infty} L_{IJ}^i \tilde{h}_J. \quad (3.8)$$

This is the Neumann series solution of the operator equation (3.1). To show that this is the WKBJ iterative solution, the Neumann series solution is written as

$$\tilde{\omega}_I = \sum_{i=0}^{\infty} \tilde{\omega}_I^{(i)}, \quad (3.9)$$

in which, according to eqs. (3.6)-(3.7),

$$\tilde{\omega}_I^{(0)} = \tilde{h}_I, \quad (3.10)$$

$$\tilde{\omega}_I^{(i)} = L_{IJ}^i \tilde{\omega}_J = L_{IJ} \tilde{\omega}_I^{(i-1)} \quad (i \geq 1). \quad (3.11)$$
Using eqs. (2.52) and (2.53), the terms of the Neumann series solution turn out to be

\[
\hat{w}_1^{(0)} = \left( \frac{1}{2} \sqrt{2} \tilde{a}_1 H(x_3 - x_3^S) \exp(-s \int_{x_3}^{x_3^S} \gamma \, d\zeta) \right),
\]

\[
\hat{w}_1^{(i)} = \left( \int_{-\infty}^{x_3} \chi(x'_3) \exp(-s \int_{x_3}^{x_3'} \gamma \, d\zeta) \hat{w}_2^{(i-1)} \, dx'_3 \right) - \left( \int_{x_3}^{\infty} \chi(x'_3) \exp(-s \int_{x_3}^{x_3'} \gamma \, d\zeta) \hat{w}_1^{(i-1)} \, dx'_3 \right), \quad (i \geq 1).
\]

These are the terms of the WKBJ iterative solution that has commonly been described in the literature. In order to obtain a more compact notation and provided that no confusion can occur, we have employed the convention that the dependence of quantities like the wavespeed, the density of mass, the vertical slowness, etc., on the vertical coordinate will not always be indicated explicitly.

It is necessary to investigate under what conditions the WKBJ iterative solution in eq. (3.9) converges. This problem has been addressed in detail by DE HOOP (1990). The convergence proof is presented in detail in appendix 3.A. With reference to this appendix we state that the WKBJ iterative solution presented in eqs. (3.9), (3.12) and (3.13) is convergent, since

- the vertical slowness

\[
\gamma(x_3) = \left[ c^{-2}(x_3) + \alpha_1^2 + \alpha_2^2 \right]^{1/2}
\]

is bounded away from zero for all real values of the Fourier transformation parameters \( \alpha_1 \) and \( \alpha_2 \), i.e., for those values that are important in view of the inverse Fourier transformation (see chapter 2);

- the inhomogeneity function

\[
\chi(x_3) = - \frac{\partial_3 c(x_3)}{2\gamma^2(x_3) c^3(x_3)} - \frac{\partial_3 \rho(x_3)}{2\rho(x_3)}
\]

is bounded since the medium parameters \( c(x_3) \) and \( \rho(x_3) \) are bounded away from zero, their derivatives \( \partial_3 c(x_3) \) and \( \partial_3 \rho(x_3) \) are bounded, and the vertical slowness \( \gamma(x_3) \) does not vanish;
the Laplace transform parameter \( s \) can be chosen sufficiently large. This follows from Lerch's theorem (Widder, 1946), which states that all Laplace transformed quantities correspond to unique and causal time domain quantities if they remain bounded for real \( s \geq s_0 > 0 \), where \( s_0 \) may be chosen as large as necessary.

In conclusion it can be stated that our WKBJ iterative solution is convergent for arbitrarily continuously layered configurations and any relevant combination of the transformation parameters. Although it is irrelevant from a physical point of view, the medium need not be homogeneous for \( z_3 \to -\infty \) or \( z_3 \to \infty \). The convergence of the transform domain WKBJ iterative solution ensures, due to Lerch's theorem, the convergence and causality of the corresponding space-time domain solution. Thus, in principle, the WKBJ iterative solution offers the possibility to find the space-time domain solution of the wave propagation problem in arbitrarily continuously layered configurations to any desired degree of accuracy.

Now that we have found the WKBJ iterative solution for the transform domain wave vector \( \tilde{w}_I \), we can apply the composition relation (2.32) to obtain the transform domain acoustic state vector \( \tilde{b}_I \). In the next section we shall give an physical interpretation of the terms that form the WKBJ iterative solution of the wavevector. In sections 3.3, 3.4 and 3.5 we will describe the transformation back to the space-time domain of the zeroth-order terms, the first-order terms, and the higher-order terms, respectively, of the acoustic state vector.

— Comparison with the frequency domain analysis —

If the temporal Fourier transformation with real transformation parameter \( \omega \) were used instead of the temporal Laplace transformation, different methods for proving the convergence of the resulting WKBJ iterative solution must be employed, depending upon the possible values of \( \gamma(z_3) \). In case of a nonzero real vertical slowness [e.g., on the real axis in between the branch points in figure 2.2(b)], the exponential functions in the WKBJ iterative solution have a purely imaginary argument. As a result, the convergence proof of appendix 3.A cannot be applied, and the proof described by Atkinson (1960) must be employed. In this case the class of profile
parameters for which convergence is guaranteed is rather limited (Sluijter, 1969). If the vertical slowness has a negative imaginary part [e.g., on the path beyond the branch points in figure 2.2(b) and on the path in figure 2.2(c)], the method from appendix 3.A can be employed with \( s \gamma \) replaced by \( \omega \text{Im} \{ \gamma \} \). But in this case the class of profile parameters that leads to a convergent solution is limited as well since \( \omega \) cannot be chosen arbitrarily large as is the case for \( s \). If the vertical slowness has a negative imaginary part and at the same time \( \omega \) is small, the convergence proof of Atkinson (1960) leads to less restrictions on the parameter profiles than the method of appendix 3.A. In case of a zero vertical slowness, the WKBJ iterative solution breaks down (Chapman, 1981). These problems can (partially) be overcome by using the solution of some of the alternative wavevector differential equations that are obtained when \( z_3 \) and the wavevector are subjected to transformations (Sluijter, 1969; Chapman, 1974b, 1981; Bates & Wall, 1976). For the one-dimensional problem, Gray (1983) has shown that after applying the inverse temporal Fourier transformation to the transform domain WKBJ iterative solution, the corresponding time domain solution is convergent for all continuously layered configurations. Due to the one-dimensional character, however, the difficulties that are caused by a zero or purely imaginary vertical slowness have not been addressed in his article. All the problems that are encountered with the temporal Fourier transformation are circumvented with our transformation scheme involving the temporal Laplace transformation with a real and positive transform parameter.

— Alternatives to the WKBJ iterative solution occurring in the literature —

For completeness we remark that in the literature alternatives to the WKBJ iterative solution have been derived by generalizing Bremmer’s approach (Sluijter, 1969). In this respect the Langer iterative solution can also be regarded as an alternative of the WKBJ iterative solution. In our case such an alternative solution would have been obtained if the eigenvalue decomposition of the system matrix \( A_{ij}(x_3) \) as described in chapter 2 would have been replaced by some other kind of decomposition. Since these alternative solutions are more intricate and less convenient in view of later manipulations, they are disregarded, however.
3.2. Physical interpretation

Starting with the zeroth-order wavevector components \( \tilde{\omega}_1^{(0)} \) and \( \tilde{\omega}_2^{(0)} \) as given in eq. (3.12), we state that these can be interpreted as zeroth-order waves or direct waves that are directly generated by the source, traveling away from it. To make this plausible, we note that from the arguments of the exponential functions it follows that \( \tilde{\omega}_1^{(0)} \) represents a wave that propagates in the direction of increasing \( z_3 \), while \( \tilde{\omega}_2^{(0)} \) represents a wave that propagates in the direction of decreasing \( z_3 \). Moreover, \( \tilde{\omega}_1^{(0)} \) only exists below the source (\( z_3 > z_3^S \)) and \( \tilde{\omega}_2^{(0)} \) is only nonzero above the source (\( z_3 < z_3^S \)). The amplitude of the waves is proportional with the source-dependent factors \( \tilde{a}_1 \) and \( \tilde{a}_2 \), respectively. From eq. (3.13) we can easily confirm that

\[
\lim_{z_3 \rightarrow z_3^S} \tilde{\omega}_i^{(0)} = \lim_{z_3 \rightarrow z_3^S} \tilde{\omega}_i^{(1)}, \quad (i = 1, 2, \ldots),
\]  

(3.16)

i.e., \( \tilde{\omega}_1^{(i)} \) and \( \tilde{\omega}_2^{(i)} \) are not directly influenced by the source if \( i \geq 1 \), and thus only \( \tilde{\omega}_1^{(0)} \) and \( \tilde{\omega}_2^{(0)} \) represent waves that are directly generated by the source. In figure 3.1 the propagation of the zeroth-order wavevector components is illustrated schematically.

As a next step in our interpretation, a zeroth-order wave generates, by continuous reflection in the inhomogeneous medium, a first-order wave, traveling in the opposite direction. Equation (3.13) forms a mathematical formulation of this process. We see that \( \tilde{\omega}_1^{(0)} \) generates \( \tilde{\omega}_2^{(1)} \) and \( \tilde{\omega}_2^{(0)} \) generates \( \tilde{\omega}_1^{(1)} \). If we substitute \( \tilde{\omega}_1^{(0)} \) and \( \tilde{\omega}_2^{(0)} \) in the expressions for \( \tilde{\omega}_2^{(1)} \) and \( \tilde{\omega}_1^{(1)} \), respectively, we see from the argument of the resulting exponential function that the partial contributions (which are represented by the integrand) to \( \tilde{\omega}_2^{(1)} \) travel upward and the partial contributions to \( \tilde{\omega}_1^{(1)} \) travel downward, i.e., in the opposite direction as compared to the waves by which they are generated. The partial contributions of the zeroth-order waves to the first-order waves depend upon \( \chi(z_3) \), which represents the local reflection factor and depends upon the degree of inhomogeneity of the medium. Finally, all partial reflections are gathered by means of an integration with respect to the vertical coordinate. In figures 3.2 and 3.3 we illustrate the generation of the first-order waves according to this process. Different situations can occur, depending upon the position of \( z_3 \) relative to \( z_3^S \). We take \( \tilde{\omega}_2^{(1)} \) as an example. Substituting the expression for \( \tilde{\omega}_1^{(0)} \) of eq. (3.12) in the expression for \( \tilde{\omega}_2^{(1)} \) of eq. (3.13), we obtain two different expressions...
Figure 3.1. Propagation of the zeroth-order wavevector components: (a) propagation of \( \tilde{w}_1^{(0)} \) in the direction of increasing \( x_3 \); (b) propagation of \( \tilde{w}_2^{(0)} \) in the direction of decreasing \( x_3 \).

for \( \tilde{w}_2^{(1)} \); these are

\[
\tilde{w}_2^{(1)} = -\frac{1}{2} \sqrt{2} \tilde{a}_1 \int_{x_3}^{\infty} \chi(x_3') \exp[-s (\int_{x_3}^{x_3'} \gamma d\zeta + \int_{x_3}^{x_3'} \gamma d\zeta')] dx_3', \quad (x_3 > x_3^S),
\]

\[
\tilde{w}_2^{(1)} = -\frac{1}{2} \sqrt{2} \tilde{a}_1 \int_{x_3}^{\infty} \chi(x_3') \exp[-s (\int_{x_3}^{x_3'} \gamma d\zeta + \int_{x_3}^{x_3'} \gamma d\zeta')] dx_3', \quad (x_3 < x_3^S).
\]

If \( x_3 > x_3^S \), eq. (3.17) holds, and the entire interval \([x_3; \infty)\) yields contributions to \( \tilde{w}_2^{(1)} \) [except for those areas where \( \chi(x_3) = 0 \)], since \( \tilde{w}_1^{(0)} \) is non-zero on \([x_3; \infty)\). This situation and the analogous situation for \( \tilde{w}_1^{(1)} \) are depicted in figure 3.2. However, if \( x_3 < x_3^S \), eq. (3.18) shows us that only the interval \([x_3^S; \infty)\) contributes to \( \tilde{w}_2^{(1)} \), since the direct wave \( \tilde{w}_1^{(0)} \) is zero on \([x_3; x_3^S]\), so this interval gives no contributions to \( \tilde{w}_2^{(1)} \). This situation and the analogous situation for \( \tilde{w}_1^{(1)} \) are sketched in figure 3.3.

The process of continuous reflection also results in higher-order waves. Generally, a wave of order \((i - 1)\) generates a \(i\)-th order wave traveling in the opposite direction by a process that is equivalent to the generation of first-order waves. This is incorporated in eq. (3.13) as well. Since \( \tilde{w}_1 \) and \( \tilde{w}_2 \) consist of the sum of all downward traveling waves represented by \( \tilde{w}_1^{(i)} \) and all upward traveling waves represented
Figure 3.2. Generation of the first order wavevector components if the first-order wave does not cross the source level: (a) generation of $\bar{w}_2^{(1)}$; (b) generation of $\bar{w}_1^{(1)}$.

Figure 3.3. Generation of the first order wavevector components if the first-order wave crosses the source level: (a) generation of $\bar{w}_2^{(1)}$; (b) generation of $\bar{w}_1^{(1)}$. 
by \( w_1^{(i)} \) and \( w_2^{(i)} \), respectively, we may state that \( \tilde{w}_1 \) and \( \tilde{w}_2 \) themselves represent downward traveling waves and upward traveling waves as well. This is an extension of the statement made at the end of section 2.4 for homogeneous layers.

The above process can also be described in terms of generalized rays (Spencer, 1960; Helberger, 1968; Wiggins & Helberger, 1974). Using this concept, we can state that the zeroth-order wave arriving at \( z_3 \) is represented by a single ray from \( z_3^S \) to \( z_3 \), in accordance with eq. (3.12). The generation of the first-order waves by continuous reflection of a zeroth-order wave is represented by rays that originate at \( z_3^S \), reflect at \( z_3' \) and finally arrive at \( z_3 \). Now a continuum of such reflecting rays must be taken into account for the complete first-order wave. This process has been described by eq. (3.13) with \( i = 1 \). In an equivalent manner \( k \)-th order waves are represented by a repeated, continuous summation of rays that originate at \( z_3^S \), are reflected at \( z_3^{(k)}, \ldots, z_3''_3, z_3' \), and finally arrive at \( z_3 \).

Upon comparing the expressions for \( w_1^{(i)} \) and \( w_2^{(i)} \), we see that in both expressions the same inhomogeneity function \( \chi(z_3) \) shows up, i.e., at a level \( z_3 \) both the upward and the downward traveling waves are reflected by the same amount. This has been arranged for by the specific normalization of our composition matrix \( N_{IJ}(z_3) \). Of course we might have used the freedom to choose some alternative normalization by multiplying the two columns of \( N_{IJ}(z_3) \) by two different constants. In that case the same expressions for \( w_1^{(i)} \) and \( w_2^{(i)} \) would have resulted as before, except that they would contain two different inhomogeneity functions. As a consequence, with the wavevector following from this alternative composition matrix, the upward traveling waves are no longer reflected by the same amount as the downward traveling waves. However, the alternative WKBJ iterative solution would still converge to the same solution as our WKBJ iterative solution. From this we may conclude that in inhomogeneous configurations there is no unique decomposition of the wavefield in up- and downgoing waves as claimed by some authors (Mirovitskii & Budagyan, 1966). This fact is in accordance with a paper by Sluijter (1970), who used an alternative Bremmer series as a counter example to show that a unique decomposition does not exist. In view of this, the statement at the beginning of this section that each term in the WKBJ iterative solution can be assigned a physical meaning does not imply that the physical picture thus obtained is unique.
3.3. Applying the Cagniard-De Hoop method to the zeroth-order terms

The third and final step in our integral transform approach consists of finding the space-time domain counterpart of the transform domain acoustic state vector, which is obtained when the composition relation (2.32) is applied to the wavevector given by eqs. (3.9), (3.12) and (3.13). The inverse transformations in eqs. (2.13) and (2.15) are not easy to perform. An analytical evaluation of these integrals is in general impossible. Although in principle one can evaluate the three inversion integrals by direct numerical integrations, our specific configuration allows for more efficient inversion methods. In the introduction four methods have been presented by which the numerical work involved with the inversion process can be reduced. These are: the spectral method with a real slowness contour, the spectral method with a complex slowness contour, the slowness method with a real slowness contour (Chapman method) and the slowness method with a complex slowness contour (Cagniard-De Hoop method). For an outline of these methods we refer to Chapman (1978) and Chapman & Orcutt (1985). One of these methods must be chosen in order to carry on with our investigation. By making the following general considerations, we select the most promising method in view of the aim of this chapter, i.e., the derivation of an integral transformation method by which it is in principle possible to obtain sufficiently accurate results for the acoustic wavefield at any time instant and in any continuously layered configuration. Firstly, it is noted that in our case the slowness methods leave only one integral to be evaluated numerically, and that the inversion process is in principle exact. Both spectral methods reduce the numerical work involved as well, but unless certain approximations are made, none of the three inversion integrals is eliminated and a considerable numerical task still remains. In view of the expected numerical efficiency and the exactness of the slowness methods, these are preferred over the spectral methods. However, a second and closer look reveals that the slowness method with a real slowness contour is most convenient when one neglects that part of the real slowness contour that lies beyond the branch points on the real slowness axis (i.e., one neglects the evanescent waves); a more complicated situation arises if an exact inverse transformation is required. For this reason the slowness method with a complex slowness contour is finally selected since
no complications arise when it is used to perform an exact inverse transformation. In the remainder of this chapter the slowness method with a complex slowness contour will be indicated by its more common name of the Cagniard-De Hoop method (De Hoop, 1960, 1961, 1988; Pao & Gajewski, 1977; Van der Hilden, 1987).

In view of the nature of the Cagniard-De Hoop method, the inverse transformation process must be carried out separately for each different order term of the acoustic state vector. The goal of this section is to illustrate the inverse transformation method for the components of the zeroth-order term of the acoustic state vector, which is explicitly given by

\[
\tilde{b}_j^{(0)} = \frac{1}{i} \tilde{a}_1 \exp(-s \int_{x_3}^{x_3^S} \gamma \, d\zeta) \left( \frac{Y^{1/2}(x_3)}{Y^{-1/2}(x_3)} \right), \quad (x_3 > x_3^S), \quad (3.19)
\]

\[
\tilde{b}_j^{(0)} = \frac{1}{i} \tilde{a}_2 \exp(-s \int_{x_3}^{x_3^S} \gamma \, d\zeta) \left( \frac{Y^{1/2}(x_3)}{-Y^{-1/2}(x_3)} \right), \quad (x_3 < x_3^S). \quad (3.20)
\]

To avoid the repetition of analogous equations for all possible zeroth-order wavefield quantities due to all possible kinds of source components, we confine ourselves in this section to a zeroth-order acoustic pressure wave traveling upwards from a source of volume injection rate (monopole source) to a receiver. This wave is indicated schematically in figure 3.4. The acoustic pressure wave is indicated by \( p^{(0)\gamma} \), where the superscript \((0)\) indicates the order of the acoustic pressure wave and the minus sign refers to its direction of propagation.

### 3.3.1. The zeroth-order Green’s function

When the wavefield is generated by a source of volume injection, according to eqs. (2.25), (2.39) and (3.20), the transform domain expression for \( \tilde{p}^{(0)\gamma} \) at the receiver level is

\[
\tilde{p}^{(0)\gamma} = \frac{1}{i} Q^S(s) Y^{-1/2}(x_3^S) Y^{-1/2}(x_3^R) \exp(-s \int_{x_3^S}^{x_3^R} \gamma \, d\zeta). \quad (3.21)
\]

As a first step in our inverse transformation process, we apply the inverse Fourier transformation (2.15) and obtain

\[
\hat{p}^{(0)\gamma} = \hat{Q}^S(s) \frac{s^2}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Pi \exp[-s(i\alpha_1 x_1^R + i\alpha_2 x_2^R + \int_{x_3^S}^{x_3^R} \gamma \, d\zeta)] \, d\alpha_1 \, d\alpha_2. \quad (3.22)
\]
Figure 3.4. Schematic picture of a zeroth-order acoustic pressure wave propagating upwards from a source to a receiver.

Here the factor

$$\Pi = Y^{-1/2}(x_3^S) Y^{-1/2}(x_3^R)$$  \hspace{1cm} (3.23)$$

represents the coupling of the acoustic pressure wave to the source and the receiver. Equation (3.22) is of the form

$$\dot{\hat{p}}^{(0)-} = s^2 \hat{Q}^S(s) \hat{G}^{(0)-}, \hspace{1cm} (3.24)$$

where we have introduced the zeroth-order space-temporal Laplace domain Green's function

$$\hat{G}^{(0)-} = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Pi \exp[-s(i\alpha_1 x_1^R + i\alpha_2 x_2^R + \int_{x_3^S}^{x_3^R} \gamma d\zeta)] \, d\alpha_1 \, d\alpha_2. \hspace{1cm} (3.25)$$

The analysis that follows will be based on the fact that in this Green's function the Laplace transform parameter $s$ only shows up as a multiplier in the argument of the exponential function; note that $s$ does not appear in $\Pi$. 
3.3.2. Transformation of the Fourier transform parameters \( \alpha_1 \) and \( \alpha_2 \) into \( p \) and \( q \)

The version of the Cagniard-De Hoop method that is commonly employed in the case of isotropic media (Van der Hadden, 1987) is characterized by the transformations

\[
\begin{align*}
\alpha_1 &= -i p \cos \theta + q \sin \theta, \\
\alpha_2 &= -i p \sin \theta - q \cos \theta,
\end{align*}
\]

where \( \theta \) is one of the cylindrical coordinates of the receiver with respect to the source. The complete set of these cylindrical coordinates consists of the horizontal offset \( r \), the polar angle \( \theta \), and the vertical separation \( z \). These are related to the Cartesian coordinates of the receiver and the source according to

\[
\begin{align*}
x_1^R &= r \cos \theta, \\
x_2^R &= r \sin \theta, \\
x_3^2 - x_3^R &= z.
\end{align*}
\]

Note that our frame of reference is chosen in such a way that the vertical axis goes through the source. Upon transforming the parameters \( \alpha_1 \) and \( \alpha_2 \) into \( p \) and \( q \), the equivalent of eq. (3.25) becomes

\[
\hat{G}^{(0)\pm} = -\frac{i}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{\Pi} \exp[-s(p r + \int_0^\infty \tilde{\gamma} d\zeta)] dp dq.
\]

(3.31)

Note that \( p \) is purely imaginary and \( q \) is real. Quantities in which \( \alpha_1 \) and \( \alpha_2 \) are replaced by \( p \) and \( q \) are indicated by an overbar. As an example, the vertical slowness is now given by

\[
\tilde{\gamma}(x_3) = \left[ c^{-2}(x_3) - p^2 + q^2 \right]^{1/2}.
\]

(3.32)

The parameter \( p \) is called the horizontal slowness.

Next we continue the integrand in eq. (3.31) analytically into the complex \( p \)-plane, and define \( \text{Re}\{\tilde{\gamma}\} \geq 0 \) in order to keep the square root in eq. (3.32) single valued. Since we subsequently want to deform the contour of integration in the complex \( p \)-plane, we will have to apply Jordan's lemma. In view of this, from now on we restrict our attention to the right half of the complex \( p \)-plane. Before the deformation of the integration contour can take place, we must investigate the
singularities of the integrand in eq. (3.31) in this area. First of all, two branch points \( p^S = \left[ c^{-2}(x_3^S) + q^2 \right]^{1/2} \) and \( p^R = \left[ c^{-2}(x_3^R) + q^2 \right]^{1/2} \) are due to \( \tilde{\gamma}(x_3^S) \) and \( \tilde{\gamma}(x_3^R) \) occurring in \( \tilde{\Pi} \). A second set of branch points is introduced by the exponential part of the integrand in eq. (3.31). Although the exponential function itself does not have branch points, its argument introduces branch points due to the integral of \( \tilde{\gamma} \). The determination of the latter branch points is investigated separately in appendix 3.B. There we find that the branch points due to the integral of \( \tilde{\gamma}(x_3) \) are the points \( p = \left[ c^{-2}(x_3) + q^2 \right]^{1/2} \) related to the levels \( x_3 \) that either form

- an endpoint of the integration interval \([x_3^R; x_3^S]\);
- a point where one or more derivatives of the wavespeed profile are discontinuous; or
- a stationary point (i.e., a point where \( \partial_3 \zeta = 0 \)) where the wavespeed profile reaches a (local) extremum.

All branch points are positioned on the real \( p \)-axes. They are supplemented with branch cuts along the positive and negative real \( p \)-axis, respectively, from the relevant branch point to infinity. Apart from the branch points mentioned, no other singularities are present in the complex \( p \)-plane.

3.3.3. The Cagniard contour

Knowing the singularities of the integrand of eq. (3.31) in the complex \( p \)-plane, we propose to deform the path of integration from the imaginary \( p \)-axis to the Cagniard contour, which consists of the complex branches of

\[
\tau = \rho r + \int_{x_3^S}^{x_3^R} \tilde{\gamma} \, d\zeta = \text{real} \tag{3.33}
\]

that are located in the right half of the complex \( p \)-plane. The features of this Cagniard contour in the case of a continuously layered configuration will now be investigated. First we define the maximum wavespeed \( c_{\text{max}} \) by

\[
c_{\text{max}} = \max_{x_3 \in [x_3^R; x_3^S]} \{c(x_3)\}. \tag{3.34}
\]

On \([x_3^R; x_3^S]\) this maximum wavespeed can be found on a level that either forms
• an endpoint of the integration interval \([x_3^R; x_3^S]\);
• a point where \(\partial_3 c\) is discontinuous; or
• a stationary point of the wavespeed profile.

These possibilities are illustrated in figure 3.5. From appendix 3.B we know that these possibilities form a subset of those that give rise to a branch point of the integrand in the complex \(p\)-plane. Consequently, the leftmost branch point \(p_e\) in the complex \(p\)-plane is always given by

\[
p_e = (c_{\text{max}}^2 + q^2)^{1/2}.
\]  

(3.35)

Although the real axis in between the origin and \(p_e\) does satisfy eq. (3.33), it will turn out that this trajectory is not a part of the final integration path in the complex \(p\)-plane, and therefore it will not be considered as a part of the Cagniard contour. For large values of \(\tau\) the complex branches of the Cagniard contour asymptotically

![Figure 3.5](image)

**Figure 3.5.** Different possibilities for the occurrence of \(c_{\text{max}}\): (a) at an endpoint of the interval; (b) at a point where \(\partial_3 c\) is discontinuous; (c) at a stationary point of the wavespeed profile.
approach the straight lines

\begin{align*}
  p & \sim \frac{\tau}{r - iz} \quad \text{if } \tau \to \infty \text{ in the first quadrant of the } p\text{-plane,} \\
  p & \sim \frac{\tau}{r + iz} \quad \text{if } \tau \to \infty \text{ in the fourth quadrant of the } p\text{-plane.}
\end{align*}

(3.36) \hspace{1cm} (3.37)

Note that these asymptotes are independent of \( q \). For continuously layered configurations the Cagniard contour can approach the real axis in two distinct ways. These depend upon the sign of the derivative

\[
\partial_p \tau = r - p \int_{x_2^R}^{x_2^F} [c^{-2}(\zeta) - p^2 + q^2]^{-1/2} \, d\zeta
\]

(3.38)

at the point \( p = p_t \), i.e., the sign of

\[
\left. \partial_p \tau \right|_{p_t} = r - \left( c_{\text{max}}^{-2} + q^2 \right)^{1/2} \int_{x_2^R}^{x_2^F} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{-1/2} \, d\zeta.
\]

(3.39)

Although the point \( p_t \) satisfies eq. (3.33), as we will see later it is not under all circumstances a point of the Cagniard contour. Nevertheless, it always is an important point in view of the determination of the type of the Cagniard contour. The conditions of occurrence and the consequences of both types of Cagniard contours will now be investigated.

---

The case \( \partial_p \tau \big|_{p_t} < 0 \) ---

To start with, we analyze the circumstances under which the case \( \partial_p \tau \big|_{p_t} < 0 \) shows up

- assume that \( c_{\text{max}} \) is arrived at in a point \( x_{3,\text{max}} \) where \( \partial_c c \neq 0 \), so \( x_{3,\text{max}} \) is either an endpoint of the interval \([x_3^L; x_3^G]\) or a point where \( \partial_c c \) is discontinuous. Now we can find a small value \( \delta > 0 \) such that either on the interval \([x_{3,\text{max}} - \delta; x_{3,\text{max}}]\) or on the interval \([x_{3,\text{max}}; x_{3,\text{max}} + \delta]\) the wavespeed profile can accurately be approximated by a linear function. Using the results of appendix 3.C, it is easily demonstrated that for \( p = p_t \) this linear approximation leads to a bounded and positive integral of \( \tilde{\gamma}^{-1} \) over these intervals. For \( p = p_t \), integration of \( \tilde{\gamma}^{-1} \) over the remaining part of \([x_3^G; x_3^F]\) yields a bounded positive
result as well since $\tilde{\gamma}^{-1}$ is bounded and positive here. As a result, the total integral in eq. (3.39) is bounded and positive. If we introduce the quantity

$$r_{sep} = \frac{1}{c_{\text{max}}} \int_{z_3^R}^{z_3^S} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{-1/2} \, d\zeta,$$

(3.40)

a closer look at eq. (3.39) shows us that $\partial_p r |_{p_t < 0}$ provided that either

- the horizontal offset satisfies $r < r_{sep}$, independently of the value of the real parameter $q$;
- the horizontal offset satisfies $r > r_{sep}$ and $q > Q_{sep}$. The quantity $Q_{sep}$ separates the intervals of $q$ where $\partial_p r |_{p_t < 0}$ and $\partial_p r |_{p_t > 0}$, so its value is found by solving $q$ from

$$\partial_p r |_{p_t} = r - (c_{\text{max}}^{-2} + q^2)^{1/2} \int_{z_3^R}^{z_3^S} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{-1/2} \, d\zeta = 0,$$

(3.41)

and is obtained as

$$Q_{sep} = \left( \frac{\tau^2 \left( \int_{z_3^R}^{z_3^S} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{-1/2} \, d\zeta \right)}{\left( \int_{z_3^R}^{z_3^S} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{-1} \, d\zeta \right)^{1/2}} \right)^{1/2};$$

(3.42)

- suppose that $c_{\text{max}}$ is reached in a point $z_{3:\text{max}}$ that forms a stationary point of the wavespeed profile. Now for any small value of $\delta > 0$ the wavespeed profile on the intervals $[z_{3:\text{max}} - \delta; z_{3:}\text{max}]$ and/or $[z_{3:}\text{max}; z_{3:}\text{max} + \delta]$ must be approximated by a quadratic or higher-degree polynomial. This implies that for $p = p_t$ the integral of $\tilde{\gamma}^{-1}(x_3)$ over at least one of these intervals is positive infinite, and so is the total integral over $[z_3^R; z_3^S]$. According to eq. (3.39) for all possible values of $r$ and $q$ we have $\partial_p r |_{p_t} < 0$. This is also indicated by the fact that in this case for all values of $r$ we find $r < r_{sep}$.

We can summarize the results of this detailed analysis by stating that the case $\partial_p r |_{p_t} < 0$ shows up if either

- the horizontal offset satisfies $r < r_{sep}$; or
- the horizontal offset satisfies $r > r_{sep}$ and the parameter $q$ satisfies $q > Q_{sep}$.
If \( \partial_p \tau |_{p_t} < 0 \) there is only a single point \( p_0 \) on the interval \([0; p_t)\) where the Cagniard contour crosses the real \( p \)-axis. This follows from the fact that the second derivative of \( \tau \), being

\[
\partial_p^2 \tau = -\int_{\sigma_R}^\sigma [c^{-2}(\zeta) - p^2 + q^2]^{-1/2} \, d\zeta - p^2 \int_{\sigma_R}^\sigma [c^{-2}(\zeta) - p^2 + q^2]^{-3/2} \, d\zeta,
\]

is always negative on \([0; p_t)\). Consequently, \( \partial_p \tau \) is monotonically decreasing when going from the origin where \( \partial_p \tau |_{p=0} = r \geq 0 \) to \( p_t \) where \( \partial_p \tau |_{p_t} < 0 \), and this proves that there exists one point on \([0; p_t)\) where \( \partial_p \tau = 0 \). Traveling along the real \( p \)-axis, \( \tau \) has a maximum value in this point. To further increase the value of \( \tau \), one must leave the real \( p \)-axis perpendicularly in \( p_0 \) and proceed along one of the complex branches of the Cagniard contour. Traveling along the Cagniard contour, \( \tau \) has a minimum value in \( p_0 \), so \( p_0 \) turns out to be a saddle point in the complex \( p \)-plane. In general, if \( \partial_p \tau |_{p_t} < 0 \) the point \( p_0 \) has to be determined numerically. Figure 3.6(a) shows an example of this type of Cagniard contour.

\[\text{Figure 3.6. Locations of the Cagniard contour and the branch points in the right half of the complex } p \text{-plane: (a) the case } \partial_p \tau |_{p_t} < 0; (b) the case } \partial_p \tau |_{p_t} > 0. \text{ For case (b) a detour around the branch point has been made.}\]
— The case $\partial_p r|_{p_t} > 0$ —

For continuously layered media we can have $\partial_p r|_{p_t} > 0$. The circumstances under which this shows up are complementary to those mentioned at the previous case, namely

- the horizontal offset must satisfy $r > r_{sep}$ and the parameter $q$ must satisfy $0 < q < Q_{sep}$. The first condition can only be met if the maximum wavespeed $c_{\text{max}}$ is reached in a point where $\partial_0 c \neq 0$ and thus $r_{sep}$ remains finite. Consequently, this point must be either an endpoint of the interval $[x_3^R; x_3^S]$ or a point where $\partial_0 c$ is discontinuous.

If $\partial_p r|_{p_t} > 0$, the Cagniard contour does not leave the real $p$-axis in a point of the interval $[0; p_t)$. To prove this, note that $\partial_p^2 r < 0$ on $[0; p_t)$ and thus $\partial_p r$ is a monotonically decreasing function on this interval. However, in the present case both $\partial_p r|_{p=0} = r \geq 0$ and $\partial_p r|_{p_t} > 0$, so $\partial_p r$ remains positive on $[0; p_t)$. As a consequence the Cagniard contour does not leave the real $p$-axis in a point of this interval. For $p > p_t$ the parameter $r$ certainly becomes complex, so this part of the real axis will not be a part of the Cagniard contour. Since $\lim_{p \to p_t} \partial_p r$ is real valued we conclude that the complex contour meets the real $p$-axis tangentially in $p_0 = p_t$. Again $r$ has a minimum in $p_0$ when traveling along the Cagniard contour.

An example of this type of Cagniard contour is shown in figure 3.6(b). [Note that, as opposed to the continuously layered configurations that are investigated in this thesis, in (piecewise) homogeneous configurations the case $\partial_p r|_{p_t} > 0$ never shows up since the integral in eq. (3.39) is always positive infinite. In view of this, when one is dealing with a continuously layered configuration, a linear interpolation function is the most simple function that conserves the continuous character of the profile and the associated properties.] This concludes the discussion of the case $\partial_p r|_{p_t} > 0$.

Now that the properties of the Cagniard contours are known, we will actually deform the integration contour into the complex $p$-plane. In the first and fourth quadrants we can form closed loops consisting of the positive or negative imaginary axis, the positive real axis from the origin to $p_0$, the upper ($P^+$) or lower ($P^-$) branches of the Cagniard contour and closing circular arcs at infinity. Applying Cauchy’s theorem and Jordan’s lemma to these loops, we find that the integration
along the imaginary $p$-axis can be replaced by an integration along the Cagniard contour, see figure 3.6. Notice that the integrations along the real $p$-axis cancel each other. If $\partial_p \tau|_{p_0} < 0$, the deformation process can be performed without special precautions. However, if $\partial_p \tau|_{p_0} > 0$ the Cagniard contour must be supplemented by a small circle with radius $\varepsilon > 0$ in order to go around the leftmost branch point, see figure 3.6(b). In the limit $\varepsilon \to 0$, this circle gives a vanishing contribution to the total integral. The Cagniard contour is symmetrical with respect to the real $p$-axis and the integrand of eq. (3.31) satisfies Schwarz' reflection principle; further the integrand is symmetrical in $q$. Using these symmetry properties, we can rewrite eq. (3.31) as

$$\hat{G}^{(0)-} = \frac{1}{2\pi^2} \int_0^{\infty} \text{Im} \left\{ \int_{P^+} \Pi \exp[-s(pr + \int_{s_R}^{s_L} \tilde{q} d\zeta)] dp \right\} dq, \quad (3.44)$$

which holds for both cases $\partial_p \tau|_{p_0} < 0$ and $\partial_p \tau|_{p_0} > 0$.

### 3.3.4. Replacing the variable of integration $p$ by $\tau$

The Cagniard contour meets the real $p$-axis in the point $p_0$. Progressing along the contour away from $p_0$, the parameter $\tau$ increases monotonically. This means that we can easily replace the integration over the Cagniard contour in the complex $p$-plane by an integration over the real parameter $\tau$. The lowest value of $\tau$ is found in the point $p_0$ and is denoted by $T(q)$. Two situations can be distinguished

- if $\partial_p \tau|_{p_0} < 0$, then $p_0 < p_\ell$. In this situation the value of $p_0$ must be found numerically. Once we have found $p_0$, we can determine $T(q)$ using the equation

$$T(q) = p_0 \tau + \int_{s_R}^{s_L} \left[ c^{-2}(\zeta) - p_0^2 + q^2 \right]^{1/2} d\zeta; \quad (3.45)$$

- if $\partial_p \tau|_{p_0} > 0$, we have $p_0 = p_\ell = (c_{\text{max}}^{-2} + q^2)^{1/2}$. In this situation eq. (3.45) becomes

$$T(q) = (c_{\text{max}}^{-2} + q^2)^{1/2} \tau + \int_{s_R}^{s_L} \left[ c^{-2}(\zeta) - c_{\text{max}}^{-2} \right]^{1/2} d\zeta. \quad (3.46)$$

The quantity $T(0)$ is equal to the arrival time of the zeroth-order contribution and is therefore denoted by $T_{\text{arr}}$. Again two possibilities exist
• if $r < r_{sep}$, even for $q = 0$ we have $\partial_p \tau|_{pt} < 0$ and the associated value of $p_0$ must be found numerically. Now $T_{arr}$ follows as

$$T_{arr} = p_0 r + \int_{x_R^d}^{x_R^e} [c^{-2}(\zeta) - p_0^2]^{1/2} d\zeta; \quad (3.47)$$

• if $r > r_{sep}$, a value $Q_{sep}$ exists and for $0 \leq q < Q_{sep}$ we have $\partial_p \tau|_{pt} > 0$. In this case the value of $p_0$ belonging to $q = 0$ simply equals $p_0 = 1/c_{max}$, and $T_{arr}$ is

$$T_{arr} = \frac{r}{c_{max}} + \int_{x_R^d}^{x_R^e} [c^{-2}(\zeta) - c_{max}^{-2}]^{1/2} d\zeta. \quad (3.48)$$

The quantity $T(Q_{sep})$ is denoted by $T_{sep}$ and equals

$$T_{sep} = \frac{r^2}{\int_{x_R^d}^{x_R^e} [c^{-2}(\zeta) - c_{max}^{-2}]^{-1/2} d\zeta} + \int_{x_R^d}^{x_R^e} [c^{-2}(\zeta) - c_{max}^{-2}]^{1/2} d\zeta. \quad (3.49)$$

In figure 3.7 a typical plot of $T(q)$ versus $q$ has been presented. Upon replacing the integration over $p$ by an integration over $\tau$, we obtain

$$\dot{G}^{(0)}_0 = \frac{1}{2\pi^2} \int_0^\infty \int_{T(q)}^\infty \text{Im} \left\{ \Pi \partial_p \right\} \exp(-\sigma \tau) d\tau dq, \quad (3.50)$$

![Figure 3.7. A typical plot of the function $T(q)$ versus $q$.](image)
where the outer integral is interpreted as
\[
\int_0^\infty \ldots dq = \int_0^{Q_{\text{sep}}} \ldots dq + \int_{Q_{\text{sep}}}^\infty \ldots dq
\]  
(3.51)
if different types of Cauchy contours are involved.

### 3.3.5. Interchanging the order of integration and recognition of the space-time domain Green’s function

Since \( \tau = T(q) \) is a monotonic function of \( q \) for \( q \geq 0 \), it possesses a unique non-negative inverse function \( q = Q(\tau) \) on \( \tau \geq T_{\text{arr}} \). The way in which the values of \( Q(\tau) \) are determined depends upon the value of \( \tau \) in the following way:

- if there is no value \( T_{\text{sep}} > T_{\text{arr}} \) (\( \tau < T_{\text{sep}} \)), the value of \( p_0 \), and thus the value of \( Q(\tau) \), must be found numerically;

- if there is a value \( T_{\text{sep}} > T_{\text{arr}} \) (\( \tau > T_{\text{sep}} \)) but \( \tau > T_{\text{sep}} \), the value of \( p_0 \), and thus the value of \( Q(\tau) \), must be found numerically;

- if there is a value of \( T_{\text{sep}} > T_{\text{arr}} \) (\( \tau > T_{\text{sep}} \)) and \( \tau \) has a value such that \( T_{\text{arr}} < \tau < T_{\text{sep}} \), the value of \( p_0 \) equals \( p_\tau \), and \( Q(\tau) \) follows from eq. (3.46) as

\[
Q(\tau) = \left( \frac{\tau - \int_{x_2^R}^{x_2^S} [c^{-2}(\zeta) - c_{\max}^{-2}]^{1/2} \, d\zeta}{\tau^2} \right)^{1/2} - c_{\max}^{-2}
\]

(3.52)

The function \( Q(\tau) \) is used when we interchange the order of integrations in eq. (3.50) according to

\[
\hat{G}^{(0)} = \frac{1}{2\pi^2} \int_{T_{\text{arr}}}^\infty \int_0^{Q(\tau)} \text{Im} \{ \tilde{P} \partial_t p \} \exp(-s\tau) \, dq \, d\tau.
\]

(3.53)

With reference to the forward Laplace transformation of eq. (2.12) and the theory of Laplace transformations we can simply recognize the required space-time domain Green’s function as

\[
G^{(0)} = \frac{1}{2\pi^2} H(t-T_{\text{arr}}) \int_0^{Q(t)} \text{Im} \{ \tilde{P} \partial_t p \} \, dq.
\]

(3.54)
This result shows that with the Cagniard-De Hoop method the number of integrals that has to be evaluated when one wants to perform the transformation back to the space-time domain has been reduced from three to only one.

### 3.3.6. Convolution with the source signature

Once we know the space-time domain Green’s function \( G^{(0)}(t) \), application of eq. (3.24) and the theory of the Laplace transformation lead to the space-time domain acoustic pressure, which is

\[
p^{(0)} = \partial_t^2 \left[ Q^S(t) \ast G^{(0)}(t) \right],
\]

(3.55)

Here, \( \ast \) indicates a convolution with respect to time. The differentiation with respect to time may act either on the source signature, or on the Green’s function, or on both, so

\[
p^{(0)} = \partial_t^2 Q^S(t) \ast \partial_t G^{(0)}(t)
\]

\[
= Q^S(t) \ast \partial_t^2 G^{(0)}(t).
\]

(3.56)

If we assume that the source signature is known in functional form, as will be the case in this thesis, the first possibility is most convenient.

### 3.4. Applying the Cagniard-De Hoop method to the first-order terms

In the present section we will illustrate the inverse transformation of the components of the first-order term of the acoustic state vector by means of the Cagniard-De Hoop method. The first-order term of the acoustic state vector is explicitly given by

\[
\begin{align*}
\vec{b}^{(1)} &= -\frac{1}{2} \tilde{a}_2 \int_{-\infty}^{x^*_3} \chi(x'_3) \exp[-s(f_{x_3}^S \gamma d\zeta + \int_{x^*_2}^{x_2} \gamma d\zeta)] \, dx'_3 \left( \frac{Y^{1/2}(x_3)}{Y^{-1/2}(x_3)} \right) \\
& \quad - \frac{1}{2} \tilde{a}_1 \int_{x^*_3}^{\infty} \chi(x'_3) \exp[-s(f_{x_3}^S \gamma d\zeta + \int_{x^*_2}^{x_2} \gamma d\zeta)] \, dx'_3 \left( \frac{-Y^{1/2}(x_3)}{Y^{-1/2}(x_3)} \right),
\end{align*}
\]

(3.57)

in which

\[
\begin{align*}
x_3^m &= \min\{x_3, x_3^S\} \quad (3.58) \\
x_3^M &= \max\{x_3, x_3^S\} \quad (3.59)
\end{align*}
\]
indicate that various situations can occur, depending on the position of the level \( z_3 \) relative to the source level \( z_3^S \). Although the method is in principle identical to the one applied to the zeroth-order components, there are two additional complications related to the inverse transformation of the first-order components. One complication is that the nature of the Cagniard-De Hoop method inhibits a single-step inverse transformation of this complete integral. Instead, the inverse transformation must act on the integrand, i.e., on the partial reflections. Since this integral is dependent on the reflection level, this level enters in the inverse transformation process as a parameter. The other complication is caused by the fact that the inhomogeneity function occurs in the integrand of the first-order terms. The inhomogeneity function has a pole in the complex \( p \)-plane, and this pole can coincide with the leftmost branchpoint at the same time that the Cagniard contour leaves the real \( p \)-axis in this point. As a result, special care must be taken during the contour deformation.

In order to avoid the repetition of analogous equations for all possible first-order field quantities due to all possible kinds of source components, in this section we confine ourselves to an upgoing first-order acoustic pressure wave, caused by continuous reflection of a downgoing zeroth-order wave due to a source of volume injection rate (monopole source). This type of wave is of major importance, e.g., in seismic prospecting based on the equivalent fluid model of the earth. This wave is indicated schematically in figure 3.8. The acoustic pressure wave is indicated by \( p^{(1)} \), where the subscript \( (1) \) indicates the order of the acoustic pressure wave and the minus sign refers to its direction of propagation.

### 3.4.1. The first-order Green's function

When the wavefield is generated by a source of volume injection, according to eqs. (2.25), (2.39), (3.57) and (3.59), the transform domain expression for \( \tilde{p}^{(1)} \) at the receiver level is

\[
\tilde{p}^{(1)\to} = -\frac{1}{4} \tilde{Q}^S(s) Y^{-1/2}(z_3^S) Y^{-1/2}(z_3^R) \int_{x_3^R}^{x_3^M} \chi(z_3') \exp[-s(J_{x_3^R}^z \gamma d\zeta + J_{x_3^S}^z \gamma d\zeta)] dz_3',
\]

(3.60)

where the lower integration boundary \( M \) depends on the relative vertical position of the source and the receiver and is given by

\[
x_3^M = \max\{x_3^R, x_3^S\}.
\]

(3.61)
Figure 3.8. Schematical picture of an upward propagating first-order partial acoustic pressure wave, caused by reflection of a downward propagating zeroth-order wave.

As a first step in our inverse transformation procedure, we apply the inverse Fourier transformation (2.15) and obtain

$$\tilde{p}^{(1)-} = \tilde{Q}^S(s) \frac{-s^2}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{x_3'}^{\infty} \Pi \chi(x_3') \exp[-s\vartheta(\alpha_1, \alpha_2, x_3')] dx_3' d\alpha_1 d\alpha_2. \quad (3.62)$$

Here the factor

$$\Pi = Y^{-1/2}(x_3^S) Y^{-1/2}(x_3^R), \quad (3.63)$$

represents the coupling of the acoustic pressure waves to the source and the receiver, and the inhomogeneity function

$$\chi(x_3') = -\frac{\partial_3 c(x_3')}{2\gamma^2(x_3') c'(x_3')} - \frac{\partial_3 \rho(x_3')}{2\rho(x_3')} \quad (3.64)$$

represents the local reflection factor at the level $x_3'$ where the partial reflection takes place. Further we have introduced the quantity $\vartheta(\alpha_1, \alpha_2, x_3')$ as

$$\vartheta(\alpha_1, \alpha_2, x_3') = i\alpha_1 x_1^R + i\alpha_2 x_2^R + \int_{\mathcal{L}(x_3')} \gamma d\zeta, \quad (3.65)$$
where \( \mathcal{L}(z'_3) \) indicates the total vertical travel path from the source, via the reflection level \( z'_3 \), to the receiver. The integral over this path should be interpreted as

\[
\int_{\mathcal{L}(z'_3)} \gamma \, d\zeta = \int_{z'_3}^{z'_4} \gamma \, d\zeta + \int_{z'_4}^{z'_3} \gamma \, d\zeta. \tag{3.66}
\]

In view of the Cagniard-De Hoop method it is necessary to change the order of integration in eq. (3.62) as to obtain

\[
\hat{p}^{(1)-} = \hat{Q}^S(s) \frac{s^2}{8\pi^2} \int_{x'_4}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Pi(x'_3) \exp[-s\theta(\alpha_1, \alpha_2, x'_3)] \, d\alpha_1 \, d\alpha_2 \, dx'_3. \tag{3.67}
\]

This implies that the inverse transformation process is applied to the integrand of \( \hat{p}^{(1)-} \). Equation (3.67) is of the form

\[
\hat{p}^{(1)-} = s^2 \hat{Q}^S(s) \hat{G}^{(1)-}, \tag{3.68}
\]

where we have introduced the first-order space-temporal Laplace domain Green's function

\[
\hat{G}^{(1)-} = \frac{-1}{8\pi^2} \int_{x'_4}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Pi(x'_3) \exp[-s\theta(\alpha_1, \alpha_2, x'_3)] \, dx'_3 \, d\alpha_1 \, d\alpha_2. \tag{3.69}
\]

Note that in this Green's function the Laplace transform parameter \( s \) only shows up as a multiplier in the argument of the exponential function and that it does not appear in \( \Pi \) and \( \chi(x'_3) \).

### 3.4.2. Transformation of the Fourier transform parameters \( \alpha_1 \) and \( \alpha_2 \) into \( p \) and \( q \)

The version of the Cagniard-De Hoop method that we employ uses the transformations

\[
\alpha_1 = -ip \cos \theta + q \sin \theta, \tag{3.70}
\]

\[
\alpha_2 = -ip \sin \theta - q \cos \theta. \tag{3.71}
\]

The angle \( \theta \) is one of the cylindrical coordinates of the receiver with respect to the source. For the first-order case these are measured by going from the source to the reflection level \( z'_3 \) and then from the reflection level to the receiver. As a consequence, the the cylindrical coordinates \( r \) (horizontal offset), \( \theta \) (polar angle),
and \( z'(z'_3) \) (vertical separation) are related to the Cartesian coordinates of the source and the receiver according to

\[
\begin{align*}
x^R_1 &= r \cos \theta, \quad (3.72) \\
x^R_2 &= r \sin \theta, \quad (3.73) \\
(z'_3 - z_3^S) + (z'_3 - z_3^R) &= z. \quad (3.74)
\end{align*}
\]

Note that \( z'(z'_3) \) is the length of the path \( \mathcal{L}(x'_3) \), and that our frame of reference is chosen in such a way that the vertical axis goes through the source. Upon transforming the parameters \( \alpha_1 \) and \( \alpha_2 \) into \( p \) and \( q \), the equivalent of eq. (3.69) becomes

\[
\hat{G}^{(0)} = \frac{i}{8\pi^2} \int_{x_3^M}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Pi \chi(x'_3) \exp[-a(pr + \int_{\mathcal{L}(x'_3)}^{\gamma} d\zeta)] \, dp \, dq \, dx'_3. \quad (3.75)
\]

Note that \( p \) is purely imaginary and \( q \) is real. Quantities in which \( \alpha_1 \) and \( \alpha_2 \) are replaced by \( p \) and \( q \) are indicated by an overbar. As an example, the vertical slowness is now given by

\[
\bar{\gamma}(x_3) = \left[ c^{-2}(x_3) - p^2 + q^2 \right]^{1/2}. \quad (3.76)
\]

The parameter \( p \) is called the horizontal slowness.

Next we continue the integrand in eq. (3.75) analytically into the complex \( p \)-plane, and define \( \text{Re} \{ \bar{\gamma} \} \geq 0 \) in order to keep \( \bar{\gamma} \) single valued. In view of the application of Jordan's lemma later on, we restrict our attention to the right half of the complex \( p \)-plane. Before a possible deformation of the integration contour into the complex \( p \)-plane can take place, the singularities of the integrand in eq. (3.75) in this area must be located. First of all, two branch points \( p^S = [c^{-2}(z^S_3) + q^2]^{1/2} \) and \( p^R = [c^{-2}(z^R_3) + q^2]^{1/2} \) are due to \( \bar{\gamma}(z^S_3) \) and \( \bar{\gamma}(z^R_3) \) occurring in \( \Pi \). A second set of branch points is introduced by the exponential part of the integrand in eq. (3.75). Although the exponential function itself does not have branch points, its argument introduces branch points due to the integral of \( \bar{\gamma} \). According to appendix 3.B we find that the branch points due to the integral of \( \bar{\gamma}(\zeta) \) are the points \( p = [c^{-2}(\zeta) + q^2]^{1/2} \) related to the levels \( \zeta \in [z^S_3; z'_3] \cup [z^R_3; z'_3] \) that either form
• an endpoint of either of the two intervals \([x_3^R; x_3']\) and \([x_3^S; x_3']\) that form the vertical integration path \(L(x_3')\);

• a point where one or more derivatives of the wavespeed profile are discontinuous; or

• a stationary point (i.e., a point where \(\partial_3 c = 0\)) where the wavespeed profile reaches a (local) extremum.

All branch points are positioned on the real \(p\)-axis. They are supplemented with branch cuts along the positive and negative real \(p\)-axes, respectively, from the relevant branch point to infinity. Apart from the branch point singularities mentioned, the occurrence of \(\widehat{\gamma}(x_3')\) in one of the denominators of \(\tilde{\chi}(x_3')\) [cf. eq. (3.64)] causes a pole in the right half of the complex \(p\)-plane that is located at \(p = [c^{-2}(x_3') + \xi^2]^{1/2}\), and thus coincides with the branch point of the integrand due to the endpoint of \([x_3^S; x_3']\) and \([x_3^R; x_3']\).

### 3.4.3. The Cagniard contour

At this point we suggest to deform the path of integration from the imaginary \(p\)-axis to the Cagniard contour, which consists of the complex branches of

\[
\tau = pr + \int_{L(x_3')} \widehat{\gamma} \, d\zeta = \text{real}
\]

that are located in the right half of the complex \(p\)-plane. An outline of the features of this Cagniard contour in case of a continuously layered configuration will now be presented. First we define the maximum wavespeed \(c_{\max}(x_3')\) by

\[
c_{\max}(x_3') = \max_{\zeta \in L(x_3')} \{c(\zeta)\}. \tag{3.78}
\]

On \(L(x_3')\) this maximum wavespeed can be found on a level that either forms

• an endpoint of either of the two intervals \([x_3^R; x_3']\) and \([x_3^S; x_3']\) that form the vertical integration path \(L(x_3')\);

• a point where \(\partial_3 c\) is discontinuous; or

• a stationary point of the wavespeed profile.
For a picture of these possibilities we refer to figure 3.5. From appendix 3.B we know that these possibilities form a subset of those that give rise to a branch point of the integrand in the complex $p$-plane. Consequently, the leftmost branch point $p_l$ in the complex $p$-plane is always given by

$$p_l(x'_3) = [c_{\max}^{-2}(x'_3) + q^2]^{1/2}. \quad (3.79)$$

Although the real axis in between the origin and $p_l(x'_3)$ does satisfy eq. (3.77), it will turn out that this trajectory is not a part of the final integration path in the complex $p$-plane, and therefore it will not be considered as a part of the Cagniard contour. For large values of $\tau$ the complex branches of the Cagniard contour asymptotically approach the straight lines

$$p \sim \frac{\tau}{r - i\tau(x'_3)} \quad \text{if } \tau \to \infty \text{ in the first quadrant of the } p\text{-plane,} \quad (3.80)$$

$$p \sim \frac{\tau}{r + i\tau(x'_3)} \quad \text{if } \tau \to \infty \text{ in the fourth quadrant of the } p\text{-plane.} \quad (3.81)$$

These asymptotes are independent of $q$. For continuously layered configurations the Cagniard contour can approach the real axis in two distinct ways. These depend upon the sign of the derivative

$$\partial_p \tau = r - p \int_{L(x'_3)} [e^{-2}(\zeta) - p^2 + q^2]^{-1/2} d\zeta \quad (3.82)$$

at the point $p = p_l(x'_3)$, i.e., the sign of

$$\partial_p \tau|_{p_l(x'_3)} = r - [c_{\max}^{-2}(x'_3) + q^2]^{1/2} \int_{L(x'_3)} [e^{-2}(\zeta) - c_{\max}^{-2}(x'_3)]^{-1/2} d\zeta. \quad (3.83)$$

Although the point $p_l(x'_3)$ satisfies eq. (3.77), as we will see later it is not under all circumstances a point of the Cagniard contour. Nevertheless, it always is an important point in view of the determination of the type of the Cagniard contour. The conditions of occurrence and the consequences of both types of Cagniard contours will now be investigated.

--- The case $\partial_p \tau|_{p_l(x'_3)} < 0$ ---

To start with, we analyze the circumstances under which the case $\partial_p \tau|_{p_l(x'_3)} < 0$ shows up
• assume that $c_{\text{max}}(z'_{3})$ is arrived at in a point $x_{3;\text{max}}$ where $\delta_{3}c \neq 0$, so $x_{3;\text{max}}$ is an endpoint of either of the two intervals $[x_{3}'; x_{3}^{R}]$ and $[x_{3}'; x_{3}^{L}]$ or a point where $\delta_{3}c$ is discontinuous. Now the same kind of reasoning as applied in section 3.3.3 leads us to the fact that for $p = p_{t}$ the integral of $\tilde{\gamma}^{-1}$ over $\mathcal{L}(z'_{3})$ yields a bounded positive result. If we introduce the quantity

$$r_{\text{sep}}(z'_{3}) = \frac{1}{c_{\text{max}}(z'_{3})} \int_{\mathcal{L}(z'_{3})} \left[ c^{-2}(\zeta) - c_{\text{max}}^{-2}(z'_{3}) \right]^{-1/2} d\zeta,$$  

(3.84)

a closer look at eq. (3.83) shows us that $\partial_{p}r|_{p_{t}(z'_{3})} < 0$ provided that either

- the horizontal offset satisfies $r < r_{\text{sep}}(z'_{3})$, independently of the value of the real parameter $q$; or
- the horizontal offset satisfies $r > r_{\text{sep}}(z'_{3})$ and $q > Q_{\text{sep}}(z'_{3})$. The quantity $Q_{\text{sep}}(z'_{3})$ separates the intervals of $q$ where $\partial_{p}r|_{p_{t}(z'_{3})} < 0$ and $\partial_{p}r|_{p_{t}(z'_{3})} > 0$, so its value is found by solving $q$ from

$$\partial_{p}r|_{p_{t}(z'_{3})} = r - [c_{\text{max}}^{-2}(z'_{3}) + q^{2}]^{1/2} \int_{\mathcal{L}(z'_{3})} \left[ c^{-2}(\zeta) - c_{\text{max}}^{-2}(z'_{3}) \right]^{-1/2} d\zeta = 0,$$  

(3.85)

and is obtained as

$$Q_{\text{sep}}(z'_{3}) = \left( \frac{r^{2}}{\left( \int_{\mathcal{L}(z'_{3})} \left[ c^{-2}(\zeta) - c_{\text{max}}^{-2}(z'_{3}) \right]^{-1/2} d\zeta \right)^{2} - c_{\text{max}}^{-2}(z'_{3})} \right)^{1/2};$$  

(3.86)

• suppose that $c_{\text{max}}(z'_{3})$ is reached in a point $x_{3;\text{max}}$ that forms a stationary point of the wavespeed profile. In an equivalent way as in section 3.3.3 it can be derived that for $p = p_{t}$ the integral of $\tilde{\gamma}^{-1}$ over $\mathcal{L}(z'_{3})$ is positive infinite. Consequently, $r_{\text{sep}}(z'_{3})$ is positive infinite, and according to eq. (3.84) for all possible values of $r$ and $q$ we have $\partial_{p}r|_{p_{t}(z'_{3})} < 0$. This is also indicated by the fact that in this case for all values of $r$ we find $r < r_{\text{sep}}(z'_{3})$.

We can summarize the results of this analysis by stating that the case $\partial_{p}r|_{p_{t}(z'_{3})} < 0$ shows up if either
• the horizontal offset satisfies \( r < r_{\text{sep}}(x'_3) \); or

• the horizontal offset satisfies \( r > r_{\text{sep}}(x'_3) \) and the parameter \( q \) satisfies \( q > Q_{\text{sep}}(x'_3) \).

If \( \partial_p \tau |_{p_0(x'_3)} < 0 \) there is only a single point \( p_0(x'_3) \) on the interval \([0;p_\ell(x'_3)]\) where the Cagniard contour crosses the real \( p \)-axis, see section 3.3.3. Traveling along the real \( p \)-axis, \( \tau \) has a maximum value in this point. To further increase the value of \( \tau \), one must leave the real \( p \)-axis perpendicularly in \( p_0(x'_3) \) and proceed along one of the complex branches of the Cagniard contour. Traveling along the Cagniard contour, \( \tau \) has a minimum value in \( p_0(x'_3) \), so \( p_0(x'_3) \) turns out to be a saddle point in the complex \( p \)-plane. In general, if \( \partial_p \tau |_{p_\ell(x'_3)} < 0 \) the point \( p_0(x'_3) \) has to be determined numerically. Figures 3.9(a) and 3.9(c) show two examples of this type of Cagniard contour.

— The case \( \partial_p \tau |_{p_\ell(x'_3)} > 0 \) —

For continuously layered media we can have \( \partial_p \tau |_{p_\ell(x'_3)} > 0 \). The circumstances under which this shows up are complementary to those mentioned at the previous case, namely

• the horizontal offset must satisfy \( r > r_{\text{sep}}(x'_3) \) and the parameter \( q \) must satisfy \( 0 \leq q < Q_{\text{sep}}(x'_3) \). The first condition can only be met if the maximum wavespeed \( c_{\text{max}}(x'_3) \) is reached in a point where \( \partial_\beta c \neq 0 \) and thus \( r_{\text{sep}}(x'_3) \) remains finite. Consequently, this point must be an endpoint of either of the two intervals \([x'_3; x'_4] \) and \([x'_3; x'_3] \), or a point where \( \partial_\beta c \) is discontinuous.

If \( \partial_p \tau |_{p_\ell(x'_3)} > 0 \), the Cagniard contour meets the real \( p \)-axis tangentially in \( p_0(x'_3) = p_\ell(x'_3) \), see section 3.3.3. Again \( \tau \) has a minimum in \( p_0(x'_3) \) when traveling along the Cagniard contour. Two examples of this type of Cagniard contour are shown in figures 3.9(b) and 3.9(d).
Figure 3.9. Locations of the Cagniard contour and the singularities in the right half of the complex p-plane. For cases (b) and (d) a detour around the leftmost branch point has been made.
Now that the properties of the Cagniard contours are known, we will actually deform the integration contour into the complex \( p \)-plane. In the first and fourth quadrants we can form closed loops consisting of the positive or negative imaginary axis, the positive real axis from the origin to \( p_0(z'_3) \), the upper \([P^+(z'_3)]\) or lower \([P^-(z'_3)]\) branches of the Cagniard contour and closing circular arcs at infinity. Applying Cauchy's theorem and Jordan's lemma to these loops, we find that the integration along the imaginary \( p \)-axis can be replaced by an integration along the Cagniard contour, see figure 3.9. Notice that the integrations along the real \( p \)-axis cancel each other. If \( \partial_\tau |_{p_0(z'_3)} < 0 \), the deformation process can be performed without special precautions. However, if \( \partial_\tau |_{p_0(z'_3)} > 0 \) the Cagniard contour must be supplemented by a small circle with radius \( \varepsilon > 0 \) in order to go around the leftmost branch point, see figures 3.9(b) and 3.9(d). Subsequently, we take the limit \( \varepsilon \to 0 \). Now the contribution of the circle to the total integral over the Cagniard contour depends on the value of the wavespeed \( c(z'_3) \) at the level \( z'_3 \) where the partial reflection takes place. The next two situations are distinguished

- if \( c(z'_3) < c_{\text{max}}(z'_3) \), we encircle the leftmost branchpoint, see figure 3.9(b). Since this branch point has a zero residue, the circle gives a vanishing contribution;

- if \( c(z'_3) = c_{\text{max}}(z'_3) \), the leftmost branchpoint coincides with the pole of the inhomogeneity function. This situation is depicted in figure 3.9(d). Now the leftmost branch point has a nonzero residue, and thus the circle yields a nonzero contribution.

The Cagniard contour is symmetrical with respect to the real \( p \)-axis and the integrand of eq. (3.75) satisfies Schwarz' reflection principle; further the integrand is symmetrical in \( q \). Using these symmetry properties, we can rewrite eq. (3.75) as

\[
\hat{G}^{(1)} = \frac{-1}{2\pi^2} \int_{z'_3}^{\infty} \int_0^\infty \text{Im} \left\{ \int_{p^+(z'_3)} \tilde{\Pi} \tilde{\chi}(z'_3) \exp[-s(p r + s \tau(z'_3) \tilde{\tau} d\zeta)] dp \right\} dq dz'_3 \\
+ \frac{1}{2\pi} \int_{z'_3}^{\infty} S(z'_3) \int_0^{\text{Res}_{p = \tau(z'_3)}} \{ \tilde{\Pi} \tilde{\chi}(z'_3) \exp[-s(p r + s \tau(z'_3) \tilde{\tau} d\zeta)] \} dq dz'_3.
\]

(3.87)

We know that the contribution of the residue must only be taken into account if \( \partial_\tau |_{\tau(z'_3)} > 0 \) and \( c(z'_3) = c_{\text{max}}(z'_3) \). The first condition is reflected by the fact that in
the residue term of the last equation the upper boundary of the integral with respect to \( q \) has been limited to \( Q_{sep}(x'_3) \), since for \( q > Q_{sep}(x'_3) \) we find \( \partial_{p_r} \tau|_{p_r(x'_3)} < 0 \). The second condition has been built into the second term of the last equation by using the function \( S(x'_3) \) that satisfies

\[
S(x'_3) = \begin{cases} 
0 & \text{if } c(x'_3) < c_{\text{max}}(x'_3) \\
1 & \text{if } c(x'_3) = c_{\text{max}}(x'_3) 
\end{cases}. \tag{3.88}
\]

The second term in eq. (3.87) is set equal to zero if we cannot obtain a value \( Q_{sep}(x'_3) > 0 \), i.e., if the horizontal offset satisfies \( r < r_{sep}(x'_3) \).

3.4.4. Replacing the variable of integration \( p \) by \( \tau \) in the non-residue term of the Green's function

The Cagniard contour meets the real \( p \)-axis in the point \( p_0(x'_3) \). Progressing along the contour away from \( p_0(x'_3) \), the parameter \( \tau \) increases monotonically. This means that in the first term of eq. (3.87) we can easily replace the integration over the Cagniard contour in the complex \( p \)-plane by an integration over the real parameter \( \tau \). The lowest value of \( \tau \) is found in the point \( p_0(x'_3) \) and is denoted by \( T(q, x'_3) \). Two situations can be distinguished:

- if \( \partial_{p_r} \tau|_{p_r(x'_3)} < 0 \), then \( p_0(x'_3) < p_L(x'_3) \). In this situation the value of \( p_0(x'_3) \) must be found numerically. Once we have found \( p_0(x'_3) \), we can determine \( T(q, x'_3) \) using the equation

\[
T(q, x'_3) = p_0(x'_3) \tau + \int_{L(x'_3)} \left[ c^{-2}(\zeta) - p_0^2(x'_3) + q^2 \right]^{1/2} \, d\zeta; \tag{3.89}
\]

- if \( \partial_{p_r} \tau|_{p_r(x'_3)} > 0 \), we have \( p_0(x'_3) = p_L(x'_3) = [c_{\text{max}}^{-2}(x'_3) + q^2]^{1/2} \). In this situation eq. (3.89) becomes

\[
T(q, x'_3) = [c_{\text{max}}^{-2}(x'_3) + q^2]^{1/2} \tau + \int_{L(x'_3)} \left[ c^{-2}(\zeta) - c_{\text{max}}^{-2}(x'_3) \right]^{1/2} \, d\zeta. \tag{3.90}
\]

The quantity \( T(0, x'_3) \) is equal to the arrival time of the first-order partial contribution coming from \( x'_3 \) and is therefore denoted by \( T_{\text{arr}}(x'_3) \). Again two possibilities exist.
• if $r < r_{\text{sep}}(x'_3)$, even for $q = 0$ we have $\partial_p \tau_{|_{p(x'_3)}} < 0$ and the associated value of $p_0(x'_3)$ must be found numerically. Now $T_{\text{arr}}(x'_3)$ follows as

$$T_{\text{arr}}(x'_3) = p_0(x'_3)r + \int_{C(x'_3)} [c^{-2}(-\zeta) - p_0^2(x'_3)]^{1/2} d\zeta; \quad (3.91)$$

• if $r > r_{\text{sep}}(x'_3)$, a value $Q_{\text{sep}}(x'_3)$ exists and for $0 \leq q < Q_{\text{sep}}(x'_3)$ we have $\partial_p \tau_{|_{p(x'_3)}} > 0$. In this case the value of $p_0(x'_3)$ belonging to $q = 0$ simply equals $p_0(x'_3) = 1/c_{\text{max}}(x'_3)$, and $T_{\text{arr}}(x'_3)$ is

$$T_{\text{arr}}(x'_3) = \frac{r}{c_{\text{max}}(x'_3)} + \int_{C(x'_3)} [c^{-2}(-\zeta) - c_{\text{max}}^{-2}(x'_3)]^{1/2} d\zeta. \quad (3.92)$$

The quantity $T[Q_{\text{sep}}(x'_3), x'_3]$ is denoted by $T_{\text{sep}}(x'_3)$ and equals

$$T_{\text{sep}}(x'_3) = \frac{r^2}{\int_{C(x'_3)} [c^{-2}(-\zeta) - c_{\text{max}}^{-2}(x'_3)]^{-1/2} d\zeta} + \int_{C(x'_3)} [c^{-2}(-\zeta) - c_{\text{max}}^{-2}(x'_3)]^{1/2} d\zeta. \quad (3.93)$$

In figure 3.10 a typical plot of $T(q, x'_3)$ versus $q$ for a fixed value of $x'_3$ has been presented. Upon replacing the integration over $p$ by an integration over $\tau$ in eq. (3.87),

![Figure 3.10. A typical plot of the function $T(q, x'_3)$ versus $q$ for a fixed value of $x'_3$.](image-url)
we obtain

\begin{align*}
\mathcal{G}^{(1)} &= -\frac{1}{2\pi^2} \int_{z_1}^{\infty} \int_{0}^{\infty} \int_{T(q_{\infty}, x_1^2)}^{\infty} \text{Im}\left\{ \Pi_\mathcal{H}(x_2^2) \partial_r p \right\} \exp(-\sigma r) \, dr \, dq \, dx_2^2 \\
&+ \frac{1}{2\pi} \int_{z_2}^{\infty} S(x_2^2) \int_{0}^{Q_{\text{sep}}(x_2^2)} \text{Res}_{p = p(x_2^2)} \left\{ \Pi_\mathcal{H}(x_2^2) \exp[-s(p r + I_{\mathcal{L}}(x_2^2) \gamma \, dq)] \right\} \, dq \, dx_2^2.
\end{align*}

(3.94)

Using eqs. (3.64), (3.76) and (3.79), the residue is found to be

\begin{align*}
\text{Res}_{p = p(x_2^2)} \left\{ \Pi_\mathcal{H}(x_2^2) \exp[-s(p r + I_{\mathcal{L}}(x_2^2) \gamma \, dq)] \right\} &= R(q, x_2^2) \exp[-sT(q, x_2^2)],
\end{align*}

(3.95)

with

\begin{align*}
R(q, x_2^2) = \frac{\partial_3 c(x_2^2) \rho(x_2^2) \rho(x_2^R)}{4c_{\text{max}}^2(x_2^2)(c_{\text{max}}^2(x_2^2) + q^2)^{1/2}[c^{-2}(x_2^2) - c_{\text{max}}^{-2}(x_2^2)]^{3/4}}.
\end{align*}

(3.96)

and where \(T(q, x_2^2)\) is given by eq. (3.90). The integral with respect to \(q\) in the first term of the right-hand side of eq. (3.87) is interpreted as

\begin{align*}
\int_{0}^{\infty} \ldots dq = \int_{0}^{Q_{\text{sep}}(x_2^2)} \ldots dq + \int_{Q_{\text{sep}}(x_2^2)}^{\infty} \ldots dq.
\end{align*}

(3.97)

if different types of Cagniard contours are involved.

3.4.5. Replacing the variable of integration \(q\) by \(\tau\) in the residue term of the Green’s function

Unlike the zeroth-order Green’s function, first-order Green’s function possesses under circumstances a residue contribution that is given by the second term of eq. (3.94). To apply the Cagniard-De Hoop method we replace the remaining integration over the variable \(q\) by an integration over the real variable \(\tau\) that is given by

\begin{align*}
\tau = T(q, x_2^2) = [c_{\text{max}}^{-2}(x_2^2) + q^2]^{1/2} + \frac{1}{I_{\mathcal{L}}(x_2^2)} \left[ c^{-2}(\zeta) - c_{\text{max}}^{-2}(x_2^2) \right]^{1/2} \, d\zeta.
\end{align*}

(3.98)

Keeping \(x_2^2\) fixed, for \(0 \leq q < Q_{\text{sep}}(x_2^2)\) the function \(T(q, x_2^2)\), and thus the parameter \(\tau\), is a monotonically increasing function, see figure 3.10. The lowest value of \(\tau\) is found at \(q = 0\) and equals \(T_{\text{arr}}(x_2^2)\), while the maximum value of \(\tau\) occurs at \(Q_{\text{sep}}(x_2^2)\).
and equals $T_{\text{sep}}(x_3')$. For $0 \leq q < Q_{\text{sep}}(x_3')$ the function $\tau = T(q, x_3')$ possesses a unique inverse $q = Q(\tau, x_3')$ that follows from eq. (3.98) as

$$Q(\tau, x_3') = \left( \frac{\tau - \int_{L(x_3')}^{\infty} [c^{-2}(\zeta) - c_{\max}^{-2}(x_3')]^{1/2} d\zeta}{r^2} - c_{\max}^{-2}(x_3') \right)^{1/2}. \quad (3.99)$$

The procedure results into

$$\hat{G}^{(1)} = \frac{-1}{2\pi^2} \int_{x_3'^2}^{\infty} \int_0^{\infty} \int_{T(q,x_3')}^{\infty} \text{Im} \left\{ \tilde{\chi}(x_3') \partial_r p \right\} \exp(-s\tau) \, d\tau \, dq \, dx_3'$$

$$+ \frac{1}{2\pi} \int_{x_3'^2}^{\infty} S(x_3') \int_{T_{\text{arr}}(x_3')}^{\infty} R(Q(\tau, x_3'), x_3') \partial_r Q(\tau, x_3') \exp(-s\tau) \, d\tau \, dx_3', \quad (3.100)$$

where

$$\delta_r Q(\tau, x_3') = \frac{1}{r} \left( 1 - \frac{r^2}{c_{\max}^2(x_3')} \left( \tau - \int_{L(x_3')}^{\infty} [c^{-2}(\zeta) - c_{\max}^{-2}(x_3')]^{1/2} d\zeta \right)^2 \right)^{-1/2}. \quad (3.101)$$

3.4.6. Interchanging the order of integration and recognition of the space-time domain Green's function

Since $\tau = T(q, x_3')$ is a monotonic function of $q$ for $q \geq 0$, it possesses a unique non-negative inverse function $q = Q(\tau, x_3')$ on $\tau \geq T_{\text{arr}}(x_3')$. The way in which the values of $Q(\tau, x_3')$ are determined depends upon the value of $\tau$ in the following way:

- if there is no value $T_{\text{sep}}(x_3') > T_{\text{arr}}(x_3')$ [$r < r_{\text{sep}}(x_3')$], the value of $p_0(x_3')$, and thus the value of $Q(\tau, x_3')$, must be found numerically;

- if there is a value $T_{\text{sep}}(x_3') > T_{\text{arr}}(x_3')$ [$r > r_{\text{sep}}(x_3')$] but $\tau > T_{\text{sep}}(x_3')$, the value of $p_0(x_3')$, and thus the value of $Q(\tau, x_3')$, must be found numerically;
if there is a value of $T_{\text{sep}}(x'_3) > T_{\text{arr}}(x'_3)$ [i.e. $\tau > \tau_{\text{sep}}(x'_3)$] and $\tau$ has a value such that $T_{\text{arr}}(x'_3) \leq \tau < T_{\text{sep}}(x'_3)$, the value of $p_0(x'_3)$ equals $p_t(x'_3)$, and just as in the previous section $Q(\tau, x'_3)$ follows from eq. (3.90) as

$$Q(\tau, x'_3) = \left( \frac{\left\{ \tau - \int_{L(x'_1)} [c^{-2}(\zeta) - c_{\text{max}}^{-2}(x'_3)]^{\frac{1}{2}} d\zeta \right\}^2}{\tau^2} - c_{\text{max}}^{-2}(x'_3) \right)^{1/2}. \quad (3.102)$$

The function $Q(\tau, x'_3)$ is used when we interchange the order of integrations in the first term of eq. (3.94) according to

$$\begin{align*}
\hat{G}^{(1) -} &= -\frac{1}{2\pi^2} \int_{x'_0}^{\infty} \int_{x'_0}^{\infty} \int_{0}^{T_{\text{arr}}(x'_3)} \Im \{ \Pi \tilde{\chi}(x'_3) \partial_t p \} \exp(-\sigma \tau) \, dq \, d\tau \, dx'_3 \\
&\quad + \frac{1}{2\pi} \int_{x'_0}^{\infty} S(x'_3) \int_{T_{\text{arr}}(x'_3)}^{T_{\text{sep}}(x'_3)} R[Q(\tau, x'_3), x'_3] \partial_t Q(\tau, x'_3) \exp(-\sigma \tau) \, d\tau \, dx'_3.
\end{align*} \quad (3.103)$$

With reference to the forward Laplace transformation of eq. (2.12) and the theory of Laplace transformations we can simply recognize the required space-time domain Green's function as

$$\begin{align*}
G^{(1) -} &= -\frac{1}{2\pi^2} \int_{x'_0}^{\infty} H[t - T_{\text{arr}}(x'_3)] \int_{0}^{Q(1, x'_3)} \Im \{ \Pi \tilde{\chi}(x'_3) \partial_t p \} \, dq \, dx'_3 \\
&\quad + \frac{1}{2\pi} \int_{x'_0}^{\infty} S(x'_3) H[t - T_{\text{arr}}(x'_3)] H[T_{\text{sep}}(x'_3) - t] R(Q(t, x'_3), x'_3) \partial_t Q(t, x'_3) \, dx'_3.
\end{align*} \quad (3.104)$$

### 3.4.7. Convolution with the source signature

Once we know the space-time domain Green's function $G^{(1) -}$, application of eq. (3.68) and the theory of the Laplace transformation lead to the space-time domain acoustic pressure, which is

$$p^{(1) -} = \partial_t^2 \left[ Q^S(t) \ast_t G^{(1) -}(t) \right], \quad (3.105)$$

Here, $\ast_t$ indicates a convolution with respect to time. The differentiation with respect to time may act either on the source signature, or on the Green's function,
or on both, so

\[ \tilde{p}^{(i)−} = \partial_1^2 Q^S(t) \ast_i G^{(1)−}(t) \]

\[ = \partial_1 Q^S(t) \ast_i \partial_2 G^{(1)−}(t) \]

\[ = Q^S(t) \ast_i \partial_2^2 G^{(1)−}(t). \]

(3.106)

If we assume that the source signature is known in functional form, as will be the case in this thesis, the first possibility is most convenient.

### 3.5. Applying the Cagniard-De Hoop method to the higher-order terms

Just as with the components of the zeroth-order and the first-order terms of the acoustic state vector, the transformation of the components of the higher-order terms of the acoustic state vector back to the space-time domain can be performed with the aid of the Cagniard-De Hoop method. In this section we will shortly indicate how the inverse transformation of the higher-order components is arranged; in view of the increased complexity of the relevant expressions we omit a full description of the inverse transformation procedure as presented in sections 3.3 and 3.4. To avoid analogous equations for all possible kinds of higher-order wavefield quantities due to all possible kinds of source components, in this section we confine ourselves to an upgoing i-th order acoustic pressure wave \( p^{(i)−} \), caused by a downgoing zeroth-order wave due to a source of volume injection rate (monopole source).

According to eqs. (2.25), (2.32), (2.39), (3.12) and (3.13), the transform domain expression for \( \tilde{p}^{(i)−} \) at the receiver level is

\[ \tilde{p}^{(i)−} = -\frac{1}{2} \tilde{Q}(s) \int_{x_3^R}^{\infty} \Pi \chi(x_{3}'') \int_{-\infty}^{x_3'} \chi(x_{3}'') \cdots \int_{x_3^{(i−1)}}^{\infty} \chi(x_{3}^{(i)}) \]

\[ \times \exp(-s \int_{x_3'}^{x_3''} \int_{x_{3}^{(i−1)} \gamma}^{x_{3}^{(i)}} d\zeta \, dz_3^{(i)} \cdots dz_3^{(i)} \, dx_3'). \]

(3.107)

Here the factor

\[ \Pi = Y^{-1/2}(x_3^S) Y^{-1/2}(x_3^R), \]

(3.108)

represents the coupling of the acoustic pressure wave to the source and the receiver,
and the inhomogeneity function

\[ \chi(z_3^{(k)}) = \frac{\partial_3 Y(z_3^{(k)})}{2Y(z_3^{(k)})} = \frac{\partial_3 c(z_3^{(k)})}{2\gamma^2(z_3^{(k)}) c^3(z_3^{(k)})} - \frac{\partial_3 \rho(z_3^{(k)})}{2\rho(z_3^{(k)})} \quad (3.109) \]

represents the local reflection factor at the level \( z_3^{(k)} \) where the \( k \)-th partial reflection takes place. The symbol \( L(z_3', z_3'', \ldots, z_3^{(i)}) \) indicates the total vertical travel path from the source, via the \( i \) reflection levels, to the receiver. The integral over this path should be interpreted as

\[ \int_{L(z_3', z_3'', \ldots, z_3^{(i)})} \gamma \, d\zeta = \int_{z_3'}^{z_3''} \gamma \, d\zeta + \int_{z_3''}^{z_3^{(i)}} \gamma \, d\zeta + \cdots + \int_{z_3^{(i-1)}}^{z_3^{(i)}} \gamma \, d\zeta. \quad (3.110) \]

The inverse transformation procedure for \( \tilde{p}^{(i)-} \) follows from the procedure for \( \tilde{p}^{(1)-} \) by replacing the single integral with respect to depth in eq. (3.60) by \( i \) repeated integrals as in eq. (3.107). As with \( \tilde{p}^{(1)-} \), one complication is that the nature of the Cagniard-De Hoop method requires the inverse transformation to act on the integrand of \( \tilde{p}^{(i)-} \) in eq. (3.107), i.e., on the partial reflections. In the inverse transformation process the \( i \) reflection levels \( z_3', z_3'', \ldots, z_3^{(i)} \) act as parameters. The total space-time domain result for \( \tilde{p}^{(i)-} \) is found by integrating the space-time domain results of all relevant \( i \)-th order partial reflections. A second and more severe complication is caused by the fact that the inhomogeneity function occurs \( i \) times in the integrand of \( \tilde{p}^{(i)-} \).

Since the inhomogeneity function has a pole in the right half complex \( p \)-plane, we have a total of \( i \) poles in this area. Depending on the values of the reflection levels \( z_3', z_3'', \ldots, z_3^{(i)} \), two or more of these poles can coalesce. Moreover, the leftmost (multiple) pole can coincide with the leftmost branchpoint at the same time that the Cagniard contour leaves the real \( p \)-axis in this point, so special care must be taken during the contour deformation.

### 3.6. Ray paths

In this section we will investigate the paths in the spatial domain for which the travel time from the source, via \( i \) levels of partial reflection, to the receiver, is minimal. This minimum travel time corresponds to the time that elapses before the associated \( i \)-th order partial contribution influences the wavefield at the receiver. In analogy with the paths of minimal travel time that can be found in case of piecewise
homogeneous media, our paths of minimal travel time will be called rays. Without loss of generality we will assume that $x_2 = 0$ in this section.

To start with, we note that from the sections 3.3 and 3.4 it can be deduced that the Cagniard contour for an $i$-th order partial contribution meets the real $p$-axis in a point $p_0$. In this point the parameter $\tau$ of the Cagniard contour has its minimum value $T(q)$. In turn, the minimum value of $T(q)$ is obtained for $q = 0$. This quantity is indicated by $T_{\text{arr}}$ and is called the arrival time, since at this time instant the first disturbance by the partial contribution arrives at the receiver. Of course, this arrival time must be equal to the time that is needed to travel from the source to the receiver along the associated ray, and thus it is plausible that there exists a relation between $p_0$ of the Cagniard contour for $q = 0$ and the ray path.

To establish this relation, we first investigate the ray associated with a zeroth-order contribution in a continuously layered medium as has been depicted in figure 3.11. The time required to travel along a ray segment $d\xi$ depends on the horizontal slowness $p_1$ and a vertical slowness $p_3$. The vertical slowness is related to the

![Figure 3.11](image)

Figure 3.11. Ray associated with a zeroth-order contribution in a continuously layered configuration: (a) wavespeed profile; (b) associated ray in the $(x_1, x_3)$-plane. The ray does not have a horizontal trajectory since $c_{\text{max}}$ occurs at a stationary point of the wavespeed profile.
Transverse acoustic waves in continuously layered media

horizontal slowness by

\[ p_3 = \left[ c^{-2}(x_3) - p_1^2 \right]^{1/2}. \]  

(3.111)

As a result, the total time needed to travel from A to B along the ray equals

\[ T_{\text{arr}} = \int_0^{x_1} p_1 \, dx_1 + \int_{x_2}^{x_3} [c^{-2}(x_3) - p_1^2]^{1/2} \, dx_3. \]  

(3.112)

Next we turn our attention to the corresponding Cagniard contour, which yields

\[ T_{\text{arr}} = p_0 r + \int_{x_2}^{x_3} [c^{-2}(x_3) - p_0^{-1/2}] \, dx_3. \]  

(3.113)

As has previously been explained, eqs. (3.112) and (3.113) must yield equal results, independently of the actual values of \( r, x_3^2 \), and \( x_3^2 \). As a consequence, we find the identity \( p_1 = p_0 \). Moreover, from figure 3.11 and eq. (3.111) we can easily deduce that

\[ \sin[\theta(x_3)] = \frac{p_1}{\sqrt{p_1^2 + p_3^2}} = p_0 \, c(x_3), \]  

(3.114)

so

\[ \frac{\sin[\theta(x_3)]}{c(x_3)} = p_0. \]  

(3.115)

The latter equation just turns out to be Snell’s law (Kline & Kay, 1965) describing the path of the ray. As one will probably expect, two cases must be distinguished, depending on the sign of \( \partial_p \tau|_{p_t} \).

— The case \( \partial_p \tau|_{p_t} < 0 \) —

In section 3.3.3 we have proved that if \( \partial_p \tau|_{p_t} < 0 \), the point \( p_0 \) is situated to the left of the point \( p_t = 1/c_{\text{max}} \), with \( c_{\text{max}} = \max_{(x_2^2,x_3^2)}(c(x_3)) \). In this case Snell’s law shows that \( \theta(x_3) < \pi/2 \), so the associated ray will nowhere have a horizontal trajectory. Figure 3.11 shows an example of a ray possessing this property since \( c_{\text{max}} \) is found at a stationary point of the wavespeed profile and thus, according to section 3.3.3, we unconditionally have \( \partial_p \tau|_{p_t} < 0 \).
— The case \( \partial_p \tau |_{p_t} > 0 \) —

If the situation arises in which \( \partial_p \tau |_{p_t} > 0 \), we have \( p_0 = p_1 = 1/c_{\text{max}} \). According to Snell's law, it is possible that \( \theta(x_3) = \pi/2 \), in which case the ray will have a horizontal trajectory. Examples of this kind of rays have been depicted in figure 3.12. In order to find the length \( h \) of the horizontal trajectory, we note that the time needed to travel from \( a \) to \( b \) along the rays in figure 3.12 equals

\[
\tau = \int_0^r p_1 \, dx_1 + \int_{x_3}^{x_3^b} p_3 \, dx_3 \\
= \int_{x_3^a}^{x_3^b} \left( p_1 \tan(\theta(x_3)) + [c^{-2}(x_3) - p_1^2]^{1/2} \right) \, dx_3 + h p_1, \tag{3.116}
\]

Differentiation of this equation with respect to \( p_1 \) yields

\[
\partial_p \tau |_{p_1} = \int_{x_3^a}^{x_3^b} \left\{ \tan(\theta(x_3)) - p_1 \left[ c^{-2}(x_3) - p_1^2 \right]^{-1/2} \right\} \, dx_3 + h = h. \tag{3.117}
\]

Since \( p_1 = p_t \) we may conclude that

\[
\partial_p \tau |_{p_t} = h, \tag{3.118}
\]

so, if the quantity \( \partial_p \tau |_{p_t} > 0 \), it indicates the length of the horizontal part of a ray.

So far we have restricted our investigation to a ray belonging to a zeroth-order contribution. However, the foregoing theory can easily be adapted to deal with a ray associated with a higher-order partial contribution, i.e., a path that runs from the source, via a number of levels of partial reflection, to the receiver. We can think of such a ray as being composed of parts that each look like a ray belonging to a zeroth-order contribution. As long as \( \partial_p \tau |_{p_t} < 0 \) a ray associated with a higher-order wave does not possess any horizontal part. An example of such a ray belonging to a first-order wave is given in figure 3.13. However, if \( \partial_p \tau |_{p_t} > 0 \), this quantity equals the total length of one or more horizontal parts of the ray. This can be seen in figure 3.14, where two rays have been depicted that are associated with a first-order partial contribution. The number of horizontal trajectories equals the number of times the ray meets the level where the maximum wavespeed is reached.

The Cagniard contour associated with any ray that does not travel horizontally anywhere, has \( \partial_p \tau = 0 \) in \( p_0 \), the point where \( \tau \) equals \( T_{\text{arr}} \). Consequently, in the
Figure 3.12. Rays associated with a zeroth-order contribution in a continuously layered configuration: (a), (c) wavespeed profiles; (b), (d) rays in the \((x_1, x_3)\)-plane that are associated with (a) and (c), respectively. Both rays have a horizontal trajectory since \(c_{\text{max}}\) is arrived at points where \(\theta_0 c \neq 0\) and \(r\) is large enough.
Figure 3.13. Ray associated with a first-order partial contribution in a continuously layered configuration: (a) wavespeed profile; (b) associated ray in the \((x_1, x_3)\)-plane. The ray does not have a horizontal trajectory since \(c_{\text{max}}\) is arrived at a stationary point of the wavespeed profile.

class of paths that begin at the receiver, travel between the subsequent levels of partial reflection, and finally arrive at the receiver, the ray is that curve for which the travel time is stationary. This is in accordance with the actual formulation of Fermat's principle (Kline & Kay, 1965). Since \(\tau\) is monotonically increasing when going along a Cagniard contour to infinity, we can be more specific and state that for the given class of paths the ray yields an absolute minimal arrival time. For a ray associated with a zeroth-order wave the latter fact has been proven by Allwright (1991) with the aid of trigonometric arguments. However, for the Cagniard contour related to a ray with a horizontal part, \(T_{\text{arr}}\) is arrived at in the point \(p_0 = p_7\), and here \(\partial_{\tau} \tau > 0\). Although for the given class of paths the ray again yields an absolute minimal travel time, it is not a stationary value in this case. It is an interesting fact that in this respect the rays with a horizontal part satisfy the original version of Fermat's principle [rays are the paths with the least travel time, see Kline & Kay (1965)], but not the actual version.
Figure 3.14. Rays associated with a first-order partial contribution in a continuously layered configuration: (a) wavespeed profile; (b) ray in the $(x_1, x_3)$-plane that is associated with (a) and that has one horizontal trajectory; (c) wavespeed profile; (d) ray in the $(x_1, x_3)$-plane that is associated with (c) and that has two horizontal trajectories. Both rays have a horizontal trajectory since $c_{\text{max}}$ occurs at points where $\delta_c \neq 0$ and $r$ is large enough.
3.7. Numerical implementation

In order to determine its practical use, we must study the numerical performance of the method that is arrived at by combining the WKBJ iterative solution and the Cagniard-De Hoop method. To make this possible, computer programs have been developed that form a numerical implementation of the expressions obtained in sections 3.3 and 3.4 for $p(0)^-$ and $p(1)^-$, the space-time domain zeroth-order and first-order acoustic pressure waves due to a source of volume injection. The goal of this section is two-fold. In the first place we will shortly describe the structure of these computer programs. Secondly we will focus on the numerical details of the most important procedures that have been used in the computer programs.

3.7.1. Structure of the computer programs

The numerical evaluation of $p(0)^-$ and $p(1)^-$ has been carried out in two steps. The first and most complex step is the evaluation of the expression for the relevant Green’s function. According to eq. (3.54) the zeroth-order Green’s function follows as

$$G^{(0)}^- = \frac{1}{2\pi^2} H(t - T_{arr}) \int_0^Q \text{Im} \left\{ \Pi \partial_t p \right\} dq,$$

while the first-order Green’s function follows from eq. (3.104) as

$$G^{(1)}^- = -\frac{1}{2\pi^2} \int_{z_3'}^{\infty} H[t - T_{arr}(z_3')] \int_0^Q \text{Im} \left\{ \Pi \bar{Q}(z_3) \partial_t p \right\} dq \, dz_3' + \frac{1}{2\pi} \int_{z_3'}^{\infty} S(z_3') H[t - T_{arr}(z_3')] H[T_{sep}(z_3') - t] R[Q(t, z_3'), z_3'] \partial_t Q(t, z_3') \, dz_3'. $$

Two separate programs have been written for the determination of these Green’s functions. The second step in the determination of $p(0)^-$ and $p(1)^-$ consists of calculating the acoustic pressure wave from a given Green’s function and a given source signature. This is performed using a separate program that froms a numerical implementation of the first lines of eqs. (3.56) and (3.106), which state that

$$p(0)^- = \delta_t Q^S(t) \ast_t G^{(0)}^-(t),$$

$$p(1)^- = \delta_t Q^S(t) \ast_t G^{(1)}^-(t).$$
respectively. We let the double time derivative act on the source signature and not on the Green's functions. This is because numerical differentiation of the Green's functions is less convenient since they possibly contain steps and/or sharp peaks, while differentiation of the source signature is relatively simple as we will only consider source signatures that are known in functional form.

The structure of the computer program for the numerical evaluation of the zeroth-order Green's function has been presented in figure 3.15. This structure is a logical consequence of the structure of the expression for \( G^{(0)}_q \). At the beginning of the program the sign of \( r - r_{sep} \) is determined; later this is used to decide in what way \( T_{arr} \) must be found. The signs of \( t - T_{arr} \) and \( T_{sep} - t \) are necessary to select the method of finding \( Q(t) \). For a theoretical description of the quantities \( r_{sep}, T_{arr}, T_{sep} \) and \( Q(t) \) we refer to subsections 3.3.3 - 3.3.5. The numerical evaluation of these quantities will be discussed in the next subsection.

In figure 3.16 the structure of the computer program for the numerical evaluation of the first-order Green's function has been depicted. Again this structure follows from the structure of the expression for \( G^{(1)}_q \). The main differences with the zeroth-

\[
\begin{array}{|c|c|}
\hline
G^{(0)}_q & \text{Determine sign of } r - r_{sep} \\
\hline
\text{For } t = T_{\text{min}} \text{ to } T_{\text{max}} \text{ step } \Delta T & \\
\hline
\text{Determine sign of } t - T_{arr} \text{ and } T_{sep} - t & \\
\hline
\text{Y} & t - T_{arr} > 0 \ ? \\
\hline
\text{N} & \text{Y} \\
\hline
G^{(0)}_q(t) := 0 & \\
\hline
G^{(0)}_q(t) := \frac{1}{2\pi} \int_0^Q Q(t) \text{QINT}(t, q) dq & \\
\hline
\text{QINT} & \text{Determine } p \text{ and } \delta p \\
\hline
\text{QINT}(t, q) = \text{Im} \{\Pi \delta p\} & \\
\hline
\end{array}
\]

Figure 3.15. Structure of the computer program for the numerical evaluation of \( G^{(0)}_q \).
### $G_q^{(1)-}$

Form intervals $I_1, \ldots, I_U$ with a constant
- functional description of $c$ and/or $\rho$;
- $S(x'_3)$;
- sign of $r - r(x'_3)$.

For $t = T_{\text{min}}$ to $T_{\text{max}}$ step $\Delta T$

Subdivide intervals $I_1, \ldots, I_U$ as to obtain intervals $J_1, \ldots, J_V$ with, in addition, a constant
- sign of $t - T_{\text{arr}}(x'_3)$;
- sign of $T_{\text{sep}}(x'_3) - t$.

$G_q^{(1)-}(t) := 0$

For $k = 1$ to $V$

<table>
<thead>
<tr>
<th>$t - T_{\text{arr}}(x'_3) &gt; 0$ on $J_k$?</th>
<th>$N$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_q^{(1)-}(t) := G_q^{(1)-}(t)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$+ \int_{J_k} XINT(t, x'_3) , dx'_3$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### XINT

Determine $Q(t, x'_3)$

$XINT(t, x'_3) := -\frac{1}{2\pi^2} \int_0^{Q(t, x'_3)} QINT(t, x'_3, q) \, dq$

<table>
<thead>
<tr>
<th>$T_{\text{sep}}(x'_3) - t &gt; 0 \land S(x'_3) = 1$?</th>
<th>$N$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$XINT(t, x'_3) := XINT(t, x'_3) + RES(t, x'_3)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### QINT

Determine $p$ and $\partial_t p$

$QINT(t, x'_3, q) := \text{Im} \{\bar{\Pi} \times \partial_t p\}$

### RES

Determine $\partial_t Q(t, x'_3)$

$RES(t, x'_3) := \frac{1}{2\pi} R(Q(t, x'_3), x'_3) \partial_t Q(t, x'_3)$

**Figure 3.16.** Structure of the computer program for the numerical evaluation of $G^{(1)-}$. 
order program are the integration with respect to the vertical coordinate \( x'_3 \) and the occurrence of a conditional contribution from the residue. In order to organize the integration with respect to depth, the program starts with dividing the vertical axis in \( U \) intervals \( J_1, \ldots, J_U \) on which

- the functional description of the wavespeed profile and the mass density profile remains the same; and
- the value of the function \( S(\tau'_3) \) remains the same; and
- the sign of \( r - r_{\text{sep}}(\tau'_3) \) remains the same.

The sign of \( r - r_{\text{sep}}(\tau'_3) \) is used to select the method of finding \( T_{\text{arr}}(\tau'_3) \). Later on, the intervals \( J_1, \ldots, J_U \) are eventually further subdivided in order to obtain \( V \) intervals \( J_1, \ldots, J_V \) on which, in addition to the items that remain constant on the intervals \( J_1, \ldots, J_U \),

- the sign of \( t - T_{\text{arr}}(\tau'_3) \) does not change; and
- the sign of \( T_{\text{sep}}(\tau'_3) - t \) does not change.

Knowledge of the latter signs is necessary in order to determine how \( Q(t, \tau'_3) \) must be found. The quantities \( S(\tau'_3), r_{\text{sep}}(\tau'_3), T_{\text{arr}}(\tau'_3), T_{\text{sep}}(\tau'_3) \) and \( Q(t, \tau'_3) \) have been treated theoretically in the subsections 3.4.3 - 3.4.6. The numerical evaluation of the quantities \( r_{\text{sep}}(\tau'_3), T_{\text{arr}}(\tau'_3), T_{\text{sep}}(\tau'_3) \) and \( Q(t, \tau'_3) \) is carried out in the same way as the evaluation of the corresponding zeroth-order quantities. The total integral with respect to the vertical coordinate is determined by summing the results of the integrals over the individual intervals \( J_1, \ldots, J_V \). To explain why this approach has been chosen we note that for a particular interval the value of \( t \) is either to small to yield a contribution to \( G^{(1)} \), or we obtain a contribution form the entire interval. In the latter case the integral with respect to \( \tau'_3 \) possesses an integrand that, depending on \( t \), either consists of the term containing the integral with respect to \( q \) only, or it is composed of the term containing the integral with respect to \( q \) plus the term due to the residue. The benefit of this piecewise integration approach is two-fold. Firstly, it is obvious that the organizational overhead during the integration over each interval \( J_1, \ldots, J_V \) is reduced to a minimum since the number of terms in the integrand is invariant within each interval. Secondly, on each interval the integrand
will be relatively smooth since the terms of the integrand do not suddenly vanish and, moreover, \( c \) and \( \rho \) do not have any discontinuous derivative. This smooth integrand is important in view of the convergence of the numerical integration process. With this in mind we have preferred the piecewise integration process just indicated over a single integral over the entire vertical range.

3.7.2. Numerical details

The numerical details of the most important procedures that have been used in the computer programs will now be discussed. We note that within the programs the continuous and piecewise continuously differentiable wavespeed profile and mass density profile are available in functional form. When procedures apply to both the zeroth-order case and the first-order case, the zeroth-order case is taken as an example; in all those cases the first-order procedure follows from the zeroth-order procedure in an obvious way, e.g., by replacing the path \([x_5^R; x_5^S]\) by the path \(\mathcal{L}(x_5^s)\).

--- Evaluation of \(\int_{x_5^s}^{x_5^R} \tilde{\gamma} \, d\zeta \) and \(\int_{x_5^R}^{x_5^S} \tilde{\gamma}^{-1} \, d\zeta \) ---

The integrals of \(\tilde{\gamma}\) and \(\tilde{\gamma}^{-1}\) show up in the expressions of many quantities that are used in relation with the Cagniard-De Hoop method. For some kinds of wavespeed profiles, such as the linear and exponential wavespeed profiles that are treated in appendix 3.C, the integrals of \(\tilde{\gamma}\) and \(\tilde{\gamma}^{-1}\) can be determined analytically. In general, however, we must use numerical methods to determine these integrals. It is important that the evaluation of the integrals is performed as fast as possible because of their frequent occurrence. In view of this an adaptive higher-order integration rule based on Gauss' quadrature (PIESSSENSET AL., 1983) leads to fast routines as the number of integrand evaluations can be kept low. Since the convergence of this type of numerical integration routines in general gets worse if the integrand possesses (higher order) discontinuities on the integration interval, it is advisory to divide \([x_5^R; x_5^S]\) in subintervals on which the functional description of the wavespeed profile does not change (in the first-order case we can apply the subintervals \(J1, \ldots, JV\)). If the integral of \(\tilde{\gamma}^{-1}\) must be evaluated on an interval where the vertical slowness
\( \hat{\tau} \) the becomes zero (or approaches zero very closely), special precautions must be taken as to avoid a large numerical error due to the (almost) singular behavior of the integrand. Two solutions exist: we can either take a numerical integration routine that can deal with the kind of singularity encountered, or we can approximate the wavespeed profile in the neighborhood of the singular point by some function that enables analytical evaluation of the integral over this neighborhood, and add the outcome to the result of the numerical integration over the remainder of the interval.

--- Determination of \( r_{sep} \) ---

In an actual case the value of \( r_{sep} \) equals the minimum value of the horizontal offset for which there exists a value \( Q_{sep} \) such that for \( 0 \leq q < Q_{sep} \) we have \( \partial_p r > 0 \) in the leftmost branch point \( p_L \) (see subsection 3.3.3). The leftmost branch point is given by \( p_L = (c_{max}^{-2} + q^2)^{1/2} \), where \( c_{max} = \max_{x_3 \in [x_3^1, x_3^2]} \{ c(x_3) \} \). According to eq. (3.40) the quantity \( r_{sep} \) is given by

\[
\begin{align*}
  r_{sep} &= \frac{1}{c_{max}} \int_{x_3^1}^{x_3^2} [c^{-2}(\zeta) - c_{max}^{-2}]^{-1/2} d\zeta.
\end{align*}

(3.123)
\]

The numerical evaluation of \( r_{sep} \) first requires the determination of \( c_{max} \). Since a functional description of the wavespeed profile is available this quantity can be found both analytically and numerically; for accuracy reasons we employ an analytical way of finding \( c_{max} \). Further the evaluation of \( r_{sep} \) mainly consists of the evaluation of the integral in eq. (3.123), which in practice means the determination of the integral of \( \hat{\tau}^{-1} \) with \( p = p_L \).

--- Determination of \( T_{arr} \) ---

The value of \( T_{arr} \) is the value of the variable \( \tau \) in the point \( p_0 \) where the Cagniard contour with parameter \( q = 0 \) crosses the real \( p \)-axis. As explained in section 3.3.4, two situations must be distinguished, which depend upon the sign of \( \tau - r_{sep} \) according to...
• if \( r - r_{\text{sep}} < 0 \), we must first numerically determine the point \( p_0 \) by searching on the real \( p \)-axis for the zero of \( \partial_p r \). According to eq. (3.38), for \( q = 0 \) the latter function is given by

\[
\partial_p r = r - p \int_{x^R}^{x^S} \left[ c^{-2}(\zeta) - p^2 \right]^{-1/2} \, d\zeta. \tag{3.124}
\]

Since \( T_{\text{arr}} \) need not be determined often, computational efficiency is unimportant and for simplicity we have chosen the regula falsi method to perform this task. The points \( p = 0 \) and \( p = p_\ell = 1/c_{\text{max}} \) are taken as starting points since \( p_0 \) will be located in between these points. Once \( p_0 \) has been found, the value of \( T_{\text{arr}} \) follows from evaluation of eq. (3.47), which states that

\[
T_{\text{arr}} = p_0 r + \int_{x^R}^{x^S} \left[ c^{-2}(\zeta) - p_0^2 \right]^{1/2} \, d\zeta; \tag{3.125}
\]

• if \( r - r_{\text{sep}} > 0 \), the point \( p_0 \) can be determined analytically since it coincides with the leftmost branch point, so \( p_0 = p_\ell = 1/c_{\text{max}} \). Now eq. (3.48) shows that determination of \( T_{\text{arr}} \) is nothing more than evaluation of the equation

\[
T_{\text{arr}} = \frac{r}{c_{\text{max}}} + \int_{x^R}^{x^S} \left[ c^{-2}(\zeta) - c_{\text{max}}^{-2} \right]^{1/2} \, d\zeta. \tag{3.126}
\]

— Determination of \( T_{\text{sep}} \) —

The value of \( T_{\text{sep}} \) equals the value of the variable \( r \) in the point \( p_0 = p_\ell \) where the Cagniard contour with parameter \( q = Q_{\text{sep}} \) meets the real \( p \)-axis. According to eq. (3.49) the value of \( T_{\text{sep}} \) follows from a straightforward evaluation of

\[
T_{\text{sep}} = \frac{r^2}{\int_{x^R}^{x^S} \left[ c^{-2}(\zeta) - c_{\text{max}}^{-2} \right]^{-1/2} \, d\zeta} + \int_{x^R}^{x^S} \left[ c^{-2}(\zeta) - c_{\text{max}}^{-2} \right]^{1/2} \, d\zeta. \tag{3.127}
\]

— Determination of \( Q(\ell) \) —

The function \( Q(\ell) \) is the inverse of the function \( T(q) \), where the value of the latter function equals the value of the parameter \( r \) in the point \( p_0 \) where the Cagniard
contour with parameter \( q \) meets the real \( p \)-axis, see subsection 3.3.5. The way in which the value of \( Q(t) \) is determined depends upon the values of \( t, T_{arr} \) and \( T_{sep} \) in the following way:

- if \( T_{sep} < T_{arr} \), this implies that \( r < r_{sep} \) and consequently the value of \( p_0 \) belonging to any Cagniard contour must be determined numerically (subsection 3.3.3). In this situation \( Q(t) \) is found with the aid of the procedure that has been depicted in figure 3.17. This procedure consists of two nested iterative loops. In the outer loop the actual value of \( Q(t) \) is determined by solving \( q \) from the equation \( F(q) = t - T(q) = 0 \). Since \( Q(t) \) must be determined frequently, we have decided to solve this equation with the aid of the Newton-Raphson rule. According to this rule, the \( \beta \)-th iterative approximation \( q^{(\beta)} \) of the equation \( F(q) = 0 \) is found from the \((\beta - 1)\)-st iterative approximation \( q^{(\beta-1)} \) as

\[
q^{(\beta)} = q^{(\beta-1)} - \frac{F(q^{(\beta-1)})}{\partial_q F(q^{(\beta-1)})}.
\]

In the inner loop \( p_0 \), indicating the point where the Cagniard contour with parameter \( q \) crosses the real \( p \)-axis, is found using the method outlined for the determination of \( T_{arr} \), case \( r - r_{sep} < 0 \). This value is used in the outer loop to determine the values of \( F(q) \) and \( \partial_q F(q) \), which are required by the Newton-Raphson rule. Using eq. (3.45), we find that

\[
F(q) = t - p_0 r - \int_{s_0^2}^{s_0^2} \left[ c^{-2}(\zeta) - p_0^2 + q^2 \right]^{1/2} \, d\zeta,
\]

while differentiation of this equation with respect to \( q \) yields

\[
\partial_q F(q) = -q \int_{s_0^2}^{s_0^2} \left[ c^{-2}(\zeta) - p_0^2 + q^2 \right]^{-1/2} \, d\zeta.
\]

As a starting value of the Newton-Raphson rule we take \( q^{(0)} = t/r \), which is the value of \( Q(t) \) in case of a homogeneous medium with infinite wavespeed. For the present application the Newton-Raphson rule turns out to converge quickly;

- if \( T_{arr} < T_{sep} < t \), this implies that \( r > r_{sep} \) and \( Q(t) > Q_{sep} \). Again the value of \( p_0 \) belonging to the Cagniard contour with parameter \( q \) in the neighborhood of \( Q(t) \) must be determined numerically (subsection 3.3.3), and the process described in the previous paragraph is invoked to determine \( Q(t) \);
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- if $T_{\text{arr}} < t < T_{\text{sep}}$, this implies that $\tau > \tau_{\text{sep}}$ and $Q(t) < Q_{\text{sep}}$. As a consequence, the point $p_0$ for the Cagniard contour with parameter $q = Q(t)$ will coincide with the leftmost branch point $p_\ell$ (subsection 3.3.3). Now the value of $Q(t)$ is found by evaluation of the analytical result

$$Q(\tau) = \left( \frac{\{{\tau - \int_{\Delta_{\max}}^{\Delta} [c^{-2}(\zeta) - c_{\max}^{-2}]^{1/2}} d\zeta\}^2}{\tau^2} - c_{\max}^{-2}\right)^{1/2}, \quad (3.131)$$

which has been presented for the first time in eq. (3.52).

<table>
<thead>
<tr>
<th>$Q(t)$</th>
<th>$q^{(0)} := \frac{t}{\tau}$, $\beta := 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>While accuracy has not been achieved</td>
<td></td>
</tr>
<tr>
<td>Take $p = 0$ and $p = p_\ell$ as starting values</td>
<td></td>
</tr>
<tr>
<td>While accuracy has not been achieved</td>
<td></td>
</tr>
<tr>
<td>Using regula falsi, determine new pair of $p$-values that better approximate the zero $p_0$ of $\partial_{\tau}t$ for given $q^{(\beta-1)}$</td>
<td></td>
</tr>
<tr>
<td>Determine $F(q^{(\beta-1)})$ and $\partial_q F(q^{(\beta-1)})$</td>
<td></td>
</tr>
<tr>
<td>$q^{(\beta)} := q^{(\beta-1)} - \frac{F(q^{(\beta-1)})}{\partial_q F(q^{(\beta-1)})}$ (Newton-Raphson)</td>
<td></td>
</tr>
<tr>
<td>$\beta := \beta + 1$</td>
<td></td>
</tr>
<tr>
<td>$Q(t) := q^{(\beta)}$</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3.17. Structure of the procedure for the numerical evaluation of $Q(t)$. 
Determination of a point on the Cagniard contour

A point \( p \) belonging to a certain value of \( t \) on a Cagniard contour with parameter \( q \) satisfies [cf. eq. (3.33)]

\[
t = pr + \int_{x^R_3}^{x^S_3} \tilde{\gamma} \, d\zeta = \text{real.} \tag{3.132}
\]

In our case we are only interested in the solution of this equation that is located in the first quadrant of the \( p \)-plane, see section 3.3.3. The procedure of finding \( p \) has to be repeated every time the integrand of the integral with respect to \( q \) must be evaluated, so this procedure operates in the heart of the program for the evaluation of the Green's functions and consequently it must be computed efficiently. We have decided to base the procedure for finding a point on the Cagniard contour on the Newton-Raphson rule, so approximations to \( p \) are found by iteratively using

\[
p^{(\alpha)} = p^{(\alpha-1)} - \frac{G(p^{(\alpha-1)})}{\partial_p G(p^{(\alpha-1)})}. \tag{3.133}
\]

With this rule we actually approximate the zero of the function

\[
G(p) = t - pr - \int_{x^R_3}^{x^S_3} \tilde{\gamma} \, d\zeta, \tag{3.134}
\]

thereby using the derivative \( \partial_p G(p) \), which is given by

\[
\partial_p G(p) = -r + \int_{x^R_3}^{x^S_3} \tilde{\gamma}^{-1} \, d\zeta. \tag{3.135}
\]

As a starting value we take

\[
p^{(0)} = \frac{r^t}{R^t} + i \frac{z}{R^t} (c_{\text{max}}^{-2} + q^2 - t^2)^{1/2}, \tag{3.136}
\]

where \( R^2 = r^2 + z^2 \) is the square of the total distance between the source and the receiver, and \( z = x^S_3 - x^R_3 \) forms the vertical separation between the source and the receiver. This starting value is the exact solution of eqs. (3.132) and (3.134) in case of a homogeneous medium with a wavespeed \( c_{\text{max}} \) (CAGNIARD, 1939; DE HOOP, 1961, 1962, 1988; VAN DER HJIDEN, 1987). The iterative process is stopped if both
the correction due to the $\beta$-th step is, in the absolute sense, smaller than some fraction $\epsilon_1$ of the value of $p^{(\beta-1)}$, and

- the correction due to the $\beta$-th step is, in the absolute sense, smaller than some fraction $\epsilon_2$ of the value of $p^{(\beta-1)} - p_t$, being the approximate distance between $p$ and the leftmost branch point.

The second part of this stopping criterion is necessary in order to ensure that the point $p$ is approximated more accurately in the neighborhood of the leftmost branch point. This is necessary since the integrand of the integral with respect to $q$ can be very sensitive to errors in $p$ when $p$ is located near the leftmost branchpoint. Under the circumstances described here the Newton-Raphson rule yields results that converge quickly.

Upon comparing eqs. (3.38) and (3.135) it turns out that $-\partial_p G(p)$ for fixed $t$ equals the derivative $\partial_p t$ in the corresponding point $p$ on the Cagniard contour. Moreover, since $t$ is monotonically increasing when traveling along the Cagniard contour away from $p_0$, in all points where $\partial_p t \neq 0$ the identity $\partial_t p = 1/\partial_p t$ holds. The derivative $\partial_t p$ occurs in the integrand of the integral with respect to $q$ and need to be determined as often as $p$. As a result, in all points except $p_0 \neq p_t$ one finds that $\partial_t p = -1/\partial_p G(p)$; in $p_0 = p_t$ the shape of the Cagniard contour reveals that $\partial_t p = \infty$. Thus, except for the latter point, the derivative $\partial_t p$ is obtained as a simple side-product of the routine that is used to find $p$.

— Evaluation of the integral with respect to $q$ —

As can be seen from eqs. (3.119) and (3.120), in both the zeroth-order case and in the first-order case we have to evaluate an integral with respect to $q$. We have already discussed the determination of the upper boundary $Q(t)$ of this integral; its lower boundary is just zero. An adaptive higher-order integration rule based on Gauss' quadrature has been selected for the evaluation of this integral since this requires a relatively small number of integrand evaluations. Because the factor $\partial_t p$ introduces a root singularity in the integrand at $q = Q(t)$ (except if $0 < Q(t) < Q_{sep}$), we have replaced the integration with respect to $q$ by an integration with respect to $q$
using the sine substitution \( q = Q(t) \sin \phi \) as to avoid numerical errors and a bad convergence due to this singularity. From a numerical point of view there are no objections to evaluate the integral as a whole. Even when there exists a value \( Q_{\text{sep}} \) such that \( 0 < Q_{\text{sep}} < Q(t) \), there is no numerical reason to implement the formal splitting as indicated in eqs. (3.51) and (3.97).

— Evaluation of the integral with respect to \( x_3' \) —

In the first-order case we have to perform an integration with respect to \( x_3' \), which is the level of reflection of the partial contributions that are represented by the integrand. The two separate integrals with respect to depth in eq. (3.120) indicate that these partial contributions are represented by a term that contains the integral with respect to \( q \) and sometimes an additional residue term. From a numerical point of view it is efficient to integrate the sum of both terms as a whole. For this integration we apply an adaptive higher-order integration rule based on Gauss' quadrature in order to keep the number of integrand evaluations low. To avoid a slow convergence of the integration process caused by discontinuities of the integrand, we subdivide the integration over the total vertical interval into piecewise integrations over the intervals \( J1, \ldots, JV \) on which the integrand is smooth.

— Determination of the value \( x_3' \) that solves equations like \( r - r_{\text{sep}}(x_3') = 0 \) —

In the first-order case intervals are created on which, among others, the sign of \( r - r_{\text{sep}}(x_3') \), \( t - T_{\text{arr}}(x_3') \) and \( T_{\text{sep}}(x_3') - t \) does not change. In order to find the boundaries of these intervals we must find the values of \( x_3' \) for which these expressions vanish. This is done in two steps, which will be explained by showing how the zeroes of \( r - r_{\text{sep}}(x_3') \) are found. First we make an array with a number of levels \( x_3' \) and the associated values of \( r - r_{\text{sep}}(x_3') \). Secondly, we step through this array until two adjacent values of \( x_3' \) are found for which the signs of \( r - r_{\text{sep}}(x_3') \) are different. This indicates that a zero must occur somewhere in between these levels (the spacing between the levels is small enough to ensure that there is not more than one zero
located in between two adjacent levels). The exact position of the zero is then determined by using the regula falsi method. Once it has been found, we proceed stepping through the array in search for another zero. The whole process is stopped when the vertical interval on which the medium is inhomogeneous has been scanned completely.

— Convolution of the Green's function with the source signature —

The final step in finding the zeroth-order or first-order acoustic pressure consists of a temporal convolution of the double time derivative of the source signature and the relevant Green's function, see eqs. (3.121) and (3.122). Since the Green's function will only be available for a fixed number of equidistant time points, for the convolution we can only apply a integration rule that requires the integrand to be evaluated for a fixed number of equidistant points. We have employed the trapezoidal rule because in general it gives results that are accurate enough while its implementation is still simple. If the Green's function has a singularity in the time domain, which we will sometimes encounter in the next section, an analytical function having the same singularity is subtracted from the Green's function. The convolution with this analytical function is then performed analytically and is added to the result of the numerical convolution with the Green's function from which the singularity has been removed.

3.8. Numerical results

In this section we will present numerical results that have been generated using the computer programs described in the previous section. With the aid of these results we intend to give an impression of the numerical performance of the method that is composed of the WKBJ iterative solution and the Cagniard-De Hoop method. The material shown will mainly consist of examples of the space-time domain zeroth-order Green's function \(^G^{(0)}\) and first-order Green's function \(^G^{(1)}\), and the related space-time domain zeroth-order acoustic pressure \(^p^{(0)}\) (see figure 3.4) and first-order acoustic pressure \(^p^{(1)}\) (see figure 3.8). Some auxiliary results will be presented as well.
Five different sets of wavespeed profiles and mass density profiles have been applied for our numerical experiments. Four sets, called LINPEAK1, LINPEAK2, EXPPEAK and LINDIP, consist of typical continuous parameter profiles. These are shown in figures 3.18 - 3.20. The LINPEAK1, LINPEAK2 and LINDIP parameter profiles are composed of piecewise linear functions of the form $ax_3 + b$, while the EXPPEAK parameter profiles are described in terms of piecewise exponential functions of the form $f \exp(gx_3)$. The fifth set of parameter profiles, called MULTI, describes a multilayer medium with homogeneous layers that are separated by linear transitions zones. These parameter profiles can be found in figure 3.21. The vertical position of the source differs for both orders of waves under investigation: all zeroth-order results have been determined for a source that has been located on a level $x^s_3 = 2000$ m, while all first-order results have been obtained using a source level $x^s_3 = 0$ m. In both the zeroth-order and the first-order case the receiver has been placed on the level $x^r_3 = 0$ m. Various values of the horizontal offset $\tau$ between the source and the receiver have been applied; for each result that will be presented the actual value of $\tau$ is indicated. All acoustic pressure waves $p^{(0)}$ and $p^{(1)}$ that will be shown belong to a source of volume injection rate with a signature that is described by a four-point optimum Blackman window function (HARRIS, 1978) of unit amplitude and a duration of 0.1 s. In functional form this signature is given by

$$Q^S(t) = \begin{cases} 
0 & (t \leq 0) \\
\sum_{k=0}^{3} b_k \cos(20\pi t) & (0 < t \leq 0.1) \\
0 & (t > 0.1)
\end{cases}$$  \hspace{1cm} (3.137)

with

$$b_0 = 0.35869 \ [m^3/s],$$  \hspace{1cm} (3.138)
$$b_1 = -0.48829 \ [m^3/s],$$  \hspace{1cm} (3.139)
$$b_2 = 0.14128 \ [m^3/s],$$  \hspace{1cm} (3.140)
$$b_3 = -0.01168 \ [m^3/s].$$  \hspace{1cm} (3.141)
Figure 3.18. The wavespeed and mass density profiles of type LINPEAK1 and LINPEAK2. The parameter profiles are composed of piecewise linear functions of the form $ax_3 + b$.

Figure 3.19. The wavespeed and mass density profiles of type EXPPEAK. The parameter profiles are composed of piecewise exponential functions of the form $f \exp(gx_3)$. 
Figure 3.20. The wavespeed and mass density profiles of type LINDIP. The parameter profiles are composed of piecewise linear functions of the form $az_3 + b$.

Figure 3.21. The wavespeed and mass density profiles of type MULTI. The parameter profiles consist of piecewise constant functions which are connected by piecewise linear functions of the form $az_3 + b$. 
A graphical representation of the double time derivative of the source signature is given in figure 3.22. Convolution of the double time derivative of the source signature with a Green's function yields the corresponding acoustic pressure wave [cf. eqs. (3.56) and (3.106)]. For this section the numerical evaluation of the expressions has been performed with a relative error that is typically less than 0.3 %. This means that even for the largest values the error in the plots is less than the line thickness of the graphs.

![Figure 3.22](image)

**Figure 3.22.** The double time derivative of the signature of the source of volume injection rate used with our numerical experiments. Convolution of this function with a Green's function yields the corresponding acoustic pressure wave.

### 3.8.1. Numerical results for the typical configurations

The zeroth-order Green's function $G^{(0)}$ and the zeroth-order acoustic pressure wave $p^{(0)}$ for the configurations with LINPEAK1, LINPEAK2, EXPPEAK and LINDIP parameters have been plotted in figures 3.23 - 3.30 for values of the horizontal offset that equal $r = 0$ m, $r = 750$ m and $r = 2500$ m. Due to the inhomogeneity of the media, the Green's functions deviate from the step function behavior that is achieved for a homogeneous medium. The Green's functions for the LINPEAK1, LINPEAK2 and EXPPEAK configurations show a rounding-off effect, while the Green's function for
Figure 3.23. The zeroth-order Green's function $G^{(0)}$ for the configuration with parameter profiles of type LINPEAK1, with different values of the horizontal offset.

Figure 3.24. The zeroth-order acoustic pressure $p^{(0)}$ for the configuration with parameter profiles of type LINPEAK1, with different values of the horizontal offset.
Figure 3.25. The zeroth-order Green's function $G^{(0)}$ for the configuration with parameter profiles of type LINPEAK2, with different values of the horizontal offset.

Figure 3.26. The zeroth-order acoustic pressure $p^{(0)}$ for the configuration with parameter profiles of type LINPEAK2, with different values of the horizontal offset.
Figure 3.27. The zeroth-order Green's function \( G^{(0)} \) for the configuration with parameter profiles of type EXPPEAK, with different values of the horizontal offset.

Figure 3.28. The zeroth-order acoustic pressure \( p^{(0)} \) for the configuration with parameter profiles of type EXPPEAK, with different values of the horizontal offset.
Figure 3.29. The zeroth-order Green's function $G^{(0)}$ for the configuration with parameter profiles of type LINDIP, with different values of the horizontal offset.

Figure 3.30. The zeroth-order acoustic pressure $p^{(0)}$ for the configuration with parameter profiles of type LINDIP, with different values of the horizontal offset.
the LINDIP configuration has an overshoot. It has been verified that in the LINPEAK1 configuration for \( r = 2500 \) m there are rays with a horizontal trajectory at a depth of 1700 m, the level where the wavespeed reaches its maximum value and where the first derivative of the wavespeed profile is discontinuous. As opposed to what one might probably expect, the occurrence of this kind of rays cannot be recognized from the Green's function. Note that the zeroth-order results for the LINPEAK2 configuration and the EXPPEAK configuration are almost the same, as could be expected since the differences between both sets of parameter profiles are only small.

The first-order Green's function \( G^{(1)} \) and the resulting acoustic pressure wave \( p^{(1)} \) for configurations with the LINPEAK1, LINPEAK2, EXPPEAK and LINDIP parameter profiles and with horizontal offsets of \( r = 0 \) m, \( r = 1500 \) m and \( r = 5000 \) m can be found in figures 3.31 - 3.38. We see that the Green's functions for \( r = 0 \) m reflect the shape of the parameter profiles. As compared to \( r = 0 \) m, for \( r = 1500 \) m the shape of the Green's function has changed in the sense that after the arrival time the Green's function has a steeper slope. This can be explained by the fact that the arrival times of the partial contributions become less dependent on the reflection level if the horizontal offset increases. This is especially the case for the partial contributions coming from shallow reflection levels. Now these arrive in a smaller time interval, thus giving rise to a faster increase of the Green's function. This effect is pronounced by the positive gradient of the wavespeed profile in case of the LINPEAK1, LINPEAK2 and EXPPEAK configurations, and is less pronounced (but not completely absent) by the negative wavespeed profile of the LINDIP configuration. It is remarked that in the configuration with LINPEAK1 parameter profiles and with \( r = 1500 \) m there are rays that have a nearly horizontal path near their reflection level. For \( r = 5000 \) m the Green's functions are dominated by a large peak. Investigation reveals that this peak is due to a singularity with a logarithmic-like behavior. A closer look at the phenomena accompanying the occurrence of this logarithmic singularity is taken in subsection 3.8.4. For the determination of the acoustic pressure wave a singular logarithmic part has been separated from the remaining part of the Green's function; the convolution of the logarithmic part with the double time derivative of the source signature is performed analytically, while the convolution of the remainder of the Green's function with the double time derivative of the source signature is performed numerically.
Figure 3.31. The first-order Green's function $G^{(1)-}$ for the configuration with parameter profiles of type LINPEAK1, with different values of the horizontal offset.

Figure 3.32. The first-order acoustic pressure $p^{(1)-}$ for the configuration with parameter profiles of type LINPEAK1, with different values of the horizontal offset.
Figure 3.33. The first-order Green’s function $G^{(1)}$ for the configuration with parameter profiles of type LINPEAK2, with different values of the horizontal offset.

Figure 3.34. The first-order acoustic pressure $p^{(1)}$ for the configuration with parameter profiles of type LINPEAK2, with different values of the horizontal offset.
Figure 3.35. The first-order Green's function $G^{(1)}$ for the configuration with parameter profiles of type EXPPEAK, with different values of the horizontal offset.

Figure 3.36. The first-order acoustic pressure $p^{(1)}$ for the configuration with parameter profiles of type EXPPEAK, with different values of the horizontal offset.
Figure 3.37. The first-order Green's function $G^{(1)}$ for the configuration with parameter profiles of type LINDIP, with different values of the horizontal offset.

Figure 3.38. The first-order acoustic pressure $p^{(1)}$ for the configuration with parameter profiles of type LINDIP, with different values of the horizontal offset.
3.8.2. Numerical results for the multilayer configuration

In figures 3.39 and 3.40 the zeroth-order Green's function and the corresponding acoustic pressure wave are plotted for the configuration with the MULTI parameter profiles and with horizontal offsets of $r = 0$ m, $r = 750$ m and $r = 2500$ m. The first-order Green's function and acoustic pressure wave for the configuration with the same type of medium and with $r = 0$ m, $r = 1500$ m and $r = 5000$ m are depicted in figures 3.41 and 3.42. The first-order Green's function for $r = 0$ m clearly reflects the shape of the parameter profiles of the MULTI configuration. For $r = 1500$ m, after the arrival time we encounter the effect that can be explained by the concentration of the partial contributions coming from the shallow layers. The first-order Green's function for $r = 5000$ m is dominated by a logarithmic-like singularity; in this case the shape of the parameter profiles can no longer be recognized from the first-order Green's function.

For comparison, in figures 3.43 and 3.44 the first-order Green's function has been plotted for both the continuously layered MULTI configuration and the piecewise homogeneous configuration that is arrived at when the slopes in the profiles of the MULTI profile are taken infinitely steep; the comparison has been made for $r = 0$ m and $r = 1500$ m, respectively. We see that for $r = 0$ m both Green's functions are much alike. The Green's function of the MULTI configuration contains slopes with a finite gradient, while in the Green's function of the piecewise homogeneous configuration discontinuous jumps occur. Again this reflects the shape of the associated parameter profiles. For $r = 1500$ m the respective Green's functions differ considerably except for long times. Here we clearly see the influence of the relatively small differences between both sets of parameter profiles. This can be explained by assuming that shortly after the arrival time, in case of the MULTI configuration, the positive partial contributions are cancelled to a larger extent by negative partial contributions than in case of the piecewise homogeneous configuration. This is due to the differences between the gradients of the slopes occurring in both wavespeed profiles. As a result, lower values of the Green's function for the MULTI configuration are obtained. From this example it becomes clear that for higher values of the horizontal offset the continuously layered configuration can give rise to results that differ significantly from the results obtained for an almost identical piecewise homogeneous configuration.
Figure 3.39. The zeroth-order Green's function $G^{(0)}$ for the configuration with parameter profiles of type MULTI, with different values of the horizontal offset.

Figure 3.40. The zeroth-order acoustic pressure $p^{(0)}$ for the configuration with parameter profiles of type MULTI, with different values of the horizontal offset.
Figure 3.41. The first-order Green's function $G^{(1)}$ for the configuration with parameter profiles of type MULTI, with different values of the horizontal offset.

Figure 3.42. The first-order acoustic pressure $p^{(1)}$ for the configuration with parameter profiles of type MULTI, with different values of the horizontal offset.
Figure 3.43. Comparison between the first-order Green's functions for the configuration with parameter profiles of type MULTI (solid line) and for the piecewise homogeneous configuration that results when the slopes of the MULTI-type parameter profiles are taken infinitely steep (dashed line). The horizontal offset is \( r = 0 \) m.

Figure 3.44. Comparison between the first-order Green's functions for the configuration with parameter profiles of type MULTI (solid line) and for the piecewise homogeneous configuration that results when the slopes of the MULTI-type parameter profiles are taken infinitely steep (dashed line). The horizontal offset is \( r = 1500 \) m.
3.8.3. Numerical results for a configuration without mass density variations

In the configurations applied up till now the mass density profile has the same shape as the wavespeed profile. Unlike the wavespeed profile, the mass density profile does not determine the arrival time of the contributions. However, together with the wavespeed profile, the mass density profile does determine the strength of the contributions. In order to show the influence on the Green’s functions of neglecting the variations of the density of mass, we have also performed our calculations for a configuration with the LINPEAK1 wavespeed profile and a constant mass density profile of 3000 kg/m. Except for a constant factor caused by the change of density of mass at the source level, the zeroth-order Green’s functions will not be different from the results in figure 3.23 and they are not presented here. However, the results for the first-order Green’s function in figure 3.45 are different from those shown in figure 3.31. Most significant is the fact that for long times the first-order Green’s function approaches the value zero instead of a constant nonzero value. Although the resulting first-order acoustic pressure waves differ from those presented in figure

![Graph showing the first-order Green's functions for different radii](image)

**Figure 3.45.** The first-order Green's functions $G^{(1)}$ for the configuration with a wavespeed profile of type LINPEAK1 and a constant density of mass $\rho = 3000$ kg/m. Different values of the horizontal offset are considered.
3.32, these differences are not as large as one would probably expect since our convolution process is insensitive for the slow variations that are mainly responsible for the change of shape of the first-order Green's function. The results for the first-order acoustic pressure waves have been omitted here.

3.8.4. Some auxiliary results

In this section we present numerical results for some quantities that play an important role in the determination of the zeroth-order and the first-order Green's functions. All results refer to a configuration with parameter profiles of the type \text{LINPEAK1}.

To begin with, in figure 3.46 the Cagniard contours for three first-order partial contributions have been plotted. The reflection level is 500 m, and the Cagniard contours are plotted for \( r = 0 \) m, \( r = 1500 \) m and \( r = 5000 \) m. As expected, the contour for \( r = 0 \) m runs along the imaginary axis since in this case this is the only path along which the real parameter \( \tau \) increases. The contour for \( r = 1500 \) m has \( \partial_\tau |_{p_\ell} < 0 \) and thus crosses the real \( p \)-axis perpendicular in a point left from the leftmost branch point \( p_\ell \). For \( r = 5000 \) m the contour has \( \partial_\tau |_{p_\ell} > 0 \) and we see that it meets the real \( p \)-axis tangentially in \( p_\ell \).

In table 3.1 the values of the zeroth-order quantities \( T_{arr} \) and \( T_{sep} \) are given for different values of the horizontal offset. For \( r = 5000 \) m and \( r = 10000 \) m we see that \( T_{arr} < T_{sep} \). For values of \( r \) that lie in between these values the situation \( \partial_\tau |_{p_\ell} > 0 \) can arise, depending on the value of the parameter \( q \). This will certainly be the case for \( q = 0 \), and thus the rays will have a horizontal trajectory at a depth of 1700 m in these cases.

Plots of the first-order quantities \( T_{arr}(x_3') \) and \( T_{sep}(x_3') \) versus \( x_3' \) for \( r = 0 \) m, \( r = 1500 \) m and \( r = 5000 \) m can be found in figure 3.47. For \( r = 0 \) m and \( r = 1500 \) m we always have \( T_{arr}(x_3') > T_{sep}(x_3') \), so we never encounter the situation \( \partial_\tau |_{p_\ell(x_3')} > 0 \) and the rays do not possess a horizontal trajectory. However, for \( r = 5000 \) m there is a vertical interval for which \( T_{arr}(x_3') < T_{sep}(x_3') \). In this interval it can occur that \( \partial_\tau |_{p_\ell(x_3')} > 0 \), depending upon the value of the parameter \( q \). The rays will have a horizontal trajectory on this interval. If \( \partial_\tau |_{p_\ell(x_3')} > 0 \), for a value \( x_3' \) in between 400 m and 1700 m we have to take into account the contribution due to the residue of the leftmost branch point.
Figure 3.46. Cagniard contours belonging to first-order partial contributions with different horizontal offsets. The reflection level is \(z_3 = 500\ m\) and the parameter \(q\) is zero. The leftmost branch point is indicated by \(p_1\) and the numbers along the contours indicate the values of \(\tau\).

<table>
<thead>
<tr>
<th>(r) m</th>
<th>(T_{arr}) s</th>
<th>(T_{sep}) s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.514</td>
<td>0.368</td>
</tr>
<tr>
<td>750</td>
<td>0.546</td>
<td>0.399</td>
</tr>
<tr>
<td>2500</td>
<td>0.785</td>
<td>0.713</td>
</tr>
<tr>
<td>5000</td>
<td>1.201</td>
<td>1.748</td>
</tr>
<tr>
<td>10000</td>
<td>2.034</td>
<td>5.890</td>
</tr>
</tbody>
</table>

Table 3.1. Table of the zeroth-order quantities \(T_{arr}\) and \(T_{sep}\) for different values of the horizontal offset.

For a horizontal offset of \(r = 5000\ m\) figure 3.48 shows which levels at a certain time instant yield a contribution to the first-order Green's function. A partial contribution is obtained for all combinations of time and depth that correspond to a point within the areas \(a\), \(b\) or \(c\). If a point lies within areas \(a\) or \(c\), the associated partial contribution always yields Cagniard contours with \(\partial_\tau |_{p(c(x))} < 0\). If, however, a point lies in the area \(b\), this means that the associated partial contribution can give rise to Cagniard contours with \(\partial_\tau |_{p(c(x))} > 0\) provided that \(0 \leq q < Q_{sep}(x_3)\). We
Figure 3.47. Plots of $T_{\text{arr}}(x'_3)$ and $T_{\text{sep}}(x'_3)$ versus $x'_3$ for different values of the horizontal offset.
Figure 3.48. Areas with points that indicate combinations of time and reflection levels giving a partial contribution to the first-order Green's function (areas a, b and c). In the areas a and c only the situation $\theta_p \tau|_{p(x)} < 0$ shows up, while in the area b the situation $\theta_p \tau|_{p(x)} > 0$ can arise as well. A close-up of the area around point A can be found in figure 3.49.

Figure 3.49. Close-up of the area around point A in figure 3.48. At point B both intervals giving rise to partial contributions merge.
see that no contributions come from levels above 400 m or below 2000 m because the configuration is homogeneous there. From this figure we also discover that the earliest partial contributions to the Green's function do not come from shallow levels, but from depths of about 1700 m. This is due to the fact that here the wavespeed has the highest values. Note that, except for the boundaries at 400 m and 2000 m, figure 3.48 in fact is a 90 degrees rotated version of figure 3.47.

If we zoom in on the square around point A in figure 3.48, an interesting detail shows up. An enlarged view of this area is given in figure 3.49. We see that there is a small time interval for which partial contributions to the first-order Green's function come from two vertical intervals that are separated by a vertical interval that does not yield partial contributions. The partial contributions from the upper interval, which begins at 400 m, yield Cagniard contours that always have \( \partial \rho \tau |_{\rho(z)} < 0 \), while for the partial contributions coming from the lower interval Cagniard contours with \( \partial \rho \tau |_{\rho(z)} > 0 \) can be obtained. If time is slightly increased, the intermediate interval becomes smaller, and at approximately a time instant of 1.799 s both intervals that give rise to partial contributions merge at point B. Investigation reveals that this point is associated with the occurrence of the logarithmic-like singularity in the first-order Green's function. The points on the curved line in between A and B (as well as on the curved line in between C and D in the previous figure) are associated with rays that nowhere have a horizontal trajectory. Points on the curved line in between B and C, however, indicate rays that have a horizontal trajectory of nonzero length. The ray belonging to B has a shape that forms the limit of both type of rays, i.e., a ray that is only horizontal in one point on the reflection level.

### 3.9. Discussion

In this chapter we have shown that the integral equation for the transform domain wavevector can be solved with the aid of the WKBJ iterative solution (Neumann series solution). Due to our transformation scheme involving the temporal Laplace transformation with real and positive transformation parameter, the transform domain vertical slowness is bounded away from zero, so difficulties that can occur due to a zero of the vertical slowness (the occurrence of turning points in the transform domain differential equation) have been avoided. Moreover, the application of the temporal Laplace transformation with real and positive transform parame-
ter has enabled us to prove the convergence of the WKBJ iterative solution in the transform domain for every continuously layered configuration; at the same time the convergence of the corresponding space-time domain solution for any time instant has been assured as well by Lerch's theorem. The transformation back to the space-time domain of the individual terms of the WKBJ iterative solution with the aid of the Cagniard-De Hoop method has been demonstrated. In principle, the integral transformation method that has been presented enables us to find sufficiently accurate results for the space-time domain acoustic wavefield at any time instant.

— Comparison with the approach in which the rotational symmetry of the configuration is employed in the forward transformation —

The reason why the WKBJ iterative solution and the Cagniard-De Hoop method form a good combination is that the mathematical form of the components of the zeroth-order term of the transform domain acoustic state vector, as well as the integrands of the components of the first-order and higher-order terms of the transform domain acoustic state vector, are ideally suited for application of the Cagniard-De Hoop method. This is due to the fact that they contain a factor in which the Laplace transform parameter does not occur and an exponential function with an argument in which the Laplace transform parameter only occurs as a multiplier. If we had employed the rotational symmetry of our configuration by using an azimuthal Fourier series expansion plus a radial Hankel transformation (Achenbach, 1973) instead of our double Fourier transformation with respect to both horizontal coordinates, the transform-domain results would have been less convenient due to the occurrence of Hankel functions instead of exponential functions.

Physically, each term in the WKBJ iterative solution can be interpreted as a wave that consists of a superposition of partial contributions that have undergone a specific number of partial reflections in the configuration. The paths of the rays that can be associated with each partial contribution have been derived as a side-product of the Cagniard-De Hoop method. In our theory a turning ray forms a special case of a "reflected" ray. Besides rays that are horizontal in one point, we also have
observed rays that have a horizontal trajectory over a non-zero distance. It has not been necessary to introduce the notion of tunneling in our theory.

Computer programs based on the derived theory have been developed and various numerical results for the zeroth-order and first-order terms of the space-time domain solution have been presented. It has been demonstrated that the combination of the WKBJ iterative solution and the Cagniard-De Hoop method is numerically robust.

A drawback of using the WKBJ iterative solution is that, although in principle a sufficiently accurate result can be obtained, the numerical evaluation of higher-order terms becomes increasingly difficult. As a consequence, the numerical convergence of the space-time domain solution can hardly be checked, and in fact one must rely on the fact that as long as the parameter profiles of the configuration do not vary too much, even with the zeroth-order plus the first-order term of the space-time domain solution a reasonable result is obtained. Especially for configurations with parameter profiles that do not vary slowly, this can be rather unsatisfactory. Clearly, the heart of the problem is the fact that the inverse transformation must act on the integrands of the multiple integrals with respect to depth that occur in the higher-order terms of the solution. This difficulty can be avoided by solving the transform domain problem applying a method that does somehow yield a solution in which the exponential function representing wave propagation is not a part of the integrand of any multiple integral with respect to depth. This implies that we must choose another method to perform the second step of our integral transformation method. Consequently, we must make a new start, with the same point of departure as at the beginning of this chapter.
Appendices to chapter 3

3.A. Convergence of the WKBJ iterative solution

In this appendix we investigate the conditions for convergence of the transform domain WKBJ iterative solution. According to eqs. (3.9), (3.12) and (3.13) in section 3.1, this solution is given by

$$\bar{w}_I = \sum_{n=0}^{\infty} \bar{w}_I^{(n)},$$  \hspace{1cm} (3.A.1)

in which the individual terms are given by

$$\bar{w}_I^{(0)} = \begin{pmatrix} \frac{1}{2} \sqrt{2} \tilde{a}_1 \, H(x_3 - x_3^S) \exp(-s \int_{x_3}^{x_3^S} \gamma \, d\zeta) \\ -\frac{1}{2} \sqrt{2} \tilde{a}_2 \, H(x_3^S - x_3) \exp(-s \int_{x_3^S}^{x_3} \gamma \, d\zeta) \end{pmatrix},$$  \hspace{1cm} (3.A.2)

$$\bar{w}_I^{(i)} = \begin{pmatrix} \int_{x_3}^{x_3^S} \chi(x_3') \exp(-s \int_{x_3'}^{x_3^S} \gamma \, d\zeta) \bar{w}_I^{(i-1)}(x_3') \, dx_3' \\ -\int_{x_3}^{x_3^S} \chi(x_3') \exp(-s \int_{x_3'}^{x_3^S} \gamma \, d\zeta) \bar{w}_I^{(i-1)}(x_3') \, dx_3' \end{pmatrix}, \quad (i \geq 1).$$  \hspace{1cm} (3.A.3)

For this moment we suppose that $\bar{w}_1^{(i)}$ and $\bar{w}_2^{(i)}$ are bounded, so

$$W^{(i)} = \sup_{-\infty < x_3 < \infty} \{|\bar{w}_I^{(i)}|\} < \infty.$$  \hspace{1cm} (3.A.4)

Moreover, we assume that $\chi(x_3)$ is bounded, i.e.,

$$\chi_{\text{sup}} = \sup_{-\infty < x_3 < \infty} \{|\chi(x_3)|\} < \infty,$$  \hspace{1cm} (3.A.5)

and that $\gamma(x_3)$ is bounded away from zero, thus

$$\gamma_{\text{inf}} = \inf_{-\infty < x_3 < \infty} \{\gamma(x_3)\} > 0.$$  \hspace{1cm} (3.A.6)

Since the Laplace transform parameter $s$ is real and positive, equation (3.A.3) gives
rise to the following inequalities

\[ |\tilde{w}_1^{(i)}| \leq W^{(i-1)} \chi_{\text{sup}} \int_{-\infty}^{x_3} \exp[-s\gamma_{\text{inf}}(x_3 - x')] \, dx' \]
\[ \leq \frac{W^{(i-1)} \chi_{\text{sup}}}{s\gamma_{\text{inf}}} \]  \hspace{1cm} (3.3.7)

\[ |\tilde{w}_2^{(i)}| \leq W^{(i-1)} \chi_{\text{sup}} \int_{x_3}^{\infty} \exp[-s\gamma_{\text{inf}}(x'_3 - x_3)] \, dx'_3 \]
\[ \leq \frac{W^{(i-1)} \chi_{\text{sup}}}{s\gamma_{\text{inf}}} . \]  \hspace{1cm} (3.3.8)

Consequently,

\[ W^{(i)} \leq \frac{W^{(i-1)} \chi_{\text{sup}}}{s\gamma_{\text{inf}}} , \]  \hspace{1cm} (3.3.9)

and thus

\[ |\tilde{w}_1^{(i)}| \leq W^{(0)} \left( \frac{\chi_{\text{sup}}}{s\gamma_{\text{inf}}} \right)^i , \]  \hspace{1cm} (3.3.10)

\[ |\tilde{w}_2^{(i)}| \leq W^{(0)} \left( \frac{\chi_{\text{sup}}}{s\gamma_{\text{inf}}} \right)^i . \]  \hspace{1cm} (3.3.11)

If the real and positive Laplace transform parameter \( s \) is chosen according to

\[ s > \frac{\chi_{\text{sup}}}{\gamma_{\text{inf}}} , \]  \hspace{1cm} (3.3.12)

summation of all the terms of the WKBJ iterative solution results in

\[ |\tilde{w}_1| \leq W^{(0)} \frac{1}{1 - \frac{\chi_{\text{sup}}}{s\gamma_{\text{inf}}}} , \]  \hspace{1cm} (3.3.13)

\[ |\tilde{w}_2| \leq W^{(0)} \frac{1}{1 - \frac{\chi_{\text{sup}}}{s\gamma_{\text{inf}}}} . \]  \hspace{1cm} (3.3.14)

Thus \( \tilde{w}_1 \) and \( \tilde{w}_2 \) are bounded if \( W^{(0)} \) is bounded. Since [cf. eq. (2.39)]

\[ \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} = \begin{pmatrix} \tilde{F}^N Y^{1/2}(x_3^S) + \tilde{Q}^N Y^{-1/2}(x_3^S) \\ \tilde{F}^N Y^{1/2}(x_3^S) - \tilde{Q}^N Y^{-1/2}(x_3^S) \end{pmatrix} , \]  \hspace{1cm} (3.3.15)

and [cf. eq. (2.25)]

\[ \begin{pmatrix} \tilde{Q}^N \\ \tilde{F}^N \end{pmatrix} = \begin{pmatrix} \tilde{Q}^S(s) + \rho^{-1}(x_3) [i\alpha_1 \tilde{F}_1^S(s) + i\alpha_2 \tilde{F}_2^S(s)] \\ \tilde{F}_3^S(s) \end{pmatrix} , \]  \hspace{1cm} (3.3.16)
from eq. (3.A.2) it follows that the final requirement is the boundedness of the transformed source signatures $\tilde{Q}^S$ and $F_k^S$. This is even true if $|\alpha_1| \to \infty$ and/or $|\alpha_2| \to \infty$, since in this case the exponential function in eq. (3.A.2) ensures that $W^{(0)}$ remains bounded even if $\tilde{Q}^N$ in eq. (3.A.16) becomes infinite. The boundedness of the transformed source signatures is met as long as the space-time domain source signatures have at most a delta function behavior in time. Under this circumstance the inequality (3.A.4) is indeed true as long as the inequalities in (3.A.5), (3.A.6) and (3.A.12) have been met. Moreover, the WKBJ iterative solution as given in eqs. (3.A.1)-(3.A.3) is absolutely convergent in that case.

3.B. The branch points of $\int_{x_3^2}^{x_3^5} \tilde{\gamma} \, dx_3$ in the complex $p$-plane

The goal of this appendix is to locate all the branch points in the complex $p$-plane that are due to the integral of $\tilde{\gamma}$. Without loss of generality we may assume that $q = 0$ in this appendix, so

$$\int_{x_3^2}^{x_3^5} \tilde{\gamma} \, dx_3 = \int_{x_3^2}^{x_3^5} [c^2(x_3) - p^2]^{1/2} \, dx_3. \quad (3.B.1)$$

In this integral, $p$ acts as a parameter. Only the square of $p$ shows up, so in the complex $p$-plane the complete integral shows point symmetry with respect to the origin. Since its branch points show the same point symmetry, we restrict our search to the right half of the complex $p$-plane. For a given value of $x_3$ the integrand introduces the branch point $p = 1/c(x_3)$ in the right half of the complex $p$-plane. Not for every $x_3 \in [x_3^2; x_3^5]$ the branch point of the integrand forms a branch point of the integral. The complete integral only depends on the integrand for specific values of $x_3$, and only the branch points associated with these values of $x_3$ are the branch points of the integral. To explain this, we recall that the wavenumber profile is a piecewise continuous function with piecewise continuous first- and higher-order derivatives on $[x_3^2; x_3^5]$. We indicate the $n$ points on $[x_3^2; x_3^5]$ where one or more derivatives are discontinuous by $x_3^{(1)}, x_3^{(2)}, \ldots, x_3^{(k)}, \ldots, x_3^{(n)}$. Now three kinds of specific values of $x_3$ will be described that determine the complete integral.
— The endpoints of the integration interval —

Suppose that the wavespeed profile has continuous first- and higher-order derivatives on \([x_3^1; x_3^3]\) and can be continued analytically into the complex \(z_3\)-plane. According to Cauchy’s theorem, the real contour of integration in eq. (3.B.1) can be deformed into an arbitrary path in the complex \(z_3\)-plane, as indicated in figure 3.B.1(a), and only \(x_3 = x_3^a\) and \(x_3 = x_3^b\), i.e., the endpoints of this arbitrary path, determine the value of the integral. Consequently, the points \(p = 1/c(x_3^a)\) and \(p = 1/c(x_3^b)\) are branch points of the integral.

\[\text{Im}(z_3)\]
\[\text{Re}(z_3)\]
\[O \quad x_3^a \quad x_3^b\]

Figure 3.B.1. Values of \(x_3\) that give rise to branch points in the complex p-plane: (a) the endpoints of the integration interval; (b) points where the wavespeed profile has a discontinuous first-order or higher-order derivative.

— Points where the wavespeed profile has one or more discontinuous derivatives —

Assume that the wavespeed profile at a point \(z_3^{(k)}\) has one or more discontinuous derivatives. This implies that on the real \(x_3\)-axis we have two distinct functions for \(z_3 < z_3^{(k)}\) and \(z_3 > z_3^{(k)}\), respectively. If we analytically continue both functions into
the complex \( x_3 \)-plane, the regions of validity of both functions must be defined. This leads to two more or less arbitrary branch cuts emanating from the real point \( x_3^{(k)} \), which prevent that one can go from one part of the real \( x_3 \)-plane to the other part without passing \( x_3^{(k)} \). The point \( x_3^{(k)} \) is the only point where we allow the transition from one function domain to the other. If the integration contour in eq. (3.B.1) is deformed from the real axis into a path in the complex \( x_3 \)-plane, this path must cross the point \( x_3^{(k)} \), according to figure 3.B.1(b). This means that the integral depends upon \( x_3^{(k)} \) as well, and thus the point \( p = 1/c(x_3^{(k)}) \) is a branch point of the complete integral.

— Stationary points where the wavespeed profile has a (local) extremum —

Consider the case in which the wavespeed profile reaches a (local) minimum or maximum value \( c_{\text{stat}} \) in a point \( x_3^{\text{stat}} \), i.e., \( \partial_3 c(x_3) = 0 \) in \( x_3^{\text{stat}} \). This point is called a stationary point. For generality further assume that the first \( (n - 1) \) derivatives are zero and the first nonzero derivative is of order \( n \). Consequently, \( n \geq 2 \) and even. Without loss of generality we further assume that \( x_3^{\text{stat}} = 0 \). Now there is a neighborhood of \( x_3^{\text{stat}} = 0 \) where the wavespeed behaves like

\[
c(x_3) \approx c_{\text{stat}} - k x_3^n > 0, \quad (x_3 \to 0), \tag{3.B.2}
\]

and for \( \text{Re}\{p\} \geq 0 \) in this neighborhood of \( x_3^{\text{stat}} = 0 \) the integrand in eq. (3.B.1) can be approximated by

\[
\bar{\gamma}(x_3) \approx \left[ \frac{1}{(c_{\text{stat}} - k x_3^n)^2} - p^2 \right]^{1/2}
= \frac{p}{c_{\text{stat}} - k x_3^n} \left[ \frac{1}{p^2} - (c_{\text{stat}} - k x_3^n)^2 \right]^{1/2}
= \frac{p}{c_{\text{stat}} - k x_3^n} \left[ \left( \frac{1}{p} + c_{\text{stat}} \right) - k x_3^n \right]^{1/2} \left[ \left( \frac{1}{p} - c_{\text{stat}} \right) + k x_3^n \right]^{1/2}
\approx \frac{p}{c_{\text{stat}}} \left( \frac{1}{p} + c_{\text{stat}} \right)^{1/2} \left[ \left( \frac{1}{p} - c_{\text{stat}} \right) + k x_3^n \right]^{1/2}, \quad (x_3 \to 0). \tag{3.B.3}
\]

Now we apply an analytical continuation in the complex \( x_3 \)-plane. If \( p \neq 1/c_{\text{stat}} \), the last factor in the last line gives rise to \( n \) branch points, regularly distributed on a
circle with a radius \(|(1/p - c_{\text{stat}})/k|^{1/n}\) and with its center in the origin of the complex \(z_3\)-plane. These will be supplemented with \(i\) branch cuts, see figure 3.B.2(a). Since the branch points in the complex \(z_3\)-plane do not coincide, the path of integration in eq. (3.B.1) can be deformed into the complex \(z_3\)-plane without the need to cross a specific point. If, however, \(p = 1/c_{\text{stat}}\), the \(n\) branch points in the complex \(z_3\)-plane coincide. Any path of integration in the complex \(z_3\)-plane that replaces the real path in eq. (3.B.1) must cross the point \(z_3 = 0\), the stationary point, according to figure 3.B.2(b). This leads to the conclusion that if the wavespeed profile has a (local) extremum \(c_{\text{stat}}\) in a stationary point \(x_{3,\text{stat}}\), the integral depends upon that stationary point, and the point \(p = 1/c_{\text{stat}}\) is a branch point of the complete integral.

![Diagram](image)

Figure 3.B.2. Situations that can arise in the complex \(z_3\)-plane if the wavespeed profile has a (local) extremum in a stationary point \(x_{3,\text{stat}}\): (a) the case \(p \neq 1/c_{\text{stat}}\); (b) the case \(p = 1/c_{\text{stat}}\).

3.C. Analytical evaluation of \(\int_{x_3^2}^{x_3^1} \tilde{\gamma} \, dx_3\) and \(\int_{x_3^2}^{x_3^1} \tilde{\gamma}^{-1} \, dx_3\) for linear and exponential wavespeed profiles

The integrals of the vertical slowness \(\tilde{\gamma} = [c^{-2}(x_3) - p^2 + q^2]^{1/2}\) and its reciprocal \(\tilde{\gamma}^{-1}\) play an important role in the Cagniard-De Hoop method of inverse transformation.
In general these integrals can only be evaluated numerically. For some wavespeed profiles, however, an analytical evaluation of these integrals is possible. For constant wavespeed profiles the evaluation of these integrals is trivial since the integrand does not depend upon the vertical coordinate. In this appendix we derive the analytical results for the integrals of \( \tilde{\gamma} \) and \( \tilde{\gamma}^{-1} \) in case of a linear and an exponential wavespeed profile. The results for the linear wavespeed profile are especially important in view of the theory of continuously layered configurations since a linear profile interpolation is the most simple one that conserves the continuous character of the configuration. Without loss of generality we take \( q = 0 \) in this appendix. Further we assume that \( \text{Re}\{p\} > 0 \).

--- Linear wavespeed profile ---

In case of a linear, non-homogeneous wavespeed profile the wavespeed satisfies

\[
c(x_3) = ax_3 + b > 0, \quad (a \neq 0).
\]

(3.C.1)

The integral of \( \tilde{\gamma} \) now follows as

\[
\int_{x_3^2}^{x_3^4} \tilde{\gamma} \, dx_3 = \int_{x_3^2}^{x_3^4} ((ax_3 + b)^{-2} - p^2)^{1/2} \, dx_3
\]

\[
= \frac{1}{a} \int_{c_a}^{c_b} (c^{-2} - p^2)^{1/2} \, dc
\]

\[
= \frac{p}{a} \int_{c_a}^{c_b} c^{-1}(p^{-2} - c^2)^{1/2} \, dc. \tag{3.C.2}
\]

Here we have applied the substitution \( c = ax_3 + b \). The new integration boundaries are \( c_a = ax_3^a + b \), the wavespeed at \( x_3^a \), and \( c_b = ax_3^b + b \), the wavespeed at \( x_3^b \). Note that the last step is made possible by the fact that \( p \) is located in the right half of the complex \( p \)-plane and the integration variable \( c \) is positive in view of eq. (3.C.1). Further we can write (CARMICHAEL & SMITH, 1962, p. 252, eq. 148)

\[
\int c^{-1} (p^{-2} - c^2) \, dc = (p^{-2} - c^2)^{1/2} - \frac{1}{p} \ln \left[ \frac{p^{-1} + (p^{-2} - c^2)^{1/2}}{c} \right]. \tag{3.C.3}
\]
As a result the integral of $\tilde{\gamma}$ turns out to be
\[
\int_{x_2^+}^{x_2^-} \tilde{\gamma} \, dx_3 \quad = \quad \frac{p}{a} (p^2 - c^2)^{1/2} \left[ \frac{c_b}{c_a} - \frac{1}{a} \ln \left( \frac{p^{-1} + (p^{-2} - c^2)^{1/2}}{c} \right) \right]_{c_a}^{c_b} \\
= \quad \frac{(1 - p^2 c_b^2)^{1/2}}{a} - \frac{(1 - p^2 c_a^2)^{1/2}}{a} + \frac{1}{a} \ln \left( \frac{c_b}{c_a} \right) \\
\quad - \frac{1}{a} \ln [1 + (1 - p^2 c_b^2)^{1/2}] + \frac{1}{a} \ln [1 + (1 - p^2 c_a^2)^{1/2}] .
\] (3.C.4)

The integral of $\tilde{\gamma}^{-1}$ is obtained by taking the derivative of the integral of $\tilde{\gamma}$ with respect to $p$. This yields
\[
\partial_p \int_{x_2^+}^{x_2^-} [c^{-2}(x_3) - p^2]' \, dx_3 = -p \int_{x_2^+}^{x_2^-} [c^{-2}(x_3) - p^2]^{-1/2} \, dx_3,
\] (3.C.5)

and thus
\[
\int_{x_2^+}^{x_2^-} \tilde{\gamma}^{-1} \, dx_3 = -\frac{1}{p} \partial_p \int_{x_2^+}^{x_2^-} \tilde{\gamma} \, dx_3 .
\] (3.C.6)

Applying this to eq. (3.C.4) we obtain
\[
\int_{x_2^+}^{x_2^-} \tilde{\gamma}^{-1} \, dx_3 = \frac{1}{a} \frac{c_b^2}{1 + (1 - p^2 c_b^2)^{1/2}} - \frac{1}{a} \frac{c_a^2}{1 + (1 - p^2 c_a^2)^{1/2}}.
\] (3.C.7)

Note that for $p = 1/c_{\text{max}}$, with $c_{\text{max}} = \max\{c_a, c_b\}$, the integral of $\tilde{\gamma}^{-1}$ remains finite, as opposed to the case of a homogeneous wavespeed profile for which this integral becomes negative infinite.

---

**Exponential wavespeed profile** ---

In case of an exponential wavespeed profile the wavespeed satisfies
\[
c(x_3) = f \exp(gx_3) > 0, \quad (g \neq 0) .
\] (3.C.8)

Under this circumstance the integral of $\tilde{\gamma}$ is given by
\[
\int_{x_2^+}^{x_2^-} \tilde{\gamma} \, dx_3 = \int_{x_2^+}^{x_2^-} \left\{ (f \exp(gx_3))^{-2} - p^2 \right\}^{1/2} \, dx_3 \\
= \quad - \frac{1}{g} \int_{c_{a}^{-1}}^{c_{b}^{-1}} y^{-1} (y^2 - p^2)^{1/2} \, dy ,
\] (3.C.9)
where we have used the substitution \( y = f^{-1} \exp(-g\omega_3) \). Further we have introduced the integration boundaries \( c_a^{-1} = f^{-1} \exp(-g\omega_3^a) \) and \( c_b^{-1} = f^{-1} \exp(-g\omega_3^b) \), where \( c_a \) and \( c_b \) denote the wavespeed at \( \omega_3^a \) and \( \omega_3^b \), respectively. It can be derived that (CARMICHAEL & SMITH, 1962, p. 253, eq. 178)

\[
\int y^{-1} (y^2 - p^2)^{1/2} \, dy = (y^2 - p^2)^{1/2} - p \arccos \left( \frac{p}{y} \right).
\]  

(3.C.10)

The integral of \( \tilde{\gamma} \) thus yields

\[
\int_{\omega_3^a}^{\omega_3^b} \tilde{\gamma} \, dz_3 = -\left. \left( \frac{(y^2 - p^2)^{1/2}}{g} \right) \right|_{c_a^{-1}}^{c_b^{-1}} + \frac{p}{g} \arccos \left( \frac{p}{y} \right) \left|_{c_a^{-1}}^{c_b^{-1}} \right.
\]

\[
= -\left. \frac{(c_b^{-2} - p^2)^{1/2}}{g} \right| + \left. \frac{(c_a^{-2} - p^2)^{1/2}}{g} \right| + \frac{p}{g} \arccos(pc_b) - \frac{p}{g} \arccos(pc_a).
\]  

(3.C.11)

The integral of \( \tilde{\gamma}^{-1} \) is evaluated using eq. (3.C.6). With the standard rules of differentiation this leads to

\[
\int_{\omega_3^a}^{\omega_3^b} \tilde{\gamma}^{-1} \, dz_3 = -\frac{1}{gp} \arccos(pc_b) + \frac{1}{gp} \arccos(pc_a).
\]  

(3.C.12)

Finally we show that no unpredicted singularities are introduced by the \( \arccos \)-function in the right half of the complex \( p \)-plane. In order to keep \( \arccos(z) \) single valued for complex arguments \( z \), branch points are introduced in the complex \( z \)-plane at \( z = 1 \) and \( z = -1 \). These are supplemented with branch cuts along the positive and negative real \( z \)-axis, respectively, from the relevant branch point to infinity. From eqs. (3.C.11) and (3.C.12) we see that in the present case \( z = pc_a \) or \( z = pc_b \), respectively. As a consequence, the branch points in the right half of the complex \( p \)-plane that are due to the \( \arccos \)-function are \( p = 1/c_a \) and \( p = 1/c_b \), respectively. The accompanying branch cuts are located along the positive real \( p \)-axis, ranging form the relevant branch point to infinity. Thus the functions \( \arccos(pc_a) \) and \( \arccos(pc_b) \) give rise to the same branchpoints and branchcuts in the right half of the complex \( p \)-plane as the functions \( (c_a^{-2} - p^2)^{1/2} \) and \( (c_b^{-2} - p^2)^{1/2} \). We note that the branch points introduced are only associated with the endpoints \( \omega_3^a \) and \( \omega_3^b \) of the integration interval. This is in accordance with appendix 3.B, since in the present case the wavespeed function does not possess any discontinuous derivatives or stationary points on \([\omega_3^a; \omega_3^b]\).
Chapter 4

Combination of WKBJ asymptotics and the Cagniard-De Hoop method

In this chapter a second integral transformation method is described. The first step, i.e., the transformation of the space-time domain problem, has already been performed in chapter 2. As the second step, in this chapter the transform domain equivalent of the wave propagation problem is solved in an approximate manner by using higher-order WKBJ asymptotic representations of the solution of the wavevector integral equation. These higher-order WKBJ asymptotic representations are valid around the point infinity of the Laplace transform parameter, i.e., they will lead to early-time asymptotic results in the space-time domain. In the space-time domain, we obtain approximations of the wavefield over a time interval of nonzero length beyond the arrival time. Due to the chosen transformation scheme, we will avoid complications that are frequently observed with WKBJ asymptotics, such as a breakdown of WKBJ asymptotic expansions at a zero of the vertical slowness or at the crossing of a Stokes line. The transformation of the approximate solutions back to the space-time domain forms the third step of the present integral transformation method. We perform this step by applying the Cagniard-De Hoop method to each term of the transform domain WKBJ asymptotic representations. Further, we show how the theory is implemented with the aid of a symbolic manipulation program. Finally, various numerical results for the approximations of the space-time domain wavefield are presented.
4.1. WKBJ asymptotics

In this chapter we will form a second integral transformation method, by which the higher-order behavior of the space-time domain acoustic wavefield right after its arrival can be determined. In other words, we form a method by which higher-order early-time asymptotic representations of the wavefield are obtained. We expect that the resulting higher-order early-time asymptotic representations will lead to accurate approximations of the wavefield over a time interval of nonzero length beyond the arrival time. Moreover, we aim at finding a method that is computationally efficient. To obtain this method, we start with the selection of an alternative method for the second step of our integral transformation method, i.e., the solution of the resulting transform domain wave propagation problem in a horizontally continuously layered configuration. This implies that we make a new start right after the first step, in other words, right after the forward transformations described in chapter 2. Several methods are available for the second step. In the introduction we have described four methods that are commonly employed: WKBJ asymptotics, Langer asymptotics, the WKBJ iterative solution and the Langer iterative solution. Reviews of these methods can be found in CHAPMAN (1981), CHAPMAN & ORCUTT (1985) and KENNEDY (1983). Here we will select one method that looks most suitable in view of the aim of this chapter. This selection will be based on the following general considerations. Firstly, we recall that with our transformation scheme involving a temporal Laplace transformation with a real and positive transformation parameter, the vertical slowness does never become zero. As a result, the WKBJ-type methods are preferred, since the Langer-type methods are more intricate, while the fact that these methods can under circumstances deal with a zero of the vertical slowness does not count. Secondly, with the asymptotic methods we can obtain transform domain results that contain a transcendental function representing the wave propagation, multiplied by a number of terms that are integer inverse powers of the Laplace transformation parameter. Due to this structure, the third step of our integral transformation method, involving the inverse transformations, can be performed very efficiently and will lead to the required higher-order early-time asymptotic representations in the space-time domain. On the other hand, the iterative solutions give rise to higher-order terms that consist of multiple integrals with integrands that
incorporate the function that represents wave propagation. As we have seen in the previous chapter, this structure causes the numerical evaluation of the higher-order terms in the space-time domain to be increasingly difficult and will not lead to an efficient way of finding the higher-order behavior right after the wavefront. Combining both considerations, the only promising possibility is to proceed our investigation using WKBJ asymptotics. Thereby we make unavoidable concessions with regard to the applicability of the WKBJ asymptotic method (restriction to configurations with parameter profiles that satisfy a certain differentiability requirement) and the exactness (the results will necessarily have an approximate nature).

In this chapter we only deal with continuously layered configurations of the type indicated in figure 4.1. Above the level $x_3^{\text{int}}$ both the parameter profiles are constant functions of $x_3$, and below $x_3^{\text{int}}$ each parameter profile is an infinitely differentiable function of $x_3$. At $x_3^{\text{int}}$ the parameter profiles of the homogeneous halfspace and the inhomogeneous halfspace match in a continuous fashion; only here the profile derivatives may show a finite jump. The point source is always located in the upper homogeneous halfspace, so $x_3^S < x_3^{\text{int}}$; the receiver may be located anywhere in the configuration except in the source. The reason for restricting the analysis in this

![Figure 4.1](image)

**Figure 4.1.** The restricted class of horizontally continuously layered configurations used when dealing with WKBJ asymptotics.
chapter to the type of configuration just described is the fact that the derivations associated with WKBJ asymptotics remain simple for this class of configurations.

In our analysis the most important item is the determination of the WKBJ asymptotic representations of the wavevector components in the inhomogeneous halfspace \( x_3 > x_3^{\text{int}} \). Once these are known, the WKBJ asymptotic representations in the homogeneous halfspace \( x_3 < x_3^{\text{int}} \) follow easily from the boundary conditions at \( x_3^{\text{int}} \). The WKBJ asymptotic representations for \( x_3 > x_3^{\text{int}} \) are found in three steps. The first step is formed by the introduction of the alternative wavevector components \( \tilde{P}_1 \) and \( \tilde{P}_2 \) according to

\[
\tilde{w}_1(x_3, s) = \frac{1}{2} \sqrt{2} \tilde{a}_1 \tilde{P}_1(x_3, s) \exp(-s \int_{x_3^{\text{int}}}^{x_3} \gamma d\zeta), \quad (x_3 > x_3^{\text{int}}), \quad (4.1)
\]

\[
\tilde{w}_2(x_3, s) = \frac{1}{2} \sqrt{2} \tilde{a}_1 \tilde{P}_2(x_3, s) \exp(-s \int_{x_3^{\text{int}}}^{x_3} \gamma d\zeta), \quad (x_3 > x_3^{\text{int}}). \quad (4.2)
\]

The source-dependent factor \( \tilde{a}_1 \) occurring in these equations follows from eqs. (2.25) and (2.39). The exponential function is equivalent to the exponential function in the WKBJ asymptotic approximations of the corresponding frequency domain problem (Chapman & Orcutt, 1985). The second step is the asymptotic expansion of the functions \( \tilde{P}_1 \) and \( \tilde{P}_2 \) according to

\[
\tilde{P}_1(x_3, s) = \sum_{n=0}^{N} s^{-n} \tilde{P}_1^{(n)}(x_3) + O(s^{-(N+1)}), \quad (x_3 > x_3^{\text{int}}, s \to \infty), \quad (4.3)
\]

\[
\tilde{P}_2(x_3, s) = \sum_{n=0}^{N} s^{-n} \tilde{P}_2^{(n)}(x_3) + O(s^{-(N+1)}), \quad (x_3 > x_3^{\text{int}}, s \to \infty), \quad (4.4)
\]

in which \( \tilde{P}_1^{(n)} \) and \( \tilde{P}_2^{(n)} \) are coefficient functions that have still to be determined. The right-hand sides of these equations are asymptotic expansions in inverse integer powers of the Laplace transform parameter \( s \). By combining eqs. (4.1) and (4.2) with eqs. (4.3) and (4.4), respectively, the WKBJ asymptotic expansions of the wavevector components are obtained. The third step consists of omitting the terms of order \( O(s^{-(N+1)}) \) from the WKBJ asymptotic expansions, which results in the \( N \)-th order WKBJ asymptotic representations of the wavevector components.

The application of WKBJ asymptotics is widespread, as can be inferred from Chapman (1973, 1974b), Richards (1971), Young (1984) and Thomson & Chapman (1984). In the literature the applied WKBJ asymptotic representations consist of the zeroth-order term or, at most, the sum of the zeroth-order term and
the first-order term of the corresponding WKBJ asymptotic expansion. This can be explained from the fact that such WKBJ asymptotic representations are mostly used to approximate the solution of a frequency domain problem at high frequencies, in which case the next lower-order term is very small. Another reason for this low number of terms is the effort involved in manually determining the coefficients of the WKBJ asymptotic representations. With the present integral transformation method the representations will be transformed back to the space-time domain, however, and we expect that the higher-order terms can have a considerable influence on the results in a time interval beyond the arrival time. Therefore, a main item in this chapter is to extend the WKBJ asymptotic representations considerably beyond their first-order terms.

An important mathematical aspect is that as \( N \to \infty \), eqs. (4.3) and (4.4) are asymptotic series. Asymptotic series need not be convergent [e.g., they can have late terms that behave like \( n!/s^n \), see WASOW (1965)]. Our asymptotic series possess the property that if we take into account only a fixed finite number of terms, the approximation is the more accurate the larger \( s \) is. For a fixed finite value of \( s \), however, it is possible that the magnitude of the terms starts to increase after a certain order, although before this order is reached the magnitude of the terms decreases. If this happens, the "best" asymptotic representations are obtained by taking that number of terms from the asymptotic series for which the terms just do not start to increase in magnitude. As long as one is satisfied with the "convergence" obtained at that time, WKBJ asymptotics are useful; if the influence of the last terms of the resulting asymptotic representations is still too large, we must take recourse to other methods.

Versions of the WKBJ asymptotic expansions with the same form as in eqs. (4.1) - (4.4) have been used, at least formally, by CHAPMAN (1973, 1981), KENNEDT (1983), CHAPMAN & ORCUTT (1985), RICHARDS (1971) and YOUNG (1984). In all these papers the application of the WKBJ asymptotic expansions has been validated by referring to CODDINGTON & LEVINSON (1955) and/or WASOW (1965). The actual coefficient functions \( \hat{\phi}_1^{(n)} \) and \( \hat{\phi}_2^{(n)} \) have been derived by substitution of the WKBJ asymptotic expansions into the relevant wavevector differential equation. With this differential equation approach some coefficient functions of the WKBJ asymptotic expansions are not given directly, but in a differentiated form that first requires inte-
gration. The problem then arises which integration constant should be applied, and in most situations a good choice cannot be made. This can be quite unsatisfactory since one knows that the WKBJ asymptotic expansions do exist, as indicated by (elaborate) general proofs. The solution of this problem is contained within these proofs, however. Authors like ERDÉLYI (1956), CODDINGTON & LEVINSON (1955) and WASOW (1965) clearly show that all proofs concerning the existence of asymptotic expansions boil down to the existence of the solution of an integral equation. Led by this observation in this chapter we employ the wavevector integral equations to derive the WKBJ asymptotic expansions of the wavevector components. According to eqs. (2.51) - (2.53), these integral equations follow from

\[ \tilde{w}_I = L_{IJ} \tilde{w}_J + \tilde{h}_I, \]  

(4.5)

in which the integral operator \( L_{IJ} \) is defined by

\[ L_{IJ} \tilde{w}_J = \left( \begin{array}{c} \int_{-\infty}^{x^3} \chi(x^3) \exp[-s \int_{x^3}^{2} \gamma(\zeta) d\zeta] \tilde{w}_2(2) \, dx^3 \ \\
- \int_{x^3}^{\infty} \chi(x^3) \exp[-s \int_{x^3}^{2} \gamma(\zeta) d\zeta] \tilde{w}_1(2) \, dx^3 \end{array} \right), \]  

(4.6)

and where the source vector \( \tilde{h}_I \) is given by

\[ \tilde{h}_I = \left( \begin{array}{c} \frac{1}{2} \sqrt{2} \tilde{a}_1 H(x_3 - x^3_2) \exp[-s \int_{x^3_2}^{x_3} \gamma(\zeta) d\zeta] \\
- \frac{1}{2} \sqrt{2} \tilde{a}_2 H(x_3 - x_3) \exp[-s \int_{x_3}^{x^3_2} \gamma(\zeta) d\zeta] \end{array} \right). \]  

(4.7)

With the integral equation approach we expect to circumvent the problem of the unknown integration constants. Using the same integral equations, for the specific configuration at hand the proof of the existence of the WKBJ asymptotic expansions for \( x_3 > x^\text{int}_3 \) is much simpler than the general proofs in the literature, and is given in appendix 4.4. It is mentioned that a more intricate but equally safe way to derive the WKBJ asymptotic expansions consists of solving the integral equations with the aid of the WKBJ iterative solution given in section 3.1 and subsequently deriving the WKBJ asymptotic expansions by repeatedly integrating the terms of this solution by parts and summing the results (YOUNG, 1984). This approach has not been used in this chapter, however. It is noted that all approaches to derive the WKBJ asymptotic expansions just mentioned, give rise to the same results.
4.1.1. WKBJ asymptotic expansions for $x_3 > x_3^{\text{int}}$

Using eqs. (4.5) - (4.7) and taking into account that the configuration under investigation is homogeneous for $x_3 < x_3^{\text{int}}$, the integral equations for the wavevector components for $x_3 > x_3^{\text{int}} > x_3^c$ (i.e., in the inhomogeneous halfspace) turn out to be

$$
\tilde{w}_1(x_3, s) = \int_{x_3^{\text{int}}}^{x_3^c} \chi(x_3') \exp(-s \int_{x_3^{\text{int}}}^{x_3'} \gamma \, d\zeta) \tilde{w}_2(x_3', s) \, dx_3' + \frac{1}{2} \sqrt{2} \hat{a}_1 \exp(-s \int_{x_3^c}^{x_3^1} \gamma \, d\zeta),
$$

(4.8)

$$
\tilde{w}_2(x_3, s) = -\int_{x_3}^{\infty} \chi(x_3') \exp(-s \int_{x_3}^{x_3'} \gamma \, d\zeta) \tilde{w}_1(x_3', s) \, dx_3'.
$$

(4.9)

To find the integral equations for the alternative wavevector components $\tilde{P}_1$ and $\tilde{P}_2$, we substitute eqs. (4.1) and (4.2) into these integral equations. This yields

$$
\tilde{P}_1 = \int_{x_3^{\text{int}}}^{x_3^c} \chi \tilde{P}_2 \, dx_3' + 1,
$$

(4.10)

$$
\tilde{P}_2 = -\int_{x_3}^{\infty} \chi \exp(-2s \int_{x_3}^{x_3'} \gamma \, d\zeta) \tilde{P}_1 \, dx_3'.
$$

(4.11)

Differentiation of eq. (4.11) with respect to $x_3$, and substitution in eq. (4.10) gives the integro-differential equation

$$
2s\gamma \tilde{P}_2 = -\chi \int_{x_3^{\text{int}}}^{x_3^c} \chi \tilde{P}_2 \, dx_3' + \chi + \partial_3 \tilde{P}_2.
$$

(4.12)

The simple form of this integro-differential equation is due to the special type of configurations that are considered; this equation would be far more intricate if it would be allowed that the source is positioned in the inhomogeneous halfspace as well. In appendix 4.A it is proved that for $x_3 > x_3^{\text{int}}$ and $s$ real and sufficiently large, asymptotic expansions of $\tilde{P}_1$ and $\tilde{P}_2$ and their derivatives $\partial_3 \tilde{P}_1$ and $\partial_3 \tilde{P}_2$ in inverse powers of $s$ exist, whatever infinitely differentiable functions (either algebraic or transcendental) may be used to represent the parameter profiles below $x_3^{\text{int}}$.

As a next step we must determine the asymptotic expansions presented in eqs. (4.3) and (4.4). We start with the substitution of the asymptotic sequence $\{s^{-n} \tilde{P}_2^{(n)} | n \in \mathbb{N} \}$ into the integro-differential equation in order to find the coefficient functions $\tilde{P}_2^{(n)}$. Equating terms with equal powers of $s$, the coefficient functions $\tilde{P}_2^{(n)}$ can be proved to satisfy the recurrence scheme.
\begin{align}
\check{P}^{(0)}_2 &= 0, \\
\check{P}^{(1)}_2 &= -\frac{\chi}{2\gamma}, \\
\check{P}^{(n)}_2 &= -\frac{\chi}{2\gamma} \int_{z_{3}^{\text{int}}}^{z_3} \chi \check{P}^{(n-1)}_2 \, dz_3' + \frac{1}{2\gamma} \partial_3 \check{P}^{(n-1)}_2, \quad (n \geq 2).
\end{align}

Subsequently, substitution of the asymptotic sequences \( \{s^{-n}\check{P}^{(n)}_1 | n \in \mathbb{N}\} \) and \( \{s^{-n}\check{P}^{(n)}_2 | n \in \mathbb{N}\} \) in eq. (4.10) and equating terms with equal powers of \( s \) reveals that the coefficient functions \( \check{P}^{(n)}_1 \) depend on the coefficient functions \( \check{P}^{(n)}_2 \) according to

\begin{align}
\check{P}^{(0)}_1 &= 1, \\
\check{P}^{(n)}_1 &= \int_{z_{3}^{\text{int}}}^{z_3} \chi \check{P}^{(n)}_2 \, dz_3', \quad (n \geq 1).
\end{align}

As expected the integrals in eq. (4.17) also show up in eq. (4.15) and follow as a side-product of the evaluation of that equation. Since \( z_3 > z_3^{\text{int}} \), the differentiations of the parameter profiles with respect to \( z_3 \) that are necessary when repeatedly evaluating eq. (4.15) need not be performed at \( z_3^{\text{int}} \), where one or more profile derivatives are discontinuous. Consequently, all coefficient functions \( \check{P}^{(n)}_1 \) and \( \check{P}^{(n)}_2 \) are bounded. This is due to the special type of configurations that are investigated; if the parameter profiles were allowed to possess one or more discontinuous derivatives in the inhomogeneous halfspace, this property would be lost and the WKBJ asymptotic expansions would break down. The implementation of the recurrence scheme for the generation of the coefficient functions will be described in section 4.4. Once the coefficient functions are known up to \( n = N \), the WKBJ asymptotic expansions of \( \check{w}_1 \) and \( \check{w}_2 \) follow from eqs. (4.1) - (4.4). Out of these WKBJ asymptotic expansions we form the \( N \)-th order WKBJ asymptotic representations of the wavevector components at a receiver level \( z_3^R > z_3^{\text{int}} \). These are given by

\begin{align}
\check{w}_1(z_3^R, s) &\sim \frac{1}{2} \sqrt{2} \check{a}_1 \sum_{n=0}^{N} s^{-n} \check{P}^{(n)}_1(z_3^R) \exp(-s \int_{z_3^{\text{int}}}^{z_3^R} \gamma \, d\zeta), \quad (z_3^R > z_3^{\text{int}}), \quad (4.18) \\
\check{w}_2(z_3^R, s) &\sim \frac{1}{2} \sqrt{2} \check{a}_1 \sum_{n=1}^{N} s^{-n} \check{P}^{(n)}_2(z_3^R) \exp(-s \int_{z_3^{\text{int}}}^{z_3^R} \gamma \, d\zeta), \quad (z_3^R > z_3^{\text{int}}). \quad (4.19)
\end{align}
4.1.2. WKBJ asymptotic expansions for $x_3 \downarrow x_3^{\text{int}}$

The limiting case $x_3 \downarrow x_3^{\text{int}}$ (i.e., the point of observation approaches $x_3^{\text{int}}$ via the inhomogeneous halfspace), is interesting from a theoretical point of view since the integrals in eqs. (4.15) and (4.17) vanish and the coefficient functions are obtained as

$$\tilde{P}_1^{(0)} = 1,$$
$$\tilde{P}_1^{(n)} = 0, \quad (n \geq 1),$$

and

$$\tilde{P}_2^{(0)} = 0,$$
$$\tilde{P}_2^{(1)} = -\frac{\chi(x_3^{\text{int}} + 0)}{2\gamma(x_3^{\text{int}})},$$
$$\tilde{P}_2^{(n)} = \frac{1}{2\gamma(x_3^{\text{int}})} \frac{\partial_3 \tilde{P}_2^{(n-1)}}{\partial x_3^{\text{int}}} \bigg|_{x_2 = x_2^{\text{int}} + 0}, \quad (n \geq 2).$$

Observe that in the last equation $\partial_3 \tilde{P}_2^{(n-1)}$ must be determined by differentiating $\tilde{P}_2^{(n)}$ given by eq. (4.15) with $n$ replaced by $n - 1$, and not by simply differentiating $\tilde{P}_2^{(n)}$ following from eq. (4.24) itself. In section 4.4 it will be shown that in this limiting case the coefficient functions can be determined exactly for all continuously differentiable parameter profiles. From the resulting WKBJ asymptotic expansions it follows that the $N$-th order WKBJ asymptotic representations to the wavevector components for a receiver level in the limiting case $x_3^R \downarrow x_3^{\text{int}}$ are given by

$$\tilde{w}_1(x_3^R, s) \sim \frac{1}{2} \sqrt{2} a_1 \exp(-s \int_{x_3^{\text{int}}}^{x_3^R} \gamma \, d\zeta), \quad (x_3^R \downarrow x_3^{\text{int}}),$$
$$\tilde{w}_2(x_3^R, s) \sim \frac{1}{2} \sqrt{2} a_1 \sum_{n=1}^{N} s^{-n} \tilde{P}_2^{(n)}(x_3^{\text{int}} + 0) \exp(-s \int_{x_3^{\text{int}}}^{x_3^R} \gamma \, d\zeta), \quad (x_3^R \downarrow x_3^{\text{int}}).$$

4.1.3. WKBJ asymptotic expansions for $x_3 < x_3^{\text{int}}$

In this subsection the case $x_3 < x_3^{\text{int}}$ (i.e., with the point of observation in the homogeneous halfspace) is considered. Physics requires that both the acoustic pressure and the vertical particle wavespeed are continuous at $x_3^{\text{int}}$. Since the parameter profiles are continuous at $x_3^{\text{int}}$, so are the components of the composition matrix in the
composition relation (2.32), and thus the wavevector components \( \tilde{\omega}_1 \) and \( \tilde{\omega}_2 \) must be continuous as well.

In the previous subsection we have shown that for \( z_3 \downarrow z_3^{\text{int}} \), eqs. (4.20) - (4.24) give rise to the \( N \)-th order WKBJ asymptotic representations

\[
\lim_{z_3 \downarrow z_3^{\text{int}}} \tilde{\omega}_1(z_3, s) \sim \frac{1}{2} \sqrt{2} \tilde{a}_1 \exp(-s \int_{z_3^{\text{int}}}^{z_3} \gamma \, d\zeta), \tag{4.27}
\]
\[
\lim_{z_3 \downarrow z_3^{\text{int}}} \tilde{\omega}_2(z_3, s) \sim \frac{1}{2} \sqrt{2} \tilde{a}_1 \sum_{n=1}^{N} \epsilon^{-n} \tilde{F}_2^{(n)}(z_3^{\text{int}} + 0) \exp(-s \int_{z_3^{\text{int}}}^{z_3} \gamma \, d\zeta). \tag{4.28}
\]

Note that all the coefficient functions \( \tilde{F}_2^{(n)}(z_3^{\text{int}} + 0) \) remain bounded.

According to eqs. (4.5) - (4.7), in the homogeneous halfspace the wavevector component \( \tilde{\omega}_1 \) only consists of the direct source contribution

\[
\tilde{\omega}_1(z_3, s) = \frac{1}{2} \sqrt{2} \tilde{a}_1 H(z_3 - z_3^S) \exp(-s \int_{z_3^S}^{z_3} \gamma \, d\zeta), \tag{4.29}
\]
which is zero above the source. Moreover, eqs. (4.5) - (4.7) show us that in the homogeneous halfspace the wavevector component \( \tilde{\omega}_2 \) equals

\[
\tilde{\omega}_2(z_3, s) = -\int_{z_3^S}^{\infty} \chi(z_3') \exp(-s \int_{z_3'}^{z_3} \gamma \, d\zeta) \tilde{\omega}_1(z_3', s) \, dz_3' \\
= \frac{1}{2} \sqrt{2} \tilde{a}_2 H(z_3^S - z_3) \exp(-s \int_{z_3}^{z_3^S} \gamma \, d\zeta). \tag{4.30}
\]

Here, the first term represents the reflection from the inhomogeneous halfspace. The second term, i.e., the direct source contribution, is zero below the source. For \( z_3 \uparrow z_3^{\text{int}} \) with \( z_3^S < z_3 < z_3^{\text{int}} \), eqs. (4.29) and (4.30) lead to

\[
\lim_{z_3 \uparrow z_3^{\text{int}}} \tilde{\omega}_1(z_3, s) = \frac{1}{2} \sqrt{2} \tilde{a}_1 \exp(-s \int_{z_3^{\text{int}}}^{z_3} \gamma \, d\zeta), \tag{4.31}
\]
\[
\lim_{z_3 \uparrow z_3^{\text{int}}} \tilde{\omega}_2(z_3, s) = -\int_{z_3^{\text{int}}}^{\infty} \chi(z_3') \exp(-s \int_{z_3'}^{z_3} \gamma \, d\zeta) \tilde{\omega}_1(z_3', s) \, dz_3'. \tag{4.32}
\]

Comparison of eqs. (4.27) and (4.31) shows us that within the order of the approximation made, \( \tilde{\omega}_1 \) is continuous at \( z_3^{\text{int}} \). As a consequence, direct comparison of eqs. (4.28) and (4.32) yields the \( N \)-th order WKBJ asymptotic representation

\[
-\int_{z_3^{\text{int}}}^{\infty} \chi(z_3') \exp(-s \int_{z_3'}^{z_3} \gamma \, d\zeta) \tilde{\omega}_1(z_3', s) \, dz_3' \\
\sim \frac{1}{2} \sqrt{2} \tilde{a}_1 \sum_{n=1}^{N} \epsilon^{-n} \tilde{F}_2^{(n)}(z_3^{\text{int}} + 0) \exp[-s(\int_{z_3^{\text{int}}}^{z_3} \gamma \, d\zeta + \int_{z_3}^{z_3^{\text{int}}} \gamma \, d\zeta)], \tag{4.33}
\]
in view of the continuity of $\tilde{w}_2$ at $x_3^{\text{int}}$. With reference to eqs. (4.27) - (4.33), the $N$-th order WKBJ asymptotic representations of the wavevector components at a receiver level $x_3^R < x_3^{\text{int}}$ turn out to be

\begin{align}
\tilde{w}_1(x_3^R, s) &\sim \frac{1}{2} \sqrt{2} \tilde{\alpha}_1 \, H(x_3^R - x_3^S) \exp(-s \int_{x_3^S}^{x_3^R} \gamma \, d\zeta), \\
\tilde{w}_2(x_3^R, s) &\sim -\frac{1}{2} \sqrt{2} \, \tilde{\alpha}_2 \, H(x_3^S - x_3^R) \exp(-s \int_{x_3^R}^{x_3^S} \gamma \, d\zeta) \\
&+ \frac{1}{2} \sqrt{2} \tilde{\alpha}_1 \sum_{n=1}^{N} s^{-n} \tilde{P}_2^{(n)}(x_3^{\text{int}} + 0) \exp[-s(\int_{x_3^R}^{x_3^{\text{int}}} \gamma \, d\zeta + \int_{x_3^{\text{int}}}^{x_3^S} \gamma \, d\zeta)].
\end{align}

The terms with the Heaviside step functions represent the waves that are directly generated by the source. The second term in the last equation forms the $N$-th order WKBJ approximation of the reflection caused by the inhomogeneous halfspace. If the first $N$ derivatives of the parameter profiles are zero for $x_3 \downarrow x_3^{\text{int}}$, i.e., if the first discontinuous derivative is of order $(N + 1)$ at $x_3^{\text{int}}$, eqs. (2.27), (2.37), (4.13) - (4.15), (4.34) and (4.35) lead to the well-known result that the reflected wave from the inhomogeneous halfspace is $O(s^{-(N+1)})$ in comparison with the incident wave.

### 4.2. Physical interpretation

In eqs. (4.1) - (4.4) of the previous section the two steps have been given that lead to the WKBJ asymptotic expansions of the wavevector components in the inhomogeneous halfspace. In this section we will show that both steps can be given a specific physical meaning.

The first step consists of the transformation of the wavevector components $\tilde{w}_1$ and $\tilde{w}_2$ into the functions $\tilde{P}_1$ and $\tilde{P}_2$ according to

\begin{align}
\tilde{w}_1(x_3, s) &= \frac{1}{2} \sqrt{2} \tilde{\alpha}_1 \, \tilde{P}_1(x_3, s) \exp(-s \int_{x_3^S}^{x_3} \gamma \, d\zeta), \quad (x_3 > x_3^{\text{int}}), \\
\tilde{w}_2(x_3, s) &= \frac{1}{2} \sqrt{2} \tilde{\alpha}_2 \, \tilde{P}_2(x_3, s) \exp(-s \int_{x_3^S}^{x_3^R} \gamma \, d\zeta), \quad (x_3 > x_3^{\text{int}}).
\end{align}

As explained in section 2.4 for a homogeneous part of the configuration and in section 3.2 for a generally continuously layered configuration, the wavevector components $\tilde{w}_1$ and $\tilde{w}_2$ represent a downward traveling wave and an upward traveling wave, respectively. In the above equations, the exponential function represents the propagation of a direct wave from the source level $x_3^S$ downward to the level $x_3$. By separating the exponential function from $\tilde{w}_1$ and $\tilde{w}_2$, we obtain $\tilde{P}_1$ and $\tilde{P}_2$. These
functions are the equivalent wavevector components in a reference frame which, relative to the fixed reference frame, moves downward with the direct wave generated by the source, or, in other words, which moves downward with the wavefront occurring in the fixed reference frame. Because the co-moving reference frame moves in the same direction as the wave represented by $\tilde{w}_1$, in the co-moving reference frame $\tilde{P}_1$ does not show a propagating behavior, and $\tilde{P}_2$ travels in the direction of negative $x_3$ with twice the speed of the wavefront. A graphical representation of the wavevector components in a fixed reference frame and a co-moving reference frame is given in figure 4.2.

The second step is formed by the asymptotic expansion of $\tilde{P}_1$ and $\tilde{P}_2$ in inverse powers of the large and positive Laplace transform parameter $s$, thus giving

$$\tilde{P}_1(x_3, s) = \sum_{n=0}^{N} s^{-n} \tilde{P}_1^{(n)}(x_3) + O(s^{-(N+1)}), \quad (x_3 > x_3^{\text{int}}, s \to \infty), \quad (4.38)$$

$$\tilde{P}_2(x_3, s) = \sum_{n=0}^{N} s^{-n} \tilde{P}_2^{(n)}(x_3) + O(s^{-(N+1)}), \quad (x_3 > x_3^{\text{int}}, s \to \infty). \quad (4.39)$$

To interpret this step, we remark that when a time-domain function $f(t)$ has the

![Diagram](image)

(a) (b)

Figure 4.2. Wavevector components: (a) in a fixed reference frame; (b) in a reference frame that is co-moving with the wavefront occurring in the fixed reference frame.
early-time asymptotic representation

\[ f(t) \sim \sum_{k=1}^{K} c_k t^{\lambda_k - 1}, \quad (0 < \lambda_1 < \lambda_2 < \ldots < \lambda_K, \ t \downarrow 0), \]  

(4.40)

then a corresponding asymptotic representation around the point infinity of \( s \) for the transform domain function \( \hat{f}(s) \) is (ERDÉLYI, 1956, p. 33)

\[ \hat{f}(s) \sim \sum_{k=1}^{K} \Gamma(\lambda_k) c_k s^{-\lambda_k}, \quad (s \to \infty), \]  

(4.41)

in which \( \Gamma \) denotes the gamma function. This implies that the coefficient functions \( \hat{P}_1^{(0)} \) and \( \hat{P}_2^{(0)} \) of the transform domain asymptotic expansions of \( \hat{P}_1 \) and \( \hat{P}_2 \) are related to the value for \( t \downarrow 0 \) of the time domain wavefield in the co-moving reference frame. Since \( \hat{P}_2^{(0)} = 0 \), only \( \hat{P}_1^{(0)} = 1 \) is associated with a nonzero wavefield at \( t \downarrow 0 \). A closer look at eqs. (4.1), (4.3) and eqs. (4.5), (4.7) reveals that \( \hat{P}_1^{(0)} \), and thus the value of the wavefield for \( t \downarrow 0 \), is caused directly by the source. The higher-order terms in the asymptotic expansions of \( \hat{P}_1 \) and \( \hat{P}_2 \) are related to the higher-order behavior for \( t \downarrow 0 \) of the wavefield in the co-moving reference frame. These terms are nonzero for \( t > 0 \) and represent modifications of the direct wavefield due to the inhomogeneous character of the configuration for \( x_3 > x_3^{\text{int}} \).

When we return to the fixed reference frame, the situation in the time domain can be described as follows. The value of the wavefield at the wavefront is only due to the wave directly generated by the source. After the arrival of the wavefront, modifications to this wavefield are made, thus accounting for the influence of the inhomogeneity of the configuration on the wavefield.

Although different exponential functions are involved, the same situation occurs in the homogeneous halfspace: the zeroth-order terms in eqs. (4.34) and (4.35) represent the wavefield at the arrival of the wavefront, which consists of waves that are directly generated by the source; the higher-order terms in eq. (4.35) play only a role after the arrival of the wavefront, and represent the reflection from the inhomogeneous halfspace.

4.3. The WKBJ paradox

An important item in the literature concerning WKBJ asymptotics is the so-called WKBJ paradox (MAHONY, 1967; MEYER, 1975, 1980; CHAPMAN & MAHONY,
1978; Gray, 1982). This paradox shows up for the frequency domain analysis of wave propagation problems in a continuously layered configuration with nonconstant parameter profiles that are infinitely differentiable on \((-\infty; \infty)\) and that tend to constant values as \(|z_3| \to \infty\). For a wave with angular frequency \(\omega\) that is incident from minus infinity, for this configuration, all terms of any order WKBJ asymptotic representation of the reflected wave at minus infinity are zero. In other words, whatever order WKBJ asymptotic representation is applied, the reflection coefficient (giving the proportionality between the incident wavefield and the reflected wavefield) at minus infinity is represented by zero. Analysis has shown that in reality the modulus of this reflection coefficient is exponentially small in \(\omega\) (Meyer, 1975; Chapman & Mahony, 1978), which explains why it remains "unseen" when using a WKBJ asymptotic representation of any order. Nevertheless, since the medium is assumed to be inhomogeneous, we expect a reflection of a part of the incident wavefield by the inhomogeneous configuration. The fact that this reflection is expected to be nonzero but is not given by a WKBJ asymptotic representation of any order is in the literature regarded as paradoxical.

In order to investigate whether in our case the WKBJ paradox shows up as well, we first formulate the conditions that lead to a configuration that is equivalent to the configuration that occurs in the description of the WKBJ paradox. To achieve this configuration, we consider the case \(z_3^S < z_3^{\text{int}} < z_3^R\) as analyzed in subsection 4.1.1, and we let \(z_3^S, z_3^{\text{int}},\) and \(z_3^R\) go to minus infinity. Further we assume that the wavespeed and mass density profiles approach constant values as \(|z_3| \to \infty\). In this configuration the inhomogeneity function \(\chi\) and its derivatives are zero at minus infinity, and according to eqs. (4.13) - (4.17), the coefficient functions \(\tilde{F}_1^{(n)}\) and \(\tilde{F}_2^{(n)}\) are zero except \(\tilde{F}_1^{(0)}\), which represents the wave directly generated by the source, i.e., the incident wavefield. Consequently, no correction of the wavefield takes place, so in our case the reflected wave at minus infinity is zero as well. Thus the WKBJ paradox seems to occur in our case as well. However, the following analysis will show that this situation is not paradoxical at all.

The item that must first be investigated is whether the exact solution does not contain exponentially small terms that remain "unseen" by any order WKBJ asymptotic representation as presented in subsections 4.1.1 - 4.1.3. To prove that such exponentially small terms are not present in the exact solution of the problem,
the first-order term of the convergent WKBJ iterative solution of the upward propagating wavevector component \( \tilde{w}_2 \) is analyzed for the case \( x_3^S < x_3^{\text{int}} \leq x_3^R \). This term follows from eq. \((3.17)\) as

\[
\tilde{w}_2^{(1)}(x_3^R, s) = -\frac{1}{2} \sqrt{2} \tilde{a}_1 \int_{x_3^R}^{\infty} \chi(x_3) \exp\left[-s \int_{x_3^R}^{\infty} x_3^{\gamma} d\zeta + \int_{x_3^R}^{\infty} \gamma d\zeta\right] dx_3. \tag{4.42}
\]

The right-hand side shows the continuous summation of all first-order partial reflections; these have first traveled from the source to a level \( x_3 \) (first term in the argument of exponential function), then have undergone a partial reflection at \( x_3 \) (the function \( \chi \)), and have finally traveled from \( x_3 \) to the receiver (second term in the argument of the exponential function). The exponential function that represents the wave propagation over the source-receiver distance can be extracted as in eq. \((4.2)\). This gives

\[
\tilde{w}_2^{(1)}(x_3^R, s) = -\frac{1}{2} \sqrt{2} \tilde{a}_1 \exp(-s \int_{x_3^R}^{\infty} \gamma d\zeta) \int_{x_3^R}^{\infty} \chi(x_3) \exp(-2s \int_{x_3^R}^{\infty} \gamma d\zeta) dx_3. \tag{4.43}
\]

The lower boundary of the integral in the argument of the second exponential function and the lower boundary of the integral with respect to \( x_3 \) are both equal to \( x_3^R \). This implies that in the space-time domain the contribution due to this first-order term starts to arrive at the receiver at the same time that the direct wave from the source reaches the receiver. Thus we have an instantaneous modification of the wavefield caused directly by the source. Equivalent arguments can be devised for all first-order and higher-order terms of \( \tilde{w}_1 \) and \( \tilde{w}_2 \) of the WKBJ iterative solution. In the transform domain neither of the terms of the WKBJ iterative solution gives rise to an additional exponential function and the only exponential function that can be extracted is the one that belongs to the direct wave generated by the source and that already appears explicitly in the WKBJ asymptotic expansions. This means that in our case there are no parts of the exact solution that remain "unseen" by any order WKBJ asymptotic representation, so there must be another reason why the reflected wavefield is zero.

To find this reason, we employ eq. \((4.42)\) to analyze the situation in which \( x_3^S, x_3^{\text{int}} \) and \( x_3^R \) are minus infinity. This yields

\[
\tilde{w}_2^{(1)}(-\infty, s) = -\frac{1}{2} \sqrt{2} \tilde{a}_1 \int_{-\infty}^{\infty} \chi \exp(-2s \int_{-\infty}^{\infty} \gamma d\zeta) dx_3. \tag{4.44}
\]

Since the parameter profiles become constant for \(|x_3| \to \infty\), only for finite values of \( x_3 \) we have a nonzero inhomogeneity function \( \chi \). As a result, the first contributions
to the reflected field at minus infinity are due to partial reflections at levels with finite $z_3$. For these values the exponential function in the integrand has a negative infinite argument, which in the space-time domain represents an infinitely long time delay before the contribution due to this first-order term arrives at the receiver. The same occurs with all higher-order terms of the convergent WKBJ iterative solution of $\tilde{w}_2$. As a consequence, the wavefield at the receiver is not corrected within a finite time to account for the inhomogeneity of the medium. In the transform domain this fact is correctly indicated by the zero terms of any order WKBJ asymptotic representation, and thus the present situation is not paradoxical.

— Comparison with the frequency domain analysis —

With reference to the foregoing analysis, at this point we will give further comments on the WKBJ paradox occurring with the frequency domain analysis. First of all, since transient wavefields require an infinitely broad frequency spectrum, the fact that only one value of $\omega$ is considered implies that the statements made in the literature refer to the case of a noncausal incident field that is monotonous and is present since $t = -\infty$. Secondly, by discarding the phase of the reflection factor, the information that it takes an infinitely long time for the reflections to return to minus infinity is lost. If we combine these two facts, we see that nothing can be said about the arrival time of the reflections since, essentially, infinity (the time the incident wave is present) minus infinity (the time needed by the reflection to return to minus infinity) is undetermined. It therefore seems more reasonable to say that the WKBJ paradox described in the literature is caused by the way in which the problem is stated instead of blaming WKBJ asymptotics that it is not suited to represent the reflected wavefield in certain physical situations.

Finally, we note that although the situation with $z_3^S$, $z_3^\text{int}$ and $z_3^R$ being minus infinity is of theoretical interest, in this paper the attention is primarily focussed on the situation in which $z_3^R$ and $z_3^S$ have finite values. If $z_3^S < z_3^\text{int} < z_3^R$, according to subsection 4.1.1 the coefficient functions $\tilde{P}_1^{(n)}$ and $\tilde{P}_2^{(n)}$ are in general nonzero and a reflected wave is obtained with WKBJ asymptotics. Although this case has
not been addressed in this chapter, it is remarked that even if \( z_3^{\text{int}} \) is minus infinity, for bounded values of \( z_3^S \) and \( z_3^R \) a reflected wave is achieved when using WKBJ asymptotics. This follows upon repeatedly performing integration by parts on the terms of the WKBJ iterative solution.

### 4.4. Implementation of the recurrence scheme

We can only successfully apply the higher-order WKBJ asymptotic representations if there exists an efficient way to determine the coefficient functions \( \tilde{P}_1^{(n)} \) and \( \tilde{P}_2^{(n)} \) for reasonably high order \( n \) and for every relevant value of the transform parameters \( \alpha_1 \) and \( \alpha_2 \). One approach is to find \( \tilde{P}_1^{(n)} \) and \( \tilde{P}_2^{(n)} \) by analytic evaluation of the recurrence scheme of eqs. (4.13) - (4.17), thereby regarding \( \alpha_1 \) and \( \alpha_2 \) as parameters. However, in view of the integral occurring in them, for arbitrary parameter profiles it is unlikely that we are able to evaluate eqs. (4.15) and (4.17) in this way. Another possibility is a purely numerical implementation of the recurrence scheme, which must be repeated for all relevant values of \( \alpha_1 \) and \( \alpha_2 \). Unfortunately, this method has the drawback of being very time consuming. The approach that we employ in this chapter consists of a symbolic implementation of the recurrence scheme to generate the analytic expressions for \( \tilde{P}_1^{(n)} \) and \( \tilde{P}_2^{(n)} \) from the analytic expression for \( \tilde{P}_2^{(n-1)} \). Nowadays several programs capable of symbolic manipulations are available. We have selected the program \textsc{Mathematica} \textsuperscript{TM} (Wolfram, 1991) for the work involved with evaluating the recurrence scheme.

#### 4.4.1. General strategy

Using symbolic manipulation, it turns out to be a good strategy to separate the determination of \( \tilde{P}_1^{(n)} \) and \( \tilde{P}_2^{(n)} \) in the following three steps.

--- **Step 1: Polynomial approximation of \( \chi/2\gamma, \chi \) and \( 1/2\gamma \)** ---

Even with a symbolic manipulation program, we can only apply the recurrence scheme a reasonable number of times if the integrations and differentiations involved remain sufficiently simple. This requires that \( \chi/2\gamma, \chi \) and \( 1/2\gamma \) are simple
functions, e.g., polynomials. For arbitrary parameter profiles this will not be the case. However, a more convenient situation is created if on \([z_3^{\text{int}}; z_3^R]\) we approximate these functions by the polynomials

\[
\frac{\chi}{2\gamma} \approx \sum_{i=0}^{M} a_i (z_3 - z_3^R)^i, \quad \text{(4.45)}
\]

\[
\chi \approx \sum_{i=0}^{M} b_i (z_3 - z_3^R)^i, \quad \text{(4.46)}
\]

\[
\frac{1}{2\gamma} \approx \sum_{i=0}^{M} c_i (z_3 - z_3^R)^i, \quad \text{(4.47)}
\]

where the coefficients \(a_i\), \(b_i\), and \(c_i\) follow from the Taylor expansions of the left-hand side functions around \(z_3^R\). Using eqs. (2.27) and (2.37) and deriving the coefficients in this way, we find that the coefficients \(a_i\) and \(b_i\) depend on the derivatives of both the wavespeed profile and the mass density profile up to order \((i + 1)\); the coefficients \(c_i\) depend on the derivatives of the wavespeed profile up to order \(i\). The approximations require that the values of these profile derivatives at the receiver level are known. The number of terms in the summations of eqs. (4.45) - (4.47) determines the number of profile derivatives that play a part in the further calculation. To avoid a loss of information about the parameter profiles, we should not take the number of terms in the summations too small. Moreover, even if the parameter profiles themselves are represented by polynomials with only a few low-order nonzero derivatives, this does not imply that higher-order terms in the summations above are zero or may be omitted. In practice the number of terms in the summations is determined by increasing \(M\) until addition of one more term does not change the result significantly (later on we will see that in the special case where \(z_3^R \downarrow z_3^{\text{int}}\) the required number of terms of the summations is directly determined by the order \(n\) of the coefficient function of a WKBJ asymptotic expansion).

All functions on the left-hand side of eqs. (4.45) - (4.47) consist of terms which are inverse powers of \(\gamma\). Since

\[
\partial_3 \gamma^{-k}(x_3) = k \gamma^{k-2}(x_3) c(x_3) \partial_0 c(x_3), \quad \text{(4.48)}
\]

it is easy to see that the coefficients \(a_i\), \(b_i\), and \(c_i\) consist of a finite sum of inverse powers of \(\gamma(z_3^R)\). The only places where the transform parameters \(a_1\) and \(a_2\) show up are in these inverse powers of \(\gamma(z_3^R)\), so it can be stated a priori that the inverse
transformation step in fact reduces to the rather simple inversion of $\gamma^{-k}(x_3^R)$. The terms in eqs. (4.45) - (4.47) are obtained by symbolic manipulation. For each particular set consisting of both parameter profiles plus the levels $x_3^{\text{int}}$ and $x_3^R$, we only need to determine the coefficients $a_i$, $b_i$ and $c_i$ once; for this we can apply symbolic manipulation as well.

— Step 2. Symbolic evaluation of the recurrence scheme —

The finite sums that are introduced above enable us to evaluate the recurrence scheme of Eqs. (4.13) - (4.17) by symbolic manipulation. During this step the formal coefficients $a_i$, $b_i$ and $c_i$ are still used, although their actual expressions are known from step 1. The first reason for this is that the application of the actual coefficients at this stage would make the symbolic evaluation of the recurrence scheme far more difficult because in general the actual coefficients themselves consist of a considerable number of terms. The second reason is that it is more convenient to first apply the recurrence scheme with the formal coefficients, since the formal result obtained in this way will be independent of the actual configuration and need only be determined once. This is important since the symbolic evaluation of the recurrence scheme, even using the formal coefficients $a_i$, $b_i$ and $c_i$, is the farmost time consuming step if we want to derive higher-order WKBJ asymptotic representations.

— Step 3. Substitution of the actual coefficients $a_i$, $b_i$ and $c_i$ into the formal results of the recurrence scheme —

The actual expressions for the coefficient functions $\hat{P}_1^{(n)}$ and $\hat{P}_2^{(n)}$ belonging to a specific configuration are found by substitution of the actual coefficients $a_i$, $b_i$ and $c_i$ (obtained in step 1) into the formal results of the recurrence scheme (generated in step 2). Again, this task is performed using symbolic manipulation. For further convenience, during this step we also let the symbolic manipulation program collect all terms with equal inverse powers of $\gamma(x_3^R)$. As a result we obtain a list of the
coefficients $c_{m,n}$ and $d_{m,n}$ from which $\tilde{P}_1^{(n)}$ and $\tilde{P}_2^{(n)}$ can be constructed according to

$$\tilde{P}_1^{(n)} = \sum_{m=n}^{L_1} c_{m,n} \gamma^{-m}(x_3^R),$$

$$\tilde{P}_2^{(n)} = \sum_{m=n}^{L_1} d_{m,n} \gamma^{-m}(x_3^R).$$

(4.49)

(4.50)

Note that $c_{m,n}$ and $d_{m,n}$ are independent of the transformation parameters $s$, $\alpha_1$ and $\alpha_2$. The values of the finite upper summation bounds $L_1$ and $L_2$ depend on $n$ and the upper summation bound $M$ used in eqs. (4.45) - (4.47), and are presented in table 4.1. As long as both parameter profiles and the levels $x_3^{\text{int}}$ and $x_3^R$ remain unchanged, the coefficients $c_{m,n}$ and $d_{m,n}$ do not alter and need only be determined once. Thus, changing the source level and/or the horizontal offset between source and receiver does not require a re-evaluation of $\tilde{P}_1^{(n)}$ and $\tilde{P}_2^{(n)}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-</td>
<td>$\tilde{P}_2^{(0)} = 0$</td>
</tr>
<tr>
<td>1</td>
<td>$5+4M$</td>
<td>$3+2M$</td>
<td>-</td>
</tr>
<tr>
<td>$\geq 2$</td>
<td>$n(5+4M)$</td>
<td>$(3+2M)+(n-1)(5+4M)$</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4.1. The values of the upper summation bounds $L_1$ and $L_2$.

4.4.2. Simplifications occurring for $x_3^R \downarrow x_3^{\text{int}}$

Although the special case $x_3^R \downarrow x_3^{\text{int}}$ (i.e., the receiver approaches $x_3^{\text{int}}$ via the inhomogeneous halfspace) is in principle treated in the same way as outlined above, we can make some additional simplifications. First of all, in the expressions for $\tilde{P}_1^{(n)}$ and $\tilde{P}_2^{(n)}$ all terms containing $(x_3^{\text{int}} - x_3^R)^i$ with $i \geq 1$ vanish. This simplifies the final result considerably: according to subsection 4.1.2, the coefficient functions $\tilde{P}_1^{(n)}$ become zero for $n \geq 1$ and the results for the coefficient functions $\tilde{P}_2^{(n)}$ become less complicated. Moreover, if the summations of eqs. (4.45) - (4.47) are substituted into the recurrence scheme given by eqs. (4.13) - (4.15) and we let $x_3^R \downarrow x_3^{\text{int}}$, an analysis shows us that $\tilde{P}_2^{(n)}$ does not depend upon those coefficients $a_i$, $b_i$ and $c_i$ for which $i \geq n$. In other words, if $x_3^R \downarrow x_3^{\text{int}}$ it is sufficient to set $M = n - 1$ in eqs. (4.45) - (4.47) in order to obtain exact results for $\tilde{P}_2^{(n)}$ for any kind of parameter profiles (either algebraic or transcendental) below $x_3^{\text{int}}$. Further analysis shows that if we
write out the coefficient function \( \tilde{p}_2^{(n)} \) in inverse powers of \( \gamma \), we obtain in this case

\[
\tilde{p}_2^{(0)} = 0, \tag{4.51}
\]
\[
\tilde{p}_2^{(n)} = \sum_{m=n}^{3n} d_{m,n} \gamma(x_3^{\text{int}}), \quad (n \geq 1), \tag{4.52}
\]

while, moreover,

\[
d_{m,n} = 0 \quad \text{if } m + n \text{ is odd}. \tag{4.53}
\]

These results are much simpler than those presented in eqs. (4.49) and (4.50). Note that for \( n \geq 1 \) the values of the upper summation bounds \( L_1 \) and \( L_2 \) denoted in table 4.1 do not hold in the present case. Firstly, the value of \( L_1 \) is irrelevant now since \( \tilde{p}_1^{(n)} = 0 \), so there is no need to write \( \tilde{p}_1^{(n)} \) as a summation. Secondly, further analysis reveals that in the present case \( L_2 = 3n \). As a consequence of the discussion following eqs. (4.45) - (4.47) and the fact that \( \tilde{p}_2^{(n)} \) does not depend upon \( a_i, b_i \) and \( c_i \) for \( i \geq n \), only profile derivatives up to order \( n \) play a role in \( \tilde{p}_2^{(n)} \) if \( x_3^R \downarrow x_3^{\text{int}} \). If the WKBJ asymptotic representations will only consist of a finite number of terms up to, say, order \( N \), for \( x_3^R \downarrow x_3^{\text{int}} \) the order of profile derivatives that can be accounted for by using these \( N \)-th order WKBJ asymptotic representations is also limited to \( N \); the higher-order profile derivatives remain "unseen". Then it is not possible to distinguish between parameter profiles that only differ after their \( N \)-th derivatives. For the case \( x_3^R \downarrow x_3^{\text{int}} \) it can therefore be stated that WKBJ asymptotics are well-suited when it is sufficient to approximate the parameter profiles by polynomials of not a too high degree. Difficulties are expected if we must represent the parameter profiles by high-order polynomials, e.g., when dealing with certain transcendental functions. Although in this subsection we have only analyzed the case \( x_3^R \downarrow x_3^{\text{int}} \), it is unlikely that the occurrence of this effect is restricted to this case; for other receiver positions we expect the same problems to exist.

### 4.5. Applying the Cagniard-De Hoop method in the case \( x_3^R > x_3^{\text{int}} \)

Now that the WKBJ asymptotic representations of the transform domain acoustic statevector components have been determined, we must perform the third and final step of our integral transformation method, which is the transformation of these
representations back to the space-time domain. In order to purely investigate the consequences of applying WKBJ asymptotic representations instead of the WKBJ iterative solution, at this stage we also require that, as in the previous chapter, the transformation back to the space-time domain is exact. Moreover, the inverse transformation method must be applicable in all situations that have been dealt with in this chapter. Following the reasoning given in section 3.3, the Cagniard-De Hoop method is selected (De Hoop, 1960, 1961, 1988; Pao & Gajewski, 1977; Van der Hulden, 1987). This is the same method of inverse transformation as has been used in the previous chapter, and this enables us to compare the efficiency by which the Cagniard-De Hoop method is applied to the WKBJ iterative solution and the WKBJ asymptotic representations.

The goal of this section is to illustrate the inverse transformation method for the WKBJ asymptotic representations. To avoid the repetition of analogous equations for all possible wavefield quantities due to all possible kinds of source components, we confine ourselves in this section to a acoustic pressure wave that is generated by a source of volume injection rate (monopole source) and that is measured at a receiver, with $x_3^S < x_3^{\text{int}} < x_3^R$, i.e., the case treated in subsection 4.1.1. This wave is indicated schematically in figure 4.3. The application of the Cagniard-De Hoop method in the cases $x_3^R \downarrow x_3^{\text{int}}$ and $x_3^R < x_3^{\text{int}}$ is demonstrated separately in section 4.6 since in these cases considerable simplifications show up in comparison with the present case.

4.5.1. The WKBJ asymptotic representation of the Green's function

When we substitute eqs. (4.49) and (4.50) in eqs. (4.18) and (4.19), we obtain the $N$-th order WKBJ asymptotic representations of the wavevector components $\vec{w}_1$ and $\vec{w}_2$ at a receiver level $x_3^R > x_3^{\text{int}}$ as

$$\vec{w}_1(x_3^R, s) \sim \frac{1}{2} \sqrt{2} \tilde{a}_1 \sum_{n=0}^{N} \sum_{m=n}^{L_1} c_{m,n} (x_3^R)^{s-n} \exp(-s \int_{x_3^R}^{x_3^S} \gamma \, d\zeta), \quad (4.54)$$

$$\vec{w}_2(x_3^R, s) \sim \frac{1}{2} \sqrt{2} \tilde{a}_1 \sum_{n=0}^{N} \sum_{m=n}^{L_2} d_{m,n} (x_3^R)^{s-n} \exp(-s \int_{x_3^R}^{x_3^S} \gamma \, d\zeta). \quad (4.55)$$
For a wavefield that is generated by a source of volume injection, application of the composition relation (2.32), eqs. (2.25) and (2.39), reveals that the $N$-th order WKBJ asymptotic representation of the transform domain acoustic pressure $\tilde{p}$ at the receiver level is given by

$$
\tilde{p} \sim \frac{1}{2} \tilde{Q}^S(s) Y^{-1/2}(x_3^S) Y^{-1/2}(x_3^R) \\
\times \sum_{n=0}^{N} \sum_{m=n}^{\max(L_1; L_2)} (c_{m,n} + d_{m,n}) \gamma^{-m}(x_3^R) s^{-n} \exp(-s \int_{x_3^R}^{x_3^S} \gamma d\zeta). \quad (4.56)
$$

This is the expression that is used to elucidate the transformation back to the space-time domain. But before the Cagniard-De Hoop method is employed, we write this expression in a form that allows for a more straightforward analysis. To this end we write the transform domain acoustic pressure as

$$
\tilde{p} = s^2 \tilde{Q}^S(s) \tilde{G}, \quad (4.57)
$$

where we have introduced the transform domain Green's function $\tilde{G}$, which has the $N$-th order WKBJ asymptotic representation

$$
\tilde{G} \sim \sum_{n=0}^{N} \sum_{m=n}^{\max(L_1; L_2)} (c_{m,n} + d_{m,n}) s^{-n} \tilde{T}(m), \quad (4.58)
$$
with

$$\tilde{T}(m) = \frac{1}{2} s^{-2} \Pi \gamma^{-m} \left( x_3^R \right) \exp(-s \int_{x_3^S}^{x_3^R} \gamma \, d\zeta).$$

(4.59)

Here the factor

$$\Pi = Y^{-1/2}(x_3^S) Y^{-1/2}(x_3^R)$$

(4.60)

represents the coupling of the acoustic pressure wave to the source and the receiver. Since \( \alpha_1 \) and \( \alpha_2 \) only occur in \( \tilde{T}(m) \), the inverse transformation problem has mainly been reduced to finding the space-time domain equivalents of the functions \( \tilde{T}(m) \) using the Cagniard-De Hoop method.

As a first step in the inverse transformation procedure, we apply the inverse Fourier transformation (2.15) and obtain

$$\tilde{T}(m) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Pi \gamma^{-m} \left( x_3^R \right) \exp[-s(i\alpha_1 x_1^R + i\alpha_2 x_2^R + \int_{x_3^S}^{x_3^R} \gamma \, d\zeta)] \, d\alpha_1 \, d\alpha_2. \quad (4.61)$$

The analysis that follows will be based on the fact that \( \tilde{T}(m) \) is only dependent on the Laplace transform parameter \( s \) through its occurrence as a multiplicative constant in the argument of the exponential function; note that \( s \) does not appear in \( \Pi \) and \( \gamma^{-m}(x_3^R) \).

### 4.5.2. Transformation of the Fourier transform parameters \( \alpha_1 \) and \( \alpha_2 \) into \( p \) and \( q \)

The version of the Cagniard-De Hoop method that we employ uses the transformations

$$\alpha_1 = -ip \cos \theta + q \sin \theta,$$

(4.62)

$$\alpha_2 = -ip \sin \theta - q \cos \theta.$$

(4.63)

The angle \( \theta \) is one of the cylindrical coordinates of the receiver with respect to the source. The complete set of these cylindrical coordinates consists of the horizontal offset \( r \), the polar angle \( \theta \), and the vertical separation \( z \). These are related to the Cartesian coordinates of the receiver and the source according to

$$x_1^R = r \cos \theta,$$

(4.64)

$$x_2^R = r \sin \theta,$$

(4.65)

$$x_3^R - x_3^S = z.$$

(4.66)
Note that our frame of reference is chosen in such a way that the vertical axis goes through the source. Upon transforming the parameters $\alpha_1$ and $\alpha_2$ into $p$ and $q$, the equivalent of eq. (4.61) becomes

$$
\hat{T}(m) = \frac{-i}{8\pi^2} \int_{-\infty}^{\infty} \int_{-i\infty}^{i\infty} \hat{\Pi} \hat{\gamma}^{-m}(x_3^R) \exp[-s(pr + \int_{x_3^R}^{x_3^S} \hat{\gamma} d\zeta)] dp dq
$$

$$
= \frac{-i}{8\pi^2} \int_{-\infty}^{\infty} \int_{-i\infty}^{i\infty} \rho^{1/2}(x_3^S) \rho^{1/2}(x_3^R) \frac{\rho}{\lambda^{1/2}(x_3^S) \lambda^{m+1/2}(x_3^R)} \exp[-s(pr + \int_{x_3^R}^{x_3^S} \hat{\gamma} d\zeta)] dp dq.
$$

(4.67)

Note that $p$ is purely imaginary and $q$ is real. Quantities in which $\alpha_1$ and $\alpha_2$ are replaced by $p$ and $q$ are indicated by an overbar. As an example, the vertical slowness is now given by

$$
\hat{\gamma}(x_3) = \left[c^{-2}(x_3) - p^2 + q^2\right]^{1/2}.
$$

(4.68)

The parameter $p$ is called the horizontal slowness.

Next we continue the integrand of eq. (4.67) analytically into the complex $p$-plane, and define $\text{Re}\{\hat{\gamma}\} \geq 0$ in order to keep $\hat{\gamma}$ single valued. In view of the application of Jordan's lemma later on, we restrict our attention to the right half of the complex $p$-plane. Before a possible deformation of the integration contour in the complex $p$-plane can take place, the singularities of the integrand in eq. (4.67) in this area must be located. First of all, branch points are introduced by the non-exponential part of the integrand. The following possibilities exist

- if $c(x_3^S) \neq c(x_3^R)$, branch points at $p^S = [c^{-2}(x_3^S) + q^2]^{1/2}$ and $p^R = [c^{-2}(x_3^R) + q^2]^{1/2}$ are introduced by $\gamma^{-1/2}(x_3^S)$ and $\gamma^{-m-1/2}(x_3^R)$;

- if $c(x_3^S) = c(x_3^R)$ (which will rarely happen but is nevertheless possible, see figure 4.4) and $m$ is even, a branch point at $p^R = [c^{-2}(x_3^R) + q^2]^{1/2}$ is introduced by $\gamma^{-m-1}(x_3^R)$.

A second set of branch points is introduced by the exponential part of the integrand in eq. (4.67). Although the exponential function itself does not have branch points, its argument introduces branch points due to the integral of $\hat{\gamma}$. According to appendix 3.B we find that the branch points due to the integral of $\hat{\gamma}(\zeta)$ are the points $p = [c^{-2}(\zeta) + q^2]^{1/2}$ related to the levels $\zeta \in [x_3^S; x_3^R]$ that either form
Figure 4.4. Two examples of the case \(c(x_3^R) = c(x_3^S)\).

- an endpoint of either of the integration interval \([x_3^S, x_3^R]\);
- a point where one or more derivatives of the wavespeed profile are discontinuous; or
- a stationary point (i.e., a point where \(\partial c = 0\)) where the wavespeed profile reaches a (local) extremum.

All branch points are positioned on the real \(p\)-axis. They are supplemented with branch cuts along the positive and negative real \(p\)-axes, respectively, from the relevant branch point to infinity. Apart from the branch point singularities mentioned

- if \(c(x_3^S) = c(x_3^R)\) (see figure 4.4) and \(m\) is odd, we obtain a pole in the right half of the complex \(p\)-plane due to the occurrence of \(\tilde{\gamma}^{-m-1}(x_3^R)\); this pole coincides with the branch point at \(p^R\).

4.5.3. The Cagniard contour

At this point we suggest to deform the path of integration from the imaginary \(p\)-axis to the Cagniard contour, which consists of the complex branches of

\[
\tau = p r + \int_{x_3^S}^{x_3^R} \tilde{\gamma} \, d\zeta = \text{real} \tag{4.69}
\]
that are located in the right half of the complex $p$-plane. An outline of the features of this Cagniard contour in case of a continuously layered configuration has been presented in subsections 3.3.3 and 3.4.3 and will now briefly be repeated. First we define the maximum wavespeed $c_{\text{max}}$

$$
c_{\text{max}} = \max_{x_3 \in [x_3^L, x_3^R]} \{c(x_3)\}.
$$

(4.70)

On $[x_3^L, x_3^R]$ this maximum wavespeed can be found on a level that either forms

- an endpoint of the integration interval $[x_3^L, x_3^R]$;
- a point where $\partial_3 c$ is discontinuous; or
- a stationary point of the wavespeed profile.

For a picture of these possibilities we refer to figure 3.5. From appendix 3.B we know that these possibilities form a subset of those that give rise to a branch point of the integrand in the complex $p$-plane. Consequently, the leftmost branch point $p_\ell$ in the complex $p$-plane is always given by

$$
p_\ell = [c_{\text{max}}^{-2} + q^2]^{1/2}.
$$

(4.71)

Although the real axis in between the origin and $p_\ell$ does satisfy eq. (4.69), it will turn out that this trajectory is not a part of the final integration path in the complex $p$-plane, and therefore it will not be considered as a part of the Cagniard contour. For large values of $\tau$ the complex branches of the Cagniard contour asymptotically approach the straight lines

$$
p \sim \frac{\tau}{r - iz} \quad \text{if} \quad \tau \to \infty \text{ in the first quadrant of the } p\text{-plane},
$$

(4.72)

$$
p \sim \frac{\tau}{r + iz} \quad \text{if} \quad \tau \to \infty \text{ in the fourth quadrant of the } p\text{-plane}.
$$

(4.73)

These asymptotes are independent of $q$. For continuously layered configurations the Cagniard contour can approach the real axis in two distinct ways. These depend upon the sign of

$$
\partial_3 \tau|_{p_\ell} = r - [c_{\text{max}}^{-2} + q^2]^{1/2} \int_{x_3^L}^{x_3^R} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{-1/2} d\zeta.
$$

(4.74)

Although the point $p_\ell$ satisfies eq. (4.69), it is not under all circumstances a point of the Cagniard contour. Nevertheless, it always is an important point in view of
the determination of the type of the Cagniard contour. With respect to \( \partial_p \tau |_{p_0} \), two cases can show up.

--- The case \( \partial_p \tau |_{p_0} < 0 \) ---

If [cf. eqs. (3.40), (3.84) and (3.42), (3.86)] we introduce \( r_{\text{sep}} \) and \( Q_{\text{sep}} \) as

\[
 r_{\text{sep}} = \frac{1}{c_{\text{max}}} \int_{x_0^2}^{x_0^2} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{-1/2} \, d\zeta, \tag{4.75}
\]

\[
 Q_{\text{sep}} = \left( \frac{r^2}{\int_{x_0^2}^{x_0^2} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{-1/2} \, d\zeta} \right)^{1/2}, \tag{4.76}
\]

the conditions of occurrence of the case \( \partial_p \tau |_{p_0} < 0 \) follow as

- the horizontal offset satisfies \( r < r_{\text{sep}} \); or
- the horizontal offset satisfies \( r > r_{\text{sep}} \) and the parameter \( q \) satisfies \( q > Q_{\text{sep}} \).

If \( \partial_p \tau |_{p_0} < 0 \) there is only a single point \( p_0 \) on the interval \( [0; p_0] \) where the Cagniard contour crosses the real \( p \)-axis. Traveling along the real \( p \)-axis, \( \tau \) has a maximum value in this point. To further increase the value of \( \tau \), one must leave the real \( p \)-axis perpendicularly in \( p_0 \) and proceed along one of the complex branches of the Cagniard contour. Traveling along the Cagniard contour, \( \tau \) has a minimum value in \( p_0 \), so \( p_0 \) turns out to be a saddle point in the complex \( p \)-plane. In general, if \( \partial_p \tau |_{p_0} < 0 \) the point \( p_0 \) has to be determined numerically. Figures 4.5(a) and 4.5(c) show two examples of this type of Cagniard contour.

--- The case \( \partial_p \tau |_{p_0} > 0 \) ---

For continuously layered media we can have \( \partial_p \tau |_{p_0} > 0 \). The circumstances under which this shows up are complementary to those mentioned at the previous case, namely
Figure 4.5. Locations of the Cagniard contour and the singularities in the right half of the complex $p$-plane, for the examples $c(x_S^R) = c(x_S^S)$ and $m$ is odd; if $c(x_S^R) \neq c(x_S^S)$ and/or $m$ is even, the situation in which $\tilde{\gamma}^{-m-1}(x_S^R)$ gives rise to a pole does not occur. For cases (b) and (d) a detour around the leftmost branch point has been made.
the horizontal offset must satisfy $r > r_{sep}$ and the parameter $q$ must satisfy $0 \leq q < Q_{sep}$. The first condition can only be met if the maximum wavespeed $c_{\text{max}}$ is reached in a point where $\partial_3 c \neq 0$ and thus $r_{sep}$ remains finite. Consequently, this point must be an endpoint of the interval $[x_3^R; x_3^S]$ or a point where $\partial_3 c$ is discontinuous.

If $\partial_p r|_{p_t} > 0$, the Cagniard contour meets the real $p$-axis tangentially in $p_0 = p_t$. Again $r$ has a minimum in $p_0$ when traveling along the Cagniard contour. Two examples of this type of Cagniard contour are shown in figures 4.5(b) and 4.5(d).

Now that the properties of the Cagniard contours are known, we will actually deform the integration contour into the complex $p$-plane. In the first and fourth quadrants we can form closed loops consisting of the positive or negative imaginary axis, the positive real axis from the origin to $p_0$, the upper ($P^+$) or lower ($P^-$) branches of the Cagniard contour and closing circular arcs at infinity. Applying Cauchy's theorem and Jordan's lemma to these loops, we find that the integration along the imaginary $p$-axis can be replaced by an integration along the Cagniard contour, see figure 4.5. Notice that the integrations along the real $p$-axis cancel each other. If $\partial_p r|_{p_t} < 0$, the deformation process can be performed without special precautions. However, if $\partial_p r|_{p_t} > 0$ the Cagniard contour must be supplemented by a small circle with radius $\varepsilon > 0$ in order to go around the leftmost branch point, see figures 4.5(b) and 4.5(d). Subsequently, we take the limit $\varepsilon \to 0$. Now the contribution of the circle to the total integral over the Cagniard contour depends on the value of the wavespeed $c$ at the receiver level $x_3^R$, and the fact whether $m$ is odd or even. The next two situations are distinguished

- if $c(x_3^R) = c(x_3^S) = c_{\text{max}}$ and $m$ is odd, the leftmost branchpoint coincides with the pole of multiplicity $\frac{1}{2}(m + 1)$ due to $\tilde{\gamma}^{-m-1}(x_3^R)$. This situation is depicted in figure 4.5(d). In this situation the leftmost branch point has a nonzero residue, and thus the circle yields a nonzero contribution;

- if the foregoing condition does not hold, we encircle the leftmost branchpoint, see figure 4.5(b). Since this branch point has a zero residue, the circle gives a vanishing contribution.
The Cagniard contour is symmetrical with respect to the real $p$-axis and the integrand of eq. (4.67) satisfies Schwarz' reflection principle; further the integrand is symmetrical in $q$. Using these symmetry properties, we can rewrite eq. (4.67) as

$$
\hat{T}(m) = \frac{1}{2\pi^2} \int_0^\infty \text{Im} \left\{ \int_{p^+}^{p^+} \frac{\rho^{1/2}(x_3^R) \rho^{1/2}(x_3^R)}{\tilde{q}^{1/2}(x_3^S) \tilde{q}^{m+1/2}(x_3^R)} \exp[-s(pr + \int_{x_3^S}^{x_3^R} \tilde{q} d\zeta)] dp \right\} dq \\
- \frac{1}{2\pi} S(x_3^R, m) \int_0^{Q_{\text{sep}}} \text{Res}_{p = p_\ell} \left\{ \frac{\rho^{1/2}(x_3^S) \rho^{1/2}(x_3^R)}{\tilde{q}^{m+1}(x_3^S)} \exp[-s(pr + \int_{x_3^S}^{x_3^R} \tilde{q} d\zeta)] \right\} dq.
$$

We know that the contribution of the residue must only be taken into account if $\partial_p \tau |_{p_\ell} > 0$ and $c(x_3^S) = c(x_3^S) = c_{\text{max}}$ and $m$ is odd. The first condition is reflected by the fact that in the residue term of the last equation the upper boundary of the integral with respect to $q$ has been limited to $Q_{\text{sep}}$, since for $q > Q_{\text{sep}}$ we find $\partial_p \tau |_{p_\ell} < 0$. The second and third condition has been built into the second term of the last equation by using the function $S(x_3^R, m)$ that satisfies

$$
S(x_3^R, m) = \begin{cases} 
1 & \text{if } c(x_3^S) = c(x_3^S) = c_{\text{max}} \text{ and } m \text{ is odd} \\
0 & \text{otherwise}
\end{cases}.
$$

(4.78)

The second term in eq. (4.77) is set equal to zero if we cannot obtain a value $Q_{\text{sep}} > 0$, i.e., if the horizontal offset satisfies $\tau < r_{\text{sep}}$.

**4.5.4. Replacing the variable of integration $p$ by $\tau$ in the non-residue term of $\hat{T}(m)$**

The Cagniard-De Hoop contour meets the real $p$-axis in the point $p_0$. Progressing along the contour away from $p_0$, the parameter $\tau$ increases monotonically. This means that in first term of eq. (4.77) we can easily replace the integration over the Cagniard contour in the complex $p$-plane by an integration over the real parameter $\tau$. The lowest value of $\tau$ is found in the point $p_0$ and is denoted by $T(q)$. Two situations can be distinguished:

- if $\partial_p \tau |_{p_\ell} < 0$, then $p_0 < p_\ell$. In this situation the value of $p_0$ must be found numerically. Once we have found $p_0$, we can determine $T(q)$ using the equation

$$
T(q) = p_0 r + \int_{x_3^S}^{x_3^R} \left[ c^{-2}(\zeta) - p_0^2 + q^2 \right]^{1/2} d\zeta;
$$

(4.79)
• if $\frac{\partial \tau}{\partial \rho} |_{\rho_t} > 0$, we have $p_0 = p_t = [c_{\text{max}}^{-2} + q^2]^{1/2}$. In this situation eq. (4.79) simplifies to

$$T(q) = [c_{\text{max}}^{-2} + q^2]^{1/2} \tau + \int_{z_3}^{z_3^R} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{1/2} d\zeta.$$  (4.80)

The quantity $T(0)$ is equal to the arrival time of the wave and is therefore denoted by $T_{\text{arr}}$. Again two possibilities exist

• if $r < r_{\text{sep}}$, even for $q = 0$ we have $\frac{\partial \tau}{\partial \rho} |_{\rho_t} < 0$ and the associated value of $p_0$ must be found numerically. Now $T_{\text{arr}}$ follows as

$$T_{\text{arr}} = p_0 \tau + \int_{z_3}^{z_3^R} [c^{-2}(\zeta) - p_0^2]^{1/2} d\zeta; \quad (4.81)$$

• if $r > r_{\text{sep}}$, a value $Q_{\text{sep}}$ exists and for $0 < q < Q_{\text{sep}}$ we have $\frac{\partial \tau}{\partial \rho} |_{\rho_t} > 0$. In this case the value of $p_0$ belonging to $q = 0$ simply equals $p_0 = 1/c_{\text{max}}$, and $T_{\text{arr}}$ is

$$T_{\text{arr}} = \frac{r}{c_{\text{max}}} + \int_{z_3}^{z_3^R} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{1/2} d\zeta. \quad (4.82)$$

The quantity $T(Q_{\text{sep}})$ is denoted by $T_{\text{sep}}$ and equals

$$T_{\text{sep}} = \frac{r^2}{\int_{z_3}^{z_3^R} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{-1/2} d\zeta} + \int_{z_3}^{z_3^R} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{1/2} d\zeta. \quad (4.83)$$

For a typical plot of $T(q)$ versus $q$ we refer to figure 3.7. Upon replacing the integration over $p$ by an integration over $\tau$ in eq. (4.77), we obtain

$$\hat{T}(m) = \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty \text{Im} \left\{ \frac{\rho^{1/2}(x_3^S)}{\tau^{1/2}(x_3^R)} \frac{\rho^{1/2}(x_3^R)}{\tau^{m+1/2}(x_3^R)} \frac{\partial \tau}{\partial \rho} \right\} \exp(-s\tau) d\tau dq$$

$$- \frac{1}{2\pi} S(x_3^R, m) \int_0^Q_{\text{sep}} \text{Res}_{p = p_t} \left\{ \frac{\rho^{1/2}(x_3^S)}{\tau^{m+1/2}(x_3^R)} \frac{\rho^{1/2}(x_3^R)}{\tau^{m+1/2}(x_3^R)} \exp[-s(\tau + \int_{z_3}^{z_3^R} \tau d\zeta)] \right\} dq.$$  (4.84)

Using eqs. (4.68) and (4.71), the residue is found to be

$$\text{Res}_{p = p_t} \left\{ \frac{\rho^{1/2}(x_3^S)}{\tau^{m+1/2}(x_3^R)} \frac{\rho^{1/2}(x_3^R)}{\tau^{m+1/2}(x_3^R)} \exp[-s(\tau + \int_{z_3}^{z_3^R} \tau d\zeta)] \right\} = R(q) \exp(-sT(q)),$$  (4.85)
with
\[ R(q) = \frac{(-1)^{\frac{1}{2}(m-1)}}{(m-1)!} \frac{\rho^{1/2}(x_3^S) \rho^{1/2}(x_3^R)}{\left[\frac{(m-1)}{2}\right]^2 2^m [c^{-2}(x_3^R) + q^2]^{m/2}}, \quad (m = \text{odd}), \]  
(4.86)
and where \( T(q) \) is given by eq. (4.80). The integral with respect to \( q \) in the first term of the right-hand side of eq. (4.77) is interpreted as
\[ \int_0^{\infty} \ldots dq = \int_0^{Q_{\text{sep}}} \ldots dq + \int_{Q_{\text{sep}}}^{\infty} \ldots dq \]  
(4.87)
if different types of Cagniard contours are involved.

### 4.5.5. Replacing the variable of integration \( q \) by \( \tau \) in the residue term of \( \hat{T}(m) \)

We have seen that under circumstances the function \( \hat{T}(m) \) possesses a residue contribution that is given by the second term of eq. (4.84). To apply the Cagniard-De Hoop method we replace the remaining integration over the variable \( q \) by an integration over the real variable \( \tau \) that is given by
\[ \tau = T(q) = [c_{\text{max}}^{-2} + q^2]^{1/2} r + \int_{x_3^S}^{x_3^R} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{1/2} d\zeta. \]  
(4.88)

For \( 0 \leq q < Q_{\text{sep}} \) the function \( T(q) \), and thus the parameter \( \tau \), is a monotonically increasing function, see figure 3.7. The lowest value of \( \tau \) is found at \( q = 0 \) and equals \( T_{\text{arr}} \), while the maximum value of \( \tau \) occurs at \( Q_{\text{sep}} \) and equals \( T_{\text{sep}} \). For \( 0 \leq q < Q_{\text{sep}} \) the function \( \tau = T(q) \) possesses a unique inverse \( q = Q(\tau) \) that follows from eq. (4.88) as
\[ Q(\tau) = \left( \frac{\left\{ \tau - \int_{x_3^S}^{x_3^R} [c^{-2}(\zeta) - c_{\text{max}}^{-2}]^{1/2} d\zeta \right\}^2}{\tau^2} - c_{\text{max}}^{-2} \right)^{1/2}. \]  
(4.89)

The procedure results into
\[ \hat{T}(m) = \frac{1}{2\pi^2} \int_0^{\infty} \int_{T(q)}^{\infty} \text{Im} \left\{ \frac{\rho^{1/2}(x_3^S) \rho^{1/2}(x_3^R)}{\gamma^{1/2}(x_3^S) \gamma^{m+1/2}(x_3^R)} \partial_r p \right\} \exp(-s\tau) d\tau dq \]
\[ - \frac{1}{2\pi} S(x_3^S, m) \int_{T_{\text{arr}}}^{T_{\text{sep}}} R[Q(\tau)] \partial_r Q(\tau) \exp(-s\tau) d\tau, \]  
(4.90)
where
\[
\partial_\tau Q(\tau) = \frac{1}{r} \left( \frac{\tau^2}{c_{max}^2 \left( \tau - \int_{r_{sep}^2}^{r^2} [c^{-2}(\zeta) - c_{max}^{-2}]^{1/2} d\zeta \right)^2} \right)^{-1/2}.
\] (4.91)

4.5.6. Interchanging the order of integration and recognition of the space-time domain function \( \Upsilon(m) \)

Since \( \tau = T(q) \) is a monotonic function in \( q \) for \( q \geq 0 \), it possesses a unique non-negative inverse function \( q = Q(\tau) \) on \( \tau \geq T_{arr} \). The way in which the values of \( Q(\tau) \) are determined depends upon the value of \( \tau \) in the following way

- if there is no value \( T_{sep} > T_{arr} (\tau < r_{sep}) \), the value of \( p_0 \), and thus the value of \( Q(\tau) \), must be found numerically;

- if there is a value \( T_{sep} > T_{arr} (\tau > r_{sep}) \) but \( \tau > T_{sep} \), the value of \( p_0 \), and thus the value of \( Q(\tau) \), must be found numerically;

- if there is a value of \( T_{sep} > T_{arr} (\tau > r_{sep}) \) and \( \tau \) has a value such that \( T_{arr} \leq \tau < T_{sep} \), the value of \( p_0 \) equals \( p_t \), and just as in the previous section \( Q(\tau) \) follows from eq. (4.80) as
\[
Q(\tau) = \left( \frac{\left\{ \tau - \int_{r_{sep}^2}^{r^2} [c^{-2}(\zeta) - c_{max}^{-2}]^{1/2} d\zeta \right\}^2}{\tau^2} - c_{max}^{-2} \right)^{1/2}.
\] (4.92)

The function \( Q(\tau) \) is used when we interchange the order of integrations in the first term of eq. (4.84) according to
\[
\Upsilon(m) = \frac{1}{2\pi^2} \int_{T_{arr}}^{\infty} \int_{0}^{Q(\tau)} \text{Im} \left\{ \frac{\rho^{1/2}(x_3^R) \rho^{1/2}(x_3^S)}{z_1^{1/2}(x_3^S) \gamma^{m+\frac{1}{2}}(x_3^R)} \partial_{\gamma} \right\} \exp(-\sigma \tau) \, dq \, d\tau
- \frac{1}{2\pi} S(x_3^R, m) \int_{T_{arr}}^{T_{sep}} \frac{R[Q(\tau)]}{\partial_\tau Q(\tau)} \exp(-\sigma \tau) \, d\tau.
\] (4.93)

With reference to the forward Laplace transformation of eq. (2.12) and the theory of Laplace transformations we can simply recognize the required space-time domain
function \( T(m) \) as

\[
T(m) = \frac{1}{2\pi^2} H(t-T_{arr}) \int_0^{Q(t)} \text{Im} \left\{ \frac{\rho^{1/2}(x_3^S) \rho^{1/2}(x_3^R)}{\gamma^{1/2}(x_3^S) \gamma^{m+\frac{1}{2}}(x_3^R)} \right\} dq - \frac{1}{2\pi} S(x_3^R, m) H(t-T_{arr}) H(T_{sep}-t) R[Q(t)] \delta, Q(t). \tag{4.94}
\]

### 4.5.7. Determination of the approximate space-time domain acoustic pressure

Once we know the space-time domain function \( T(m) \), eq. (4.58) and the theory of the Laplace transformation show us that the \( N \)-th order approximation of the space-time domain Green’s function \( G(t) \) is given by

\[
G(t) \sim \sum_{n=0}^{N} \sum_{m=n}^{\max(L_1;L_2)} (c_{m,n} + d_{m,n}) H(t) \frac{t^{n-1}}{(n-1)!} *_t T(m). \tag{4.95}
\]

Here, \( *_t \) indicates a convolution with respect to time. Finally, according to the time domain equivalent of eq. (4.57), the space-time domain acoustic pressure \( p \) follows from

\[
p = \partial_t^2 \left[ Q^S(t) *_t G(t) \right], \tag{4.96}
\]

and we will apply the same equation to relate the approximate versions of the space-time domain acoustic pressure and the Green’s function. The differentiation with respect to time may act either on the source signature, or on the Green’s function, or on both, so

\[
p = \partial_t^2 Q^S(t) *_t G(t)
= \partial_t Q^S(t) *_t \partial_t G(t)
= Q^S(t) *_t \partial_t^2 G(t). \tag{4.97}
\]

If we assume that the source signature is known in functional form, as will be the case in this thesis, the first possibility is most convenient.

### 4.6. Applying the Cagniard-De Hoop method in the cases \( x_3^R \perp x_3^{\text{int}} \) and \( x_3^R < x_3^{\text{int}} \)

In the cases \( x_3^R \perp x_3^{\text{int}} \) and \( x_3^R < x_3^{\text{int}} \) the vertical layer between \( x_3^S \) and \( x_3^R \) has completely been located in the upper homogeneous halfspace. Besides the possibility
to employ the less intricate version of the Cagniard-De Hoop method that applies to homogeneous configurations (De Hoop, 1960, 1961, 1988; Van der Hinden, 1987), it turns out that we can analytically perform all steps needed to obtain the inverse transforms of \( \tilde{T}(m) \), \( \tilde{G} \), and, under circumstances, \( \tilde{p} \).

When we substitute eqs. (4.51) and (4.52) in eqs. (4.25) and (4.26), the \( N \)-th order WKBJ asymptotic representations of the wavevector components \( \tilde{w}_1 \) and \( \tilde{w}_2 \) at the receiver level \( x_3^R \leq x_3^{int} \) are found as

\[
\tilde{w}_1(x_3^R, s) \sim \frac{1}{2} \sqrt{2} \tilde{a}_1 \exp(-s \int_{x_3^R}^{x_3^{int}} \gamma \, d\zeta),
\]

\[
\tilde{w}_2(x_3^R, s) \sim \frac{1}{2} \sqrt{2} \tilde{a}_1 \sum_{n=1}^{N} \sum_{m=1}^{N} d_{m,n} \gamma^{-m}(x_3^{int}) s^{-n} \exp(-s \int_{x_3^R}^{x_3^{int}} \gamma \, d\zeta).
\]

In the same way, using eqs. (4.34) and (4.35) we obtain the \( N \)-th order WKBJ asymptotic representations of the wavevector components at a receiver level \( x_3^R < x_3^{int} \) as

\[
\tilde{w}_1(x_3^R, s) \sim \frac{1}{2} \sqrt{2} \tilde{a}_1 H(x_3^R - x_3^{int}) \exp(-s \int_{x_3^R}^{x_3^{int}} \gamma \, d\zeta),
\]

\[
\tilde{w}_2(x_3^R, s) \sim -\frac{1}{2} \sqrt{2} H(x_3^{int} - x_3^R) \exp(-s \int_{x_3^R}^{x_3^{int}} \gamma \, d\zeta)
\]

\[
+ \frac{1}{2} \sqrt{2} \tilde{a}_1 \sum_{n=1}^{N} \sum_{m=1}^{N} d_{m,n} \gamma^{-m}(x_3^{int}) s^{-n} \exp[-s(\int_{x_3^R}^{x_3^{int}} \gamma \, d\zeta + \int_{x_3^{int}}^{x_3^R} \gamma \, d\zeta)].
\]

For the cases \( x_3^R \leq x_3^{int} \) and \( x_3^R < x_3^{int} \) we essentially perform the same manipulations as those carried out at the beginning of subsection 4.5.1. As a result, functions \( \tilde{T}(m) \) of the form

\[
\tilde{T}(m) = \frac{1}{2} s^{-2} \frac{\rho}{\gamma_{-m-1}} \exp(-sz \gamma)
\]

show up [cf. eq. (4.59)]. In this equation, the vertical slowness is given by

\[
\gamma = (c^{-2} + \alpha_1^2 + \alpha_2^2)^{1/2}.
\]

Both the wavespeed \( c \) and the density of mass \( \rho \) that play a role in \( \tilde{T}(m) \) are constant and possess the value that holds in the homogeneous halfspace. The parameter \( z \) is the vertical separation between the source and the receiver, which is always positive.

Proceeding along the lines of the previous section, the space-time domain function \( \tilde{T}(m) \) is found as

\[
\tilde{T}(m) = \frac{1}{2\pi} \frac{H(t - T_{arr})}{\rho} \int_{0}^{Q(t)} \text{Im} \left\{ \frac{\rho}{\gamma_{-m-1}} \partial_{zp} \right\} dq.
\]
This is in fact the same result as given by eq. (4.94), since the residue term is in this case equal to zero. This follows from the fact that in a homogeneous layer $\partial c = 0$ for $c_{\text{max}}$, and thus the Cagniard contour does not meet the leftmost branch point. Unlike the right-hand side in eq. (4.94), the present expression for $T(m)$ can be evaluated analytically. Since the wavespeed is constant, the Cagniard contour now consists of the complex branches of

$$t = pr + z(c^{-2} - p^2 + q^2)^{1/2},$$

which are given by

$$p = \frac{rt}{R^2} \pm i \frac{z}{R^2} [t^2 - T^2(q)]^{1/2}$$

(Cagniard, 1939; De Hoop, 1960). Here,

$$R = (r^2 + z^2)^{1/2}$$

is the total distance between the source and the receiver and

$$T(q) = R(c^{-2} + q^2)^{1/2}$$

is the lowest value of $t$ on the Cagniard contour for a given $q$. If we apply the fact that for the first quadrant part $P^+$ of this Cagniard contour $\partial_t p$ is given by

$$\partial_t p = \frac{iz}{[t^2 - T^2(q)]^{1/2}}$$

(De Hoop, 1960, 1961), and if we introduce a new variable of integration $\psi$ by

$$q = \left(\frac{t^2}{R^2} - c^{-2}\right)^{1/2} \sin \psi$$

(De Hoop, 1961), equation (4.103) turns into

$$T(m) = \frac{1}{2\pi^2 R} H(t - T_{arr}) \int_0^{\pi/2} \text{Re} \left\{ \frac{\rho}{\gamma - m} \right\} d\psi.$$  

(4.111)

From this equation we immediately see that

$$T(0) = \frac{\rho}{4\pi R} H(t - T_{arr}).$$  

(4.112)
Upon finding $\mathcal{T}(1)$, manipulations with the constant wavespeed version of eq. (4.68), and with eqs. (4.106) - (4.108) and (4.110) yield
\[ \bar{\eta} = a - i b \cos \psi, \]  
(4.113)

where
\[ a = \frac{\kappa t}{R^2}, \]  
(4.114)
\[ b = \frac{r}{R^2} \left( t^2 - \frac{R^2}{c^2} \right)^{1/2}. \]  
(4.115)

Now $\mathcal{T}(1)$ follows from
\[ \mathcal{T}(1) = \frac{\rho}{2\pi^2 R} \int_0^{\pi/2} \Re \left\{ \frac{1}{a - ib \cos \psi} \right\} \, d\psi \]
\[ = \frac{\rho}{2\pi^2 R} \int_0^{\pi/2} \frac{a}{a^2 + b^2 \cos^2 \psi} \, d\psi \]
\[ = \frac{\rho}{4\pi R} (a^2 + b^2)^{-1/2}. \]  
(4.116)

Substitution of eqs. (4.114) and (4.115) finally gives
\[ \mathcal{T}(1) = \frac{\rho}{4\pi} \left( t^2 - \frac{r^2}{c^2} \right)^{-1/2}. \]  
(4.117)

When $m \geq 2$, we find the function $\mathcal{T}(m)$ by differentiating both sides of eq. (4.116) $(m - 1)$ times with respect to $a$, followed by a substitution of eqs. (4.114) and (4.115) into the resulting expression. Using a symbolic manipulation program to perform these tasks, the analytical expressions for space-time domain functions $\mathcal{T}(m)$ for all relevant values of $m$ can be derived in an easy manner. In table 4.2 the expressions for $\mathcal{T}(m)$ are listed for $m = 0$ to 4.

Finally, we note that it is possible to analytically perform the convolution of $\mathcal{T}(m)$ with $t^{n-1}/(n-1)!$ that shows up in eq. (4.95). In view of the increasing complexity for growing values of $m$ and $n$ it is convenient to apply a symbolic manipulation program for this analytical convolution. For source signatures $Q^S(t)$ that have a suitable mathematical form, it is even possible to analytically perform the last convolution, i.e., the convolution of the approximate Green's function with the double time derivative of the source signature. This implies that in that case the complete exact inverse transformation process can be performed analytically.
Table 4.2. The expressions for the space-time domain functions $\Upsilon(m)$ for $m = 0$ to 4.

### 4.7. Numerical results

Implementation of the theory developed in this chapter using the symbolic manipulation program MATHEMATICA™ has enabled us to generate numerical results, which will be presented in this section. With the aid of these results we intend to give an impression of the numerical performance of the method that consists of a combination of WKBJ asymptotics and the Cagniard-De Hoop method. The material shown will mainly consist of examples of approximations of the space-time domain Green’s function $G$ and related approximations of the space-time domain acoustic pressure wave $p$. A comparison with results obtained in section 3.8 will be presented as well.

Six different sets of typical continuous wavespeed profiles and mass density profiles have been applied. These sets, called LINEAR1, LINEAR2, EXPONENTIAL, QUADRATIC1, QUADRATIC2 and GAUSSIAN, are shown in figures 4.6 - 4.9. Above the level $z_3^{\text{int}} = 400$ m all parameter profiles are described by constant functions. Below $z_3^{\text{int}}$ the LINEAR1 and LINEAR2 parameter profiles are described by functions of the form $az_3 + b$, while the EXPONENTIAL parameter profiles are represented by functions of the form $f \exp(gz_3)$. Note that up to $z_3 = 1700$ m these parameter profiles are identical to the parameter profiles LINPEAK2, LINPEAK1 and EXPPEAK,
Figure 4.6. The wavespeed and mass density profiles of type LINEAR1 and LINEAR2. In the inhomogeneous halfspace the parameter profiles are described by functions of the form $ax_3 + b$.

Figure 4.7. The wavespeed and mass density profiles of type EXPONENTIAL. In the inhomogeneous halfspace the parameter profiles are described by functions of the form $f \exp(gx_3)$. 
Figure 4.8. The wavespeed and mass density profiles of type QUADRATIC1 and QUADRATIC2. In the inhomogeneous halfspace the parameter profiles are described by functions of the form $ax_3^2 + bx_3 + c$. The QUADRATIC2 profiles have $\partial_3c = 0$ and $\partial_3\rho = 0$ at $x_3^{\text{int}}$.

Figure 4.9. The wavespeed and mass density profiles of type GAUSSIAN. In the inhomogeneous halfspace the parameter profiles are described by functions of the form $f \exp[g(x_3 - 800)^2] + h$ and have $\sigma = 200$ m.
respectively, which have been applied in section 3.8. Below \( z^\text{int}_3 \) the QUADRATIC1 and QUADRATIC2 parameter profiles are described by functions of the form \( az_3^2 + bz_3 + c \). In contrast to the other parameter profiles, the QUADRATIC2 parameter profiles have \( \partial_3 c = 0 \) and \( \partial_3 \rho = 0 \) at \( z^\text{int}_3 \). Finally, below \( z^\text{int}_3 \) the GAUSSIAN parameter profiles are represented by functions of the form \( f \exp[g(x_3 - 800)^2] + h \) having points of inflection at a distance of \( \sigma = 200 \text{ m} \) from the maximum. For all numerical experiments the source and receiver have been located at a level of \( x_3^S = 200 \text{ m} \) and \( x_3^R = 0 \text{ m} \), respectively. For the horizontal offset \( r \) between the source and the receiver the two different values \( r = 0 \text{ m} \) and \( r = 1500 \text{ m} \) have been applied; for each result that will be presented the actual value of \( r \) is indicated. Every approximate acoustic pressure wave \( p \) that will be shown, belongs to a source of volume injection rate with a signature that is described by a four-point optimum Blackman window function (HARRIS, 1978) of unit amplitude and a duration of 0.1 s. In functional form this signature is given by

\[
Q^S(t) = \begin{cases} 
0 & (t \leq 0) \\
\sum_{k=0}^{3} b_k \cos(20k\pi t) & (0 < t \leq 0.1) \\
0 & (t > 0.1) 
\end{cases} \tag{4.118}
\]

with

\[
b_0 = 0.35869 \text{ [m}^3/\text{s}], \tag{4.119}
\]

\[
b_1 = -0.48829 \text{ [m}^3/\text{s}], \tag{4.120}
\]

\[
b_2 = 0.14128 \text{ [m}^3/\text{s}], \tag{4.121}
\]

\[
b_3 = -0.01168 \text{ [m}^3/\text{s}]. \tag{4.122}
\]

A graphical representation of the double time derivative of the source signature is given in figure 4.10. Convolution of this double time derivative with an approximation of the Green's function yields the corresponding approximation of the acoustic pressure wave [cf. eq. (4.97)]. In the results that will be presented, the contribution of the zeroth-order terms of the WKBJ asymptotic representations, i.e., the contribution of the wavefield that is directly generated by the source, has been omitted in order to avoid its dominance over the more interesting contribution of the wavefield that is reflected by the inhomogeneous halfspace. For completeness we note that for all examples with a horizontal offset of \( r = 0 \text{ m} \) occurring in this section the
Figure 4.10. The double time derivative of the signature of the source of volume injection rate used with our numerical experiments. Convolution of this function with a Green's function yields the corresponding acoustic pressure wave.

direct wavefield gives a step function contribution of $0.398 \, H(t - 0.214) \, \text{kg/m}^4$ to the Green's function; in those cases in which the horizontal offset equals $r = 1500$ m this contribution is $0.148 \, H(t - 0.577) \, \text{kg/m}^4$. The highest order of the terms used in our WKBJ asymptotic representations, i.e., the highest order of approximation $N$, is eleven. Since we are dealing with the case $z_3^R < z_3^{\text{int}}$, according to eq. (4.35) we only have to determine the coefficient functions $\tilde{F}_2^{(n)}(z_3^{\text{int}} + 0)$. With reference to section 4.4.2, we set the upper summation bound occurring in eqs. (4.45) - (4.47) equal to $M = N - 1$, which suffices to obtain exact expressions for the coefficient functions $\tilde{F}_2^{(n)}(z_3^{\text{int}} + 0)$. Thus the only approximation that plays a role in the presented results is the omission of terms higher than the order $N$ from the WKBJ asymptotic expansions when forming an $N$-th order WKBJ asymptotic representation. For this section the numerical evaluation of the expressions has been performed with a relative error that is typically less than $0.3\%$. This means that even for the largest values the error in the plots is less than the line thickness of the graphs.
4.7.1. Typical behavior of the approximations of the space-time domain Green's function

The configuration with the LINEAR1 parameter profiles and a horizontal offset of \( r = 0 \) m will be used to show the typical behavior of the approximations of the space-time domain Green's function \( G \) with an increasing order \( N \). We remind that in all the approximations the terms of order zero, representing the contribution of the direct wavefield, have been omitted. In figure 4.11 the approximate Green's functions of order \( N = 1 \) to order \( N = 11 \) have been plotted for \( 0 \leq t \leq 2 \) s. We clearly see that on this time interval the terms of order \( n = 1 \) to order \( n = 4 \) of

![Graph showing the behavior of Green's function for different orders](image)

**Figure 4.11.** The approximate Green's functions for a configuration with parameter profiles of type LINEAR1, with a zero horizontal offset. The approximations of order \( N = 4 \) to order \( N = 11 \) coincide on the whole time interval considered.

the WKBJ asymptotic representations influence the result; the terms of order \( n = 5 \) to order \( n = 11 \) do not visibly alter the approximation obtained for \( N = 4 \). This can be explained by assuming that in the transform domain only large values of the Laplace transform parameter \( s \) are relevant, for which the sum of the first few terms of the WKBJ asymptotic representations forms a good approximation of the final result, and the next term and a number of terms onwards do not significantly alter the approximation. In view of the theorem given in eqs. (4.40) and (4.41),
this implies that in the space-time domain for reasonably small values of $t$ this next term and a number of terms onwards are insignificant. Although it forms no proof, with reference to this reasoning, from now on we take the freedom to assume that a number of almost coinciding subsequent approximations are nearly indistinguishable from the exact result. As a consequence, in figure 4.11 the approximations of order $N = 4$ to order $N = 11$ of the Green's function are assumed to be visibly coinciding with the exact Green's function.

It is interesting to see that the approximation obtained by only taking into account the first-order term is much worse than the result that follows when the first-order term and the second-order term are added. From other numerical experiments (not all of which have been presented here) we have observed that this is almost always the case. Unfortunately, in the majority of cases encountered in the literature at most the first-order term has been incorporated in the WKBJ asymptotic representations.

A typical phenomenon is that approximations that almost coincide for reasonably early times can significantly differ for later times. An example of this is presented in figure 4.12, where we see that the approximations of order $N = 4$ and order $N = 5$, which are almost coinciding for $0 \leq t \leq 2$ s, start to diverge for $t > 2$ s. The explanation is simple: taking into account longer times in the space-time domain implies that in the transform domain smaller values of $s$ become relevant as well. Probably, the transform domain asymptotic series are non-convergent, and the transform domain asymptotic representations that for larger values of $s$ belong to the (quasi)-convergent regime of the associated asymptotic series become part of the divergent regime when $s$ becomes small enough. As a consequence, space-time domain approximations of subsequent order that are nearly equal for early times can differ considerably for later times.

4.7.2. Numerical results for the typical configurations

In figures 4.13 - 4.20 some approximations to the Green's function $G$ and the acoustic pressure wave $p$ have been presented for the configurations with LINEAR1, LINEAR2, EXPONENTIAL and QUADRATIC1 parameter profiles and horizontal offsets of $r = 0$ m and $r = 1500$ m. The approximations of the Green's functions that are shown have been selected by taking from each complete set of approximations (with orders
Figure 4.12. The approximate Green's functions for a configuration with parameter profiles of type LINEAR1, with a zero horizontal offset. The time interval is three times as long as in figure 4.11 and the approximations diverge for $t > 2.5$ s.

$N = 1$ to $N = 11$) only those subsequent approximations that show the least mutual differences on the time interval under investigation. For each approximation of the Green's function the order $N$ is given. Note that for the Green's functions considered the time interval on which the subsequent approximations are visibly equal ranges from $T_{arr}$ to at least $6T_{arr}$ when $r = 0$ m, and from $T_{arr}$ to at least $2T_{arr}$ when $r = 1500$ m.

The approximate acoustic pressure has been determined by using one of the presented approximations of the Green's function. The order $N$ of this specific approximation is denoted on the right side of the graph of the acoustic pressure wave. The applied source signature has a relatively short duration compared to the behavior of the differences between the approximations of the Green's functions. For this reason the differences between the approximate acoustic pressure waves is far less than the differences between the approximate Green's functions. This is why only one approximation of the acoustic pressure wave is given for each configuration. With hindsight, with the present source signature the approximate Green's function that has been applied to determine the approximate acoustic pressure wave could have been replaced by a less accurate approximation without obtaining a sig-
Figure 4.13. The approximate Green's functions for configurations with parameter profiles of type LINEAR1, with horizontal offsets $r = 0$ m and $r = 1500$ m.

Figure 4.14. The approximate acoustic pressure for the configuration with parameter profiles of type LINEAR1, with horizontal offsets of $r = 0$ m and $r = 1500$ m.
Figure 4.15. The approximate Green's functions for configurations with parameter profiles of type LINEAR2, with horizontal offsets \( r = 0 \) m and \( r = 1500 \) m.

Figure 4.16. The approximate acoustic pressure for the configuration with parameter profiles of type LINEAR2, with horizontal offsets of \( r = 0 \) m and \( r = 1500 \) m.
Figure 4.17. The approximate Green's functions for configurations with parameter profiles of type EXPONENTIAL, with horizontal offsets $r = 0 \text{ m}$ and $r = 1500 \text{ m}$.

Figure 4.18. The approximate acoustic pressure for the configuration with parameter profiles of type EXPONENTIAL, with horizontal offsets of $r = 0 \text{ m}$ and $r = 1500 \text{ m}$.
Figure 4.19. The approximate Green's functions for configurations with parameter profiles of type QUADRATIC1, with horizontal offsets $r = 0 \text{ m}$ and $r = 1500 \text{ m}$.

Figure 4.20. The approximate acoustic pressure for the configuration with parameter profiles of type QUADRATIC1 and horizontal offsets of $r = 0 \text{ m}$ and $r = 1500 \text{ m}$. 
nificantly worse approximation for the acoustic pressure wave. Obviously, with a source signature of longer duration, the differences between the approximate acoustic pressure waves would have been larger and the importance of taking an accurate approximation of the Green's function would become more important.

The approximate Green's functions for a configuration with QUADRATIC2 parameter profiles and \( r = 0 \) m are given in figure 4.21. Here we encounter the phenomenon that all approximations with an odd order \( N \) are equal to the approximations with an even order \( N - 1 \). The reason for this is that the QUADRATIC2 parameter profiles have \( \delta_3 c = 0 \) and \( \delta_3 \rho = 0 \) at \( z_3^{\text{int}} \), and thus we find \( \chi = 0 \) there as well. Consequently, in the recurrence scheme given by eqs. (4.20) - (4.24) all coefficient functions \( P_2^{(n)} \) with odd \( n \) are zero, so all odd order terms in the WKBJ asymptotic representations of the transform domain wavevector components are zero as well. Although fewer different approximations are obtained in this case, it is still possible to select a set of subsequent approximations that are almost equal from \( T_{\text{arr}} \) up to approximately \( 6 T_{\text{arr}} \).

For the results presented up to now we may state that the time after which the approximate Green's functions start to diverge is sufficiently large to contain all the interesting features of the Green's function. Unfortunately, this is not always the case, as figure 4.22 shows. Here we see that for a configuration with GAUSSIAN parameter profiles and \( r = 0 \) m the divergence of the approximations starts at, roughly, \( 2 T_{\text{arr}} \), and not all significant features of the Green's function are in the regime where the approximations are nearly equal.

In order to give an impression of the computation speed achieved when using the method presented in this chapter it is noted that on a VAXstation 3100 the determination of the approximate Green's functions with \( N = 1 \) to \( N = 11 \) takes a few minutes. For comparison: on the same machine the determination of the first-order term of the WKBJ iterative solution presented in section 3.8 in general takes several hours.

4.7.3. Numerical results for a configuration without density of mass variations

In the configurations considered up to now the mass density profile has the same shape as the wavespeed profile. In order to show the influence of neglecting the
Figure 4.21. The approximate Green's functions for configurations with parameter profiles of type QUADRATIC2, with a zero horizontal offset.

Figure 4.22. The approximate Green's functions for a configuration with parameter profiles of type GAUSSIAN, with a zero horizontal offset.
variations of the density of mass, we also have performed our calculations for a configuration with the LINEAR1 wavespeed profile, a constant mass density profile of 3000 kg/m and \( r = 0 \) m. The results have been depicted in figure 4.23. We see that the results in this figure differ from the results shown in figure 4.13. Most significant is the fact that for long times the approximate Green's functions tend to go to zero instead of to a constant nonzero value. This is consistent with the behavior of the first order Green's function \( G^{(1)} \) that has been presented in figure 3.45.

![Graph](image)

Figure 4.23. The approximate Green's functions for the configurations with a wavespeed profile of type LINEAR1, a constant density of mass \( \rho = 3000 \) kg/m, and a zero horizontal offset.

4.7.4. Comparison between the results obtained by using WKBJ asymptotics and the WKBJ iterative solution

Now we have available two methods for the determination of the space-time domain acoustic wavefield in horizontally continuously layered media, it is interesting to compare the results generated by both methods. In figures 4.24 and 4.25 we have presented approximate Green's functions of the total reflected acoustic pressure wave as obtained by using WKBJ asymptotics, and the first-order Green's function \( G^{(1)} \) of the WKBJ iterative solution. The results presented in both figures belong to con-
Figure 4.24. Comparison between the approximate Green's functions as obtained by using WKBJ asymptotics (solid line), and the first-order Green's function as follows from the WKBJ iterative solution (dashed line). The results belong to configurations with parameter profiles of type LINEAR1, with horizontal offsets $r = 0$ m and $r = 1500$ m, respectively.

Figure 4.25. Comparison between the approximate Green's functions as obtained by using WKBJ asymptotics (solid line), and the first-order Green's function as follows from the WKBJ iterative solution (dashed line). The results belong to configurations with parameter profiles of type LINEAR2, with horizontal offsets $r = 0$ m and $r = 1500$ m, respectively.
figurations with parameter profiles of the type $\text{LINEAR1}$ and $\text{LINEAR2}$, respectively, and in each case the horizontal offsets $r = 0 \text{ m}$ and $r = 1500 \text{ m}$ have been considered. No comparison has been made for $r = 5000 \text{ m}$. For this value of the horizontal offset and the $\text{LINEAR1}$ and the $\text{LINEAR2}$ parameter profiles, it often takes less time for the reflection from the inhomogeneous halfspace to arrive at the receiver than it takes to travel from the source via the boundary of the inhomogeneous halfspace to the receiver (cf. section 3.8). Since the exponential function in second term of the WKBJ asymptotic representation in eq. (4.35) gives rise to the latter time delay, this WKBJ asymptotic representation is not applicable in this case. It turns out that if it is still applied, meaningless results are generated. We recall that in both cases the inverse transformation method is exact and that the differences between the results are only caused by using different methods to solve the transform domain problem.

In all cases the results for the first-order reflections as given by the WKBJ iterative solution are larger than the approximations for the total reflected wavefield according to the WKBJ asymptotic representations. This can be explained by the fact that the higher-order terms of the WKBJ iterative solution have been omitted. In the present case the most significant of these omitted terms is the third-order term, which would have decreased the value of the total result [cf. eq. (3.13)]. On the time intervals considered the differences between the results obtained by using both methods is less than $5 \%$. From this we may infer that the higher-order terms of the WKBJ iterative solution are rather insignificant as compared to the first-order term. i.e., the first-order reflections are highly dominant in these cases. This is consistent with the fact that the inhomogeneity function $\chi$, which determines the amplitude of the partial reflections, is small (a maximum value of $\chi = 5.128 \cdot 10^{-4}$ is obtained for a configuration with $\text{LINEAR1}$ parameter profiles and a perpendicular incident plane wave having $\alpha_1 = \alpha_2 = 0$). As expected, the absolute differences between the results are smaller for the configurations with $\text{LINEAR1}$ profiles than for the configurations with $\text{LINEAR2}$ profiles.

4.8. Discussion

In this chapter the integral equation for the transform domain wavevector has been solved in an approximate manner using higher-order WKBJ representations around
the point infinity of the Laplace transform parameter. Firstly, WKBJ asymptotic
expansions of the transform domain wavevector components have been derived. As
a consequence of our transformation scheme, which involves the temporal Laplace
transformation with a real and arbitrarily large positive transformation parameter,
the existence of WKBJ asymptotic expansions in inverse powers of the temporal
Laplace transform parameter can be proved; neither a breakdown of the WKBJ
asymptotics caused by a zero of the vertical slowness does occur, nor problems im-
posed by the crossing of a Stokes line (ERDÉLYI, 1956; WASOW, 1965) arise. Once
the WKBJ asymptotic expansions have been determined, approximations of the
transform domain wavevector components have been derived in the form of N-th
order WKBJ asymptotic representations, consisting of terms up to order N. The
coefficient functions that occur in the WKBJ asymptotic representations satisfy a
recurrence scheme, and we have shown how this recurrence scheme can be evalu-
ated with the aid of symbolic manipulation. The inverse transformation of the
approximate transform domain wavefield quantities has been performed using the
Cagniard-De Hoop method. Since one and the same exponential function occurs
in all terms of the transform domain WKBJ asymptotic representations, an effi-
cient use of the Cagniard-De Hoop method is achieved. For the case in which the
source and the receiver are located in the same homogeneous halfspace, a substantial
part of the steps in the Cagniard-De Hoop method has been performed analytically.
The integral transformation method that has been presented leads to higher-order
eyrly-time asymptotic representations of the space-time domain acoustic wavefield.

Physically, the terms of order zero represent the wavefield that is directly gener-
ated by the source, while the sum of the higher-order terms represents the wavefield
that is generated by the reflection of the direct wavefield by the inhomogeneous fluid.
The WKBJ paradox has been discussed, and it has turned out that this paradox
does not show up in our case.

Various numerical results have been generated for the approximate space-time
domain Green’s function and the related approximate space-time domain acoustic
pressure. The results show that, starting with the arrival of the reflected wavefield,
there is a time interval on which the higher-order approximations of the Green’s
function nearly coincide. We have assumed that these nearly coinciding approxi-
mations are almost equal to the exact Green’s function. Beyond this interval we
have a regime where the subsequent approximations diverge. The point separating both intervals depends on the parameter profiles and the position of the source and receiver in the configuration. By incorporating higher-order terms in the approximate Green's function, we have obtained results that are in general better than the low-order results obtained in the literature. Since we have applied an exact inverse transformation method it has been possible to purely investigate the influence of the application of the WKBJ asymptotic representations in comparison with the first-order term of the WKBJ iterative solution. For those configurations for which a comparison has taken place, a good agreement between the results of both kinds of results has been observed. A major benefit of the application of WKBJ asymptotics in combination with the Cagniard-De Hoop method is that the resulting computer implementation is very fast as compared to other methods.

Although the method gives the best results in the most interesting part of the time axis, i.e., the interval directly following the arrival of the reflected wavefield, the subsequent space-time domain approximations diverge for later times and the method is not generally suitable to predict the late time behavior of the space-time domain wavefields. This behavior is a fundamental property of the present integral transformation method. Another drawback of the method presented in this chapter is that the assumption that almost coinciding approximations are nearly equal to the exact wavefield is made plausible using a reasoning that follows a numerical observation, but that does not form a mathematical proof. However, observation of the numerical results raises the question whether for each configuration there exists a characteristic time instant, before which the subsequent space-time domain approximations converge absolutely. If this is the case, the assumption that nearly coinciding subsequent approximations are almost equal to the exact solution is justified for the time interval ranging from the arrival of the reflected wavefield up to this characteristic time instant. Further, in its present form the method can only be applied to continuously layered configurations consisting of a homogeneous halfspace and an inhomogeneous halfspace; the medium parameters of the inhomogeneous halfspace must be infinitely often differentiable, and they are connected to the medium parameters of the homogeneous halfspace in a continuous manner. This difficulty can be overcome by firstly solving the transform domain problem using the WKBJ iterative solution (which converges for all continuously layered configurations), and
secondly deriving asymptotic representations by integrating the terms of the WKBJ iterative solution by parts. Finally, it is noted that the method cannot be applied if the reflection from an inhomogeneous halfspace requires less time to arrive at the receiver than needed to travel from the source, via the boundary of the inhomogeneous halfspace, to the receiver. The latter delay equals the time delay introduced by the exponential function in the relevant WKBJ asymptotic representation.

A number of the shortcomings mentioned above will not impose a severe constraint on the applicability of the method. In this respect we mention the divergence of the approximations for later times, the lack of the mathematical proof relating the subsequent approximations and the exact solution, and the problems that might occur at high horizontal offsets. Although the restriction of the class of continuously layered media will impose a certain limitation on its application, in its present form the method can be useful for the generation of synthetic seismograms for testing purposes, and the calculation of the wavefield in background media or macro models that are used with imaging and inversion methods.
Appendix to chapter 4

4.A. Proof of the existence of the asymptotic expansions of $\tilde{P}_1$, $\tilde{P}_2$ and $\partial_3 \tilde{P}_1$, $\partial_3 \tilde{P}_2$ for $x_3 > x_3^{\text{int}}$

In this appendix we prove for $x_3 > x_3^{\text{int}}$ the existence of the asymptotic expansions of the functions $\tilde{P}_1$ and $\tilde{P}_2$ and their derivatives $\partial_3 \tilde{P}_1$ and $\partial_3 \tilde{P}_2$ in inverse integer powers of the Laplace transformation parameter $s$. The functions $\tilde{P}_1$ and $\tilde{P}_2$ are the solutions of the integral equations

\begin{align*}
\tilde{P}_1 &= \int_{x_3^{\text{int}}}^{x_3} \chi \tilde{P}_2 \, dx_3' + 1, \quad (4.A.1) \\
\tilde{P}_2 &= -\int_{x_3}^{\infty} \chi \exp(-2s \int_{x_3}^{x_3'} \gamma \, d\zeta) \, \tilde{P}_1 \, dx_3' \quad (4.A.2)
\end{align*}

[cf. eqs. (4.10) and (4.11)]. In the proof that will follow the fact that $x_3 > x_3^{\text{int}}$ is understood and will not be mentioned explicitly.

— The asymptotic expansions of $\tilde{P}_1$ and $\tilde{P}_2$ —

To prove the existence of the asymptotic expansions of $\tilde{P}_1$ and $\tilde{P}_2$ in inverse powers of $s$, we recall that these functions are a solution of the coupled integral equations (4.A.1) and (4.A.2). According to appendix 3.A and eqs. (2.25), (2.39), (4.1) and (4.2), these solutions can be determined using the absolutely convergent Neumann series

\begin{align*}
\tilde{P}_1(x_3) &= \sum_{i=0}^{\infty} \tilde{Q}_1^{(i)}(x_3), \quad (4.A.3) \\
\tilde{P}_2(x_3) &= \sum_{i=0}^{\infty} \tilde{Q}_2^{(i)}(x_3), \quad (4.A.4)
\end{align*}

with

\begin{align*}
\tilde{Q}_1^{(0)} &= 1, \quad (4.A.5) \\
\tilde{Q}_2^{(0)} &= 0, \quad (4.A.6)
\end{align*}
and

$$
\tilde{Q}_1^{(i)} = \int_{z_3}^{z_2} \chi \tilde{Q}_2^{(i-1)} \, dx_3',
$$

(4.A.7)

$$
\tilde{Q}_2^{(i)} = - \int_{z_3}^{z_2} \chi \exp(-2s \int_{z_2}^{z_3} \gamma d\zeta) \tilde{Q}_1^{(i-1)} \, dx_3'.
$$

(4.A.8)

It is sufficient to prove that the individual terms $\tilde{Q}_1^{(i)}$ and $\tilde{Q}_2^{(i)}$ possess asymptotic expansions in inverse powers of $s$; the sums of these asymptotic expansions then yield the asymptotic expansions of $\tilde{P}_1$ and $\tilde{P}_2$ (ERDÉLYI, 1956, p. 19).

The proof that $\tilde{Q}_1^{(i)}$ and $\tilde{Q}_2^{(i)}$ possess an asymptotic expansion in inverse powers of $s$ is carried out by induction. Suppose that $\tilde{Q}_1^{(i-1)}$ and $\tilde{Q}_2^{(i-1)}$ both have an asymptotic expansion in inverse powers of $s$. Then $\tilde{Q}_1^{(i)}$ and $\tilde{Q}_2^{(i)}$ have an asymptotic expansion in inverse powers of $s$ as well. The proof of this statement consists of two parts. Firstly, we observe that the asymptotic expansion of $\tilde{Q}_1^{(i)}$ follows from substitution of the assumed asymptotic expansion

$$
\tilde{Q}_2^{(i-1)}(x_3) = \sum_{k=0}^{N} s^{-k} q_2^{(k)}(x_3) + O(s^{-(N+1)}), \quad (s \to \infty),
$$

(4.A.9)

in eq. (4.A.7). This yields

$$
\tilde{Q}_1^{(i)}(x_3) = \int_{z_3}^{z_2} \chi \left[ \sum_{k=0}^{N} s^{-k} q_2^{(k)} + O(s^{-(N+1)}) \right] \, dx_3', \quad (s \to \infty).
$$

(4.A.10)

We are allowed to interchange the summation and integration (ERDÉLYI, 1956, p. 16). This gives

$$
\tilde{Q}_1^{(i)}(x_3) = \sum_{k=0}^{N} s^{-k} \int_{z_3}^{z_2} \chi q_2^{(k)} \, dx_3' + O(s^{-(N+1)}), \quad (s \to \infty),
$$

(4.A.11)

which proves that $\tilde{Q}_1^{(i)}$ has an asymptotic expansion in inverse powers of $s$. Secondly, the asymptotic expansion of $\tilde{Q}_2^{(i)}$ follows from substitution of the assumed asymptotic expansion

$$
\tilde{Q}_1^{(i-1)}(x_3) = \sum_{k=0}^{N} s^{-k} q_1^{(k)}(x_3) + O(s^{-(N+1)}), \quad (s \to \infty),
$$

(4.A.12)
in eq. (4.1.8). This equation can then be written as

\[
\bar{Q}_2^{(i)} = - \int_{x_3}^{0} \chi \exp(-2s \int_{x_3}^{x_3'} y \, d\zeta) \left[ \sum_{k=0}^{N} s^{-k} q_{1}^{(k)} + O(s^{-(N+1)}) \right] \, dx_3' \\
= - \sum_{k=0}^{N} s^{-k} \int_{x_3}^{0} \chi q_{1}^{(k)} \exp(-2s \int_{x_3}^{x_3'} y \, d\zeta) \, dx_3' + O \left[ s^{-(N+1)} \int_{x_3}^{0} \chi \exp(-2s \int_{x_3}^{x_3'} y \, d\zeta) \, dx_3' \right], \quad (s \to \infty). \tag{4.1.13}
\]

Next, the substitution \( \gamma = 2 \int_{x_3}^{x_3'} \chi \, d\zeta \) is performed. Provided that the parameter profiles have sufficiently high order derivatives, it follows from a repeated integration by parts that the integrals with respect to \( x_3' \) in eq. (4.1.13) have an asymptotic expansion in powers of \( s \) starting with \( s^{-1} \). As a consequence, \( \bar{Q}_2^{(i)} \) will have an asymptotic expansion in inverse powers of \( s \) as well. Since both \( \bar{Q}_2^{(0)} \) and \( \bar{Q}_2^{(0)} \) have (trivial) asymptotic expansions in inverse powers of \( s \), we have proven the existence of asymptotic expansions in inverse powers of \( s \) for \( \bar{Q}_2^{(i)} \) and \( \bar{Q}_2^{(i)} \). Note that this proof is valid for all kinds of parameter profiles for \( x_3 > x_3^{\text{int}} \) (either represented by algebraic or transcendental functions), as long as they possess sufficiently high continuous derivatives; the proof breaks down at the moment that a discontinuity of a parameter profile is encountered for \( x_3 > x_3^{\text{int}} \).

---

The asymptotic expansions of \( \partial_3 \bar{P}_1 \) and \( \partial_3 \bar{P}_2 \) ---

Substitution of the asymptotic expansions for \( \bar{P}_1 \) and \( \bar{P}_2 \) in

\[
\partial_3 \bar{P}_1 = \chi \bar{P}_2, \quad \tag{4.1.14}
\]
\[
\partial_3 \bar{P}_2 = 2s \gamma \bar{P}_2 + \chi \bar{P}_1, \quad \tag{4.1.15}
\]

where \( \gamma \) and \( \chi \) are bounded functions, yields the asymptotic expansions for \( \partial_3 \bar{P}_1 \) and \( \partial_3 \bar{P}_2 \). Although the first term on the right-hand side of eq. (4.1.15) is proportional to \( s \), the asymptotic expansion of \( \partial_3 \bar{P}_2 \) starts with a term of order \( s^0 \). This is due to the fact that the asymptotic expansion of \( \bar{P}_2 \) starts with a term that is proportional to \( s^{-1} \) since \( \bar{Q}_2^{(0)} = 0 \).
Chapter 5
Conclusions

In this chapter we present some conclusions with respect to both integral transformation methods that have been developed in this thesis. In general, the features of both methods strongly depend on the mathematical methods that we have selected for their three basic steps. Our choice of the temporal Laplace transformation with a real and positive transformation parameter has an essential influence on the further theory of both methods (e.g., no turning point problems arise in the transform domain). Our first integral transformation method is applicable to every horizontally continuously layered configurations and can in principle be used to obtain sufficiently accurate results for the acoustic wavefield at any time instant. The method is numerically robust, but a drawback of the method is that the numerical effort needed for the evaluation of the higher-order terms will be large. Our second integral transformation method yields the higher-order early-time behavior of the acoustic wavefield. The computer implementation of this method is fast and gives subsequent approximations that nearly coincide on a time interval of nonzero length beyond the arrival time.
5.1. The overall approach

In this thesis we have presented two integral transformation methods for the solution of the space-time domain acoustic wave propagation problem in a horizontally continuously layered and isotropic configuration. The way in which the integral transformation methods have been developed is in both cases the same. First the aims of the method to be developed have been stated and the mathematical methods for the basic steps of the method have been selected. Next, the theory has been developed. After the numerical implementation of the theory, numerical results have been generated, showing the numerical performance of the method. As expected, the features of both integral transformation methods strongly depend on the mathematical methods that have been selected to perform their basic steps.

The first step of our integral transformation methods, being the forward transformation of the space-time domain basic acoustic equations, is in both cases the same, and consists of the temporal Laplace transformation and the spatial Fourier transformation. The fact that during our analysis we keep the integration parameter of the temporal Laplace transformation real and positive, has proven to be important in view of later parts of the theory (e.g., no turning point problems arise in the transform domain, as would be the case with the temporal Fourier transformation with a real transformation parameter).

5.2. Combination of the WKBJ iterative solution and the Cagniard-De Hoop method

Our first aim has been to develop an integral transformation method by which it is in principle possible to obtain sufficiently accurate results for the acoustic wavefield at any time instant and in every continuously layered configuration. This has lead to the selection of the WKBJ iterative solution (Neumann series solution) for solving the transform domain problem, and the application of the Cagniard-De Hoop method (slowness method with a complex slowness contour) for the transformation back to the space-time domain. It has turned out that with these choices, our goal can indeed be met: we have been able to prove the convergence of the transform domain solution, as well as the space-time domain solution, for every continuously layered configuration. This implies that in principle results of sufficient accuracy
can be obtained by taking into account a sufficient number of terms of the WKBJ iterative solution. By employing the numerical implementation for the zeroth-order term and the first-order term of the space-time domain solution, we have shown that from a numerical point of view the method is robust.

Some additional items have been investigated as well. As a first result, we mention a physical interpretation of the WKBJ iterative solution in terms of upgoing and downgoing waves. A second result is that the paths of the rays (minimum travel time trajectories between a number of levels) have been analyzed. It has become clear that in our case a turning ray is a special case of a ray with a reflection point, and that the rays can have horizontal trajectories of nonzero length.

A drawback of the method derived is that the numerical evaluation of the higher-order terms of the space-time domain solution becomes increasingly difficult. This is caused by the fact that the inverse transformation must act on the individual integrands of the multiple integrals that form the higher-order terms of the transform domain solution. As a result of this, the numerical convergence of the space-time domain solution can hardly be checked. It has been stated that in order to avoid this problem, another integral transformation method must be derived, starting with the second step, in which a more suitable transform domain solution is obtained. Adaptation of the theory to other kinds of waves (elastodynamic waves or electromagnetic waves) forms a possible subject of further research.

5.3. Combination of WKBJ asymptotics and the Cagniard-De Hoop method

During the development of our second integral transformation method, the aim has been to determine higher-order early-time representations of the wavefield right after the arrival time. Moreover, we have assumed that these representations lead to approximations of the wavefield over a time interval of nonzero length beyond the arrival time. In view of this, we have chosen higher-order WKBJ asymptotics around the point infinity of the Laplace transformation parameter for the determination of subsequent approximations of the transform domain solution. The Cagniard-De Hoop method (slowness method with a complex slowness contour) has been selected to perform the transformation of these approximations back to the space-time domain. We have shown that with these choices, our goal has been met for
the restricted class of horizontally continuously layered fluid-filled configurations that are homogeneous above a certain level and that are inhomogeneous below this level, and that have discontinuous derivatives of the medium parameters at this level only. The method is mathematically straightforward, and we have been able to use the Cagniard-De Hoop method in an efficient way. We have shown that the numerical implementation of the theory can profitably be performed using a symbolic manipulation program. Numerical results have shown that, for a time interval of nonzero length beyond the arrival time of the wavefield, we obtain nearly coinciding subsequent approximations; these have been considered as a good approximation of the exact result. The computer implementation is very fast in comparison with other integral transformation methods. For configurations that have been investigated, a good agreement between the results of both integral transformation methods has been observed.

Further, some additional items have been investigated. To start with, we mention that the physical interpretation of the zeroth-order term that has been obtained by WKBJ asymptotics represents the wave that is directly generated by the source; the sum of the higher-order terms that play a role in the WKBJ asymptotics forms a correction of the wavefield after the arrival of this direct wave. Further, during a discussion of the WKBJ paradox it has turned out that this paradox does not apply in our case. Moreover, the three steps have been presented that lead to the numerical implementation of the theory using a symbolic manipulation program.

Unfortunately, it has turned out that the method possesses some shortcomings as well. First of all, the method is not suitable for the determination of the wavefield at late time instants, since the subsequent approximations diverge there. This seems to be a fundamental property of the method. Moreover, there is as yet no strict mathematical proof that nearly coinciding subsequent approximations are good approximations to the exact solution. Another point is that in its present form the method can only deal with a restricted class of continuously layered configurations. Finally, the method cannot be applied in certain configurations with a high horizontal offset, i.e., if both the source and the receiver have been located in the homogeneous halfspace, and the time needed for the reflections from the inhomogeneous halfspace to arrive is less than the time needed to travel from the source, via the boundary of the inhomogeneous halfspace, to the receiver.
CONCLUSIONS

A number of difficulties sketched above, e.g., the divergence of the approximations for later times, the lack of the mathematical proof relating the subsequent approximation and the exact solution, and the problems that might occur at high horizontal offsets, will not form a severe constraint on the applicability of the method. On the other side, the restriction on the class of continuously layered media will impose a certain constraint on its application. However, a straightforward solution of this problem seems possible. In its present form the method is already suitable for the generation of synthetical seismograms that can be used for testing purposes, and for calculation of the wavefield in background media or macro models that are used with imaging and inversion methods. Investigation of the difficulties related to the method and the application of the theory to other kinds of waves (elastodynamic waves or electromagnetic waves) and to configurations with less simple media (including losses and/or anisotropy) form possible subjects for further research.
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TRANSIENT ACOUSTIC WAVES IN CONTINUOUSLY LAYERED MEDIA
Summary

In the first chapter of this thesis it is explained how physical fields of different nature and time behavior are used as a tool for the determination of the distribution of one or more medium parameters that characterize the materials inside the Earth. From these medium parameter distributions, a prediction can be made about the composition of the Earth and, as in the case of exploration geophysics, about the possible location of natural resources (oil, gas, minerals, ores, water). However, in order to deduce the distribution of medium parameters from a measured field, it is often necessary to be able to determine the field in a known configuration. The subject of this thesis is the development of methods for the determination of the space-time domain acoustic wavefield in an horizontally continuously layered and isotropic medium; although there are situations in which such methods can be useful, up till now these are rare. Since the configuration is shift invariant in the horizontal directions as well as in the time, integral transformation methods are considered suitable for this. Each integral transformation method consists of three basic steps: the transformation of the space-time domain problem using integral transformations, the solution of the resulting transform domain problem, and the transformation of this solution back to the space-time domain. For each of these steps several mathematical methods exist, from which we make specific choices in this thesis in order to derive two complete integral transformation methods.

In chapter 2 the continuously layered configuration that will be used in our investigations is described in detail. Next, the space-time domain linearized equations for the acoustic wavefield quantities (the particle velocity and the acoustic pressure) are given. These basic acoustic equations form a system of coupled partial differential equations in which derivatives with respect to the three spatial coordinates and the time occur. As the first step of both integral transformation methods to be devel-
oped, we reduce the number of derivatives in this system by applying a one-sided Laplace transformation with respect to the time and a double-sided Fourier transformation with respect to both horizontal directions. As a result, the dimension of the problem is reduced to one by parametrization, and an ordinary differential equation for the acoustic state vector (with the transformed vertical particle velocity and the transformed acoustic pressure as its components) is obtained. By subjecting the acoustic state vector to a linear transformation, the acoustic wavevector is arrived at, whose components represent either downwardly or upwardly propagating waves. Subsequently, the wavevector differential equation and the equivalent wavevector integral equation are derived. It is of fundamental importance that in this thesis we apply a real and positive Laplace transform parameter, since in this way many difficulties are avoided that are commonly encountered during the further analysis (e.g., the occurrence of turning points in the transform domain differential equations).

In chapter 3 a first complete integral transformation method is presented. The first step, i.e., the transformation of the space-time domain problem, has already been performed in the chapter 2. As the second step, in this chapter the transform domain equivalent of the wave propagation problem in a horizontally continuously layered configuration is solved by means of the WKBJ iterative solution (Neumann series) of the wavevector integral equation. Due to the chosen transformation scheme, we can prove the convergence of the transform domain solution for every horizontally continuously layered configuration. Moreover, we show that this implies the convergence of the space-time domain solution for any time instant as well. Thus, it is in principle possible to obtain sufficiently accurate results for every horizontally continuously layered configuration and for any time instant. The transformation of the derived solution back to the space-time domain forms the third step of the present integral transformation method. This step is performed by applying the Cagniard-De Hoop method to the individual terms of the transform domain solution. It is shown how the Cagniard-De Hoop method is adapted in order to deal with continuously layered configurations. Further, the numerical implementation of the zeroth-order term and the first-order term of the space-time domain solution is discussed and for these terms various numerical results are presented.

In chapter 4 a second integral transformation method is described. The first step, i.e., the transformation of the space-time domain problem, has already been
performed in chapter 2. As the second step, in this chapter the transform domain equivalent of the wave propagation problem is solved in an approximate manner by using higher-order WKBJ asymptotic representations of the solution of the wavevector integral equation. These higher-order WKBJ asymptotic representations are valid around the point infinity of the Laplace transform parameter, i.e., they will lead to early-time asymptotic results in the space-time domain. In the space-time domain, we obtain approximations of the wavefield over a time interval of nonzero length beyond the arrival time. Due to the chosen transformation scheme, we will avoid complications that are frequently observed with WKBJ asymptotics, such as a breakdown of WKBJ asymptotic expansions at a zero of the vertical slowness or at the crossing of a Stokes line. The transformation of the approximate solutions back to the space-time domain forms the third step of the present integral transformation method. We perform this step by applying the Cagniard-De Hoop method to each term of the transform domain WKBJ asymptotic representations. Further, we show how the theory is implemented with the aid of a symbolic manipulation program. At the end, various numerical results for the approximations of the space-time domain wavefield are presented.

In the final chapter we present some conclusions with respect to both integral transformation methods that have been developed in this thesis. In general, the features of both methods strongly depend on the mathematical methods that we have selected for their three basic steps. Our choice of the temporal Laplace transformation with a real and positive transformation parameter has an essential influence on the further theory of both methods. Our first integral transformation method is applicable to every horizontally continuously layered configurations and can in principle be used to obtain sufficiently accurate results for the acoustic wavefield at any time instant. The method is numerically robust, but a drawback of the method is that the numerical effort needed for the evaluation of the higher-order terms will be large. Our second integral transformation method yields the higher-order early-time behavior of the acoustic wavefield. The computer implementation of this method is fast and gives subsequent approximations that nearly coincide on a time interval of nonzero length beyond the arrival time.
Transiënte akoestische golven in continu gelaagde media
Samenvatting

In het eerste hoofdstuk van dit proefschrift wordt uitgelegd hoe fysische velden van verschillende aard en met verschillend tijdgedrag worden gebruikt om de verdeling van één of meer mediumparameters, die de materialen in de aarde karakteriseren, te bepalen. Uit deze verdeling van mediumparameters kan een voorspelling worden gemaakt omtrent de samenstelling van de aarde en, zoals bij de exploratiegeophysica, de mogelijke lokatie van natuurlijke hulpbronnen (olie, gas, mineralen, erts, water). Om uit een gemeten veld de verdeling van mediumparameters af te leiden, is het vaak noodzakelijk om het veld in een bekende configuratie te kunnen bepalen. Het onderwerp van dit proefschrift is het ontwikkelen van methoden voor het bepalen van het ruimte-tijddomein akoestische golfveld in een horizontaal continu gelaagd en isotroop medium; ondanks dat er zich situaties voordoen waarin zulke methoden hun nut kunnen bewijzen, zijn deze methoden tot op heden vrij zeldzaam. Omdat de configuratie invariant is in zowel de horizontale ruimtelijke richtingen als in de tijd, is voor de oplossing hiervan uitgegaan van de zogenaamde integraaltransformatie-methoden. Elke integraaltransformatie-methode bestaat uit drie basisstappen: transformatie van het ruimte-tijddomein probleem met behulp van integraaltransformaties, oplossing van het resulterende probleem in het getransformeerde domein, en terugtransformatie van deze oplossing naar het ruimte-tijddomein. Voor elk van deze stappen bestaan verscheidene wiskundige methoden, waaruit verder in dit proefschrift specifieke keuzen worden gemaakt teneinde twee complete integraaltransformatie-methoden te realiseren.

In hoofdstuk 2 wordt in detail de continu gelaagde configuratie, die in het verdere onderzoek wordt gebruikt, beschreven. Vervolgens worden de gelinearizeerde ruimte-tijddomein vergelijkingen voor de grootheden van het akoestische golfveld gegeven. Deze akoestische basisvergelijkingen vormen een stelsel gekoppelde partiële diffe-
rentiaalvergelijkingen waarin afgeleiden naar de drie ruimtelijke coördinaten en de tijd voorkomen. Als eerste stap van de beide integraaltransformatie-methoden die zullen worden ontwikkeld, wordt het aantal afgeleiden in dit stelsel gereduceerd door toepassing van de enkelsljdige Laplacetransformatie met betrekking tot de tijd en de dubbelzijdige Fouriertransformatie met betrekking tot de horizontale coördinaten. Op deze manier wordt de dimensie van het probleem door middel van parametrisatie tot één gereduceerd, en wordt een gewone differentiaalvergelijking voor de akoestische toestandsvektor (met als componenten de getransformeerde vertikale deeltjesnelheid en de getransformeerde akoestische druk) verkregen. Door vervolgens een lineaire transformatie op de akoestische toestandsvektor toe te passen, verkrijgen wij de akoestische golfvektor, waarvan de componenten een neergaande of een opgaande golf voorstellen. Hierna worden de differentiaalvergelijking en de hiermee equivalentte integraalvergelijking voor de akoestische golfvektor afgeleid. Het is zeer belangrijk dat wij in dit proefschrift een reële en positieve Laplacetransformatieparameter toepassen, omdat hiermee in de verdere analyse vaak voorkomende problemen (zoals het aanwezig zijn van "turning points" in de differentiaalvergelijking in het getransformeerde domain) worden vermijden.

In hoofdstuk 3 wordt een eerste complete integraaltransformatie-methode gepresenteerd. De eerste stap, de transformatie van het ruimte-tijddomein probleem, is reeds uitgevoerd in hoofdstuk 2. Als tweede stap wordt in dit hoofdstuk de getransformeerde versie van het golfpropagatieprobleem in een horizontaal continu gelaagd medium opgelost door gebruik te maken van de WKBJ iteratieve oplossing (Neumannreeks) van de integraalvergelijking voor de golfvektor. Door het eerder gekozen transformatieschema kunnen wij bewijzen dat de oplossing in het getransformeerde domein convergent is voor elke horizontaal continu gelaagde configuratie. Bovendien houdt dit in dat de oplossing in het ruimte-tijddomein convergent is voor elk tijdstip. Zodoende is het in principe mogelijk om voldoende nauwkeurige resultaten te verkrijgen voor elke horizontaal continu gelaagde configuratie en voor elk tijdstip. De terugtransformatie van de gevonden oplossing naar het ruimte-tijddomein is de derde stap van deze integraaltransformatie-methode. Deze stap wordt uitgevoerd door de Cagniard-De Hoop-methode toe te passen op de individuele termen van de oplossing in het getransformeerde domein. Getoond wordt hoe de Cagniard-De Hoop-methode kan worden aangepast voor gebruik in continu gelaagde
configuraties. Vervolgens is de numerieke implementatie van de nulde- en eerste-orde term van de ruimte-tijddomein oplossing behandeld, en zijn verschillende numerieke resultaten voor deze termen gepresenteerd.

In hoofdstuk 4 wordt een tweede complete integraaltransformatie-methode gepresenteerd. De eerste stap, de transformatie van het ruimte-tijddomein-probleem, is reeds uitgevoerd in hoofdstuk 2. Als tweede stap wordt in dit hoofdstuk de getransformeerde versie van het golfpropagatieprobleem benaderend opgelost met behulp van hogere-orde WKBJ asymptotische representaties van de oplossing van de integraalvergelijking voor de golfvektor. Deze hogere-orde WKBJ asymptotische representaties zijn geldig rondom het punt oneindig van de Laplacetransformatie-parameter, d.w.z., deze geven in het ruimte-tijddomein aanleiding tot asymptotische resultaten voor kleine tijden. In het ruimte-tijddomein worden op deze manier benaderingen gevonden voor het golfveld na de aankomsttijd over een tijdsinterval met een zekere lengte. Door de keuze van de toegepaste transformaties blijven complicaties welke vaak optreden bij toepassing van WKBJ asymptotiek, zoals het niet langer geldig zijn van de WKBJ asymptotische ontwikkelingen bij een nulpunt van de vertikale ”slowness” of bij het passeren van een Stokeslijn, achterwege. De terugtransformatie van de benaderende oplossingen naar het ruimte-tijddomein is de derde stap van deze integraaltransformatie-methode. Wij voeren deze stap uit door de Cagniard-De Hoop-methode toe te passen op de individuele termen van de WKBJ asymptotische representaties van de golfveld-grootheden in het getransformeerde domein. Verder laten wij de implementatie van de theorie met behulp van een programma voor symbolische manipulatie zien. Tenslotte zijn verschillende numerieke resultaten voor het benaderde ruimte-tijddomein golfveld getoond.

In het laatste hoofdstuk worden enkele conclusies getrokken met betrekking tot de beide gepresenteerde integraaltransformatie-methoden die in dit proefschrift zijn ontwikkeld. In het algemeen valt op dat de eigenschappen van beide methoden sterk afhangen van de wiskundige methoden die zijn toegepast voor de drie basisstappen. De keuze van de Laplacetransformatie met betrekking tot de tijd, waarbij de transformatie-parameter positieve reële getallen kan aannemen, is van essentiële invloed op het vervolg van de theorie van beide methoden. De eerste integraaltransformatie-methode is toepasbaar in elke horizontaal continu gelaagde configuratie en kan in principe worden gebruikt om voldoend nauwkeurige resul-
taten voor het akoestische golven veld op elk gewenst tijdstip te bepalen. De methode is numeriek robuust, maar een nadeel van de methode is dat het numeriek gezien veel inspanning kost om de hogere-orde termen te berekenen. De tweede integraal-transformatie-methode levert het hogere-orde korte-tijd gedrag van het akoestische golven veld. De computerimplementatie van deze methode blijkt snel te zijn en levert opeenvolgende benaderingen op na de aankomsttijd over een tijdsinterval met een zekere lengte.
About the author

Martin Verweij was born in Alphen aan den Rijn, The Netherlands, on May 9, 1961. He followed secondary education at the Christelijk Lyceum in Alphen aan den Rijn, from which he obtained the diploma Atheneum B in 1979. In 1983 he received the B.Sc. degree in Electrical Engineering from the Municipal Polytechnical School in The Hague. From 1983 to 1984 he served in the Dutch army as a sergeant. In 1984 he became a student of the Faculty of Electrical Engineering of the Delft University of Technology. For his master's thesis he carried out theoretical research on the propagation of space-time domain acoustic wavefields in continuously layered media. This research was carried out at the Laboratory of Electromagnetic Research under the supervision of Prof.dr. P.M. van den Berg. In 1988 he received the M.Sc. degree (cum laude) in Electrical Engineering from the Delft University of Technology. In the same year Martin started his Ph.D. research at the Laboratory of Electromagnetic Research under the supervision of Prof.dr. P.M. van den Berg. This research consisted of an advanced study of the propagation of space-time domain acoustic wavefields in continuously layered media. The results of this research have been described in the present thesis. He has written four papers (three as a first author and one as a co-author) related to his research subject. In addition, he has presented his work on four international conferences in the U.S.A. Two of these conference visits have been sponsored by NWO (the Dutch Association for Scientific Research) and Royal/Dutch Shell, respectively. Besides his research activities, from 1991 till 1992 he has been active as an elected member of the Council of the Faculty of Electrical Engineering.