THE ROAD-PICTURE AS A TOUCHSTONE FOR THE THREEDIMENSIONAL DESIGN OF ROADS

PART I: TEXT

BY

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AND

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Symbols that are used

Preface

Introduction

The necessity for harmony in the road picture

The use of perspective

The picture of a straight section of road

The picture of a curved road section

The vertical curve

The horizontal curve

The composite curve

Successive elements

The transition-curve

Properties of curves in the picture

The vertical curve

The radius of the concave curve

The turning point in the convex curve

The horizontal curve

The radius of the outer curve

The turning point in the inner curve

The composite curve

The radius of the composite curve

The turning point in the inner curve

The relationship between the tangent points in the horizontal and longitudinal sections

Criteria for the change of the picture when approaching a series of successive elements

Causes of a disturbing road-impression

Kinks
7.1.1 Minor changes of direction
7.1.2 Straight sections in hilly regions
7.1.3 A horizontal curve in front of a horizontally straight elevation

7.2 Short S-shapes
7.2.1 Horizontal displacements
7.2.2 The transition to the superelevation
7.2.3 Minor changes of level in (almost) straight sections
7.2.4 A composite curve in front of a curve in horizontal sense

7.3 Straight sections
7.4 Double S-shapes
7.4.1 Local horizontal displacements
7.4.2 Local dips in a horizontally straight section
7.4.3 A terrace in a change of elevation
7.4.4 An elevation beginning in a horizontal curve
7.4.5 A local dip in a horizontal curve

8 The presentation of the road-impression
8.1 Staking out in the field
8.2 Calculated perspectives
8.2.1 The tangents method
8.2.2 The coordinates method
8.2.3 Computer-drawn perspectives
8.3 Scale models
8.4 The study of existing roads

9 Explanations
Symbols that are used

A parameter of the clothoid
\(a_L\) distance from the observer to the left and right side of the road
\(a_R\) origin of picture
b road width
d distance from the eye to the picture plane
e eye-height
H vertex
L length
M centre
\(M_S\) centre situated in the side of the road
m direction factor of a straight line
O eye
P point in space
p point in the picture
\(R_V\) radius of the vertical curve
\(R_H\) radius of the horizontal curve
\(R_S\) resultant radius of the composite curve
S division point in the tangents method
T point of contact with tangent
u local shift
V vanishing point
X abcis in space
x abcis in the picture
Y ordinate in space
y ordinate in the picture
Z distance to a point in space, length of an element
\(Z_K\) distance to a turning point
\(Z_B\) length of a curve
\(\alpha\) alpha horizontal direction factor of a straight line in space
\(\beta\) beta vertical direction factor of a straight line in space
\(\gamma\) gamma real change of direction
\(\delta\) delta visual change of direction
\(\theta\) theta change of direction in the transition curve of the third degree
\(\lambda\) lambda angle of the main axis of an ellipse with the X-axis
\(\rho\) rho proportional factor between A of the clothoid and R of the circular curve
\(\tau\) tau change of direction in the clothoid
\(\phi\) phi angle between the asymptotes
\(\omega\) omega angle in the picture between the side of the road and the x-axis
Les ingénieurs font de l'architecture car ils emploient le calcul issu des lois de la nature, et leurs œuvres nous font sentir l'harmonie.
Il y a donc une esthétique de l'ingénieur, puisque il faut, en calculant, qualifier certains termes de l'équation, et c'est le goût qui intervient.

Le Corbusier 1924
Preface

All objects created by man, be they buildings or bridges, cars, airplanes and ships, weapons and utensils, have since long ago not only been shaped for usefulness, but also to please the eye.

In road design this consciousness of beauty should also play a part. It is difficult to estimate how much attention this received in the past. But its importance increased strongly with the advent of the motorcar with its greater speed, which was the cause of long sections of road beginning to draw the attention of the road-user.

As a result of this, consciousness of the influence of threedimensional design on the road-picture is rather new. It was Dr. H. Lorenz, lastly Regierungsbaudirektor in Bavaria, who as the first, already in the nineteen-thirties, was engaged with this.

In the Netherlands ir. H. B. Bakker began over twenty years ago, mainly using empirical methods, supported by a great experience. The results of this were published in 1959 as "Aesthetic Aspects of the Road" by "Het Nederlandsche Wegencongres". The first theoretical principles were published in the Netherlands in a series of papers by ir. Bakker and ir. Springer, titled "De Belijning van de Weg", which pointed at the importance of optical guidance and elegance of the alignment. Nowadays this elegance is no longer considered to be the main interest, since science on mathematical principles has begun to dominate the artistic viewpoint. This science, dealt with in this treatise, connects a clear and harmonic road picture with the data of the three-dimensional design. The changing of the picture while driving, a characteristic of motor traffic, is taken into consideration as well. The reader is reminded, that the greater part of the roads in the Netherlands is mainly horizontal, so the majority of elevations is caused by crossing other roads or waterways. The authors do therefore not claim to have dealt completely with roads in mountainous area's, although they feel that their propositions are applicable there as well.

We thank Dr. Lorenz and Mr. L. Roberts, F.I.C., in Birmingham for their kind assistance, and Mr. Th. ten Brummelaar, B.E., M.Eng.Sc., M.I.E.Aust., senior lecturer at the University of New South Wales, for his critical contributions.

It is hoped that this treatise will induce the designers of roads to give utmost attention to the future road-picture.

The picture of the road is the display of the design. If it is not admired by everyone, the purpose is failed. Let the objective be that the roads are worth the admiration of the many generations of people who will use them.
1 Introduction

The impression of the road on the user is related to its design data. Road design involves three dimensions. This treatise gives rules for such design, aiming at a pleasing impression for the observer, without reducing traffic-safety, in such a way that in every point a clear and harmonious road-picture will arise. The theoretical foundations are explained separately in chapter 9.
2 The necessity for harmony in the road picture

The first phase of road design consists mainly of locating a line on a topographic map. Obstacles to be avoided are indicated, as well as the location of intersections. It is of the utmost importance that the vertical curves are located simultaneously with the horizontal curves. The choice between under- or overpasses should be made at this early stage.

When fixing the plan-centre-line and the longitudinal section, the picture the road-user will see should immediately be considered. The road-user observes from a continually changing view-point; he sees a moving picture of several typical lines, i.e. the edge-lines of the carriage-way or carriage-ways. This is more or less influenced by road-markings, lighting columns, trees and shrubs, ditches, fences, buildings and, if driving in the dark, the lighting on and beside the road.

If this picture is ugly, confusing or even misleading, because of an unsuitable combination of longitudinal and horizontal curves, the road-user will be disturbed, puzzled and distracted. A correct picture of the road is harmonious, it has a flowing alignment, and improves the drivers satisfaction and even his safety.

It is supposed by some that because of the increased number of vehicles, the road itself will become invisible. This will not be the case. There will always be periods in which long stretches of road can be overseen.

The impression which the road-user forms of the road-picture strongly depends on his eye-level, particularly in a flat region. Moreover, speed influences what the road-user sees. It means that the character of the road, be it motorway or country-road, plays an important part in the appreciation of proportions and shapes.

On motorways especially the driver's attention will be directed to the road itself, in contrast to other roads, because the impression of the motorway thrusts itself more strongly on the observer.

A clear impression results also in reduction of the road-furniture required to improve optical guidance.

White line-markings at the edges of the carriage-way mainly define the road-picture. They form a three-dimensional alignment, the combination of the horizontal and vertical curves.

There is a difference between macro-alignment, concerning the situation and relation of elements in a road-section, and micro-alignment, concerning small deviations in height or width over a short distance, mainly in the transition from one element to the other. However a sharp distinction between these conceptions is of no great importance.
Each section of road is influenced by the sections on either side of it. The whole road is a unity, and no individual part should ever be designed independent of adjacent sections.

The intention is to obtain at the outset a design which achieves a harmonious combination of horizontal and vertical curves, avoiding all haphazardly arisen disturbing features.
3 The use of perspective

Figure 3.1 The study and criticism of road-pictures demands a knowledge of perspective (see page 46). This is a special form of central projection, the eye being the projection centre. Objects in space are projected onto a usually flat picture-plane. Provided it is observed from the correct view-point, the picture gives the same impression as the objects in reality.

Unlike architectural perspective drawings, where a certain scale is fixed, this form of central projection results in pictures that, viewed from the correct angle and position, show the objects in their true dimensions, scale 1 : 1.

As figure 3.1 shows, a point in space is defined by three rectangular coordinates, the absciss X, the ordinate Y and the distance Z, originating in the projection-centre, in this case the eye of the observer.

The XY-plane is called the vanishing plane, the XZ-plane is the eye-level and the YZ-plane has no particular name.

The Z-axis is the axis perpendicular to the picture-plane and is called the picture-plane axis. This axis intersects the picture-plane in the origin of the coordinates of the picture in this plane, the point B.

The length of the picture-plane axis between the eye and the picture plane is the distance d. Increasing this distance results in a larger picture of exactly the same proportions.

To obtain a correct impression of the reality, the perspective picture should be observed from the same position as was used in its construction.

To construct the picture of a point P(x, y, z), this point is connected with the eye. The connection-line is called the eye-line from P. The intersection-point p(x, y) of this eye-line with the picture-plane is the picture of the point P. Its coordinates with regard to the origin B are defined by

\[ x = d \frac{X}{Z} \]

and

\[ y = d \frac{Y}{Z} \]

In these formulae \( \frac{X}{Z} \) and \( \frac{Y}{Z} \) are without dimension (metres divided by metres).
The distance \( d \) defines the coordinates of the picture and hence its size.

**Figure 3.2** The picture of a straight line can be obtained by connecting two of its points in the picture.

**Figure 3.3** A straight line may also be fixed in space by one of its points and its direction.

It can be shown that all lines running in the same direction meet in one point of the picture, the **vanishing point** \( V \). This point is the intersection-point of the eye-line in that direction with the picture plane (see page 46).

If the horizontal angle between the direction of the line and the YZ-plane is \( \alpha \) and the vertical angle with the XZ-plane \( \beta \), the coordinates of the vanishing point will be

\[
x_v = d \tan \alpha
\]

and

\[
y_v = d \tan \beta
\]

By connecting \( p \) with \( V \) the picture of the straight line will appear.

It will also appear that in figure 3.2 the coordinates of the vanishing point can be obtained from the coordinates of the two points by the formulae

\[
x_v = d \frac{X_2 - X_1}{Z_2 - Z_1}
\]

and

\[
y_v = d \frac{Y_2 - Y_1}{Z_2 - Z_1}
\]

(page 47)

**Figure 3.4** The picture of a part of a straight line with a length \( L \), beginning in \( P \) at a distance \( Z \) from the observer, can be constructed by drawing two parallel lines through \( p \) and \( V \), in an arbitrarily chosen direction. From \( p \) and from \( V \), but in opposite directions, parts are staked out proportionally to \( Z \) resp. \( L \). The line connecting the end of these parts intersects the line \( pV \) in a point representing the distance \( L \) from \( P \).
4 The picture of a straight section of road

Figure 4.1 The picture of a straight section of road is the cross-section of a triangular prism in the picture plane. This prism consists of the surface of the road and the planes through the eye and both of the edge-lines of the road. The intersection-line of these two planes is the eye-line in the direction of the road. The point of intersection of this eye-line with the picture-plane is the vanishing point of the edge lines of the road (page 48).

So the picture of the part of the road which lies behind the picture-plane is a triangle, the top is the vanishing point, the base is the width of the road and the height is the eye-height $h$, i.e. the distance from the eye to the road-surface, in the direction of the Y-axis. The entire picture, so including the part in front of the picture-plane, appears as two straight lines, continuing into the infinite. It will be clear that the lines, connecting the points of intersection of the road-edges in the vanishing plane with the eye, are parallel to the picture-plane, so the pictures of these intersection-points will be in the infinite. The road-surface is part of the plane through the edge-lines of the road. The lines of intersection of this surface with the vanishing plane, the picture-plane and the plane in the infinite are parallel, and in this case, horizontal lines.

The projection of the eye on the base divides the width of the triangle of the road in a negative part $a_L$ and a positive part $a_R$. The eye-height $h$ has a negative value if the road-surface lies below the eye-level, and a positive value if measured from the eye upwards, for instance under a bridge.

Figure 4.2 Figure 4.2a shows the influence of the distance on the size of the picture. Figure 4.2b and c show the influence of resp. the eye-level and the width of the road on the picture's shape.

Figure 4.3 Figure 4.3 makes clear that, with the direction of the picture-plane axis remaining the same, a change of place of the observer in the cross-section does not affect the distance between the origin $B$ (i.e. the picture of the mill) and the vanishing point, but alters only the shape of the picture.

Figure 4.4 Figure 4.4 on the contrary shows that, when the place of the observer remains the same, changing of the direction of the picture-plane-axis, i.e. the distance between the vanishing point and the origin $B$, does not affect the shape of the impression on the observer. The shape of the picture however does change (page 48).

Figure 4.5 Figure 4.5 gives a calculated example.
Attention is drawn to the fact that most of the pictures in this paper represent a road width of 8 metres, the observer is supposed to be at 3 metres to the left of the right-hand side, and the eye-height equals $-1.20$ metres. For simple calculation the distance to the picture-plane is supposed to be 1 metre, so $d$ no more appears in the formulae.
5 The picture of a curved road section

5.1 The vertical curve

In a flat country, like the main part of the Netherlands, vertical curves appear especially on approaches to viaducts and bridges, (abbreviated in the following as "approaches"). The slopes seldom exceed 4.5%. Consequently the vertical curves are short, and when observed from a great distance will give the impression of a kink.

In hilly and mountainous regions, in which roads have to be built elsewhere in the world, the vertical curves of motorways also have only small changes in the gradient. Because of this, their length will be relatively short.

In the picture the concave curves are especially important. The convex curves are of less interest.

The small length of the curves and their slight change of direction cause a similarity between circular and parabolic vertical curves. As the parabola has a simpler formula, in practice the parabolic shape is preferred.

This formula is

\[ Y = \frac{Z^2}{2R_v}, \]

in which \( Z \) is the length of the tangent to the curve in its point of contact in the vanishing plane, \( Y \) the elevation and \( R_v \) the radius of the curve at its beginning. This use of the parabola instead of the circle has neither influence on the driving nor is it visible.

When approaching a vertical curve or driving in it, the road-user will observe the edge-lines of the road as hyperbolas (page 49).

Figure 5.1 This figure shows that the common centre \( M \) of these hyperbolas is the vanishing point of the picture of the tangents of the concave curves at the side of the observer, in other words, in the XY-plane or vanishing plane of figure 3.1.

As remarked in paragraph 1.2, the XY-plane is called the vanishing plane, because the picture of every point in this plane is vanished in the infinite. For, every line connecting the eye with such a point, its eye-line, is parallel to the picture-plane.

So it is clear that the tangent-points of the asymptote — 0 with the hyperbola, which lie in the infinite, are the pictures of the intersection-points of the curves with the vanishing plane.
In the road-picture these tangents appear to be the asymptotes $-0$ of the hyperbolas on both sides of the road.
The other asymptote, i.e. the asymptote $-\infty$, which both hyperbolas have in common, is the picture of the vertical tangent in the infinite, and at the same time the vanishing line of the planes in which both curves are situated. This asymptote $-\infty$ coincides with the y-axis. The centre M is the picture of the vertex of both curves when they are supposed to be extended into the infinite.

Figure 5.2 If there is a straight road-section with a length of $Z_1$ between the observer and the beginning of the vertical curve, the centre $M_2$ lies at a distance of $\frac{Z_1}{R_v}$ below the vanishing point $V_1 = V_0 = M_1$ of the straight or nearly straight section in front of it, which should be considered as a degenerated hyperbole. The radius of concave curves is supposed to be positive, that of convex curves negative. This distance $\frac{Z_1}{R_v}$ is the picture of the change of direction of the extension of the curve in the direction of the observer unto the vanishing plane.
The asymptote $-\infty$ is the vertical line in the centre, the lines connecting the centre with the points $H_L$ and $H_R$ on both sides of the road at a distance $\frac{1}{2}Z$ from the observer are the asymptotes $-0$.
The points $H_L$ and $H_R$ are the pictures of the intersection-points of the tangents of the extended part of the curve, between the beginning of the curve and the vanishing plane.
The coordinates in the picture of points in the vertical curve results from the formulae

$$x = \frac{a}{Z}$$

and

$$y = \frac{(Z - Z_1)^2}{2R_v} - \frac{Z}{Z}$$

The side distance $a$ is measured from the observer to the edge-lines of the road in the cross-section, positive to the right and negative to the left.
$(Z - Z_1)$ is the distance measured from the tangent-point at the beginning of the curve. This means that $Z_1 = 0$ when the observer is situated in the curve.

5.2 The horizontal curve

Often a horizontal curve is composed of a succession of circular curves with different radii, eventually connected by transition-curves, generally clothoids.
Sometimes superelevation is required. The influence of this on the picture belongs to the micro-alignment and will not be discussed here. Only the picture of circular curves and clothoids are considered.

It is desirable that the driver, when approaching a curve, has a general conception of its course, enabling him to act accordingly. The question is what is the mathematical shape of the horizontal curve in the picture.

Many horizontal curves have too large a change of direction to allow the replacement of circles by parabolas. This means that horizontal curves usually are real circles, in contrast to vertical curves.

In the picture a horizontal curve shows up as an ellipse if the circle lies entirely in front of the observer; if not, it becomes a hyperbola, with the parabola as transition (see page 51).

Usually the shape of a horizontal curve is not observed from distances greater than the customary radii. As the radii on motorways are at least 750 metres, observation from 300 metres results in a hyperbolic picture. If this hyperbolic picture, originated by a circle, could be replaced by the hyperbolic picture of a parabolic curve, this would simplify its study.

This replacement appears to be allowed to a maximum change of direction of 12°. Greater changes of direction are seldom of importance in the road-picture (page 55). This means that in drawing perspectives, horizontal circles can safely be substituted by parabolas.

Figure 5.3 The formula of this parabola is $X = \frac{Z^2}{2R_H}$ in which $Z$ is the length of the tangent to the curve in its point of contact in the vanishing plane, $X$ the horizontal deviation, and $R_H$ the radius of the curve at its beginning. $R_H$ is positive when the curve is to the right, negative when to the left.

As in vertical curves, the road-user usually sees the lines of the road as hyperbolas. Such a hyperbola is here the picture of the part of the horizontal curve from the vanishing plane to the infinite. The tangents at both sides of this curve appear to be the asymptote $-0$ and the asymptote $-\infty$ of this hyperbola.

The asymptote $-0$ is the picture of the tangent in the observer’s cross-section, i.e. the intersection-point of the curve and the vanishing plane.

The asymptote $-\infty$ is a horizontal line touching the curve at an infinite distance, and at the same time the vanishing line of the plane in which the curve is situated.

The centre $M$ is the picture of the intersection-point of the tangents of the entire part of the curve that can be seen.
**Figure 5.4** When observed from a distance $Z_1$ in front of a horizontal curve, the centre $M_2$ is at a distance $\frac{Z_1}{R_H}$ from the vanishing point $V_1 = V_0 = M_1$ of the (almost) straight part in front of the curve, on the convex side of the curve (See page 54 and page 55).

In the picture this distance $\frac{Z_1}{R_H}$ corresponds with the change of direction of the curve when it is extended from its beginning to the vanishing plane.

The asymptote $-\infty$ will be found in the horizontal line through the centre, the asymptotes $-0$ in the lines connecting the centre with the points $H_L$ and $H_R$ on the edge-lines, at a distance $\frac{1}{2}Z_1$ from the observer.

In this case also, the asymptotes $-0$ in the picture are the tangents of the extended curves at their intersection-points with the vanishing plane, on each side of the observer.

The coordinates of a point in the picture of a horizontal curve are defined by:

$$a + \frac{(Z - Z_1)^2}{2R_H} = \frac{Z_1}{Z}$$

and

$$y = \frac{h}{Z}$$

**5.3 The composite curve**

**Figure 5.5-5.8** The composite curve bends horizontally as well as vertically. Similar characteristics apply as for horizontal and vertical curves. Again the pictures of the edge-lines appear to be hyperbolas (See page 56).

If the observer is in a straight section at a distance $Z_1$ from the beginning of the curve, the centre $M_2$ of the hyperbola is found at a horizontal distance $\frac{Z_1}{R_H}$ from the vanishing point $V_1 = V_0 = M_1$ of the tangent, on the convex side of the horizontal curve, and a vertical distance $\frac{Z_1}{R_V}$ from the vanishing point of the tangent, on the convex side of the vertical curve.

The asymptote $-\infty$, which was horizontal or vertical, becomes an inclined line through the centre. The tangent of the angle of this line with the x-axis is $\frac{R_H}{R_V}$. This asympt-
tote—∞ is again the vanishing line of the planes in which the edge-lines of the road are situated.

The coordinates of a point in the picture are

\[ x = a + \frac{(Z - Z_1)^2}{2R_H} \]

\[ y = h + \frac{(Z - Z_1)^2}{2R_v} \]

and

For the remainder the figures are self-explanatory.

If the picture of the centre M of the hyperbolas lies between the road-edges, the vertical curve dominates. If the picture of M lies outside of these edges, the horizontal curve is dominant.

*Figure 5.9* A straight line on the outside is seen when the centre M of the hyperbolas is situated on the outer edge-line.

### 5.4 Successive elements

A road-picture is composed of a series of elements, e.g. vertical curves, horizontal curves, composite curves, transition-curves and straight or almost straight sections. The transition-curve, chiefly influencing the micro-alignment is not considered in this respect. One of the properties of the parabola is that the tangents at two of its points intersect each other half-way along their distance, measured along these tangents. Consequently a system of asymptotes in a series of elements can easily be constructed, in order to obtain an idea of the shape of the picture that the road-user will observe.

*Figure 5.10* As a result of the property of the parabola it appears that when two elements are coupled, three lines intersect each other in one point, namely
1. the asymptote—0 of the hyperbola that is the picture of the preceding element.
2. the tangent in the transition-point from the preceding to the next element.
3. the asymptote—0 of the hyperbola that is the picture of the next element.

As an example a concave curve is assumed, with a radius \( R_v \), beginning in \( T_{1l} \) and \( T_{1r} \) at a distance \( Z_1 \) from the observer, with a length \( Z_2 \), followed by a curve in horizontal sense to the right with a radius \( R_H \) and a length \( Z_3 \) (page 59).
Figure 5.11 As mentioned above, and as shown in figure 5.11, for each road-edge the asymptote —0 of the hyperbola representing the vertical curve, the asymptote —0 of the hyperbola representing the horizontal curve, and the common tangent in the transition-point, intersect each other in one point for each road-edge. In figure 5.11 the points $H_{2L}$ and $H_{2R}$.

The construction of the picture of the vertical curve, after drawing the straight section, starts with the definition of the centre of the vertical curve $M_2$, at a distance $\frac{Z_1}{R_v}$ below the vanishing point of the straight section, $V_i = V_0 = M_1$. The two asymptotes —0 intersect the edge-lines of the road, which converge in the vanishing point $V_1$ on the x-axis. The intersection-points $H_{1L}$ and $H_{1R}$ lie at a distance $\frac{h}{\frac{1}{2}Z_1}$ below the x-axis, because of the property of the parabola.

The tangent at the end of the vertical curve at a distance $Z_2$ from the beginning has its vanishing point $V_2$ in a point at $\frac{Z_2}{R_v}$ above $V_1$.

The coordinates of the transition-points $T_{2L}$ and $T_{2R}$ between the vertical curve and the curve in horizontal sense are defined by the formulae

\[ x_{T_2} = \frac{a}{Z_1 + Z_2} \]

and

\[ y_{T_2} = \frac{h + \frac{Z_2^2}{2R_v}}{Z_1 + Z_2} \]

Connecting these points with $V_2$ gives the tangents in the transition-points $T_{2L}$ and $T_{2R}$.

These tangents intersect the asymptotes —0 of the vertical curve in $H_{2L}$ and $H_{2R}$, while $T_{2L}$ and $T_{2R}$ lie in the middle of $H_{2L}V_2$ and $H_{2R}V_2$ because of the property of the hyperbola.

The centre $M_3$ of the curve in horizontal sense is horizontally at a distance $-\frac{Z_1 + Z_2}{R_H}$ from the vanishing point $V_2$.

By connecting this centre $M_3$ with the points $H_{2L}$ and $H_{2R}$ the asymptotes —0 of the curve in horizontal sense are obtained.

The vanishing point $V_3$ of the direction at the end of this curve is found at a horizontal distance $\frac{Z_3}{R_H}$ from $V_2$. 

21
The coordinates at the end of both road-edges can be calculated from

\[ x = \frac{a + \frac{Z_3^2}{2R_H}}{Z_1 + Z_2 + Z_3} \]

and

\[ y = \frac{h + \frac{Z_2^2 + Z_3}{2R_V} + \frac{Z_2}{R_V}}{Z_1 + Z_2 + Z_3} \]

**Figure 5.12-5.21** Some examples are shown in figures 5.12-5.21.

In figure 5.15, 5.17 en 5.18 a horizontally and vertically straight section appears. The ends of its edges are calculated by

\[ x = \frac{a + \frac{Z_2^2 + (Z_3 + Z_4)}{2R_H} \frac{Z_2}{R_H}}{Z_1 + Z_2 + Z_3 + Z_4} \]

and

\[ y = \frac{h + \frac{(Z_2 + Z_3)^2}{2R_{V_{1}}} + \frac{Z_4}{R_{V_{1}}} \frac{Z_2 + Z_3}{Z_1 + Z_2 + Z_3 + Z_4}}{(R_{V_{1}} = R_{V_{2}})} \]

The vertex \( H_{4r} \) is found by measuring out the double length of \( T_{4r}V_4 \) from \( V_4 \). The asymptote \( -0 \) of the fifth element is then \( M_5H_{4r} \).

It appears that the figure of this system of asymptotes leads to an easy recognition and location of faults in the alignment. Attention is drawn to the fact that the centre of the hyperbola always lies on its convex side.

If the asymptote \( -\infty \) lies above the asymptote \( -0 \) a convex curve appears, in the other case a concave curve.

It is possible from the pattern of the asymptotes to determine whether or not S-shapes in the picture will occur, without actually drawing it.

**5.5 The transition-curve**

For the construction of the picture of a transition-curve, the clothoid is approximated by means of the cubic parabola \( X = \frac{Z^3}{6A^2} \) (page 59).
Figure 5.22 Ruling out the transition of the superelevation, the coordinates in the picture are defined by the formulae

\[ a \pm \frac{(Z - Z_1)^3}{6A^2} \]

\[ x = \frac{Z}{Z} \]

and

\[ y = \frac{h}{Z} \]

Again it appears possible to construct the picture of the transition-curve by defining the centre and the asymptotes without complicated calculations.

Two of the three asymptotes of the pictures of the two branches of the cubic parabola, of which the transition-curve is a part, coincide with the horizontal axis, and are called the asymptotes \(-\infty\).

The intersection-point of the asymptotes is situated on this axis at a distance of \[ \frac{Z_1^2}{2A^2} \]

from the vanishing point, in the direction of the curve. The asymptote \(-0\), the third one, on the left side respectively the right side of the road, is constructed by connecting the centre with the points on the edge-lines of the road at \[ \frac{1}{3} Z_1 \] from the observer.

In figure 5.22 these points are \( H_L \) and \( H_R \).

The entire cubic parabola has an S-shape. Of this only a small part of the remote half appears in the picture when seen from a distance. Because of this, the transition-curve is not dealt with in the following paragraphs.
6 Properties of curves in the picture

6.1 The vertical curve

6.1.1 The radius of the concave curve

The concave curve plays an important part in the picture of the road. The radius of a convex curve is mainly defined by demands of traffic-safety, and is very rarely important for the road picture. The question is, which radius should be used in an almost straight or straight approach, to avoid a disturbing kink. There are two problems; at what distance will the road-user become aware that the curve draws near, and which radius produces an acceptable shape at that particular distance.

If observed from a great distance in a long straight section, every vertical change of direction seems to be a kink, though not a very significant one. Experience has proved that the road-user becomes conscious of the beginning of an approach at a distance of about 300 metres, approximately 10 seconds when driving on a motorway (with a design speed of 120 km/h), even if it is already visible from a much larger distance.

Figure 6.1-6.3 Whether or not a kink is observed, depends on the strongest bend of the hyperbola in the picture, and the measure of its curvature. The figures 6.1, 6.2 and 6.3 show that a kink appears when the angle between the asymptotes $- \infty$ and $0$ of both the hyperbola is less than 90°. When the angle is greater, a more weakly curved line is visible.

If a right angle is accepted as the limit, this means that the centre of the hyperbola is situated on the connecting line between the intersection points of the asymptotes $- \infty$ and $0$ with the edge-lines of the road, or

$$\frac{Z_1}{R_V} = \frac{h}{\frac{1}{2}Z_1}$$

or

$$R_V = \frac{Z_1^2}{2h}$$

For $Z_1 = 300$ m and $h = -1.20$ m a radius of 37.500 m suffices. In practice a radius of 30 000 m is recommended as a minimum for a vertical curve at the end of a straight or an almost straight section (See page 62).
6.1.2 The turning point in the convex curve

A turning point in the picture of a convex curve arises at the spot where the road becomes invisible, at a distance $Z_K$ which depends on the position of the observer. This distance has no connection to the ground plan, but is only defined by the gradient.

Figure 6.4 If the observer is in a straight section at a distance $Z_1$ in front of the concave curve at the foot of the approach with a length $Z_2$, the formula for the distance $Z_K$ to the turning point of the convex curve will be:

$$Z_K^2 = (Z_1 + Z_2)^2 - 2hR_{V_2} + \frac{R_{V_2}}{R_{V_1}} (2Z_1 + Z_2)Z_2$$

(1)

$R_{V_1}$ is the radius of the concave curve, $R_{V_2}$ of the convex curve at the top.

Figure 6.5 When the observer has entered the convex curve, resulting in $Z_1 = 0$, and the remaining length $Z_2$ is gradually decreasing, this means

$$Z_K^2 = -2hR_{V_2} + Z_2^2 \cdot \frac{R_{V_1}}{R_{V_1}} + \frac{R_{V_2}}{R_{V_1}}$$

(2)

If the observer is at the beginning of an approach of height $H$ and the radii for the foot and the top of the approach are $R_{V_1}$ and $R_{V_2}$ respectively, formula (2) can also be written as:

$$Z_K^2 = 2HR_{V_1} - 2hR_{V_2}$$ (See page 63)

(3)

If the observer is in the top-curve, then

$$Z_K^2 = -2hR_{V_2}$$

(4)

6.2 The horizontal curve

When horizontal curves are preceded by a long straight or almost straight section, their presence is noticeable from a great distance. Because of the strong change of direction, at this distance every curve will appear as a kink, even when a large radius is used. In order to obtain an impression of the change of direction it is just as well to draw a kink instead of the very short curve in the picture.

Figure 6.6 It appears that the visual change of direction, because of the strong foreshortening, is much greater than the actual change of direction. The relation between the actual change of direction $\gamma$ and the visual one $\delta$ is defined by

$$\frac{\tan \gamma}{\tan \delta} = \frac{h}{Z_1}.$$
This shows that a small real change of direction in a road already at a relatively short distance results in a visual change of direction of nearly 90°.

Figure 6.7 Nearing the curve, the visual change decreases, but will always remain a multiple of the real one. Figure 6.7 shows that when nearing the curve, the distance of the vertex from the horizontal axis increases. The distance between the vanishing points of the direction at the beginning and the end of the curve does not change when the observer moves. Trying to soften the contrast between the straight section and the curve by means of an introduction to the curve, i.e. by another curve with twice the radius, or a clothoid of great length, will have no result. Over long distances the contrast will remain and the kink will remain visible. If possible (almost) straight sections should be avoided and replaced by real curves having large radii.

6.2.1 The radius of the outer curve

If a straight section is unavoidable, it is essential that there is a gradual appreciation of the shape of the curve. It is desirable that this is achieved before the driver is at 10 seconds from the beginning of the curve. For 120 km/h this means about 300 m.

Figure 6.8-6.9 Practice has proved that a radius of 2000 m at this distance gives a good impression of the course of the curve.

Figure 6.10 The introduction of a transition-curve will give a further improvement of this impression. Figure 6.10 shows a curve with a radius of 2000 m, without and with a transition-curve, in both cases seen from 300 m front in of their beginning.

Figure 6.11 The experience of a good impression of a radius of 2000 m when seen from 300 m means that the angle between the asymptotes of the hyperbola representing the picture of the outer curve may be taken as the critical angle when nearing a horizontal curve. This angle is about 4°. It appears that

\[
\tan \varphi = \frac{h}{Z_1 \div \frac{1}{2}} \div \frac{a}{Z_1 - \frac{1}{2}Z_1} = \frac{2hR_H}{Z_1^2 - 2aR_H}.
\]

On roads having two lanes the distance \(a\) from the edge-line of the outer curve is supposed to be 5 m. With an eye-height of \(-1,20\) m and \(\varphi = 4^\circ\) this means:

\[Z_1^2 = 45R_H\]

(See page 63)
When RH is under 200 m, it is clear that the distance $Z_1$ will become less than 300 m, which is undesirable. By contrast, larger radii give a correct understanding at a greater distance and so these are to be recommended. It should be considered that twice the radius will give an increase of the perception-distance with a factor $\sqrt{2}$, so enlarging the radius for this purpose is not very effective.

6.2.2 The turning point in the inner curve

When driving in a horizontal curve, the picture of the outer curve appears to be a weakly bent line, while usually the inner curve is sharply bent and contains a turning point.

The distance $Z_k$ from the vanishing plane to this turning point depends on the distance $a$ from the observer to the inner side of the road and the radius $R_H$ of the curve. It is defined by the formula $Z_k = \sqrt{2aR_H}$ if the curve is supposed to be a parabola. Thus, when driving at a given distance from the inner side, this distance $Z_k$ yields a measure of the radius.

If a turning point is visible, the road appears to be curved.

Supposing there is a sight-distance of 1 km and the observer is driving at 3 m from the inner side, a radius of 160 000 m will be observed as a curved road. Often longer distances can be seen, so radii of 200 km and above can cause curved sections in the picture.

Figure 6.12 It is assumed that, if a turning point is observed at a distance of over 500 m, the section is an "almost straight" section. Turning points at a distance less than 500 m, that is when the radius is smaller than about 40 000 m, cause a real curve in the picture. In the turning point itself the real horizontal deviation is $a$.

Figure 6.13 Nearing a curve in a straight section without a transition-curve, the distance from the beginning of the curve to the turning point is

$$Z_k = \sqrt{Z_1^2 + 2aR_H} - Z_1$$

(See page 64).

Supposing a distance $Z_1$ of 100 m to the tangent-point of the curve and $a = 3$ m, the distance $Z_k$ from the tangent-point of the turning point will be for

<table>
<thead>
<tr>
<th>$R_H$</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
<th>10 000 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_k$</td>
<td>48</td>
<td>67</td>
<td>85</td>
<td>100</td>
<td>164 m</td>
</tr>
</tbody>
</table>

As the formula shows, a shorter distance from the observer to the beginning of the curve causes a greater distance from the beginning of the curve to the turning point.
If $Z_1 = 0$, that is at the entrance of the curve, the maximum distance is reached, and this remains the same while driving through the curve.

In the same way the turning point in the clothoid-transition-curve, when observed from its beginning, can be found from $Z_K = \frac{3}{\sqrt{3}} a A^2$ (See page 66).

The actual outward deviation in the turning point is $\frac{1}{2} a$ in this case. This turning point is of importance to distinguish the shape of the transition-curve. If the transition-curve is to be effective, the turning point should, when observed from the beginning of the curve, be situated in the picture of this curve, not in the circular curve behind it. In order to be sure of this in all cases the choice of $A = \frac{1}{3} R_H$ appears to be justifiable.

6.3 The composite curve

6.3.1 The radius of the composite curve

It can be shown that the resultant radius $R_S$ of a composite curve is smaller than the horizontal and the vertical radius, as this resultant radius is defined by

$$\frac{1}{R_v^2} + \frac{1}{R_H^2} = \frac{1}{R_S^2}$$

6.3.2 The turning point in the inner curve

The distance from the beginning of a concave composite curve to its turning point is equal to the distance to the turning point in the horizontal component. In the case of a convex composite curve, where several turning points may arise, the problem is more complex. For a worked example see chapter 9, Explanations, under 7.4.4.

6.4 The relationship between the tangent-points in the horizontal and longitudinal sections

Figure 6.14 and 6.15 If the tangent-points of horizontal and vertical curves coincide, this results in a fluent alignment, but does not guarantee an elegant and clear road-picture. An unfavourable result may be achieved, when the design includes horizontal and vertical curves in succession.

Figure 6.16 and 6.17 When this should be the case, increasing the radii to get an overlap causes a less capricious picture.
Figure 6.18 A coincidence of both tangent-points of horizontal and vertical curves is the ideal solution (See also 7.2.2 and 7.2.4).

Figure 6.19-6.21 The shape of the picture is defined by the proportion of the horizontal to the vertical radii.

6.5 Criteria for the change of the picture when approaching a series of successive elements

Figure 6.22 An impression of the course of the centre $M_2$ is given in figure 6.22. During the approach it moves vertically, and arrives at $V_1$ when $Z_1 = 0$. Similarly, the centre $M_3$ of the curve in horizontal sense moves horizontally and arrives at $V_2$ when $Z_2 = 0$.
As long as the path along which a centre moves during the approach does not intersect the edges of the road in the picture, the curve remains bent in the same sense.

Figure 6.23 If the path of a centre intersects the edge of the road, an S-shape appears or disappears while the vehicle approaches. The distance between $M_5$, the point of intersection of the path with the edge of the road, and the vanishing point $V_0$ of the direction at the beginning, coinciding with $M_1$, defines the place of this alteration, in such a way that $Z = M_5V_0$ in which $M_5V_0$ is expressed in metres.
At the moment when the centre is situated on the edge of the road, an apparent straight line is observed for a moment, which may be somewhat disturbing between two equi-directional curves.
For a test of the approach it is important to examine the motion of the centres from at least 300 m before the beginning of the element concerned. As a result of this, in many cases further examination of the picture will be superfluous.
7 Causes of a disturbing road-impression

The road-impression is defined by entirely different factors than could be expected from the design-data.
Dimensions that are acceptable on the basis of traffic dynamics are often too small to produce a clear and attractive impression of the road.
The most common disturbing or misleading elements in the road picture are kinks, short S-shapes, straight sections and double S-shapes. The main reasons for these four errors are visually incorrect dimensions or incorrect proportions of the design-elements.

7.1 Kinks

7.1.1 Minor changes of direction

*Figure 7.1* Enlarging the radius if a minor change of direction takes place results in a softening of the picture only if pictures are compared which are seen from the same distance $Z_H$ to the vertex of the curve.

*Figure 7.2* If observed at the same distance $Z_1$ in front of the beginning of the curve, a larger radius will cause a more disturbing kink in the inner curve.
In this case the turning point lies at a greater distance from the observer. Fortunately it can be noted that in extremely small changes of direction there is no turning point at all within a normal distance of observation.
The change of direction of a vertical curve (in a flat stretch of country) is usually not more than 2,5\%.
Since the angle between the asymptotes $-0$ and $-\infty$ is always larger than $90^\circ$, smaller radii can be used for vertical curves than for horizontal curves with the same change of direction, where this angle is always smaller than $90^\circ$.
In order to avoid a disturbing kink in the picture, i.e. the case of a small change of direction (less than $6^\circ$) the curve ought to have at least a length of twice the length of a transition-curve with $A = \frac{1}{3} R_H$ corresponding to the maximum radius for which transition-curves are at all applied, i.e. $R_H = 4000$ m. The length of this transition-curve, when using a parameter of $A = \frac{1}{3} R_H$, is $\frac{1}{6} R_H$ or about 450 m, the double length will be 900 m.
So 900 m is the minimum length for a horizontal curve with a change of direction at an angle smaller than $6^\circ$. This gives

$$R_H = \frac{900}{\tan \alpha}.$$
7.1.2 Straight sections in hilly regions

Sufficiently large radii in vertical curves, especially when concave, result in a pleasant picture of the road. This may, however, cause an elevated section or a cut, which clashes with the landscape.

It is generally more desirable to avoid horizontally straight sections and to design horizontally curved roads, which will accord with the surroundings.

7.1.3 A horizontal curve in front of a horizontally straight elevation

Figure 7.3 The danger will arise here of a kink in the picture of the convex curve, combined with a grotesque shape of the inner curve; this is more disturbing if the radius of the horizontal curve is large.

In horizontal radii larger than 15 000 m this danger will not occur at a distance less than 300 m to the beginning of the elevation (See page 67).

Increasing the radius of the convex curve will give no perceptible correction, nor will a greater radius for the concave curve, causing a less steep slope, have good results.

If possible, the elevation should be located elsewhere, if not, a combination of the horizontal curve and the concave vertical curve will give the best result.

7.2 Short S-shapes

7.2.1 Horizontal displacements

Figure 7.4 When moving the axis horizontally, it is desirable to choose the largest possible radii in the horizontal S-curve. A displacement over a small width is disturbing.

If possible, increasing the shift in such a way that the road-section beyond it is observed as a separate element, will cause an attractive variation. But it is better to avoid this entire problem by designing a curved section.

7.2.2 The transition to the superelevation

Short S-shapes in the picture of the outer curve in the superelevation can be avoided or softened by situating the alteration in height in the transition-curve.

Figure 7.5 If the elevation does not rise above a plane through the eye and the (nearly) straight section in front of the outer curve, no bulge in the picture will be seen. The most unfavourable place for the eye of the road-user in this respect is above the edge-line at the side of the inner curve. From this it follows that it is desirable that the proportion between the horizontal displacement and the elevation is larger than the proportion between road-width and eye-height. As the straight section usually has a cross-fall to the right, the right side is lower than the middle of the road. This allows for a larger eye-height than normal at the road-edge, which may be set at 1,50 m.
The horizontal displacement of a clothoid is very small at its beginning. For sufficient drainage the change of height should, however, be rather steep. In order to observe the proportion $\frac{b}{h}$, the beginning of the transition to the superelevation should be put at some distance behind the beginning of the transition-curve, at about $\frac{1}{3}$ of its length.

In curves with small radii, where the demands of friction and drainage dominate and the entire length of the clothoid will be needed for the transition to the superelevation, an irregularity in the shape of the outer curve cannot be avoided. If, however, a bulge appears, it will not be very conspicuous because of the sharp change of direction and, moreover, if it should be seen, it will act as a warning in exposing the small radius.

7.2.3 Minor changes of level in (almost) straight sections

Figure 7.6-7.9 Minor changes of level, less than eye-height, show short S-shapes, because the concave curve does not dominate in the picture. It appears that the rate of slope determines its acceptability. Rates of slope below 0.1% hide the difference in height rather well. This depends on the place of the observer in the cross-section and his eye-level. Moreover a small dip is more disturbing than a small elevation. In this case a straight slope with small radii at the top and the foot will cause no objections.

Figure 7.10 To use a linking concave and convex curve with the same length results in the double rate of slope.

7.2.4 A composite curve in front of a curve in horizontal sense

To prevent a short S-shape arising in the picture of a composite curve preceded by an (almost) straight section and succeeded by a horizontal curve, it is desirable to choose the proportion between the vertical and the horizontal radii of the composite curve larger than the proportion between total road-width $b$ and eye-height $h$.

In this way a counter-curve in the outer side caused by the vertical curve is overrun by the horizontal one (See page 68).

In a transition-curve the elevation should begin at about 0.4 of the length of the transition-curve.

7.3 Straight sections

Figure 7.11-7.14 Straight sections, especially short ones, horizontal, vertical as well as in three dimensions, are tedious and rigid and should be avoided if possible. They prevent a view of the traffic in front, and are only justifiable if aiming at a distinct
object, like a tower. They are extremely disturbing when observed between two curves bent in the same sense, which is accentuated by optical illusion. In motorways straight sections, especially between two concave curves, are often visible over a distance of 1.5 to 2 km, even then showing the disturbing picture of a straight between two curves in the same sense. If possible, and in any case in short straight sections, a fluent connection of the curves by means of a concave curve with a large radius, for instance 100,000 m, should be opted for.

7.4 Double S-shapes

7.4.1 Local horizontal displacements

In straight or almost straight sections a local lateral shift of small dimension, for instance in a widening or narrowing of the central area over a short distance, less than the sight distance, will give a disturbing picture, notwithstanding the application of large radii (See page 69). Because of this, local horizontal shifts ought to be hidden in a clearly bent section, for instance by a series of curves in the same direction with different radii but with equal lengths.

Figure 7.15 The relationship between the dimensions of these radii can be calculated from the equations

\[
\frac{3}{32R_{H1}} + \frac{1}{32R_{H2}} = \frac{1}{8R_H} \pm \frac{u}{L^2}
\]

and

\[
\frac{1}{2R_{H1}} + \frac{1}{2R_{H2}} = \frac{1}{R_H}
\]

If L is chosen larger than \(4 \sqrt{R_H} \cdot u\) no protuberance will be observed.

7.4.2 Local dips in a horizontally straight section

A local dip less than about 1.5 m deep

Local dips of this kind which disappear within sight distance show a shallow depression in the road. The same rules should be followed as for a single change of level of less than eye-height. The transitions between concave and convex curves should have a slope as flat as possible, so as to distribute the difference in height over a great length. The concave
curve itself must have a large radius to avoid a kink in the picture. The radii of the convex curves remain visually unimportant.

A local dip, more than about 1.5 m deep

A flat central part and horizontally straight slopes cause a disturbing straight line between two curves in the same direction.

Figure 7.16 and 7.17 There ought to be a fluent transition between all the curves. Increasing the radii of the convex curves will not improve the picture. Large radii of the concave curves however, are of great importance. In deep tunnels the definition of the radii will not leave much choice. They are fixed by demands of economy and traffic-safety. Unless there is a composite curve, the road-picture causes no problems because in that case three elements are never observed simultaneously. Composite curves on the contrary need special attention.

7.4.3 A terrace in a change of elevation

Figure 7.18 and 7.19 A straight slope between the concave and the convex curve of a change of elevation is undesirable, but this cannot always be avoided. More than one element between the curve at the foot and the top, creating a horizontal or a sloping terrace, will cause a disturbing picture in elevations that can be seen in their entirety. The lower the terrace is situated and the shorter it is, the more disturbing it will be to the eye, when seen from the foot, because it can be observed from a shorter distance. Observed from the top, every terrace is disturbing. A horizontally curved elevation with obstacles in the inner curve, i.e. buildings, trees or shrubs, etc. is less vulnerable in this respect.

7.4.4 An elevation beginning in a horizontal curve

If it is necessary to start an elevation at a given distance after the start of a horizontal curve there is a danger of a countercurve in the picture. Considerably larger radii in the concave curve should be chosen, as is done in a composite curve behind a long (almost) straight section (See 7.2.4). When the horizontal curve begins at a distance $Z_l$ from the observer and the elevation at a distance $Z_H$ in the horizontal curve, the radius of the concave curve is defined by:

$$R_v > \frac{bR_H}{Z_H} + Z_H(\frac{1}{2}Z_H + Z_l)$$

(See page 69).
If the observer is in the horizontal curve, $Z_1 = 0$. This means, if $Z_H = 300$ m, for instance for $R_H < 1000, 2000$ and $5000$ m that $R_Y$ should be over $45000, 50000$ and $70000$ m respectively.

In the case where the elevation begins behind the turning point of the inner curve and the observer is at approximately $300$ m from the start of the horizontal curve (which means horizontal radii larger than $15000$ m), very large radii for the concave curve are required in order to avoid an S-shape or an apparent by straight section. If this is impossible, it is desirable that the view of the inner curve is hidden by earthworks, trees etc. (without impairing traffic-safety) to make the S-curve invisible. *Figure 7.20* Observation from the top will cause a disturbing picture in every case where more than two elements are visible at the same time, or a shift in the picture appears.

7.4.5 *A local dip in a horizontal curve*

**A dip, less than about 1.5 m deep**

This can be admitted in horizontal curves with radii between $4000$ m and $10000$ m when the concave curve is at least $30000$ m. If the radius of the horizontal curve is less than $4000$ m, these dips are ugly if observed in their entirety. Horizontal curves over $10000$ m require similar treatment as dips in straight sections, see 7.2.3.

**A dip, more than 1.5 m deep**

A satisfactory picture can be obtained in horizontal curves with radii over $3000$ m, if the concave curve has a radius of more than $30000$ m. Where the radii of horizontal curves are under $3000$ m the concave curves require at least $20000$ m.
8 The presentation of the road impression

A three-dimensional presentation combining horizontal and vertical alignment is often required in order to criticise the design of a road, especially with regard to its visual qualities.

By experience and by design-data referred to earlier, it is in many cases possible to make an immediate appraisal.

Often however, imagination falls short, and a visual representation of the road is indispensable. There are different methods by which this can be obtained.

8.1 Staking out in the field

The practice of staking-out in the field, provided a sufficient number of stakes are used, often will give a convincing impression of the road-design. However, this method is complicated and relatively expensive, in addition to which observation in the exact eye-level is often difficult. Because of this the use of this method is somewhat restricted.

8.2 Calculated perspectives

In cases where a poor road-picture may be expected it often suffices to make a few hand-sketches to prove the point. If it is not known where critical sites may occur, many sketches may be necessary and for this it is often convenient to use a computer.

The size of the road-picture depends on the distance to the picture-plane. If this is fixed at 1 m, the calculations are simplified. These drawings should also be observed at a distance of 1 m, but with some experience, a reliable presentation can also be obtained at a shorter distance.

If several people are making an observation together, the distance of the picture-plane has to be increased to a few metres. It should be borne in mind that although a larger distance of the picture-plane allows more accurate drawing, this has no actual influence on the reliability of the picture itself.

8.2.1 The tangents method

As already discussed, nearly every curve, whether it is vertical, horizontal or composite, appears in the picture as a hyperbola. Its centre and asymptotes can be constructed in a simple manner.
One of the asymptotes (the asymptote $-\infty$) is the picture of the tangent of the curve at an infinite distance, supposing this curve to be a parabola.

The other asymptote (the asymptote $-0$) is the picture of the tangent in the intersection-point of the curve with the vanishing plane.

The centre $M$ is the intersection-point in the picture of both tangents, i.e. the picture of the vertex of the part of the curve that lies in front of the vanishing plane and is thought to extend to infinity.

One of the properties of the hyperbola is that the point of contact of every tangent lies in the middle of that part of the tangent which is cut off by the asymptotes.

The purpose of the tangents method is to produce an enveloping figure by constructing a number of tangents and their points of contact with the hyperbola, achieved with little calculation. This is possible by defining the intersection-points of the enveloping tangents with the asymptotes.

A simple construction of this system of intersection-points can be achieved by using two properties of the parabola (which is pictured as a hyperbola) as follows:

1. The change of direction of a part of the parabola is directly proportional to the distance $Z$ from its beginning i.e.

\[
\frac{dX}{dZ} = \frac{Z}{R_H}
\]

and

\[
\frac{dY}{dZ} = \frac{Z}{R_V}
\]

2. The tangent to the parabola at a distance $Z$ from the vanishing plane intersects the tangent at the vanishing plane at a distance $\frac{1}{2}Z$. This intersection-point is the vertex of the pictured part of the curve.

With these two properties the intersection-points of each tangent with the asymptotes can be constructed.

Figure 8.1 As an example a section is examined, which is horizontally curved to the right and has a radius $R_H$. The observer is supposed to be situated at an eye-level of $h$ m above and at a distance of $a$ m to the left of the curved section. The curve is supposed to end at a distance $Z$ m from the observer. The distance to the picture-plane is defined at $1$ m.

The right edge of the road is constructed first.

For convenience the centre $M$ of the hyperbola, which is observed, is chosen as the origin of the picture, i.e. the centre of the system of coordinates. It is also the intersection-point of the asymptotes.

The asymptote $-\infty$ coincides with the x-axis.
The asymptote $-0$ is the picture of the tangent to the curve in its intersection-point with the vanishing plane.

The choice of the picture-origin means that this tangent intersects the picture-plane perpendicularly at $h$ m below and $a$ m to the right of the picture-origin. The asymptote $-0$ connects this intersection-point with the centre $M$, which coincides with the vanishing point $V_0$. Its angle with the x-axis has a tangent of $\frac{h}{a}$.

A part with a length $Z'$ is chosen to be constructed first.

The vertex $H'$ of this part of the curve is situated on the asymptote $-0$ at $\frac{h}{2Z'}$ below the x-axis or $\frac{a}{2Z'}$ to the right of $V_0$.

The change of direction of this first part of the curve is $\frac{Z'}{R_H}$. The vanishing point $V'$ of the direction at the end of this part will be found on the asymptote $-\infty$ at $\frac{Z'}{R_H}$ m to the right of $V_0$.

The change of direction of the entire curve is $\frac{Z}{R_H}$.

The vanishing point $V$ of the direction at the end of the curve is therefore situated on the asymptote $-\infty$ at $\frac{Z}{R_H}$ m to the right of $V_0$.

On the asymptote $-\infty$ several vanishing points, $V''$, $V'''$ and $V''''$ are plotted between $V'$ and $V$, representing parts of the curve with increasing lengths.

The vertices of these parts are situated on the asymptote $-0$. In order to construct these, an auxiliary line is drawn through $H'$ parallel to the asymptote $-\infty$.

On this line the auxiliary points $S''$, $S'''$, $S''''$ and $S$ are measured out, at equal distances to those between the vanishing points, but in the opposite direction, in such a way that $H'S'' = V'V''$, $H'S''' = V'V'''$ etc.

Connecting these auxiliary points with $V'$ will produce the vertices $H''$, $H'''$, $H''''$ and $H$ on the asymptote $-0$.

This auxiliary line through $H'$ parallel to the asymptote $-\infty$ is the picture of the intersection of the plane which contains the curve, with the plane parallel to the vanishing plane at a distance $\frac{1}{2}Z'$.

The connection-lines are pictures of parallel lines in the direction of $V'$, giving, when intersecting the asymptote $-0$, perspective pictures proportional to the sections $H'S''$, $S''S'''$ or to $V'V''$, $V''V'''$ etc.

The points of contact $T'$, $T''$, $T'''$ etc. are situated in the middle of $H'V'$, $H''V''$, $H'''V'''$.
H''\text{'}V''\text{'} etc., respectively, in which the tangents indicate the directions in the points of contact.

The lines HV result in an enveloping curve.

**Figure 8.2** The asymptote \(-0\) of the picture of the outer curve naturally has the same vanishing point \(V_0\) coinciding with \(M\).

By transportation parallel to the cross-fall of the road, the vertices \(H', H'', H''\text{'}\) etc. of the outer curve can be found on the asymptote \(-0\). In the middle of the connecting lines of these vertices with the vanishing points \(V', V'', V''\text{'}\) etc. lie the points of contact of the outer curve \(T', T'', T''\text{'}\) etc.

**Figure 8.3** This figure concerns the case where the horizontal curve with a length \(Z_2\) begins at a distance \(Z_1\) m in front of the observer, while the connecting section is straight. The same method is used, in which the horizontal curve, which is now the second element, is imagined to be extended unto the vanishing plane. The picture of the straight section has an angle with the x-axis with a tangent \(-\frac{h}{a}\). The vanishing point \(V_0\) coincides with \(V_1\), and, moreover, is also \(M_1\). The picture-origin is chosen in this same point.

The centre \(M_2\) of the hyperbolic picture of the curve will be found it on the asymptote \(-\infty\) at a distance \(\frac{Z_1}{R_H}\) m from \(V_1\) in the direction opposite to the bend of the curve.

The vertex \(H_1\) of the imaginary extended part of the curve between the real beginning of the curve and the vanishing plane will be found on the edge of the straight section at a distance \(\frac{h}{2Z_1}\) m from the observer, i.e. at \(\frac{h}{2Z_1}\) m below the x-axis. The asymptote \(-0\) connects \(M_2\) with \(H_1\).

On the asymptote \(-\infty\), coinciding with the x-axis, a sufficient number of vanishing points \(V_2', V_2''\), etc. is plotted between \(V_1\) and \(V_2\). This \(V_2\) is the vanishing point of the direction at the end of the curved section, i.e. at a distance \(\frac{Z_2}{R_H}\) m from \(V_1\).

In the point \(H_1\) an auxiliary line is drawn parallel to the asymptote \(-\infty\), on which the auxiliary points \(S_2', S_2''\) etc. are indicated, at the same distances as those between the vanishing points, but in the opposite direction.

The vertices \(H_2', H_2''\) etc. of the parts of the curved second element are found as the intersection-points with the asymptote \(-0\) of the connecting lines of \(S_2', S_2''\) etc. with \(V_1\).

When these vertices are connected with their corresponding vanishing points, the points of contact \(T_2', T_2''\) etc. will be lying in the middle of these connecting lines.
Figure 8.4 and 8.5  Figure 8.4 and 8.5 show the corresponding construction of vertical and composite curves.

Needless to say that the asymptote $-\infty$ of a composite curve is not horizontal. Its direction is defined by the angle with the x-axis, with a tangent $\frac{R_H}{R_V}$. This asymptote is the intersection-line of the plane which contains the curve with the plane parallel to the vanishing plane in the infinite.

A vertical curve has an asymptote $-\infty$ coinciding with the y-axis.

Figure 8.6  A worked example is presented in figure 8.6. Here there is a succession of:

1. a horizontal straight section, with a length $Z_1 = 100$ m,
2. a composite curve with a radius in horizontal sense $R_{H1} = 5000$ m to the right, and a vertical radius $R_{V2} = 10000$ m upwards, with a length $Z_2 = 400$ m,
3. a straight top-curve with radius $R_{V3} = 10000$ m and a length $Z_3 = 800$ m,
4. a straight concave curve with $R_{V4} = 10000$ m and a length $Z_4 = 400$ m.

The picture is, with a distance to the picture-plane of 1 m, constructed as follows:

1. The picture-plane is chosen in the direction of the straight section, so the origin coincides with the vanishing point $V_0 = V_1$ of the straight section.
   This point can also be considered as the centre $M_1$ of the straight section.
   The edges of the straight section have an angle with a tangent $\frac{h}{a}$ with the x-axis, this for the right hand side is $-\frac{1.2}{3}$ and for the left hand side is $-\frac{1.2}{5}$.
   They start at the margin of the picture and finish in the points $T_1$ at the right and the left, at $d = 1 \cdot \frac{h}{Z_1} = 1 \cdot \frac{-1.2}{100} = 0.012$ m below the x-axis.
   The right side of the carriage-way is developed first.

2. The part of the composite curve, continued from the end of the straight section unto the vanishing plane, has its vertex $H_1$ at a distance of $\frac{1}{2}Z_1$ in front of the beginning of the curve, on the tangent in $T_1$, therefore on the straight part at $d = \frac{h}{\frac{1}{2}Z_1} = 1 \cdot \frac{-1.2}{20} = 0.024$ m below the x-axis.
   The centre $M_2$ in the picture of the composite curve lies at $d = \frac{Z_1}{R_{H2}} = 1 \cdot \frac{1.0000}{5000} = 0.02$ m to the left of $V_1$ and at $d = \frac{Z_1}{R_{V2}} = 1 \cdot \frac{1.0000}{10000} = 0.01$ m below it.
   These distances show the changes of direction of the imaginary extension of the composite curve between its beginning and the observer.
Connection of \( M_2 \) and \( H_1 \) results in the asymptote \(-0\) of the composite curve. The angle between the asymptote \(-\infty\), which goes through \( M_2 \), and the x-axis has a tangent \( \frac{R_{H_1}}{R_{V_2}} = \frac{5000}{10000} = 0.5 \).

The composite curve with a length \( Z_2 = 400 \) m is in this example divided into four parts of 100 m each. The change of direction after 100 m appears in \( V_2' \) on the asymptote \(-\infty\) horizontally at \( d \cdot \frac{Z_1 + 100}{R_{H_2}} = 1 \cdot \frac{100 + 100}{5000} = 0.04 \) m to the right of \( M_2 \).

and vertically at \( d \cdot \frac{Z_1 + 100}{R_{V_2}} = 1 \cdot \frac{100 + 100}{10000} = 0.02 \) m upwards from \( M_2 \).

The vanishing points \( V_2'', V_2''' \) and \( V_2 \) are just as \( V_2' \) situated on the asymptote \(-\infty\) and at equal distances apart as \( V_1 V_2' \). Now on an auxiliary line, parallel to the asymptote \(-\infty\) in \( H_1 \), but in the opposite direction, the distances \( V_1 V_2', V_2' V_2'' \) etc. are plotted, giving the auxiliary points \( S_2', S_2'', S_2''' \) and \( S_2 \).

The intersection-points with the asymptote \(-0\) of the connecting lines between \( V_1 \) and these auxiliary points, are the vertices \( H_2', H_2'', H_2''' \) and \( H_2 \) of the series of the curved sections.

Connection of these vertices to their corresponding vanishing points results in the tangents, in the middle of which are the contact-points \( T_2', T_2'', T_2''' \) and \( T_2 \) of the inner curve. \( T_1 \) also lies in the middle of \( H_1 V_1 \).

3. In the vanishing point \( V_2 \) the direction of the asymptote \(-\infty\) changes and now belongs to the top-curve. Because this curve is vertical, the asymptote \(-\infty\) has a vertical direction.

The centre \( M_3 \) lies, according to the formula

\[
d \cdot \frac{Z_1 + Z_2}{R_{V_3}} = 1 \cdot \frac{100 + 400}{10000} = 0.05 \text{ m above } V_2.
\]

The vertical curve is divided in 8 parts, 100 m long each, so a series of vanishing points \( V_3' - V_3 \) appears, at \( d \cdot \frac{100}{R_{V_3}} = 1 \cdot \frac{100}{10000} = 0.01 \) m apart, below \( V_2 \).

Connecting \( M_3 \) and \( H_2 \) gives the asymptote \(-0\) of the top curve.

In \( H_2 \) an auxiliary parallel line is constructed, on which, equal to these distances between the vanishing points, but in upward direction, the auxiliary points \( S_3' - S_3 \).
Connecting $V_2$ to these auxiliary points gives the vertices $H_3' - H_3$ as intersection-points with the asymptote $-0$.

As in the curves already constructed, the tangents of the top-curve with their contact-points $T_3' - T_3$ are defined by connecting $H_3' - H_3$ with $V_3' - V_3$ and dividing into two equal parts. $V'''$ lies on the x-axis, so $T'''$ is the top of the curve.

4. The vertical curve at the end of the elevation is constructed in the same way as the top-curve.

The left side of the road can of course be constructed by exactly the same method as the right side.

However this is rather complicated, it demands more space in this case, and is not very accurate, as several lines intersect at a very small angle. It is more simple to define the vertices $H_2, H_3$ etc. at the same level as $H_2, H_3$ etc., on the asymptote $-0$ at the left, running through $M_2$ and vertex $H_1$ on the left.

If the cross section of the carriage-way is not horizontal, there is a difference in the eye-height for both sides of the road. $H_1$ is in that case on the left higher or lower than on the right. The connecting lines of the other vertices to left and right are parallel to the cross-fall.

The tangents and their contact-points are defined in the same way as for the right side of the road.

The tangents method is particularly suitable for giving to the draughtsman a quick and correct impression of the road during the design of longitudinal and horizontal alignment.

At the same time the causes of defects will be apparent and indicate how to modify the design and so obtain a more acceptable appearance.

*Figure 8.7- 8.9* Further examples are given in figure 8.7, 8.8 and 8.9.

In figure 8.8 the tangents method is applied to construct a picture of a road which has as its third element a horizontally and vertically straight section. This part can be defined as a parabola degenerated to a straight line, the picture of which is a hyperbola that is in the same way degenerated to a straight line. This means that in the figure $V_2 = M_3 = V_3$.

This coincidence causes the impossibility of using the tangents method for the construction of points in the picture of this third element.

In order to find the end $T_3$ and the vertex $H_3$ of the straight part, an auxiliary line is drawn through $H_2$, parallel to the asymptote $- \infty M_2 V_2$, which is the vanishing line of the plane containing the preceding composite curve $T_1 T_2$. 

42
This auxiliary line is the picture of the intersection-line of the plane that contains the straight part of the road, with a plane parallel to the vanishing plane at the distance of \( H_2 \), i.e. \( \frac{Z_1 + Z_2}{2} = 150 \text{ m} \).

The line through \( V_1 \) and the beginning of the straight section, \( T_2 \), intersects the auxiliary line in \( C_1 \).

Given the parallelism, shown by the common vanishing point, the part \( H_2C_1 \) is the picture of the projection of \( H_2T_2 \) on the vertical plane through the auxiliary line.

The proportion of \( H_2T_2 \) to \( T_2T_3 \) is exactly represented by the lengths of \( H_2C_1 \) and \( C_1C_2 \) if

\[
\frac{H_2C_1}{C_1C_2} = \frac{2}{Z_3} = \frac{150}{200} = \frac{3}{4}.
\]

So \( C_2 \) is found by measuring out \( \frac{4}{3} \) of the length of \( H_2C_1 \) beyond \( C_1 \). By connecting \( C_2 \) with \( V_1 \), the end of the straight section, \( T_3 \), is found as the intersection-point with \( H_2V_2 \).

The vertex \( H_3 \) is required for the construction of the fourth element, a convex curve. This point \( H_3 \) lies on the extension of \( T_2T_3 \), at half the distance from \( T_3 \) to the vanishing plane, so \( H_3V_3 \) is twice \( T_3V_3 \).

The vanishing point \( V_4 \) lies at \( \frac{Z_4}{R_{V_2}} \) below \( V_3 \), in this case on the x-axis.

The centre \( M_4 \) lies at a distance \( \frac{Z_1 + Z_2 + Z_3}{R_{V_2}} \) above \( V_3 \).

Connecting \( H_3 \) with \( M_4 \) gives the asymptote \(-0\) of the fourth element. In \( H_3 \) another auxiliary line is drawn, parallel to \( M_4V_4 \), in the opposite direction, in order to construct the convex curve.

### 8.2.2 The coordinates method

**Figure 8.10** When, in designing a road or examining an existing road the coordinates in space are known, it is possible to make a picture by means of the coordinates method.

In this case the points in the picture are calculated by the formulae

\[
x = d \frac{X}{Z}
\]

and

\[
y = d \frac{Y}{Z}.
\]
The picture can be achieved by calculating a sufficient number of its points and connecting them. First of all the transition-points between the elements are defined and after this the necessary intermediate points, for convenience at roughly estimated distances. Figure 8.10 with its table of coordinates gives an example. A disadvantage of this method lies in the extensive calculations, whereas no insight is obtained of possible improvements.

8.2.3 Computer-drawn perspectives

If it is desirable to contemplate a design from a number of station-points, when it is uncertain where a disturbing picture can be expected, the computer will give a series of perspective drawings, up to now by means of the coordinates method only. The data used for these pictures are the same as are used in calculating the edge-lines of the carriage-way in the design. A complete impression of the future road can be obtained by a series of pictures. It is even possible to create a moving presentation on a screen, similar to the sensation experienced when driving. These pictures by the coordinates methods may be interesting as a study but they will give no information at all about the cause of defects. Correction with a light-pen on the screen will not be successful, for improvement in one place may cause new faults elsewhere. Only an analysis of the design-data by means of the tangents method will make it clear where the faults are, and the means to improve upon them. It is expected that in the near future soft-ware for this will be available.

Not before it becomes possible to create a computer-road-design system which would exclude visual faults from the start, will it be feasible to make full use of the computer. The required criteria have not been completely achieved but some have been pointed out in this treatise.

8.3 Scale models

Models are suitable to illustrate the alignment in difficult situations, for instance at interchanges, which cannot be explained by means of perspective drawings, and for an exact definition of faults as well as a convincing medium for correction. This special purpose demands great accuracy, also to accentuate the micro-structure and the relationships between the road-edges. Length, height and width should have the same scale, to avoid deception. Common scales are 1 : 250 or 1 : 200 for interchanges and 1 : 100 for details, i.e. of over-bridges and under-bridges.
Building with an accuracy of 0.1 mm is sufficient, provided the models are of solid construction (See page 72). Models of this kind seem expensive, but because of their solidity they are durable and useful for purposes other than the study of alignment, such as road-marking, lighting, sign-posting and landscaping, and during the construction. It is absolutely necessary to observe such models at the correct eye-height. In individual observation prisms or mirrors are used, but for group-observation, photographs, films or television, using a periscope, give good results.

8.4 The study of existing roads

In order to achieve a better understanding of how alignment-faults arise it is important to examine actual roads on site and to compare the design problems with the finished result. The use of photography and similar methods is suitable for discussion with others.

Figure 8.11 To obtain a clear relationship between the vertical and the horizontal alignments the use of a curvature diagram is recommended \( \frac{1}{R} \) versus \( L \).
9 Explanations

To 3 Perspective

The concept "perspective" comes from the latin "perspicere", which means "to look through". Simon Stevin (1548-1620) called it "the art of visual penetration". Curiously enough, objects were originally not pictured as observed but as they were supposed to appear.

Gradually it became clear that an object near the observer seems larger than further away, and that parallel lines seem to converge at a point in the distance. Techniques of perspective were developed particularly during the Renaissance.

Among others, Da Vinci (1452-1519) and Dürer (1471-1528) have intensively dealt with this subject. The latter often used a so-called perspectograph, an instrument that still is used with good results in our time when designing roads, especially in hilly regions.

The first study of road-perspective was developed in Germany by V. J. Ch. von Ranke and published in 1943 by Dr. H. Lorenz.

The mathematical properties of the road-picture and also of the three-dimensional road-design were up to now virtually unknown. This paper describes these properties in particular, and they are expounded and proved in these explanations.

In this paper the word "picture" means the figure shown in the picture plane. The word "impression" is used for what is shown in reality and observed in the picture at its correct distance and place.

THE PICTURE OF A STRAIGHT LINE

The picture of a straight line continuing into the infinite ends in a point, the vanishing point.

The coordinates of a vanishing point in a picture-plane are defined as follows:

It is supposed that the line has a direction with an angle \( \alpha \) in the XZ-plane, and \( \beta \) in the YZ-plane.

If two points of this straight line are fixed:

\[
P_1(x_1y_1z_1)
\]

and

\[
P_2(x_2y_2z_2)
\]
these directions are defined by

\[ \tan \alpha = \frac{X_2 - X_1}{Z_2 - Z_1} \]

and

\[ \tan \beta = \frac{Y_2 - Y_1}{Z_2 - Z_1} \]

When \( \alpha \) and \( \beta \) are known, and point \( P(X_pY_pZ_p) \), the equations of the straight line in three-dimensional coordinates are:

\[ X - X_p = (Z - Z_p) \tan \alpha \]

and

\[ Y - Y_p = (Z - Z_p) \tan \beta \]

or

\[ X = (Z - Z_p) \tan \alpha + X_p \]

and

\[ Y = (Z - Z_p) \tan \beta + Y_p \]

The equations of the picture of the straight line in the picture plane are

\[ x = \frac{d}{Z} \{ (Z - Z_p) \tan \alpha + X_p \} \]

and

\[ y = \frac{d}{Z} \{ (Z - Z_p) \tan \beta + Y_p \} \]

or

\[ x = d \left\{ \left( 1 - \frac{Z_p}{Z} \right) \tan \alpha + \frac{X_p}{Z} \right\} \]

and

\[ y = d \left\{ \left( 1 - \frac{Z_p}{Z} \right) \tan \beta + \frac{Y_p}{Z} \right\} \]

47
The vanishing point $V$ is the picture at an infinite distance of the observed straight line, so $Z = \infty$, which means

$$x_v = d \tan \alpha$$

and

$$y_v = d \tan \beta$$

or for a straight through two points

$$x_v = d \frac{X_2 - X_1}{Z_2 - Z_1}$$

and

$$y_v = d \frac{Y_2 - Y_1}{Z_2 - Z_1}$$

It can also be shown, that the coordinates of the picture of all points in the vanishing plane are infinitely large. This is because $Z = 0$, and so

$$x = \frac{X}{0} = \infty$$

and

$$y = \frac{Y}{0} = \infty$$

The picture of the part of the straight line in front of the vanishing plane runs from infinity to the vanishing point $V$.

To 4  The picture of a straight section

The picture-plane-axis is generally parallel to the road-axis. Figure 4.1 shows that this is not particularly necessary but more convenient. In figure 4.4 the observer is situated in such a way that both roads give the same impression.

The picture of road 2, the direction of which is not perpendicular to the picture-plane, is different from the picture of road 1, but their impression is identical, if observed in the correct place.
The curve in the picture; the calculation of the hyperbola, its asymptotes and centre

The proof that the picture of a curve generally is a hyperbola, and the calculation of its asymptotes and its centre are given below.

To 5.1 The vertical curve

As chapter 5.1 mentions, the equations defining the coordinates of a point are

\[ x = \frac{a}{Z} \]

and

\[ y = \frac{h + \frac{(Z - Z_1)^2}{2R_v}}{Z} \]

Elimination of \( Z \) from these equations results in the equation of the curve that represents the vertical parabola in the picture. This equation is

\[(2hR_v + Z_1^2)x^2 - 2aR_vxy - 2aZ_1x + a^2 = 0\]

This is the equation of a conic section. The type of conic section which this equation represents, can be defined by the product of the coefficient of \( x^2 \) and \( y^2 \), less the square of half the coefficient of \( xy \). When the result appears to be positive, zero or negative, the equation represents respectively an ellipse, a parabola or a hyperbola. In this particular case the calculation results in the factor \(-a^2R_v^2\). This is always negative, so the parabola is represented in the picture by a hyperbola to which the side-line of the straight section in front of it is the tangent.

The centre of the hyperbola is defined by fixing the differentials of the equations with respect to \( x \) and \( y \) equal 0. This means:

\[ 2(2hR_v + Z_1^2)x - 2aR_vy - 2aZ_1 = 0 \]

and

\[ -2aR_vx = 0 \]

or

\[ x_M = 0 \]

and

\[ y_M = -\frac{Z_1}{R_v}. \]
As already explained, the coordinates of the centre relating to a given R_v depend on the distance Z_I from the vanishing plane to the beginning of the vertical curve.

If Z_I = 0, the observer is at the beginning of the curve or in it. Then the centre coincides with the vanishing point of the picture of the tangent in the vanishing plane, which means that this tangent is parallel to the picture-plane axis.

When the observer is at a certain distance in front of the curve and the curve is extended in its direction to the vanishing plane, the change of direction of this extended part of the curve is \( \frac{Z_I}{R_v} \).

This explains why in the picture plane (when d = 1) the centre lies at \( \frac{Z_I}{R_v} \) below the vanishing point.

The equations for the direction of the asymptotes can be obtained by making the factors of the second degree in the equation of the curve equal to 0 and resolving them. This results in the equations of the asymptotes as

\[
x = 0 \tag{1}
\]

and

\[
(2hR_v + Z_I^2)x - 2aR_v y - 2aZ_I = 0 \tag{2}
\]

The first is the asymptote \(-\infty\).

The asymptote \(-0\) represented by equation (2) is the picture of the tangent in the intersection-point with the XY-plane or vanishing plane, in which it touches the extended curve.

The picture of this tangent-point is in the infinite, because the vanishing plane is parallel to the picture-plane. The edge of the straight section in front of the curve has in the picture the equation \( \frac{x}{y} = \frac{a}{h} \) or \( x = \frac{a}{h} y \).

Substitution in (2) gives as the picture the intersection-point of this asymptote \(-0\) with the edge of the road \( y = \frac{h}{2Z_I} \).

This means that the asymptote \(-0\) intersects the road-edge half-way between the observer and the beginning of the curve.

This is also understandable if it is remembered that for the parabola \( Y = \frac{Z^2}{2R_v} \) the direction of the tangent is

\[
\frac{dY}{dZ} = \frac{Z}{R_v}.
\]
This results for the tangent in the intersection-point with the vanishing plane, with coordinates

\[ Y_o = h + \frac{(0 - Z_1)^2}{2R_v} \]

and

\[ Z_o = 0 \]

in the equation

\[ Y - h - \frac{(0 - Z_1)^2}{2R_v} = \frac{0 - Z_1}{R_v} (Z - 0) \]

The intersection-point with the (horizontal) edge of the road has \( Y = h \), which means for \( Z \):

\[ \frac{Z \cdot Z_1}{R_v} - \frac{Z_1^2}{2R_v} = 0 \]

or

\[ Z = \frac{Z_1}{2} \]

As previously shown, these properties make it possible to define the hyperbola in the picture in a simple way.

To 5.2  \textit{The horizontal curve}

In road-design horizontal curves, contrary to vertical curves, will not appear as parabolas but as circular curves. The perspective picture of the circular curve however does not differ much from that of the parabola, at least if the change of direction is relatively low, as explained in the following pages.

Figure 9.1- 9.3 At first a circular curve is considered. The coordinates of a point are:

\[ x = \frac{a + R_h - \sqrt{R_h^2 - (Z - Z_1)^2}}{Z} \]

and

\[ y = \frac{h}{Z} \]
Elimination of $Z$ results in the equation of a curve in the picture, representing the circular horizontal curve. This is the equation of a conic section.

$$h^2x^2 - 2h(a + R_H)xy + (a^2 + 2aR_H + Z_1^2)y^2 - 2hZ_1y + h^2 = 0$$

The resultant conic section depends on the product of the coefficients of $x^2$ and $y^2$, less the square of half the coefficient of $xy$, or

$$h^2(a^2 + 2aR_H + Z_1^2) - h^2(a + R_H)^2 = 0$$

or

$$Z_1^2 = R_H^2$$

It will appear that:

1. If the circular curve is completely in front of the vanishing plane, the picture will be an ellipse (Figure 9.1).
2. If the circular curve touches the vanishing plane the picture is a parabola (Figure 9.2).
3. If the circular curve intersects the vanishing plane, the picture will be a hyperbola (Figure 9.3).

This means that the curve is pictured as part of an ellipse, parabola or hyperbola respectively, beginning at the point of contact with the section in front of it.

In practice one is not consciously aware of the shape of the horizontal curve until a few hundred metres away from its beginning, while normally the radii in motorways will not be less than 750 metres. *So on the road the picture of most circular curves will be a hyperbola.*

Even if smaller radii are concerned, for instance in interchanges, the picture is usually a hyperbola, because one will not be aware of its shape when observing it from a greater distance.

*Figure 9.4* In the less usual case when the picture of the horizontal curve is an *ellipse*, the coordinates of its centre in the picture can be calculated by the formulae:
and

\[ x_M = \frac{(a + R_H)Z_1}{Z_1^2 - R_H^2} \]

and

\[ y_M = \frac{hZ_1}{Z_1^2 - R_H^2} \]

The lines through the centre in the direction of the road section in front of it and parallel to the vanishing plane are the directions of two conjugate axes.

Their lengths are respectively

\[ A_1 = \frac{2R_H}{Z_1^2 - R_H^2} \sqrt{h^2 + (a + R_H)^2} \]

and

\[ B_1 = \frac{2R_HZ_1}{Z_1^2 - R_H^2} \]

while the sine of the angle between these lines can be found as

\[ \sin \theta = \frac{h}{\sqrt{h^2 + (a + R_H)^2}} \]

The rectangular axes can be defined by

\[ AB = A_1B_1 \sin \theta = \frac{4hR_H^2Z_1}{(Z_1^2 - R_H^2)^2} \]

and

\[ A^2 + B^2 = \frac{4R_H^2}{(Z_1^2 - R_H^2)^2} \{h^2 + (a + R_H)^2 + Z_1^2\} \]

The direction of the rectangular axes results from

\[ \tan 2\lambda = \frac{2h(a + R_H)}{h^2 - a^2 - 2aR_H - Z_1^2} \]

In practice this calculation of the principal axes is not important, as an extremely elongated ellipse is pictured.

In the more usual case of a hyperbola the directions of the asymptotes and the centre can be defined similarly to the vertical curve.
The coefficients defining the directions of the asymptotes will appear as:

\[ y = - \frac{h(a + R_H) \pm \sqrt{R_H^2 - Z_1^2}}{a^2 + 2aR_H + Z_1^2} \cdot x. \]

The centre M can be found by the equations

\[ 2h^2x_M - 2h(a + R_H)y_M = 0 \]

and

\[ 2h(a + R_H)x_M - 2(a^2 + 2aR_H + Z_1^2)y_M - 2hZ_1 = 0 \]

or

\[ x_M = \frac{(a + R_H)Z_1}{Z_1^2 - R_H^2} \]

and

\[ y_M = \frac{hZ_1}{Z_1^2 - R_H^2}. \]

The asymptotes of the hyperbola which represents the circular curve, are the pictures of the tangents touching the circular curve in the vanishing plane.

When the centre of the circular curve lies in the vanishing plane, i.e. when driving in the circular curve, the picture of this centre will be in the infinite, so the tangents are parallel.

In all other cases the intersection-point of the asymptotes, being the picture of the intersection-point of the tangents touching the circular curve in the vanishing plane, is situated sometimes in front of, mostly however behind the vanishing plane. This means that its picture is usually above the x-axis.

When instead of this equation of the picture of a circular curve the equation of the picture of a parabola is defined from the coordinates of a point

\[ a + \frac{(Z - Z_1)^2}{2R_H} \]

\[ x = - \frac{2R_H}{Z} \]

and

\[ y = \frac{h}{Z} \]

the elimination of Z gives:

\[ (2aR_H + Z_1^2)y^2 - 2hR_Hxy - 2hZ_1y + h^2 = 0 \]
This again will be a hyperbola, of which the asymptotic directions can be defined by

\[ y = 0 \quad (1) \]

and

\[ y = \frac{2hR_H}{2aR_H + Z_t^2} x \quad (2) \]

The centre \( M \) is then fixed by

\[ x_M = -\frac{Z_t}{R_H} \]

and

\[ y_M = 0 \]

As explained in the case of vertical curves it can be shown that the asymptote – 0 will intersect the straight section in front of it at a distance \( \frac{1}{2}Z_t \) from the vanishing plane.

**THE LIMIT OF APPROXIMATION OF A CIRCULAR CURVE BY A PARABOLA**

Changing the formula of the picture of the circular curve into that of the parabola, means omitting three factors: \( h^2x^2 \), \( 2axy \) and \( a^2y^2 \). This emission is permitted because \( a, h \) and \( y \) are proportionally small, compared with \( RH' \). Only for large values of \( x \), i.e. in the case of an important change of direction, do these factors become more important.

To which change of direction this substitution should be limited, will appear when comparing the parabola formula

\[ X = \frac{Z^2}{2R_H} \]

with the circular formula

\[ X = R_H - \sqrt{R_H^2 - Z^2}. \]

The latter can be written as

\[ \frac{Z^2}{2R_H} + \frac{2}{1 + \sqrt{1 - \frac{Z^2}{R_H^2}}}, \]
the difference is

\[
\frac{Z^2}{2R_H} \left( 1 - \frac{2}{1 + \sqrt{1 - \frac{Z^2}{R_H^2}}} \right).
\]

If a deviation of 1% of the value of \( x \) is permitted, which will be sufficient in practice for the picture to be criticised, this means that

\[
1 - \frac{2}{1 + \sqrt{1 - \frac{Z^2}{R_H^2}}} = 0.01,
\]

which results in

\[
\frac{Z}{R_H} = \frac{2}{10.1} \approx \frac{1}{5}.
\]

Up to a change of direction with tangent \( \frac{1}{5} \) \( (= 11.30°) \), the parabola-formula may be used in calculating horizontal curves instead of the circle-formula.

To 5.3  *The composite curve*

The composite curve is a combination of a horizontal and a vertical parabolic curve lying in a flat plane. The coordinates of a point in the picture are defined by the formulae:

\[
x = a + \frac{(Z - Z_i)^2}{2R_H},
\]

and

\[
y = h + \frac{(Z - Z_i)^2}{2R_V},
\]

Elimination of \( Z \) from these equations results in the equation of the curve in the picture that represents the composite curve:

\[
- R_H^2 (2hR_V + Z_i^2)x^2 + 2R_H R_V (Z_i^2 + hR_V + aR_H)xy
- R_V^2 (2aR_H + Z_i^2)y^2 - 2Z_i R_H (hR_V - aR_H)x
+ 2Z_i R_V (hR_V - aR_H)y - (hR_V - aR_H)^2 = 0
\]
This curve once more is a hyperbola, since the product of the coefficients of \(x^2\) and \(y^2\) less the square of half the coefficient of \(xy\) is again negative, namely \(-(hR_v-aR_h)^2\).

When \(aR_h = hR_v\), the curve degenerates into a straight line, which means \(\frac{R_v}{R_h} = \frac{a}{h}\).

The transformation-formulae of the coordinates of any point of a given curve in which the horizontal angle between the picture-plane-axis and the straight section in front is \(\alpha\), and the vertical one is \(\beta\), are:

\[
x_1 = \tan \alpha + \frac{(Z - Z_1)^2}{2R_h} (1 + \tan^2 \alpha) \frac{1}{Z}
\]

and

\[
y_1 = \tan \beta + \frac{(Z - Z_1)^2}{2R_v} (1 + \tan^2 \beta) \frac{1}{Z}
\]

Or

\[
x_1 - \tan \alpha = (1 + \tan^2 \alpha)x
\]

and

\[
y_1 - \tan \beta = (1 + \tan^2 \beta)y
\]

Supposing \(\tan \alpha = m\) and \(\tan \beta = n\),

\[
x = \frac{x_1 - m}{1 + m^2}
\]

and

\[
y = \frac{y_1 - n}{1 + n^2}
\]

The general equation of the hyperbola can be written as

\[
a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0
\]

The equation of the general curve is found by substitution of the above calculated solutions for \(x\) and \(y\), resulting in:

\[
a_{11} \frac{x_1^2 - 2x_1m + m^2}{(1 + m^2)^2} + 2a_{12} \frac{x_1y_1 - my_1 - nx_1 + mn}{(1 + m^2)(1 + n^2)} +
\]

\[
+ \ a_{22} \frac{y_1^2 - 2y_1n + n^2}{(1 + n^2)^2} + 2a_{13} \frac{x_1 - m}{1 + m^2} + 2a_{23} \frac{y_1 - n}{1 + n^2} + a_{33} = 0.
\]
The product of the coefficients of \( x_1^2 \) and \( y_1^2 \) diminished by the square of half the coefficient of \( x_1y_1 \) becomes:

\[
\frac{a_{11} \cdot a_{22}}{(1 + m^2)^2 (1 + n^2)^2} - \frac{a_{12}^2}{(1 + m^2)^2 (1 + n^2)^2} = \frac{a_{11} \cdot a_{22} - a_{12}^2}{(1 + m^2)^2 (1 + n^2)^2}
\]

Because \( a_{11} \cdot a_{22} - a_{12}^2 \) was already negative, this picture is again a hyperbola.

If the equation of the composite curve is divided by \( R_v^2 \) and supposing that \( R_v = \infty \), the equation of the horizontal curve is obtained. Equally the equation of the vertical curve can be obtained by dividing by \( R_h^2 \) and supposing \( R_h = \infty \).

The asymptotic directions can be defined from the equations

\[
(2hR_v + Z_1^2)R_hx = (Z_1^2 + 2aR_h)R_vy
\]

and

\[
R_hx = RVy.
\]

The coordinates of the centre appear in the same way as

\[
x_M = -\frac{Z_1}{R_h}
\]

and

\[
y_M = -\frac{Z_1}{R_v}
\]

The equations of the asymptotes can be found from the asymptotic directions through the centre.

It will appear that once more the asymptotes \(-0\) intersect the concerning edge-lines of the straight section in front of the curve at a distance of \( \frac{1}{2}Z_1 \) from the vanishing plane.
So it will also be possible to fix the centre and the asymptotes of the hyperbola representing the picture of a composite curve in a simple way.

To 5.4 Successive elements

Figure 9.5 It can be proved that there exists an intersection in one point of:
1. the asymptote \(-0\) of the hyperbola representing the vertical curve;
2. the asymptote \(-0\) belonging to the curve in horizontal sense;
3. the tangent to both these curves in their transition-point.
It is the point $H_2$ in figure 9.5.

For this the vertical curve $T_1T_2$ is assumed to be extended in the direction of the observer unto the intersection point $B_1$ with the vanishing plane. If a picture-plane is supposed in front of the observer, the tangent of the vertical curve in $B_1$ is the asymptote $-0$ of the hyperbola representing the picture of the vertical curve. This tangent intersects the straight line $A_1T_1$ in $H_1$ at a distance $\frac{1}{2}Z_1$, according to the property of the parabola.

If the tangent $T_2A_2$ is constructed in the end $T_2$ of the vertical curve, this tangent will lie in the same plane as the tangent $B_1H_1$ and so intersect it in the point $H_2$.

Because of the property of the parabola $H_2$ also lies in the middle of $T_2A_2$, so $A_2H_2 = H_2T_2$.

Now the curve in horizontal sense situated behind the vertical curve is also extended unto its intersection point $B_2$ with the vanishing plane. On a picture-plane the tangent in $B_2$ of the curve in horizontal sense is the asymptote $-0$ of the hyperbola, representing this curve.

This tangent also intersects the line $T_2A_2$ in $H_2$, which proves the proposition.

Attention is drawn to the fact that the line $T_2A_2$, being the common tangent of both curves, is also the intersection-line of the planes in which the curves are situated.

To 5.5 The transition curve

Figure 9.6 The clothoid $RL = A^2$ is used in transition-curves. Figure 9.6 shows:

\[ R \, d\tau = dL \]
\[ d\tau = \frac{1}{R} \, dL = \frac{L}{A^2} \, dL \]
\[ \int_0^\tau d\tau = \frac{1}{A^2} \int_0^L L \, dL \]
\[ \tau = \frac{1}{A^2} \cdot \frac{1}{2} L^2 \]
\[ 2A^2 \tau = L^2 \]
\[ \tau = \frac{L^2}{2A^2} = \frac{L^2}{2RL} = \frac{L}{2R} \]

The X-coordinate is approximately

\[ X_{\text{cloth}} \approx \frac{\sqrt{2}}{3} \left[ \frac{L}{2R} \right]^3 = \frac{L^2}{6R} = \frac{L^3}{6A^2} \]

59
Assuming \( L \approx Z \), this equation becomes

\[
X = \frac{Z^3}{6A^2}
\]

This is a cubic parabola, which is more or less similar to a clothoid with parameter \( A \).

The change of direction of its tangents over a distance \( Z \) will be

\[
\frac{dX}{dZ} = \frac{Z^2}{2A^2}
\]

The defining of the coordinates of a point in the picture of a transition-curve which begins at a distance \( Z_1 \) in front of the observer is based on the formulae

\[
x = \frac{a \pm (Z - Z_1)^3}{6A^2}
\]

and

\[
y = \frac{h}{Z}
\]

Elimination of \( Z \) from these equations results in the equation of the curve in the picture that represents the transition-curve to the right:

\[
(6A^2a - Z_1^3)y^3 - 6A^2hxy^2 + 3hZ_1^2y^2 - 3h^2Z_1y + h^3 = 0
\]

This means that the picture is part of a curve of the third degree.

The asymptotes are found by substituting \( y = mx + c \) in this equation, fixing the coefficients of the third and second degree at 0.

Then it appears to be possible to construct the asymptotes without complicated calculations.

They are:

two coinciding lines in the x-axis; \( y = 0 \), and the line connecting the point \( x = \frac{Z_1^2}{2A^2} \), \( y = 0 \) with the point on the edge-line of the road at \( \frac{1}{3}Z_1 \) in front of the observer, so at \( \frac{h}{\frac{1}{3}Z_1} \) from the x-axis.
This will also be clear if it is remembered that the tangent in a point of a cubic parabola will intersect the tangent at its beginning at a distance of \( \frac{1}{3} \) of the considered length.

The point in which the part of the transition-curve, that is extended in the direction of the observer, intersects the vanishing plane is defined by:

\[
X = a - \frac{Z_1^3}{6A^2}
\]

and

\[
Z = 0
\]

The equation of the tangent of the transition-curve in this point is

\[
X - a + \frac{Z_1^3}{6A^2} = \frac{Z_1^2}{2A^2} Z
\]

The intersection-point with the edge-line of the road has \( X = a \), so

\[
\frac{Z_1^3}{6A^2} = \frac{Z_1^2}{2A^2} Z
\]

or

\[
Z = \frac{1}{3}Z_1
\]

**Figure 9.7** The transition-curve, the picture of which is represented by a curve of the third degree, cannot be constructed so easily as is the case with the tangents method in a parabolic curve.

Still, the change of direction of the entire transition-curve can be made use of, which has as its factor of proportionality \( p = \frac{A}{R} \), and is defined by \( \frac{1}{2}p^2 \).

The length of the transition curve is \( p^2R \), while the tangent at the end of the curve intersects the tangent in the vanishing plane at a distance of \( \frac{3}{4}p^2R \) in front of this plane. This point \( H_e \) is connected with the vanishing point of the direction at the end of the curve, \( V_e \), and \( V_0V_e = \frac{1}{2}p^2 \).

The end of the transition-curve is found at \( \frac{1}{3} \) of the connection-line \( H_eV_e \).

A second point of the transition-curve can be constructed by connecting the point \( V_1 \) on \( V_0V_e \), at \( \frac{1}{4} \) of its length, with the point \( H_0 \), so that \( H_1V_0 = 2H_eV_0 \). The required point \( T_1 \) lies on \( H_1V_0 \), with \( T_1H_1 = \frac{1}{3}V_1H_1 \).

Usually no further points nearer to the observer will be required, because of their very short distance.

This method makes the drawing of a transition-curve possible, in order to obtain an impression of its course.
Generally the impression of a transition-curve is only of importance when seen at a short distance. If further away, its length in the picture becomes negligible.

To 6.1.1 The radius of the concave curve

When radii are smaller than 30000 m, there is a danger that a certain distance the beginning of the vertical curve in the picture is situated in the part where the hyperbola has its smallest radius. If a radius of a concave-curve of 10000 m is observed from a distance of 300 m, the top of the hyperbola will coincide with the beginning of the curve.

Figure 9.8 This may be proved as follows:
In figure 9.8 the point T is the beginning of the vertical curve. If this point in the picture is the top of the hyperbola, it is clear that the angle VTM is a right angle and so the angle VMT = VTN = ω.

\[ \tan \omega = \frac{a}{h} = \frac{Z_1 Z_1^2 - h R_v}{Z_1} \]

or

\[ R_v = \frac{h Z_1^2}{h^2 + a^2} \]

Supposing \( h = -1.2\) m, \( a = 3\) m and \( Z_1 = 300\) m, \( R_v \) will be about 10000 m.
When \( R_v = 20000\) m, this relation leads to \( Z_1 = 400\) m, and with \( R_v = 30000\) m, \( Z_1 = 500\) m.

In order to create a gradual introduction to the minimum radius of the hyperbola and to avoid a disturbing kink becoming visible at 300 m, the radius of the concave curve should be at least 30000 m.

Figure 9.9 and 9.10 An almost straight section is a section with such a horizontal radius, that, at a distance 300 m, the picture remains nearly identical to that of a really straight section.
The critical minimum radius is 40000 m.

Smaller radii of horizontal curves cause a turning-point, therefore a true horizontal curve, and because of this in combination with the vertical curve behind it, a bizarre shape.
To 6.1.2  *The turning point in the convex curve*

The formula $Z_k^2 = 2HR_{V_1} - 2hR_{V_2}$ can be obtained as follows: In figure 6.4 the ordinate of $V_1$ is the change of the gradient of the concave curve, i.e. $Z_2$ if $Z_2$ is the length of the concave curve.

Similarly the difference between the ordinates of $V_2$ and $V_k$ is $\frac{Z_k - Z_2}{R_{V_2}}$, while the ordinate of $V_k$, which is its height in the picture at a distance $Z_k$, is

$$h + \frac{Z_2^2}{2R_{V_1}} + \frac{Z_2}{R_{V_1}} (Z_k - Z_2) - \frac{(Z_k - Z_2)^2}{2R_{V_2}}.$$

From the figure it is clear that

$$\frac{Z_2}{R_{V_1}} = \frac{Z_k - Z_2}{R_{V_2}} + \frac{Z_2^2}{2R_{V_1}} + \frac{Z_2}{R_{V_1}} (Z_k - Z_2) - \frac{(Z_k - Z_2)^2}{2R_{V_2}} \quad (1)$$

or

$$\frac{Z_k^2}{2R_{V_2}} = h + \frac{Z_2^2}{2} \cdot \frac{R_{V_1} + R_{V_2}}{R_{V_1}R_{V_2}} \quad (2)$$

Now is:

$$\frac{Z_2^2}{2R_{V_2}} = H \frac{R_{V_1}}{R_{V_1} + R_{V_2}}$$

Substitution in (2) results in the formula

$$Z_k^2 = 2HR_{V_1} - 2hR_{V_2} \quad (3)$$

In a similar way formula (1) can be developed in which the observer is situated at a distance $Z_1$ in front of the beginning of the concave curve.

To 6.2.1  *The picture of the outer curve*

The formula $Z_1^2 = 45R_{H}$ is based, in the case of a curve to the right, where the eye-height of the observer at 5 m from the outer edge is $-1.20$ m, on the height of the outer curve itself of $-1.10$ m, caused by a crossfall of 2% over 5 m. This is true for all curves turning to the right with radii larger than 2000 m (having no superelevation).
In the case of a curve to the left, $a = 3$ and the eye-height will be 1.14 m. This is more unfavourable, also because of the transition to the superelevation.

The picture of the transition-curve to a radius of 2000 m, usually taking a length of little more than 225 m, will appear so short at a distance of 300 m, that it is hardly noticeable.

To 6.2.2 Definition of the turning point

1 THE TURNING POINT WHEN DRIVING IN A CURVE

Figure 9.11 If the picture-plane-axis is fixed in the direction of the tangent in the vanishing plane, the $x$-coordinate of the picture of the inner curve is defined by the equation

$$x = \frac{a + \frac{Z^2}{2RH}}{Z}$$

In a turning point the direction of the road in the picture is perpendicular to the $x$-axis and has its vanishing point on this axis.

The distance between this vanishing point $V_K$ and the vanishing point $V_0$ of the direction of the road in the vanishing plane is $\frac{Z_K}{RH}$, so

$$a + \frac{Z_K^2}{2RH} = \frac{Z_K}{RH}$$

or

$$Z_K = \sqrt{2aRH}$$

The real deviation on the site where the turning point is seen, is:

$$\frac{Z_K^2}{2RH} = \frac{2aRH}{2RH} = a$$

in the picture this is:

$$\frac{a}{Z_K} = \sqrt{\frac{a}{2RH}}$$
If the observer is on the road-axis, observing the outer curve, it will be clear that the intersection-point of the outer curve with the eye-line in the direction of the road is situated in the same cross-section as the turning point of the inner curve.

In this case \( a \) is half the road-width = \( \frac{1}{2}b \), and the distance of the intersection-point of the outer curve with the eye-line appears from the equation

\[
\frac{1}{2}b - \frac{Z_K^2}{2R_{\parallel}} = 0
\]

or

\[
Z_K = \sqrt{bR_{\parallel}}.
\]

The road-width in the picture on the site of the turning point in this case is:

\[
\frac{b}{\sqrt{bR_{\parallel}}} = \sqrt{\frac{b}{R_{\parallel}}}
\]

The distance from the vanishing point \( V_0 \) of the direction of the road in the vanishing plane to the vanishing point \( V_K \) of the direction of the road in the turning point is

\[
\frac{Z_K}{R_{\parallel}} = \sqrt{\frac{bR_{\parallel}}{R_{\parallel}}} = \sqrt{\frac{b}{R_{\parallel}}}
\]

It may be generally concluded that the entire-road-surface can be overlooked up to the turning point of the inner curve. Beyond this turning point only the course of the outer curve can be observed over a distance \( \sqrt{2bR_{\parallel}} \). Its total length appears to be 300 m when \( R_{\parallel} = 2000 \) m.

2  THE TURNING POINT, WHEN THE CURVE BEGINS AT SOME DISTANCE FROM THE OBSERVER

If the distance from the observer to the beginning of the curve is \( Z_I \), while he is at a distance \( a \) from the edge of the road, the distance \( Z_K \) between the observer and the turning point can be defined by the equation:

\[
a + \frac{(Z_K - Z_I)^2}{2R_{\parallel}} = \frac{Z_K - Z_I}{R_{\parallel}}
\]

or

\[(Z_K - Z_I)^2 + 2Z_K(Z_K - Z_I) - 2aR_{\parallel} = 0\]

This means

\[Z_K = \sqrt{Z_I^2 + 2aR_{\parallel}}\]
from the beginning of the curve.

3 THE TURNING POINT IN THE TRANSITION-CURVE

Figure 9.12 Similarly to the fixing of the turning point in parabolic curves, the turning point in the picture of a transition-curve, observed from its beginning, can be found by the equation

\[ a + \frac{Z_k^3}{6A^2} = \frac{Z_k^2}{2A^2} \]

or

\[ Z_k = \frac{3\sqrt{3}aA^2}{6A^2} \]

The deviation on the site of the turning point, when observed from the beginning is

\[ \frac{Z_k^3}{6A^2} = \frac{3aA^2}{6A^2} = \frac{a}{2} \]

in the picture:

\[ a = \frac{a}{2Z_k} \]

Only if the picture of the transition-curve itself shows a turning point when it is observed in front of and at its beginning, it will function as a fluent transition to the circular curve behind it.

The distance of the turning point to the beginning of the curve is maximal at the instant of entering the curve.

If the parameter of the clothoid in relation to the circular curve behind it can be defined as \( A = \rho R_H \), it is clear that the length of the clothoid will be \( L = \rho^2 R_H \).

The distance to the turning point, \( Z_k = \frac{3\sqrt{3}aA^2}{\sqrt{3} \rho^2 R_H^2} \) must be less than \( \rho^2 R_H \), in order to obtain a turning point in the transition-curve. If \( a = 3 \) m, this will mean

\[ \rho > \frac{\sqrt{3}}{4 \sqrt{R_H}} \]
For an \( R_H \) of at least 750 m, \( p \) should be \( > \frac{1}{3} \), which means that the parameter of the clothoid must not be fixed smaller than one third of the radius of the circular curve.

**Figure 9.13** The deviation at the end of the transition-curve, expressed in \( p \) and \( R \), is

\[
X = \frac{Z^3}{6A^2} = \frac{\rho^6 R_H^3}{6 \rho^2 R_H^2} = \frac{1}{6} \rho^4 R_H
\]

It appears that \( X \) depends of the fourth power of \( \rho \). In figure 9.13 it is shown that this deviation has importance only when \( \rho > \frac{1}{3} \) or \( A > \frac{1}{3} R_H \).

**4 THE TURNING POINT IN A TRANSITION-CURVE BEGINNING AT SOME DISTANCE IN FRONT OF THE OBSERVER**

If the distance from the observer to the beginning of the transition-curve is \( Z_1 \) and the observer is \( m \) from the edge of the road, this distance \( Z_K \) to the turning point can be defined from the equation of the transition-curve

\[
a + \frac{(Z_K - Z_1)^3}{6A^2} = \frac{(Z_K - Z_1)^2}{2A^2}
\]

This results in an equation of the third degree:

\[
2(Z_K - Z_1)^3 + 3(Z_K - Z_1)^2 Z_K - 6aA^2 = 0
\]

By means of a graph the turning point can be found when \( a \), \( Z \) and \( A \) are known.

To 7.1.3 *A horizontal curve in front of a horizontally straight approach*

When in driving to a straight approach beyond a horizontal curve, the connecting line of the centres of the hyperbola that picture the vertical curves, lies between the edges of the road, the approach will not be observed from the side.

**Figure 9.14 and 9.15** If there is a turning point and the connecting line of the centres is outside of the road, the "approach" will be observed from the side, while the inner curve shows a grotesque shape.

The limit will appear when the turning point of the horizontal curve lies on the connecting line of the centres. When it is supposed that the "approach" begins to be consciously noticed at a distance of 300 m, it will appear that \( \sqrt{2AR_H} = 300 \) or \( 6R_H = 90000 \) m or \( R_H = 15000 \) m.
To 7.2.4 Composite curves

If a straight or nearly straight section without a transition-curve is followed by a composite curve with a horizontal radius $R_H$ and a vertical radius $R_V$, it will be necessary to avoid a bulge in the picture in all positions, so that $\frac{X}{Y} \geq \frac{b}{h}$.

When $\frac{X}{Y} = \frac{b}{h}$ a straight line appears on the outside, causing a stiff impression. So $\frac{X}{Y} > \frac{b}{h}$ is to be recommended. Now $X = \frac{Z^2}{2R_H}$ and $Y = \frac{Z^2}{2R_V}$ so $\frac{R_V}{R_H} > \frac{b}{h}$ is a condition for obtaining a harmonic picture of the road.

It will be proved that an elevation in a transition-curve should begin at about 0.4 of the length of this curve.

The picture of a curve will show no fault if the absciss of the end of the transition-curve and the ordinate of the elevation in this end are proportional to the road-width and the eye-height. This means, supposing that the equation of the clothoid can be approximated by $X = \frac{Z^3}{6A^2}$ and the length of the transition-curve $L \approx Z = \frac{A^2}{R_H}$, the absciss at the end will be

$$X = \left(\frac{A^2}{R_H}\right)^3 = \frac{A^4}{6A^2} = \frac{A^4}{6R_H^3}.$$

The ordinate of the elevation beginning at a distance $Z_e$ in front of the end of the transition-curve is

$$Y = \frac{Z_e^2}{2R_V} = \frac{Z_e^2}{2b} = \frac{hZ_e^2}{2bR_H}.$$

However $\frac{X}{Y} > \frac{b}{h}$ is a condition.

Calculation results in

$$Z_e = \frac{A^2\sqrt{3}}{3R_H}.$$

The length of the transition-curve is $L = \frac{A^2}{R_H}$, so the elevation should begin at a distance $> L - Z_e$, i.e. at about 0.4 of the length of the transition-curve.
To 7.4.1  Local horizontal shifts

The formulae to solve this problem are founded on the supposition that the shift exists of two curves with different radii but equal length.

The calculation gives

\[ \frac{1}{16R_{H1}} = \frac{1}{16R_H} \pm \frac{u}{L^2}. \]

To avoid a bulge, i.e. a curve in the opposite sense, \( \frac{1}{16R_{H1}} \) should be positive, or

\[ \frac{u}{L^2} < \frac{1}{16R_H} \text{ or } L > 4\sqrt{R_Hu}. \]

The shift of the road-axis over \( u \) in a radius \( R_H \) demands a length of the transition larger than \( 4\sqrt{R_Hu} \).

This value itself should not be used, because it would show a straight line between two curves in the same direction.

To 7.4.4  An elevation beginning in a horizontal curve

Figure 9.16  The horizontal curve begins at a distance \( Z_1 \) from the observer. If the beginning of the elevation is situated at a distance \( Z_1 + Z_H \) from the observer, the vertex \( H_1 \) of the horizontal curve in front of the elevation lies at a distance of \( Z_1 + \frac{1}{2}Z_H \) from the observer.

When the vanishing point of the horizontal curve in front of the elevation is \( V_2 \), the tangent \( H_1V_2 \) will touch the curve at the beginning of the elevation.

In order that the elevation does not cause a S-curve in the picture, it should not rise above the tangent \( H_1V_2 \) in the picture, so

\[ \frac{R_H}{R_V} < \frac{h}{Z_1 + \frac{1}{2}Z_H} \]

\[ \frac{b}{Z_1 + \frac{1}{2}Z_H} + \frac{Z_H}{R_H} \]

which means

\[ R_V > \frac{bR_H + Z_H(Z_1 + \frac{1}{2}Z_H)}{h}. \]

Figure 9.17  The following situation is an example: when approaching an elevation, with a concave vertical curve of \( R_V = 20000 \) and a horizontal curve of \( R_H = 2000 \), unto a distance of 400 m in front of the elevation, the situation of the centres and asymptotes in the picture is represented in figure 9.17.
The asymptote $-\infty$ of the vertical curve lies below the corresponding asymptote $-0$ of this curve, for the inner curve as well as for the outer curve. This means that inner and outer curves will show as concave curves in the picture. On both sides an S-shape will arise.

**Figure 9.18** At 200 m in front of the elevation the asymptote $-\infty$ is situated above the asymptote $-0$. The picture of the inner curve looks like a horizontal curve, no S-shape will appear. The outer curve however still retains its S-shape.

**Figure 9.19** Nearing unto 100 m causes the S-shape in the outer curve to disappear.

**Figure 9.20** The distance at which the S-shape disappears from the inner curve is fixed by the equation

$$
\frac{Z_1}{R_V} - \frac{2h}{2a_R} = \frac{R_H}{R_V} \frac{Z_1}{R_V}
$$

or

$$
Z_1 = \sqrt{2hR_V + 2a_R R_H}.
$$

**Figure 9.21** In the same way the equation for the outer curve will be

$$
\frac{2h}{Z_1} - \frac{Z_1}{R_V} = \frac{R_H}{R_V} \frac{2a_R}{Z_1}
$$

or

$$
Z_1 = \sqrt{2hR_V - 2a_R R_H}.
$$

If the observer is in front of, or at the foot of a horizontally curved elevation, a turning point will appear as a result of the horizontal bend, preceding the turning point of the vertical bend in the convex curve.

**Figure 9.22** Approaching the top-curve, the distance shortens between both turning points. At the moment of coincidence the inner curve will show a seemingly straight line.
This is the case if \( Z_K^2 = 2aR_H \) is substituted in the formula (2) of 6.1.2. This gives

\[
Z_2^2 = \frac{2R_{V_1}(aR_H + hR_{V_2})}{R_{V_1} + R_{V_2}}.
\]

*Figure 9.23*  This picture will arise at one point only.

This is not the case when having passed the transition point between the foot and the top curve. In that case a seemingly straight line will arise that is observed for some time, depending on the length of the top curve, and defined by \( Z_K = \sqrt{2aR_H} = \sqrt{2hR_V} \).

*Figure 9.24*  It may also occur that the composite curve at the top shows two turning points, while the composite curve at the end has a third turning point. In this case an (invisible) loop will arise in the inner curve.

This case only appears when large radii for the horizontal curve or small radii for the convex curve at the top are used.

In moving up and down in a horizontal curve the centres of the hyperboles \( M_1 \) and \( M_2 \) will begin to move towards each other.

At the moment of coincidence the seemingly straight line in the inner curve will arise in the picture.

As the advance proceeds, the centres will pass each other, and the (invisible) loop will arise, while the concave curve at the end of the approach becomes visible.

Since the radius at the top is generally smaller than the radius at the foot, it often happens that, in the case of radii for the horizontal curve between 6000 and 15 000 m, the horizontal curve will dominate at the foot, while at the top a seemingly straight line or a vertical curve will be seen.

Considering this, and in relation to 6.1.2, 7.1.3 and 7.1.4, it may be established that:

1. In horizontal curves with \( R > 15 000 \) m, elevations are not to be recommended. Where they do occur, a straight section is preferable for the elevation.
2. If \( R_H < 6000 \) m, an elevation is permissible. Should this start at a 100 m or more past the beginning of the horizontal curve, the radius of the concave curve should be at least 45 000 m.
3. If horizontal curves with \( R_H \) between 6000 m and 15 000 m are used, either the radius at the top ought to be chosen larger than 15 000 m or the horizontal change of direction ought to be divided in different curves, in which the elevation is situated in a curve with \( R_H < 6000 \) m.

*It may now be clear that, in order to obtain a harmonious threedimensional design, use must be made of mathematical analysis of the perspective picture. This requires simultaneous consideration of horizontal and vertical curves.*
To 8.3 Models

Models in which carriage-ways made of cardboard or plywood are supported by pins standing in plastic foam or by little plastic pillars, are suitable on a scale of 1 : 200 or more for a rough appraisal of the three-dimensional design, the visibility of lighting columns or signposts etc. For a reliable representation of alignment-details the scale applied for this construction cannot be less than 1 : 100. On that scale a complete intersection would become unwieldy, and its centre unattainable for corrections, so studies can be made of parts only. Besides this, models of this type are easily damaged by shocks.

Figure 9.25 For models which up to a scale of 1 : 250 have an accuracy that makes them similar to reality with respect to the superelevation and the length of vision, a way of building as seen in figure 9.25 has, after many experiments, proved to be satisfactory.

On hardwood girders, supported by 3-mm diameter steel bolts, carriage-ways are glued that have been sawed out of waterproof hardboard. For board of 8 mm the distance between the girders may be about 16 cm in more or less straight sections, rising to 20 cm if there is a superelevation. For board of 6 mm these distances are 12 and 15 cm respectively. In curves with a radius smaller than about 100 m no thicker board than 6 mm can be used for a scale of 1 : 250 because of the tension caused by the superelevation.

Models of this type can stand transportation, and one can walk on them, as may be necessary for corrections or photography. Bolts thinner than 3 mm are not strong enough for this use, thicker ones do not allow the bending which is unavoidable in setting the exact height. The height is measured from the table-top with a vernier gauge to an accuracy of 1/10 of a mm, and fixed with brass nuts, steel nuts not being available in a good quality.

The tables are built of 2.5 mm plywood, strengthened with deals and mounted horizontally with the help of wedges on light, but stout supports which preferably are screwed to the floor. If required, abutments of bridges and earthworks are partly built of wood and modelled in detail with wood-granite.

Figure 9.26 Elaborating the model into further details with buildings, planting etc. may be useful, but gives cause to a rise of cost. In practice, however, it appears time and again that these models are worth the time and money spent on them, by their showing all sorts of defects in all stages of the design. On a smaller scale, it is difficult to obtain the same reliability with this building system. These smaller models are, however, very efficient for general information.
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