TECHNOLOGICAL UNIVERSITY DELFT

DEPARTMENT OF AERONAUTICAL ENGINEERING

Report VTH - 118

**PHASEPLANE REPRESENTATION OF THE INCOMPRESSIBLE
VISCOUS FLOW BETWEEN NON-PARALLEL PLANE WALLS**

by

J. L. van Ingen

Delft - the Netherlands

September, 1964

Phaseplane representation of the incompressible
viscous flow between non-parallel plane walls.

J.L. van Ingen

Department of Aeronautical Engineering
Technological University
Delft

Summary

In this paper the flow of an incompressible viscous fluid between non-parallel plane walls, usually called after Jeffery and Hamel is considered. The phase plane concept, known from the theory of non-linear oscillations, is employed to clarify the flow problem.

In the theory of oscillations for a system with one degree of freedom the phase plane is defined as the plane in which speed is plotted versus displacement. In this plane the equation of motion reduces to a first order differential equation. The singular points of the equation correspond to equilibrium positions. The type of the singularities determines the form of stability or instability of the motion near the equilibrium positions.

In the present paper the phase plane is defined as the plane in which shear is plotted vs. speed. Singular points are shown to correspond to the edge of a boundary layer. Boundary layer type velocity profiles - with practically constant speed at large distances from the wall - can be obtained only when saddle points or stable nodes occur.

The treatment includes the effect of suction at one wall and blowing at the other.

Notation

a,b,c,d	coefficients in eq. (2.2)
f	nondimensional streamfunction defined by eq. (3.5)
$F = \frac{ur}{\nu}$	
H,K	integration constants in chapter 4
p	static pressure
r	coordinate defined in fig. 11
R	Reynoldsnumber defined by eq. (4.14)
t	time
u_1	constant in eq. (3.3)
u	velocity component parallel to wall in chapter 3 (fig. 2) also: radial velocity component in chapter 4 (fig. 11)
v	velocity component normal to wall in chapter 3 (fig. 2) also: velocity component in θ -direction in chapter 4 (fig. 11)
v_o	normal velocity at wall
U	velocity at the edge of the boundary layer
$\bar{u} = \frac{u}{U}$	
x,y	coordinates defined in fig. 2 also: independent and dependent variable in chapter 2
2α	angle between walls (fig. 11)
γ	parameter in chapter 4 related to the suction velocity
λ	parameter in chapter 3 related to the suction velocity
θ	angular distance from centerline of channel (fig. 11)
$\eta = \frac{y}{x} \sqrt{\frac{ u_1 }{\nu}}$	
$\xi = \frac{dF}{d\theta}$	
$\bar{\tau} = \frac{d\bar{u}}{d\eta}$	
ν	coefficient of kinematic viscosity
ρ	density
ψ	streamfunction
$\gamma_1, \theta_1, F_1, \xi_1$	reduced variables defined in table 3.

1. Introduction and outline of paper.

It is well known that the behaviour of steady viscous flows around bodies depends strongly upon the streamwise pressure gradient and the normal velocity at the walls. An important example of this is given by the viscous flow between non-parallel plane walls. The theory of this flow has been given already in 1915 and 1916 independently by Jefferyy [1]¹⁾ and Hamel [2] (see also [3] and [4]. For the case of inflow between converging walls a boundary layer type solution can occur. This means that at large distances from the walls the radial velocity becomes practically constant. In this case the boundary layer equations can be solved in a closed form given by Pöhlhausen in 1921 [5].

For outflow between diverging walls no boundary layer type solution can be found unless a sufficient amount of suction at the wall is applied. This fact has been known for several decades from studies of the asymptotic behaviour of the solutions of the boundary layer equations (Mangler [6,7]).

The non-linear terms in the equations are responsible for this change in character which occurs when the direction of flow is reversed. This point stands out clearly when the equations are studied in a "phase plane" where shear is plotted versus radial velocity.

Such a procedure is analogous to the use of the phase plane in the theory of oscillations of (non-linear) autonomous systems with one degree of freedom where speed is plotted vs. displacement. These oscillations are governed by a second order ordinary differential equation - in the variables displacement and time - which only implicitly depends on the time. Selection of displacement and speed as new variables reduces this equation to one of the first order.

Singular points of this differential equation correspond to points of equilibrium of the motion. The type of the singularities determines the character of the stability or instability of the motion near the equilibrium positions.

The general theory of this equation was given by Poincaré [8] . Except in some special cases, this theory determines the type of the singularities of the equation without actually solving it. In section 2 of the paper the main results of Poincaré's theory will be collected for later use.

Section 3 shows that the equation, governing the boundary layer flow through the converging or diverging channel between non-parallel plane walls, is of the same type as the equation of motion of an autonomous system with one degree of freedom. Hence the phase plane concept can also be used in the flow problem. It will be shown that singular points correspond now to the edge of the boundary

---

¹⁾ Numbers between square brackets [ ] denote references listed at the end of the paper

layer. Boundary layer type solutions - for which the radial velocity component becomes practically constant at large distances from the wall - are only possible when the singular point is a saddle point or a stable node. For the case of outflow between diverging walls no boundary layer type solution occurs unless a sufficient amount of suction is applied at the walls. For inflow there is always a boundary layer type solution irrespective of the amount of suction or blowing.

It might be argued that the strange behaviour of the solutions of the boundary layer equations is caused by the approximate nature of these equations. In section 4 of this paper it is shown however that the same behaviour is exhibited by the solutions of the Navier-Stokes equations. It is even possible in some cases to transform the Navier-Stokes equations into the boundary layer equations by introducing suitable new variables. If the Reynoldsnumber becomes very large also the boundary conditions of the Navier-Stokes equations tend to those of the boundary layer problem.

The only other phase-plane representation of viscous flow, known to the author, is that given by Ku [9]. He discusses the boundary layer flow for the flat plate and the plane stagnation point. These flows are described by the Falkner-Skan boundary layer equation which is of the third order; hence a phase space instead of a phase plane is needed.

Because a general theory for the singularities of a third order equation is much more complicated than for the equation of second order, Ku was unable to establish a relation between the types of the singularities and the behaviour of the solutions of the boundary layer equations.

2. General theory of the differential equation $\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$.

In this section some results of the classical theory of the differential equation

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)} \quad (2.1)$$

will be collected. A more detailed treatment can be found in the books by Minorsky [10] and Stoker [11]

Points where both $P(x,y)$ and $Q(x,y)$ become zero are called singular points of (2.1); all other points are called regular. It is assumed in what follows that the origin of a Cartesian coordinate system coincides with the singular point under investigation. Then equation (2.1) may be written in the form

$$\frac{dy}{dx} = \frac{ax + by + P_1(x,y)}{cx + dy + Q_1(x,y)} \quad (2.2)$$

where P_1 and Q_1 do not contain constant terms or terms linear in x and y . Poincaré has shown [8] that the only types of singularities occurring in (2.2) are those of the much simpler equation

$$\frac{dy}{dx} = \frac{ax + by}{cx + dy} \quad (2.3)$$

provided $ad - bc \neq 0$ (2.4)

and P_1 and Q_1 vanish like $x^2 + y^2$ when x and y tend to zero. Therefore (2.3) will be considered first.

Some special cases of (2.3), obtained by assigning suitable values to a, b, c , and d , are reviewed in fig. 1. For these cases the equation can be integrated easily in closed form giving the solutions indicated in the figure. It can be shown (see for instance Stoker [11]) that for arbitrary values of a, b, c and d which comply with (2.4) equation (2.3) can be reduced to one of these special cases by applying a suitable linear transformation. Hence the only singularities occurring in (2.3) are those of fig. 1 provided $ad - bc \neq 0$. (Of course in the general case the curves can be referred to oblique axes.) This means that the type of the singularity of (2.3) in the origin is determined by the values of a, b, c and d .

Table 1, taken from Stoker [11], allows the determination of the type of the singularity. If both x and y are thought to depend on a variable t - for instance time - equation (2.3) can be replaced by the system:

$$\begin{aligned} \frac{dy}{dt} &= ax + by \\ \frac{dx}{dt} &= cx + dy \end{aligned} \quad (2.5)$$

It becomes possible then to introduce the terms stability and instability. A node or spiral is called stable if for increasing t all the trajectories through the singular point are traversed in the direction of this point. Instability occurs if the direction is reversed. The stability is determined by the sign of

(b+c), according to the scheme shown in table 1.

It has been remarked already that the types of singularities for (2.2) and (2.3) are the same as long as $ad - bc \neq 0$ and for so far P_1 and Q_1 do not contain terms of lower degree than the second. Hence table 1 is valid for both (2.2) and (2.3). The only exception is, that for $D < 0$ it can not be ascertained from the value of $(b + c)$ whether a singularity of (2,2) is a center or a spiral. In this case the terms of higher order have to be considered also. In general symmetry considerations will help in determining the type of the singularity.

3. The boundary layer approximation for the flow between non-parallel plane walls.

3.1. General.

For the calculation of steady two-dimensional laminar boundary layers along flat or slightly curved walls use is made of the boundary layer equation

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \gamma \frac{\partial^2 u}{\partial y^2} \quad (3.1)$$

and the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3.2)$$

(see for instance Schlichting [12], chapter 7). Here x and y are the coordinates along and normal to the wall respectively; u and v are the velocity components in these directions (fig. 2) and γ the coefficient of kinematic viscosity. The velocity U at the edge of the boundary layer only depends on x . In boundary layer calculations generally U is assumed to be equal to the velocity component tangential to the body which would occur for inviscid flow. For the flow between non-parallel plane walls this leads to

$$U = \frac{u_1}{x} \quad (3.3)$$

For $u_1 > 0$ the expression (3.3) represents outflow from a line source at the origin; $u_1 < 0$ corresponds to flow into a line-sink. Hence (3.3) together with (3.1) and (3.2) describes the boundary layer flow due to the flow between non-parallel plane walls. In what follows both outflow between diverging walls

and inflow between converging walls will be considered. These two cases will be denoted by "outflow" and "inflow" respectively.

As no specific length occurs in the problem it may be surmised that a similarity solution will be possible. To show that the equations are compatible with a similarity assumption the non-dimensional wall-distance η is introduced with

$$\eta = \frac{y}{x} \sqrt{\frac{|u_1|}{\nu}} \quad (3.4)$$

and the non-dimensional streamfunction $f(\eta)$ by

$$\Psi = \sqrt{\nu |u_1|} f(\eta) - \lambda \sqrt{\nu |u_1|} \ln x \quad (3.5)$$

Here Ψ is the dimensional streamfunction and λ is a constant still to be specified. On straight lines through the origin clearly η is constant. The velocity components u and v are determined from the streamfunction ψ by

$$u = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = - \frac{\partial \Psi}{\partial x} \quad (3.6)$$

so that equation (3.2) is satisfied. Further it follows from (3.4), (3.5) and (3.6) that

$$u = \frac{|u_1|}{x} f'(\eta) = |U| f'(\eta) \quad \text{or} \quad \bar{u} = \frac{u}{|U|} = f'(\eta) \quad (3.7)$$

$$v = \sqrt{\nu |u_1|} \left(\frac{\eta}{x} f'(\eta) + \frac{\lambda}{x} \sqrt{\nu |u_1|} \right) \quad (3.8)$$

where primes denote differentiation with respect to η .

Introducing (3.7) and (3.8) into (3.1) gives

$$f'''' - \lambda f'' + (f')^2 - 1 = 0 \quad (3.9)$$

Substitution of $f' = \bar{u}$ in this non-linear differential equation reduces the order by one giving

$$\bar{u}'' - \lambda \bar{u}' + \bar{u}^2 - 1 = 0 \quad (3.10)$$

Solutions of (3.10) should fulfill the no-slip condition at the wall:

$$\bar{u} = 0 \quad \text{for} \quad \eta = 0 \quad (3.11)$$

Moreover to obtain boundary layer type solutions $|\bar{u}|$ should tend to 1 at large distances from the wall. This leads to the following boundary conditions for $\eta \rightarrow \infty$:

$$\left. \begin{aligned} \bar{u} &\longrightarrow +1 \quad \text{for outflow } (u_1 > 0) \\ \bar{u} &\longrightarrow -1 \quad \text{for inflow } (u_1 < 0) \\ \bar{u}' &\longrightarrow 0 \quad \text{both for outflow and inflow} \end{aligned} \right\} \quad (3.12)$$

From (3.8) the normal velocity at the wall ($\eta = 0$) is found to be

$$v_o = \frac{\lambda}{x} \sqrt{\nu |u_1|} \quad (3.13)$$

Hence to obtain similar solutions v_o should be inversely proportional to x . For $\lambda > 0$ blowing occurs and suction for $\lambda < 0$. For a non-porous wall λ is equal to zero.

3.2. The non-porous wall ($\lambda = 0$).

3.2.1. The classical approach.

When $\lambda = 0$ equation (3.10) can be integrated once after multiplication by $2\bar{u}'$. The result is

$$(\bar{u}')^2 + \frac{2}{3} \bar{u}^3 - 2\bar{u} = A \quad (3.14)$$

where A is an integration constant.

For inflow the boundary conditions (3.12) require that $A = +4/3$; condition (3.11) then leads to $(\bar{u}')^2 = 4/3$ for $\eta = 0$. Hence the non-dimensional shear stress at the wall is $\bar{u}'(0) = \frac{-2}{\sqrt{3}}$. The negative sign for the square root is taken because for inflow the velocity and shear stress at the wall are both negative (fig. 2).

For outflow the conditions (3.12) at the edge of the boundary layer require $A = -4/3$. A real value of the skin friction at the wall however requires a non-negative value of A according to (3.14). Hence no boundary layer type solution is possible for outflow between non-porous plane walls.

The solution of (3.14) for the case of inflow has been given by Pohlhausen in 1921 [5]. (See also [12], chapter 9). Using $A = +4/3$ equation (3.14) leads to

$$\frac{d\bar{u}}{d\eta} = \bar{u}' = - \sqrt{\frac{4}{3} + 2\bar{u} - \frac{2}{3}(\bar{u})^3} \quad (3.15)$$

The negative sign for the square root in (3.15) is taken because both $\bar{u}$ and $\bar{u}'$ are negative for inflow in the present coordinate system. From (3.15) it follows that

$$\eta = - \int_0^{\bar{u}} \frac{d\bar{u}}{\sqrt{\frac{4}{3} + 2\bar{u} - \frac{2}{3}\bar{u}^3}} \quad (3.16)$$

Upon integration of (3.16) the non-dimensional velocity profile is found to be

$$-\bar{u} = 3 \operatorname{tgh}^2 \left\{ \frac{\eta}{\sqrt{2}} + 1,146 \right\} - 2 \quad (3.17)$$

A graph of $\bar{u}$ versus η is given in fig. 3.

It has been stated already that no corresponding solution for outflow can be found. This can be clarified further by studying equation (3.14) in the phase plane.

3.2.2. Phase plane description.

By introducing

$$\bar{\tau} = \frac{d\bar{u}}{d\eta} \quad (3.18)$$

it follows that equation (3.10), for $\lambda = 0$, may be written in the form

$$\frac{d\bar{\tau}}{d\eta} = 1 - \bar{u}^2 \quad (3.19)$$

Elimination of η between (3.18) and (3.19) gives the following first order equation

$$\frac{d\bar{\tau}}{d\bar{u}} = \frac{1-\bar{u}^2}{\bar{\tau}} \quad (3.20)$$

This equation is of the form (2.1) and hence the general theory described in section 2 can be applied.

Singular points, where both the nominator and denominator of the right hand side of (3.20) become zero, are $\bar{\tau} = 0$, $\bar{u} = \pm 1$ and hence correspond to the edge of the boundary layer for inflow and outflow respectively. Shifting the origin in turn to each of these singular points and applying the criteria of table 1 it is easily found that $(-1,0)$ is a saddle point and $(+1,0)$ a center or a spiral point. The nominator of (3.20) contains non-linear terms and hence no distinction can be made between a spiral or a center. From the symmetry of the equation however it is easily inferred that $(+1,0)$ is a center. This follows also from the phase plane portrait given in fig. 4.

The trajectories in the phase plane are given by the first integral of (3.20)

$$\frac{-2}{\bar{\tau}} = 2\bar{u} - \frac{2}{3}\bar{u}^3 + A \quad (3.21)$$

where A denotes an integration constant. Arrows in fig. 4 indicate the direction in which the trajectories are traversed with increasing distance from the wall η . This direction follows from (3.18) which shows that η increases with $\bar{u}$ when $\bar{\tau}$ is positive. The curves in fig. 4 were drawn by a plotting machine coupled to one of the electronic analog computers of the Department for Aeronautical Engineering of Delft Technological University. On the scale of fig. 4 no difference can be noticed between the results of the analog computation and a direct computation using (3.21).

To obtain a solution which satisfies the boundary conditions, a trajectory has to be chosen which connects some point on the $\bar{\tau}$ - axis (representing the wall) with the singularity representing the edge of the boundary layer. It is easily seen from fig. 4 that only for inflow there is a possible solution, represented by the trajectory PS.

Of course this leads again to Pohlhausen's solution. The velocity profile follows from

$$\eta = \int_0^{\bar{u}} \frac{d\bar{u}}{\bar{\tau}} \quad (3.22)$$

The analog computer can easily be arranged to produce η as a function of $\bar{u}$. The velocity profile thus obtained is shown in fig. 5 together with the

velocity profiles corresponding to some adjacent trajectories in fig. 4. This clearly shows that the Pohlhausen solution is a unique one. There is no trajectory connecting a point on the $\bar{\tau}$ - axis with the isolated singular point (+1,0); hence for the case of outflow no boundary layer type solution can be found. This confirms the conclusion reached in section 3.2.1. It will be shown however in the next section that for outflow boundary layer type solutions may be obtained if a sufficient amount of suction is applied at the wall.

3.3. The effects of suction and blowing ($\lambda \neq 0$).

For $\lambda \neq 0$ the walls of the channel should be porous giving a normal velocity distribution in accordance with (3.13). This means that at one wall there is suction with a normal velocity varying as x^{-1} . At the other wall there is blowing with the same distribution of the normal velocity. For $\lambda > 0$ blowing at the lower wall is obtained and suction at the upper wall. In what follows only the lower wall will be considered; results for the upper wall follow immediately from those for the lower wall by reversing the sign of λ and η .

A closed form solution of equation (3.10) cannot be found for $\lambda \neq 0$. According to Mangler [7] numerical solutions have been obtained by Holstein [13] for $\lambda = 0, +2, +4$ and $+10^{(1)}$. Detailed numerical results will not be given in the present paper. Instead of this (3.10) will be studied in the phase plane giving better insight into the structure of the equation.

Introducing $\bar{\tau}$ by (3.18) and eliminating η between (3.10) and (3.18) the following equation is obtained

$$\frac{d\bar{\tau}}{d\bar{u}} = \frac{\lambda\bar{\tau} + 1 - \bar{u}^2}{\bar{\tau}} \quad (3.23)$$

This equation is again of the type (2.1) and can therefore be studied by applying the theorie of section 2. The singular points are $\bar{\tau} = 0, \bar{u} = \pm 1$ for all values of λ and, as in section 3.2.2., correspond to the edge of the boundary layer for outflow and inflow respectively. From table 1 it is easily found that (-1,0) is always a saddle point irrespective of the value of λ . The type of the singularity at (+1,0) depends on λ according to table 2.

---

¹⁾As [13] was not available to the present author this work could not be consulted.

According to section 2 no distinction can be made between a center or a spiral for $-2\sqrt{2} < \lambda < +2\sqrt{2}$. From section 3.2.2. it is known already that for $\lambda = 0$ a center is obtained. For $\lambda \neq 0$ a spiral should occur because a center would conflict with the property that $\lambda \bar{\tau}$ is odd in $\bar{\tau}$. It also follows from the phase plane portraits which were obtained on the analog computer. They are shown in fig. 6 for several values of λ .

The singularity at $(-1,0)$ corresponding to the edge of the boundary layer for the case of inflow remains a saddle for all values of λ . Hence a unique boundary layer type solution is obtained irrespective of the amount of suction or blowing at the wall (compare section 3.2.2. for $\lambda = 0$). These solutions correspond to the trajectories connecting the $-\bar{\tau}$ - axis with $(-1,0)$. The velocity profiles are shown in fig. 7 for different values of λ .

It may be shown that the trajectories through the saddle can be represented by a strongly convergent power series in $(\bar{u} + 1)$. Hence a good approximation of the flow can be obtained assuming a polynomial expression for $\bar{\tau}(\bar{u})$.

A boundary layer type solution for outflow would require a trajectory in the phase plane connecting some point on the $+\bar{\tau}$ - axis with the singularity $(+1,0)$. For $\lambda > 0$ (blowing) all trajectories lead away from the singular point. Hence no boundary layer type solution is possible for outflow with blowing at the wall. It was shown already in section 3.2.2. that also for $\lambda = 0$ no solution can be obtained.

If $\lambda < 0$ there is an infinite number of trajectories between the $+\bar{\tau}$ - axis and $(+1,0)$. For $-2\sqrt{2} < \lambda < 0$ the singularity at $(+1,0)$ is a spiral leading to velocity profiles of the type shown in fig. 8. Here $\bar{u}$ approaches 1 in an oscillatory manner and not asymptotically when $\eta \rightarrow \infty$. This type of velocity profile conflicts with our physical picture of a boundary layer and therefore has to be rejected.

For relatively strong suction ($\lambda \leq -2\sqrt{2}$) a stable node is obtained (fig. 6). Again there is an infinite number of trajectories connecting the $+\bar{\tau}$ - axis with $(+1,0)$. Part of these pass through the region $\bar{u} > 1$ and hence produce velocity profiles with an "overshoot" for which $\bar{u}$ approaches 1 asymptotically but from above. Again this type of profile is rejected as physically unacceptable. There remains however an infinity of trajectories approaching the singularity from the left. They all lead to velocity profiles without an overshoot and hence would be physically acceptable. (See fig. 6 for $\lambda = -4$ for instance). It is not clear on physical grounds which of the profiles - if any - should be chosen as the required solution.

A similar problem was encountered by Hartree [14] in his numerical study of the boundary layer flows connected with the potential flow $U = u_1 x^m$. For $m \geq 0$ he found numerically a unique solution, much the same as for inflow in the present problem. For $m < 0$ an infinite number of solutions was possible, part of them showing a velocity profile with an overshoot. In this case Hartree selected the solution approaching 1 as fast as possible without making an overshoot. This choice made the skin friction at the wall as a function of m continuous at $m = 0$.

The distance from the wall in the present description is given by (3.22) indicating that $\bar{u}$ tends to 1 as fast as possible when $\bar{\tau}$ is as high as possible. Following Hartree's choice in the present case it follows that from the trajectories with $\bar{u} \leq 1$ the one with the highest shear has to be selected. The displacement thickness δ^* of the boundary layer is given by

$$\delta^* = \int_0^{\infty} (1 - \bar{u}) dy \quad (3.24)$$

or in non-dimensional form

$$\frac{\delta^*}{x} \sqrt{\frac{|u_1|}{y}} = \int_0^{\infty} (1 - \bar{u}) dy = \int_0^1 \frac{(1 - \bar{u})}{\bar{\tau}} d\bar{u} \quad (3.25)$$

Similarly the momentum thickness Θ is given by

$$\frac{\Theta}{x} \sqrt{\frac{|u_1|}{y}} = \int_0^1 \frac{\bar{u}(1 - \bar{u})}{\bar{\tau}} d\bar{u} \quad (3.26)$$

The local viscous dissipation is given, in non-dimensional form, by

$$2 \int_0^{\infty} \left(\frac{d\bar{u}}{d\eta} \right)^2 d\eta = 2 \int_0^1 \bar{\tau} d\bar{u} \quad (3.27)$$

It follows then that the selected boundary layer solution is distinguished from all the other solutions without overshoot, by the properties that it has the smallest thickness and the highest local viscous dissipation.

Some further arguments in favour of Hartree's choice can be produced for the present problem.

If it is required that $\bar{u}$ approaches 1 exponentially it should be possible to develop $\bar{\tau}$ in a power series in $(\bar{u} - 1)$, starting with a term of the first degree. This is possible for both main branches of the node. The other trajectories through the node can only be represented by a non-analytical series. The power series for the steeper main branch is useful for all values of $\lambda \leq -2\sqrt{2}$. For the other main branch some of the coefficients in the series may become infinite at some values of λ . Hence if $\bar{u}$ tends to 1 exponentially with η for all values of λ only the steeper main branch of the node can be used. This agrees with Hartree's choice.

A further argument results from an inspection of fig. 6. This shows that for decreasing λ (stronger suction) the steeper main branch moves into a region of higher shear and hence produces a thinner boundary layer. On the other hand for the second main branch the shear becomes smaller and hence the boundary layer thickness increases when the intensity of suction is increased. This conflicts with our physical ideas about the effect of suction on boundary layer flows.

In view of these arguments it is then finally decided to select the steeper main branch of the node as the required boundary layer type solution.

The velocity profiles obtained in this way for different values of λ are shown in fig. 7 together with those for inflow. When $\lambda \rightarrow -\infty$ the boundary layer thickness tends to zero and $|\bar{\tau}|$ at the wall $\rightarrow \infty$ both for inflow and outflow. Therefore it is useful to introduce new variables η_1 and $\bar{\tau}_1$ according to

$$\eta_1 = -\lambda \eta \quad \text{and} \quad \bar{\tau}_1 = \frac{\bar{\tau}}{-\lambda} = \frac{d\bar{u}}{d\eta_1} \quad (3.28)$$

Substituting this into (3.23) gives

$$\frac{d\bar{\tau}_1}{d\bar{u}} = \frac{-\bar{\tau}_1 + \frac{1-\bar{u}^2}{\lambda^2}}{\bar{\tau}_1} \quad (3.29)$$

For $\lambda \rightarrow -\infty$ this leads to

$$\frac{d\bar{\tau}_1}{d\bar{u}} = -1 \quad (3.30)$$

explaining the shape of the trajectories for large values of $-\lambda$. Integrating equation (3.30) twice and using the boundary conditions $\bar{u} = 0$ for $\eta_1 = 0$ and $\bar{u}' = 0$ for $|\bar{u}| = 1$ the following result is obtained.

$$|\bar{u}| = 1 - e^{-\eta_1} = 1 - e^{\lambda\eta} \quad (3.31)$$

With (3.4) and (3.13) this leads to

$$|\bar{u}| = 1 - e^{-\frac{v_o y}{\nu}} \quad (3.32)$$

This is the well known asymptotic suction velocity profile which is found far from the leading edge of a flat plate with constant suction at the wall (Iglisch [15]). Hence both for inflow and outflow the velocity profile tends to the asymptotic suction profile when $\lambda \rightarrow -\infty$. This result has been obtained already by Pretsch [16] (See Mangler [7]).

It might be supposed that the apparent anomalous behaviour of the solutions of the boundary layer equations for outflow is due to the boundary layer approximation itself. It will be shown in the next section however that the same behaviour is displayed by the solutions of the Navier-Stokes equations. Therefore it is more likely that the behaviour is dictated by the fact that for outflow the fluid has passed through the singularity in the physical plane at $x = 0$.

4. Description of the flow between non-parallel plane walls using the Navier-Stokes equations.

4.1. General.

In this section the viscous flow between non-parallel plane walls will be studied starting from the Navier-Stokes equations. This problem has been studied already in 1915 and 1916 independently by Jeffery [1] and Hamel [2]. (See also [3] and [4]). A related problem concerning the flow in symmetrical channels with slightly curved walls has been studied by Fraenkel [19].

The geometry of the problem suggests the use of plane polar coordinates r and θ with the walls denoted by $\theta = \pm \alpha$ (fig. 11). The velocity components in the r and θ directions will be denoted by u and v respectively. The Navier-Stokes equations and the continuity equation in these coordinates read (see [12], chapter 3):

$$u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right) \quad (4.1)$$

$$u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{uv}{r} = - \frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u}{\partial \theta} \right) \quad (4.2)$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} = 0 \quad (4.3)$$

In the classical approach to this problem solutions of the equations are sought in which the flow is purely radial (so that no suction or blowing occurs) and for which moreover the velocity profiles are similar for all values of r . To satisfy these requirements the expressions for the velocity components have to assume the form

$$u = G(r) F(\theta) \quad (4.4)$$

$$v = 0 \quad (4.5)$$

where F and G are functions of θ and r alone. From the continuity equation it follows that $G(r)$ should be inversely proportional to r . Hence (4.4) can be written as

$$u = \frac{\nu}{r} F(\theta) \quad (4.6)$$

where the coefficient of kinematic viscosity ν is added to make $F(\theta)$ non-dimensional.

In what follows a more general problem will be treated for which the flow is no longer required to be purely radial so that suction or blowing may occur.

The assumption of flow similarity is retained however. If the expression (4.6) for u is retained also it follows from continuity arguments that v should be independent of θ . It can easily be shown that the Navier-Stokes equations become independent of r and hence that similar solutions may be obtained if v is inversely proportional to r . Therefore solutions can be expected of the following form

$$u = \frac{v}{r} F(\theta) \quad (4.6)$$

$$v = \frac{\gamma}{r} \quad (4.7)$$

where γ is a constant determining the normal velocity at the walls. For $\gamma = 0$ the classical results should be obtained while for $\gamma > 0$ suction is applied at the upper wall and blowing at the lower wall (fig. 11). The normal velocity is inversely proportional to r in the same way as for the boundary layer problem of section 3.

By choosing u and v according to (4.6) and (4.7) the continuity equation is satisfied already. Substitution of (4.6) and (4.7) into (4.1) and (4.2) leads to

$$-F''' + \gamma F'' - F^2 - \gamma^2 = \frac{-r^3}{\rho \gamma^2} \frac{\partial p}{\partial r} \quad (4.8)$$

$$0 = \frac{-r^2}{\rho \gamma^2} \frac{\partial p}{\partial \theta} + 2 F \quad (4.9)$$

where primes denote differentiation with respect to θ . The pressure p can be eliminated from (4.8) and (4.9) by differentiating these equations with respect to θ and r respectively and subtracting the results. This leads to:

$$F'''' - \gamma F''' + 4F'' + 2FF' = 0 \quad (4.10)$$

and after a single integration to:

$$F''' - \gamma F'' + 4F + F^2 + K = 0 \quad (4.11)$$

where K is an integration constant. The no-slip condition requires that at the walls $u = 0$, and the boundary conditions for (4.11) then become:

$$F = 0 \quad \text{for} \quad \theta = \pm \alpha \quad (4.12)$$

From (4.8) and (4.11) it follows that

$$K = \frac{-r^3}{2\rho\nu} \frac{\partial p}{\partial r} - 4F + \gamma^2 \quad (4.13)$$

From (4.13) it is seen that K is related to the radial pressure gradient and the non-dimensional suction velocity γ . In order to determine K there should be another boundary condition besides (4.12). For instance the total flux through the channel or the velocity at the centerline ($\theta = 0$) may be prescribed. The last possibility will be used here. It can be seen from (4.6) that this condition should take the form

$$\left(\frac{ur}{y}\right)_{\theta=0} = F(0) = \text{constant} = R \quad (4.14)$$

The constant R in (4.14) will be called the Reynolds number.

For later use the following observation may be made here. If there is a potential flow through the channel $\frac{ur}{y} = F(\theta)$ is also independent of θ and equal to the Reynoldsnumber R . Then it follows from (4.11) and (4.14) that

$$K = -R^2 - 4R \quad (4.15)$$

If the absolute value of R approaches infinity (4.15) reduces to

$$\frac{-K}{R^2} = 1 \quad (4.16)$$

4.2. The classical approach.

4.2.1. Impervious walls ($\gamma = 0$).

For $\gamma = 0$ equation (4.11) can be integrated once after multiplication by $2F'$ giving:

$$(F')^2 = \frac{2}{3} (H - 3KF - 6F^2 - F^3) \quad (4.17)$$

in which H is a second integration constant. Solutions of (4.17) can be given in terms of elliptic functions. Due to the periodicity of these functions the

solutions are in general not unique but a multiplicity of solutions may exist. This analysis will not be repeated here since excellent reviews of it are available elsewhere. (See for instance the work of Millsaps and Pohlhausen [17], Rosenhead [18]). The main results of these investigations however will be summarised below. In this summary the terms "pure inflow" and "pure outflow" will denote flow patterns for which F is negative respectively positive throughout the cross section of the channel. For these cases the Reynoldsnumber R also is negative or positive. (In the general case velocity profiles with local backflow may occur for which it is even possible that $R \gtrless 0$ while the total flux through the channel is $\lesseqgtr 0$).

Summary of known results.

1. Pure outflow is only possible for values of 2α less than a limiting value $(2\alpha)_{\max}$ which depends on R (fig. 12). For $R \rightarrow \infty$ the relation between $(2\alpha)_{\max}$ and R is given by:

$$\frac{(2\alpha)_{\max}}{\pi} = \frac{6.422}{\pi \sqrt{R}} \quad (4.18)$$

For $R = 0$ one finds $(2\alpha)_{\max} = \pi$.

If $2\alpha > (2\alpha)_{\max}$ the solutions of (4.17) lead to velocity profiles with symmetrical or asymmetrical backflow. The velocity profile becomes more irregular the more $(2\alpha)_{\max}$ is exceeded. (Rosenhead [18]). Fig. 13 taken from [17] shows the velocity profiles for different values of R in a channel with $2\alpha = 10^\circ$.

2. Pure inflow is possible for all values of 2α , provided $|R| > 3$. If $|R| < 3$ again a limiting value of 2α exists (fig. 12).

If, for pure inflow, $R \rightarrow -\infty$ the velocity profile - if plotted in the form $F(\theta)/F(0)$ versus $|\theta - \theta_{\text{wall}}| \sqrt{R}$ - tends to the Pohlhausen solution of the boundary layer problem discussed in section 3. This means that in the middle of the channel a core with nearly constant velocity is obtained.

Fig. 13 shows some velocity profiles for inflow into a channel with $2\alpha = 10^\circ$.

3. In addition to solutions with pure inflow or outflow solutions with partial backflow may be obtained. Outside the limits shown in fig. 12 these are the only possible solutions.

2.2. Porous walls ($\gamma \neq 0$).

Equation (4.11) cannot be integrated in closed form for $\gamma \neq 0$. To obtain numerical results the equation has to be integrated numerically. It is not the purpose of the present paper to provide such results and this course will not be pursued further. Instead it will be shown in the next section that a great deal about the qualitative behaviour of the solutions can be learnt from a study in the phase plane.

3. Phase plane description.

3.1. General.

Following the procedure of section 3 a parameter ξ is defined by

$$\frac{dF}{d\theta} = \xi \quad (4.19)$$

Introducing this parameter into (4.11) and eliminating θ between (4.11) and (4.19) the following differential equation is obtained

$$\frac{d\xi}{dF} = \frac{\gamma \xi - 4F - F^2 - K}{\xi} \quad (4.20)$$

Singular points of this equation are $(-2 - \sqrt{4-K}, 0)$ and $(-2 + \sqrt{4-K}, 0)$ irrespective of the value of γ . These points are real only for $K \leq 4$; for $K = 4$ both singularities merge at $(-2, 0)$.

For $K < 4$ the singular point at $(-2 - \sqrt{4-K}, 0)$ is a saddle for all possible values of γ . The type of the singularity at $(-2 + \sqrt{4-K}, 0)$ is related to the value of $\gamma_1 = \gamma(4-K)^{-1/4}$ according to table 2.

For $0 \leq |\gamma_1| < 2\sqrt{2}$ no distinction can be made between a center or a spiral because non-linear terms occur in the nominator of (4.20). It is easily inferred from symmetry considerations however that a center should occur for $\gamma_1 = 0$ and a spiral for $0 < |\gamma_1| < 2\sqrt{2}$.

The type of the singularity for $K = 4$ cannot be found using table 1 because the special case $ad - bc = 0$ occurs here. From the phase plane portraits to be discussed in sections 4.3.2. and 4.4. however it can be seen that a cusp is found.

In the next section the case with $\gamma = 0$ will be considered in greater detail.

4.3.2. Impervious walls ($\gamma = 0$).

For the non-porous wall (4.20) reduces to

$$\frac{d\zeta}{dF} = \frac{-4F - F^2 - K}{\zeta} \quad (4.21)$$

This equation can be integrated once, leading to the following equation for the trajectories in the phase plane

$$\zeta^2 = -4F^2 - \frac{2}{3} F^3 - 2KF + \frac{2}{3} H \quad (4.22)$$

in which H is an integration constant. Of course, this equation is the same as (4.17) of section 4.2.1. The phase planes are given in fig. 14 for different values of K. Arrows in these figures indicate the direction of increasing θ . Equation (4.22) may be written in the following form

$$\zeta^2 = -\frac{2}{3} (F - e_1)(F - e_2)(F - e_3) \quad (4.23)$$

where e_1 , e_2 and e_3 are the roots of

$$F^3 + 6F^2 + 3KF - H = 0 \quad (4.24)$$

and hence denote the points where a given trajectory intersects the horizontal axis. It may be assumed without loss of generality that $e_1 \geq e_2 \geq e_3$ if all the roots are real. If (4.24) has only one real root it will be denoted by e_1 (fig. 15). It easily follows from a comparison of (4.22) and (4.23) that

$$\begin{aligned} e_1 + e_2 + e_3 &= -6 \\ e_1 e_2 + e_2 e_3 + e_1 e_3 &= 3K \\ e_1 e_2 e_3 &= H \end{aligned} \quad (4.25)$$

To obtain a solution of (4.22), values have to be assigned to K and H which define a trajectory in the phase plane such that the boundary conditions are satisfied. If the Reynoldsnumber is prescribed $F(0)$ is known. To satisfy the no-slip condition the following relation, obtained from (4.19) should hold

$$2\alpha = \int_{F=0}^{F=0} \frac{dF}{\zeta} \quad (4.26)$$

where the integration is taken along a phase curve from one point with $F = 0$ to another.

If the trajectory is closed it is, of course, possible to follow it more than once. This leads to a velocity profile which is periodic in the θ -direction. If the curve is not followed a complete number of periods asymmetric velocity profiles are obtained.

Using (4.23) and the boundary conditions the velocity profile may be expressed in terms of elliptic functions. This is essentially the same procedure as in the classical approach and hence this course will not be pursued further. Only some remarks, peculiar to the phase plane representation, will be made.

An important problem in the present study is the occurrence of boundary layer type solutions. Such a solution is obtained when in the core of the channel a practically constant value of F is found. This means that both $\frac{dF}{d\theta} = \zeta$ and $\frac{d^2F}{d\theta^2} = \frac{d\zeta}{d\theta}$ tend to zero at the same time. This is equivalent to the requirement that the trajectory passes through a singular point.

Fig. 14 shows that this is only possible for inflow when $K \leq 3$. In what follows the two cases of inflow and outflow will be discussed separately.

Inflow. For inflow the trajectories in the phase plane should be closed at the left hand side and hence only the case of three real roots e_1 , e_2 and e_3 will occur (fig. 15). To satisfy the boundary condition $F = 0$ for $\theta = \pm \alpha$ the trajectory should cross or at least touch the ζ -axis; this requires $K \leq 3$ (fig. 14).

If e_2 is put equal to R it follows from (4.25) that

$$e_1 = -(3 + \frac{R}{2}) + (-\frac{3}{4}R^2 - 3R + 9 - 3K)^{\frac{1}{2}} \quad (4.27)$$

$$e_3 = -(3 + \frac{R}{2}) - (-\frac{3}{4}R^2 - 3R + 9 - 3K)^{\frac{1}{2}} \quad (4.28)$$

From numerical calculations, which are not repeated here, it is found that for $R \rightarrow -\infty$

$$\frac{-K}{R} \rightarrow 1, \quad \frac{e_1}{-R} \rightarrow 2 \quad \text{and} \quad \frac{e_3}{-R} \rightarrow -1 \quad (4.29)$$

From the last of these relations and $e_2 = R$ it follows that $e_3 \longrightarrow e_2$. This means that the corresponding trajectory in the phase plane tends to the separatrix passing through the saddle point. Hence a boundary layer type solution is obtained for inflow when $R \longrightarrow -\infty$.

In this case the elliptic functions degenerate into the hyperbolic functions leading to Pohlhausens velocity profile, given in section 3.2.1. (See for instance [17]).

Outflow. For outflow the Reynolds number R is positive. To satisfy also the condition $F(+\infty) = 0$ the solution of (4.22) should be represented by a trajectory intersecting the F -axis at $e_1 > 0$ and having at least one point in common with the ξ -axis (fig. 15).

For outflow with $\gamma = 0$ no boundary layer type solutions can occur. This follows from an inspection of the phase planes, given in fig. 14, which shows that there are no trajectories passing through a singularity at $F(\theta) > 0$.

4.3.3. Porous walls ($\gamma \neq 0$).

Studying (4.20) in the phase plane for porous walls has the disadvantage that for each combination of values for K and γ a different phase plane portrait is obtained.

Instead of pursuing this course any further it will be shown in the next section that the equations can be made independent of K by applying a suitable transformation. This leads to a considerable reduction in the number of phase plane portraits to be studied.

4.4. The phaseplane description in reduced variables.

4.4.1. General.

By selecting suitable new variables the equations (4.11) and (4.21) can be made independent of K . The necessary transformations and the resulting differential equations are listed in table 3. They are different for $K < 4$, $K = 4$ and $K > 4$, depending on whether the singular points are real and different, real and merged into one point or complex. Also the boundary conditions in the new variables are given in table 3. Formerly they required

$F = 0$ for $\Theta = \pm \alpha$. To satisfy the new boundary conditions F_1 should assume a certain value, depending on K , for a value of Θ_1 which depends both on α and K .

The singular points of the new equation for $K < 4$ are found at $F_1 = \pm 1$, $\zeta_1 = 0$; for $K = 4$ there is one singular point at $(0,0)$, while for $K > 4$ there are no real singular points.

Fig. 16 shows the phaseplane portraits of the new equations for different values of γ_1 . In what follows the cases $K < 4$, $K = 4$ and $K > 4$ will be discussed separately.

$K < 4$. In this case the equations in the reduced variables are exactly the same as those for the boundary layer problem of section 3 if λ is replaced by γ_1 , $\bar{u}$ by F_1 , $\bar{\tau}$ by ζ_1 and $d\eta$ by $d\Theta_1$. Hence the results of section 3 can directly be applied to the present problem. This means that the singularity at $(-1,0)$ is always a saddle. The type of the singularity at $(+1,0)$ can be found from table 2.

Also the phase plane portraits are the same as for the boundary layer problem. Hence the effect of suction and blowing on the occurrence of boundary layer type solutions can be deduced directly from the results in section 3.

The only difference with the boundary layer problem is given by the boundary conditions, both at and far from the walls.

At the walls it is required in the present problem that F_1 assumes the value $2(4-K)^{-\frac{1}{2}}$ while in chapter 3 the condition was $\bar{u} = 0$ at the wall.

In the boundary layer problem the boundary condition far from the wall was $|\bar{u}| \rightarrow 1$ while in the present problem F_1 in the middle of the channel follows from

$$F_1 = \frac{R + 2}{\sqrt{4-K}} \quad (4.30)$$

The constant K is determined by the values of R and α .

These differences in boundary conditions disappear however when boundary layer type solutions occur for large values of $|R|$. This can be shown as follows.

For boundary layer type solutions the corresponding trajectory in the phase plane crosses the horizontal axis near a singular point. As the singular points are given by $|F_1| = 1$ this removes the differences between the boundary

conditions in the middle of the channel. From $|F_1| \approx 1$ and (4.30) it follows that

$$\sqrt{4-K} \approx |R+2| \quad (4.31)$$

and hence the value of F_1 at the wall reduces to

$$F_1 = \frac{2}{|R+2|} \quad (4.32)$$

or for $|R| \rightarrow \infty \quad F_1 \rightarrow 0 \quad (4.33)$

Hence for boundary layer type solutions at large values of $|R|$ the boundary conditions of the present problem - both at and far from the walls - become identical to those of the boundary layer problem of chapter 3.

The consequence of this is that velocity profiles of the boundary layer type which are obtained from the Navier-Stokes equations differ from those obtained from the boundary layer equations only for very small values of $|R|$. For the case of inflow without suction this had been shown already by Reeves and Kippenhan [20] from results of numerical calculations. This conclusion is reached here directly from the form of the transformed equations and boundary conditions.

From the preceding remarks the success of the transformations for $K < 4$ can be understood more clearly. For boundary layer type flows at large values of $|R|$ equation (4.31) shows that

$$(4-K)^{\frac{1}{2}} \approx (-K)^{\frac{1}{2}} \approx |R| \quad (4.34)$$

Eq. (4.34) is in agreement with (4.16) which is valid for a potential flow through the channel. From (4.34) it follows that $\theta_1 = \theta(4-K)^{1/4} \approx \theta |R|^{\frac{1}{2}}$.

Transforming in the same way the angular distance from the wall gives

$$|\theta - \theta_{\text{wall}}| (|R|)^{\frac{1}{2}}$$

Near the walls this parameter becomes identical to the variable η used in chapter 3. In the same way F_1 and γ_1 replace $\bar{u}$ and λ respectively.

K = 4 For $K = 4$ both singular points merge at $(0,0)$. The type of this singularity cannot be deduced from the theory of chapter 2 because the special case $ad - bc = 0$ occurs. From fig. 16 it is seen that for $\gamma_1 = 0$ a cusp is obtained; for $\gamma_1 \neq 0$ the singularity is of the same kind. It follows from fig. 16 that for $K = 4$ only solutions for outflow can be obtained. No boundary layer type solutions occur.

K > 4 For $K > 4$ there are no real singular points. From fig. 16 it follows that only solutions for outflow - not of the boundary layer type - can be obtained.

Acknowledgements.

It is a pleasure to thank prof. dr. ir. H.J. van der Maas for proposing and continuously stimulating the authors work in boundary layer research. Prof.dr.ir.J.A.Steketee gave much of his time for discussions on the subject of this paper and for reading critically a first draft of the manuscript. Thanks are also due to mr. P.J.Cornelissen and mr. A.J.Blom for their advice on the use of the analog computer.

References

1. Jeffery, G.B. : "The two-dimensional steady motion of a viscous fluid".
Phil. Mag. Ser. 6, vol 29, april 1915,
p 455-465.
2. Hamel, G. : "Spiralförmige Bewegungen zäher Flüssigkeiten".
Jahrbuch deutsch. Math. Ver. 25, 1916, p 34-60.
3. Goldstein, S. (ed) : "Modern Developments in Fluid Dynamics".
Clarendon Press, Oxford, 1938.
4. Rosenhead, L. (ed) : "Laminar boundary layer theory".
Clarendon Press, Oxford, 1963.
5. Pohlhausen, K. : "Zur näherungsweise Integration der Differentialgleichung der laminaren Grenzschicht". ZAMM, vol 1, 1921, p 252.
6. Mangler, W. : "Die ähnlichen Lösungen der Prandtlschen Grenzschichtgleichungen".
ZAMM, vol 23, nr 5, 1943, p 241-251.
7. Mangler, W. : Göttinger Monographien, part B.1.
"Stationäre laminare Grenzschichten".
B.1.2. "Spezielle exakte Lösungen". 1945.
8. Poincaré, H. : "Sur les courbes définies par une equation differentielle".
Oeuvre, Gauthier-Villars, Paris, Vol 1, 1892.
9. Ku, Y.K. : "Boundary layer problems solved by the method of non-linear mechanics".
Proc. 9th Intern. Congr. Appl. Mech.,
Brussels 1956, p 132-144.
10. Minorsky, N. : "Nonlinear oscillations". Van Nostrand Co.,
Princeton, 1962.
11. Stoker, J.J. : "Nonlinear vibrations in Mechanical and electrical systems". Intersc. Publ. 1950.
12. Schlichting, H. : "Boundary layer theory" Englisch ed, Pergamon Press-Verlag Braun, 1955.
13. Holstein, H. : "Ähnliche laminare Reibungsschichten an durchlässigen Wänden". UM 3050, 1943.

14. Hartree, D.R. : "On an equation occurring in Falkner and Skans approximate treatment of the equations of the boundary layer". Proc. Cambr. Phil.Soc., Vol 33, pt II, 1937, p 223-239.
15. Iglisch, R. : "Exakte Berechnung der laminaren Reibungsschicht an der längsangeströmten ebenen Platte mit homogener Absaugung". Schriften d. deutsch. Akad. d. Luftfahrtforschung, 8B, nr 1, 1944. Also: NACA TM 1205, 1949.
16. Pretsch, J. : Die laminare Grenzschicht bei starkem Absaugen und Ausblasen". UM 3091, 1944.
17. Millsaps, K. and Pohlhausen, K. : Thermal distributions in Jeffery-Hamel flows between non-parallel plane walls". Journ. Aeron. Sc. Vol 20, nr 3, 1953, p 187-196.
18. Rosenhead, L. : "The steady two-dimensional radial flow of viscous fluid between two inclined walls". Proc. Roy. Soc. London, series A, vol 175, 1940, p 436.
19. Fraenkel, L.E. : "Laminar flow in symmetrical channels with slightly curved walls". Proc. Roy. Soc. London, 267 A (1328), 1962, p 119-138.
20. Reeves, B.L. and Kippenhan, C.J. : On a particular class of similar solutions of the equations of motion and energy of a viscous fluid" Journ. Aero-Space Sc. Vol 29, jan. 1962, p 38-47.

$D=(b-c)^2 + 4ad$	$ad-bc$	$b+c$	type	stability
> 0	< 0	< 0	node	stable
		> 0	node	unstable
	> 0	-	saddle	-
< 0		0	center	
		< 0	spiral	stable
		> 0	spiral	unstable
0		< 0	node	stable
		> 0	node	unstable

Table 1: Classification of the singularities of eq (2.3);

If P_1 or $Q_1 \neq 0$ when $D < 0$ no distinction can be made between a center and a spiral

λ or r_1	Type of singularity
$\leq -2\sqrt{2}$ $-2\sqrt{2} < \lambda(r_1) < 0$ 0 $0 < \lambda(r_1) < 2\sqrt{2}$ $\geq 2\sqrt{2}$	stable node (center or) stable spiral center (or spiral) (center or) unstable spiral unstable node

Table 2: Type of the singularity of eq (3.23) at $(+1,0)$ or eq (4.20) at $(-2 + \sqrt{4-k},0)$ or the first eq. of table 3 at $(+1,0)$.

K	New variables defined by	Transformed equations	Boundary conditions
$K < 4$	$F = -2 + F_1(4-K)^{\frac{1}{2}}$ $\theta_1 = \theta(4-K)^{\frac{1}{4}}$ $\gamma_1 = \gamma(4-K)^{-1/4}$	$F_1'' - \gamma_1 F_1' + F_1^2 - 1 = 0$ $\frac{d\zeta_1}{dF_1} = \frac{\gamma_1 \zeta_1 - F_1^2 + 1}{\zeta_1}$	$F_1 = 2(4-K)^{-\frac{1}{2}}$ for $\theta_1 = \pm \alpha(4-K)^{\frac{1}{4}}$
$K = 4$	$F = -2 + F_1$ $\theta_1 = \theta$ $\gamma_1 = \gamma$	$F_1'' - \gamma_1 F_1' + F_1^2 = 0$ $\frac{d\zeta_1}{dF_1} = \frac{\gamma_1 \zeta_1 - F_1^2}{\zeta_1}$	$F_1 = 2$ for $\theta_1 = \pm \alpha$
$K > 4$	$F = -2 + F_1(K-4)^{\frac{1}{2}}$ $\theta_1 = \theta(K-4)^{\frac{1}{4}}$ $\gamma_1 = \gamma(K-4)^{-1/4}$	$F_1'' - \gamma_1 F_1' + F_1^2 + 1 = 0$ $\frac{d\zeta_1}{dF_1} = \frac{\gamma_1 \zeta_1 - F_1^2 - 1}{\zeta_1}$	$F_1 = 2(K-4)^{-\frac{1}{2}}$ for $\theta_1 = \pm \alpha(K-4)^{\frac{1}{4}}$
all values	$\zeta_1 = \frac{dF_1}{d\theta_1}$		

Table 3: Transformation of the equations (4.11) and (4.20)

fig. 1 : Solutions for special cases of eq (2.3) ; A denotes an integration constant.

a) inflow

b) outflow

fig. 2 : Coordinate system for chapter 3.

fig. 3 : Velocity profile for Pohlhausens solution of the boundary layer equations (eq 3.17).

fig. 4 : Phase plane portrait for the boundary layer flow between impervious non-parallel plane walls ($\lambda = 0$).

fig. 5 : Velocity profiles for inflow ($\lambda = 0$).

fig. 6 : Phase plane portraits for the boundary layer flow between porous walls.

fig. 7 : Velocity profiles for inflow and outflow at different values of λ .

fig. 8 : Velocity profiles for outflow corresponding to a stable spiral in the phase plane ($\lambda = -1$).

fig. 9. Velocity profiles for outflow corresponding to a stable node in the phase plane ($\lambda = -4$).

fig. 10 : Characteristics of a stable node.

fig. 11 : Coordinate system for chapter 4.

fig. 12 : $(2\alpha)_{\max}$ as a function of R
both for inflow ($R < 0$) and outflow ($R > 0$)

fig. 13 : Velocity profiles at different values of R for a channel with
 $2\alpha = 10^\circ$ (taken from [17] .)

$R_A = -5000$; $R_B = -1342$; $R_C = -684$; $R_D = +684$; $R_E = +1342$; $R_F = +5000$

fig. 14 : Phase plane portraits for $\gamma = 0$ at different values of K . (eq 4.21).

a) three real roots

b) one real root

fig. 15 : Numbering of roots e_1 , e_2 , e_3 .

fig. 16 : Phase plane portraits for the reduced equations of table 3.

Rapport 118

60141010253