A Minimax Optimal Control Analysis of Lateral Escape Maneuvers for Microburst Encounters

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Title: A Minimax Optimal Control Analysis of Lateral Escape Maneuvers for Microburst Encounters

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Abstract: This report summarizes the status of a continuing research effort involving the optimization of lateral escape trajectories in a microburst wind flow field for an aircraft on final approach. The performance index being minimized is the maximum value of altitude drop at any point along the trajectory. In contrast to earlier work, the Chebyshev minimax performance index has not been approximated by a Bolza integral performance index. Rather, "exact" Chebyshev solutions have been established by transforming the minimax problem into an equivalent optimal control problem with state variable inequality constraints. A composite performance index has been formulated, containing a weighted combination of a Bolza term, a Chebyshev term and an end-cost term involving specific energy. An extensive numerical effort has been undertaken to investigate the main characteristics of open-loop extremal solutions for a range of values of the weight factors in the performance index and for different values of the aerodynamic roll angle limit. In comparison with Bolza solutions, Chebyshev solutions demonstrated a marked improvement in reducing the peak value of altitude drop. In Chebyshev solutions typically altitude is traded for airspeed in the initial phase of the encounter, such as to place the aircraft in a region of relatively low downdraft. Consequently, Chebyshev solutions no longer exhibit an initial climb in a lateral escape maneuver. The results cast some doubts on the validity of the Bolza performance index approximation in this particular application. On a more positive note, the overall benefits of lateral escape vis-a-vis nonturning escape were clearly reconfirmed in this study. Including a small end-cost term involving specific energy in the performance index turned out to significantly improve trajectory behavior in the after-shear region, without affecting the minimum altitude performance. On the other hand, a large specific energy end-cost term results in a huge gain in the terminal value of specific energy, but this comes at the expense of minimum altitude performance.

Keyword(s): Trajectory optimization, minimax Chebyshev optimal control problems, multiple shooting, windshear, microburst escape strategies.
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### Subscripts

- $o$: initial value
- $c$: center of microburst
- $f$: final value
- $\text{max}$: maximum value
- $\text{ref}$: reference value
- $\text{uc}$: unconstrained

### Superscript

- $^*$: optimal value
1. Introduction

When an aircraft is proceeding along the glide slope during final approach, a pilot may decide to abort the landing and initiate an escape maneuver, in case an inadvertent microburst is encountered. Optimal control strategies for executing this type of escape procedure have been studied for some time now\(^6\). Early and widely known work in this area is due to Miele et al. In addition to considering control strategies to improve the take-off and penetration landing performance during microburst encounter\(^2\),\(^3\), Miele also specifically deals with the abort landing\(^4\). In Ref. 4, Miele et al. consider optimal (open-loop) trajectories through windshears and downdraft that minimize the peak value of the altitude drop. However, these trajectories are confined to a vertical plane only. In Ref.5, the work of Miele et al. has been extended to flight in three dimensions, i.e., to optimal trajectories that feature lateral maneuvering.

In both Refs. 4 and 5, optimal escape maneuvers were established by solving a so-called minimax (or Chebyshev) optimal control problem. More specifically, for the abort landing the objective is to maximize the minimum altitude reached by an aircraft at any point along the trajectory, or, in other words, to minimize the peak value of the altitude drop (see Fig. 1):

\[
I^* = \min I = \min \left[ \max_t \left( h_{\text{ref}} - h(t) \right) \right], \quad 0 \leq t \leq t_f, \quad (1)
\]

where \(h_{\text{ref}}\) is a constant reference altitude and \([0,t_f]\) is the fixed flight time interval. Based on a well-known result obtained using functional analysis\(^6\), i.e.,

\[
\lim_{k \to \infty} \left[ \int_0^{t_f} (h_{\text{ref}} - h(t))^{2k} \, dt \right]^{1/2k} = \max_t (h_{\text{ref}} - h(t)), \quad (2)
\]

the minimax criterion in Eq.(1) (Chebyshev performance index), can be approximated by a Bolza performance index:

\[
J^* = \min J = \min \int_0^{t_f} (h_{\text{ref}} - h)^n \, dt, \quad (3)
\]

where \(n\) is a large positive, even exponent. Note that for the best possible computational results, the reference altitude \(h_{\text{ref}}\) should be chosen as small as possible, but such that the
right-hand side of Eq.(1) remains positive at all times. The numerical value for the exponent in Eq.(3) is typically taken as: $n = 6$.

In Ref.5 it was already noted that in addition to the above performance index approximation, it is also possible to apply another transformation technique which allows to solve the original minimax problem (1). More specifically, the minimax problem can be converted into an equivalent optimal control problem with state variable inequality constraints$^{7,8,9,10,11}$. In conjunction with the multiple-shooting algorithm$^{12}$ that already has been used in Ref.5 to solve the Bolza trajectory optimization problems, this technique appears to be particularly suited for numerical treatment of the Multi-Point-Boundary-Value-Problem (MPBVP), which arises from the minimax optimal control analysis. Although this approach is mathematically rather complicated, the results in Refs. 7 and 8 were found to be sufficiently intriguing to warrant a closer examination of this approach. The purpose of the present report therefore is to present the results of the application of this approach such as to establish the solution to the original minimax problem ("Chebyshev solution"), rather than to the problem based on the Bolza performance index approximation ("Bolza solution").

![Figure 1: Illustration of Performance Index](image)

In Ref.13, Zhao and Bryson propose an alternative formulation for the optimization of flight paths through microbursts, using a different performance measure. More specifically, paths are determined through windshears and downdrafts that maximize the final value of specific energy, while taking into account a minimum altitude constraint. It will be shown herein that this particular formulation can be included in the present minimax analysis, without significant modification. Indeed, in the present work a "composite"
performance criterion will be considered.

In Ref. 5, the exploration of alternative optimization criteria was viewed as one of the major issues to be addressed in future research. However, some other potential problem refinements were identified as well. Most notably, the issue of the fairly crude model for the angle-of-attack response was raised. In Ref.5, instantaneous angle-of-attack and roll angle response has been assumed. This contrasts with the work of Miele et al., where not only a bound is imposed on the maximum value the angle-of-attack, but also the rate of change of this variable is limited. Although neglecting the $\alpha$-constraint seems to be justified due to the fairly large time scale of flight, a validation effort which involves including the aircraft rotational inertia in roll and pitch in the system model was clearly warranted. However, this important issue regarding the response characteristics of the current control variables angle-of-attack and roll angle, will not be addressed herein. This issue is dealt with in Ref. 14, where the advantages of the lateral escape vis-a-vis the longitudinal escape are examined for a system model that does incorporate the aircraft rotational inertia in roll and pitch. Other relevant issues addressed in Ref.14 include, the imposition of terminal boundary conditions, the adoption of a more realistic microburst wind flow field model (Oseguera-Bowles model), and the implementation of a small business-jet type aircraft (Cessna Citation II). Moreover, a direct numerical optimization method has been tested for its suitability as an alternative to the (indirect) multiple shooting algorithm. More specifically, the nonlinear programming/collocation algorithm of Ref.15 has been used for this purpose. Although this easy-to-use method turned out to converge fairly slowly, it also exhibited very favorable robustness characteristics. For this reason, this particular algorithm has also been utilized in the present investigation, primarily to serve as a "path-finder" for generating good starting solutions for the very high-fidelity multiple-shooting approach.

The dynamic model as well as the microburst model in this study are exactly the same as in Refs. 1 and 5. This has the advantage that numerical results can be directly compared. It is important to note that in this report, we have only concerned ourselves with computing open-loop optimal trajectories. Although in Ref.1 some preliminary results are reported on closed-loop guidance strategies that closely approximate the open-loop optimal trajectories, the outcome of the present research effort will make clear that it is probably necessary to "re-think" the envisioned constant-pitch technique.

Due to recent advances made in the area of forward looking windshear detection and warning systems, alert times up to 60-90 seconds can be obtained in a microburst encounter. In view of this fairly large warning time, a real need for escape guidance strategies seems questionable. However, as evidenced by some recent windshear accidents\(^{(16)}\), a microburst may form and dissipate within a relatively short time span. Indeed, a microburst may develop well within the 60-90 sec. time period covered by advanced detection and warning systems. Consequently, the need for guidance strategies will always remain, no matter how far forward looking systems can look ahead of the aircraft. For this reason, the ultimate goal of the research still remains the development of a near-optimal feedback escape strategy for microburst encounters. The present open-loop optimal control results primarily serve to set "ideal" standards against which simulated closed-loop guidance solutions can be compared.
2. Microburst Encounter Modeling

Using the familiar relative wind-axes reference-frame defined in Ref. 5, the equations of motion, describing the aircraft dynamics (represented by a point-mass model) in the three-dimensional space can be written as:

\[ x = V \cos \gamma \cos \psi + W_x \]  
(4)

\[ y = V \cos \gamma \sin \psi + W_y \]  
(5)

\[ h = V \sin \gamma + W_h \]  
(6)

\[ \dot{E} = \frac{[T (1 - (\alpha + \delta)^2) - D]}{2} V + W_h \]  
\[ - \frac{V}{g} [W_x \cos \gamma \cos \psi + W_y \cos \gamma \sin \psi + W_h \sin \gamma] \]  
(7)

\[ \dot{\gamma} = \frac{g}{V} \left[ \frac{L + T (\alpha + \delta)}{W} \cos \mu - \cos \gamma \right] \]  
\[ + \frac{1}{V} [W_x \sin \gamma \cos \psi + W_y \sin \gamma \sin \psi - W_h \cos \gamma] \]  
(8)

\[ \dot{\psi} = \frac{g}{V \cos \gamma} \frac{L + T (\alpha + \delta)}{W} \sin \mu + \frac{1}{V \cos \gamma} [W_x \sin \psi - W_y \cos \psi] \]  
(9)

\[ \dot{\beta} = \frac{1}{\tau} [\beta_t - \beta] \]  
(10)

where \( x, y \) and \( h \) are the position coordinates, \( E \) is the specific energy, \( \gamma \) is the flight path angle, \( \psi \) is the heading angle and \( \beta \) the throttle response. The wind velocity vector has three components, viz., \( W_x, W_y \) and \( W_h \). The above equations embody the following assumptions: (i) the wind flow field is steady, (ii) zero angle-of-sideslip, (iii) a flat non-rotating earth, and (iv) the aircraft weight is constant. The thrust \( T \) is assumed to have a fixed inclination \( \delta \) relative to the zero-lift axis. To simplify the optimal control analysis, a small angle approximation has been employed in the equations of motion for the thrust vector components along and perpendicular to the airspeed vector respectively, i.e.,
\[ \cos(\alpha + \delta) = 1 - \frac{1}{2} (\alpha + \delta)^2 \text{ and } \sin(\alpha + \delta) = (\alpha + \delta). \] The throttle response is modeled as a first-order lag with a time constant \( \tau \).

In the mathematical model the controls are:

(i) The throttle setting \( \beta_i \) constrained by:

\[ 0 \leq \beta_i \leq 1 \quad (11) \]

(ii) The aerodynamic roll angle \( \mu \) which is limited by:

\[ |\mu| \leq \mu_{\text{max}} \quad (12) \]

(iii) The angle-of-attack \( \alpha \) which is forced to remain within the range:

\[ 0 \leq \alpha \leq \alpha_{\text{max}} \quad (13) \]

The aerodynamic forces (lift \( L \) and drag \( D \)) are modeled as functions of airspeed \( V \), altitude \( h \) and the angle-of-attack \( \alpha \):

\[ L = C_L(\alpha) q S = \left[ L_0 + L_1 \alpha + L_2 (\alpha - \alpha_{\text{ref}})^2 \right] \frac{1}{2} \rho V^2 S \quad (14) \]

\[ D = C_D(\alpha) q S = \left[ D_0 + D_1 \alpha + D_2 \alpha^2 \right] \frac{1}{2} \rho V^2 S \quad (15) \]

The maximum thrust is assumed to be a function of airspeed only, i.e.:

\[ T = \beta T_{\text{max}}(V) = \beta \left( T_0 + T_1 V + T_2 V^2 \right) \quad (16) \]

The aircraft type used in the investigation is a Boeing 727. Details of the aerodynamic and thrust data for this aircraft type (in landing configuration) are given in Ref.17. Reference 17 also presents a detailed description of the microburst wind field flow model. For this reason details are omitted here. However, it is recalled that in view of the axisymmetric character of the microburst model, polar coordinates have been used to describe the flow field in a horizontal plane (see Fig. 2). The employed analytic approximation of the flow field characteristics actually features separate models for the radial flow \( W_r \) (which may lead to horizontal shear) and the downdraft \( W_h \). Note that the horizontal wind component \( W_r \) is only a function of the radial distance to the microburst center, whereas the vertical wind component \( W_h \) also depends on altitude to ensure that \( W_h \) decreases with decreasing altitude and has a stagnation point (i.e., zero vertical windspeed) at ground level. Clearly, the vertical windspeed model (at least in the neighborhood of the microburst center) is
valid at low altitudes only.

![Diagram showing the geometry of microburst encounter.](image)

Figure 2: Geometry of microburst encounter.

For a given aircraft position \((x, y)\), it is readily clear from Figure 2 that the radial distance \(r\) from the microburst center (axis of symmetry) located at the point \((x_0, y_0)\), can be computed from the relation:

\[
r = \sqrt{(x - x_0)^2 + (y - y_0)^2}
\]  

(17)

Also observe that in the present study the origin of the coordinate frame is located at the runway threshold. Using polar coordinates, the horizontal wind components \(W_x\) and \(W_y\) can be readily related to the radial wind velocity \(W_r\):

\[
W_x = \cos \psi_w \, W_r(r) \quad ; \quad W_y = \sin \psi_w \, W_r(r)
\]  

(18)
where $\psi_w$ is the direction of the radial wind velocity vector. In view of assumption (iii) in the previous Section, the total derivatives of these wind velocity components satisfy the relations:

$$W_x = \frac{\partial W_x}{\partial x} x + \frac{\partial W_x}{\partial y} y$$
$$W_y = \frac{\partial W_y}{\partial x} x + \frac{\partial W_y}{\partial y} y$$

(19)

$$W_h = \frac{\partial W_h}{\partial x} x + \frac{\partial W_h}{\partial y} y + \frac{\partial W_h}{\partial h} h$$

(20)

An important characteristic parameter used in the evaluation of windshear performance is the F-factor. The F-factor can be defined in various ways; here the definition introduced in Ref. 5 will be employed:

$$F \triangleq \frac{(T - D)}{W} - \frac{\dot{E}}{\dot{V}}$$

(21)

Defining the F-factor in this particular fashion permits its use in the analysis of three-dimensional windshear encounters. It is recalled that the F-factor can be interpreted as the loss or gain in available excess thrust-to-weight ratio due to the combined effect of downdraft and horizontal windshear\(^{(5,17)}\). The F-factor therefore represents a direct measure of the degradation of an aircraft's capability to gain energy due to the windshear. Note that positive values of the F-factor indicate a performance decreasing situation.

Although the F-factor does not merely depend on the spatial location within the flow field, but rather on all state variables, it was found in Refs. 5 and 17 that for the present wind model the peak value of the F-factor is not generally reached at the center of the microburst, but rather at some distance away from the center (see Figure 2 in Ref. 5). It is conjectured here that this particular feature is responsible for the fact that, in addition to optimal lateral escape trajectories that turn away from the microburst core, also extremals can be found that pass right through the microburst center. The findings in Ref. 14, where the Oseguera-Bowles microburst model was used, appear to support this conjecture.
3. Optimal Control Formulation

3.1 Transforming a Minimax Problem into a Standard Optimal Control Problem

Although minimax problems arise in many applications, they are relatively difficult to handle. One of the most commonly employed approaches is to apply the transformation technique due to Warga\(^{(11)}\). Using this technique, the minimax problem can be converted into an equivalent optimal control problem with a mixed state/control inequality constraint. Let us consider the following functional that consists of a standard Bolza performance index (path integral plus end-cost term \(\phi\)), to which a minimax (also called Chebyshev) term is added:

\[
J = \phi [x(t_f), t_f] + \int_0^{t_f} L[x(t), u(t)] \, dt + \max_t C[x(t), u(t)]
\]  
(22)

The problem now is to find the control \(u(t) \in U, 0 \leq t \leq t_p\) that minimizes the functional \(J\) in Eq.(22), while the state variables \(x(t)\) satisfy the non-linear differential equations, along with the appropriate boundary conditions, i.e.:

\[
\dot{x} = f[x(t), u(t)] \quad ; \quad x(0) = x_0
\]  
(23)

where \(x(t)\), an \(n\)-vector function, and \(u(t)\), an \(m\)-vector function. The final time \(t_f\) is assumed to be fixed. The windshear trajectory optimization problem considered herein adheres to the above stated problem statement, as we will see, but more general problem formulations are possible.

Assuming that an optimal solution (not necessarily a unique minimum) exists, then the Chebyshev function \(C[x(t), u(t)]\) takes on its maximum value along the optimal trajectory, at a finite number of compact subintervals only. For the Chebyshev function two standard types of maxima can be distinguished (see Figure 3), (i) a unique maximum (adopted at some point \(t_{cp}\)), and, (ii) a flat maximum (adopted at some interval \([t_{en}, t_{ex}]\)). However, a combination of unique and flat maxima along an optimal trajectory may also occur.

Following the approach of Ref.11, we define

\[
\zeta(t) \triangleq \max_t C[x(t), u(t)]
\]  
(24)

The above defined minimax problem can now be shown to be equivalent to the following standard optimal control problem with a mixed state/control inequality constraint: find the functions \(u(t) \in U\) that minimize the following functional \(J\):
\[ J = \phi(x(t_f), t_f) + \zeta(t_f) + \int_0^{t_f} \left[ L(x(t), u(t)) \right] dt \]  

subject to the constraints:

\[ \dot{x} = f(x(t), u(t)); \quad x(0), t_f \text{ given}, \]  

\[ \zeta = 0 \]  

\[ S[x, \zeta, u] \Delta C[x(t), u(t)] - \zeta(t) \leq 0 \]  

It can be concluded that we end up with a Bolza-type problem, subject to an augmented state system, and a mixed control/state inequality constraint. It is important to note that the Chebyshev function does not necessarily involve the control explicitly. In the latter case, the constraint (28) will be a pure state constraint.

![Diagram](a) unique maximum  

![Diagram](b) flat maximum

**Figure 3**: Two standard types of Chebyshev Functions.

Now that the optimization problem is in standard form, the basic necessary conditions of optimal control theory for problems involving state/control inequality constraints can be applied. Here we follow the approach presented in Chapter 3 of Ref.18.

Let us, for the purpose of the discussion, assume that the constraint (28) does not explicitly contain the control u:
\[ S[\bar{x}, \zeta] \triangleq C[\bar{x}(t)] - \zeta(t) \leq 0, \quad (29) \]

and that \( q \) successive total time derivatives of \( S \) are required until we obtain an expression that explicitly depends on the control \( u \), i.e.:

\[
\frac{\partial}{\partial u} \left( \frac{d^k S}{dt^k} \right) = 0, \quad (k = 0, 1, \ldots, q - 1) \quad (30)
\]

\[
\frac{\partial}{\partial u} \left( \frac{d^q S}{dt^q} \right) \neq 0
\]

If \( q \) derivatives are required, the constraint (29) is called a \( q \)-th order state variable inequality constraint. The \( q \)-th total time derivative of \( S \) will be denoted here as \( S^{(q)} \). The lower order derivatives are collected in the tangency vector \( N(\bar{x}, \zeta) \):

\[
N[\bar{x}(t), \zeta(t)] = \begin{pmatrix}
S[\bar{x}, \zeta] \\
S^{(1)}[\bar{x}] \\
\vdots \\
S^{(q - 1)}[\bar{x}]
\end{pmatrix} = 0, \quad (31)
\]

Introducing the Kuhn-Tucker multiplier \( \eta \), the augmented Hamiltonian can be defined as:

\[
\bar{H} \triangleq H + \eta S^{(q)} = L[\bar{x}, u] + \lambda^T f[\bar{x}, u] + \eta S^{(q)}[\bar{x}, u], \quad (32)
\]

where:

\[
S^{(q)} = 0 \quad on \ the \ constraint \ boundary, \quad S = 0 \quad (33)
\]

\[
\eta = 0 \quad off \ the \ constraint \ boundary, \quad S < 0
\]

The adjoint differential equations and Minimum Principle can now be obtained in a similar fashion to the unconstrained case:
\[ \dot{\lambda}^T = -\frac{\partial H}{\partial x} = - \frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x} - \eta \frac{\partial S^{(q)}}{\partial x} \quad (34) \]

\[ \dot{\lambda}_\xi = 0 \quad (35) \]

\[ u^* = \arg \left[ \min_{u \in \mathcal{U}} \tilde{H}(x, \lambda, u, \eta) \right] \quad (36) \]

A necessary (sign) condition for the multiplier function \( \eta(t) \) is:

\[ \eta(t) \geq 0 \quad \text{on } S = 0 \quad \text{if minimizing } J \quad (37) \]

For \( S < 0 \), we have \( \eta = 0 \) and Eq.(36) determines \( u^*(t) \). For \( S = 0 \), Eq.(36) along with \( S^{(q)[x,u]} = 0 \) determine both \( u^*(t) \) and \( \eta^*(t) \). Note that it is assumed here that the minimax optimal problem is non-singular, i.e., we assume that the Hamiltonian is nonlinear with respect to its control variables.

Assuming that the problem is also non-degenerate, the transversality conditions produce the following natural boundary conditions:

\[ \left( -\lambda^T + \frac{\partial \phi}{\partial x} \right)_{t = t_f} = 0 \quad (38) \]

\[ \lambda_\xi(0) = 0 \quad ; \quad \lambda_\xi(t_f) = 1 \quad (39) \]

Note that the first of Eqs.(39) results from the fact that no initial condition on \( \xi \) is imposed.

In correspondence with the two standard types of Chebyshev functions, a state constraint can become active in two different fashions, namely at an isolated point (contact point), or on some subinterval of \([0,t_f]\). Indeed, we are now ready to complete the MPBVP by considering the so-called "jump" and "switching" conditions for the two basic types of active state constraints, viz. the touch point and the constrained subarc.

**constrained subarc**

A path entering onto a constrained boundary at the entry point \( t_{en} \) has to meet the tangency constraint (31) in order to enable the system to remain on the constraint boundary, i.e.:
\[ N[\bar{x}(t_{en}), \zeta(t_{en})] = 0 \quad , \] (40)

Obviously, the same tangency conditions also need to apply to the path leaving the constraint boundary at the point \( t_{en} \). Mathematically, Eq.(40) is treated as a set of interior boundary conditions at the entry point \( t_{en} \). Consequently, the adjoint variables are, in general, discontinuous at the entry junction. Assuming that \( q > 0 \), a \( q \)-component vector of constant Lagrange multipliers \( \sigma = [\sigma_0, \sigma_1, \ldots, \sigma_q] \) is introduced to accommodate the set of interior point conditions. The following set of "jump conditions" is then obtained:

\[ \lambda^T(t_{en}^+) = \lambda^T(t_{en}^-) - \sigma^T \frac{\partial N}{\partial x(t_{en})} \] (41)

\[ \lambda_\zeta(t_{en}^+) = \lambda_\zeta(t_{en}^-) + \sigma_0 \] (42)

\[ \sigma_i \geq 0 \quad (i = 0, \ldots, q - 1) \] (43)

Here we let \( t_{en}^- \) signify just before \( t_{en} \), and \( t_{en}^+ \) just after \( t_{en} \). Note that at the exit point, the adjoints are continuous. Due to the assumed regularity of the Hamiltonian, the "switching conditions" are fairly simple:

\[ u(t_{en}^+) = u(t_{en}^-) \quad ; \quad u(t_{ex}^+) = u(t_{ex}^-) \] (44)

This continuity of the controls at the junctions can be directly related to the continuity of the Hamiltonian at the junctions, i.e.:

\[ H|_{t_{en}^+} = H|_{t_{en}^-} \quad ; \quad H|_{t_{ex}^+} = H|_{t_{ex}^-} \] (45)

At this point it is probably prudent to point out that the above condition only applies to autonomous systems (systems that do no explicitly depend on time \( t \)), as considered here. However, condition (45) can be modified to handle non-autonomous systems, if required.

Further necessary (sign) conditions involving the function \( \eta(t) \) can be applied to a posteriori test the optimality of a MPBVP solution\(^{(19)}\). Sets of sharp general sign conditions have been established for this purpose. Since these sets are fairly elaborate, they will not be reiterated here. In the next Section, however, we will state the appropriate conditions pertaining to the specific situation of the windshear trajectory optimization problem under consideration.
touch point

A touch point is a contact point (an isolated root $t_{ip} \in (0,t_{p})$ of $S[\bar{z}(t),\zeta(t)] = 0$), that also is a stationary point of $S$. In other words, $S^{(1)}[\bar{z}(t_{ip}),\zeta(t_{ip})] = 0$. A touch point thus obeys the following conditions:

$$S[\bar{z}(t_{ip}),\zeta(t_{ip})] = 0 \quad ,$$

$$S^{(1)}[\bar{z}(t_{ip})] = 0 \quad ,$$

$$\lambda_{T}^{+}(t_{ip}) = \lambda_{T}^{-}(t_{ip}) - \sigma_{0} \frac{\partial S}{\partial \bar{z}(t_{ip})} \quad (48)$$

$$\lambda_{\zeta}(t_{ip}) = \lambda_{\zeta}(t_{ip}) + \sigma_{0} \quad (49)$$

$$\sigma_{0} \geq 0 \quad ,$$

where $\sigma_{0}$ is a constant multiplier. It needs to be noted that a nontrivial touch point can occur for $q \geq 2$ only. For $q = 1$, an active state constraint can only occur as a constrained subarc. On the other hand, it has also been proven that for odd-ordered state constraints starting from $q = 3$ (i.e., $q = 3, 5, 7, ..., $), the occurrence of constrained subarcs is impossible.

Although, we have only considered the two individual standard types of active state constraints, an optimal path may actually contain multiple touchpoints and constrained subarcs. The necessary conditions can be easily adapted to cover any possible combination of touchpoints and constrained arcs. However, in most practical applications it is generally rather difficult to a priori know the number and relative location of touch points and constrained subarcs. As a result some trial-and-error efforts will be typically required.

### 3.2 Specification of the Performance Index for the Minimax Windshear Problem

In order to be able to apply the necessary conditions stated in the previous section to the windshear trajectory optimization problem we first need to formally state the performance index that we seek to optimize. As already pointed out in the introduction, the objective in this study is to minimize the peak value of the altitude drop ($h_{ref} - h$), as expressed by the Chebyshev functional in Eq. (1). The Bolza functional that approximates the Chebyshev index is stated in Eq.(3). In normalized form it will be denoted here as $J_{1}$:

$$J_{1} = \int_{0}^{t_{p}} \left( 1 - \frac{h}{h_{ref}} \right)^{6} dt \quad (51)$$
Following the approach of the previous Section, the minimax problem can be transformed into an equivalent optimal control problem with a state variable inequality constraint. Let us introduce the altitude drop $\zeta(t)$ as a new state variable:

$$\zeta(t) \triangleq \max_{t} \left( h_{\text{ref}} - h(t) \right), \quad 0 \leq t \leq t_f$$  \hspace{1cm} (52)

We then obtain the end-cost function $J_2$:

$$J_2 = \zeta(t_f), \hspace{1cm} (53)$$

subject to the constraints:

$$\dot{\zeta} = 0$$  \hspace{1cm} (54)

$$h_{\text{ref}} - h(t) - \zeta(t) \leq 0$$  \hspace{1cm} (55)

A third criterion involves the maximization of specific energy at termination:

$$J_3 = -E(t_f)$$  \hspace{1cm} (56)

Obviously, the minus sign in Eq.(56) results from the fact that we seek to minimize the performance index $J_3$. In order to permit a trade-off between the various performance criteria, the following "composite" index is formed:

$$J = \Lambda_1 \int_{0}^{t_f} \left( 1 - \frac{h}{h_{\text{ref}}} \right)^6 dt + \Lambda_2 \zeta(t_f) - \Lambda_3 E(t_f)$$  \hspace{1cm} (57)

$$= \Lambda_1 J_1 + \Lambda_2 J_2 + \Lambda_3 J_3,$$

where $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ are (positive) scaling factors. In addition, the following weight factors are introduced, to represent the tradeoff between the three performance measures:
\[ K_1 \triangleq \frac{\Lambda_2}{\Lambda_1} ; \quad K_2 \triangleq \frac{\Lambda_3}{\Lambda_1} \]  

(58)

Dividing Eq.(57) by \( \Lambda_1 \), we obtain:

\[ \bar{J} = \int_0^{t_f} \left( 1 - \frac{h}{h_{ref}} \right)^6 dt + \frac{\Lambda_2}{\Lambda_1} \zeta(t_f) - \frac{\Lambda_3}{\Lambda_1} E(t_f) \]

(59)

\[ = J_1 + K_1 J_2 + K_2 J_3 \]

For the numerical computation of the optimal trajectories, a continuation process has been used, in which the parameters \( K_1 \) and \( K_2 \) served as homotopy parameters. Obviously, the homotopy chain was started with \( K_1 = K_2 = 0 \), which corresponds to the previously established results involving the integral performance measure only. In small steps the parameters \( K_1 \) and \( K_2 \) were gradually increased to large numbers. Note that only if the scaling factors \( \Lambda_1 \) and \( \Lambda_2 \) are set to zero, the "exact" minimax problem is obtained. However, the introduction of the parameters \( K_1 \) and \( K_2 \) precludes the specification of a "pure" Chebyshev performance measure, since letting \( \Lambda_1 \to 0 \) implies \( K_1 \to \infty \). In other words, for any finite value of \( K_1 \) there will always be some integral part \( J_1 \) in the composite performance measure (59). In the numerical results it will be revealed that in practical terms the influence of the integral part can be made negligibly small, simply by specifying a very large value for \( K_1 \).

The main reason for adopting the present definition of the performance index (59) is that we actually seek to avoid the formulation of a pure Chebyshev optimal control problem. It is well known that many minimax problem formulations are plagued by serious concerns regarding the existence and uniqueness of the solution. As we will see, this type of problem is particularly acute in the present windshear trajectory optimization problem, that has been formulated as a fixed final time problem.

A second motivation for the present definition of the weight parameters \( K_1 \) and \( K_2 \) relates to the objective of maximizing final energy. As outlined in Ref.13, the maximum energy performance index makes sense from a physical point of view only if some minimum safe altitude constraint is included as well. With respect to the performance index (57), this implies that we can never simultaneously specify \( \Lambda_1 \) and \( \Lambda_2 \) as zero. However, if the composite index (59) is used this particular case is excluded by definition. Maximum final energy solutions can always be found, even if \( K_1 = 0 \). However, if in that case \( K_2 \) is gradually increased, the actually achieved maximum altitude drop will also gradually increase. In other words, an increase in final energy comes at the expense of a lower minimum altitude. Consequently, unlike for \( K_1 \), there will be a maximum value that can be specified for \( K_2 \), if a certain safe minimum altitude is not to be exceeded. This conclusion also holds true in case \( K_1 > 0 \), but obviously the maximum value of \( K_2 \) that can then be specified will be higher.
3.3 Specification of the Boundary Conditions for the Minimax Windshear Problem

Similar to Ref.1 a single set of initial conditions has been specified in the present study. The following initial conditions (at which the escape procedure is commenced) have been assumed here:

\[
\begin{align*}
    x(0) &= x_0 = -2500 \text{ m}, \\
    y(0) &= y_0 = 0 \text{ m}, \\
    h(0) &= h_0 = 131 \text{ m}, \\
    E(0) &= E_0 = 384.326 \text{ m}, \\
    \gamma(0) &= \gamma_0 = -3^\circ, \\
    \chi(0) &= \chi_0 = 0^\circ, \\
    \beta(0) &= \beta_0 = 0.333
\end{align*}
\]

These values correspond to a situation in which an aircraft would fly during a stabilized approach (\(V = 70.5 \text{ m/s}\)) without winds or windshear. It needs to be realized that in the presence of winds, the required values for \(\gamma\) and \(\beta\) will be somewhat different. The final time \(t_f\) has been set to 50 seconds, which is sufficiently long to allow a transition of the shear region. No terminal boundary conditions on the state variables have been imposed. As shown in Ref. 14, such conditions would mainly affect the extremal solution in the after-shear region. The inclusion of even a small final energy term in the performance index (i.e. \(K_2 > 0\) in Eq.(59)) has a profound impact on the behavior of the state variables in the after-shear region, near termination. If the weight parameter \(K_2\) is increased, this influence will gradually extend forward into the high-shear region.

In order to assess the influence on the escape procedures, different locations of the microburst have been considered in the numerical examples of Ref.5. However, in this study the main interest is on comparing solutions for different weight factors \(K_1\) and \(K_2\). For this reason, the numerical examples included in this report have been limited to the reference scenario defined in Ref.1. In other words, we will only consider a "symmetric" reference situation, with the microburst center located at (-1500 m, 0 m). This implies that for the given initial conditions, an aircraft in straight flight will fly exactly along the x-axis of the reference frame (see Fig.2), passing right through the microburst center.

3.4 Necessary Conditions of Optimality for the Minimax Windshear Problem

To summarize, the optimal control problem to be solved is to determine the optimal controls \(\beta^*, \mu^*\) and \(\alpha^*\) such that starting from the initial conditions, the performance index of Eq.(59) is minimized for a given final time \(t_f\), while satisfying all imposed constraints.

Let us first examine the nature of the state constraint (55). The inequality constraint (55) is of second order, which can be easily concluded from observing consecutive time derivatives of \(S(t)\):

\[
S(t) \Delta h_{ref} - h(t) - \zeta(t) \leq 0 \tag{60}
\]
\[ S^{(1)}(t) = -\dot{h}(t) - \dot{\zeta}(t) = -\dot{h}(t) \quad (61) \]

\[ S^{(2)}(t) = -\ddot{h}(t) \]

\[ = -[V \cos \gamma \dot{\gamma} + \dot{V} \sin \gamma + \dot{W}_h] \]

\[ = -g \left[ \frac{T [1 - \frac{1}{2}(\alpha + \delta)^2] - D}{W} + \cos \gamma \frac{L + T(\alpha + \delta)}{W} \cos \mu - 1 \right], \quad (62) \]

where we have used Eqs.(54), (6) and (7) and the relationship:

\[ E \triangleq h + \frac{V^2}{2g} \Rightarrow \dot{V} = \frac{g}{V} [\dot{E} - \dot{h}] \quad (63) \]

Note that \( S^{(2)} \) in Eq.(62) explicitly contains the control variables, angle-of-attack \( \alpha \) and aerodynamic roll angle \( \mu \), but that no wind terms are present.

In the following the first-order necessary conditions for optimality are given. First, the augmented Hamiltonian of the problem is defined as:

\[ \bar{H} = H + \eta S^{(2)} \]

\[ = \left(1 - \frac{h}{h_{ref}}\right)^6 + \lambda_x \{V \cos \gamma \cos \psi + W_x\} + \lambda_y \{V \cos \gamma \sin \psi + W_y\} \]

\[ + \lambda_h \{V \sin \gamma + W_h\} + \lambda_g \left\{ \frac{T [1 - \frac{1}{2}(\alpha + \delta)^2] - D}{W} \right\} \frac{V}{\dot{V}} \]

\[ + \frac{W_h - V}{g} [\dot{W}_x \cos \gamma \cos \psi + \dot{W}_y \cos \gamma \sin \psi + \dot{W}_h \sin \gamma] \}

\[ + \lambda_\gamma \left\{ \frac{g}{V} \frac{L + T(\alpha + \delta)}{W} \cos \mu - \cos \gamma \right\} \]

\[ + \frac{1}{V} [\dot{W}_x \sin \gamma \cos \psi + \dot{W}_y \sin \gamma \sin \psi - \dot{W}_h \cos \gamma] \}

\[ + \lambda_\psi \left\{ \frac{g}{V \cos \gamma} \frac{L + T(\alpha + \delta)}{W} \sin \mu + \frac{1}{V \cos \gamma} [\dot{W}_x \sin \psi - \dot{W}_y \cos \psi] \right\} \]

\[ + \lambda_\beta \left\{ \frac{1}{\tau} [\dot{\beta} - \beta] \right\} + \eta S^{(2)}, \quad (64) \]

where \( \eta(t) \) is a multiplier function. For the sake of conciseness it needs to be pointed out that the above defined Hamiltonian corresponds to the usage in the Minimum Principle.

The system of adjoint equations and corresponding transversality conditions arises
from the necessary conditions for optimality, as summarized in Section 3.1:

\[ \lambda_x = -\frac{\partial \bar{H}}{\partial x} \quad ; \quad \lambda_x(t_f) = 0 \]  \hspace{1cm} (65)

\[ \lambda_y = -\frac{\partial \bar{H}}{\partial y} \quad ; \quad \lambda_y(t_f) = 0 \]  \hspace{1cm} (66)

\[ \lambda_h = -\frac{\partial \bar{H}}{\partial h} \bigg|_E = -\frac{\partial \bar{H}}{\partial h} \bigg|_V + g \frac{\partial \bar{H}}{\partial V} \bigg|_h \quad ; \quad \lambda_h(t_f) = 0 \]  \hspace{1cm} (67)

\[ \lambda_E = -\frac{\partial \bar{H}}{\partial E} \bigg|_h = -g \frac{\partial \bar{H}}{\partial V} \bigg|_h \quad ; \quad \lambda_E(t_f) = -K_2 \]  \hspace{1cm} (68)

\[ \lambda_\gamma = -\frac{\partial \bar{H}}{\partial \gamma} \quad ; \quad \lambda_\gamma(t_f) = 0 \]  \hspace{1cm} (69)

\[ \lambda_\psi = -\frac{\partial \bar{H}}{\partial \psi} \quad ; \quad \lambda_\psi(t_f) = 0 \]  \hspace{1cm} (70)

\[ \lambda_\beta = -\frac{\partial \bar{H}}{\partial \beta} \quad ; \quad \lambda_\beta(t_f) = 0 \]  \hspace{1cm} (71)

\[ \lambda_\zeta = 0 \quad ; \quad \lambda_\zeta(0) = 0 \quad , \quad \lambda_\zeta(t_f) = K_1 \]  \hspace{1cm} (72)

Note that the Appendix presents a complete overview of all the partial derivatives needed to evaluate the right-hand-sides of Eqs.(65) through (71).

The optimal control functions \( \mu^* \) and \( \alpha^* \) are found by applying the Minimum Principle. Assuming these "optimal aerodynamic controls" are within the interior of their admissible range and the state constraint (60) is inactive, the following conditions apply:
\[
\frac{\partial H}{\partial \mu} = -\lambda \frac{g}{V} L + T (\alpha + \delta) \sin \mu + \lambda \frac{g}{V \cos \gamma} \frac{L + T (\alpha + \delta)}{W} \cos \mu = 0
\]

\[
\Rightarrow \tan \mu_{uc} = \frac{(\lambda \sin \gamma)}{\lambda}
\]

\[
\frac{\partial H}{\partial \alpha} = \lambda \frac{g}{V} \frac{1}{W} [qS \frac{\partial C_L}{\partial \alpha} + T] \cos \mu + \lambda \frac{g}{V \cos \gamma} \frac{1}{W} [qS \frac{\partial C_D}{\partial \alpha} + T] \sin \mu - \lambda \frac{V}{W} [T (\alpha + \delta) + qS \frac{\partial C_D}{\partial \alpha}] = 0 ,
\]

where the subscript uc is used to denote an "unconstrained" solution. Using Eqs.(14) and (15), the following terms are readily evaluated:

\[
\frac{\partial C_L}{\partial \alpha} = L_1 + 2L_2 (\alpha - \alpha_{ref}) , \quad \frac{\partial C_D}{\partial \alpha} = D_1 + 2D_2 \alpha
\]

Let us define \( \Phi(\mu) \) as:

\[
\Phi(\mu) \triangleq \lambda \cos \mu + (\lambda \sin \gamma) \sin \mu
\]

Substitution of Eqs.(75) and (76) into Eq.(74) allows to solve for \( \alpha_{uc} \):

\[
\alpha_{uc} = \frac{\lambda \frac{V^2}{g} [D_1 + \frac{T \delta}{qS}] - [\frac{T \delta}{qS} + L_1 - 2L_2 \alpha_{ref}] \Phi}{-2 \lambda \frac{V^2}{g} [D_2 + \frac{T \delta}{qS}] + 2L_2 \Phi},
\]

It needs to be realized that the control solution of Eq.(77) has an ambiguity in the sense that it has multiple roots. The Legendre-Clebsch Condition can be used to solve this ambiguity. In particular, for a minimum of \( H \) we must have that:

\[
\frac{\partial^2 H}{\partial \mu^2} = -\frac{g}{V W} \left[ \lambda \cos \mu + (\lambda \sin \gamma) \sin \mu \right] = -\frac{g}{V W} \Phi \geq 0 \Rightarrow \Phi \leq 0 ,
\]

where \( \Phi \leq 0 \) must apply since physically it is clear that \( L \geq 0 \). From Eq.(78) it can be
seen that:

\[
\tan^2 \mu = -1 + \frac{1}{\cos^2 \mu} = \frac{(\lambda_\psi/\cos \gamma)^2}{\lambda_\gamma^2} \quad \Rightarrow \\
\cos^2 \mu = \frac{\lambda_\gamma^2}{(\lambda_\psi/\cos \gamma)^2 + \lambda_\gamma^2} \quad ; \quad \sin^2 \mu = \frac{(\lambda_\psi/\cos \gamma)^2}{(\lambda_\psi/\cos \gamma)^2 + \lambda_\gamma^2}
\]

(79)

The appropriate roots of Eqs.(79) are:

\[
\cos \mu = \frac{-\lambda_\gamma}{\sqrt{(\lambda_\psi/\cos \gamma)^2 + \lambda_\gamma^2}} \quad ; \quad \sin \mu = \frac{-\lambda_\psi/\cos \gamma}{\sqrt{(\lambda_\psi/\cos \gamma)^2 + \lambda_\gamma^2}}
\]

(80)

which can be readily verified by substitution of Eq.(80) into Eq.(78):

\[
\Phi = -\sqrt{(\lambda_\psi/\cos \gamma)^2 + \lambda_\gamma^2} \leq 0
\]

(81)

If the control constraints in Eqs.(12) and (13) are to be obeyed, the optimal control solutions \( \mu^* \) and \( \alpha^* \) are ultimately given by:

\[
\mu^* = \min \{ \mu_{\text{max}}, |\mu_{\text{uc}}| \} \cdot \text{sign}(\lambda_\psi)
\]

(82)

\[
\alpha^* = \max \{ 0, \min \{ \alpha_{\text{max}}, \alpha_{\text{uc}} \} \}
\]

(83)

The third control variable is the throttle setting \( \beta \). Again, from physical considerations it is clear that full throttle should be applied during a microburst escape maneuver. Mathematically this fact needs to be checked by verifying that:

\[
\lambda_\beta \leq 0
\]

(84)

It is recalled that the control solution (82), (83) has been derived by assuming that the state constraint is not active, at least not for some finite period of time. In other words, on a constrained subarc the control solution needs to be modified.
constrained subarc

On a constrained subarc, the optimal control needs to keep the aircraft on the constraint boundary. To achieve this, the constraint (62) is adjoined to the Hamiltonian using the multiplier $\eta$, to form the augmented Hamiltonian given by Eq.(64). As expressed by Eqs.(33) and (37), the multiplier $\eta(t)$ is nonnegative on the constraint boundary, but zero off the constraint boundary.

If the control limits (12) and (13) are disregarded, for the time being, the optimal controls can be established by evaluating the stationary point of the augmented Hamiltonian:

$$\frac{\partial \bar{H}}{\partial \mu} = -\lambda_\gamma \frac{g}{V} \frac{L + T(\alpha + \delta)}{W} \sin \mu + \lambda_\gamma \frac{g}{V \cos \gamma} \frac{L + T(\alpha + \delta)}{W} \cos \mu$$

$$+ \eta V \cos \gamma \frac{g}{V} \frac{L + T(\alpha + \delta)}{W} \sin \mu = 0$$

$$\frac{\partial \bar{H}}{\partial \alpha} = \lambda_\gamma \frac{g}{V} \frac{1}{W} \left[ q S \frac{\partial C_L}{\partial \alpha} + T \right] \cos \mu + \lambda_\gamma \frac{g}{V \cos \gamma} \frac{1}{W} \left[ q S \frac{\partial C_L}{\partial \alpha} + T \right] \sin \mu$$

$$- \lambda_\gamma \frac{V}{W} \left[ T(\alpha + \delta) + q S \frac{\partial C_D}{\partial \alpha} \right] + \eta \left\{ \sin \gamma \frac{g}{V} \frac{1}{W} \left[ T(\alpha + \delta) + q S \frac{\partial C_D}{\partial \alpha} \right] \cos \mu \right\}$$

Equations (85), (86) together with Eq.(62) form a set of three algebraic relations that allow to solve for the two controls $\mu$ and $\alpha$ and the single multiplier $\eta$ in terms of the state and adjoint variables. Unfortunately, however, the roots of this system of nonlinear equations can not be found in closed form. Consequently, we need to resort to a numerical approach. In the present study a simple one-dimensional numerical search has been employed, which will be briefly described here. However, for the sake of transparency, let us first introduce the following variables:

$$\bar{\lambda}_E \triangleq \lambda_E - \eta \frac{g}{V} \sin \gamma$$

$$\bar{\lambda}_\gamma \triangleq \lambda_\gamma - \eta V \cos \gamma$$

$$\bar{\Phi} \triangleq \bar{\lambda}_\gamma \cos \mu + (\lambda_\gamma / \cos \gamma) \sin \mu$$

Substitution of these new variables into Eqs.(85) and (86) leads to optimal control
conditions that are very similar to Eqs.(73) and (77):

\[ \tan \mu_{cs} = \frac{(\lambda_\psi/cos \gamma)}{\lambda_y} \]  

(90)

\[ \alpha_{cs} = \frac{\lambda_E \frac{V^2}{g} [D_1 + \frac{T \delta}{qS}] - [\frac{T \delta}{qS} + L_1 - 2L_2 \alpha_{ref}] \Phi}{-2\lambda_E \frac{V^2}{g} [D_2 + \frac{T \delta}{qS}] + 2L_2 \Phi} \]  

(91)

where the subscript cs signifies a constrained subarc. Note that Eqs.(90) and (91) are completely identical to Eqs.(73) and (77) off the constraint boundary, where \( \eta(t) = 0 \).

In the present implementation of the optimal control conditions in the MSA algorithm, a numerical one-dimensional search is performed to find the multiplier \( \eta \) at each point on the constrained subarc. Using a guessed value for the multiplier \( \eta \), the optimal aerodynamic roll angle \( \mu \) can be computed from Eq.(90). Using the obtained result, the angle-of-attack \( \alpha \) can be computed from Eq.(91), as well as from the control constraint Eq.(62). Excepting a lucky guess for \( \eta \), the roots of Eqs.(91) and (62) will be different. The guessed value of \( \eta \) therefore needs to be iteratively adjusted until the angle-of-attack evaluated from Eq.(91) matches the root of Eq.(62) within a specified tolerance.

In the analysis of the optimal control on the constraint subarc we have up to this point ignored the control constraints (12) and (13). Fortunately, however, the above numerical procedure is hardly complicated by the inclusion of the control constraints (12) and (13). As a matter of fact, to enforce the aerodynamic roll angle limitations, we merely need to modify Eq.(90) along the lines of Eq.(82):

\[ \mu^* = \min \{ \mu_{max}, |\mu_{cs}| \} \cdot \text{sign}(-\lambda_\psi) \]  

(92)

In the present algorithm, the maximum angle-of-attack limit has not been enforced. The simple reason for this is that for the aircraft type considered here and with realistic roll angle limitations, it is just not possible to keep the aircraft on the state constraint boundary when maximum angle-of-attack is used. Indeed, in all numerical examples we found values well below the permissible maximum value of angle-of-attack, on a constrained subarc (if present).

A few remarks regarding the above numerical procedure are in place. First of all, note that for a given value of the roll angle limit, Eq.(62) represents a quadratic polynomial in the angle-of-attack \( \alpha \), thus permitting an analytic evaluation of the root. The numerical search procedure that has been employed relies on the subroutine DZREAL from the IMSL library(20), which implements Müller's method. The initial guess for \( \eta \) only needs to be given for the entry point \( t_e \) of the constrained subarc. The obtained result \( \eta(t_e) \) can then be used as the initial guess for the next point on the constrained subarc. The initial guess for \( \eta \) at time \( t_e \) is actually easily obtained by making use of one of the
necessary sign conditions:  

\[ \eta(t_{en}^+) = \sigma_1 \]  

(93)

where \( \sigma_1 \) is one of the multipliers used to enforce the entry jump conditions, which we will discuss next. In other words, the guess of \( \eta(t_{en}) \) can be related to the guess of \( \sigma_1 \).

Application of Eq.(31) to the present 2nd-order state constraint, results in the following tangency vector \( N(x, \zeta) \) that needs to be satisfied on a constraint subarc:

\[
N[x, y, h, E, \gamma, \zeta] = \begin{pmatrix}
    h_{ref} - h - \zeta \\
    -V\sin\gamma - W_h(x, y, h)
\end{pmatrix} = 0
\]  

(94)

where Eqs.(61), (62) and (6) were used. The jump conditions (41) and (42) are now readily evaluated:

\[
\lambda_x(t_{en}^+) = \lambda_x(t_{en}^-) + \sigma_1 \frac{\partial W_h}{\partial x(t_{en})}
\]  

(95)

\[
\lambda_y(t_{en}^+) = \lambda_y(t_{en}^-) + \sigma_1 \frac{\partial W_h}{\partial y(t_{en})}
\]  

(96)

\[
\lambda_h(t_{en}^+) = \lambda_h(t_{en}^-) + \sigma_0 + \sigma_1 [\frac{\partial V}{\partial h} \sin\gamma + \frac{\partial W_h}{\partial h}]_{t=t_{en}}
\]  

(97)

\[
\lambda_E(t_{en}^+) = \lambda_E(t_{en}^-) + \sigma_1 \frac{\partial V}{\partial E} \sin\gamma |_{t=t_{en}}
\]  

(98)

\[
\lambda_{\gamma}(t_{en}^+) = \lambda_{\gamma}(t_{en}^-) + \sigma_1 \left[ V \cos\gamma \right] |_{t=t_{en}}
\]  

(99)

\[
\lambda_{\zeta}(t_{en}^+) = \lambda_{\zeta}(t_{en}^-) + \sigma_0
\]  

(100)

Moreover, the multipliers \( \sigma_0 \) and \( \sigma_1 \) need to satisfy the sign conditions (43):

\[
\sigma_0 \geq 0 \quad ; \quad \sigma_1 \geq 0
\]  

(101)
Note that the partial derivatives of speed in Eqs.(97) and (98) readily follow from the
definition of specific energy in Eq.(63):

$$\frac{\partial V}{\partial E} = -\frac{\partial V}{\partial h} = \frac{g}{V}$$  \hspace{1cm} (102)

In addition to the sign condition (19), there are some more sign conditions which can be
used to check candidate solutions after they have been computed:\(^\text{19}\):

$$\dot{\eta}(t) \leq 0 ; \quad \ddot{\eta}(t) \geq 0 \quad , \quad t \in [t_{\text{en}}, t_{\text{ex}}]$$  \hspace{1cm} (103)

$$\eta(t_{\text{ex}}^-) = 0 ; \quad \dot{\eta}(t_{\text{en}}^-) \geq -\sigma_0$$  \hspace{1cm} (104)

In particular, the first condition in Eq.(104) is not only useful for an a posteriori test of the
solution, but also in setting up the MPBVP (prior to the solution).

Since the state constraint is of order two, we know from Section 3.1 that a touch
point may also occur on an optimal trajectory.

touch point

The optimal control conditions established for the case of an inactive state
constraint, i.e., Eqs.(73) and (77), are also applicable in case of a touch point. However, as
in the case of a constrained subarc, the adjoint are generally discontinuous at the touch
point. Application of the conditions (46) through (50) yields the following conditions :

$$\zeta(t_{ip}) = h_{\text{ref}} - h(t_{ip})$$  \hspace{1cm} (105)

$$\dot{h}(t_{ip}) = 0$$  \hspace{1cm} (106)

$$\lambda_h(t_{ip}^+) = \lambda_h(t_{ip}^-) + \bar{\sigma}_0$$  \hspace{1cm} (107)

$$\lambda_{\zeta}(t_{ip}^+) = \lambda_{\zeta}(t_{ip}^-) + \bar{\sigma}_0$$  \hspace{1cm} (108)

$$\bar{\sigma}_0 \geq 0 \quad ,$$  \hspace{1cm} (109)

It is clear that the jump and switch conditions for a touch point are much simpler than
those for a constrained subarc.

As already mentioned in Section 3.1, there is also a possibility of multiple constrained subarcs and touch points. For each specific combination of constrained subarcs and touch points, the necessary conditions need to be adapted. For the present windshear problem, four different possibilities of active state constraints have been found, including the two standard cases, viz. (i) a single touch point and, (ii) a single constrained subarc. The two other cases are, (iii) a constrained arc, followed by a touch point, and (iv) two consecutive touch points. Of course, at this stage we can not really rule out the possibility of other combinations. Before analyzing each case separately, we will first show the possibility of simplifying the MPBVP by eliminating some variables.

### 3.5 Simplification of the Minimax Windshear MPBVP

In this section we will show that it is generally possible to eliminate the variable $\zeta$ and its adjoint $\lambda_\zeta$. This possibility arises as a result of the fact that $\lambda_\zeta$ is piecewise constant on constrained and unconstrained subarcs, which immediately follows from inspection of Eq.(72). More specifically, the adjoint $\lambda_\zeta$ can only jump when entering a constrained arc, when reaching a touch point. If for example in the case of a single touch point, Eq.(72) is combined with Eq.(108), it is readily found that:

$$
\lambda_\zeta(t^-) = \lambda_\zeta(0) = 0 \quad ; \quad \lambda_\zeta(t^+) = \lambda_\zeta(t_f) = K_1
$$

$$
\Rightarrow \lambda_\zeta(t^+) = K_1 = \lambda_\zeta(t^-) + \sigma_0 = \bar{\sigma}_0
$$

The adjoint $\lambda_\zeta$ has now been completely eliminated. The two unknowns $t_{ip}$ and $\bar{\sigma}_0$ are determined by Eqs.(105) and (106). However, we have just established in Eq.(110) the result that $\bar{\sigma}_0 = K_1$. This means that $\zeta(t)$, which is constant according to Eq.(54), can be decoupled from the system. More specifically, $\zeta(t)$ is removed from the problem by dropping Eq.(54) along with condition (105). Of course, Eq.(105) can still be used to a posteriori compute $\zeta$.

For the case of a single constrained arc (case 2), the above analysis is virtually unchanged, except that instead of Eq.(108), we need to consider Eq.(100). This leads to the obvious result:

$$
\lambda_\zeta(t^-) = \lambda_\zeta(0) = 0 \quad ; \quad \lambda_\zeta(t^+) = \lambda_\zeta(t_f) = K_1
$$

$$
\Rightarrow \lambda_\zeta(t^+) = K_1 = \lambda_\zeta(t^-) + \sigma_0 = \sigma_0
$$

The simplification for the cases 3 and 4 proceeds in a very similar fashion as well. In case 3 jumps in the adjoint $\lambda_\zeta$ occur at the entry point of the constrained arc and at the touch point. Combining Eqs.(100),(108) and (72) then yields:
\[ \lambda_\zeta(t_{en}^-) = \lambda_\zeta(0) = 0 \quad ; \quad \lambda_\zeta(t_{ip}^+) = \lambda_\zeta(t_f) = K_1 \]

\[ \Rightarrow \lambda_\zeta(t_{ip}^-) = \lambda_\zeta(t_{en}^+) = \lambda_\zeta(t_{en}^-) + \sigma_0 \]

\[ \lambda_\zeta(t_{ip}^+) = K_1 = \lambda_\zeta(t_{ip}^-) + \sigma_0 = \sigma_0 + \tilde{\sigma}_0 \] (112)

Case 4 represents the situation where we have two touch points. Obviously, in order to be able to distinguish between these two touch points it is useful to number them. Here we will use subscripts with Roman numerals for this purpose:

\[ \lambda_\zeta(t_{ip}^+) = \lambda_\zeta(0) = 0 \quad ; \quad \lambda_\zeta(t_{ip_{II}}^+) = \lambda_\zeta(t_f) = K_1 \]

\[ \Rightarrow \lambda_\zeta(t_{ip_{II}}^-) = \lambda_\zeta(t_{ip}^+) = \lambda_\zeta(t_{ip}^-) + (\tilde{\sigma}_0)_I = (\tilde{\sigma}_0)_I \]

\[ \lambda_\zeta(t_{ip_{II}}^+) = K_1 = \lambda_\zeta(t_{ip_{II}}^-) + (\tilde{\sigma}_0)_{II} = (\tilde{\sigma}_0)_I + (\tilde{\sigma}_0)_{II} \] (113)

It can be concluded that in all four cases of active state constraints the multipliers associated to the jump conditions can be related to the parameter \( K_1 \).

3.6 Numerical Implementation of the Minimax Windshear MPBVP

The standard transformation of the minimax optimal control state constrained optimal control problem is convenient in enabling the numerical treatment of the resulting MPBVP using the multiple-shooting technique. For the resolution of the problems formulated above we have employed the well-known BOUNDSCO(12) code that has also been used in previous work(5).

Each of the four different cases of active state constraints, described in the previous section, requires a specific implementation in BOUNDSCO. Unfortunately, for a given windshear encounter scenario it is not generally possible to a priori tell which is the appropriate case. As a result, some trial and error is typically required. However, as already mentioned in the introduction, an alternative direct numerical optimization method has been utilized in the research effort as well. The nonlinear programming/collocation algorithm of Ref.15 proved to be very a very effective tool for examining the activity of the state constraint. As a result, the trial and error effort to establish optimal solutions with BOUNDSCO could be significantly alleviated. Nevertheless, the effort was still fairly laborious, as a result of the fact that the MPBVP associated to the windshear problem is highly sensitive and therefore rather difficult to solve.

The MPBVP’s as implemented in BOUNDSCO will now be summarized here for each of the four cases.
Case 1: a single touch point

The MPBVP for case 1 is described by the 7 state equations (4) through (10), the 7 adjoint equations (65) through (71), with the optimal evaluated using Eqs.(73) and (77). Moreover, full throttle is assumed. The system of 14 differential equations is augmented with a trivial differential equation for $\bar{\sigma}_0$, with corresponding initial condition:

$$\frac{d\bar{\sigma}_0}{dt} = 0 \ ; \ \bar{\sigma}_0(0) = K_1 ,$$

(114)

where Eq.(112) was used to generate the initial condition. To solve the 15 first-order differential equations 15 integration constants are required. Also the instant $t_p$ at which the touch point is reached is unknown. To resolve these 16 unknowns 16 conditions are needed. Obviously, we have 7 initial boundary conditions as well as 7 natural boundary conditions at the terminal time $t_f$, given in Eqs.(65) through (71). The remaining 2 conditions are the initial condition in Eq.(114) and the switching condition (106). Note that $\bar{\sigma}_0$ is coupled to the MPBVP only through the jump condition (107), the only one for the case of a single touch point. This completes the MPBVP for case 1. Once the solution to the MPBVP has been established, Eq.(105) can be used to evaluate $\zeta$ afterwards.

Case 2: a single constrained subarc

The MPBVP for case 2 is again described by the 14 state and adjoint equations, but this time 2 additional trivial differential equations for the multipliers $\sigma_0$ and $\sigma_1$ are also required:

$$\frac{d\sigma_0}{dt} = 0 \ ; \ \sigma_0(0) = K_1$$

(115)

$$\frac{d\sigma_1}{dt} = 0$$

(116)

In other words, in this particular case we have a total of 16 differential equations. For resolution the specification of 16 integration constants is required. The optimal controls in the state-adjoint system is obtained from Eqs.(91) and (92), in conjunction with $\eta(t) = 0$ off the constrained subarc and Eq.(62) on the constrained arc. In addition to the 16 integration constants, both the entry time $t_{en}$ and the exit time $t_{ex}$ on the constrained arc are unknowns as well. We thus need 18 conditions to properly define the MPBVP. Again we have 7 initial boundary conditions along with 7 natural boundary conditions at the terminal time $t_f$ and the initial condition in Eq.(115). The three remaining conditions are, $\bar{h}(t_{en}) = 0$ (Eq.(61)), $\bar{h}(t_{ex}) = 0$ (Eq.(62)) and $\eta(t_{ex}) = 0$ (Eq.(104)). At the entry point $t_{en}$, the 5 jump conditions (95) through (99) need to be satisfied.
Case 3: a single constrained subarc, followed by a single touch point

Case 3 is more or less a combination of cases 1 and 2. The 14 state and adjoint equations are now augmented by three trivial differential equations:

\[
\frac{d\tilde{\sigma}_0}{dt} = 0 \\
\frac{d\sigma_0}{dt} = 0 \\
\frac{d\sigma_1}{dt} = 0
\]  
(117)  
(118)  
(119)

The 17 integration constants together with the unknown entry time \( t_{en} \), exit time \( t_{ex} \), and the time \( t_p \) at which the touch point is reached add up to a total number of 20 unknowns. These 20 unknowns call for 20 conditions, 14 of which are provided by the forced and natural boundary conditions of the original problem. The remaining 6 conditions are the following, \( \sigma_0 + \tilde{\sigma}_0 = K_1 \) (Eq.(112)), \( h(t_{en}) = 0 \) (Eq.(61)), \( \dot{h}(t_{en}) = 0 \) (Eq.(62)), \( \eta(t_{en}) = 0 \) (Eq.(104)), \( h(t_p) = 0 \) (Eq.(106)) and \( h(t_{en}) = h(t_p) \). The last condition obviously ensures that at the touch point the same minimum altitude is achieved as on the constrained subarc. The MBPVP for case 3 features no less than 6 jump conditions, viz. 5 at the entry point of the constrained subarc (Eqs.(95) through (99)) and 1 at the touch point (Eq.(107)).

Case 4: two consecutive touch points

The case of two consecutive touch points is actually very similar to case 3, but it has only 18 unknowns. The reason for this that in case 3 we have only two trivial differential equations:

\[
\left( \frac{d\tilde{\sigma}_0}{dt} \right)_I = 0 \\
\left( \frac{d\tilde{\sigma}_0}{dt} \right)_II = 0
\]  
(120)  
(121)

and two unknown instances \( t_{pI} \) and \( t_{pII} \). The 18 conditions required for the MBPVP include again the 14 initial and final boundary conditions on states and adjoints, along with the following 4 conditions are the following, \( \tilde{\sigma}_0 \) + \( \sigma_0 = K_1 \) (Eq.(113)), \( \dot{h}(t_{pI}) = 0 \) and \( \dot{h}(t_{pII}) = 0 \) (Eq.(106)), and \( h(t_{pI}) = h(t_{pII}) \). There are only 2 jump conditions to be satisfied, one at each touch point (Eq.(107)).
4. Computational Results

4.1 Reference Scenario

In order to investigate the characteristic features of the optimal escape trajectories, the principal parameters that have been varied in this study are the weight factors $K_1$ and $K_2$ in the performance index (59). The reference situation that has been selected here to serve as a baseline concerns a non-turning flight, with $K_2 = 0$. This implies that an aircraft will fly exactly along the x-axis of the reference frame (see Fig.2), passing right through the microburst center.

In Figs. 4 the results pertaining to the reference situation have been summarized. Three different values for $K_1$ are considered in Figs. 4, viz. $K_1 = 0, 0.025$ and 25. The case $K_1 = 0$ represents the original Bolza problem, whereas $K_1 = 25$ represents virtually a pure Chebyshev problem. Figure 4a shows the time-history for the control variable angle-of-attack for each of the three cases. It can be seen that the angle-of-attack behavior for $K_1 = 0$ and $K_1 = 0.025$ is very similar. Nevertheless, the resulting altitude profiles are quite different, as can be observed in Figure 4b. In particular, the minimum altitude achieved is significantly higher for $K_1 = 0.025$ (about 17m !). Furthermore, we note that a massive increase in the parameter $K_1$ by a factor 1000 (from $K_1 = 0.025$ to $K_1 = 25$) leads to only a modest improvement in the actually achieved minimum altitude (about 5.5m).

However, the angle-of-attack behavior is markedly different for the latter case, especially in the initial phase of the encounter. A striking feature that can be observed in the angle-of-attack behavior for the two cases $K_1 = 0.025$ and $K_1 = 25$, concerns the discontinuity in $\frac{dc}{dt}$ at (approximately) $t = 10$ sec. and $t = 20$ sec., respectively. These instances actually represent the entry times of the constrained subarcs that occur in both trajectories. The duration of these constrained subarcs is about 6.1 and 14.7 seconds, respectively. The trajectory for $K_1 = 25$ also features a touch point in the after-shear region.

Figures 4b, 4c and 4d show that for an increasing value of the weight parameter $K_1$, altitude is traded for airspeed in the initial phase of the trajectory. The variation in specific energy is very similar for all trajectories. The point of minimum energy is reached near the end of the high-shear region, which occurs when the values of the F-factor start to come down rapidly (see Figure 4f). Observe in Figure 4f that an increase in $K_1$ leads to a reduction in the first peak in the F-factor time-history, but to an increase in the second peak. This is not really a surprising phenomenon, since the higher the value of $K_1$ the earlier the minimum altitude is reached. Since the magnitude of vertical windspeed is proportional to altitude, this implies that initially the downdraft will be lower, only to become higher at a later stage. Figure 4g, which shows the time-histories of downdraft, confirms these observations. On the other hand, it is clear from Figure 4h that the horizontal windspeed variations are very similar for all three cases. Due to the prevailing downdraft, the constant-altitude constrained subarcs are flown with positive values for the flight path angle, as evidenced by Figure 4e. Figure 4e also reveals a very steep climb in the aftershear-region. This is a direct result from the fact that during this phase maximum angle-of-attack is utilized (see Figure 4a). This, unfortunately, also leads to very low values for airspeed in the terminal phase. As mentioned before, this situation arises as a result of the fact that no terminal boundary conditions have been specified. In later examples we will show that this undesirable situation can be easily corrected, without adversely affecting the minimum altitude performance.
Figure 4: Comparison of extremal solutions for various values of $K_1$. No lateral maneuvering ($\mu_{\text{max}} = 0$); no terminal boundary conditions on the state variables; the weight factor $K_2 = 0$. 
Figure 4: (concluded)
4.2 Lateral Escape Maneuvers

In this section lateral escape trajectories will be examined. We start off by considering the same scenario as in Section 4.1, however, this time lateral maneuvering is facilitated by raising the maximum aerodynamic roll angle limit from 0° to 10°. In Figs. 5 the results for the lateral escape trajectories are shown. Again three different values for $K_1$ are considered. The time-histories for the control variable angle-of-attack are shown in Figure 5a. A comparison with the nonturning results in Figure 4a learns that initially the angle-of-attack is much larger in a lateral escape maneuver. The relatively high initial angle-of-attack for a lateral maneuver results in a relatively high lift. The increase in lift is needed to produce a high turn-rate which is then used to direct the aircraft away from the microburst center (see Figs 5g, 5h and 5i). One of the most striking results established in the previous study\(^5\) concerned the initial climb in a lateral escape maneuver. Indeed, Figure 5b shows the presence of such an initial climb in the "Bolza trajectory", with the weight parameter $K_1 = 0$. Apparently, the relatively high lift which is needed to produce a high turn-rate also leads to an initial climb in this particular lateral escape maneuver. Surprisingly, the initial climb completely disappears in the optimal trajectories when the weight factor $K_1$ is increased. As a matter of fact, for $K_1 = 25$ the altitude profiles of the nonturning and turning escape maneuvers are very similar, as can be seen by comparing Figs. 4b and 5b. Nevertheless, the advantage of the lateral escape maneuver remains preserved: in the lateral escape trajectory the minimum altitude is some 5m higher than in the nonturning flight path.

In addition to improving the turn-rate, drag considerations also play an important role in the optimization process. Low drag results in a low energy bleed-off rate which is needed to maintain climb-gradient capability. Although an instantaneous reduction in the angle-of-attack results in an immediate decrease in drag, it also leads to higher airspeeds at a later stage, which, in turn, will lead to an increase in drag. Figures 5b and 5c confirm that an increase in $K_1$ indeed leads to trajectories with a higher speed in the initial phase, clearly achieved at the expense of altitude. As observed earlier, a lower altitude helps to negate the downdrift. In the optimization process a fairly complex compromise between conflicting requirements is thus established. Apparently, reducing the downdrift has priority in an optimal Chebyshev solution.

Similar to the reference situation, the lateral escape trajectories contain a constrained subarc for $K_1 = 0.025$ and $K_1 = 25$. Moreover, the latter trajectory also has a touch point somewhere in the after-shear region. Overall, in a lateral escape maneuver the distribution of kinetic and potential energy seems to be influenced by the weight parameter $K_1$ to a somewhat greater extent than in the straight maneuvers (compare Figures 4a, 4b and 4c with 5a, 5b and 5c respectively). This relatively large influence of the weight factor $K_1$ on the performance, as well as on the trajectory behavior in general, appears to indicate that the Bolza problem formulation really is not all that good. Indeed, slightly raising the factor $K_1$ from $K_1 = 0$ in the nonturning baseline problem has a much greater impact on performance than raising the roll angle limit with a few degrees, to permit a lateral escape.

The lateral maneuver aspects are illustrated in Figs. 5g, 5h and 5i. Similar to the control variable angle-of-attack $\alpha$, the behavior of the aerodynamic roll angle $\mu$ is not significantly affected by a slight increase in $K_1$. Only when $K_1$ is increased significantly, to such an extent that the Chebyshev part in the performance index becomes dominant, the aerodynamic roll angle behavior is affected.
Figure 5: Comparison of extremal solutions for various values of $K_1$. Lateral maneuvering, with $\mu_{\text{max}} = 10^0$; no terminal boundary conditions on the state variables; the weight factor $K_2 = 0$. 
Figure 5:
(continued)
Indeed, for $K_1 = 25$ the aerodynamic roll angle leaves it limit much earlier and at approximately $t = 33.6$ sec. the decay seems to be completed. However, soon after this instance the aerodynamic roll angle rapidly increases again and subsequently behaves similar to the trajectories with low values of $K_1$. Obviously, the sudden rise in aerodynamic roll angle explains the "wiggle" in the corresponding time-history of heading angle $\psi$ as shown in Figure 5h. The ground tracks shown in Figure 5i make clear that the three trajectories only start to diverge more or less outside the shear-region, but that the differences are not really dramatic.

How can this peculiar roll behavior be explained? It is important to realize that the instance where this strange roll angle behavior occurs happens to be the final touch point to the constrained subarc. Furthermore, in a pure Chebyshev problem the behavior of the controls is completely irrelevant in the interval $[t_{fp}, t_f]$. Indeed, once the final touch point (which occurs already outside the shear-region) has been passed, the setting of the controls can be chosen more or less arbitrarily without affecting the minimum altitude performance (state constraint). This is a direct consequence of the fact that the adjoints tend to vanish on the interval $[t_{fp}, t_f]$. Obviously, this may lead to difficulty in computing the optimal solution. One way to resolve this issue is to simply regard the last touch point $t_{fp}$ as the unspecified final time $t_f$. Since we would like to keep a fixed final time, numerical complications are avoided here by precluding "pure" Chebyshev problems through the specification of the performance index (59), which inherently contains an integral part. The roll angle behavior shown in Figure 5g can now be fully understood. On the interval $[0, t_{fp}]$ the emphasis in the optimization process is on the Chebyshev part of the
performance index, while on the interval \([t_{n-1}, t_n]\) this is no longer required and the attention can therefore be shifted towards minimizing the integral part of the performance index. Evidently, the truly interesting part of the trajectory concerns the interval \([0, t_1]\), where it is to be understood that \(t_{n-1}\) refers to the final touch point. In this particular example, there is only one touch point (in addition to a constrained subarc), but in a later example we will see that for large values of the specified roll angle limit, trajectories with two consecutive touch points may also occur (for large values of \(K_1\)).

Up to this point, we have more or less implicitly assumed that \(K_1 = 25\) is sufficiently large to represent a Chebyshev solution. In order to verify this assumption, we have computed optimal lateral escape trajectories for a range of values of \(K_1\), rather than just the three values indicated in Figs. 5. The results are summarized in Figure 6. Figure 6 shows how the integral part \(J_1\) and the Chebyshev part \(J_2\) in the performance index vary with \(K_1\) in the optimal solution.

![Graph showing \(J_1\) and \(J_2\) vs. ratio \(K_1\)](image)

**Figure 6:** Variation of optimal value of the integral part \(J_1\) and Chebyshev part \(J_2\) with the weight factor \(K_1\). The performance index \(\bar{J} = J_1 + K_1J_2\); weight factor \(K_2 = 0\); lateral maneuvering, with \(\mu_{\text{max}} = 10^\circ\); no terminal boundary conditions on the state variables.
Note in Figure 6 that a logarithmic scale is used for the parameter $K_1$. It is recalled that $K_1 = 0$ represents the solution in which only the integral part $J_1$ is minimized, while for $K_1 \to \infty$ the Chebyshev part $J_2$ in the performance index (59) is minimized. The Chebyshev part here represents the maximum altitude drop at any point along the trajectory, measured relative to an altitude $h_{ref} = 400m$. Note that the actually achieved minimum altitude is thus given by $(400 - J_2)$. As expected, the value of $K_1$ monotonically increases with $K_1$, while $J_2$ monotonically decreases with $K_1$. What really comes as a surprise is the magnitude in the variations of the two performance measures. By gradually moving from a Bolza problem to a Chebyshev problem, the minimum altitude performance can be improved by as much as 25m! It seems safe to state that the Bolza approximation, as presently employed, is indeed rather poor. Attempts have been made to improve the Bolza approximation by increasing the exponent in the integrand of $J_1$ from 6 to 8 or higher, however this proved to be futile: the higher the exponent, the greater the numerical complications in establishing solutions.

Another conclusion that can be drawn from Figure 6 is that the absolute minimum altitude performance can be fairly closely approximated by solving the optimal control problem for any value of $K_1 > 0.1$. In other words, the value $K_1 = 25$ that was adopted in the previous two examples is indeed representative for a Chebyshev solution. Table I presents some numerical values corresponding to Figure 6. As was to be expected, the value $J_1$ is lowest for $K_1 = 0$, while $J_1 + 0.1J_2$ is lowest for $K_1 = 0.1$.

<table>
<thead>
<tr>
<th></th>
<th>$K_1 = 0$</th>
<th>$K_1 = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_1$</td>
<td>8.8300</td>
<td>9.2262</td>
</tr>
<tr>
<td>$J_2$</td>
<td>345.61</td>
<td>321.84</td>
</tr>
<tr>
<td>$J_1 + 0.1J_2$</td>
<td>43.391</td>
<td>41.410</td>
</tr>
</tbody>
</table>

A striking feature that can be observed in the curves $J_1$, $J_2$ versus $K_1$ concerns the slight "cusp" around $K_1 = 0.03$. The presence of this cusp can be explained from the fact that for values $K_1 > 0.03$ trajectories start featuring a touch point in addition to a constrained subarc. For values of $K_1 < 0.015$, the trajectories only feature a single touch point, rather than a constrained subarc. In all, Figure 6 thus comprises three different types of extremal trajectories, viz. trajectories featuring a single touch point, trajectories featuring a single constrained subarc, and trajectories featuring both a constrained subarc and a touch point. It needs to be noted that these particular three types of trajectories only occur for values of the maximum aerodynamic roll angle limit lower than about 15°. Indeed, for values of $\mu_{max}$ of 15° or higher, a single constrained subarc will not occur.
For larger values of the roll angle limit (let's say, from 15° up to 25°), a plot similar to Figure 6 will contain the following three types of extremal solutions: trajectories featuring a single touch point, trajectories featuring two consecutive touch points, and trajectories featuring both a constrained subarc as well as a touch point. The reason why for lateral maneuvers with a large roll angle limit, an increase in the parameter $K_1$ may result in a touch point splitting up into two touch points rather than a constrained subarc, is the fact that this type of lateral maneuvers typically calls for high angles-of-attack in the initial phase. At very high angles-of-attack it is simply not possible to remain on a constrained subarc. On the other hand, as may have become clear from the previous examples, increasing $K_1$ typically leads to lower angles-of-attack in the initial phase. As a consequence a constrained subarc may still occur for very large values of $K_1$.

The next numerical example serves to illustrate an optimal trajectory featuring two consecutive touch points. In this example, the following set of parameters has been used: $K_1 = 0.02$, $K_2 = 0$ and $\mu_{\text{max}} = 15°$. For reference purposes also the corresponding nonturning trajectory ($\mu_{\text{max}} = 0°$) is included. The results for this particular example are summarized in Figs. 7. Figure 7b reveals that the nonturning trajectory contains a constrained subarc, whereas the lateral escape trajectory features two touch points. From Figure 7a it can be learned that the first touch point occurs at a relatively high angle-of-attack, but still just below the limit. On the other hand, the second touch point takes place when angle-of-attack is at its limit. In contrast to the first touch point, the second touch point occurs when the aircraft is already in the after-shear region, as can be seen in Figure 7h. Another interesting observation can be with regard to roll angle behavior. Inspection of Figure 7f makes clear that at the first touch point roll angle is at its maximum limit, while the wings are virtually again at the second touch point.

Although the lateral escape trajectory features an initial climb (see Figure 7b), it needs to be realized that the occurrence of this phenomenon is largely a consequence of the fairly low value of $K_1$ that has been adopted in this example. A further increase in $K_1$ will eventually eliminate this initial climb. Nevertheless, all examples make clear that lateral maneuvering is beneficial, regardless of the value of $K_1$. In this specific example, lateral maneuvering leads to an increase in minimum altitude of more than 12m! Observe that the nonturning trajectory already exhibits a tendency towards a touch point in the after-shear region.

The higher minimum altitude in the lateral maneuver obviously leads to higher values of the downdraft in the shear region (see Figure 7g) and consequently to a higher first peak in the F-factor time-history (see Figure 7h). Indeed, the performance improvement offered by lateral maneuvering is a direct consequence of the fact that the shear region is simply transited much quicker. This implies that in a lateral maneuver the energy drain due to windshear is not really less, but merely halted much earlier. Consequently, the lowest value of specific energy value is also higher in a lateral maneuver (see Figure 7d). Note that, as evidenced by Figure 7c, in a lateral maneuver airspeed is typically lower in the shear region.

At this stage, it is important to point out that it proved to be very difficult to actually compute optimal trajectories for relatively high values of both $K_1$ and $\mu_{\text{max}}$. Reducing the fixed final time $t_f$ slightly relaxes the computational difficulties, but can not eliminate them altogether. As a matter of fact, reduction of $t_f$ is mandatory for maneuvers with a roll angle limit in excess of 15°, if it is desired to keep the reference altitude unaltered. Cutting short the shear region obviously results in a higher final altitude. Yet, final altitude most remain below reference altitude at any time such as to ensure that the integral part $J_1$ of the performance index remains valid.
Figure 7: Comparison of extremal solutions for two values of $\mu_{\text{max}}$. The weight factor $K_1 = 0.02$; the weight factor $K_2 = 0$; no terminal boundary conditions on the state variables.
Figure 7:
(concluded)
Figure 8 presents more or less the complete inventory of optimal trajectories (up to $\mu_{\text{max}} = 20^\circ$) that we have been able to compute using the multiple shooting algorithm (indirect approach). In contrast, the direct optimization technique permits the computation of optimal trajectories for any (reasonable) value of $K_i$ and $\mu_{\text{max}}$.

A problem of indirect methods in general is that there is always a danger that the algorithm converges to a nonoptimal solution. In the present application, where indirect solutions are readily available, this phenomenon is less likely to manifest itself. Indirectly computed optimal solutions can not only be used as starting solutions in the direct optimization process, but may also be used to validate the resulting trajectories. The direct method performed above expectations, in the sense that virtual identical results were obtained for both methods. In particular, altitude deviations between directly and indirectly computed solutions never were found to exceed 0.5m. Obviously, a direct optimization method gives more "crude" results than the multiple shooting approach due to the discretized nature of the problem. Typically, 26 grid points have been used here in the collocation process (2 second intervals). Using a homotopy approach, direct optimal solutions could also be computed in situations where it proved to be impossible to obtain converged solutions with the multiple shooting approach. However, a close examination of the resulting trajectories did not reveal any peculiarities that could explain the convergence difficulties of the multiple shooting algorithm in those situations.

For example, in the case of a maximum aerodynamic roll angle limit of 25°, the "pure" Chebyshev solution (i.e. $K_i \to \infty$) obtained using a direct method demonstrated a minimum altitude of 108m. The "best" indirect solution that could be found for this particular value of the roll angle limit was for $K_i = 0.01$, resulting in a minimum altitude of 104m. In the first case, the optimal trajectory contains a constrained subarc, followed by a touch point. The latter case features two consecutive touch points.

![Figure 8](image)

**Figure 8:** Variation of minimum altitude with the specified value of the aerodynamic roll angle limit, for various values of the weight factor $K_i$. The weight factor $K_2 = 0$; no terminal boundary conditions on the state variables.
4.3 Maximizing Specific Energy at Termination

In this section (lateral) escape trajectories will be examined that not only attempt to minimize the maximum altitude drop, but also aim at maximizing specific energy at termination. In other words, in contrast to the numerical examples in the previous sections, in this section optimal trajectories will be presented for non-zero values of $K_2$. More specifically, the value of $K_2$ will be gradually increased, but of course such that the terrain limit $h \geq 0$ is always obeyed.

Figures 9 present the results for a typical example, in which the following parameters have been adopted: $K_1 = 0.05$ and $v_{\text{max}} = 10^5$. In Figs.9 some results are shown for four different values of $K_2$. The most striking common feature of optimal solutions with non-zero values for $K_2$, is the fact that they all terminate with zero angle-of-attack (see Figure 9a). This particular feature is not really difficult to understand. First of all, recall from Eq.(68) that $\lambda_E(t_f) = -K_2$. Evaluation of Eq.(77) at $t = t_f$ makes now readily clear that, for any finite value of $\lambda_E$ and with $\Phi \to 0$, $\alpha(t_f)$ attains a fixed negative value. Application of the control constraints in Eq.(83) then results in $\alpha'(t_f) = 0$.

For relatively small values of $K_2$, the emphasis in the optimization process obviously still is on improving the minimum altitude performance. For increasing values of $K_2$, the emphasis shifts towards the maximum final energy criterion. This behavior is particularly clearly demonstrated in Figs. 9a and 9b. For a small non-zero value of $K_2$, i.e., $K_2 = 0.005$, the angle-of-attack behavior within the shear region is virtually the same as that for the trajectory with $K_2 = 0$. Consequently, also the minimum altitude achieved is virtually the same for both trajectories. Figure 9d shows that the effort to gain specific energy is confined to the after-shear region only. As the parameter $K_2$ is increased, angle-of-attack will leave its limit earlier in the after-shear region. As a matter of fact, when $K_2$ is increased sufficiently, angle-of-attack will no longer reach its limit. This is evident in the solution for $K_2 = 0.05$ (the same value as used for $K_1$). In this particular case, the controls are noticeably influenced, already within the shear region. Indeed, it is clear that minimum altitude is traded for an additional gain in specific energy. Note that the loss in minimum altitude is relatively modest in comparison with the gain in specific energy. Figure 9b also reveals that the optimal trajectory now no longer has a touch point in the after-shear region, i.e., only a constrained subarc is present.

A further increase in $K_2$ leads to significant increases in specific energy at the expense of a further reduction in minimum altitude. For $K_2 = 0.2$, the minimum altitude is only 6m. Clearly, the solution exhibits a single touch point for this particular value of $K_2$. From a practical point of view the resulting trajectory is hardly realistic, however, it does illustrate the tradeoff between maximizing minimum altitude and maximizing final specific energy. The time-history of angle-of-attack in Figure 9a shows that for large values of $K_2$, fairly low values of angle-of-attack are used throughout the maneuver. Gaining energy apparently calls for low altitudes and high airspeeds (see Figure 9c). A high airspeed has the advantage that the high-shear region is passed quicker (see Figure 9h), while a low altitude results in a moderate downdraft only (see Figure 9g).

Even the specification of only a modest value of $K_2$ already helps to avoid low airspeeds in the after-shear region. Nevertheless, the behavior in the terminal phase is still far from desirable. The gain in airspeed in this phase comes at the expense of altitude. Indeed, in the final phase trajectories with non-zero values for $K_2$ exhibit a steep dive and terminate with a negative value for the flight path angle (see Figure 9e). Evidently, this can be attributed to the reduction of angle-of-attack to zero in the final phase.
Figure 9: Comparison of extremal solutions for various values of $K_2$. Lateral maneuvering, with $\mu_{\text{max}} = 10^3$; no terminal boundary conditions on the state variables; the weight factor $K_1 = 0.05$. 
Figure 9: (concluded)
A second negative consequence of this angle-of-attack behavior can be observed in Figure 9f. Unlike in the previous examples, aerodynamic roll angle does not decay to zero, but rather diverges in the final few seconds. Although it does not really have an impact on the energy and altitude performance, it does complicate the numerical process. The negative consequences of the angle-of-attack behavior near termination can be simply avoided by specifying a single terminal boundary condition, namely, a specified final value for the flight path angle. The final example in this report serves to illustrate this point.

This final example actually involves one of trajectories considered in the previous example, namely, the trajectory computed for \( K_2 = 0.005 \). However, in addition to this trajectory which has been computed without a terminal boundary condition on the flight path angle, we also consider two trajectories with a prescribed terminal value for the flight path angle, viz. \( \gamma(t_f) = 0^\circ \) and \( \gamma(t_f) = 5^\circ \). Although Miele et al.\(^4\) specify the flight path angle corresponding to the steepest climb (in steady state) as the terminal condition, we prefer a parametric evaluation. The simple reason for this is that the aircraft does not terminate its trajectory in steady state. Consequently, any specification of boundary conditions is therefore somewhat arbitrary. The zero terminal value for the flight path angle reflects the fact that we seek to avoid a terminal dive, while keeping the impact on the performance to a minimum. The condition \( \gamma(t_f) = 5^\circ \) is probably more realistic, in the sense that the aircraft will end up in a state where it is able to gain both airspeed and altitude.

The results for this example have been summarized in Figure 10. Note that since terminal boundary conditions only affect the solution in the after-shear region, the plots concern the final 25 seconds on the escape maneuver only. As a matter of fact, inspection of Figs. 10 makes clear that the influence is limited to final 15 seconds only. Consequently, the minimum altitude performance remains almost unaffected. From Figure 10f it can be concluded that the influence of boundary conditions on the final specific energy level is also marginal.

The impact of specifying a terminal boundary condition on flight path angle can be best observed in Figure 10a. In contrast to the situation where no terminal boundary conditions are enforced, angle-of-attack now terminates with non-zero values, well below the maximum limit. This behavior is fairly transparent as well. Due to specification of the terminal value for flight path angle, the transversality condition (69) involving the flight path angle adjoint now no longer generally applies. Therefore, the terminal value of the flight path angle adjoint \( \lambda_\gamma(t_f) \) will be non-zero as well. In turn, a finite value of \( \lambda_\gamma \) implies a finite value of \( \Phi \), according to Eq.(76). Finally, finite values of both \( \Phi \) and \( \lambda_\gamma \) ensure that \( \alpha_{uc} \) in Eq.(77) remains finite, and, apparently, this finite value remains within the permitted range \([0,\alpha_{max}]\) for the present example trajectories.

Due to the changes in the angle-of-attack behavior, the overall airspeed and altitude behavior has also become markedly better, from a practical perspective. In particular, the airspeed oscillations in the terminal phase are smoothed (see Figure 10c) and the terminal dive has disappeared. Airspeed remains well above 50m/s throughout the maneuvers considered in the present scenario.

From the time-histories of aerodynamic roll angle in Figure 10c it can be concluded that specifying a terminal boundary condition for flight path angle completely eliminates the divergent behavior near termination. Including a final specific energy term in the performance index, along with adopting a terminal boundary condition has helped the numerical resolution of the minimax trajectory, in the sense that the tendency of the adjoint variables to vanish in the after-shear region has been attenuated.
Figure 10: Comparison of extremal solutions for various values of the final boundary condition on flight path angle. Weight factor $K_1 = 0.05$; weight factor $K_2 = 0.005$; lateral maneuvering, with $\mu_{\text{max}} = 10^\circ$.
Figure 10:
(concluded)
5. Concluding Remarks

Optimal lateral escape trajectories in a microburst wind field were studied for an aircraft on final approach. The performance index being minimized was the peak value of altitude drop, with the additional consideration of maximizing final specific energy. Minimizing the altitude drop can be achieved by directly solving the associated Chebyshev problem, but also by approximating the minimax performance index by a Bolza integral. The latter approach is mathematically much simpler and therefore has generally been the method of choice up to this point. A clear disadvantage of this approach is that it only offers an approximation to the original optimal solution, and we do not really have a good idea of the accuracy of the approximation. For this reason a new composite performance index is introduced, which allows the problem to be solved as a combination of a Chebyshev and a Bolza approach. With the weight factor \( K_1 \) set to zero in the performance index, the Bolza problem is addressed, while with \( K_1 \to \infty \) the Chebyshev problem is obtained. The parameter \( K_2 \) thus more or less serves as an interpolation parameter between the two solutions. In the numerical procedure, the weight factor \( K_1 \) has also been used as a homotopy parameter. A second weight factor \( K_2 \) was introduced to embed the maximum terminal specific energy term in the performance index. In this report, the MPBVP corresponding to the composite performance index has been derived and implemented in a multiple shooting code.

Unfortunately, for large values of the parameter \( K_1 \), it proved to be rather difficult to generate optimal lateral escape trajectories with a specified roll angle limit larger than 15°, with the multiple shooting approach. An alternative direct optimization method based on collocation and nonlinear programming did not exhibit such difficulties, so that a good insight into optimal trajectory behavior could still be achieved. Depending on the specified value of the roll angle limit, typically three different types of extremal solutions can be found when the parameter \( K_1 \) is varied between zero and a large value. For roll angle limits up to 15°, trajectories may contain a single touch point, a single constrained subarc, or a constrained subarc and a touch point. For roll angle limits in excess of 15°, trajectories may contain a single touch point, two touch points, or a constrained subarc and a touch point.

Probably the most striking result obtained in this report relates to the dramatic improvement in minimum altitude performance that can be obtained by increasing the weight factor \( K_1 \). This leads us to believe that the previously employed Bolza performance index approximation is not really all that useful. As a matter of fact, application if this approximation in its present form appears no longer justified. Since this Bolza approximation is fairly widely used, this conclusion is likely to have a much wider bearing than just the present windshear problem.

A close examination of the numerical results has revealed that the improvement in minimum altitude performance with increasing values of \( K_1 \), is primarily realized by reducing angle-of-attack in the initial phase of the microburst encounter. An additional consequence turned out to be the complete removal of a peculiar phenomenon discovered in earlier work involving the Bolza problem, namely a climb in the initial phase of the encounter. In contrast, a Chebyshev solution appears to rapidly trade altitude for speed, such as to reduce the exposure to downdraft.

Inclusion of a final specific energy term in the performance criterion turned out to simplify, rather than to complicate the computational process, provided that a terminal boundary condition on flight path angle is imposed as well. It is conjectured that inclusion
of this term helps to avoid existence and uniqueness problems which frequently plague Chebyshev optimization problems, especially when formulated as fixed final time problems. If a small weight factor $K_2$ is selected (i.e., $K_2 < K_1$), the influence of the maximum specific energy part in the performance index is limited to the after-shear region only. In particular, the minimum altitude performance remains unaffected. Through the appropriate choice of $K_2$ and the terminal boundary condition on flight path angle, airspeed and altitude can be made to behave desirably in the after-shear region, without affecting performance. However, if a larger value is specified for $K_2$ (i.e., $K_2 > K_1$), the entire trajectory is influenced. More specifically, minimum altitude performance is sacrificed for the benefit of gaining specific energy.

In the literature an alternative performance index has been proposed, namely the maximization of specific energy, subject to a minimum safe altitude constraint. By using the composite performance index introduced in this study, this alternative performance index is completely covered as well. In a given scenario, the actually achieved minimum altitude is a function of the weight factor $K_2$: the larger $K_2$, the lower the achieved minimum altitude. The problem thus simply is to find the value of $K_2$ that produces the trajectory for which the actually achieved minimum altitude matches the desired safe minimum altitude. Evidently, a slight modification of the MPBVP should allow the desired result to be produced directly.

A very important finding in the present study is that the benefits of lateral maneuvering remain unchallenged. Whatever the values adopted for $K_1$ and $K_2$, lateral maneuvering was always found to be beneficial relative to straight flight in the present study. Although only a single scenario for the encounter has been considered herein, we do not expect that our conclusions need to be revised due to the outcome of a more comprehensive study.

Although this issue has not been addressed here, the ultimate aim of the research effort is the development of feedback strategy for microburst encounter on final approach. A preliminary research effort to develop a feedback guidance law that approximates the open-loop optimal lateral escape trajectories has already been undertaken. Preliminary results based on constant pitch guidance seemed encouraging, in the sense that the Bolza trajectories were fairly closely approximated. However, the Chebyshev results established herein actually force us to reassess this constant pitch technique. An alternative and, possibly, superior technique might be to guide the aircraft towards a fixed safe minimum altitude, while attempting to gain specific energy in the process. In a truly severe microburst this approach probably makes sense. However, if during a microburst encounter the hazard turns out to be less severe than expected, the aircraft will still descend towards the minimum altitude, despite the fact that this is not really necessary. Clearly, this is undesirable from an operational perspective. In contrast, if a constant pitch technique is used, a positive climb gradient will be maintained in such a scenario.

It can be concluded that although the present research has provided some meaningful results, at the same time some entirely new issues have been raised, for which answers will need to be found in future research.
References


Appendix A: Evaluation of some Partial Derivatives

The following presents an overview of the right-hand sides of the adjoint equations (65) through (71):

\[
\begin{align*}
\frac{\partial H}{\partial x} &= \lambda_x \frac{\partial W_x}{\partial x} + \lambda_y \frac{\partial W_y}{\partial x} + \lambda_h \frac{\partial W_h}{\partial x} + \lambda_e \left( \frac{\partial W_e}{\partial x} \right) \\
- \frac{V \cos \gamma \cos \psi}{g} &\left( \frac{\partial^2 W_x}{\partial x^2} \frac{\partial}{\partial x} + \frac{\partial^2 W_x}{\partial x \partial y} \frac{\partial}{\partial y} \right) \\
- \frac{V \cos \gamma \sin \psi}{g} &\left( \frac{\partial^2 W_y}{\partial x^2} \frac{\partial}{\partial x} + \frac{\partial^2 W_y}{\partial x \partial y} \frac{\partial}{\partial y} \right) \\
- \frac{V \sin \gamma}{g} &\left( \frac{\partial^2 W_h}{\partial x^2} \frac{\partial}{\partial x} + \frac{\partial^2 W_h}{\partial x \partial y} \frac{\partial}{\partial y} \right) \\
+ \frac{\partial^2 W_h}{\partial x \partial h} \frac{\partial}{\partial h} \left( \frac{\partial W_h}{\partial h} + \frac{\partial W_h}{\partial W_h} \right)
\end{align*}
\]

(A.1)
\[
\frac{\partial H}{\partial y} = \lambda_x \frac{\partial W_x}{\partial y} + \lambda_y \frac{\partial W_y}{\partial y} + \lambda_h \frac{\partial W_h}{\partial y} + \lambda_E \frac{\partial W_E}{\partial y}
- \frac{V \cos \psi \cos \psi}{g} \left[ \frac{\partial^2 W_x}{\partial x \partial y} \frac{\partial W_x}{\partial x} + \frac{\partial^2 W_x}{\partial x \partial y} \frac{\partial W_x}{\partial y} + \frac{\partial^2 W_x}{\partial y^2} \frac{\partial W_x}{\partial y} + \frac{\partial W_x}{\partial y} \frac{\partial W_y}{\partial y} \right]
- \frac{V \sin \psi}{g} \left[ \frac{\partial^2 W_y}{\partial x \partial y} \frac{\partial W_y}{\partial x} + \frac{\partial^2 W_y}{\partial x \partial y} \frac{\partial W_y}{\partial y} + \frac{\partial^2 W_y}{\partial y^2} \frac{\partial W_y}{\partial y} + \frac{\partial W_y}{\partial y} \frac{\partial W_y}{\partial y} \right]
+ \frac{\partial^2 W_h}{\partial y \partial h} \frac{\partial W_h}{\partial y} + \frac{\partial W_h}{\partial y} \frac{\partial W_h}{\partial y}
\] (A.2)

\[
\frac{\partial \tilde{H}}{\partial h} \bigg|_V = \frac{6}{h_{ref}} \left( 1 - \frac{h}{h_{ref}} \right)^5 + \lambda_h \frac{\partial W_h}{\partial h} + \lambda_E \left\{ \frac{V}{W} \frac{\partial D}{\partial h} 
+ \frac{\partial W_h}{\partial h} \frac{V}{g} \sin \psi \left[ \frac{\partial^2 W_x}{\partial x \partial h} \frac{\partial W_x}{\partial x} + \frac{\partial^2 W_x}{\partial x \partial h} \frac{\partial W_x}{\partial y} + \frac{\partial^2 W_x}{\partial y^2} \frac{\partial W_x}{\partial y} + \frac{\partial W_x}{\partial y} \frac{\partial W_y}{\partial y} \right]
' \right. 
\left. + \lambda_y \left\{ \frac{g}{V} \frac{\partial V}{\partial h} \cos \psi \left[ \frac{\partial^2 W_x}{\partial x \partial h} \frac{\partial W_x}{\partial x} + \frac{\partial^2 W_x}{\partial x \partial h} \frac{\partial W_x}{\partial y} + \frac{\partial^2 W_x}{\partial y^2} \frac{\partial W_x}{\partial y} + \frac{\partial W_x}{\partial y} \frac{\partial W_y}{\partial y} \right] \right. 
\right. 
\left. \right. 
\right. 
\left. + \lambda_E \left\{ \frac{g}{V \cos \gamma} \frac{\partial W_x}{\partial h} \cos \psi \right\} \right\} + \eta \left\{ \frac{g}{W} \left[ \sin \psi \frac{\partial D}{\partial h} - \cos \psi \frac{\partial L}{\partial h} \cos \psi \right] \right\} \right) \] (A.3)
\[ \frac{\partial H}{\partial \nabla} \bigg|_h = \lambda_x \cos \gamma \cos \psi \cos \psi + \lambda_y \cos \gamma \sin \psi + \lambda_z \sin \gamma \]

\[ + \frac{T}{2} \left( 1 - \frac{(\alpha + \delta)^2}{2} \right) - D + \frac{V}{W} \left[ \frac{\partial T}{\partial \nabla} \left( 1 - \frac{(\alpha + \delta)^2}{2} \right) - \frac{\partial D}{\partial \nabla} \right] \]

\[ - \frac{\cos \gamma \cos \psi}{g} \left[ \frac{\partial W_x}{\partial x} (2V \cos \gamma \cos \psi + W_x) + \frac{\partial W_y}{\partial y} (2V \cos \gamma \cos \psi + W_y) \right] \]

\[ - \frac{\cos \gamma \sin \psi}{g} \left[ \frac{\partial W_y}{\partial x} (2V \cos \gamma \cos \psi + W_x) + \frac{\partial W_y}{\partial y} (2V \cos \gamma \sin \psi + W_y) \right] \]

\[ - \frac{\sin \gamma}{g} \left[ \frac{\partial W_x}{\partial x} (2V \cos \gamma \cos \psi + W_x) + \frac{\partial W_y}{\partial y} (2V \cos \gamma \sin \psi + W_y) \right] \]

\[ + \frac{\partial W}{\partial h} (2V \sin \gamma + W_h) \]

\[ L + (V \frac{\partial T}{\partial \nabla} - T)(\alpha + \delta) \quad \text{(A.4)} \]

\[ + \lambda \left( \frac{g}{V^2} \frac{\cos \mu + \cos \gamma}{W} \left[ \frac{\sin \gamma \cos \psi}{V^2} \left[ \frac{\partial W_x}{\partial x} W_x + \frac{\partial W_y}{\partial y} W_y \right] - \frac{\sin \gamma \sin \psi}{V^2} \left[ \frac{\partial W_x}{\partial x} W_x + \frac{\partial W_y}{\partial y} W_y \right] \right] \right. \]

\[ + \frac{\cos \gamma}{V^2} \left[ \frac{\partial W_x}{\partial x} W_x + \frac{\partial W_y}{\partial y} W_y + \frac{\partial W_y}{\partial h} W_h \right] \]

\[ + \lambda \left( \frac{g}{V^2 \cos \gamma} \frac{\sin \mu}{W} \left[ \frac{\sin \gamma}{V^2 \cos \gamma} \left[ \frac{\partial W_x}{\partial x} W_x + \frac{\partial W_y}{\partial y} W_y \right] + \frac{\cos \gamma}{V^2 \cos \gamma} \left[ \frac{\partial W_x}{\partial x} W_x + \frac{\partial W_y}{\partial y} W_y \right] \right] \right. \]

\[ - \eta \frac{g}{W} \sin \gamma \left[ \frac{\partial T}{\partial \nabla} \left( 1 - \frac{(\alpha + \delta)^2}{2} \right) - \frac{\partial D}{\partial \nabla} \right] + \cos \gamma \left[ \frac{\partial L}{\partial \nabla} + \frac{\partial T}{\partial \nabla} (\alpha + \delta) \right] \cos \mu \]
\begin{align*}
\frac{\partial H}{\partial y} &= -\lambda_x V \sin \gamma \cos \gamma - \lambda_y V \sin \gamma \sin \gamma + \lambda_h V \cos \gamma \\
&\quad + \lambda_x \left( \frac{V \sin \gamma \cos \gamma}{g} \frac{\partial W_h^x}{\partial x} (2V \cos \gamma \cos \psi + W_x) + \frac{\partial W_h^x}{\partial y} (2V \cos \gamma \sin \psi + W_y) \right) \\
&\quad + \frac{V \sin \gamma \sin \psi}{g} \frac{\partial W_h^y}{\partial x} (2V \cos \gamma \cos \psi + W_x) + \frac{\partial W_h^y}{\partial y} (2V \cos \gamma \sin \psi + W_y) \\
&\quad - \frac{V}{g} \frac{\partial W_h^h}{\partial x} [V \cos \psi (\cos^2 \gamma - \sin^2 \gamma) + W_x \cos \gamma] \\
&\quad - \frac{V}{g} \frac{\partial W_h^h}{\partial y} [V \sin \psi (\cos^2 \gamma - \sin^2 \gamma) + W_y \cos \gamma] \\
&\quad - \frac{V \cos \gamma}{g} \frac{\partial W_h^h}{\partial h} (2V \sin \gamma + W_h) \\
&\quad + \lambda_x \left( \frac{g \sin \gamma}{V} \frac{\partial W_h^x}{\partial x} [V \cos \psi (\cos^2 \gamma - \sin^2 \gamma) + W_x \cos \gamma] \right) \\
&\quad + \frac{\cos \psi}{V} \frac{\partial W_h^x}{\partial y} [V \sin \psi (\cos^2 \gamma - \sin^2 \gamma) + W_y \cos \gamma] \\
&\quad + \frac{\sin \psi}{V} \frac{\partial W_h^x}{\partial x} [V \cos \psi (\cos^2 \gamma - \sin^2 \gamma) + W_x \cos \gamma] \\
&\quad + \frac{\sin \psi}{V} \frac{\partial W_h^x}{\partial y} [V \sin \psi (\cos^2 \gamma - \sin^2 \gamma) + W_y \cos \gamma] \\
&\quad + \frac{\sin \gamma}{V} \frac{\partial W_h^h}{\partial x} [(2V \cos \gamma \cos \psi + W_x) + \frac{\partial W_h^h}{\partial y} (2V \cos \gamma \sin \psi + W_y)] \\
&\quad + \frac{1}{V} \frac{\partial W_h^h}{\partial h} [V (\sin^2 \gamma - \cos^2 \gamma) + W_h \sin \gamma] \\
&\quad + \frac{L}{\sin \gamma} \left( \frac{1 + \frac{(\alpha + \delta)^2}{2} \sin \mu}{V \cos^2 \gamma W} \right) \\
&\quad + \frac{\sin \gamma}{V \cos^2 \gamma} \left( \frac{\partial W_h^x}{\partial x} W_x + \frac{\partial W_h^y}{\partial y} W_y \sin \psi - \frac{\partial W_h^x}{\partial y} W_x + \frac{\partial W_h^y}{\partial x} W_y \cos \psi \right) \\
&\quad + \eta \frac{g}{W} \left( - \cos \gamma [T (1 - \frac{(\alpha + \delta)^2}{2}) - D] + \sin \gamma [L + T (\alpha + \delta)] \cos \mu \right)
\end{align*}
\[ \frac{\partial \tilde{H}}{\partial \psi} = -\lambda_x V \cos \psi \sin \psi + \lambda_y V \cos \gamma \cos \psi \]

\[ + \lambda_x \left\{ \frac{V \cos \psi \sin \psi}{g} \frac{\partial W}{\partial x} (2V \cos \gamma \cos \psi + W_x) \right\} \]

\[- \frac{V \cos \psi}{g} \frac{\partial W}{\partial y} \left[ V \cos \psi (\cos^2 \gamma - \sin^2 \gamma) - W_y \sin \psi \right] \]

\[- \frac{V \cos \gamma \cos \psi}{g} \frac{\partial W}{\partial x} \left[ V \cos \psi (\cos^2 \gamma - \sin^2 \gamma) + W_x \cos \psi \right] \]

\[- \frac{V \cos \psi \cos \gamma}{g} \frac{\partial W}{\partial y} \left( 2V \cos \gamma \sin \psi + W_y \right) \]

\[ + \frac{V^2 \cos \gamma \sin \psi}{g} \left( \frac{\partial W}{\partial x} \sin \psi - \frac{\partial W}{\partial y} \cos \psi \right) \]

\[ + \lambda_y \left\{ \frac{\sin \gamma \sin \psi}{V} \frac{\partial W}{\partial x} (2V \cos \gamma \cos \psi + W_x) \right\} \]

\[ + \frac{\sin \gamma}{V} \frac{\partial W}{\partial y} \left[ V \cos \gamma (\cos^2 \psi - \sin^2 \psi) - W_y \sin \psi \right] \]

\[ + \frac{\sin \gamma}{V} \frac{\partial W}{\partial x} \left[ V \cos \gamma (\cos^2 \psi - \sin^2 \psi) + W_x \cos \psi \right] \]

\[ + \frac{\sin \gamma \cos \psi}{V} \frac{\partial W}{\partial y} \left( 2V \cos \gamma \sin \psi + W_y \right) \]

\[ + \cos^2 \gamma \left( \frac{\partial W}{\partial x} \sin \psi - \frac{\partial W}{\partial y} \cos \psi \right) \]

\[ + \lambda_y \left\{ \frac{1}{V \cos \gamma} \frac{\partial W}{\partial x} \left[ V \cos \gamma (\cos^2 \psi - \sin^2 \psi) + W_x \cos \psi \right] \right\} \]

\[ + \frac{\cos \gamma}{V \cos \psi} \frac{\partial W}{\partial y} (2V \cos \gamma \sin \psi + W_y) \]

\[ + \frac{\sin \psi}{V \cos \gamma} \frac{\partial W}{\partial x} (2V \cos \gamma \cos \psi + W_x) \]

\[ - \frac{1}{V \cos \gamma} \frac{\partial W}{\partial y} \left[ V \cos \gamma (\cos^2 \psi - \sin^2 \psi) - W_y \sin \psi \right] \]

\[ \frac{\partial \tilde{H}}{\partial \beta} = \lambda_E \frac{V}{W} \frac{\partial T}{\partial \beta} \left( 1 - \frac{(\alpha + \delta)^2}{2} \right) + \lambda_y \frac{1}{V} \frac{g}{W} \frac{\partial T}{\partial \beta} (\alpha + \delta) \cos \mu \]

\[ + \lambda_x \frac{1}{V} \frac{g}{W} \frac{\partial T}{\partial \beta} (\alpha + \delta) \sin \mu - \lambda_y \frac{1}{\tau} \]

\[ - \frac{g}{W} \left( \sin \gamma \frac{\partial T}{\partial \beta} \left( 1 - \frac{(\alpha + \delta)^2}{2} \right) + \cos \gamma \frac{\partial T}{\partial \beta} (\alpha + \delta) \cos \mu \right) \]