Nonlinear Vibrations of Anisotropic Cylindrical Shells

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Eelco Luc JANSEN
ingenieur luchtvaart- en ruimtevaarttechniek
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Dit proefschrift is goedgekeurd door de promotor:
Prof. dr. J. Arbocz

Samenstelling promotiecommissie:

Rector Magnificus, voorzitter
Prof. dr. J. Arbocz, Technische Universiteit Delft, promotor
Prof. dr. ir. D.H. van Campen, Technische Universiteit Eindhoven
Prof. dr. ir. A. de Boer, Universiteit Twente
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Prof. dr. ir. A. van Keulen, Technische Universiteit Delft
Prof. dr. ir. D.J. Rixen, MSc, Technische Universiteit Delft

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Voor mijn ouders
Summary

This thesis presents a theoretical investigation of the vibration and dynamic stability behaviour of thin-walled anisotropic cylindrical shells. The main emphasis is placed on the vibration behaviour when the vibration amplitude is of the order of the wall thickness of the shell, generally referred to as the large amplitude flexural vibrations or nonlinear flexural vibrations. The research focuses on several characteristic aspects of the nonlinear vibration behaviour, namely the coupling between asymmetric and axisymmetric modes and the possibility of circumferentially traveling waves. The influence of important parameters, such as geometric imperfections, static loading and boundary conditions, receives particular attention. In the present approach, semi-analytical (i.e. analytical/numerical) models with different levels of complexity are developed. In a Level-1 Analysis (or Simplified Analysis) a small number of assumed modes which approximately satisfy "simply supported" boundary conditions at the shell edges, are used in a Galerkin procedure or variational method. In a Level-2 Analysis (or Extended Analysis) the specified boundary conditions are accurately satisfied by means of the numerical solution of corresponding two-point boundary value problems. Nonlinear Donnell-type governing equations are adopted in combination with classical lamination theory. It is assumed that the cylindrical shell is statically loaded by axial compression, radial pressure, and torsion.

The main conclusions of the research can be summarized as follows. Several models have been developed to simulate the nonlinear vibration behaviour of cylindrical shells. The models are in accordance with a strategy for the analysis of complicated shell problems. The models can simulate the characteristics of the behaviour and are capable of capturing the effects of the relevant parameters indicated earlier. An adequate modelling of the secondary coordinates is essential. The results of the models developed are qualitatively in agreement with available experimental results.

The thesis can be divided into three main themes. First, the stationary nonlinear flexural vibrations of imperfect anisotropic cylindrical shells under harmonic radial excitation are studied via a Level-1 Analysis. The assumed deflection function includes two asymmetric modes. The \textit{driven mode} is excited directly by the applied harmonic radial pressure. The \textit{companion mode} is in shape identical to the driven mode, but circumferentially ninety degrees out-of-phase. The axisymmetric mode satisfying a relevant coupling condition with the asymmetric modes is included in the assumed deflection function. The static state response is assumed to be affine to the given two-mode imperfection, which consists of an axisymmetric and an asymmetric mode. A possible skewedness of the asymmetric modes is taken into account.
Approximate solutions for the dynamic state equations are obtained by applying Galerkin's method and the method of averaging in sequence. Frequency-amplitude curves for free and forced nonlinear vibrations are obtained from the resulting set of nonlinear algebraic equations in the averaged vibration amplitudes. Due to an internal parametric resonance, for a large excitation in the spectral neighbourhood of the linear natural frequency a coupled mode response can occur. This response may be interpreted as a travelling wave pattern in the circumferential direction of the shell. Results of the nonlinear vibration analysis of a specific anisotropic cylindrical shell are presented. The effect of imperfections and axial loading on the vibrations of this anisotropic cylinder is discussed.

The second part of this thesis is also within the framework of a Level-1 Analysis. The transient nonlinear flexural vibrations and several dynamic stability problems of imperfect anisotropic cylindrical shells, namely dynamic buckling and nonlinear parametric excitation, are analysed via numerical time-integration. The in-plane inertia of both the fundamental axial and the fundamental torsional mode, including the inertia effect of a ring or disk at the loaded end of the shell, is taken into account approximately. Viscous modal damping is included in the analysis. Hamilton’s principle is applied as an approximate solution method in order to obtain a set of ordinary differential equations in the coefficients of the assumed deflection modes. The dynamic response is obtained via numerical time-integration of the resulting differential equations. The coupled mode nonlinear vibration behaviour of an isotropic shell is simulated using this approach. Nonstationary vibrations where the response drifts between single mode and different types of coupled mode solutions, are observed in a small frequency region near resonance. In addition, characteristic results for dynamic buckling and nonlinear parametric excitation are discussed for several specific isotropic and anisotropic cylindrical shells.

Finally, in the third part of this thesis, the effect of the boundary conditions at the shell edges on the flexural vibration behaviour of anisotropic cylindrical shells is assessed via a Level-2 Analysis. The effect of finite amplitudes is investigated via a perturbation expansion for both the frequency parameter and the dependent variables. Imperfections and a nonlinear static deformation are included in the formulation. The perturbation procedure eliminates the time dependence and leads to sets of boundary value problems with the spatial coordinates as independent variables. A Fourier decomposition in the circumferential direction of the shell is used in order to eliminate the dependence of the solution on the circumferential coordinate. Hereby the partial differential equations are reduced to ordinary differential equations. The elastic boundary conditions are satisfied accurately by solving the resulting two-point boundary value problems numerically via the parallel shooting method. Numerical results show the effects of different sets of boundary conditions on the nonlinear vibration behaviour of several specific isotropic and anisotropic cylindrical shells. The results of the Level-2 analysis are compared with results of Level-1 analyses and results from the literature.
Samenvatting

Dit proefschrift beschrijft een theoretisch onderzoek over het trillingsgedrag en de dynamische stabiliteit van dunwandige anisotrope cylinderschalen. De nadruk wordt gelegd op het trillingsgedrag wanneer de amplitude van de trilling van de orde van de wanddikte van de schaal is, algemeen aangeduid als “buigtrillingen met grote amplitude” of “niet-lineaire buigtrillingen”. Het onderzoek richt zich vooral op een aantal karakteristieke aspecten van het niet-lineaire trillingsgedrag, namelijk de koppeling tussen “asymmetrische” en “axiaal-symmetrische” trillingsvormen en de mogelijkheid van lopende golven in omtreksrichting van de schaal. De invloed van belangrijke parameters zoals geometrische imperfecties, statische belasting en randvoorwaarden krijgt speciale aandacht. In de huidige aanpak worden semi-analytische (d.w.z. analytisch/numerieke) modellen met verschillende niveau’s van complexiteit ontwikkeld. In een Niveau-1 Analyse (of “Simplified Analysis”) wordt een klein aantal aangenomen verplaatsingsvormen die bij benadering aan de klassieke randvoorwaarde voor een oplegging voldoen, gebruikt in een Galerkin procedure of een variatiemethode. In een Niveau-2 Analyse (of “Extended Analysis”) wordt nauwkeurig aan de gespecificeerde randvoorwaarden voldaan door middel van het numeriek oplossen van corresponderende tweepunts randvoorwaarde problemen. Om het probleem te beschrijven worden niet-lineaire vergelijkingen van het Donnell-type gebruikt in combinatie met klassieke laminaat theorie. Er wordt verondersteld dat de cylinderschaal statisch belast wordt door axiale compressie, radiale druk, en torsie.

De belangrijkste conclusies van het onderzoek kunnen als volgt worden samengevat. Er zijn verschillende modellen ontwikkeld om het niet-lineaire trillingsgedrag van cylinderschalen te beschrijven. De modellen passen binnen een strategie voor het analyseren van complexe schaalproblemen. De modellen kunnen de karakteristieke aspecten van het gedrag simuleren en zijn in staat om de effecten van de eerder genoemde relevante parameters te kunnen beschrijven. Het op een adequate wijze modelleren van de secundaire coördinaten is essentieel. De resultaten van de ontwikkelde modellen zijn kwalitatief in overeenstemming met beschikbare experimentele resultaten.

Het proefschrift kan in drie belangrijke thema’s worden verdeeld. In de eerste plaats worden de stationaire niet-lineaire trillingen van imperfecte anisotrope cylinderschalen onder harmonische radiale excitatie bestudeerd via een Niveau-1 Analyse. De aangenomen verplaatsingsfunctie bevat twee asymmetrische trillingsvormen. De driven mode wordt direct aangedreven door de aangebrachte harmonische radiale druk. De companion mode is identiek wat vorm betreft maar negentig graden uit
fase in de omtreksrichting. De axiaalsymmetrische trillingsvorm die aan een belangrijke koppelingvoorwaarde met de asymmetrische trillingsvormen voldoet is opgenomen in de aangenomen verplaatsingsfunctie. De responsie in de “statische toestand” wordt verondersteld gelijkvormig te zijn aan de gegeven “twee-modes” imperfectie, die bestaat uit een axiaalsymmetrische en een asymmetrische “mode”. Er wordt rekening gehouden met een mogelijke torsieverdraaiing van de asymmetrische “modes”. Benaderingsoplossingen voor de “dynamische toestand” worden verkregen door Galerkin’s methode en de “method of averaging” na elkaar toe te passen. Frequentie-amplitude krommes voor vrije en gedwongen trillingen worden verkregen uit het resulterende stelsel van niet-lineaire algebraïsche vergelijkingen in de gemiddelde trillingsamplitudes. Ten gevolge van een interne parametrische resonantie kan voor een hoog excitatieniveau in de buurt van de lineaire eigenfrequentie een “gekoppelde modes” responsie optreden. Deze responsie kan geïnterpreteerd worden als een lopende golf in de omtreksrichting van de schaal. Resultaten van de niet-lineaire trillingsanalyse worden gepresenteerd voor een specifieke anisotrope schaal. Er wordt ingegaan op het effect van imperfecties en een axiale belasting op de trillingen van deze anisotrope schaal.


Tenslotte wordt in het derde deel van dit proefschrift het effect van randvoorwaarden aan de schaaleinden op het buigtrillingsgedrag van anisotrope cilinderschalen bekeken via een Niveau-2 Analyse. Het effect van eindige amplitudes wordt onderzocht via een perturbatie expansie voor zowel de frequentie parameter als de afhankelijke variabelen. Imperfecties en een niet-lineaire statische vervorming worden meegenomen in de formulering. De perturbatie procedure elimineert de tijdsvariabiliteit en leidt tot stelens randvoorwaarde problemen met de ruimtelijke coördinaten als onafhankelijke variabelen. Een Fourier decompositie in de omtreksrichting van de schaal wordt gebruikt om de afhankelijkheid van de oplossing van de
ontreksrichtingscoördinaat te elimineren. Hierbij worden de partiële differentiaalvergelijkingen gereduceerd tot gewone differentiaalvergelijkingen. Aan de elastische randvoorwaarden wordt nauwkeurig voldaan door het resulterende tweepunts randvoorwaarde probleem numeriek op te lossen via de “parallel shooting method”. Numerieke resultaten laten het effect zien van verschillende types randvoorwaarden op het niet-lineaire trillingsgedrag van een aantal specifieke isotrope en anisotrope cirkelschalen. De resultaten van de Niveau-2 analyse worden vergeleken met resultaten van Niveau-1 analyses en met resultaten uit de literatuur.
Samenvatting

Het boek is geschreven in het Nederlands en behandelt het onderwerp "Analyse". De tekst gaat in op de fundamentele aspecten van de dynamische stabiliteit en is geschreven voor studenten met een basis in de parameterische ecologie. De boek bespreekt de belangrijkste principes en toepassingen van de dynamische stabiliteit en biedt een gids voor de veralgemening van de wetenschap. Het boek is geschreven voor studenten die een basis hebben in de parameterische ecologie en de toepassingen van deze wetenschap.
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## Nomenclature

### General

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<th>Description</th>
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<tr>
<td>$A_{ij}$</td>
<td>extensional stiffness matrix, App. A</td>
</tr>
<tr>
<td>$A_{ij}^*$</td>
<td>semi-inverted extensional stiffness matrix, App. A</td>
</tr>
<tr>
<td>$\bar{A}_{ij}^*$</td>
<td>nondimensional $A_{ij}^<em>$ ($\bar{A}_{ij}^</em> = E_h A_{ij}^*$)</td>
</tr>
<tr>
<td>$B_{ij}$</td>
<td>bending-stretching coupling matrix, App. A</td>
</tr>
<tr>
<td>$B_{ij}^*$</td>
<td>semi-inverted bending-stretching coupling matrix, App. A</td>
</tr>
<tr>
<td>$\bar{B}_{ij}^*$</td>
<td>nondimensional $B_{ij}^<em>$ ($\bar{B}_{ij}^</em> = (2c/h) B_{ij}^*$)</td>
</tr>
<tr>
<td>$c$</td>
<td>flexural stiffness matrix, App. A</td>
</tr>
<tr>
<td>$D_{ij}$</td>
<td>semi-inverted flexural stiffness matrix, App. A</td>
</tr>
<tr>
<td>$D_{ij}^*$</td>
<td>nondimensional $D_{ij}^<em>$ ($D_{ij}^</em> = (4c^2)/(E_h^3) D_{ij}^*$)</td>
</tr>
<tr>
<td>$E$</td>
<td>reference Young’s modulus</td>
</tr>
<tr>
<td>$F$</td>
<td>Airy stress function</td>
</tr>
<tr>
<td>$h$</td>
<td>reference shell wall thickness</td>
</tr>
<tr>
<td>$L$</td>
<td>shell length</td>
</tr>
<tr>
<td>$L_{A}, L_{B}, L_{D}$</td>
<td>linear operators, Eq. 2.5</td>
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<td>$L_{NL}$</td>
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<tr>
<td>$M_x, M_y, M_{xy}, M_{yx}$</td>
<td>moment resultants</td>
</tr>
<tr>
<td>$n$</td>
<td>number of full waves in circumferential direction</td>
</tr>
<tr>
<td>$N_x, N_y, N_{xy}$</td>
<td>stress resultants</td>
</tr>
<tr>
<td>$N_0$</td>
<td>applied axial load ($N_0 = -N_x(x = L)$)</td>
</tr>
<tr>
<td>$N_{cl}$</td>
<td>classical buckling load ($N_{cl} = (E_h^2)/(cR)$)</td>
</tr>
<tr>
<td>$p$</td>
<td>applied external pressure</td>
</tr>
<tr>
<td>$\bar{p}$</td>
<td>nondimensional external pressure ($\bar{p} = (cR^2)/(E_h^3)p$)</td>
</tr>
<tr>
<td>$\bar{p}_\infty$</td>
<td>nondimensional freestream static pressure ($\bar{p}<em>\infty = (cR^2)/(E_h^3)p</em>\infty$)</td>
</tr>
<tr>
<td>$q$</td>
<td>axial load eccentricity (measured from shell midsurface)</td>
</tr>
<tr>
<td>$\bar{q}$</td>
<td>nondimensional load eccentricity ($\bar{q} = (4cR/h^2)q$)</td>
</tr>
<tr>
<td>$Q_x, Q_y$</td>
<td>transverse shear stress resultants</td>
</tr>
<tr>
<td>$R$</td>
<td>radius of shell</td>
</tr>
<tr>
<td>$T$</td>
<td>kinetic energy</td>
</tr>
<tr>
<td>$T_0$</td>
<td>applied torque ($T_0 = N_{xy}(x = L)$)</td>
</tr>
<tr>
<td>$u, v, W$</td>
<td>axial, circumferential and radial displacement</td>
</tr>
<tr>
<td>$V$</td>
<td>strain energy</td>
</tr>
<tr>
<td>$W_{nc}$</td>
<td>work done by nonconservative loads</td>
</tr>
</tbody>
</table>
Nomenclature

\( \tilde{W} \) initial radial imperfection

\( x, y, z \) axial, circumferential and radial coordinate, respectively

\( \tilde{Z} \) modified Batdorf parameter (\( \tilde{Z} = L^2/(Rh) \))

\( \gamma \) 1) gas constant
2) shear ratio parameter (\( \gamma = \frac{G_{ex}}{E} (1 - \nu_{xy} \nu_{yz}) \)), Chapter 3

\( \epsilon_x, \epsilon_y, \gamma_{xy} \) strain components

\( \kappa_x, \kappa_y, \kappa_{xy} \) curvature changes and twist, respectively

\( \lambda \) nondimensional axial load (\( \lambda = (cR)/(Eh^2)N_0 \))

\( \nu \) reference Poisson's ratio

\( \Pi \) potential energy

\( \rho \) specific mass

\( \bar{\rho} \) averaged specific mass

\( \sigma_x, \sigma_y, \tau_{xy} \) in-plane stresses

\( \tau \) nondimensional torque (\( \tau = (cR)/(Eh^2)T_0 \))

\( \omega \) radial frequency

\( \tilde{\omega} \) normalized radial frequency (\( \tilde{\omega} = R\sqrt{(ph/A_{22})/\omega} \))

\( \omega_0 \) normalized radial frequency (\( \omega_0 = \sqrt{(cR^3)/(Eh)} \omega \))

\( (\cdot) \) partial differentiation w.r.t. coordinate following the comma

\( (\cdot)' \) differentiation w.r.t. \( \tilde{x} \), Chapter 5

\( (\cdot) \) static state variable

\( (\cdot) \) dynamic state variable

Simplified Analysis (Chapters 3 and 4)

\( a_1, a_2, a_3, a_4 \) stress function coefficients, defined by

\[
\begin{align*}
    a_1 &= \frac{a^2 Eh}{R(a^2 + \beta^2)^2} \\
    a_2 &= \frac{a^2 Eh}{2(a^2 + \beta^2)} \\
    a_3 &= \frac{a^2 Eh}{16(a^2 + \beta^2)} \\
    a_4 &= \frac{a^4}{4} \beta^4 R E h \left[ \frac{1}{9(a^2 + \beta^2)^2} - \frac{1}{(a^2 + \beta^2)^2} \right]
\end{align*}
\]

\( A(t), B(t) \) time-dependent coefficients of asymmetric vibration modes

\( A_1(t), B_1(t) \) slowly varying amplitudes of asymmetric vibration modes

\( A, B \) average amplitudes of asymmetric vibration modes

\( C_0(t), C_1(t) \) time-dependent coefficients of axisymmetric vibration mode

\( F_p \) particular solution of stress function

\( \ell \) number of circumferential waves of vibration mode

\( m \) number of axial half waves

\( n \) number of circumferential waves of imperfection / static state mode

\( N_x^*, N_y^*, N_{xy}^* \) spatially constant stresses

\( q \) radial excitation
Extended Analysis (Chapter 5)

\( a_d \) first nonlinearity coefficient, Eq. (5.7)
\( A \) coefficient matrix, Eq. (5.69)
\( b_d \) second nonlinearity coefficient, Eq. (5.7)
\( b_{ijk} \) coefficients in expansion, Eq. (5.7)
\( B_1 - B_{20} \) constants in boundary conditions, App. D
\( B_{11} \) coefficient matrix in boundary conditions
\( B_1 - B_{35} \) constants in boundary conditions, App. D
\( C_1 - C_{31} \) constants in first-order state equations, App. D
\( C_1 - C_3 \) constants in Eq. (5.63)
\( D_1 - D_{32} \) constants in second-order state equations, App. D
\( f_0 \) fundamental state stress function component
\( f_1, f_2 \) first-order stress function components
\( f_1, f_2, f_{\gamma} \) second-order stress function components
\( F^{(0)}, F^{(1)}, F^{(2)} \) stress functions for 0th-, 1st-, and 2nd-order state
\( n \) number of circumferential waves vibration mode
\( w \) nondimensional radial displacement \((w = W/h)\)
\( w_0 \) fundamental state radial displacement component
\( w_1, w_2 \) first-order radial displacement components
\( w_3, w_4 \) second-order radial displacement components
\( W_{p}, W_{t} \) generalized Poisson's expansions
\( W^{(0)}, W^{(1)}, W^{(2)} \) radial displacements for 0th-, 1st-, and 2nd-order state
\( \bar{x} \) nondimensional axial coordinate \((\bar{x} = x/R)\)
\( Y \) vector of dependent variables
\( Y_0, Y_1, Y_2 \) vector of dependent variables, 0th-, 1st-, and 2nd-order state
\( \theta \) nondimensional circumferential coordinate \((\theta = \gamma/R)\)
\( \Lambda \) eigenvalue parameter
\( \xi, \xi_l, \xi_m \) perturbation parameter
\( \Delta, \Delta_{11}, \Delta_2 \) constants, App. D

Basic Equations (Appendix A)

\( E_{11}, E_{22} \) Young's moduli orthotropic layer
\( G_{12} \) shear modulus orthotropic layer
\( h_k \) thickness of k-th layer
\( N \) number of layers
\( Q_{ij} \) stiffness matrix orthotropic layer w.r.t. lamina principal axes
\( Q_{ij} \) stiffness matrix orthotropic layer w.r.t. shell reference axes
Nomenclature

\( \epsilon_1, \epsilon_2, \gamma_{12} \)  
strain components orthotropic layer

\( \theta_k \)  
orientation of \( k \)-th layer

\( \nu_{12}, \nu_{21} \)  
Poisson's ratios orthotropic layer

\( \sigma_1, \sigma_2, \tau_{12} \)  
in-plane stresses orthotropic layer

Parallel Shooting Method (Appendix E)

\( f, \dot{f} \)  
vector functions

\( J, \dot{J} \)  
Jacobian matrices

\( N \)  
half of the number of intervals in shooting method

\( s, t \)  
initial guess vectors

\( s_0, s_2 \)  
initial condition vectors

\( S, \dot{S} \)  
initial guess vectors

\( U, V \)  
solution vectors

\( \dot{U}, \dot{V} \)  
solution vectors

\( W(\bar{x}) \)  
solution matrix of variational system

\( W_i, W_i \)  
solution vectors of variational equations (forward integration)

\( W_{ij} \)  
submatrices of \( W(\bar{x}) \)

\( x_1 \)  
matching point in double shooting method

\( Z_i, \dot{Z}_i \)  
solution vectors of variational equations (backward integration)

\( \gamma \)  
vector of inhomogeneous terms in matching condition

\( \Phi, \dot{\Phi} \)  
vectors of matching conditions
Chapter 1

Introduction

1.1 Background

Because of their favourable strength-over-weight and stiffness-over-weight ratio, stiffened and unstiffened shells are widely used as primary components in (weight-critical) structural applications in aerospace and other fields of engineering. These thin-walled structures are prone to static and dynamic buckling instabilities. The circular cylindrical shell is of special interest for several reasons. Firstly, this configuration is significant from a practical viewpoint since it is commonly applied. It should be noted that the results of an analysis for cylindrical shells may also be characteristic for a more general class of shell structures, such as shells of revolution. Secondly, the severe instability of a cylinder under axial compressive loading may result in a catastrophic failure. Finally, its basic, simple geometry makes the cylindrical shell extremely suited for a theoretical analysis.

1.1.1 Vibration research related to shell buckling

The notorious discrepancy between the experimental results and the theoretical predictions (based on linearized small deflection theory) for the buckling load of a cylindrical shell under axial compression led to an enormous research effort in the 1960s and 1970s (Fig. 1.1). For a review of these investigations see Arbocz (1981).

The main reason for the difference is attributed to the presence of unavoidable small deviations from the perfect cylindrical form due to the fabrication process, the so-called initial geometric imperfections. A second factor which can explain the discrepancy is the effect of the boundary conditions (including the effect of the edge restraint). It was realized that these factors, which can have such an important effect on the buckling behaviour, could also influence the dynamic behaviour of shell structures. The influence of geometric imperfections on the natural frequencies was studied (Singer and Rosen, 1976; Singer and Prucz, 1982), and later also their influence on the nonlinear vibrations (Watawala and Nash, 1982; Liu, 1988). The effects of boundary conditions were also investigated (Liu, 1988). Evensen (1974) stated that nonlinear static previbration loads can make the nonlinear effects much more pronounced. In his thesis, Liu (1988) also studied the effect of a static axial
Introduction

Figure 1.1: Discrepancy between theoretical buckling load and experimental results for cylindrical shells (from Arbocz (1981)). Normalized buckling load \( P/P_{cl} \) versus radius to thickness ratio \( R/h \).

compressive load on the nonlinear vibrations.

The formal analogy between buckling and vibration has stimulated the use of vibration tests to obtain information which is important to assess the buckling behaviour. Firstly, it has been suggested that one could use vibration tests to establish the actual boundary conditions, the so-called vibration correlation technique (Singer, 1983). Another possibility is to estimate the buckling load from vibration tests as a way of nondestructive testing (e.g., Schneider Jr. et al., 1991). A good understanding of nonlinear effects (due to imperfections, large amplitudes, etc.) on the vibration behaviour is indispensable when such methods are applied. There is also an analogy between post-buckling and nonlinear vibrations. The unstable post-buckling behaviour of shells, or in other words, the negative slope of the post-buckling path, corresponds with the softening vibration behaviour of shells, i.e., the frequency decreases with increasing vibration amplitude.

Cylindrical shells form the primary structure of missiles and launch vehicles. During powered flight, these thin-walled structures are often forced to vibrate at large amplitudes by their environment. This observation provided a direct reason to study this problem in the beginning of the space age (about 1960). If the vibration amplitude is sufficiently small, the dynamic behaviour of a shell may be described adequately by a linear analysis, but when the amplitude is of the order of the shell thickness, nonlinear effects have to be taken into account. These large amplitude vibrations are therefore generally referred to as nonlinear vibrations. The nonlinearity is clearly demonstrated by two phenomena, 1) the shape of the response-frequency relationship in the vicinity of a resonant frequency, and 2) the occurrence of a traveling wave response in the circumferential direction of the shell (Fig. 1.2). The early developments in the research field of nonlinear vibrations of cylindrical shells (and related work) can be found in a review by Evensen (1974).
A second reason to study the nonlinear vibrations of shell structures was that the subject is important from a theoretical viewpoint. The nonlinear (free or forced) vibration problem of shell structures can be seen as one of the fundamental topics in the dynamic stability analysis of shells. The term ‘dynamic stability’ here denotes a number of different phenomena in the dynamics of shells which are related to the structural stability (Herrmann, 1967) and which have in common that they are investigated by means of Newton’s law or equivalent methods (i.e. inertia is taken into account in the formulation). Important fields in the dynamic stability of shell structures include dynamic buckling (buckling under step loading or impulsive loading), parametric excitation (vibration buckling under pulsating loads) and flutter (instability induced by a gas flow). It is noted that a discrepancy between theory and experiment has not only occupied researchers working on shell buckling, but also investigators in the field of shell flutter (Dowell, 1970, 1975) and nonlinear vibrations (Evensen, 1974).

A long debated question concerned the type of nonlinearity of the flexural vibrations of cylindrical shells. The earliest theoretical investigations predicted a
hardening nonlinearity, that is, the frequency increases with the amplitude of vibration, whereas experiments showed the contrary, a softening nonlinearity (i.e., the frequency decreases with amplitude). The omission of the appropriate axisymmetric deflection functions in the analysis (which may lead to not satisfying the circumferential periodicity condition) was identified as the principal shortcoming of these theoretical investigations and a softening nonlinearity was obtained for the practically important cases (Evensen, 1965, 1967). The work by Prathap and Pandalai (1978) was motivated by the lack of a clear and consistent model to describe the nonlinear vibrations of axisymmetric structures. Their paper gives insight into the background of the type of nonlinearity of the vibrations of these structures, and highlighted the importance of the median surface curvature in particular.

The reasons stated earlier to study besides the linear vibrations also the nonlinear vibrations of shells are related to structural stability problems. If sonic fatigue is a design consideration one may also need a nonlinear analysis rather than a linear one. In the analysis of composite plates and panels subjected to high intensity acoustic fatigue loading, the effect of large amplitudes can be taken into account (e.g., Locke, 1991). Another example of an application not directly related to stability is the work by Brown et al. (1991) dealing with nonlinear acoustic response in fluid-structural interactions on a thin cylindrical shell.

1.1.2 Status of research

In the Structures Group of the Faculty of Aerospace Engineering at Delft there has been a continuing effort in the field of shell vibrations. Both theoretical and experimental work has been conducted on the (nonlinear) vibration behaviour of cylindrical shells (Hol, 1983; Liu, 1988; Zaal, 1989; Jansen, 1992; Jansen and Gunawan, 1996; Gunawan, 1998). The vibration research is strongly connected with the principal research topic in this group, the (static) buckling of cylindrical shells. For this reason, particular attention was paid to the effects of axial loading, the effect of boundary conditions, and the effect of initial imperfections on the vibration behaviour.

The methods which have been developed in the Structures Group for the nonlinear vibration analysis (Liu, 1988; Jansen, 1992) are based on semi-analytical methods of two levels of complexity. The simplest analyses are based on the use of a limited number of assumed modes which approximately satisfy “simply supported” boundary conditions at the shell edges. Following the classification of buckling analyses used by Arbocz (1993), these analyses are referred to as Level-1 Analyses. The effects of boundary conditions can be taken into account in a more complicated analysis involving the numerical solution of a two-point boundary value problem, a so-called Level-2 Analysis.

In the analyses of Liu (1988), Donnell-type governing equations have been used (Donnell, 1933; Brush and Almroth, 1975). The Donnell equations permit the reduction of the number of dependent variables in the governing equations to two by introducing an Airy stress function. This has obvious advantages for analytical manipulations.
1.1 Background

At the start of the present research (about 1990), the linearized and nonlinear vibration behaviour of orthotropic shells under axial compression had been modelled by Liu (1988) using both a Level-1 and a Level-2 Analysis. Liu addressed the discrepancy between results regarding the effect of asymmetric imperfections on the linearized vibration behaviour of isotropic shells. Liu confirmed the results obtained by Hol (1983), who predicted a decrease in frequency for a shell with an asymmetric imperfection affine to the vibration mode (see Fig. 1.3). Liu further studied the effects of imperfections and axial loading on the coupled mode response of isotropic shells, and he investigated the effect of boundary conditions on the single mode vibrations.

Around 1990, the agreement between theoretical predictions for nonlinear vibrations of cylindrical shells and the (scarce) experimental results (e.g., Chen and Babcock, 1975; Yamaki, 1983) was by far complete. Improvement of the theoretical models was necessary to clarify the various aspects of nonlinear shell vibration. Parallel to the present theoretical research, an experimental study of the nonlinear vibrations of isotropic cylindrical shells was started (Gunawan, 1998).

Both in the theoretical and in the experimental results for nonlinear shell vibrations phenomena had been observed, for which no satisfactory explanation could be given at that time, like regions in the frequency-response diagram where no stable solution exists (Liu, 1988), and “beating” or “nonstationary” responses (Chen and Babcock, 1975). The nonlinear dynamic behaviour of mechanical systems received around 1990 considerable attention due to the interest in the chaotic behaviour of these systems. The theories and tools developed to analyse nonlinear dynamical systems and chaos (e.g., Thompson and Stewart, 1986; Moon, 1987) provided the
possibility to understand the phenomena observed in shell vibrations.

The introduction of fibre reinforced composite materials in the seventies had provided new opportunities in the design of lightweight structures. The designer can try to find optimal structural configurations by choosing different materials and varying the fibre orientations and stacking sequence of the layers. The use of laminated structures requires an analysis which takes into account the possible couplings between bending and stretching deformations. These couplings have already been taken into account in static stability analyses (e.g., Booton, 1976; Arbocz and Hol, 1989). The nonlinear vibration behaviour of composite cylinders has been studied by Iu and Chia (1988b) using a multi-mode analysis. However, about 1990, the attention given to the vibration behaviour of composite shells was still limited, and the development of methods capable of analysing the dynamic behaviour of composite shells was desirable at that time.

1.2 Objectives and scope

Practical problems in the stability and vibration analysis of shell structures can be very complex due to geometry, anisotropy, nonlinearity, and number of degrees of freedom, and their solution may require a powerful discretization method, a Level-3 Analysis in the terminology used by Arbocz (1993). The finite element method can be used for the spatial discretization, in combination with numerical time-integration to determine the temporal behaviour.

A prerequisite for the application of a Level-3 analysis is to have insight in the structural behaviour. The mechanisms which play a role must be understood in advance for the model definition and for the interpretation of the results. For the cylindrical shell (an important complicated problem with a simple geometry) it is possible to devise semi-analytical (i.e. analytical/numerical) methods which emphasize characteristic aspects of the structural behaviour. These models can provide reference solutions, and since they are in general efficient with respect to computational time, they are suited for parametrical studies. Therefore such models are indispensable for a good understanding of the structural behaviour. Methods with different levels of complexity are complementary in the analysis of complicated structural behaviour.

Analogous to the analysis strategy for buckling problems proposed by Arbocz (1993), for nonlinear shell vibration problems the following methods should be available:

- In a so-called Level-1 Analysis or Simplified Analysis the system is modelled with a small number of degrees of freedom using a Galerkin or variational approach. The assumed deflection modes approximately satisfy “simply supported” boundary conditions at the shell edges. In this method, the effect of imperfections is included. The term “simplified” refers to the use of a limited number of assumed modes which (approximately) satisfy simply supported boundary conditions, and to the use of a simple imperfection shape. A Level-1 analysis forms the first step in the characterization of the nonlinear behaviour.
of the shell, and should in general be accompanied by more accurate analysis methods.

- In a Level-2 Analysis or Extended Analysis, a Fourier decomposition of the dependent variables in the circumferential direction of the shell is used to eliminate the dependence of the circumferential coordinate. The problem is formulated as a set of 2-point boundary value problems for the axial direction, which is solved numerically via the parallel shooting method. In this approach, the specified boundary conditions can be satisfied rigorously.

In the light of the considerations stated in the previous sections, the objectives of the present research were the following:

- to develop semi-analytical methods with different levels of complexity for the nonlinear vibration analysis of anisotropic (layered composite) cylindrical shells. The models should be capable of capturing the effects of
  - anisotropy (e.g. the influence of the stacking sequence and layer orientation on the vibration characteristics)
  - initial geometric imperfections
  - different boundary conditions
  - static preloading

- to verify other theoretical results available and interpret experimental results available, and to explain the discrepancies between the different results.

- to shed light on the phenomena observed in the experiments which have not been explained as yet, such as regions in the frequency-response diagram where no stable solution exists, and ‘beating’ or ‘nonstationary’ responses.

- to interpret the results of the parallel experimental research in the Structures Group on nonlinear shell vibrations carried out by Gunawan (1998).

1.3 Outline

The organization of this thesis on the nonlinear vibration behaviour of anisotropic cylindrical shells is as follows. Chapter 2 contains the problem definition and mathematical formulations. The governing equations for the nonlinear vibrations of composite cylindrical shells are presented. Chapter 3 treats the vibration behaviour of a composite cylinder using a Simplified Analysis. The effect of imperfections on the linearized vibrations of a statically loaded shell is investigated. Amplitude-frequency relations are derived using the method of averaging. Chapter 4 deals with a Simplified Analysis for several vibration and dynamic stability problems, namely nonlinear vibrations, nonlinear parametric excitation, and dynamic buckling under step loading. Numerical time-integration is used to obtain the dynamic response.
In Chapter 5, the effect of the boundary conditions at the shell edges is taken into account in an Extended Analysis. A perturbation method is used to determine the dependence of the frequency on the vibration amplitude and imperfection amplitude. Several conclusions and suggestions for future work are given in the final Chapter 6.
Chapter 2
Problem Definition and Governing Equations

2.1 Problem Definition

In the present study, the nonlinear vibration problem of a laminated (composite) circular cylindrical shell is considered. The shell is stiffened by closely spaced rings and stringers. The shell geometry is characterized by the shell radius $R$, the length $L$, and the thickness $h$ (see Fig. 2.1).

The static loading consists of the three basic (axisymmetric) loads, axial compression $\bar{P}$, uniform radial pressure $\bar{p}$, and torsion $\bar{T}$ (Fig. 2.1). In addition, the shell can also be loaded dynamically by the three basic loads $\bar{P}(t)$, $\bar{p}(t)$, and $\bar{T}(t)$, specified as functions of time. Further, the shell is subjected to a (spatially varying) radial harmonic loading $q$, and to the radial aerodynamic loading due to a supersonic flow $p_{ae}$. The shell loading will be discussed in more detail in Section 2.3.

A class of deformations is considered, which is intermediate in the sense that the geometric nonlinearity is limited to moderately small rotations (Sanders, 1963; Koiter, 1966; Brush and Almroth, 1975). In the present analysis, the usual Kirchhoff assumptions are employed. The nonlinear equations used in the present work are based on a Donnell-type thin shell theory. The basic assumptions are (cf. Yamaki, 1984):

- the shell is thin, i.e., $h/R \ll 1$, $h/L \ll 1$,
- strains are small (of order $\epsilon$ where $\epsilon \ll 1$),
- displacements $u$, $v$ are infinitesimal, $W$ is of the order of the shell thickness,
- flexural rotations of shell elements are moderately small ($W_{xz}$ and $W_{yz}$ are of order $\epsilon$),
- the Kirchhoff assumptions are used, that is
  
  1. the transverse normal stress is small compared to the other normal stress components and may be neglected,
Problem Definition and Governing Equations

In Chapter 5, the effect of the load conditions at the shell edge is taken into account in an Extended Analytical Load-Transmission method to determine the dependence of the in-plane bending and circumferential shear restraint on the geometrical conditions at the edges for several configurations and materials. Results are given in the final Chapter 6.

Figure 2.1: Shell geometry, coordinate system and applied loading

2. normals to the undeformed middle surface remain straight and normal to the deformed middle surface and suffer no extension.

For shells consisting of composite laminae it may be necessary to include the effect of transverse shear deformation, because the transverse shear stiffness is usually very small as compared to the in-plane stiffness (Vinson and Chou, 1974). The possibility of incorporating transverse shear deformation within the framework of the present approach, is outlined in Appendix A.

In addition, the governing equations are based on the assumptions made by Donnell (1933):
- the term containing the transverse shear stress resultant $Q_y$ is neglected in the force equilibrium equation in $y$-direction, or equivalently,
- the displacement term $v$ is neglected in the expression for the rotation $\beta_y$ (see Appendix A).

For static problems Koiter (1966) has shown that under the Donnell assumptions, terms containing quadratic displacements can be omitted in the potential energy expression of the applied load, i.e., the applied loads can be treated as dead loads. It is well known for buckling problems that if the number of circumferential waves $n$ of the buckling pattern becomes small ($n \leq 4$, say), while also the circumferential bending contribution in the strain energy is important as compared to the axial bending contribution (Calladine, 1983), the Donnell equations lose accuracy. The restrictions of the Donnell equations in the nonlinear range have been discussed for static (buckling) problems by Koiter (1966) (see also Hunt et al., 1986). Using Donnell equations one may miss certain relevant quartic terms in the potential energy expression, which contribute to the stabilizing behaviour at large deflections.
2.1 Problem Definition

The error in the linear frequency due to the use of Donnell equations in dynamic problems is small for high circumferential wave numbers $n$. It should be noted that for dynamic problems the maximum error does not necessarily occur for the lowest $n$ (El-Raheb and Babcock, 1981). Donnell-type equations for both linear and nonlinear dynamic problems have been discussed by Bogdanovich (1993).

Other assumptions used in the analysis are:

- In-plane inertia of the (predominantly) radial modes is neglected in the governing equations. The error in the frequency due to neglecting in-plane inertia is proportional to $1/n^2$ for high $n$. In the analyses in Chapter 4 the in-plane inertia of both the fundamental axial and the fundamental torsional mode is taken into account approximately. Rotatory inertia is also neglected in the analysis. Rotatory inertia may be important for deformations with small wavelengths, i.e. in the cases in which transverse shear deformation may also be significant (Appendix A).

- Classical lamination theory is employed (Ashton et al., 1969). The constitutive equations for a layered anisotropic shell are used, where the layers are assumed to be orthotropic and their principal axes can be oriented in arbitrary directions. The effect of stiffening elements, rings in circumferential direction and stringers in axial direction, is included via a smeared stiffener approach (Baruch and Singer, 1963).

- Linearly elastic material behaviour is assumed.

- Thermal and hygrothermal effects, i.e. moisture effects at high temperatures (Vinson and Sierakowski, 1987), are not considered.
Problem Definition and Governing Equations

• In the analysis of flutter in supersonic flow (Chapter 5), piston theory is used (see e.g. Olson and Fung, 1966). At high supersonic speeds, there will be an increased temperature due to aerodynamic heating. The influence of (nonuniform) aerodynamic heating, which can be important for the static response (e.g. Birman et al., 1988), is not taken into account in the present work.

The basic equations for a layered anisotropic shell, namely

• the nonlinear strain-displacement relations,
• the constitutive equations,
• the equations of motion,

can be found in Appendix A. The equations of motion can be derived from the strain-displacement relations and the appropriate energy and work expressions by application of Hamilton’s principle. The equations obtained are reduced to a set of governing equations in terms of the radial displacement $W$ and a stress function $F$. These equations, together with the corresponding boundary conditions, form the basis of the present analyses.

2.2 Governing equations

The equations governing the nonlinear dynamic behaviour of a cylindrical shell vibrating about a nonlinear static state will be presented in this section. It is assumed that the radial displacement $W$ is positive inward (see Fig. 2.1). Introducing an Airy stress function $F$ as $N_x = F_{yy}$, $N_y = F_{xx}$, and $N_{xy} = -F_{xy}$ (see Fig. 2.2), then the Donnell-type nonlinear imperfect shell equations for a general anisotropic material can be written as (see Appendix A)

$$L_{A^*}(F) - L_{B^*}(W) = -\frac{1}{R}W_{xx} - \frac{1}{2}L_{NL}(W, W + 2\bar{W})$$

(2.1)

$$L_{B^*}(F) + L_{D^*}(W) = L_{NL}(F, W + \bar{W}) + p - \rho h W_{tt}$$

(2.2)

where the variables $W$ and $F$ depend on $x$, $y$ and the time $t$, $R$ is the shell radius, $\rho h W_{tt}$, is the radial inertia term, $\bar{\rho}$ is the (averaged) specific mass of the laminate, $h$ is the (reference) shell thickness, and $p$ is the (effective) radial pressure (positive inward), which can depend on time. The 4th order linear differential operators

$$L_{A^*}(\cdot) = A_{22}^*(\cdot)_{xxxx} - 2A_{26}^*(\cdot)_{xxyz} + (2A_{12}^* + A_{66}^*)\cdot_{xyy}$$

(2.3)

$$L_{B^*}(\cdot) = B_{21}^*(\cdot)_{xxxx} + (2B_{26}^* - B_{66}^*)\cdot_{xxyz} + (B_{12}^* + B_{22}^* - 2B_{66}^*)\cdot_{xyy}$$

(2.4)

$$L_{D^*}(\cdot) = D_{11}^*(\cdot)_{xxxx} + (2D_{16}^* - D_{66}^*)\cdot_{xxyz} + (D_{12}^* + D_{21}^* - 2D_{66}^*)\cdot_{xyy}$$

(2.5)
2.2 Governing equations

depend on the stiffness properties of the laminate, and the nonlinear operator defined by

\[
L_{NL}(S, T) = S_{xx} T_{yy} - 2S_{xy} T_{xxy} + S_{yy} T_{xxx}
\]  

(2.6)

reflects the geometric nonlinearity. The stiffness parameters \( A_{ij}, B_{ij}, \) and \( D_{ij} \) \((i, j = 1, 2, 6)\) are given in Appendix A. Equation (2.1) guarantees the compatibility of the strains and the radial displacement field. Equation (2.2) is the equation of motion (dynamic equilibrium equation) in radial direction.

The shell can be loaded both statically and dynamically by the basic loads axial compression, radial pressure and torsion. Both \( W \) and \( F \) are expressed as a superposition of two states of displacement and stress

\[
W = \bar{W} + \hat{W}
\]

(2.7)

\[
F = \bar{F} + \hat{F}
\]

(2.8)

where \( \bar{F} \) and \( \bar{W} \) are the stress function and normal displacement of the static, geometrically nonlinear state which develops under the application of a static load on the imperfect shell, while \( \hat{F} \) and \( \hat{W} \) are the stress function and normal displacement of the dynamic state corresponding to the large amplitude vibration about the static state. The Donnell-type equations governing the nonlinear static state of an imperfect anisotropic cylindrical shell become

\[
LA^*(\bar{F}) - LB^*(\bar{W}) = -\frac{1}{R} \bar{W}_{xxx} - \frac{1}{2} L_{NL}(\bar{W}, \bar{W} + 2\bar{W})
\]

(2.9)

\[
LB^*(\bar{F}) + LD^*(\bar{W}) = -\frac{1}{R} \bar{F}_{xxx} + L_{NL}(\bar{F}, \bar{W} + \bar{W}) + \bar{p}
\]

(2.10)

where \( \bar{p} \) is the static radial loading, and the equations governing the nonlinear dynamic state can be written as

\[
LA^*(\hat{F}) - LB^*(\hat{W}) = -\frac{1}{R} \hat{W}_{xxx} - \frac{1}{2} L_{NL}(\hat{W}, \hat{W})
- \frac{1}{2} L_{NL}(\hat{W}, \hat{W} + 2\hat{W}) - \frac{1}{2} L_{NL}(\bar{W}, \bar{W})
\]

(2.11)

\[
LB^*(\hat{F}) + LD^*(\hat{W}) = -\frac{1}{R} \hat{F}_{xxx} + L_{NL}(\hat{F}, \hat{W}) + \bar{L}_{NL}(\hat{F}, \hat{W} + \hat{W})
+ L_{NL}(\bar{F}, \bar{W}) - \bar{p}\tilde{h}\tilde{W}_{x} + \hat{p}
\]

(2.12)

where \( \hat{p} \) is the dynamic radial loading. This may be a specified load, explicitly given as a function of time, or a load which implicitly depends on time, as in the case of flutter (see the next section). In the dynamic (i.e. time-dependent, in general nonconservative) case we distinguish between nonstationary loads (loads which are specified functions of time) and stationary loads (loads which do not explicitly depend on time).
Damping terms have not been included in the dynamic equations presented. To solve the above sets of partial differential equations, in subsequent chapters the differential equations will be reduced to a system with a finite number of degrees of freedom by using a number of assumed spatial modes in a Galerkin or variational procedure. Damping is introduced via viscous modal damping terms in the equations of motion of the discretized system.

2.3 Boundary conditions and applied loading

In this section, the following cases are distinguished:

- static state (prebuckling, previbration, preflutter)
- dynamic buckling (buckling under step loading)
- parametric excitation (vibration buckling under pulsating loads)
- nonlinear vibrations
- flutter (instability in supersonic flow)

2.3.1 Pressure loading

In the static state, i.e. the time-independent case, the radial pressure load is conservative. The pressure loading is assumed to be constant over the shell surface and is therefore axisymmetric. The net pressure (positive inwards) is the difference between the external and internal pressure and will be denoted by $p_e$, hence

$$\bar{p} = p_e$$ (2.13)

If the net pressure is directed outwards (i.e. is negative) then $p_e = -p_i$, where $p_i$ is the net internal pressure.

The nonstationary load in the case of nonlinear vibrations is given by

$$\dot{p} = q = \dot{q}(x, y) \cos \omega t$$ (2.14)

where $q$ is the specified radial loading (with spatial distribution of the vibration mode).

In the case of dynamic buckling and parametric excitation the nonstationary pressure (constant over the shell surface) is given by

$$\bar{p} = p_0 f(t)$$ (2.15)

where $p_0$ is a constant and $f(t)$ a specified function of time. For step loading (the dynamic buckling case) $f(t) = u(t)$, where $u(t)$ is the unit step function, and for a pulsating load (the parametric excitation case) $f(t) = \cos \Omega_e t$, where $\Omega_e$ is the excitation frequency.
2.3 Boundary conditions and applied loading

In the case of flutter, the outer surface of the shell is exposed to a high Mach number supersonic flow directed parallel to its axis. In the present analysis, the aerodynamic pressure is obtained from linear piston theory, which is usually applied and the simplest theory available (see e.g. Olson and Fung, 1966). An even simpler formulation, the Ackeret theory, is obtained if in the piston theory aerodynamic damping is neglected.

The aerodynamic loading is a stationary loading. The flow conditions are characterized by $p_{\infty}$, the free-stream static pressure, the free-stream speed of sound $a_{\infty}$, the free-stream Mach number of the airstream (the ratio of the free-stream velocity $U_{\infty}$ to its speed of sound) $M_{\infty} = U_{\infty}/a_{\infty}$, and the gas constant $\gamma$. The aerodynamic pressure loading due to the external supersonic flow in positive axial direction of the shell depends on the velocity and deflection of the shell. In the first-order, high Mach number approximation of the linear potential flow theory, the aerodynamic loading $p_{ae}$ is given by (Barr and Stearman, 1969)

$$p_{ae} = \frac{2q}{\beta} \left( W_{xz} + W_{z} \right) + \frac{1}{U_{\infty}} \frac{\beta^2 - 1}{\beta^2} W_{t} - \left[ \frac{1}{2\beta R} W \right]$$

where $q = \frac{\gamma}{2} p_{\infty} M_{\infty}^2$ is the dynamic pressure and $\beta^2 = M_{\infty}^2 - 1$.

Neglecting the curvature correction term in square brackets, this expression reduces to linear piston theory for a Mach number sufficiently large as compared to one (cf. Barr and Stearman, 1969),

$$p_{ae} = \gamma p_{\infty} \left\{ \frac{1}{a_{\infty}} W_{st} + M_{\infty} (W_{xz} + W_{z}) \right\}$$

The time-independent and time-dependent component of the aerodynamic loading become, respectively,

$$\tilde{p}_{ae} = \gamma p_{\infty} M_{\infty} (\tilde{W}_{xz} + \tilde{W}_{z})$$

$$\dot{p}_{ae} = \gamma p_{\infty} (\frac{1}{a_{\infty}} \tilde{W}_{st} + M_{\infty} \tilde{W}_{z})$$

Notice that the aerodynamic loading has a static component which depends on the imperfection and deflection of the shell. The time-dependent component consists of a velocity-dependent, dissipative load, and a velocity-independent, circulatory load. For the static radial pressure we obtain

$$\tilde{p} = \tilde{p}_{ae} + p_c = \tilde{p}_{ae} - p_i$$

where $p_i$ is the net (i.e. relative to $p_{\infty}$) shell internal pressure, while for the dynamic loading we have

$$\dot{p} = \dot{p}_{ae}$$

In the case of flutter, the character of the governing equations changes since the problem is no longer self-adjoint. This is reflected by the odd-derivative term proportional to the aerodynamic pressure.
Problem Definition and Governing Equations

The possibilities in the different cases for the radial loading \( p \) are summarized in Table 2.1.

<table>
<thead>
<tr>
<th>case</th>
<th>( N_0 )</th>
<th>( T_0 )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>static state</td>
<td>( \tilde{N}_0 )</td>
<td>( \tilde{T}_0 )</td>
<td>( p_e )</td>
</tr>
<tr>
<td>preflutter</td>
<td>( \tilde{N}_0 )</td>
<td>( \tilde{T}_0 )</td>
<td>( \tilde{p}_{ae} + p_e )</td>
</tr>
<tr>
<td>dynamic buckling</td>
<td>( \tilde{N}_0 u(t) )</td>
<td>( \tilde{T}_0 u(t) )</td>
<td>( \tilde{p}_0 u(t) )</td>
</tr>
<tr>
<td>parametric excitation</td>
<td>( \tilde{N}_0 \cos \Omega_c t )</td>
<td>( \tilde{T}_0 \cos \Omega_c t )</td>
<td>( \dot{q}(x, y) \cos \omega t )</td>
</tr>
<tr>
<td>nonlinear vibration</td>
<td>( \tilde{N}_0 )</td>
<td>( \tilde{T}_0 )</td>
<td>( \tilde{p}_{ae} )</td>
</tr>
<tr>
<td>flutter</td>
<td>( \tilde{N}_0 )</td>
<td>( \tilde{T}_0 )</td>
<td>( \tilde{p}_{ae} )</td>
</tr>
</tbody>
</table>

*Table 2.1: Summary of loads \( u(t) = \text{unit step function} \)*

2.3.2 In-plane loading and boundary conditions

In the static state, at the shell edges the basic axisymmetric in-plane loads, axial compression and torsion, are applied. They are assumed to be uniform, i.e. axisymmetric in the sense that they do not depend on the circumferential coordinate,

\[
N_0 = \tilde{N}_0 \\
T_0 = \tilde{T}_0
\]

In the case of dynamic buckling and parametric excitation the nonstationary loading consists of the basic loads, axial compression and torsion, specified as functions of time,

\[
N_0 = \tilde{N}_0 f(t) \\
T_0 = \tilde{T}_0 f(t)
\]

where \( \tilde{N}_0 \) and \( \tilde{T}_0 \) are constants and \( f(t) \) a specified function of time. For step loading (the dynamic buckling case) \( f(t) = u(t) \), where \( u(t) \) is the unit step function, and for a pulsating load (the parametric excitation case) \( f(t) = \cos \Omega_c t \), where \( \Omega_c \) is the excitation frequency. The different possibilities for the in-plane loading are summarized in Table 2.1.

The in-plane displacements \( u^e \) and \( v^e \) at the shell edges can be written in the following form:

\[
\begin{bmatrix}
  u^e \\
  v^e
\end{bmatrix} =
\begin{bmatrix}
  u_0^e \\
  v_0^e
\end{bmatrix} + \sum_{n=1}^{\infty} \left\{ \begin{bmatrix}
  u_{1n}^e \\
  v_{1n}^e
\end{bmatrix} \cos n\theta + \begin{bmatrix}
  u_{2n}^e \\
  v_{2n}^e
\end{bmatrix} \sin n\theta \right\}
\]

while the stress resultant \( N_x^e \) and \( N_{xy}^e \) at the shell edges can be written as
2.3 Boundary conditions and applied loading

\[
\begin{bmatrix}
N_{e}^x \\
N_{e}^y \\
N_{e}^{xy}
\end{bmatrix} = \begin{bmatrix}
N_{x0}^e \\
N_{y0}^e \\
N_{y0}^e
\end{bmatrix} + \sum_{n=1}^{\infty} \left\{ \begin{bmatrix}
N_{x1n}^e \\
N_{y1n}^e \\
N_{y2n}^e
\end{bmatrix} \cos n\theta + \begin{bmatrix}
N_{x2n}^e \\
N_{y2n}^e
\end{bmatrix} \sin n\theta \right\}
\] (2.27)

By definition, the applied compressive force and torque at the shell edges correspond to in-plane stress resultants at the edges which have been averaged over the edge plane,

\[
N_0 = -\frac{1}{2\pi R} \int_0^{2\pi R} N_x dy = \text{specified}
\] (2.28)

\[
T_0 = \frac{1}{2\pi R} \int_0^{2\pi R} N_{xy} dy = \text{specified}
\] (2.29)

In the present work, these stress resultants are assumed to be prescribed. They correspond with the constant parts (uniform in the circumferential direction) in the general expressions for the stress resultants (2.27).

There will be an averaged displacement \(u_0\) and an averaged twist \(v_0/R\) of the edge planes \(x = 0\) and \(x = L\) relative to each other:

\[
u_0 = \frac{1}{2\pi R} \int_0^{2\pi R} \int_0^L \nu_x dx dy
\] (2.30)

\[
\frac{v_0}{R} = \frac{1}{2\pi R} \int_0^{2\pi R} \int_0^L \nu_x dx dy
\] (2.31)

These displacements correspond with the constant parts (uniform in the circumferential direction) of the in-plane displacements in (2.26). The associated boundary conditions are often referred to in the literature as the ‘movability’ conditions of the edge planes. Since the averaged in-plane membrane stresses are prescribed, the average displacements of the shell edges, relative to each other, are not constrained. For definiteness, the loads are assumed to be applied at \(x = L\), and the circumferentially averaged edge plane displacements at \(x = 0\) are assumed to be zero. The in-plane loading is assumed to be specified. If the value of the axial compression or torsional load is not given, it is implicitly assumed to be zero.

| SS-1 | \(N_x = -N_0\) | \(N_{xy} = T_0\) | \(W = 0\) | \(M_x = -N_0q\) |
| SS-2 | \(u = 0\) | \(N_{xy} = T_0\) | \(W = 0\) | \(M_x = -N_0q\) |
| SS-3 | \(N_x = -N_0\) | \(v = 0\) | \(W = 0\) | \(M_x = -N_0q\) |
| SS-4 | \(u = 0\) | \(v = 0\) | \(W = 0\) | \(M_x = -N_0q\) |
| C-1 | \(N_x = -N_0\) | \(N_{xy} = T_0\) | \(W = 0\) | \(W_{ix} = 0\) |
| C-2 | \(u = 0\) | \(N_{xy} = T_0\) | \(W = 0\) | \(W_{ix} = 0\) |
| C-3 | \(N_x = -N_0\) | \(v = 0\) | \(W = 0\) | \(W_{ix} = 0\) |
| C-4 | \(u = 0\) | \(v = 0\) | \(W = 0\) | \(W_{ix} = 0\) |

Table 2.2: Standard boundary conditions
The standard boundary conditions are denoted as in Table 2.2, where \( q \) is the axial load eccentricity, measured from the shell midsurface (positive inward). For the first-order state problems treated in Chapter 5, the (linear) buckling or linear vibration problem, the boundary conditions in Table 2.2 become homogeneous. For the in-plane boundary conditions a distinction is made between circumferentially constant and circumferentially varying in-plane displacements. The zero displacements in the standard boundary conditions are not to be interpreted as “immovability” conditions, in the aforestated sense, but as constraints for the circumferentially varying displacements, relative to the average displacement of the ‘moving’ edge plane. It is noted that the conditions \( u = 0 \) and \( v = 0 \) in the standard boundary conditions are equivalent to \( u_{yy} = 0 \) and \( v_{yy} = 0 \), respectively, which can be used to express these conditions in terms of the variables \( W \) and \( F \) (the so-called reduced boundary conditions). Average in-plane loads are assumed to be specified in the fundamental state. Average in-plane loads which correspond to an incremental (first-order) or higher-order state are implicitly assumed to vanish.

Two consequences of the distinction between constant and circumferentially varying displacements deserve particular attention. First, when axisymmetric deformation occurs, for example for the zero-th and the second-order state in a nonlinear (perturbation) analysis (see Chapter 5), or in the linearized vibration of a cylinder with asymmetric imperfection with the shape of the vibration mode, this will involve either in-plane displacements or in-plane stresses at the shell edges. Secondly, for anisotropic shells there may be a coupling between the axial and torsional deformation. For both cases it should be realized that in the present formulation the circumferentially constant displacements of the edge planes are assumed to be free.

The more general case of elastic edge restraint is modelled by introducing the elastic stiffness parameters \( k_u, k_v, k_w \) and \( k_{ww} \) in the boundary conditions (for circumferentially varying displacements) as follows:

\[
N_x + k_u u = 0 \\
N_{xy} + k_v v = 0 \\
M_{xx} + (M_{xy} + M_{yx})_y + N_x(W_{xx} + \bar{W}_{1x}) + N_{xy}(W_{yy} + \bar{W}_{1y}) + k_w W = 0 \\
M_x + k_{ww} W_{sx} = 0
\]

where the sign of the stiffness parameters depends on the shell edge (\( x = 0 \) or \( x = L \)). Finally, it is noted that the cylindrical shell geometry requires the periodicity of all variables in the circumferential direction of the shell.
Chapter 3

Level-1 Steady State Vibration Analysis

3.1 Introduction

In this chapter, the steady state nonlinear flexural vibration behaviour of imperfect anisotropic cylindrical shells under harmonic lateral excitation is analysed via a Level-1 Analysis or Simplified Analysis (cf. Chapter 1). The early investigations of the nonlinear vibration behaviour of shells belong to the class of Level-1 Analyses. The present work forms an extension of these investigations from isotropic and orthotropic shells to laminated (anisotropic) shells.

Deviations from the perfect cylindrical form due to the fabrication process (the so-called geometric imperfections) can have a considerable influence on the dynamic behaviour of cylindrical shells. The same holds for a stress in the previbration state due to an applied (static) loading. Watawala and Nash (1982) studied the effect of asymmetric imperfections on the nonlinear vibrations of isotropic cylinders. Hoi (1983) developed a model to investigate the effect of axisymmetric and asymmetric imperfections on the linearized vibrations of axially loaded ring- and stringer-stiffened cylindrical shells, while Liu (1988) studied the nonlinear vibrations of such shells.

To study characteristic aspects of the nonlinear behaviour, a small number of deflection modes is used in a Galerkin procedure. Two asymmetric modes are included in the deflection function which are circumferentially ninety degrees out-of-phase between each other, the directly excited 'driven mode' and its 'companion mode'. The coupled mode response of these modes can be interpreted as a travelling wave pattern in the circumferential direction of the shell. The two assumed asymmetric modes have $m$ axial waves. The axisymmetric mode $C_0 \cos \frac{m \pi z}{L}$, which satisfies the strong coupling condition $i = 2m$ with the two asymmetric modes (cf. Liu, 1988), is included in the assumed deflection function. This axisymmetric mode plays an essential role in the nonlinear behaviour.

The shell is statically loaded by axial compression, radial pressure, and torsion. The static state response is assumed to be affine to the given two-mode imperfection, which consists of an axisymmetric and an asymmetric mode. Nonlinear Donnell-type
governing equations are used and classical lamination theory is employed. Khot’s formulation (Khot and Venkayya, 1970) is used to account for a possible skewedness of the asymmetric modes. The modes approximately satisfy “simply supported” boundary conditions. Galerkin’s method is applied to solve for the static state. Galerkin’s method and the method of averaging are used in sequence to obtain frequency-amplitude curves for free and forced nonlinear vibrations.

3.2 Nonlinear static state

In this section a simplified two-mode analysis is presented for the static behaviour of anisotropic shells under axial compression, external pressure, and torsion, which will subsequently be used in combination with a simplified model for (nonlinear) vibrations. Assuming displacement modes, affine to the two-mode imperfection shape, the corresponding stress function is determined from the compatibility equation. Substituting the assumed displacements and the corresponding stress function in the out-of-plane equilibrium equation, and application of Galerkin’s method, gives a set of nonlinear algebraic equations for the unknown amplitudes. To gain insight into the effect of imperfections on the static and dynamic behaviour of shells, an idealized imperfection model is used consisting of one axisymmetric mode $\dot{\xi}_1 \cos \frac{2m \pi x}{L}$ and one asymmetric mode having $m$ axial waves. These modes satisfy the relevant coupling condition $i = 2m$ between each other. The radial displacement is assumed to be affine to the given two-mode imperfection, imperfection:

$$W/h = \dot{\xi}_1 \cos \frac{2m \pi x}{L} + \dot{\xi}_2 \sin \frac{m \pi x}{L} \cos \frac{n}{R} (y - \tau_K x)$$

(3.1)

static state:

$$W/h = \ddot{\xi}_0 + \ddot{\xi}_1 \cos \frac{2m \pi x}{L} + \ddot{\xi}_2 \sin \frac{m \pi x}{L} \cos \frac{n}{R} (y - \tau_K x)$$

(3.2)

where $m$ denotes the number of half waves in axial direction, $n$ is the number of full waves in the circumferential direction of the imperfection mode and static response mode, and $\tau_K$ and $\tau_K$ are Khot’s skewedness parameters (Khot and Venkayya, 1970), introduced to account for the skewedness of the asymmetric modes which appears under torsional loading and which may occur due to torsion-bending coupling in the constitutive equations. In these cases, the deflection modes have circumferential nodal lines which spiral from one end of the shell to the other. In the following, the imperfection and response are assumed to be affine, and hence the skewedness parameter of the imperfection is taken to be equal to the skewedness parameter of the response, $\tau_K = \tau_K$. Notice that the response mode of the static state satisfies the classical “simply supported” (SS-3) boundary conditions only approximately because of the inclusion of the axisymmetric terms. Moreover, this boundary condition is not satisfied for cases in which the skewedness parameter $\tau_K$ is not equal to zero. By substituting the assumed displacement modes into the compatibility equation
(2.9) and solving the equation for a particular solution $\tilde{F}_p$ of the stress function $\tilde{F}$ one obtains

\[
\tilde{F}_p = \tilde{f}_{n,1} \sin(\ell_m x - \ell_n y) + \tilde{f}_{n,2} \sin(\ell_p x + \ell_n y) + \tilde{f}_{m+1} \sin(\ell_{m+2} x - \ell_n y) + \tilde{f}_{2n} \cos(\ell_{m-p} x - \ell_{2n} y) + \tilde{f}_0 \cos(\ell_m x + \ell_{2p} y)
\]  

(3.3)

where $\ell_m = \frac{m \pi}{L} + \frac{m \pi}{L'}$ and $\ell_n = \frac{n \pi}{L}$, $\ell_{m+2p} = \ell_m + \ell_{2p}$, etc. and where the coefficients $\tilde{f}_{n,i}$, $\tilde{f}_{2n}$ and $\tilde{f}_0$ depend on the radial displacement amplitudes, the imperfections, the wave numbers, and the geometry and stiffness parameters. The coefficients are listed in Appendix G (see Section G.3). The stress function $\tilde{F}$ can now be written as

\[
\tilde{F} = \frac{1}{2} \tilde{N}_x^* y^2 + \frac{1}{2} \tilde{N}_y^* x^2 - \tilde{N}_{xy}^* xy + \tilde{F}_p
\]  

(3.4)

The constant stresses $\tilde{N}_x^*$ and $\tilde{N}_{xy}^*$ can be related via the boundary conditions to the averaged in-plane loads,

\[
\tilde{N}_x^* = -\tilde{N}_0, \tilde{N}_{xy}^* = \tilde{T}_0
\]  

(3.5)

One may use the linear relation between $\tilde{N}_y^*$ and $\tilde{p}$ obtained from equilibrium considerations a priori to relate $\tilde{p}$ to $\tilde{\xi}_2$ and the applied loads immediately. Alternatively, the corresponding equilibrium equation can be derived consistently via a Galerkin procedure. Both methods will be presented in the following.

In the first method, $\tilde{N}_y^*$ is obtained a priori from the equilibrium condition

\[
\tilde{N}_y^* = -pR
\]  

(3.6)

For a complete cylindrical shell it is necessary that all variables satisfy circumferential continuity conditions. It is important to remember that it is not sufficient that the stress function $\tilde{F}$ is continuous and periodic in the $y$-direction. The continuity requirement for $\tilde{v}$ is given by

\[
\tilde{v}(x,y) = \tilde{v}(x,y + 2\pi R)
\]  

(3.7)

This condition can be written as

\[
\int_0^{2\pi R} \tilde{v}_{xy} dy = 0
\]  

(3.8)

This relation is called the circumferential periodicity condition. Using the kinematic relations and the constitutive equations one can express $\tilde{v}_{xy}$ in terms of $\tilde{F}$ and $\tilde{W}$, yielding

\[
\int_0^{2\pi R} \{\tilde{\xi}_y + \frac{\tilde{W}}{R} - \frac{1}{2}(\tilde{W}_{yy} + 2\tilde{W}_{xy})\tilde{W}_{yy} dy\} = 0
\]  

(3.9)
The term $\tilde{\xi}_0$ can be obtained from the circumferential periodicity condition by substituting for $\tilde{W}$, $\tilde{W}$, and $\tilde{F}$ from Eqs. (3.1) to (3.4) and carrying out the $y$-integration. This yields

$$\tilde{\xi}_0 = C_{f2}\tilde{\xi}_2 + C_{f1}\tilde{\xi}_1 + C_{f0}$$  (3.11)

In the second, alternative approach this last equation is derived consistently via Galerkin's method (cf. Chapter 4). In this method, $N_y^*$ is obtained from the circumferential periodicity condition in terms of $\xi_0$, $\xi_1$, $\xi_2$, and $N_y^*$ (see Appendix G, Section G.3). Substitution of the given imperfection $\tilde{W}$ (Eq. (3.1)), assumed radial displacement $\tilde{W}$ (Eq. (3.2)), and the solution of the stress function (Eq. (3.4)) into the out-of-plane equilibrium equation (2.10), and application of Galerkin's method, with $\partial\tilde{W}/\partial\xi_0$, $\partial\tilde{W}/\partial\xi_1$, and $\partial\tilde{W}/\partial\xi_2$ as weighting functions, gives equations for the unknown amplitudes $\xi_0$, $\xi_1$, and $\xi_2$ of the following form:

$$\tilde{\xi}_0 = C_{f2}\tilde{\xi}_2 + C_{f1}\tilde{\xi}_1 + C_{f0}$$  (3.12)

$$\tilde{C}_{10}\tilde{\xi}_1 + \tilde{C}_{01}\tilde{\xi}_2 + \tilde{C}_{11}\tilde{\xi}_1\tilde{\xi}_2 + \tilde{C}_{02}\tilde{\xi}_2^2 + \tilde{C}_{12}\tilde{\xi}_1\tilde{\xi}_2 + \tilde{C}_{00} = 0$$  (3.13)

$$\tilde{D}_{10}\tilde{\xi}_1 + \tilde{D}_{01}\tilde{\xi}_2 + \tilde{D}_{20}\tilde{\xi}_2^2 + \tilde{D}_{11}\tilde{\xi}_1\tilde{\xi}_2 + \tilde{D}_{21}\tilde{\xi}_1^2\tilde{\xi}_2 + \tilde{D}_{00} = 0$$  (3.14)

where the constant coefficients $\tilde{C}_{ij}$, $\tilde{D}_{ij}$, and $\tilde{D}_{ij}$ depend on the shell geometry, stiffness parameters, imperfections and wave numbers. The coefficients are listed in Appendix G (see Eqs. (G.32) and (G.37)). Notice that the first equilibrium equation, Eq. (3.12), is identical to the relation obtained in the first method presented, but in this alternative method appears as the result of the Galerkin procedure, without using the equilibrium equation (3.6) a priori. The second and third equation constitute a set of two coupled algebraic equations in the amplitudes $\xi_1$ and $\xi_2$. For a specified imperfection and static loading one can solve this set for the unknown amplitudes.

### 3.3 Nonlinear vibrations about a nonlinear static state

In this section, the governing equations for the nonlinear dynamic state will be presented. The vibration behaviour is modelled by assuming an axisymmetric and two asymmetric vibration modes. The static state response is assumed to be affine to the given two-mode imperfection. Application of Galerkin's procedure to eliminate the
3.3 Nonlinear vibrations about a nonlinear static state

spatial dependence and the method of averaging to eliminate the time dependence results in two coupled nonlinear algebraic equations for the average vibration amplitudes $\hat{A}$ and $\hat{B}$. Results for nonlinear vibrations which have been published earlier by other investigators using analogous methods are reproduced to check the computer program based on the present formulation. Finally, results for the linearized and nonlinear vibrations of imperfect anisotropic shells are presented.

To investigate the important phenomena of the nonlinear (large amplitude) vibrations of statically loaded imperfect anisotropic cylindrical shells, the following expressions for the imperfection and response modes are used (cf. Eqs. (3.1) and (3.2)):

imperfection:

$$W/h = \xi_1 \cos \frac{2m\pi x}{L} + \xi_2 \sin \frac{m\pi x}{L} \cos \frac{n}{R}(y - \tau_K x)$$

static state:

$$W/h = \hat{\xi_1} + \hat{\xi_1} \cos \frac{2m\pi x}{L} + \hat{\xi_2} \sin \frac{m\pi x}{L} \cos \frac{n}{R}(y - \tau_K x)$$

dynamic state:

$$W/h = C_0(t) + C_1(t) \cos \frac{2m\pi x}{L} + A(t) \sin \frac{m\pi x}{L} \cos \frac{\ell}{R}(y - \tau_K x) + B(t) \sin \frac{m\pi x}{L} \sin \frac{\ell}{R}(y - \tau_K x)$$

(3.15)

where $C_0(t)$, $C_1(t)$, $A(t)$, and $B(t)$ are the unknown time-dependent coefficients of the displacement modes, and where $n$ and $\ell$ are the number of full waves in the circumferential direction of the imperfection and the vibration mode, respectively. Notice that, as for the static case, the response modes of the dynamic state do not satisfy the classical simply supported boundary conditions exactly. The imperfection and the static and dynamic response are assumed to be affine, and hence the skewedness parameter of the imperfection is taken to be equal to the skewedness parameter of the response, $\tau_K = \tau_K$.

The radial displacement Eq. (3.15) contains two asymmetric modes. The mode with time-dependent coefficient $A(t)$ (driven mode) is excited directly by the external excitation, which has the same spatial distribution and is assumed to be harmonic in time,

$$q = Q_{m\ell r} \sin \frac{m\pi x}{L} \cos \frac{\ell}{R}(y - \tau_K x) \cos \omega t$$

(3.16)

where $\omega$ is the excitation frequency, and $Q_{m\ell r}$ is the excitation amplitude (a constant). The mode with time-dependent coefficient $B(t)$ (companion mode) can respond due to a nonlinear coupling with the driven mode. This coupling is classified in nonlinear vibration terminology as a one-to-one internal (autoparametric) resonance (Chin and Nayfeh, 1996). The physical background is that the stress in
the circumferential direction due to the large amplitude motion (parametrically) excites the companion mode. The driven mode and companion together give a coupled mode response, which can be interpreted as a traveling wave pattern in the circumferential direction of the shell (Evensen, 1965, 1967).

The displacement dependent part of the compatibility equation (2.11) consists of several contributions,

1. a linear part, \(-\frac{1}{2} W_{xx}\), corresponding to the change of Gaussian curvature which occurs because of the coupling between i) the curvature with respect to the x-direction due to the (dynamic) deflection and ii) the initial curvature of the shell,

2. a linear part, \(L_B(W)\), due to the coupling of stretching and bending which occurs for orthotropic and anisotropic shells and

3. nonlinear parts, \(-\frac{1}{2} L_{NL}(W, \dot{W}), -\frac{1}{2} L_{NL}(\dot{W}, \ddot{W})\), and \(-\frac{1}{2} L_{NL}(W, \dddot{W})\) associated with the change of Gaussian curvature due to the interactions of i) the static deflection curvature, ii) the curvature due to the imperfection, and iii) the dynamic deflection curvature.

Substituting the given imperfection \(W\) (Eq. (3.1)), the static solution \(\dot{W}\) (Eq. (3.2)) obtained earlier and the dynamic response \(W\) (Eq. (3.15)) into the dynamic compatibility equation (2.11) one obtains an inhomogeneous linear partial differential equation for the dynamic stress function \(F\). A particular solution can be obtained by substituting the following particular solution for \(F\), denoted as \(F_p\),

\[
F_p = \tilde{f}_{n,1} \sin(\ell_n y - \ell_{n+1} y) + \tilde{f}_{n,2} \sin(\ell_{n+1} y - \ell_{n} y) \\
+ \tilde{f}_{n,3} \sin(\ell_{n+2} y + \ell_{n} y) + \tilde{f}_{n,4} \sin(\ell_{n+2} y - \ell_{n} y) \\
+ \tilde{f}_{n,5} \sin(\ell_{n+3} y + \ell_{n} y) + \tilde{f}_{n,6} \sin(\ell_{n+3} y - \ell_{n} y) \\
+ \tilde{f}_{n,7} \cos(\ell_{n+1} y - \ell_{n+2} y) + \tilde{f}_{n,8} \cos(\ell_{n+1} y + \ell_{n+2} y) \\
+ \tilde{f}_{n,9} \cos(\ell_{n+2} y - \ell_{n+1} y) + \tilde{f}_{n,10} \cos(\ell_{n+2} y + \ell_{n+1} y) \\
+ \tilde{f}_{n,11} \cos(\ell_{n+3} y - \ell_{n+2} y) + \tilde{f}_{n,12} \cos(\ell_{n+3} y + \ell_{n+2} y) \\
+ \tilde{f}_{n,13} \cos(\ell_{n+4} y - \ell_{n+3} y) + \tilde{f}_{n,14} \cos(\ell_{n+4} y + \ell_{n+3} y) \\
+ \tilde{f}_{n,15} \cos(\ell_{n+5} y - \ell_{n+4} y) + \tilde{f}_{n,16} \cos(\ell_{n+5} y + \ell_{n+4} y) \\
+ \tilde{f}_{n,17} \cos(\ell_{n+6} y - \ell_{n+5} y) + \tilde{f}_{n,18} \cos(\ell_{n+6} y + \ell_{n+5} y) \\
+ \tilde{f}_{n,19} \cos(\ell_{n+7} y - \ell_{n+6} y) + \tilde{f}_{n,20} \cos(\ell_{n+7} y + \ell_{n+6} y) \\
+ \tilde{f}_{n,21} \cos(\ell_{n+8} y - \ell_{n+7} y) + \tilde{f}_{n,22} \cos(\ell_{n+8} y + \ell_{n+7} y) \\
+ \tilde{f}_{n,23} \cos(\ell_{n+9} y - \ell_{n+8} y) + \tilde{f}_{n,24} \cos(\ell_{n+9} y + \ell_{n+8} y) \\
+ \tilde{f}_{n,25} \cos(\ell_{n+10} y - \ell_{n+9} y) + \tilde{f}_{n,26} \cos(\ell_{n+10} y + \ell_{n+9} y) \\
+ \tilde{f}_{n,27} \cos(\ell_{n+11} y - \ell_{n+10} y) + \tilde{f}_{n,28} \cos(\ell_{n+11} y + \ell_{n+10} y) \\
+ \tilde{f}_{n,29} \cos(\ell_{n+12} y - \ell_{n+11} y) + \tilde{f}_{n,30} \cos(\ell_{n+12} y + \ell_{n+11} y) \\
+ \tilde{f}_{n,31} \cos(\ell_{n+13} y - \ell_{n+12} y) + \tilde{f}_{n,32} \cos(\ell_{n+13} y + \ell_{n+12} y) \\
+ \tilde{f}_{n,33} \cos(\ell_{n+14} y - \ell_{n+13} y) + \tilde{f}_{n,34} \cos(\ell_{n+14} y + \ell_{n+13} y) \\
+ \tilde{f}_{n,35} \cos(\ell_{n+15} y - \ell_{n+14} y) + \tilde{f}_{n,36} \cos(\ell_{n+15} y + \ell_{n+14} y) \\
+ \tilde{f}_{n,37} \cos(\ell_{n+16} y - \ell_{n+15} y) + \tilde{f}_{n,38} \cos(\ell_{n+16} y + \ell_{n+15} y) \\
+ \tilde{f}_{n,39} \cos(\ell_{n+17} y - \ell_{n+16} y) + \tilde{f}_{n,40} \cos(\ell_{n+17} y + \ell_{n+16} y) \\
+ \tilde{f}_{n,41} \cos(\ell_{n+18} y - \ell_{n+17} y) + \tilde{f}_{n,42} \cos(\ell_{n+18} y + \ell_{n+17} y) \\
+ \tilde{f}_{n,43} \cos(\ell_{n+19} y - \ell_{n+18} y) + \tilde{f}_{n,44} \cos(\ell_{n+19} y + \ell_{n+18} y)
\]

where \(\ell_n = \frac{m \pi}{L}, \ell_{n+1} = \ell_n + \ell_p = \ell_n + 2m \pi / L\), etc. into the compatibility equation and equating coefficients of like goniometric terms. The resulting linear
3.3 Nonlinear vibrations about a nonlinear static state

equations for the unknown coefficients of the stress function, which contain terms which are linear and quadratic in the time-dependent parts of the displacement modes, can be solved routinely. The coefficients are listed in Appendix G (see Section G.4). It is noted that one can obtain the same results for the stresses via a formulation with the in-plane equilibrium equations using the in-plane displacements $u$ and $v$ (Tamura, 1973). For a perfect isotropic shell, the particular solution for the stress function $\tilde{F}_p$ from Eq. (3.17) should reduce to the expression given by Evensen (1967),

$$\tilde{F}_p = a_1 \sin \alpha x (A \cos \beta y + B \sin \beta y) - a_2 (A^2 - B^2) \cos 2\beta y$$

$$- a_3 AB \sin 2\beta y + a_4 (A^2 + B^2) \sin 3\alpha x (A \cos \beta y + B \sin \beta y)$$

(3.18)

where $\alpha = m\pi / L$, $\beta = n / R$ and the coefficients $a_1, a_2, a_3$, and $a_4$ are defined in the Nomenclature. Notice that the form of the particular solution given by Evensen (1967) consists of a term, linear in the time-dependent part of the displacement mode and with an $(m, \ell)$ spatial distribution, a quadratic term with a $(0, 2\ell)$ distribution, and there is a cubic term with a $(3m, \ell)$ spatial distribution. A cubic term with an $(m, \ell)$ distribution is missing in this expression. However, the final expressions for the amplitude frequency relations given by Evensen seem to agree with the present ones.

The dynamic stress function $\tilde{F}$ can now be written as

$$\tilde{F} = \frac{1}{2} \tilde{N}_x^* y^2 + \frac{1}{2} \tilde{N}_y^* x^2 - \tilde{N}_{xy}^* xy + \tilde{F}_p$$

(3.19)

where the constant stress terms $\tilde{N}_x^*$ and $\tilde{N}_y^*$ are equal to prescribed averaged stress resultants at the shell edge,

$$\tilde{N}_x^* = -\tilde{N}_0 = 0, \tilde{N}_y^* = \tilde{T}_0 = 0$$

(3.20)

while in the present approach it will be assumed that

$$\tilde{N}_y^* = 0$$

(3.21)

The background of this assumption will be discussed later in this section.

It is important that all dynamic variables satisfy the circumferential periodicity condition. The continuity requirement for $\dot{v}$ can be derived by using the superposition of a static and a dynamic state discussed in Section 2.2, Eqs. (2.7) and (2.7). This procedure yields (see also the corresponding equations for the static state in Section 3.2)

$$\int_0^{2\pi R} \left\{ \dot{\varepsilon}_y + \frac{\dot{\mathbf{W}}}{R} - (\dot{\mathbf{W}}_{xy} + \dot{\mathbf{W}}_{yy}) \dot{\mathbf{W}}_{xy} - \frac{1}{2} \dot{\mathbf{W}}_{yy}^2 \right\} dy = 0$$

(3.22)

or
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\[ \int_0^{2\pi R} \left( A_{11} \dot{F}_{yy} + A_{22} \dot{F}_{xx} + A_{12} \dot{F}_{xy} - D_{11} \ddot{W}_{xx} - D_{12} \ddot{W}_{xy} - 2D_{22} \ddot{W}_{yy} \right) + \frac{\ddot{W}}{R} - (\dot{W}_{yy} + \dot{W}_{xy}) \dot{W}_{yy} - \frac{1}{2} \ddot{W}_{yy}^2 \, dy = 0 \]  
(3.23)

Substituting for \( \ddot{W}, \dddot{W}, \ddot{W} \) and \( \dddot{W} \) from Eqs. (3.1), (3.2), (3.15) and (3.17) and carrying out the \( y \)-integration yields

\[ C_0(t) = \frac{1}{8} \ell^2 \rho h \left\{ A(t)^2 + B(t)^2 + 2\delta_{n,t} A(t)(\hat{\xi}_2 + \hat{\xi}_3) \right\} \]  
(3.24)

Notice that this equation does not contain the coefficient \( C_1(t) \).

Essential for a satisfactory description of the nonlinear behaviour is the inclusion of certain axisymmetric modes in the assumed displacement function. In the present approach, following Evensen (1967), the time-dependent coefficients of both axisymmetric modes, \( C_0 \) and \( C_1 \), are eliminated by expressing them in terms of the time-dependent coefficients of the asymmetric modes, \( A \) and \( B \). To eliminate \( C_1 \), in the present approach it will be assumed that

\[ C_1(t) = -h^2 \rho h \left\{ A(t)^2 + B(t)^2 + 2\delta_{n,t} A(t)(\hat{\xi}_2 + \hat{\xi}_3) \right\} \]  
(3.25)

The background of this assumption will be explained later in this section. Notice that under this assumption, since

\[ C_1(t) = -C_0(t) \]  
(3.26)

Eq. (3.15) becomes

\[ \ddot{W}/h = C_0(t)(1 - \cos \frac{2m\pi x}{L}) + A(t) \sin \frac{m\pi x}{L} \cos \frac{\ell}{R} (y - \tau_k x) + B(t) \sin \frac{m\pi x}{L} \cos \frac{\ell}{R} (y - \tau_k x) \]  
(3.27)

which, introducing \( C(t) = 2C_0(t) \), can also be written as

\[ \ddot{W}/h = C(t) \sin \frac{m\pi x}{L} + A(t) \sin \frac{m\pi x}{L} \cos \frac{\ell}{R} (y - \tau_k x) + B(t) \sin \frac{m\pi x}{L} \cos \frac{\ell}{R} (y - \tau_k x) \]  
(3.28)

Thus the boundary condition \( \ddot{W} = 0 \) at the shell edges \( x = 0 \) and \( x = L \) is satisfied. This is exactly the form used by Evensen (1967).

It should be noted that in his analysis procedure, Evensen (1967) a priori relates \( C_1 \) to \( C_0 \) by satisfying the boundary condition \( \ddot{W} = 0 \) at the shell edge, and subsequently satisfies the periodicity condition, implicitly assuming that the constant stress term \( \tilde{N}_y = 0 \).
3.3 Nonlinear vibrations about a nonlinear static state

An alternative reasoning will be given in the following. In this line of thought the relations which have been used to eliminate the time-dependent coefficients of the axisymmetric modes, $C_0$ and $C_1$, are based on assumptions concerning the coefficients of the stress which are constant in the circumferential direction of the shell, $N^*_y$ and $f_0^* \cos \ell_{m+p} x$ (see Eq. (3.17)). The justification of these assumptions will be discussed in the following.

As shown earlier, expressions for the nonlinear stress terms in terms of the assumed displacement functions can be obtained directly by substituting an expression for the stress function (with underdetermined coefficients) into the periodicity condition or the compatibility equation and solving for the unknown coefficients of the stress function. The circumferentially constant term $N^*_y$ is determined from the circumferential periodicity condition, while the coefficient $f_0^*$ can be obtained from the compatibility equation.

In the following, the assumptions used in the present approach to eliminate $C_0$ and $C_1$ will be discussed, and the connection with the assumptions used in Evensen’s approach will be explained.

- **Assumption used to eliminate $C_0$**

Firstly, the validity of the assumption $N^*_y = 0$, which is used implicitly by Evensen, will be discussed. The circumferential periodicity condition can be formulated as a relation between the (circumferentially) constant term $N^*_y$, $C_0$, and the time-dependent coefficients of the asymmetric modes $A$ and $B$, and can be written as follows (see Appendix G, Section G.4 for further details):

$$N^*_y = \frac{1}{C_N} \left\{ A^2 + B^2 + 2 \delta_{n,\ell} A (\xi_2 + \bar{\xi}_2) - \epsilon C_0 \right\} \quad (3.29)$$

where $C_N = A_{22}^*/(\frac{1}{2} h^2 \ell_2^2)$ and $\epsilon = 8/(\ell_2^2 Rh)$.

Under the assumption that $N^*_y$ is equal to zero, the spatially constant term $C_0$ of the axisymmetric mode can be expressed in terms of the time-dependent coefficients of the asymmetric modes $A$ and $B$,

$$C_0 = (1/\epsilon) \left\{ A^2 + B^2 + 2 \delta_{n,\ell} A (\xi_2 + \bar{\xi}_2) \right\} \quad (3.30)$$

In other words, $C_0/\epsilon h$ is assumed to be such that the associated stress $-\frac{1}{C_N} \epsilon C_0$, linear in $C_0$, compensates the constant (nonlinear) circumferential stress due to large deformation bending, $\frac{1}{C_N} \left\{ A^2 + B^2 + 2 \delta_{n,\ell} A (\xi_2 + \bar{\xi}_2) \right\}$. Hereby the constant circumferential stress term $N^*_y$ is annihilated. Notice that for perfect shells $C_0$ is proportional to the square of $A$ and $B$.

The interpretation of the assumption $N^*_y = 0$ can be given via a comparison with the equations of a ‘consistent’ approach (Prathap and Pandalai, 1978), which retains the axisymmetric mode amplitudes as independent coordinates.
The resulting equilibrium equations in this approach are denoted as the $C_0$-equation, $C_1$-equation, and $A$-equation, respectively.

The $C_0$-equation is of the following form:

$$ C_C \dot{C}_0 - \frac{1}{R} \frac{1}{C_N} \left\{ (A^2 + B^2 + 2\delta_{n,\ell} A (\xi_2 + \xi_2)) - \epsilon C_0 \right\} = 0 \tag{3.31} $$

where $C_C = \rho h^2$ and where $\delta_{n,\ell} = 1$ if $n = \ell$ and $\delta_{n,\ell} = 0$ otherwise. Comparing (3.29) with (3.31) it is seen that the constant stress resultant contribution $N^* / R$ must balance the inertia term $C_C \dot{C}_0$. Therefore, the assumption $N^*_y = 0$ can be interpreted as neglecting the inertia term of $C_0$ in the $C_0$-equation, $|C_C \dot{C}_0| < |\epsilon C_0|$. Restricting the discussion to perfect shells, this is justified if the natural frequency of the axisymmetric mode is much larger than the double frequency of the asymmetric mode.

• **Assumption used to eliminate $C_1$**

In Evensen’s approach, the coefficients $C_0$ and $C_1$ were related to each other in order to satisfy the boundary condition $\dot{W} = 0$. However, it has been pointed out that this approach yields a special relation between $C_1$ and the time-dependent parts of the asymmetric modes $A$ and $B$. The interpretation of this relation will be discussed in the following.

To express the time-dependent part $C_1$ of the axisymmetric mode in terms of the time-dependent parts of the asymmetric modes $A$ and $B$, one observes that the coefficient $\dot{f}_0$ of the $\cos (m+p\pi x)$ term of the stress function can be written as follows (see Appendix G, Section G.4),

$$ \dot{f}_0 = \dot{f}_{01} A^2 + \dot{f}_{02} B^2 + \dot{f}_{03} C_1 \tag{3.32} $$

Inspection of the coefficients $\dot{f}_{0j}$ ($j = 1, 2, 3$) shows that, if $B_{21}^*$ is equal to zero, a relation between the time-dependent part of the axisymmetric mode $C_1$ and the time-dependent parts of the asymmetric modes $A$ and $B$ can be chosen such that the coefficient $\dot{f}_0$ of the $\cos (m+p\pi x)$ term of the stress function vanishes. This yields

$$ C_1 = -\frac{1}{\epsilon} \left\{ A^2 + B^2 + 2\delta_{n,\ell} A (\xi_2 + \xi_2) \right\} \tag{3.33} $$

Arguments which are similar to the case of the elimination of $C_0$ can be used to interpret the elimination of the axisymmetric amplitude $C_1$. The amplitude $C_1$ is also expressed in terms of the amplitudes of the asymmetric modes, whereby the $(2m)$th harmonic in axial direction of the axisymmetric circumferential stress vanishes. This corresponds to the assumption that $\dot{f}_0$ is equal to zero. For a perfect shell, the interpretation is that in the $C_1$-equation, the inertia term of the $C_1$ mode, the axial bending contribution for this mode, and the contribution due to the interaction of the linear stresses and linear curvatures, are neglected.
It should be emphasized that the periodicity condition by itself does not impose a constraint on \( C_1 \). However, the relation between \( C_1 \) and \( C_0 \) obtained, adopting the aforestated assumptions regarding the stresses in the circumferential direction of the shell, turns out to be the same as the relation used in Evensen’s approach. For the time-dependent parts of the axisymmetric modes \( C_0 \) and \( C_1 \) the following relationship with Evensen’s and Liu’s time-dependent part \( C \) of the axisymmetric modes (Evensen, 1967; Liu, 1988) can be established: \( C_0 = -C_1 = \frac{1}{2} C \).

Substitution of the given imperfection mode, the assumed radial deflections of the static state and the solutions obtained for the stress functions \( F \) and \( \tilde{F} \) into the dynamic out-of-plane equilibrium equation (2.12) yields a “residual” equation and application of Galerkin’s method leads to a coupled set of nonlinear ordinary differential equations in the time-dependent parts \( A \) and \( B \) of the asymmetric vibration modes. The weighting functions used in the Galerkin method are

\[
\begin{align*}
\frac{\partial W}{\partial A} &= h\left\{ \frac{m\pi x}{L} \cos \frac{n}{R} (y - \tau_K x) + (h^2 R^2) [A + \delta_{n\ell}(\xi_2 + \xi_2)] \sin^2 \left( \frac{m\pi x}{L} \right) \right\} \quad (3.34) \\
\frac{\partial W}{\partial B} &= h\left\{ \frac{m\pi x}{L} \sin \frac{n}{R} (y - \tau_K x) + (h^2 R^2) B \sin^2 \left( \frac{m\pi x}{L} \right) \right\} \quad (3.35)
\end{align*}
\]

The resulting set of differential equations is of the following form

\[
\gamma_0 \frac{d^2 A}{dt^2} + \gamma_{11} A \frac{d^2 A}{dt^2} + \gamma_{12} \left( \frac{dA}{dt} \right)^2 + \gamma_{13} B \frac{d^2 B}{dt^2} + \gamma_{14} \left( \frac{dB}{dt} \right)^2 \\
+ \gamma_{11} A \frac{d^2 A}{dt^2} + \gamma_{12} A \left( \frac{dA}{dt} \right)^2 + \gamma_{13} AB \frac{d^2 B}{dt^2} + \gamma_{14} A \left( \frac{dB}{dt} \right)^2 \\
+ c_{10} A + c_{20} A^2 + c_{02} B^2 + c_{30} A^3 + c_{12} AB^2 \\
+ c_{40} A^4 + c_{22} A^2 B^2 + c_{04} B^4 \\
+ c_{50} A^5 + c_{32} A^3 B^2 + c_{14} AB^4 = c_{exc} Q_{\delta \ell r} \cos \omega t \quad (3.36)
\]

\[
\delta_{01} \frac{d^2 B}{dt^2} + \delta_{11} AB \frac{d^2 A}{dt^2} + \delta_{12} B \left( \frac{dA}{dt} \right)^2 + \delta_{13} A B \frac{d^2 B}{dt^2} + \delta_{14} \left( \frac{dB}{dt} \right)^2 + \delta_{15} B \frac{d^2 A}{dt^2} \\
+ d_{01} B + d_{11} AB + d_{21} A^2 B + d_{03} B^3 \\
+ d_{31} A^3 B + d_{13} AB^3 + d_{41} A^4 B + d_{23} A^2 B^3 + d_{05} B^5 = 0 \quad (3.37)
\]

where the coefficients \( \gamma_{ij}, c_{ij}, c_{exc}, \delta_{ij}, d_{ij} \) are constant coefficients depending on the geometry and stiffness parameters, and on the wave numbers of the imperfection and deflection modes. The coefficients are listed in Appendix G (see Section G.4). These equations include Liu’s equations for orthotropic shells as a special case. The asymmetry for the two equations with respect to \( A \) and \( B \) is due to the assumed imperfection shape. It is noted that in the present approach the equations contain nonlinear inertia terms (terms of the form \( \frac{d^2 A}{dt^2} \), etc.; the terminology is from Bolotin (1963)). For steady state vibrations, the time dependence
can be eliminated via the method of averaging (Evensen, 1965). It is well known
(see e.g. Nayfeh and Mook, 1979) that the first order approximation of averaging is
not consistent when even-order (quadratic, quartic etc.) terms are present. This is
the case for shells with asymmetric imperfections. To obtain in this case a consistent
approximation of the higher order corrections to the amplitude-frequency relation,
a higher order approximation or numerical time integration should be employed.
The higher order corrections to the amplitude-frequency relation due to the effect
of asymmetric imperfections are expected to be small \((O(\epsilon^2 A^2))\), cf. Chapter 4 and
Appendix B.

Finally, it should be noted that in the present approach the higher order nonlinear
effects are taken into account only approximately. For a consistent approximation
of the higher order terms in Eqs. (3.36) and (3.37), corresponding to the shift to
hardening behaviour for larger vibration amplitudes, also higher order harmonics
in the temporal description should be included (Evensen, 1977). A more accurate
approximation may also require more terms in the assumed spatial mode.

To apply the first order approximation of the method of averaging we assume

\[
A = A_1(t) \cos \omega t \quad (3.38) \\
B = B_1(t) \sin \omega t \quad (3.39)
\]

Substituting into the governing equations (3.36) and (3.37) and applying the averag­
ing procedure (for details of this procedure the reader is referred to Evensen (1965)
or Liu (1988)) we obtain two coupled nonlinear algebraic equations of the following
form:

\[
(a_{10} - \alpha_{10} \Omega^2) \ddot{A} + (a_{31} - \alpha_{31} \Omega^2) \ddot{A}^3 + (a_{12} - \alpha_{12} \Omega^2) \dot{A} \dot{B}^2
+ a_{50} \dot{A}^5 + a_{32} \dot{A}^3 \dot{B}^2 + a_{14} \ddot{A} \dot{B}^4 = G_{m\tau} \quad (3.40)
\]

\[
(b_{01} - \beta_{01} \Omega^2) \ddot{B} + (b_{21} - \beta_{21} \Omega^2) \ddot{B}^3 + (b_{03} - \beta_{03} \Omega^2) \dot{B}^3
+ b_{14} \dot{B}^4 + b_{23} \dot{A} \dot{B}^3 + b_{05} \ddot{B}^5 = 0 \quad (3.41)
\]

where \(\ddot{A} \) and \(\ddot{B} \) are average values (over one period) of \(A_t \) and \(B_t \), and
where \(a_{ij} , \alpha_{ij} \), \(b_{ij} \), and \(\beta_{ij} \) are constant coefficients depending on the geometry, stiffness parameters
etc., and where \(G_{m\tau} \) is the generalized dynamic excitation. The coefficients are
listed in Appendix G (Section G.4). The normalized frequency parameter \(\Omega \) is
defined by

\[
\Omega = \frac{\omega}{\omega_{lin}} \quad (3.42)
\]

where

\[
\omega_{lin} = \sqrt{\frac{a_{10}}{\alpha_{10}}} \quad (3.43)
\]

is the small amplitude ("linearized") frequency for the given shell properties, im­
perfection, vibration mode, and applied loading. Eqs. (3.40) and (3.41) can be used
3.3 Nonlinear vibrations about a nonlinear static state

to calculate amplitude-frequency curves for nonlinear free or forced vibrations of statically loaded imperfect anisotropic cylindrical shells. If $B = 0$, it is possible to solve Eq. (3.40) for the unknown averaged amplitude $\bar{A}$. This corresponds to the case that only the directly driven mode responds, while the companion mode is quiescent. However, if the single-mode response becomes unstable with respect to perturbations in the companion mode, it is necessary to compute both $\bar{A}$ and $\bar{B}$ by solving Eqs. (3.40) and (3.41) simultaneously. The equations can be used to determine the nonlinear free vibration behaviour by setting $G_{m\ell r}$ equal to zero in Eq. (3.40). It is noted that for perfect unloaded isotropic shells the equations reduce to those given by Evensen (1967). For orthotropic shells, the equations correspond to the equations derived by Liu (Liu, 1988), who also used Evensen's formulation. For free single mode vibrations of isotropic shells the following amplitude-frequency relation is obtained (Evensen, 1967; Liu, 1988):

$$\Omega^2(1 + \beta_{c1}) = \beta_{c2} + \beta_{c3} \bar{A}^2 + O(\bar{A}^4)$$

(3.44)

The coefficients are defined as follows,

$$\begin{align*}
\beta_{c1} &= \frac{3}{16} \epsilon_o \\
\beta_{c2} &= \frac{2\epsilon_o}{12(1 - \nu^2)}(\xi^2 + 1)^2 + \frac{\xi^4}{(\xi^2 + 1)^2} \\
\beta_{c3} &= \frac{3}{32} \xi^4 \epsilon_o - \frac{3}{2} \epsilon_o (\xi^2 + 1)^2 + \frac{3}{2} \xi^4 \frac{\epsilon_o^2}{12(1 - \nu^2)}
\end{align*}$$

(3.45)

where $\epsilon_o = (\ell^2 h/R)^2$ and $\xi = \frac{\pi R/\ell}{L/m}$.

The term in Eq. (3.44) with coefficient $\beta_{c1}$ stems from the nonlinear inertia terms in the governing equations and plays a decisive role with respect to the type of the initial nonlinearity. The term with $\beta_{c2}$ contributes to a softening behaviour. The other significant term, the term with $\beta_{c3}$, originates from several effects. For small values of $\xi = \frac{\pi R/\ell}{L/m}$, the nonlinearity is softening. For larger values of $\xi$, the initial nonlinearity shifts to hardening (see p.12 of Evensen (1967)).

The interaction of displacement and stress terms which are linear and quadratic in the amplitude $\bar{A}$ of the asymmetric mode, and the cubic stress term, contribute to the $\beta_{c3} \bar{A}^2$ term. It is interesting to identify the origin of the different contributions to the $\beta_{c3} \bar{A}^2$ term. For this purpose the "residual" equation corresponding to Eq. (2.12) will be considered. Two types of terms can be distinguished in this equation, $(m, \ell)$ terms, terms of the form $C_{m1} \sin \frac{m\pi x}{L} \cos \frac{\ell R}{R}$ and $(2m, 0)$ terms, terms of the form $C_{m2} \cos \frac{2m\pi x}{L}$. These two types of terms in the 'residual' equation will, after application of Galerkin's method, give a contribution to the resulting set of differential equations (3.36) and (3.37):

$(m, \ell)$ contributions are contributions to the residual of the out-of-plane dynamic equilibrium equation, Eq. (2.12), in the form of an $(m, \ell)$ term. The first two $(m, \ell)$ contributions are associated with the initial shell curvature $1/R$, etc.
and with the interaction of the asymmetric and the axisymmetric mode. The third contribution is a hardening 'plate-like' contribution which stems from the interaction between the stress due to large deformation and the deformation itself.

1. The first \((m, \ell)\) contribution is softening and corresponds to \(N_y/R\) and hence to the initial shell curvature. It originates from the interaction of the \((m, \ell)\) mode with the axisymmetric contraction mode. The change of Gaussian curvature due to the curvatures of (i) the linear \((m, \ell)\) mode and (ii) the contractive \((2m, 0)\) mode gives a cubic term with an \((m, \ell)\) spatial distribution. Since this change of Gaussian curvature is negative for a positive panel \((m, \ell)\) displacement mode, this stress is destabilizing in combination with the initial curvature \(1/R\).

2. The second \((m, \ell)\) contribution is softening and corresponds to the term \(N_x W_{xx}\) in the dynamic out-of-plane equilibrium equation and hence to the curvature due to deformation. It stems from the interaction of the linear \((m, \ell)\) stress term with the quadratic displacement (curvature) term in the (contractive) \((2m, 0)\) mode. In combination with the curvature of the \((2m, 0)\) mode the linear stress \(N_x\) gives a destabilizing (softening) contribution in the dynamic out-of-plane equilibrium equation. For the outward deflecting part, \(N_x\) is tensile, and the stress will combine with the \((2m, 0)\) mode to give a \((m, \ell)\) membrane stress contribution directed outwards for the outward deflecting part. For the inward deflecting part, the stress is compressive and the contribution will be directed inwards. Hence these contributions are softening.

3. The third contribution is hardening and corresponds to the large amplitude curvature of the \((m, \ell)\) mode. The change of Gaussian curvature due to the curvatures of the \((m, \ell)\) mode gives a quadratic term with a \((0, 2\ell)\) spatial distribution. The interaction of this stress with the \((m, \ell)\) displacement mode gives a hardening membrane stress contribution to the \((m, \ell)\) terms in the dynamic equilibrium equation. Physical explanation is that this stress has positive maxima (is tensile) at the maxima of the \((m, \ell)\) mode, and has negative maxima (is compressive) at the nodes of the \((m, \ell)\) mode. This effect corresponds to the familiar stabilizing postbuckling (large bending) plate behaviour. It is noted that the corresponding \((2m, 0)\) terms are cancelled out in the present formulation due to the axisymmetric deformation contribution of the \((2m, 0)\) mode.

\((2m, 0)\) contributions are contributions to the out-of-plane dynamic equilibrium equation, Eq. (2.12), in the form of a \((2m, 0)\) term. These \((2m, 0)\) contributions are coupled to the \((m, \ell)\) contributions and are essential for the nonlinear behaviour. They give a contribution to the resulting set of differential equations (3.36) and (3.37) corresponding to the 'nonlinear modes' in Evensen's approach.
1. The first contribution is hardening and stems from the restoring bending moments due to the quadratic \((2m,0)\) axisymmetric displacements.

2. The second contribution is softening and stems from the \((\text{nonlinear})\) interaction of the linear stresses with the linear displacements (curvatures), which results in a contribution to \((2m,0)\) distributions in the out-of-plane dynamic equilibrium equation. Since the sign of the linear membrane stresses is such that in the linear case their contribution is inwardly directed for the outwardly deflecting parts and vice versa, the interaction gives membrane stress contributions in the \((2m,0)\) mode, directed in the deflection direction and hence giving a softening contribution. This effect is due to the occurrence of a large amplitude curvature and the interaction of two modes (see also the second \((m,\ell)\) contribution described earlier).

Summarizing, we note that the first \((m,\ell)\) contribution corresponds to the quadratic membrane stress terms which occur for closed shells due to the interaction of the asymmetric mode with the axisymmetric mode. The second contribution is associated with the quadratic terms due to the interaction of the linear membrane stresses and the quadratic displacements of the secondary mode (closed shell effect). The third one corresponds to the cubic terms which occur due to the interaction of the quadratic membrane stress terms (plate-effect) with the linear displacements. The first \((2m,0)\) contribution is due to bending moments (shell effect). The second one stems from the interaction of the linear stresses with the linear displacements.

The large bending interaction of the asymmetric modes with the resulting membrane stress in the circumferential direction is essential with respect to the type of nonlinearity. It is noted that in the present approach (based on Evensen’s method) a decisive contribution in the frequency-amplitude relation is provided by the nonlinear inertia term. A hardening contribution stems from the familiar midplane stretching due to large bending deformation. For linear vibrations, there is the familiar stabilizing effect of positive Gaussian curvature. For nonlinear vibrations of (axisymmetric) curved closed structures, there is a softening contribution due the additional contraction mode. In the case of shells, the effect of curvature is reflected by a linear stabilizing stress, and by the interactions leading to softening behaviour for the nonlinear behaviour.

It is noted that for beams, the immovability condition of the supports results in a constant quadratic stress term in the resulting equations. For plates (and shells), quadratic stress terms arise irrespective of the movability/immovability of the boundary.

An important effect which occurs for ‘open’ shells (panels), is due to the quadratic softening term corresponding to the interaction of three terms, namely the linear stress and the linear displacement term, and the initial curvature of the structure. For a closed shell, this ‘open shell’ phenomenon is replaced by the softening interaction of the linear modes/stresses with the contraction modes. In the present formulation, we encounter this effect if we include initial imperfections with the same shape as the (linear) vibration mode.

It is noted that the effect of the curvature leads to quadratic terms of the type
AC in the resulting time-dependent ordinary differential equations, characteristic for curved closed structures (cf. Prathap and Pandalai, 1978). Asymmetric imperfections result in terms of the form $\xi^2 A^2$ in the resulting ordinary differential equations, quadratic in the asymmetric deflection amplitude (Eqs. (3.36) and (3.37)). This corresponds to the quadratic terms which occur for curved open structures. For open structures which can be modelled by one degree of freedom systems, there is the familiar stabilizing effect of curvature in the linear case, and a softening effect of curvature in the nonlinear case, provided by the quadratic term. In such a case, the shell does not spend equal time intervals during outward and inward displacement. The fundamental characteristic of a curved structure is its unequal stiffness with respect to outward and inward deflections. For closed structures, complicating effect is the interaction of two modes which contributes to a softening behaviour (Prathap and Pandalai, 1978). For very large vibration amplitudes the behaviour in general becomes hardening.

The present approach can be extended to include the effect of viscous damping (Liu, 1988). Further, Liu (1988) investigated the stability of the response via a perturbation method in combination with the method of slowly varying parameters. The stability of the solutions will be investigated in the next chapter.

### 3.4 Results and discussion

To be able to perform a parametric investigation of the linear and nonlinear vibrations of statically loaded imperfect laminated (anisotropic) shells, a FORTRAN program, SILVANA (Simplified Large amplitude Vibration Analysis including Anisotropy), has been developed. The analysis presented in this chapter was implemented in this program. Extensive use of the symbolic manipulation program REDUCE (Hearn, 1993) was made in order to execute the detailed derivations described in Appendix G.

At first, to verify the correctness of both the derivations of the equations and the implementation into the program, several results of earlier investigations will be reproduced. In the first and second example the nonlinear vibration behaviour of an isotropic and an orthotropic shell, respectively, are illustrated. Finally, results for the linearized and nonlinear vibrations of an imperfect anisotropic shell will be discussed. The data of the shells used in the calculations are given in Appendix F.

The first example concerns the nonlinear vibrations of an isotropic shell studied by Evensen (1967). The vibration is characterized by the nonlinearity parameter $\epsilon = (k^2 h/R)^2$ and the aspect ratio $\xi = \frac{\pi R/\ell}{L/m}$. Results are presented for $\epsilon = 0.01$, $\xi = 0.1$, and Poisson's ratio $\nu = 0.3$. This corresponds for example with a shell for which $h/R = 0.004$ and $L/R = 2\pi$ and a vibration mode given by

$$\begin{align*}
\hat{W}/h = \frac{\ell^2}{4R}[A(t) \sin \frac{\pi x}{L} + A(t) \sin \frac{\pi x}{L} \cos \frac{\ell}{R} y]
\end{align*}$$

with $\ell = 5$. Notice that in this vibration mode, as discussed previously in Section 3.3, the coefficients of the axisymmetric modes, $C_0$ and $C_1$ in Eq. (3.15), have been
3.4 Results and discussion

Figure 3.1: Coupled mode nonlinear vibrations of isotropic shell. ES-2 shell (Evensen, 1967). a) Driven mode, amplitude $|A|$; b) Companion mode, amplitude $|B|$.

eliminated by expressing them in terms of the coefficient of the asymmetric mode $A$ by means of Eqs. (3.30) and (3.33). For single mode vibrations, the initial nonlinearity is softening, which is caused by the softening contributions due to the interaction between the axisymmetric contraction mode and the asymmetric deformation. It is noted that the present model predicts hardening behaviour if the aspect ratio $\xi$ is large, say, $\xi > \frac{5}{2}$ (see p. 12 of Evensen (1967)). In the latter case the effect of large bending deformation already prevails for the initial nonlinear vibration behaviour.

The amplitude-frequency curves for the case when both asymmetric modes (driven and companion mode) vibrate, the coupled mode response, are shown in Fig. 3.1. Both modes initially (for amplitudes not too large) show a softening behaviour. For larger amplitudes a shift to hardening behaviour takes place. To determine which response form will actually occur the stability of the response has to be investigated (Liu, 1988). Since the present analysis includes Evensen’s formulation for isotropic shells, Evensen’s results are matched exactly. Comparing the backbone curves of single mode and coupled mode response, it can be concluded that the combination of the asymmetric modes leads to a reduction of the softening nonlinearity as compared to the single mode case. The contraction term is proportional to the sum of the squares of the time-dependent parts of the asymmetric vibration modes, $A^2 + B^2$.

As $A \sim \cos \omega t$ and $B \sim \sin \omega t$ the time dependent parts (the 2nd harmonic content) of the contraction modes, which contribute to the softening behaviour, cancel (cf. Evensen, 1965, 1967).

The second example concerns the single mode free vibration of an orthotropic shell with the vibration parameters $\epsilon = 0.01$, $\xi = 1.0$. The vibration mode is assumed to be

$$W/h = \frac{l^2}{4R} [A(t) \sin \frac{\pi x}{L}]^2 + A(t) \sin \frac{\pi x}{L} \cos \frac{\ell}{R} y$$

with $\ell = 5$. This shell was used by El-Zaouk and Dym (1973). The stiffness properties of the shell are $E_y/E_x = 1$, $\nu_{xy} = 0.3$, and $G_{xy}/E_x = \frac{\nu_{xy}}{1-\nu_{xy}}$, where $E_x,$
$E_y$, etc. are the usual stiffness parameters for an orthotropic shell, and $\gamma$ is the so-called shear ratio parameter. In Fig. 3.2 amplitude-frequency curves are shown for varying shear ratio $\gamma$. For smaller values of $\gamma$, the slope of the $|A|$ versus $\Omega$ curve becomes less negative, i.e. the softening character of the (initial) nonlinear behaviour decreases (El-Zaouk and Dym, 1973). It must be noted that the frequencies are normalized with respect to their linear frequencies. Consequently, in comparing the nonlinearity for cases with different linear frequencies a ‘scaling’ effect occurs. The present results match the results obtained by El-Zaouk and Dym since they use a similar formulation, namely, the extension of Evensen’s method from isotropic to orthotropic shells.

Having demonstrated that the current analysis operational in the FORTRAN program SILVANA is able to reproduce results obtained earlier via a similar (but less general) analysis, in the following the linearized and nonlinear vibration behaviour of a specific anisotropic composite shell will be investigated. The data of this shell are given in Appendix F. This shell has been used earlier in static stability investigations by Booton (1976) and Arbocz and Hol (1989). In this case the shell length is $L = 3.776$ in., the shell radius $R = 2.67$ in., and the total shell thickness $h = 0.0267$ in. The lay-up of Booton’s shell is $[\theta_1, 0, -\theta_1] = [30, 0, -30]$. Notice that because the laminate of this shell is unbalanced, torsion-bending coupling exists. This is reflected by the nonvanishing terms $B_{16}$ and $B_{26}$ in the ABD-matrix. Results will be presented illustrating the effect of axial loading and imperfections on the linearized and nonlinear vibrations of this anisotropic shell.

The effect of varying the layer orientation $\theta_1$ of Booton’s shell on the lowest natural frequency and the corresponding vibration mode is shown in Fig. 3.3. The frequency has been normalized with respect to $\omega_{ref} = \sqrt{\frac{E}{2\rho h^2}}$, where $E$ is a reference value, and $E = E_{11}$ is Young’s modulus of a layer in the 1-direction ($\omega_{ref}\sqrt{\rho} = 639.45\sqrt{Nm^{-2}}$ in the present case). The vibration mode
3.4 Results and discussion

Figure 3.3: Effect of layer orientation on lowest natural frequency and corresponding vibration mode of anisotropic shell: Booton-type shell, lay-up $[\theta_1, 0, -\theta_1]$.

The effect of axisymmetric imperfections of the form

$$\hat{W}/h = \xi_1 \cos \frac{2m\pi x}{L}$$

on the lowest natural frequency is depicted in Fig. 3.4. The frequency is normalized with respect to the (linear) frequency of the unloaded perfect shell $\omega_{ltr}$, which can be obtained from the equation for the linearized frequency $\omega_{lin}$ (Eq. (3.43), evaluated for the case that $\xi_1 = \xi_2 = 0$ ($\omega_{lin} = 608.16\sqrt{V/\rho}$ in the present case). The axisymmetric imperfection mode satisfies a strong coupling condition with the asymmetric vibration mode (cf. Liu, 1988), the number of axisymmetric half waves $i$ in the $C_1 \cos(i\pi x/L)$ mode is equal to $2m$. For small values of the imperfection amplitude the frequency decreases with increasing amplitude. The membrane stresses which correspond to the given deflection mode can be obtained directly from the dynamic state compatibility equation. The stabilizing membrane stress with a spatial distribution of the first-order asymmetric deflection mode $m, \ell$ is decreased...
by the axisymmetric imperfections (inward at the shell midlength). This effect is initially predominant. At a certain imperfection amplitude the frequency begins to increase with growing imperfection amplitude due to the stabilizing effect of the different geometry of the imperfect shell. This stabilizing curvature effect (corresponding to the membrane stress contribution $N_x\bar{W}_{xz}$ in the dynamic equilibrium equation) predominates for larger imperfection amplitudes. The trends observed for axisymmetric imperfections were reported earlier for isotropic and orthotropic shells by Singer and Prucz (1982).

The effect of asymmetric imperfections in the form of the vibration mode $(n = \ell)$

$$\bar{W}/h = \bar{\xi}_2 \sin \frac{m\pi x}{L} \cos \frac{\ell}{R}(y - \bar{r}_K x)$$

is also shown in Fig. 3.4. The frequency decreases with the imperfection amplitude. The interaction of the asymmetric mode with the accompanying axisymmetric mode contributes to a moderate reduction of the frequency with increasing imperfection amplitude. For increasing imperfection amplitudes the effect of the other terms in the equilibrium equation involving curvature (corresponding to the membrane stress contribution $L_{NL}(F, \bar{W})$) comes into play. This shift to an increasing frequency occurs at a larger value of the imperfection amplitude than the maximum ($\bar{\xi}_2 = 2.0$) shown in this figure. It is noted that the analysis of Singer and Prucz (1982) does not include the axisymmetric modes in the assumed displacement function which are necessary to satisfy the circumferential periodicity condition. For the case $n = \ell$ the trends they predict for orthotropic shells do not agree with the predictions of the present analysis (see Fig. 8 in Singer and Prucz (1982)). The discrepancy for isotropic shells between the trends predicted by the analyses of Singer and Rosen (1976) and Liu (1988) has also been attributed to the fact that in the former case the authors did not satisfy the periodicity condition (Liu, 1988). For $n \neq \ell$ both the present analysis and Singer and Prucz in general predict an increase in frequency with increasing imperfection amplitude.
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Figure 3.5: Influence of axial load on linearized frequency of lowest vibration mode of imperfect anisotropic shell; Booton's shell, axisymmetric imperfection amplitude $\xi_1 = -0.25$, asymmetric imperfection amplitude $\xi_2 = 0.25$.

Figure 3.6: Static asymmetric mode response of imperfect anisotropic shell; Booton's shell, response mode is lowest vibration mode.
Figure 3.1: Influence of axial load on linearized frequency of anisotropic shell; Booton’s shell, axisymmetric imperfection amplitude $\xi_1 = -0.04$, asymmetric imperfection amplitude $\xi_2 = 0.05$.

In Fig. 3.5, for Booton’s shell the effect of an axial compressive load on the frequency of the mode with $m = 1$, $\ell = 6$ and $\tau_K = -0.002$, is depicted. The mode considered is the “lowest vibration mode”, i.e., the mode corresponding to the lowest natural frequency of the perfect shell. The axial load parameter $\lambda$ is defined by $\lambda = (cR)/(Eh^2)N_0$, where $c = \sqrt{3(1 - \nu^2)}$ and $E = E_{11}$ and $\nu = \nu_{12}$ (Young’s modulus and Poisson’s ratio of a layer) are reference values. The axial load reduces the frequency corresponding to this mode until at a load level of $\lambda = 0.555$, the frequency becomes zero. The static behaviour of the shell for the mode considered is shown in Fig. 3.6 both for a small and for a relatively large imperfection amplitude.

The effect of relatively large imperfections on the frequency a loaded shell is illustrated in Fig. 3.7 by considering also an asymmetric imperfection, an axisymmetric imperfection and the combination of these two imperfections. The frequency of the loaded imperfect shell has been normalized with respect to the frequency of the unloaded perfect shell $\omega_{m\ell\tau}$ ($\omega_{m\ell\tau}\sqrt{\rho} = 608.16\sqrt{N/m^2}$). Notice that the effect of the axisymmetric imperfection is important also at zero axial load (Singer and Rosen, 1976).

It must be noted that at $\lambda = \lambda_{m\ell\tau} = 0.40691$, the lowest buckling load of this shell occurs for the mode $m = 3$, $\ell = 5$, and $\tau_K = -1.56$ (“lowest buckling mode”), and the frequency corresponding to this buckling mode becomes zero. Above this load level, the static state is unstable. This is illustrated in Fig. 3.7, where the frequency of the loaded shell is plotted both for the lowest vibration mode and for the lowest buckling mode. The frequency has been normalized with respect to $\omega_{ref} = \frac{E}{2\rho h^2}$, where $E = E_{11}$. Moreover, the effect of an initial imperfection, affine to the vibration or buckling mode, is shown. The imperfection amplitudes are
3.4 Results and discussion

Figure 3.8: Influence of axial load on linearized frequency of anisotropic shell with axisymmetric imperfection; Booton's shell, lowest vibration mode; $W/h = \xi_1 \cos \frac{2\pi x}{L}$; $W/h = \xi_1 \cos \left(\frac{2\pi x}{L}\right) + A(t) \sin \frac{2\pi x}{L} \cos \left(\frac{\pi}{L} y - \tau_K x\right)$; $\tau_K = -0.002$, $\ell = 6$.

$\xi_1 = -0.04$ and $\xi_2 = -0.05$. Notice that since the lowest buckling mode has a stable postbuckling behaviour, the frequency of the imperfect shell does not become zero when the load is increased, but it reaches a minimum below the buckling load of the perfect shell and then starts to increase again. The static asymmetric response of Booton’s shell corresponding to the lowest buckling mode is discussed in the next chapter (see Fig. 4.5).

In the following, the effect of axial loading on the frequency of the mode corresponding to the lowest natural frequency of the perfect shell (“lowest vibration mode”) will be analysed. The variation of the frequency with the loading will be shown up to the load at which the frequency becomes zero.

In Figs. 3.8 and 3.9 the effect of the imperfection amplitude on the load versus frequency curves are given. In Fig. 3.8, the curves are plotted for various values of the axisymmetric imperfection amplitude $\xi_1$, where

$$W/h = \xi_1 \cos \frac{2\pi x}{L}$$

and in Fig. 3.9 for various values of the asymmetric imperfection amplitude $\xi_2$, where

$$W/h = \xi_2 \sin \frac{\pi x}{L} \cos \frac{5}{R} (y - \tau_K x)$$

with $\tau_K = -0.002$. The frequency has been normalized with respect to the frequency of the unloaded perfect shell $\omega_{mfr}$ ($\omega_{mfr} \sqrt{\rho} = 608.16 \sqrt{N m^{-2}}$). As stated earlier, the effect of axisymmetric imperfections on the frequency is significant also at zero
Figure 3.9: Influence of axial load on linearized frequency of anisotropic shell with asymmetric imperfection; Booton’s shell, lowest vibration mode. 

\[
\dot{W}/h = \xi_2 \sin \frac{\pi e}{L} \cos \frac{\ell}{R} (y - \tau_K x); \quad W/h = \frac{E^2}{4K} |A(t) \sin \frac{\pi e}{L}|^2 + A(t) \sin \frac{\pi e}{L} \cos \frac{\ell}{R} (y - \tau_K x); \quad \tau_K = -0.002, \ell = 6.
\]

Figure 3.10: Influence of axial load on frequency of perfect anisotropic shell for finite vibration amplitudes; Booton’s shell, lowest vibration mode. 

\[
\dot{W}/h = \frac{E^2}{4K} |A(t) \sin \frac{\pi e}{L}|^2 + A(t) \sin \frac{\pi e}{L} \cos \frac{\ell}{R} (y - \tau_K x); \quad \tau_K = -0.002, \ell = 6.
\]
3.4 Results and discussion

Figure 3.11: Influence of axial load on frequency of anisotropic shell with axisymmetric imperfection for finite vibration amplitudes; Booton’s shell, lowest vibration mode, $W/h = \xi_1 \cos \frac{\pi x}{L}$; $W/h = \frac{\ell^2}{4R} [A(t) \sin \frac{\pi x}{L}]^2 + A(t) \sin \frac{\pi x}{L} \cos \frac{\ell}{L} (y - \tau_K x); \tau_K = -0.002, \ell = 6; \xi_1 = -0.25.$

loading. Notice further the strongly nonlinear behaviour when the load reaches the limit-point load in the case of asymmetric imperfections.

In Figs. 3.10 and 3.11 the effect of the vibration amplitude on the load versus frequency curves are given. In Fig. 3.10, the curves are plotted for various values of the vibration amplitude for a perfect shell, in Fig. 3.11 for a shell with a relatively large axisymmetric imperfection.

The effect of axisymmetric imperfections on the nonlinear vibrations in the mode with $m = 1, \ell = 6$ and $\tau_K = -0.002$ is depicted in Fig. 3.12. For this case $\epsilon = 0.1296$ and $\xi = \frac{\pi R L}{\ell} = 0.3702$. The frequency has been normalized by $\omega_{lin}$ ($\Omega = \frac{\omega}{\omega_{lin}}$), the linearized frequency of the imperfect shell. The vibration of the perfect shell shows a softening behaviour for small vibration amplitudes and shifts to a hardening behaviour for larger amplitudes. For larger imperfection amplitudes the softening nonlinearity decreases (the nonlinear stiffness increases) due to the stabilizing curvature effect of the imperfections. It is noted that there is a scaling effect because the linear frequency decreases with increasing imperfection amplitude and may be considerably lower than the frequency of the perfect shell. The effect of asymmetric imperfections on the nonlinear vibrations in the mode with $m = 1, \ell = 6$ and $\tau_K = -0.002$ is depicted in Fig. 3.13. Also in this case the frequency has been normalized by $\omega_{lin}$, the linearized frequency of the imperfect shell. As stated earlier, when discussing the accuracy of the Eqs. (3.36) and (3.37), the higher order effect of asymmetric imperfections on the frequency-amplitude relations is generally small ($O(\xi_2^2 A^2)$), cf. Chapter 4 and Appendix B. The zeroth order effect is predominant.
Figure 3.12: Amplitude-frequency curves of anisotropic shell for various axisymmetric imperfection amplitudes; Booton's shell. \( \frac{W}{h} = \xi_1 \cos \frac{2\pi x}{L}; \frac{W}{h} = \frac{\ell^2}{4R}(A(t) \sin \frac{\pi y}{L})^2 + A(t) \sin \frac{\pi y}{L} \cos \frac{\pi y}{L}(y - \tau_K x); \tau_K = -0.002, \ell = 6. \)

Figure 3.13: Amplitude-frequency curves of anisotropic shell for various asymmetric imperfection amplitudes; Booton's shell. 
\( \frac{W}{h} = \xi_2 \sin \frac{\pi y}{L} \cos \frac{\pi y}{L}(y - \tau_K x); \frac{W}{h} = \frac{\ell^2}{4R}(A(t) \sin \frac{\pi y}{L})^2 + A(t) \sin \frac{\pi y}{L} \cos \frac{\pi y}{L}(y - \tau_K x); \tau_K = -0.002, \ell = 6. \)
3.5 Conclusions

The vibrations of composite shells were studied by Iu and Chia (1988b), who used a multi-mode analysis for the nonlinear vibration and postbuckling of unsymmetric cross-ply cylindrical shells. They contributed the considerable discrepancy between their results and those of El-Zaouk and Dym (1973) to the fact that El-Zaouk and Dym’s analysis does not satisfy simply supported boundary conditions and to the constraining of the axisymmetric mode. Since the present formulation is similar in this respect to the one used by El-Zaouk and Dym, these issues will be investigated further in the following chapters.

3.5 Conclusions

An idealized model to investigate the linearized and nonlinear vibrations of statically loaded imperfect anisotropic cylindrical shells was presented. The analysis presented in this chapter is a Level-1 analysis in the terminology introduced in Chapter 1 and fits into the analysis strategy suggested there. This model can be used to investigate the effect of different parameters, such as stiffness properties of the laminate, imperfections, and static loading, on the vibration behaviour.

Results for the nonlinear vibrations of isotropic, orthotropic, and anisotropic cylinders have been presented. The effect of imperfections on the vibrations of a specific anisotropic cylinder has been discussed. Imperfections can strongly influence the linearized vibrations of shells when certain coupling conditions between the imperfection mode and the vibration mode are satisfied.

When one evaluates the results of the present Level-1 analysis, the restrictions imposed by the underlying theory should be kept in mind. In general it will be necessary to analyse the nonlinear behaviour with more accurate models. Several simplifying assumptions which limit the applicability of the method or which limit the range of validity of the results are addressed in subsequent chapters.

Restrictions of the present analysis stem firstly from the governing equations due to the shell theory used (Donnell-type equations). In addition, the following comments can be made about the accuracy and applicability of the present Level-1 model:

- In order to eliminate the spatial dependence and the time dependence, approximate solution methods are used (Galerkin’s method and the method of averaging, respectively).

The (secondary) axisymmetric modes are constrained in the sense that they are related \textit{a priori} to the asymmetric vibration modes. In the procedures used in Chapters 4 and 5, the axisymmetric modes are not constrained in this sense. Further, the double harmonic in the circumferential direction is not included in the present formulation. The effect of this secondary response mode will be investigated in Chapter 5.

Only “single mode” or “coupled mode” vibrations are included in the present model. Other modal interactions are not taken into account.
The "simply supported" (SS-3) boundary condition is not exactly satisfied in the present procedure. In the method used in Chapter 5 the SS-3 boundary condition can be satisfied accurately.

The influence of the actual boundary condition on the (nonlinear) vibration behaviour may be significant. The influence of different types of boundary conditions will also be addressed in Chapter 5.
Chapter 4

Level-1 Transient Analysis

4.1 Introduction

In this chapter, the transient nonlinear flexural vibration behaviour of imperfect anisotropic cylindrical shells under harmonic lateral excitation is investigated via a Level-1 Analysis or Simplified Analysis. Two types of dynamic stability problems of cylindrical shells will also be dealt with, namely dynamic buckling and parametric excitation. The present investigation is based on models which are simplified in the sense that a small number of modes is used to describe the dynamic response, and can be seen as an extension of earlier work in this area on isotropic and orthotropic shells to the case of anisotropic shells.

Liu (1988) used numerical time-integration to verify the method of averaging used in his analysis of the nonlinear vibrations of orthotropic shells. Liu’s analysis was based on Evensen’s approach (see Chapter 3), in which the time-dependent coefficients of the axisymmetric vibration modes in the assumed displacement function are constrained by relating them to the time-dependent coefficients of the asymmetric modes. In the present study the coefficients of the axisymmetric mode are used as additional generalized coordinates.

Important aspects of the dynamic buckling behaviour under step loading can be studied via the model developed by Tamura (1973) for imperfect isotropic shells, which includes the effect of in-plane inertia in an approximate way. In this model, dynamic buckling can occur as a phenomenon, parametrically induced by the axial vibrations of the shell. Van Houten (1992) used a multi-mode method to study the dynamic buckling of isotropic cylindrical shells. In the present investigation, the effect of anisotropy is taken into account. The dynamic buckling load is defined as the load for which a large increase in the maximum response occurs (cf. Budiansky and Roth, 1962).

The linear and nonlinear parametric excitation problem of isotropic shells was investigated by Yao (1963, 1965). For finite amplitudes nonlinear effects, corresponding with the nonlinear terms in the differential equation, come into play and modify the growth. In the present approach, the nonlinear parametric excitation behaviour of anisotropic shells is modelled including the effects of imperfections and of a nonlinear static state.
In Section 4.2 a general simplified analysis is presented for the analysis of the dynamic stability behaviour of composite cylindrical shells. In the subsequent section specific cases are treated (nonlinear vibrations, parametric excitation, and dynamic buckling). In the final section, specific results of these analyses are presented. The analysis is based on assumed mode shapes which can describe the travelling wave vibration. In the nonlinear vibration case, a coupled mode response may occur, which can be interpreted as travelling waves in the circumferential direction of the shell. To describe this response, which involves the directly excited ‘driven mode’ and its ‘companion mode’, two asymmetric modes are used in a variational formulation in combination with numerical time integration of the resulting modal amplitude equations. The two assumed asymmetric modes have \( m \) axial waves. The axisymmetric mode \( C_1 \cos \frac{\theta}{l} \), which satisfies the strong coupling condition \( i = 2m \) with the two asymmetric modes (cf. Liu, 1988), is included in the assumed deflection function. Imperfections and static loading are taken into account. The shell is statically loaded by axial compression, radial pressure, and torsion. The static response is assumed to be affine to the given two-mode imperfection, consisting of an axisymmetric and an asymmetric mode. Donnell-type governing equations are used and classical lamination theory is employed. Khot’s formulation is used to account for “simply supported” boundary conditions. The effect of axial and torsional inertia of the fundamental in-plane modes, including the inertia effect of a ring or disk at the loaded end of the shell, is taken into account approximately. Viscous modal damping is included in the analysis.

4.2 General analysis

To find approximate solutions for the equations governing the dynamic state of anisotropic cylindrical shells, based on a small number of assumed deflection modes, we use Hamilton’s (extended) variational principle

\[
\int_{t_1}^{t_2} \delta (T - V) dt + \int_{t_1}^{t_2} \delta (W_{nc}) dt = 0
\]

where \( T \) is the total kinetic energy, \( V \) is the potential energy, and \( W_{nc} \) is the work done by nonconservative forces.

The lateral pressure is split into a conservative part and a nonconservative part. The in-plane loads (axial compressive load and counter-clockwise torque) are applied at \( x = L \) and correspond to averaged stress resultants. They are also split into a conservative part and nonconservative part:

\[
\hat{p} = \hat{p}^c + \hat{p}^{nc}; \quad \hat{N}_x|_{x=L} = \hat{N}_x^c + \hat{N}_x^{nc}; \quad \hat{N}_{xy}|_{x=L} = \hat{N}_{xy}^c + \hat{N}_{xy}^{nc}
\]

where the superscript \( c \) denotes the conservative part, and \( nc \) the nonconservative part. The conservative parts of the loading correspond to an applied step loading.
4.2 General analysis

At \( x = 0 \) the in-plane displacements (averaged in the circumferential direction) are assumed to be zero.

The potential energy for the dynamic state then becomes

\[
V = V_0 + V_1
\]

where \( V_0 \) is the strain energy, and \( V_1 \) is the potential energy of the applied conservative loads,

\[
V_0 = \frac{1}{2} \int_0^{2\pi R} \int_0^L \{ \dot{N}_x \dot{e}_x + \dot{N}_y \dot{e}_y + \dot{N}_{xy} \dot{\gamma}_{xy} \\
+ \dot{M}_x \dot{\kappa}_x + \dot{M}_y \dot{\kappa}_y + \frac{\dot{M}_xy}{2} \dot{\kappa}_{xy} \} \, dx \, dy
\]

\[
V_1 = -\int_0^{2\pi R} \int_0^L \hat{N}_x \hat{u}_{xx} \, dx \, dy - \int_0^{2\pi R} \int_0^L \hat{p} \hat{W} \, dx \, dy - \int_0^{2\pi R} \int_0^L \hat{N}_{xy} \hat{v}_{xy} \, dx \, dy
\]

where \( \hat{N}_x = -\hat{N}_0 u(t) \) is the applied axial step load, \( \hat{p} = \hat{p}_u(t) \) is the applied external pressure step load, and \( \hat{N}_{xy} = \hat{T}_0 \) is the applied counter-clockwise torque. Here, \( u(t) \) is the Heaviside unit step function. It is noted that the dynamic state variables depend on the solution of the static state problem.

The virtual work done by the nonconservative forces is

\[
\delta W_{nc} = \int_0^{2\pi R} \int_0^L \hat{N}_x \delta u_{xx} \, dx \, dy + \int_0^{2\pi R} \int_0^L \hat{p} \delta W \, dx \, dy + \int_0^{2\pi R} \int_0^L \hat{N}_{xy} \delta v_{xy} \, dx \, dy
\]

where \( \hat{N}_x, \hat{p}, \) and \( \hat{N}_{xy} \) are the nonconservative loads.

The kinetic energy of the shell is given by

\[
T_s = T_o + T_i
\]

where the out-of-plane inertia contribution \( T_o \) is given by

\[
T_o = \frac{1}{2} \hat{\rho} h \int_0^{2\pi R} \int_0^L \left( \frac{\partial \hat{W}}{\partial t} \right)^2 \, dx \, dy
\]

and the in-plane inertia contribution \( T_i \) is given by

\[
T_i = \frac{1}{2} \hat{\rho} h \int_0^{2\pi R} \int_0^L \left\{ \left( \frac{\partial \hat{u}}{\partial t} \right)^2 + \left( \frac{\partial \hat{v}}{\partial t} \right)^2 \right\} \, dx \, dy
\]

For a discretized system with \( N \) generalized coordinates (degrees of freedom) \( q_i \), application of Hamilton's principle leads to the Lagrange's equations of motion:
where the generalized forces $Q_i$, corresponding to the generalized coordinates $q_i$, are defined by $\delta W_{nc} = \sum_i Q_i \delta q_i$. Substitution of the given imperfection mode, the assumed radial deflections of the static state and the particular solutions for the stress functions $\bar{F}$ and $\bar{F}$ into the energy expression, leads to a coupled set of nonlinear ordinary differential equations in the unknown vibration amplitudes $q_i$ of the following general form (for normal coordinates):

$$
\ddot{q}_i + k_i q_i + \sum_j \sum_k a_{ijk} q_j q_k + \sum_j \sum_k \sum_l b_{ijkl} q_j q_k q_l = g_i
$$

where $c_i$, $a_{ijk}$, and $b_{ijkl}$ are coefficients, which can in general depend on time, and $g_i$ is the forcing term. The coefficients are listed in Appendix G (see Section G.5).

The lateral displacement is assumed as a function of $q_i$, $\bar{W} = \bar{W}(q_i)$, and a particular solution for the stress function $\bar{F}_p = \bar{F}_p(q_i)$ can be obtained directly from the compatibility equation. It is noted that the use of in-plane displacements $u$ and $v$ in combination with the in-plane equilibrium equations (Tamura, 1973), instead of using the stress function with the compatibility condition, leads to the same expressions for the coefficients of the stress function.

The solution for $\bar{F}$ becomes

$$
\bar{F} = \bar{F}_p + \bar{F}^*
$$

where the complementary solution $\bar{F}^*$, corresponding to stress resultants constant over the shell, can be written as

$$
\bar{F}^* = \frac{1}{2} \bar{N}_x^* y^2 + \frac{1}{2} \bar{N}_y^* x^2 - \bar{N}_{xy}^* xy
$$

The corresponding complementary in-plane displacements $\bar{u}^*$ and $\bar{v}^*$, are assumed, for a fixed end at $x = 0$, as

$$
\bar{u}^* = -\left( \frac{C_i h}{L} \right) x, \quad \bar{v}^* = -\left( \frac{C_i h}{L} \right) x
$$

where the introduced generalized coordinates $C_i$ and $C_j$ can be related to the spatially constant stress resultants via the boundary conditions for averaged in-plane stress resultants and the periodicity condition:

$$
\int_0^{2\pi R} \int_0^L \bar{u}_{x} \, dx \, dy = -2\pi R h C_a = f_{1,lin}(\bar{N}_x^*, \bar{N}_y^*, \bar{N}_{xy}^*) + f_{1,nl}(q_i)
$$

$$
\int_0^{2\pi R} \int_0^L \bar{v}_{y} \, dy = 0 = f_{2,lin}(\bar{N}_x^*, \bar{N}_y^*, \bar{N}_{xy}^*) + f_{2,nl}(q_i)
$$

$$
\int_0^{2\pi R} \int_0^L \bar{v}_{x} \, dx \, dy = -2\pi R h C_l = f_{3,lin}(\bar{N}_x^*, \bar{N}_y^*, \bar{N}_{xy}^*) + f_{3,nl}(q_i)
$$
where \( f_{j,lin} \) are linear functions of the constant stresses, and \( f_{j,nl} \) are nonlinear functions of the generalized coordinates \( q_i \). The functions are obtained by using the kinematic relations, Eqs. (A.22) in combination with the partially inverted constitutive equations, Eqs. (A.16). Inverting the relations (4.15) to (4.17) we can obtain the unknown (spatially) constant stress resultants as functions of \( C_a \), \( C_i \) and \( q_i \),

\[
\begin{align*}
\bar{N}_x^* &= g_1(C_a, C_i, q_i); \\
\bar{N}_y^* &= g_2(C_a, C_i, q_i); \\
\bar{N}_{xy}^* &= g_3(C_a, C_i, q_i)
\end{align*}
\]  

These expressions can be found in Appendix G (see Section G.5). In the present approach, the boundary conditions for averaged stress resultants are prescribed. It is noted that for a static analysis (cf. Section 3.2), the constant stresses \( \bar{N}_x^* \) and \( \bar{N}_{xy}^* \) can be related directly to the applied averaged in-plane loads \( N_0 \) and \( T_0 \), and the in-plane displacements \( C_a \) and \( C_i \) can be eliminated. Further, in the static case \( \bar{N}_y^* \) can be found directly from the circumferential periodicity condition (4.16). The inertia forces are obtained from the kinetic energy expressions for the in-plane contribution \( T_i \) and the out-of-plane contribution \( T_o \). The effect of the shell in-plane inertia is taken into account approximately by including the complementary in-plane displacements \( u = \bar{u}^* \), and \( v = \bar{v}^* \) in \( T_i \). The kinetic energy due to a ring or disk at the loaded end of the shell is given by

\[
T_m = \frac{1}{2} m \left( \frac{\partial \bar{u}^*}{\partial t} \right)^2|_{x=L} + \frac{1}{2} \frac{I_p}{R^2} \left( \frac{\partial \bar{v}^*}{\partial t} \right)^2|_{x=L}
\]

where \( m \) is the mass, and \( I_p \) the polar moment of inertia of the ring or disk. The total kinetic energy is given by

\[
T = T_i + T_o + T_m
\]

Finally, it is noted that modal viscous damping terms are introduced in the equations of motion by adding terms of the form \( c_i q_i \) in (4.11).

### 4.3 Application to specific cases

In this section, the general theory presented in the previous section is applied to specific cases, namely to 1) nonlinear vibrations, 2) nonlinear parametric excitation, and 3) dynamic buckling. Equations (4.11) can be written in the following form,

\[
\ddot{\bar{C}}_a = \{-e^{10} - e^{10} \cos(\Omega_a t) - e^{11} \dot{\bar{C}}_a + e^{10m}_a C_a + e^{10m}_i C_i + e^{1m}_0 C_0 \\
+ e^{21}_a C_1 + e^{21}_0 A + e^{22} C_1^2 + e^{2m} A^2 + e^{2n} B^2\}
\]  

\[
(4.21)
\]
\[ \dot{C}_1 = \{-e^{30}C_0 - e^{30}C_1 + e^{24m}C_a + e^{24m}C_0 + e^{24m}C_0 \}
+ e^{24m}C_1 + \{e^{24m}C_0 + e^{24m}A + e^{24m}A \}
+ e^{24m}A + e^{24m}A \} \] (4.22)

\[ \dot{C}_0 = \{-e^{30}C_0 - e^{30}C_1 + e^{24m}C_a + e^{24m}C_0 + e^{24m}C_0 \}
+ e^{24m}C_1 + \{e^{24m}C_0 + e^{24m}A + e^{24m}A \}
+ e^{24m}A + e^{24m}A \} \] (4.23)

\[ \dot{C}_1 = \{-e^{30}C_0 - e^{30}C_1 + e^{24m}C_a + e^{24m}C_0 + e^{24m}C_0 \}
+ e^{24m}C_1 + \{e^{24m}C_0 + e^{24m}A + e^{24m}A \}
+ e^{24m}A + e^{24m}A \} \] (4.24)

\[ \dot{A} = \{-e^{30}A - e^{30}A + e^{24m}A_0 + e^{24m}A_0 \}
+ e^{24m}A + \{e^{24m}A_0 + e^{24m}A_1 + e^{24m}A_0 \}
+ e^{24m}A + e^{24m}A \} \] (4.25)

\[ \dot{B} = \{-e^{30}B - e^{30}B + e^{24m}B + e^{24m}B \}
+ e^{24m}B + \{e^{24m}B + e^{24m}B_0 + e^{24m}B_0 \}
+ e^{24m}B + e^{24m}B \} \] (4.26)

The coefficients of this equation (\(e^{30}, e^{24m}, \text{etc.}\)) can be found in Appendix G (see Section G.5, Eq. (G.63)).

### 4.3.1 Nonlinear vibrations

To investigate several important phenomena of the nonlinear (i.e., large amplitude) vibrations of statically loaded imperfect anisotropic cylindrical shells under a harmonic lateral loading of the form (3.16), the expressions (3.1) and (3.2) for the imperfection and static response mode, respectively, are used:

**imperfection:**

\[
\tilde{W}/h = \tilde{\xi}_1 \cos \frac{2m\pi x}{L} + \tilde{\xi}_2 \sin \frac{m\pi x}{L} \cos \frac{n}{L} (y - \tau_k x)
\]

**static state:**

\[
\tilde{W}/h = \tilde{\xi}_0 + \tilde{\xi}_1 \cos \frac{2m\pi x}{L} + \tilde{\xi}_2 \sin \frac{m\pi x}{L} \cos \frac{n}{L} (y - \tau_k x)
\]

In the method used in Chapter 3, the time-dependent parts \(C_0\) and \(C_1\) of the axisymmetric vibration modes in the assumed displacement function (3.15) are constrained by relating them a priori to the time-dependent parts \(A\) and \(B\) of the
asymmetric modes by imposing the periodicity condition. In this way, the nonlinear vibration problem was reduced to a problem with two generalized coordinates, \( A \) and \( B \). This procedure has been discussed in depth in the previous chapter and corresponds with Evensen’s approach (Evensen, 1967). The constraining of the axisymmetric modes limits the range of validity of this approach. In this chapter, a “consistent” approach (cf. Prathap and Pandalai, 1978) is used, which retains the axisymmetric mode amplitudes \( C_0 \) and \( C_1 \) as independent coordinates. The following displacement field is assumed

\[
W/h = C_0(t) + C_1(t) \cos \frac{2m\pi x}{L} + A(t) \sin \frac{m\pi x}{L} \cos \frac{\ell}{R} (y - \tau_k x) \\
+ B(t) \sin \frac{m\pi x}{L} \sin \frac{\ell}{R} (y - \tau_k x) \tag{4.29}
\]

where the generalized coordinates \( A, B, C_0, \) and \( C_1 \) depend on the time \( t \).

### 4.3.2 Nonlinear parametric excitation

The imperfection and static response modes, Eqs. (3.1) and (3.2), are used to describe the behaviour of the anisotropic cylindrical shell under a static loading consisting of the three basic axisymmetric loads. In addition, the shell is loaded by the parametric axial, torsional, and radial loading given by

\[
\begin{align*}
N_0 \cos n_0 t; & \quad N_{xy} \cos n_0 t; & \quad \rho_0 c \cos \Omega_c t; \\
N_0 \cos n_0 t; & \quad N_{xy} \cos n_0 t; & \quad \rho_0 c \cos \Omega_c t \tag{4.30}
\end{align*}
\]

where \( N_0, T_0, \) and \( \rho_0 \) are constants, and where \( \Omega_c \) is the frequency of the applied load. The radial response under parametric excitation is described by the displacement function in Eq. (4.29).

### 4.3.3 Dynamic buckling

For dynamic buckling under step loading the displacement functions for imperfection (3.1) and static response (3.2) are again used.

The applied loads are the following step loads:

\[
\begin{align*}
\hat{N}_c &= -N_0 u(t); & \quad \hat{N}_{xy} &= \hat{T}_0 u(t); & \quad \hat{\rho}_c &= \hat{\rho}_0 u(t) \tag{4.31}
\end{align*}
\]

where \( u(t) \) is the unit step function. A general transient loading is given by

\[
\begin{align*}
\hat{N}_c &= -N_0 f(t); & \quad \hat{N}_{xy} &= \hat{T}_0 f(t); & \quad \hat{\rho}_c &= \hat{\rho}_0 f(t) \tag{4.32}
\end{align*}
\]

where \( f(t) \) is a specified function of time. For both cases \( \hat{N}_0, \hat{T}_0, \) and \( \hat{\rho}_0 \) are constants. It is assumed that the radial displacement of the dynamic state can be described by
the deflection function in Eq. (4.29). The effect of the in-plane inertia of axial and in-plane shear modes is taken into account via the approach discussed in Section 4.2. In this method, introduced by Tamura (1973), the complementary in-plane displacements (4.14) are included in the formulation,

\[ \ddot{u}^* = -\left(\frac{C_u h}{L}\right)x, \quad \ddot{v}^* = -\left(\frac{C_v h}{L}\right)x \]

The kinetic energy in Eq. (4.20) is the sum of the kinetic energy of the shell, Eq. (4.7), and the kinetic energy of the end ring, Eq. (4.19).

4.4 Results and discussion

The detailed derivations of the analyses described in the previous sections have been carried out using the symbolic manipulation program REDUCE (Hearn, 1993) and can be found in Appendix G. The results were coded in the FORTRAN computer program SILVANA (Simplified Large amplitude Vibration Analysis including Anisotropy). In this section, results obtained via this code will be presented. The Adams-Moulton method with variable step size is used for the numerical integration of the ordinary differential equations. The data of the shells used in the calculations have been compiled in Appendix F.

Characteristic results will be shown for shells under axial parametric excitation, and for the dynamic buckling of shells under axial step loading. For these two dynamic stability problems, examples for specific isotropic and anisotropic shells will be presented.

Further, the nonlinear vibration behaviour of a specific isotropic shell which has been used in experiments is investigated in detail. A comparison is made between the present analysis, based on numerical time integration, and Evensen’s approach (Chapter 3), in which the method of averaging is used, and in which the axisymmetric modes are constrained by relating them to the asymmetric modes (Eqs. (3.30) and (3.33)).

**Parametric excitation**

The first example for parametric excitation concerns the type of nonlinearity for moderately large amplitudes. It will be shown that for “single” (primary) mode vibrations it is necessary to include axisymmetric modes in the assumed response.

The shell is subjected to a parametric axial loading,

\[ \dot{N}_x^{nc} = -\ddot{N}_0 \cos \Omega_d t; \quad \dot{N}_x^{nc} = 0; \quad \ddot{p}_e^{nc} = 0 \]

The amplitude of the excitation is \( \lambda = \ddot{N}_0 / \dot{N}_d = 0.35 \). In addition, the shell is subjected to static axial previbration loading \( \dot{N}_x(x = L) = -\ddot{N}_0 \). The shell has an asymmetric imperfection.
4.4 Results and discussion

Figure 4.1: Softening behaviour of isotropic shell (Bogdanovich' shell) for 'single' mode vibration under parametric excitation. Frequency-response curves for different static axial loads $\lambda$. Forcing frequency $\Omega = \Omega_0/\omega_{im}$.

\[
W/H = 0.01 \sin \frac{\pi x}{L} \cos \frac{5}{R} y
\]

It is noted that for an imperfect shell, the applied axial load will induce static deformations

\[
\dot{W}/h = \tilde{\xi}_0 + \tilde{\xi}_1 \cos \frac{2\pi x}{L} + \tilde{\xi}_2 \sin \frac{\pi x}{L} \cos \frac{5}{R} y
\]

and the corresponding stresses.

The dynamic response mode is given by

\[
\dot{W}/h = C_0(t) + C_1(t) \cos \frac{2\pi x}{L} + A(t) \sin \frac{\pi x}{L} \cos \frac{5}{R} y
\]

which, in addition to the asymmetric mode, includes the corresponding axisymmetric modes. For the "single" (primary) mode vibration of this isotropic shell vibrating with $m = 1$ and $\ell = 5$, Bogdanovich (1993) obtains hardening behaviour, whereas the present model predicts a softening behaviour. This is illustrated in Fig. 4.1, which shows the response of Bogdanovich' shell under parametric axial loading (Eq. 4.30).

Frequency-response curves have been obtained via numerical time integration. Damping was neglected ($\zeta_A = 0$). The response amplitude increases in time, and after a rapid build-up has occurred it reaches a maximum. An integration time of 50 forcing periods $T_0 = 2\pi/\Omega_0$ was sufficient to capture this maximum. The maximum response during the integration interval has been plotted as a function of the forcing
Figure 4.2: Parametric excitation of Booton’s anisotropic shell, (a) frequency-response curve, (b) time history at $\omega = \Omega/\omega_{\text{lin}} = 1.95$.

The forcing frequency is normalized with respect to $\omega_{\text{lin}}$, the linear frequency of the statically loaded imperfect shell (for $\lambda = 0.5$, $\omega_{\text{lin}} = 346.92$ rad/s, for $\lambda = 0.25$, $\omega_{\text{lin}} = 424.26$ rad/s, for $\lambda = 0.1$, $\omega_{\text{lin}} = 464.56$ rad/s, and for $\lambda = 0$, $\omega_{\text{lin}} = 489.59$ rad/s).

For static loads which are small as compared to the classical buckling load, a Hopf bifurcation occurs near the double linear frequency of the vibration mode considered. The associated jump phenomenon corresponds to a softening behaviour. The response curves “bend backwards”, resulting in upward jumps for the upward sweep and downward jumps for the downward sweep, typical of a softening behaviour.

At the static axial load level $\lambda = 0.5$, a dynamic buckling phenomenon occurs. It is noted that both the softening behaviour observed and the dynamic buckling phenomenon are captured by the present model because of the inclusion of axisymmetric modes in the assumed deflection function (4.36). If the axisymmetric modes are not included, one misses these phenomena.

A characteristic result for the parametric vibrations of the anisotropic Booton’s shell is depicted in Fig. 4.2. The figure shows the frequency-response curve of the vibration mode

$$\frac{\ddot{W}(t)}{\ddot{h}} = C_0(t) + C_1(t) \cos \frac{2\pi x}{L} + A(t) \sin \frac{\pi x}{L} \cos \frac{6}{R} \left(y - \tau_K x\right)$$

with $\tau_K = -0.002$.

The shell is subjected to a pulsating axial load,

$$\ddot{N}_x^e = -\dot{N}_0 \cos \Omega_t t; \quad \dot{N}_y^e = 0; \quad \dot{N}_z^e = 0$$

with amplitude $\dot{\lambda} = \dot{N}_0/N_d = 0.1$. The following two-mode imperfection is assumed:
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Figure 4.3: Maximum response under axial step loading, Tamura’s isotropic shell (Tamura, 1973).

\[ \tilde{W}/h = -0.04 \cos \frac{2\pi x}{L} + 0.05 \sin \frac{\pi x}{L} \cos \frac{6}{R} (y - \tau_K x) \]

with \( \tau_K = -0.002 \). Damping is neglected (\( \zeta_A = \zeta_C = \zeta_C' = 0 \)). The maximum response during the integration time of 200 forcing periods \( T_0 \) (where \( T_0 = 2\pi/\Omega_0 \)) is plotted as a function of the forcing frequency. The forcing frequency is normalized with respect to \( \omega_{lin} \), the linear frequency of the imperfect shell (\( \omega_{lin} \sqrt{\rho} = 597.9\sqrt{N/m^2} \)).

The response in time is shown for a forcing frequency \( \Omega = \Omega_0/\omega_{lin} = 1.95 \). The initial conditions were \( A(t = 0) = \dot{A}(t = 0) = 0 \). Initially, the response amplitude remains small for several forcing periods, until between \( \tau = (1/R)\sqrt{(E/\rho)t = 2000} \) and \( \tau = 3000 \) a rapid build-up occurs. It is noted that if damping is present, depending on the value of the damping parameter, gradually the beating character of the response disappears in time, and the amplitude tends to go to a constant value (cf. Yao, 1965).

**Dynamic buckling**

In Fig. 4.3, results for the dynamic buckling under axial step loading are depicted,

\[ \ddot{N}_x = -\dot{N}_0 u(t); \quad \ddot{N}_{xy} = 0; \quad \ddot{p}_c = 0 \]  

(4.40)

First example is the dynamic buckling of Tamura’s isotropic shell (Tamura, 1973). The inertia of the axial mode

\[ \ddot{u} = -\left( \frac{C_a h}{L} \right) x \]

is included (see Eq. (4.14)). The damping parameter corresponding to the fundamental axial mode \( \zeta_{c_x} = 0.2 \) (Appendix G).

Bending modes with a high axial wave number may be important for the dynamic buckling behaviour under axial step loading. Tamura (1973) has shown, that these
Figure 4.4: Time histories of the different displacement components near the dynamic buckling load. a) asymmetric mode, b) axial mode, c) constant axisymmetric mode and d) double harmonic axisymmetric mode; Tamura's shell (Tamura, 1973).
modes can be sensitive to buckling, parametrically induced by the vibration in the axial mode.

In this example, the shell is assumed to buckle in the mode \((m, \ell) = (20, 29)\),

\[
W(t)/h = C_0(t) + C_1(t) \cos \frac{40\pi x}{L} + A(t) \sin \frac{20\pi x}{L} \cos \frac{29}{R} y
\]  

(4.42)

For this case, dynamic buckling is induced by the parametric axial loading which corresponds to the vibration in the axial mode. The maximum response of the asymmetric mode during an integration time \(\tau_{fin} = 400\) \((\tau = (1/R)\sqrt{(E/\rho)t})\) has been used to plot the load-response curve, which shows a clear jump when the load is increased from \(\lambda = \lambda_0/N_d = 0.80\) to \(\lambda = 0.805\). The latter load level can be defined as the dynamic buckling load.

For \(\lambda\) greater than 0.825, the applied step load will always result in an escape to the remote equilibrium path. Due to interactions of the different modes involved, it is possible that for certain load values below this level (from \(\lambda = 0.805\)), the shell buckles dynamically, i.e. the shell escapes to the remote equilibrium path, while for a slightly different load value the response remains small during the integration time considered. This is illustrated in Fig. 4.3, where for \(\lambda = 0.810\) and \(\lambda = 0.820\) the response remains small, while for \(\lambda = 0.815\) dynamic buckling occurs. This type of behaviour was also found by Tamura (1973). Another example of this phenomenon will be shown later for Booton’s shell (see Fig. 4.6). Time histories \((\tau = 200)\) of the different displacement components for two characteristic values of the loading near the dynamic buckling load are shown in Fig. 4.4.

The second example for dynamic buckling concerns the behaviour of Booton’s anisotropic shell under axial step loading. In Fig. 4.5, static response curves of Booton’s shell are shown for the mode \((3, 5, -1.56)\),

\[
\tilde{W}/h = \tilde{\xi}_0 + \tilde{\xi}_1 \cos \frac{6\pi x}{L} + \tilde{\xi}_2 \sin \frac{3\pi x}{L} \cos \frac{5}{R} (y - \tau_K x)
\]

with \(\tau_K = -1.56\), the mode corresponding to the lowest static bifurcation buckling load \(\lambda_{as} = 0.407\). The response curve corresponding to the mode with the same \(m\) and \(\ell\) but without skewedness, \((m, \ell, \tau_K) = (3, 5, 0)\), is also shown in this figure. The static bifurcation buckling load of this mode occurs at \(\lambda = 0.829\). The buckling load of the corresponding axisymmetric mode \((2m, 0) = (6, 0)\) occurs at \(\lambda = 0.501\). In both cases, the imperfection amplitudes assumed are \(\tilde{\xi}_1 = -0.04\) and \(\tilde{\xi}_2 = 0.05\). Static response curves for the mode corresponding to the lowest vibration frequency \((m, \ell, \tau_K) = (1, 6, -0.002)\) can be found in Fig. 3.6.

After characterizing the static behaviour of the cases \((m, \ell, \tau_K) = (3, 5, -1.56)\), \((m, \ell, \tau_K) = (3, 5, 0)\), and \((m, \ell, \tau_K) = (1, 6, -0.002)\), results for Booton’s anisotropic shell under axial step loading are depicted in Fig. 4.6. The maximum response of the asymmetric mode during an integration time \(\tau_{fin} = 2000\) \((\tau = (1/R)\sqrt{(E/\rho)t})\) was used to plot the load-response curve.

In Fig. 4.6a, the shell is assumed to buckle dynamically in the mode \((m, \ell, \tau_K) = (3, 5, -1.56)\),
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Figure 4.5: Static asymmetric mode response of imperfect anisotropic shell; Booton’s shell; Response mode (3,5,\(\tau_K\)).

\[
W(t)/h = C_0(t) + C_1(t) \cos \frac{6\pi x}{L} + A(t) \sin \frac{3\pi x}{L} \cos \frac{5}{R}(y - \tau_K x) \tag{4.44}
\]

with \(\tau_K = -1.56\). This mode corresponds to static buckling. The inertia of the in-plane modes has not been taken into account. The imperfection amplitudes are \(\xi_1 = -0.04\) and \(\xi_2 = 0.05\),

\[
\bar{W}/h = -0.04 \cos \frac{6\pi x}{L} + 0.05 \sin \frac{3\pi x}{L} \cos \frac{5}{R}(y - \tau_K x)
\]

with \(\tau_K = -1.56\), and damping is neglected \((\zeta = 0)\). The maximum response is given as a function of the applied step load. It is noted that a clear jump in this curve does not occur in this case, since the buckling mode considered has a stable (static) postbuckling behaviour.

The mode corresponding to the lowest vibration mode is imperfection sensitive and the response curve shows a distinct jump in this case at the dynamic buckling load \(\lambda = 0.445\) (see Fig.4.6b). A distinct jump also occurs for the mode (3,5,0), in this case at \(\lambda = 0.415\) (see Fig.4.6c). However, similar to the phenomenon encountered in the example of Tamura’s shell (Fig. 4.3), also at lower loads a jump to the remote branch already may occur. For the deflection functions chosen, at the applied step loads \(\lambda = 0.385\) and at \(\lambda = 0.388\) the shell buckles dynamically.

Comparison between Method of Averaging and Numerical Time-integration

In Figs. 4.7 and 4.8, results for the free nonlinear vibration behaviour obtained with the method presented in the previous chapter are compared with the present
4.4 Results and discussion

Figure 4.6: a) Maximum response under axial step loading: Booton’s anisotropic shell (Booton, 1976); a) Mode (3,5,-1.56), b) Mode (1,6,-0.002), c) Mode (3,5,0).
approach. In the previous chapter, the method of averaging was used in order to eliminate the time dependence. The axisymmetric modes of the assumed displacement function are a priori related to the asymmetric mode, hereby reducing the number of degrees of freedom.

Results are shown for an isotropic shell (Olson’s shell), vibrating in the (1,10) mode. It should be noted that to describe the vibration behaviour properly one should include the double harmonic in the circumferential direction of the shell in the assumed deflection function. This issue will be addressed in the next chapter.

The results of the approaches are compared for a large vibration amplitude and for large imperfection amplitudes. Backbone curves obtained using the method of averaging are compared with numerical integration results for a specific large vibration amplitude ($\bar{A} = 3$). The frequencies have been normalized with respect to the frequency of the perfect shell, $\Omega = \omega/\omega_{ntr}$. In Fig. 4.7, results are shown for an isotropic shell with axisymmetric imperfections. The results using numerical integration are seen virtually to coincide with the results obtained via the method of averaging.

As stated in the previous chapter, the zeroth order effect of the imperfections on the amplitude-frequency relation is predominant. In Fig. 4.8, it can be seen that in this case even for large asymmetric imperfection amplitudes the results obtained by numerical time integration deviate only slightly from the results obtained via the method of averaging.

Nonlinear forced vibrations

In the following, the nonlinear coupled mode vibration behaviour of an axially loaded
isotropic (aluminum) shell is studied. The shell has been used in the experiments of Gunawan (Gunawan, 1998) \((R = 125 \text{ mm, } L = 240 \text{ mm, } h = 0.253 \text{ mm, specific mass } \rho = 2700 \text{ kg/m}^3, \text{ Young’s modulus } E = 72000 \cdot 10^6 \text{ N/m}^2)\). The results of a response calculation will be presented, based on the assumed displacement in Eq. (4.29) for Gunawan’s shell vibrating in the \((m = 1, \ell = 11)\) mode,

\[
\hat{W}/h = C_0(t) + C_1(t) \cos \frac{2\pi x}{L} + A(t) \sin \frac{\pi x}{L} \cos \frac{11}{R} y + B(t) \sin \frac{\pi x}{L} \sin \frac{11}{R} y \quad (4.46)
\]

The damping parameters corresponding to the asymmetric modes are \(\zeta_A = \zeta_B = 0.0023\). The shell is loaded by a static axial compressive load \(P = 2 \text{ kN} \quad (\lambda = (cR)/(\rho h^2)N_0 = 0.1174)\), which reduces the linear frequency from 518.0 Hz (3255 rad/s) to 501.7 Hz (3152 rad/s). The excitation pressure \(Q_{\text{exc}} = 22 \text{ Pa}\) results in response amplitudes of about two times the wall thickness. This pressure level corresponds to the acoustic pressure in the experiments created by two acoustic drivers, positioned at opposite shell circumferential locations.

The response was calculated by integrating over a sufficiently large number of forcing periods for a sequence of frequencies near the natural frequency. The excitation frequency was both increased (“upward sweep”) and decreased (“downward sweep”) with small increments \(\Delta \omega\). The end conditions after the numerical integration at a specific excitation frequency were used as initial conditions for the response calculation at the incremented frequency. For each excitation frequency, the vibration amplitudes of the driven and companion mode during the last forcing period were plotted. They are denoted as \(|A_{\text{fin}}|\) and \(|B_{\text{fin}}|\), respectively. The amplitudes are defined here as the maximum absolute values of the coefficients \(A\) and \(B\) during one forcing cycle.

Figure 4.8: Backbone curves of isotropic shell with asymmetric imperfection. Comparison between Method of Averaging and Numerical Integration; Olson’s shell. \(\hat{W}/h = \xi_2 \sin \frac{\pi x}{L} \cos \frac{10}{R} y\)
When the quiescent companion mode is not disturbed, one can obtain the single mode response. In the single mode response calculations ($\Delta \omega = 5 \text{ rad/s}$), for each excitation frequency, numerical integration over 500 forcing periods was performed. This integration interval was sufficient to reach a stationary response. The response exhibits a softening behaviour (Fig. 4.9). The well-known jump phenomenon is observed both for the upward frequency sweep and the downward sweep.

However, the single mode response is unstable with respect to perturbations in the companion mode in a region around resonance. In this region, the companion mode participates in the vibration, and a coupled mode response occurs. The response calculations for the coupled mode response region have been carried out with a frequency step $\Delta \omega = 1 \text{ rad/s}$ and integration times of 1500 forcing periods. The result is depicted in Fig. 4.10. The integration time used was sufficient in the main part of the frequency interval to reach a stationary response. However, the coupled mode response region includes an interval, in which modulated responses are observed where the response drifts between (the neighbourhood of) the single mode and coupled mode solutions, and the vibration amplitude is no longer constant in time. This phenomenon corresponds to the results obtained by Liu (Liu, 1988) using Evensen's steady-state approach (Evensen, 1967). For the isotropic shell investigated in that study Liu found via a stability analysis (based on a perturbation procedure in combination with the method of averaging) that there is a region where both the single mode and the coupled mode response are unstable.
Figure 4.10: Coupled mode response of isotropic shell for high excitation level, a) driven mode, b) companion mode. Gunawan’s shell, mode (1,11), $Q_{\text{mfr}} = 22$ Pa, $\zeta_A = 0.0023$, $P = 2$ kN ($\lambda = 0.1174$).
Figure 4.11: Poincaré map for various excitation frequencies; Driven mode (left) and companion mode (right).
4.4 Results and discussion

\[ n = 3120.0 \, \text{rad/s} \]

\[ n = 3115 \, \text{rad/s} \]

Figure 4.12: Poincaré map for various excitation frequencies; Driven mode (left) and companion mode (right).
Figure 4.13: Poincaré map for various excitation frequencies; Driven mode versus companion mode.
Figure 4.14: Poincaré map for various excitation amplitudes; Driven mode (left) and companion mode (right). Excitation frequency $\omega = 3115 \text{ rad/s}$. 
Figure 4.15: Poincaré map for various excitation amplitudes; Driven mode (left) and companion mode (right). Excitation frequency $\omega = 3115 \text{ rad/s}$. 

$Q_{mfr} = 21 \text{ Pa}$

$Q_{mfr} = 22 \text{ Pa}$
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Figure 4.16: Poincaré map for various excitation amplitudes; Driven mode versus companion mode. Excitation frequency $\omega = 3115 \text{ rad/s}$.
Figure 4.17: Time history of response, sampled at each forcing period for various forcing amplitudes. Driven mode (solid line) and companion mode (dashed line). Excitation frequency $\omega = 3115$ rad/s.
4.4 Results and discussion

The different possibilities of the coupled mode response behaviour will now be described for decreasing excitation frequency. The response can be represented in Poincaré maps, i.e. in plots in which response variables have been sampled at discrete times $t = n(2\pi/\omega)$. In the following, for both the driven mode and the companion mode, the responses $(A, A)$ and $(B, B)$ have been sampled at discrete times $t = n(2\pi/\omega)$ for post-transient motion ($n = 3000$ - $5000$), see Figs. 4.11 and 4.12. Fig. 4.13 shows Poincaré maps for the driven mode versus the companion mode.

Recall that the (linear) natural frequency occurs at $f = 501.68$ Hz ($\omega = 3152.12$ rad/s). Decreasing the excitation frequency, slightly above the natural frequency a bifurcation in the single mode frequency-response curve occurs at $f = 503.1$ Hz ($\omega = 3161$ rad/s), cf. Figs. 4.9 and 4.10. The single mode response becomes unstable with respect to perturbations in the companion mode, which at that frequency begins to participate in the vibration. A steady state coupled mode response appears. The Poincaré map at $f = 503.1$ Hz ($\omega = 3161$ rad/s) consists of a single point in the $(A, A)$ plane and a corresponding single point in the $(B, B)$ plane. The driven mode solution curve bifurcates (slightly) upward, i.e. the slope of the curve becomes steeper (more negative) (cf. Evensen, 1974). Decreasing the frequency further, at $f = 498.95$ Hz ($\omega = 3135$ rad/s) the steady state coupled mode responses $A$ and $B$ jump down to their lower branches (Fig. 4.10). Subsequently, there is a region where both the steady state single mode response and the coupled mode response are unstable (Liu, 1988). This starts with a slight beating at $f = 497.76$ Hz ($\omega = 3127.5$ rad/s), see Fig. 4.11. The Poincaré map shows small closed curves, indicating quasiperiodic motion. At $f = 497.36$ Hz ($\omega = 3125$ rad/s), the system
has obviously also a tendency to respond in a single (driven) mode (Fig. 4.11). This results in a strong beating response where the solution drifts between a “single” mode and a “coupled” mode response. The Poincaré map still shows closed curves, characteristic of quasiperiodic motion. At \( f = 496.56 \text{ Hz } (\omega = 3120 \text{ rad/s}) \) this drifting behaviour is accompanied by another drifting tendency, namely the drift between two companion mode solutions of opposite sign (see Fig. 4.12). It should be noted that in the steady state case, the companion mode solution is determined to within a sign. An alternative interpretation is that there are two possible directions of the circumferentially travelling wave. For steady state vibrations, the actual sign in the (numerical) experiment is decided by the initial conditions. In the nonstationary case, there is a wandering from one companion mode solution to a companion mode with the opposite sign, corresponding to reversals of the travelling wave direction during the vibration. At \( f = 496.56 \text{ Hz } (\omega = 3120 \text{ rad/s}) \) a gradual transition to the opposite sign of the companion mode results in a regular beating character. The corresponding time history of the displacements shows modulated coupled mode responses. It is noted that a similar beating/nonstationary response has been observed earlier in experiments (Chen, 1972). At \( f = 495.77 \text{ Hz } (\omega = 3115 \text{ rad/s}) \) this regular beating character has disappeared and a chaotic-like modulation is observed. Figs. 4.12 and 4.13 shows that for the forcing frequency \( f = 495.77 \text{ Hz } (\omega = 3115 \text{ rad/s}) \) the companion mode, wandering between positive and negative companion mode solutions, approaches the origin of the phase-plane (or Poincaré map). Increasing the excitation level \( Q_m/s \) from zero to 22 Pa for the fixed frequency \( \omega = 3115 \text{ rad/s} \) confirms this observation (see Figs. 4.14 to 4.16). The chaotic-like behaviour in the modulation is associated with the alternative paths which are followed in an irregular sequence. The corresponding response displacements, sampled at every forcing period, have been plotted as a function of time in Fig. 4.17. Here, the nondimensional time \( \tau = \frac{1}{R} \sqrt{\left( \frac{E}{\rho} \right) t} \) is used. Fig. 4.18 shows the chaotic-like modulation at 22 Pa over a larger time interval.

Results obtained with SILVANA have been compared with experimental results found by Gunawan in Jansen and Gunawan (1996) and Gunawan (1998). The softening behaviour and the occurrence of the companion mode were found both in the physical experiment and in the numerical simulation. Beating responses have also been observed in the physical experiments. For further details see Jansen and Gunawan (1996) and Gunawan (1998).

4.5 Conclusions

A Level-1 (‘Simplified’) Analysis has been presented, which is capable of analysing the dynamic stability and nonlinear flexural vibration behaviour of anisotropic cylindrical shells. The method is based on a variational formulation using a small number of modes in combination with numerical time-integration of the resulting set of ordinary differential equations.

Characteristics of the nonlinear vibration behaviour of an axially loaded, radially excited isotropic shell have been highlighted, namely, the softening behaviour,
4.5 Conclusions

the coupled mode (travelling wave) response, and the nonstationary vibrations in
which the response drifts between single mode and different types of coupled mode
solutions. Transitions occur between single mode response and different types of
coupled mode responses. Chaos-like modulated responses have been observed in
the numerical experiments. In future investigations, the qualitative aspects of the
nonstationary responses may be studied in more detail via the present model using
tools of the theory of nonlinear dynamics and chaos (see e.g. Thompson and Stewart
(1986); Moon (1987)).

Moreover, the model developed was used to analyse several dynamic stability
problems of cylindrical shells, such as dynamic buckling and nonlinear parametric
excitation. Examples were shown, in which mode interactions strongly influence the
nonlinear behaviour.

Referring to the remarks on the accuracy and applicability of the Simplified
Analysis based on Evensen's method stated in the previous chapter, the following
the following observations are made with regard to the present analysis:

• Hamilton's principle is applied as an approximate solution method in order to
obtain a set of ordinary differential equations in the amplitudes of the assumed
deflection modes. Numerical time-integration gives an accurate solution of the
resulting dynamic response problem.

The (secondary) axisymmetric modes are included in the formulation as addi­tional
generalized coordinates. The double harmonic in the circumferential
direction is not included in the present formulation. The effect of this sec­ondary
response mode will be investigated in Chapter 5.

The interaction of the fundamental axial and torsional mode with "single
mode" or "coupled mode" vibrations can be taken into account in the present
model. The analysis presented may be extended to multi-mode analyses, which
take the coupling between different circumferential and axial harmonics into
account. In particular, this may be useful for dynamic buckling analyses and
parametric excitation problems. Applying a multi-mode analysis may also
influence the nonlinear vibration results presented in this chapter, but the
main characteristics of the nonlinear coupled mode response are believed to
be captured by the present model.

• To obtain a good (quantitative) correlation between theory and experiment
it is necessary to use discretization models which include the effect of the
boundary conditions at the shell edges. This is possible by using Level-2 type
methods (see the next chapter) or Level-3 type methods (methods based on
finite elements).
Level-1 Transient Analysis

A Level-1 (Simplified) Analysis has been presented, which demonstrates simulating an
the dynamic stability and nonlinear free-vibration behaviour of anisotropic-cylindrical shellmodelced
(simplified) model. The method is based on a variational formulation using a small number
of modes in combination with numerical time-integration of the resulting set of
ordinary differential equations.

Characteristics of the nonlinear vibration behaviour of an axially loaded, radially
excited isotropic shell have been highlighted, namely, the softening behaviour.
Chapter 5

Level-2 Perturbation Analysis

5.1 Introduction

In this chapter, the effect of boundary conditions at the shell edges (including the effect of a nonlinear prebuckling or previbration state) on the buckling and vibration of an anisotropic shell is considered. The method can be classified as a Level-2 Analysis or Extended Analysis (cf. Section 1.2).

It is well known that boundary conditions influence the restoring (incremental) bending moments and the stabilizing (incremental) membrane stresses during buckling and vibration. Moreover, the prebuckling or previbration bending deformation due to the edge restraint will be accompanied by destabilizing compressive membrane stresses in the circumferential direction of the shell. For shorter shells, the fundamental state (prebuckling or previbration state) may affect the buckling or vibration behaviour considerably. In the following, a formulation will be presented for the nonlinear vibration analysis of composite cylindrical shells, in which the (uniform) elastic boundary conditions at the shell edge are rigorously satisfied.

A method often applied for the analysis of buckling and vibration problems of shells of revolution under axisymmetric loads (including torsion) is to use a Fourier decomposition in the circumferential direction of the shell for the dependent variables, in order to eliminate the dependence of the solution on the circumferential coordinate. The resulting boundary value problem of ordinary differential equations in the meridional direction is solved numerically. Initial value (shooting) techniques have often been applied for this purpose (Kalnins, 1964; Cohen, 1968a; Arbocz and Sechler, 1976; Cohen, 1981), also in problems directly related to the present work (Liu, 1988; Arbocz and Hol, 1989). Via shooting methods, accurate solutions can be obtained by the numerical integration of the differential equations using an initial value solver. In the present study the vibration behaviour of composite cylindrical shells is investigated. In combination with a perturbation method to describe the temporal behaviour, the parallel shooting method (Keller, 1968; Hall and Watt, 1976; Ascher et al., 1988) is employed to solve the spatial two-point boundary value problems resulting from the Fourier decomposition of the dependent variables.

In Section 5.2, the equations governing the static and dynamic state of composite cylindrical shells will be presented. By means of separation of variables the
problems governing the various states are reduced to a two-point boundary value problem (with the axial coordinate as independent variable). The static and dynamic first-order states form eigenvalue problems. Values of the eigenvalue parameter are sought, for which a nontrivial solution exists. The (linearized) buckling and vibration problem are treated in Section 5.5.1. Flutter in supersonic flow is discussed in Section 5.5.2.

In Appendix B, a general theory is given for the vibration of structures about a static nonlinear fundamental state, based on a perturbation expansion for both the frequency parameter and the dependent variables. The theory includes the effects of finite amplitudes, imperfections, and a nonlinear static deformation (orthogonal to the fundamental state). The dependence of the frequency on these parameters is given for free vibrations. It is noted that the formulae derived in Appendix B are generally applicable. They can also be used in a Level-1 ("Simplified") Analysis, i.e. an analysis in which the displacement satisfies "simply supported" boundary conditions (cf. the Level-1 buckling "b-factor" analysis for composite cylinders by Arbocz (1992)). In the present analysis, this perturbation theory is applied to the nonlinear vibration problem of composite cylindrical shells including edge effects. It should be noted that the starting point differs from that in the previous chapters. The starting point of the present analysis are the Donnell-type differential equations of a shell with an axisymmetric imperfection only, and not the governing nonlinear equations including imperfections. The first-order state problem is an eigenvalue problem for the unknown eigenfrequency and vibration modes. The associated higher order state problems are response problems which depend on the solution of the corresponding first-order state problem.

The perturbation procedure leads to boundary value problems for partial differential equations with the two spatial coordinates as independent variables. A Fourier decomposition in the circumferential direction of the shell is used for the dependent variables to reduce these problems to sets of ordinary differential equations for the axial direction. The specified boundary conditions are satisfied rigorously by solving the resulting two-point boundary value problems numerically via the parallel shooting method. Nonlinear Donnell-type equations formulated in terms of the radial displacement $W$ and an Airy stress function $F$ are used, and classical lamination theory is employed. The numerical solution procedure used to solve the governing equations is discussed in Section 5.8. Numerical results are presented in Section 5.9.

The perturbation theory used is exact in an asymptotic sense (Rehfield, 1974). In its original form, only the lowest order effects of the nonlinearity are taken into account. In Appendix B, the theory is extended to a higher order analysis of the nonlinear vibrations of (perfect) structures. Further, the result of a multi-mode analysis in Appendix B will be applied to the two-mode case of 'travelling wave' coupled mode nonlinear vibrations, which involves two asymmetric modes which are circumferentially ninety degrees out-of-phase between each other.

The theory developed in this chapter is related to the initial post-buckling and imperfection sensitivity theory for composite shells (Arbocz and Hol, 1989), in which the dependence of the load parameter on the deflection and imperfection amplitudes is obtained. It is noted that the initial post-buckling theory can be extended to
deal with dynamic buckling problems (Budiansky and Hutchinson, 1964; Budiansky, 1967). Furthermore, the linearized vibration problem forms the basis of the analysis to determine the dynamic instability regions under parametric axisymmetric or torsional loading.

Finally, it is noted that the theory presented for free vibrations can be extended to forced vibrations (Chen, 1972; Rehfield, 1974). Chen (1972) also includes the effect of viscous damping. Ginsberg (1972) investigated the stability of the response via a perturbation analysis.

5.2 Governing equations

The Donnell-type nonlinear equations of a perfect shell for a general anisotropic material can be deduced from Eqs. (2.1) and (2.2) and can be written for the static state as

\[
L_A \cdot \{f\} - L_B \cdot \{w\} = -\frac{1}{R} \frac{\partial^2}{\partial x^2} W_{xx} - \frac{1}{2} L_{NL}(\tilde{W}, \tilde{W}) \quad (5.1)
\]

\[
L_B \cdot \{f\} + L_D \cdot \{w\} = \frac{1}{R} \frac{\partial^2}{\partial x^2} F_{xx} + L_{NL}(\tilde{F}, \tilde{W}) + \bar{p} \quad (5.2)
\]

where \(\bar{p}\) is the static lateral loading. The corresponding equations governing the nonlinear dynamic state are

\[
L_A \cdot \{f\} - L_B \cdot \{w\} = -\frac{1}{R} \frac{\partial^2}{\partial x^2} W_{xx} - L_{NL}(\tilde{W}, \tilde{W}) - \frac{1}{2} L_{NL}(\tilde{W}, \tilde{W}) \quad (5.3)
\]

\[
L_B \cdot \{f\} + L_D \cdot \{w\} = \frac{1}{R} \frac{\partial^2}{\partial x^2} F_{xx} + L_{NL}(\tilde{F}, \tilde{W}) + L_{NL}(\tilde{F}, \tilde{W})
+ L_{NL}(\tilde{F}, \tilde{W}) - \bar{p} W_{tt} + \bar{p} \quad (5.4)
\]

where \(\bar{p}\) is the dynamic lateral loading.

5.3 Perturbation expansion for buckling and vibrations

For the static displacement the following perturbation expansion is assumed:

\[
\tilde{W} = \tilde{W}^{(0)} + \xi \tilde{W}^{(1)} + \xi^2 \tilde{W}^{(2)} + \ldots \quad (5.5)
\]

\[
\tilde{F} = \tilde{F}^{(0)} + \xi \tilde{F}^{(1)} + \xi^2 \tilde{F}^{(2)} + \ldots \quad (5.6)
\]

where \(\xi\) is a measure of the static asymmetric displacement amplitude. In the case of free vibrations, the dynamic lateral excitation is equal to zero (\(\bar{p} = q = 0\)). Assuming that a single vibration mode is associated with the (linear) natural frequency \(\omega_c\), the following perturbation expansion for the frequency \(\omega\) is considered.
Level-2 Perturbation Analysis

\[(\frac{\omega}{\omega_c})^2 = 1 + a_d \xi_v + b_d \xi_v^2 + \ldots \]
\[+ (b_{110} \xi_t + b_{101} \xi) + (b_{210} \xi_t + b_{201} \xi) \xi_v + \ldots \]
\[+ (b_{120} \xi_t^2 + b_{111} \xi + b_{102} \xi^2) + \ldots \]
\[= 1 + a_d \xi_v + b_d \xi_v^2 + \ldots \]
\[+ \sum_{i,j,k} b_{ijk} \xi_v^{i-1} \xi_t^j \xi_k^k + \ldots \]  

and the corresponding solution is assumed as

\[\hat{W} = \xi_v \hat{W}^{(1)} + \xi_v^2 \hat{W}^{(2)} + \ldots \]
\[+ \xi_t \xi_v \hat{W}^{(11)} + \xi_t \xi_v^2 \hat{W}^{(12)} + \ldots \]
\[+ \xi_t^2 \xi_v \hat{W}^{(21)} + \xi_t^2 \xi_v^2 \hat{W}^{(22)} + \ldots \]
\[+ \ldots \]  

\[\hat{F} = \xi_v \hat{F}^{(1)} + \xi_v^2 \hat{F}^{(2)} + \ldots \]
\[+ \xi_t \xi_v \hat{F}^{(11)} + \xi_t \xi_v^2 \hat{F}^{(12)} + \ldots \]
\[+ \xi_t^2 \xi_v \hat{F}^{(21)} + \xi_t^2 \xi_v^2 \hat{F}^{(22)} + \ldots \]
\[+ \ldots \]

where \(\xi_t = \xi + \xi\), and where \(\hat{W}^{(1)}\) will be normalized with respect to the shell thickness \(h\) and \(\hat{W}^{(2)}\) is orthogonal to \(\hat{W}^{(1)}\) in an appropriate sense (cf. Appendix B, Section B.3); \(\xi_v\) is a measure of the displacement amplitude, and \(\xi\) is the asymmetric imperfection amplitude. A formal substitution of these expansions into the nonlinear governing equations (5.1) and (5.2) yields a sequence of equations for the functions appearing in the expansions. The coefficients \(a_d, b_d, b_{ijk}\) are derived in Section B.3 of Appendix B for general structures and they will be evaluated for the present case in Section 5.7. Notice that for the present approach there is an analogy with the well-known “b-factor method” used for static buckling problems (e.g. Arbocz and Hol, 1989). In view of this analogy, the coefficients of the perturbation expansion \(b_d\) and \(b_{ijk}\) used in the present method will be denoted as “dynamic b-factors”. A negative \(b_d\) corresponds to an initially softening nonlinear behaviour.

5.4 Static axisymmetric fundamental state

The set of governing equations for \(\hat{W}^{(0)}\) and \(\hat{F}^{(0)}\), the static axisymmetric fundamental state (0th-order state) is

\[L_A(\hat{F}^{(0)}) - L_B(\hat{W}^{(0)}) = -\frac{1}{R} \hat{W}_{zx}^{(0)} - \frac{1}{2} L_{NL}(\hat{W}^{(0)}, \hat{W}^{(0)})\]  

\[L_B(\hat{F}^{(0)}) + L_D(\hat{W}^{(0)}) = \frac{1}{R} \hat{F}_{zx}^{(0)} + L_{NL}(\hat{F}^{(0)}, \hat{W}^{(0)}) + \hat{p}\]
5.4 Static axisymmetric fundamental state

where the static lateral loading consists of a uniform external pressure \( \bar{p} = p_e \). The axisymmetric fundamental state can be represented as

\[
\dot{W}^{(0)} = h(W_w + W_p + W_t) + \dot{h}w_0(x) \tag{5.12}
\]

\[
\dot{P}^{(0)} = \frac{Eh^2}{cR} \left\{ -\frac{1}{2} \dot{\lambda} - \frac{1}{2} \dot{\bar{p}}x^2 - \ddot{\tau}xy + \dddot{R}f_0(x) \right\} \tag{5.13}
\]

where \( c, E, \) and \( h \) are reference values (Appendix A). The constant term in the radial displacement (a generalized ‘Poisson’s expansion’) can be related to the applied loads (cf. Section 2.3) by enforcing the circumferential periodicity condition (Appendix C) and consists of the following three contributions,

\[
\dot{W} = \frac{A^*_{12}}{c} \lambda; \quad \dot{W}_p = \frac{A^*_{22}}{c} \bar{p}; \quad \dot{W}_t = -\frac{A^*_{20}}{c} \bar{f} \tag{5.14}
\]

where the loading parameters are defined by

\[
\lambda = \frac{cR}{Eh^2} \tilde{N}_0; \quad \bar{p} = \frac{cR^2}{Eh^2} p_e; \quad \bar{f} = \frac{cR}{Eh^2} \tilde{T}_0 \tag{5.15}
\]

where \( \tilde{N}_0 = -\tilde{N}_e(x = L) \) is the applied axial compression, \( p_e \) is the applied (net) external pressure, and \( \tilde{T}_0 = \tilde{T}_{xy}(x = L) \) the applied counter-clockwise torque. Since the in-plane loads at the shell edges, \( \tilde{N}_0 \) and \( \tilde{T}_0 \), are assumed to be prescribed, the corresponding in-plane displacements at the edges are free (cf. Section 2.3). The nondimensional stiffness parameters \( \tilde{A}^*_{ij}, \tilde{B}^*_{ij}, \) and \( \tilde{D}^*_{ij} \) are defined in Appendix A.

Introduction into the governing equations and rearrangement gives two 4th-order differential equations in \( \dot{w}_0 \) and \( f_0 '' \):

\[
\tilde{A}^*_{22} f_0 '' - \frac{h}{2R} \tilde{B}^*_{21} w_0'' = -c\dot{w}_0'' \tag{5.16}
\]

\[
2\tilde{B}^*_{21} f_0'' - \frac{h}{R} \tilde{D}^*_{11} w_0'' = \frac{4cR}{h} f_0'' - 4c\lambda \dot{w}_0'' \tag{5.17}
\]

where \( \dot{)} = d(\)/\( dx \) and \( x = \bar{x}/R \). The compatibility equation can be integrated twice to obtain an expression for \( f_0 '' \),

\[
f_0'' = \frac{h}{2R} \tilde{B}^*_{21} w_0'' - \frac{c}{\tilde{A}^*_{22}} w_0 + \tilde{C}_1 \bar{x} + \tilde{C}_2 \tag{5.18}
\]

where the constants of integration \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are set equal to zero because, in combination with the definition of the Poisson’s expansions, the periodicity condition is then satisfied (Appendix C). The equations governing the fundamental state can now be combined into one equation.
\[
(D_1 + \frac{B_2^2}{A_{22}^2})w_0'' + \frac{4cR}{h}(\lambda - \frac{B_2^2}{A_{22}^2})w_0'' + \frac{4c^2 R^2}{h^2} \frac{1}{A_{22}^2} w_0 = 0
\]  
(5.19)

It is noted that the theory can easily be extended for axisymmetric imperfections by including the appropriate terms in the governing equations. Eq. (5.17) is then modified to

\[
2B_2 f_0'' - \frac{h}{R} D_{11} w_0'' = \frac{4cR}{h} f_0'' - 4c\lambda (w_0'' + \bar{w}_0'')
\]  
(5.20)

while Eq. (5.19) becomes

\[
(D_1 + \frac{B_2^2}{A_{22}^2})w_0'' + \frac{4cR}{h}(\lambda - \frac{B_2^2}{A_{22}^2})(w_0'' + \bar{w}_0'') + \frac{4c^2 R^2}{h^2} \frac{1}{A_{22}^2} w_0 = 0
\]  
(5.21)

Finally, it is noted that in the case of a flutter analysis, the fundamental state equation (5.11) includes the effect of the additional static aerodynamic loading term \( \bar{p}_{ae} \),

\[
\bar{p} = \bar{p}_{ae} + p_e = \gamma p_\infty M_\infty (\bar{W}_{x_1}^{(0)} + \bar{W}_{x_2}^{(0)}) + p_e
\]  
(5.22)

### 5.5 First-order states

#### 5.5.1 Vibration and buckling

The equations governing the first order dynamic state are given by

\[
L_{A^*} (\dot{F}^{(1)}) - L_{B^*} (\dot{W}^{(1)}) = -\frac{1}{R} \dot{W}_{x_1}^{(1)} - \dot{W}_{y_1}^{(1)} \dot{W}_{x_2}^{(0)}
\]  
(5.23)

\[
L_{B^*} (\dot{F}^{(1)}) + L_{D^*} (\dot{W}^{(1)}) = \frac{1}{R} \dot{F}_{x_1}^{(1)} + \dot{F}_{y_1}^{(0)} \dot{W}_{x_1}^{(1)} - 2\dot{F}_{x_2y}^{(0)} \dot{W}_{x_2y}^{(1)}
\]

\[
+ \dot{F}_{x_2}^{(0)} \dot{W}_{x_2}^{(1)} + \dot{F}_{y_2}^{(0)} \dot{W}_{y_2}^{(0)} - \rho h \dot{W}_{rt}^{(1)}
\]  
(5.24)

The first order equations admit separable solutions of the form

**Dynamic 1st-order state**

\[
\dot{W}^{(1)} = W^{(1)} \cos \omega t = h \{ \bar{w}_1(x) \cos n\theta + \bar{w}_2(x) \sin n\theta \} \cos \omega t
\]  
(5.25)

\[
\dot{F}^{(1)} = F^{(1)} \cos \omega t = \frac{ERh^2}{c} \left\{ \bar{f}_1(x) \cos n\theta + \bar{f}_2(x) \sin n\theta \right\} \cos \omega t
\]  
(5.26)

where \( \theta = y/R \), and \( n \) is the number of circumferential waves.
5.5 First-order states

The corresponding equations for the static first order state \((\bar{W}^{(1)}, \bar{F}^{(1)})\) are similar, apart from the inertia term, which vanishes for the static case,

\[
L_A(\bar{F}^{(1)}) - L_B(\bar{W}^{(1)}) = -\frac{1}{R} \bar{W}^{(1)}_{,xx} - \bar{W}^{(1)}_{yy} \bar{W}^{(0)}_{,xx} \tag{5.27}
\]

\[
L_{B^*}(\bar{F}^{(1)}) + L_{D^*}(\bar{W}^{(1)}) = \frac{1}{R} \bar{F}^{(1)}_{,xx} + \bar{F}^{(0)}_{,xx} \bar{W}^{(1)}_{,yy} - 2 \bar{F}^{(0)}_{,xy} \bar{W}^{(1)}_{,xy} + \bar{F}^{(0)}_{,yy} \bar{W}^{(1)}_{,xy} + \bar{F}^{(1)}_{,xy} \bar{W}^{(0)}_{,xx} \tag{5.28}
\]

They admit separable solutions of the form

**Static 1st-order state**

\[
\bar{W}^{(1)} = h \{\bar{w}_1(x) \cos n\theta + \bar{w}_2(x) \sin n\theta\} \tag{5.29}
\]

\[
\bar{F}^{(1)} = \frac{ERh^2}{c} \{\bar{f}_1(x) \cos n\theta + \bar{f}_2(x) \sin n\theta\} \tag{5.30}
\]

Introduction of the assumed solutions into the governing equations and equating coefficients of like trigonometric terms, gives a set of four 4th-order differential equations in \(f_1, f_2, w_1\) and \(w_2\) for the associated first-order state (either the static state or the dynamic state). The equations, in which the fourth order derivatives \(f_1'', f_2'', w_1''\) and \(w_2''\), have been expressed as functions of \(f_1, f_2, w_1, w_2\) and their derivatives up to third order, can be written in a concise form, applying the summation convention, as

\[
f_1'' = c_{111} f_1 + c_{121} f_2 + c_{131} w_1 + c_{141} w_2 \]
\[
f_2'' = c_{211} f_1 + c_{221} f_2 + c_{231} w_1 + c_{241} w_2 \]
\[
w_1'' = c_{311} f_1 + c_{321} f_2 + c_{331} w_1 + c_{341} w_2 \]
\[
w_2'' = c_{411} f_1 + c_{421} f_2 + c_{431} w_1 + c_{441} w_2 \tag{5.31}
\]

where \(a^i\) denotes the \(i\)-th derivative of \(a\) \((i = 0, 1, 2, 3)\), and where the coefficients \(c_{jki}\) \((j = 1, 2, 3; k = 1, 2, 3, 4)\) depend on the geometric parameters, the stiffness parameters, and the circumferential wave number. The coefficients also depend on the solution of the fundamental state, which results in variable coefficients. The coefficients \(c_{jki}\) are listed in Appendix G, Section G.6.1. The “reduced” boundary conditions for the 1st-order state can be found in Appendix G, Section G.6.2. The case of axisymmetric (bifurcation) buckling and vibration can be derived analogously, or is readily obtained by setting \(n\) equal to zero in the present formulation, and requiring that \(v = 0\) \((v_{,y} = 0\) in the “reduced” boundary conditions) at the boundaries. This guarantees that the periodicity condition is identically satisfied for all \(x\) (Appendix C, Section C.1).
5.5.2 Flutter in supersonic flow

This section deals with an analysis for the linearized aeroelastic stability problem of a cylindrical shell in supersonic flow (see Chapter 2). Piston theory is used (see Section 2.3). Neglecting further the aerodynamic damping results in the Ackeret aerodynamic model. The equations governing the static axisymmetric fundamental state are given in Section 5.4. It is noted that there is a static aerodynamic loading component. The linearized flutter behaviour is governed by

\[ L_A^{(1)} \Delta \dot{W}^{(1)} - L_B^{(1)} \Delta \dot{W}^{(1)} = -\frac{1}{R} \dot{W}^{(1)}_{zy} - \dot{W}^{(1)}_{yy} \dot{W}^{(1)}_{zx} \]

\[ L_B^{(1)} \Delta \dot{W}^{(1)} + L_D^{(1)} \Delta \dot{W}^{(1)} = \frac{1}{R} \dot{F}^{(1)}_{zx} + \dot{F}^{(0)}_{xy} \dot{W}^{(1)}_{zy} - 2 \dot{F}^{(0)}_{xy} \dot{W}^{(1)}_{zy} \]

\[ + \dot{F}^{(0)}_{yy} \dot{W}^{(1)}_{zx} + \dot{F}^{(1)}_{yy} \dot{W}^{(0)}_{yx} + \Delta \theta - \rho h \dot{W}^{(1)}_{zt} \]

where the aerodynamic loading is given by

\[ \Delta \theta = \gamma \rho_{\infty} M^2 \dot{W}^{(1)}_{zt} \]

The variables involved are defined in Chapter 2, Section 2.3. The first order equations admit separable solutions of the form

\[ \dot{W}^{(1)} = W^{(1)} e^{i\omega t} = h \{ \dot{w}_1(x) \cos n\theta + \dot{w}_2(x) \sin n\theta \} e^{i\omega t} \]

\[ \dot{F}^{(1)} = F^{(1)} e^{i\omega t} = \frac{E R h^2}{c} \left\{ \dot{f}_1(x) \cos n\theta + \dot{f}_2(x) \sin n\theta \right\} e^{i\omega t} \]

where \( \omega \) is the (complex) vibration frequency. The governing equations can be cast into the same form as the equations governing vibration and buckling. The flutter equations are given in Appendix G, Section G.6.1.

5.6 Second-order states

The initial nonlinearity of the large amplitude vibrations is governed by the equations of the dynamic second-order state (\( \xi^2 \)-terms). The equations governing the dynamic imperfect state (\( \xi \xi \)-terms) and the coupled mode dynamic state are given in Appendix G, Section G.6.1.

The equations governing the dynamic second-order state can be written as
5.6 Second-order states

\[ L_A - (F^{(2)}) - L_B - (W^{(2)}) = -h w_{0,xx} W_{yy}^{(2)} - h \ddot{w}_{0,xx} \dot{W}_{yy}^{(2)} \]
\[ + \ddot{W}_{yy}^{(2)} - W_{xx}^{(1)} \dot{W}_{yy}^{(1)} - \dot{W}_{xx}^{(2)} (1/R) \]  
\[ (5.37) \]
\[ L_B - (F^{(2)}) + L_D - (W^{(2)}) = f_{0,xx} W_{yy}^{(2)} (1/c) E h^2 R \]
\[ - 2 \ddot{F}_{xy}^{(1)} \dot{W}_{xx}^{(1)} + \ddot{F}_{xx}^{(1)} \dot{W}_{yy}^{(1)} + \ddot{F}_{yy}^{(1)} \dot{W}_{xx}^{(1)} \]
\[ + \ddot{F}_{yy}^{(1)} (1/R) + \ddot{F}_{yy}^{(2)} w_{0,xx} h + \ddot{F}_{yy}^{(2)} \ddot{w}_{xx} h + \rho h \ddot{W}_{yy}^{(2)} \]
\[ + 2 \ddot{W}_{xy}^{(2)} (1/c) E h^2 (1/R) \dot{r} - \ddot{W}_{xx}^{(2)} (1/c) E h^2 (1/R) \lambda \]
\[ - \ddot{W}_{yy}^{(2)} (1/c) E h^2 (1/R) \ddot{p} + a_d \omega \rho h \dot{w} \]  
\[ (5.38) \]

These equations admit separable solutions of the form

**Dynamic 2nd-order state (\( \xi^2 \)-terms)**

\[ \dot{W}^{(2)} = h(W_{\nu}^{(20)} + W_{\nu}^{(22)}) + h(W_{\nu}^{(22)} + W_{\nu}^{(22)}) \cos 2\omega t \]
\[ + h \{ \dot{w}_{\alpha,20}(x) + \dot{w}_{\beta,20}(x) \cos 2n\theta + \dot{w}_{\gamma,20}(x) \sin 2n\theta \} \]
\[ + h \{ \dot{w}_{\alpha,22}(x) + \dot{w}_{\beta,22}(x) \cos 2n\theta + \dot{w}_{\gamma,22}(x) \sin 2n\theta \} \cos 2\omega t \]  
\[ (5.39) \]

\[ \ddot{F}^{(2)} = \frac{ERh^2}{c} \left\{ - \frac{1}{2} \lambda^{(20)} y^2 - \ddot{\pi}^{(20)} xy \right\} \]
\[ + \frac{ERh^2}{c} \left\{ - \frac{1}{2} \lambda^{(22)} y^2 - \ddot{\pi}^{(22)} xy \right\} \cos 2\omega t \]
\[ + \frac{ERh^2}{c} \left\{ \dot{f}_{\alpha,20}(x) + \dot{f}_{\beta,20}(x) \cos 2n\theta + \dot{f}_{\gamma,20}(x) \sin 2n\theta \right\} \]
\[ + \frac{ERh^2}{c} \left\{ \dot{f}_{\alpha,22}(x) + \dot{f}_{\beta,22}(x) \cos 2n\theta + \dot{f}_{\gamma,22}(x) \sin 2n\theta \right\} \cos 2\omega t \]  
\[ (5.40) \]

The 2nd-order dynamic state solution contains the ‘drift’ term (0-th harmonic in time) and the 2nd harmonic in time. Introduction of the assumed solutions into the corresponding governing equations and equating coefficients of like trigonometric terms, gives sets of four 4th-order differential equations in \( f_{\beta}, f_{\gamma}, w_{\beta}, \) and \( w_{\gamma} \) of the associated 2nd-order state. The two 4th-order equations for \( w_{\alpha} \) and \( f_{\alpha} \) can be reduced to one 4th-order equation. The equations governing the 2nd order states can be written in the following concise form (applying the summation convention):

\[ w_{\alpha}^{iv} = d_{00} w_{\alpha}^{i} + d_{00} \]
\[ (5.41) \]
\[ f_{\beta}^{iv} = d_{11} f_{\beta}^{i} + d_{11} + d_{12} f_{\gamma}^{i} + d_{13} w_{\beta}^{i} + d_{14} w_{\gamma}^{i} + d_{10} \]
\[ (5.42) \]
\[ f_{\gamma}^{iv} = d_{21} f_{\beta}^{i} + d_{22} f_{\gamma}^{i} + d_{23} w_{\beta}^{i} + d_{24} w_{\gamma}^{i} + d_{20} \]
\[ (5.43) \]
\[ w_{\beta}^{iv} = d_{31} f_{\beta}^{i} + d_{32} f_{\gamma}^{i} + d_{33} w_{\beta}^{i} + d_{34} w_{\gamma}^{i} + d_{30} \]
\[ (5.44) \]
\[ w_{\gamma}^{iv} = d_{41} f_{\beta}^{i} + d_{42} f_{\gamma}^{i} + d_{43} w_{\beta}^{i} + d_{44} w_{\gamma}^{i} + d_{40} \]  
\[ (5.45) \]
where $a^i$ denotes, as in the previous section, the $i$-th derivative of $a$ ($i = 0, 1, 2, 3$), and where the coefficients $d_{jki}$ ($j = 1, 2, 3, 4; k = 1, 2, 3, 4$) depend on the geometric parameters, the stiffness parameters, the circumferential wave number, and the solution of the fundamental state. The $d_{j0}$ are the forcing functions, depending on the solution of the corresponding lower order states. The coefficients $d_{jki}$ are listed in Appendix G, Section G.6.1.

5.7 Coefficients of perturbation expansion

5.7.1 Single mode cases

The first order static deflection and the imperfection shape are assumed to be affine to the vibration mode shape:

$$\hat{W}^{(1)} = \hat{W} = W^{(1)}$$ (5.46)

The coefficients in the perturbation expansion, Eq. (5.7), are given for general structures in a functional notation in Appendix B, Section B.3. Evaluating the coefficients in the expansion for an 'asymmetric' mode, i.e. a mode with a circumferential wave number $n$ not equal to zero, one obtains

$$a_d = 0$$ (5.47)

The 'dynamic b-factor' becomes

$$b_d = -\frac{1}{\omega^2 \Delta_d} \left\{ 2\hat{F}^{(1)} \ast (\hat{W}^{(1)}, \hat{W}^{(1)}) + \hat{F}^{(2)} \ast (\hat{W}^{(1)}, \hat{W}^{(1)}) \right\}$$ (5.48)

The b-factors corresponding to the 'imperfect dynamic' state become, for an asymmetric mode,

$$b_{110} = b_{101} = b_{210} = b_{201} = 0$$ (5.49)

$$b_{120} = \frac{1}{\omega^2 \Delta_d} \left\{ \hat{F}^{(1)} \ast (\hat{W}^{(1)}, \hat{W}^{(1)}) + \hat{F}^{(2)} \ast (\hat{W}^{(1)}, \hat{W}^{(1)}) + \hat{F}^{(11)} \ast (\hat{W}^{(1)}, \hat{W}^{(1)}) \right\}$$ (5.50)

$$b_{111} = \frac{1}{\omega^2 \Delta_d} \left\{ \hat{L}_1^{(1)}(\hat{W}^{(1)}, \hat{W}^{(1)}), \hat{W}^{(11)}, \hat{W}^{(1)} \right\} - \hat{F}^{(1)} \ast (\hat{W}^{(11)}, \hat{W}^{(1)}) - 2\hat{F}^{(2)} \ast (\hat{W}^{(1)}, \hat{W}^{(1)}) - 4\hat{F}^{(1)} \ast (\hat{W}^{(2)}, \hat{W}^{(1)}) \right\}$$ (5.51)

$$b_{102} = \frac{1}{\omega^2 \Delta_d} \left\{ \hat{F}^{(2)} \ast (\hat{W}^{(1)}, \hat{W}^{(1)}) + 2\hat{F}^{(1)} \ast (\hat{W}^{(2)}, \hat{W}^{(1)}) \right\}$$ (5.52)

where
5.7 Coefficients of perturbation expansion

\[ \Delta_d = \int_0^{2\pi} \int_0^{2\pi R} \int_0^L \rho W^{(1)}^2 \, dx \, dy \, d\tau \]  
(5.53)

and where the shorthand notation

\[ A \ast (B, C) = \int_0^{2\pi} \int_0^{2\pi R} \int_0^L \{ A_{xx} B_{xy} C_{y} + A_{yy} B_{yx} C_{x} - A_{xy} (B_{xy} C_{x} + B_{yx} C_{y}) \} \, dx \, dy \, d\tau \]  
(5.54)

is used. Further, the operator \( L_1 \) with four arguments used in Eq. (5.51) is defined as follows,

\[ L_1(A, B, b, c) = \int_0^{2\pi} \int_0^{2\pi R} \int_0^L \{ L_{11y}(A, B) b_{xy} c_{y} + L_{11x}(A, B) b_{yx} c_{x} + L_{12y}(A, B) (b_{xy} c_{y} + b_{yx} c_{x}) \} \, dx \, dy \, d\tau \]  
(5.55)

where

\[ L_{11x}(A, B) = \int_0^{2\pi} \int_0^{2\pi R} \int_0^L \{ A_{11} A_{x} B_{x} + A_{12} A_{y} B_{y} + A_{16}(A_{xx} B_{xy} + A_{yy} B_{yx}) \} \, dx \, dy \, d\tau \]  
(5.56)

\[ L_{11y}(A, B) = \int_0^{2\pi} \int_0^{2\pi R} \int_0^L \{ A_{12} A_{x} B_{x} + A_{22} A_{y} B_{y} + A_{26}(A_{xx} B_{xy} + A_{yy} B_{yx}) \} \, dx \, dy \, d\tau \]  
(5.57)

\[ L_{12y}(A, B) = \int_0^{2\pi} \int_0^{2\pi R} \int_0^L \{ A_{16} A_{x} B_{x} + A_{26} A_{y} B_{y} + A_{66}(A_{xx} B_{xy} + A_{yy} B_{yx}) \} \, dx \, dy \, d\tau \]  
(5.58)

The notation defined in Eq. (5.54) is related to the notation used in Appendix B as follows:

\[ F^{(1)} \ast (W^{(1)}, W^{(2)}) = \sigma_i \cdot L_{11}(u_j, u_k) \]  
(5.59)

The static analogue of the above b-factor formula can be found in Cohen (1968b) and in Arbocz and Hol (1989). In these references, the effect of a nonlinear prebuckling state is included.
5.7.2 Coupled Mode vibrations

The expression for the perturbation expansion of the frequency for the multi-mode case derived in Appendix B is applied in this section to the coupled mode vibration case involving an asymmetric mode, the driven mode, and a mode which is circumferentially 90 degrees out-of-phase with respect to this driven mode, the companion mode (cf. Section 3.3). The perturbation expansion associated with $\xi_1$, the amplitude of the driven mode, becomes

$$
\left( \frac{\omega}{\omega_c} \right)^2 \xi_1 = \xi_1 + a_{111} \xi_1^2 + (a_{121} + a_{211}) \xi_1 \xi_2 + a_{221} \xi_2^2 \\
+ b_{1111} \xi_1^3 + (b_{1121} + b_{1211} + b_{2111}) \xi_1^2 \xi_2 \\
+ (b_{1221} + b_{2121} + b_{2211}) \xi_1 \xi_2^2 \\
+ b_{2221} \xi_2^3 + \ldots
$$

while the expansion associated with $\xi_2$ has a similar form,

$$
\left( \frac{\omega}{\omega_c} \right)^2 \xi_2 = \xi_2 + a_{112} \xi_1^2 + (a_{122} + a_{212}) \xi_1 \xi_2 + a_{222} \xi_2^2 \\
+ b_{1112} \xi_1^3 + (b_{1122} + b_{1212} + b_{2112}) \xi_1^2 \xi_2 \\
+ (b_{1222} + b_{2122} + b_{2212}) \xi_1 \xi_2^2 \\
+ b_{2222} \xi_2^3 + \ldots
$$

The coefficients in the expansions are given by

$$
b_{i,j,k,l} = - \frac{1}{2a^2 \Delta d} \{ F^{(I,i)} * (W^{(j)}, W^{(k)}) + F^{(I,j)} * (W^{(k)}, W^{(l)}) \\
+ F^{(I,k)} * (W^{(j)}, W^{(k,l)}) + F^{(I,j,k)} * (W^{(l)}, W^{(k,l)}) \\
+ 2F^{(I,j)} * (W^{(l)}, W^{(k,l)}) \}$$

where $I = 1, 2$ denotes the equation associated with $\xi_1$ and $\xi_2$, respectively. For the coefficients in Eq. (5.60) one obtains $a_{111} = a_{121} = a_{211} = a_{221} = b_{1121} = b_{1211} = b_{2111} = b_{2221} = 0$.

5.8 Numerical analysis via parallel shooting

The sets of differential equations with variable coefficients for the various first-order and second-order states, together with the appropriate boundary conditions, are solved numerically via the parallel shooting method (Keller, 1968; Hall and Watt, 1976; Ascher et al., 1988). The boundary conditions simulating elastic supports as defined in Section 2.3 can be prescribed in the present formulation. This section deals with the details of the numerical analysis.
5.8 Numerical analysis via parallel shooting

5.8.1 Static fundamental state problem

The equation governing the axisymmetric static fundamental state (5.19) can be written in the form

\[ w_0'' = (\dot{C}_1 - \dot{C}_2 \lambda)w_0'' - \dot{C}_3 w_0 \]  
(5.63)

where \( \dot{C}_1, \dot{C}_2 \) and \( \dot{C}_3 \) are constants, depending on the stiffness parameters (see Appendix D, Section D.1.1). The boundary conditions correspond to either ‘simply supported’ or ‘clamped’. The problem is solved numerically by employing parallel shooting. The fundamental state is nonlinear in the sense that the radial displacement is a nonlinear function of the applied load. It must be noted that, although Eq. (5.63) describes a nonlinear equilibrium path, for a fixed value of the load parameter \( \lambda \) the differential equation is linear and a closed form solution can be obtained (Booton, 1976; Arbocz and Hol, 1989). Equation (5.63) is the governing nonlinear bending equation. For small axial loads, the equation admits exponential solutions with ‘boundary layer’ like zones close to the shell edges. These ‘bending zones’ however, spread over the whole cylinder length for increasing load. At a certain load level, the solution becomes purely sinusoidal. Increasing the load further, the solution starts to grow without bounds when the load level is reached at which the denominator of the multiplicative constant of the solution of Eq. (5.63) becomes zero (cf. Flügge, 1973). The linear membrane fundamental state corresponds to \( w_0 = 0 \), since \( \dot{W}(0) = h(W_0 + W_p + W_t) \), a constant displacement which can be evaluated by enforcing the circumferential periodicity condition. Finally, it is noted that including an axisymmetric imperfection and a static aerodynamic loading requires a modification of Eq. (5.63), see Section 5.4.

5.8.2 First-order and second-order state problems

The equations governing the first-order states are given by Eqs. (5.31). For a buckling analysis, the applied loading, consisting of a combination of axial compression, radial pressure and torsion, is divided into a fixed and a variable part. The variable part is characterized by a nondimensional load parameter \( \Lambda \). For a vibration analysis, \( \Lambda \) denotes a frequency parameter. To be able to use the shooting method, the equations are cast in the form of a set of first-order differential equations. Introducing the 16-dimensional vector variable \( \mathbf{Y}_1 \) as

\[
\begin{align*}
Y_1 &= f_1 \\
Y_2 &= f_2 \\
Y_3 &= w_1 \\
Y_4 &= w_2 \\
Y_5 &= f_1'^* \\
Y_6 &= f_2'^* \\
Y_7 &= w_1'^* \\
Y_8 &= w_2'^* \\
Y_9 &= f_1'' \\
Y_{10} &= f_2'' \\
Y_{11} &= w_1'' \\
Y_{12} &= w_2'' \\
Y_{13} &= f_1''' \\
Y_{14} &= f_2''' \\
Y_{15} &= w_1''' \\
Y_{16} &= w_2''' \\
\end{align*}
\]  
(5.64)

the governing equations can be written in the following general form...
\[
\frac{d\mathbf{Y}_3}{dx} = \mathbf{f}_1(\bar{x}, \mathbf{Y}_0, \mathbf{Y}_1; \Lambda) = \mathbf{A}_1(\bar{x}, \mathbf{Y}_0; \Lambda)\mathbf{Y}_1
\]

where \(\mathbf{A}_1\) is a 16 \times 16 coefficient matrix, and \(\mathbf{Y}_0\) is the solution of the fundamental state. The first-order state constitutes an eigenvalue problem.

For the equations governing the second-order states the 20-dimensional vector variable \(\mathbf{Y}_2\) is introduced as follows:

\[
\begin{align*}
\mathbf{Y}_1 &= f_\beta \\
\mathbf{Y}_2 &= f'_\beta \\
\mathbf{Y}_3 &= w_x \\
\mathbf{Y}_4 &= w_\beta \\
\mathbf{Y}_5 &= w_r \\
\mathbf{Y}_6 &= f'_\gamma \\
\mathbf{Y}_7 &= f''_\gamma \\
\mathbf{Y}_8 &= w'_\alpha \\
\mathbf{Y}_9 &= w''_\alpha \\
\mathbf{Y}_{10} &= w'_\gamma \\
\mathbf{Y}_{11} &= w''_\gamma \\
\mathbf{Y}_{12} &= w''_\gamma \\
\mathbf{Y}_{13} &= w''_\gamma \\
\mathbf{Y}_{14} &= w''_\gamma \\
\mathbf{Y}_{15} &= w''_\gamma \\
\mathbf{Y}_{16} &= f''_\beta \\
\mathbf{Y}_{17} &= f''_\gamma \\
\mathbf{Y}_{18} &= w''_\alpha \\
\mathbf{Y}_{19} &= w''_\gamma \\
\mathbf{Y}_{20} &= w''_\gamma
\end{align*}
\]

The second-order state equations can be written in the following general form,

\[
\frac{d\mathbf{Y}_2}{dx} = \mathbf{A}_2(\bar{x}, \mathbf{Y}_0)\mathbf{Y}_2 + \mathbf{g}(\mathbf{Y}_1)
\]

The second-order equations form response problems with the first-order solutions appearing in the forcing function \(\mathbf{g}\).

### 5.8.3 Computer program

The two-point boundary value problems defined in the previous sections are solved numerically via parallel shooting (Appendix E). The numerical computations were carried out by means of the FORTRAN program BIANCA (Bifurcation ANalysis of Cylinders including Anisotropy). The Extended Analysis outlined in this chapter relies on derivations via the symbolic manipulation package REDUCE (see Appendix G, Section G.6.1).

In the shooting procedure for the buckling and vibration problem of composite shells, a reduction of the computational work of the problem is possible due to the special structure of the governing equations (cf. Booton, 1976). Following Booton (1976), for the first-order equations the 16-dimensional vector variable \(\mathbf{Y}\) can be introduced as:

\[
\begin{align*}
\mathbf{Y}_1 &= w_1 \\
\mathbf{Y}_2 &= f_1 \\
\mathbf{Y}_3 &= w'_2 \\
\mathbf{Y}_4 &= f'_2 \\
\mathbf{Y}_5 &= w''_1 \\
\mathbf{Y}_6 &= f''_1 \\
\mathbf{Y}_7 &= w''_2 \\
\mathbf{Y}_8 &= f''_2 \\
\mathbf{Y}_9 &= w_2 \\
\mathbf{Y}_{10} &= f_2 \\
\mathbf{Y}_{11} &= -w'_1 \\
\mathbf{Y}_{12} &= -f'_1 \\
\mathbf{Y}_{13} &= w''_2 \\
\mathbf{Y}_{14} &= f''_2 \\
\mathbf{Y}_{15} &= w''_1 \\
\mathbf{Y}_{16} &= -f''_1
\end{align*}
\]

Using this ordering of the variables, the governing equations can be written in matrix notation as follows:
5.8 Numerical analysis via parallel shooting

\[
d\mathbf{Y} = f(\bar{x}, \mathbf{Y}_0, \mathbf{Y}; \Lambda) = \mathbf{A}(\bar{x}, \mathbf{Y}_0; \Lambda)\mathbf{Y}
\]  

(5.69)

where \( \mathbf{A} \) is a 16 \( \times \) 16 coefficient matrix, and \( \mathbf{Y}_0 \) is the solution of the fundamental state. The boundary conditions at \( x = 0 \) can be written in the following form

\[
\mathbf{B}_{11}\mathbf{Y}_1(0) = 0 \tag{5.70}
\]

\[
\mathbf{B}_{12}\mathbf{Y}_2(0) = 0 \tag{5.71}
\]

where \( \mathbf{B}_{11} \) is a 4 \( \times \) 8 coefficient matrix and \( \mathbf{Y}^T = \{\mathbf{Y}_1^T, \mathbf{Y}_2^T\} \), and similar expressions can be given for the boundary conditions at \( x = L \). The shooting procedure which is used to solve these equations is briefly discussed in Appendix E. For the buckling and vibration problem the matrix \( \mathbf{A} \) has the following form:

\[
\mathbf{A} = \begin{bmatrix}
0 & \mathbf{A}_{12} \\
-\mathbf{A}_{12} & 0
\end{bmatrix}
\]  

(5.72)

where

\[
\mathbf{A}_{12} = \begin{bmatrix}
-1 & 0 & 1 \\
0 & 1 \\
-\bar{D}_1 & \bar{D}_4 & \bar{D}_2 & C_4 & \bar{D}_3 & C_1 & -C_7 & -C_3 \\
-\bar{D}_5 & \bar{D}_8 & \bar{D}_6 & C_{20} & \bar{D}_7 & C_{17} & C_{23} & C_{19}
\end{bmatrix}
\]  

(5.73)

The coefficients of this matrix can be found in Appendix D. In this case, it can be shown (Booton, 1976) that if the boundary conditions at \( x = 0 \) and \( x = L \) are identical, \( w_1 \) and \( f_1 \) can be chosen as even functions of \( x \), and \( w_2 \) and \( f_2 \) as odd functions (or vice versa) with respect to the midlength of the cylinder. Therefore the eigenvalues can be obtained from a homogeneous boundary value problem defined over only one half of the cylinder length. This property does not hold for the problem of flutter in supersonic flow. It is noted further that describing the displacement pattern in the form (5.25) renders the solution in principle undetermined with respect to its circumferential position. However, prescribing symmetry and/or antimony conditions at the shell midlength fixes the circumferential position of the mode. It is noted that Booton (1976) also indicates another reduction of the computational work of the problem. In the present approach, for the buckling and vibration case, the variational equations

\[
\mathbf{W}' = \mathbf{A}\mathbf{W}
\]  

(5.74)
where $W = [W_1, W_2, \ldots, W_{16}]$, with the corresponding initial conditions, have a special form which permits that only half of the number of variational sets have to be solved (see Appendix E).

In the program, there are three options available to solve the eigenvalue problem: 1) determinant plotting, 2) mode iteration, and 3) Keller’s method. In the determinant plotting method (e.g. Booton, 1976) a determinantal function of the eigenvalue parameter $\Lambda$ is evaluated. By increasing $\Lambda$ in small steps, and monitoring the characteristic determinant, the eigenvalues can be located. Vanishing of the determinant is, for simple eigenvalues, revealed by a sign change. The mode iteration method (e.g. Cohen, 1968a) yields a sequence of inhomogeneous boundary value problems which are solved by means of the shooting method. This method, properly formulated, gives convergence to the mode corresponding to the lowest eigenvalue. The general shooting method described by Keller (1968) is based on adjoining a normalization condition to the homogeneous equations (see Appendix E).

The mode iteration method is an efficient method to find the lowest eigenvalue. For a linear eigenvalue problem, convergence to the lowest eigenvalue is in principle guaranteed. Bathe (1982) shows for the corresponding matrix eigenvalue problem, that the method converges very rapidly (cubically) if one uses eigenvalue shifting based on Rayleigh quotient corrections. In the present case, the eigenvalue problem is nonlinear since the fundamental state has a nonlinear dependence on the applied load. Solution of this nonlinear problem involves the solution of a sequence of linearized eigenvalue problems (see Appendix E).

The solution of the initial value problems was done by an Adams-Moulton predictor-corrector scheme, with starting values obtained by the method of Runge-Kutta-Gill. A variable step size with automatic truncation error control can be used. It is noted that the shooting method can both be parallelized and vectorized (Deerenberg and Jansen, 1990). The integration routines are suited for vectorization, while the integration over the different subintervals can be done in parallel.

5.9 Results and discussion

In this section, numerical results of the Level-2 analyses are presented. The data of the shells used in the calculations are given in Appendix F. The notation used to specify the boundary conditions is given in Section 2.3. It is noted that they correspond to displacements relative to the ‘moving’ edge plane. In all examples to be presented, the boundary conditions are symmetric with respect to the shell midlength. Except for the flutter analysis results, the mode shapes are therefore plotted for $0 < \tilde{x} < L/(2R)$. It is noted that the nondimensional displacement $w = W/h$ is used. The (nondimensional) linear buckling and vibration modes are normalized by setting the maximum displacement equal to one.
Buckling and linear vibration of anisotropic shells

In Table 5.1, buckling loads are tabulated for Booton’s anisotropic shell (with \( L/R = 2 \)) for eight standard boundary conditions. The following (loading) cases are considered:

1. axial compression (\( \lambda = (cR/Eh^2)N_0 \)), where \( c \) and \( E \) are the reference values given in the Nomenclature.

2. hydrostatic pressure (\( \bar{p} = (cR^2/Eh^2)p \) and \( \lambda = \frac{1}{2}\bar{p} \), i.e., a uniform pressure applied to the lateral surface as well as to the ends of the cylinder, and

3. torsion (\( \tau = (cR/ Eh^2)T_0 \)), both counter-clockwise (corresponding to a positive sign), and clockwise.

4. Further, the lowest natural frequency (\( \omega = R\sqrt{(\rho h/A_{22})} \omega \)) has been tabulated.

For the weak boundary support \( N_{xy} = 0 \) (the circumferential displacement \( v \) is unconstrained) the buckling load under axial compression is drastically reduced (Hoff, 1965; Yamaki, 1983). This support condition, however, is not likely to be encountered in practical applications (Almroth, 1966). Both for the hydrostatic pressure case and for the natural frequency, the axial restraint \( u = 0 \) has a strong influence on the incremental in-plane displacements \( u^{(1)} \) and \( v^{(1)} \), and thereby on the important (stabilizing) incremental membrane stress \( N_z^{(1)} \).

An interesting consequence of the unbalanced lay-up is observed for torsion. The buckling load depends on the sense of the applied torque (Booton, 1976). The equivalent phenomenon that reversing the lay-up results in a different buckling load for a specified sense of the applied torque will be considered here. The phenomenon is due to linear effects, and is associated with the difference in membrane stress.
resultants at buckling for the two cases. This can be elucidated as follows. The compatibility equation relates the strains to the lateral displacement. For this case (because of the $B_{16}$ and $B_{26}$ terms) there is also a coupling between the strains and the displacement through the constitutive relations. The membrane stresses occurring at buckling can be directly found from a given buckling mode shape via the compatibility equation. Therefore, assuming that for both lay-ups the same buckling mode occurs, the difference in buckling load can be attributed to the modification of the membrane stresses due to the $B_{16}$ and $B_{26}$ terms. This affects both the restoring bending moments and the stabilizing contribution of the membrane stress due to the initial shell curvature. For counter-clockwise torsion, as compared with the lay-up corresponding to $B_{16} = B_{26} = 0$, the restoring forces for the $[+30,0,-30]$ lay-up are larger, whereas for the $[-30,0,+30]$ lay-up the restoring forces are smaller.

**Nonlinear vibrations of isotropic and orthotropic shells**

A comparison between the backbone curves obtained via the Simplified Analysis based on Evensen's approach (Chapter 3) and the present Extended Analysis is shown in Fig. (5.1) for four different isotropic shells. The backbone curves for the Extended Analysis are based on the value of the dynamic b-factor in the perturbation.
5.9 Results and discussion

expansion of Eq. (5.7). The dynamic b-factor for the Extended Analysis follows from Eq. (5.48). For the different cases considered the dynamic b-factors can be found in Table 5.2, where they are compared with the b-factors corresponding to the Simplified Analysis based on Evensen’s approach.

<table>
<thead>
<tr>
<th></th>
<th>Simplified</th>
<th>Extended</th>
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<tbody>
<tr>
<td>ES2</td>
<td>-2.5412 \cdot 10^{-3}</td>
<td>-2.5772 \cdot 10^{-3}</td>
</tr>
<tr>
<td>Chen</td>
<td>-6.1875 \cdot 10^{-3}</td>
<td>-7.4243 \cdot 10^{-3}</td>
</tr>
<tr>
<td>Olson</td>
<td>-2.0476 \cdot 10^{-3}</td>
<td>-4.6079 \cdot 10^{-3}</td>
</tr>
<tr>
<td>WN</td>
<td>-0.28587</td>
<td>0.12072</td>
</tr>
</tbody>
</table>

Table 5.2: Dynamic b-factors of isotropic shells. Comparison between Simplified Analysis (Averaging) and Extended Analysis (Perturbation method).

The results for the ES2-shell compare well with the simplified analysis. For both Olson’s shell and Chen’s shell, the effect of the second order double (2n-th) harmonic in the circumferential direction of the shell is important. For the WN-shell, a vibration mode of relatively high order (5 axial half waves) is analysed. For this case the internally forcing double frequency of the asymmetric mode is not small as compared to the frequency of the axisymmetric mode. The effect of the axisymmetric mode has a drastic influence on the nonlinearity, which becomes hardening, whereas Evensen’s approach with the constrained axisymmetric mode predicts softening behaviour. In the latter approach, the amplitudes of the axisymmetric modes are constrained by relating them \textit{a priori} to the amplitudes of the asymmetric modes via the circumferential periodicity condition. A Galerkin-type Level-2 nonlinear vibration analysis, i.e. an analysis in which the effect of boundary conditions is taken into account, was presented by Liu (1988). Liu did not include the double harmonic in the circumferential direction in the assumed deflection function. Furthermore, in Liu’s analysis the axisymmetric mode is constrained by relating it \textit{a priori} to the corresponding asymmetric modes via enforcement of the circumferential periodicity condition.

To shed more light on the behaviour of the WN-shell, the simplified analysis using numerical time integration developed in Chapter 4 was used to obtain frequency-response curves for representative values of excitation amplitude and damping parameter. The analysis is based on the vibration mode

\[ \ddot{W}(t)/h = C_0(t) + C_1(t) \cos \frac{2m\pi x}{L} + A(t) \sin \frac{m\pi x}{L} \cos \frac{n}{R} (y - \tau_K x) \]  

(5.75)

The result in Fig. 5.2 shows that in the present case a single mode analysis is not adequate to describe the response in this region. The axisymmetric mode should be included in the analysis as an additional generalized coordinate.

To check the influence of the double (2n-th) harmonic in the circumferential direction, a new “Simplified Analysis” was implemented in SILVANA (see Chapter 4).
For the “single” (primary) mode analysis of a perfect isotropic or orthotropic shell, a deflection mode was used of the form

\[ \frac{\Delta(t)}{h} = C_0(t) + C_1(t) \cos \frac{2m \pi x}{L} + \cos \frac{n}{R} (y - \tau K x) + A_2(t) \cos \frac{2n}{R} y \]

where \( n \) denotes the number of circumferential full waves of the vibration mode. This deflection mode includes the ‘rosette’ mode \( \cos \frac{2n}{R} y \). The results of this analysis are also included in Fig. 5.2.

Two isotropic shells have often been used in the literature as reference cases, Chen’s shell (Chen, 1972) and Olson’s shell (Olson, 1965):

- Chen’s shell: \( R = 4 \) in., \( h = 0.01 \) in., \( L = 8 \) in., \( \nu = 0.31 \) (\( L/R = 2 \), \( R/h = 400 \)). The vibration mode considered has one half wave in axial direction and 6 circumferential full waves.

- Olson’s shell: \( L = 15\frac{\text{3}}{\text{2}} \) in., \( R = 8 \) in., \( h = 0.0044 \) in., \( \nu = 0.30 \) (\( L/R = 1.922 \), \( R/h = 1818 \)). The vibration mode considered has one half wave in axial direction and 10 circumferential full waves.

In Fig. 5.3 backbone curves for Chen’s shell (Chen, 1972) obtained from the Simplified Analysis based on the method of averaging (Chapter 3) are compared with
5.9 Results and discussion

The present (Extended Analysis) result for SS-3 boundary conditions. Response calculations obtained with the Simplified Analysis via numerical time-integration (Chapter 4) are also shown in this figure for typical values of excitation amplitude and damping parameter. This analysis is based on the deflection mode from Eq. (5.75). Further, response calculations obtained via numerical integration in time using the 4-mode analysis based on Eq. (5.76), which includes the double harmonic in the circumferential direction of the shell, are shown in Fig. 5.3. For Chen’s shell, there is a fair agreement between the results of the different analyses, and for this case they are reasonably close to the theoretical results obtained by Chen (Chen, 1972). Chen’s theoretical backbone curves for this shell are in fair agreement with his experimental results (Chen, 1972).

In Fig. 5.4 backbone curves for Olson’s shell (Chen, 1972) obtained from Evensen’s Simplified Analysis based on the method of averaging (Chapter 3) are compared with the present (Extended Analysis) result for SS-3 boundary conditions. Response calculations obtained via numerical integration in time using the 4-mode analysis based on Eq. (5.76), are also shown in Fig. 5.3 for typical values of excitation amplitude and damping parameter.

In the case of Olson’s shell, the double (2n-th) harmonic in the circumferential direction should be included in the assumed displacement function. The frequency of this “secondary” mode is of the same order as the internally forcing double frequency of the “primary” asymmetric mode, resulting in a strong interaction between the

Figure 5.3: Backbone curves of isotropic shell. Comparison between Simplified Analyses and Extended Analysis. Chen’s shell.
Figure 5.4: Backbone curves of isotropic shell. Comparison between Simplified Analyses and Extended Analysis. Olson's shell.

Figure 5.5: Backbone curves of isotropic shell. Comparison between different analyses. Olson's shell.
5.9 Results and discussion

first-order $n$-mode and the second-order ‘double harmonic’ $(2n)$-mode. This explains
the large discrepancy between the Simplified Analysis based on Averaging and the
Extended Analysis results. The length of Olson’s shell is 15.375 in. If one varies the
shell length while the other geometry parameters are fixed, one obtains at $L = 14.8$
in., relatively close to the length of Olson’s shell, internal resonance of the double
harmonic mode.

It is noted that Chen’s results (Chen, 1972), which satisfy SS-3 boundary con-
ditions exactly, and the present Extended Analysis results correspond to similar
formulations. Chen’s results for Olson’s shell are compared with results of the Ex-
tended analysis in Fig. 5.5. Solutions obtained via numerical time-integration, both
for the simplified analysis based on Eq. (5.75) and for the analysis based on the
4-mode deflection given by Eq. (5.76), have been plotted for two vibration ampli-
tudes ($\bar{A} = 2$ and $\bar{A} = 3$). The reason for the discrepancy between the present
Extended Analysis results and Chen’s results is unknown. In this context, a brief
overview will be given of results for cylindrical shells which have appeared in the
literature. As stated earlier, Chen’s theoretical backbone curves (Chen, 1972) are
in fair agreement with the results he obtained in his experiments (“Chen’s shell”).
Padovan (1980) uses a finite element discretization in combination with a pertur-
bation method to describe the temporal behaviour. His results are close to Chen’s
(theoretical) solutions (for “Chen’s shell”). The same holds for the backbone curves
of Ganapathi and Varadan (1996), who use finite elements with numerical time in-
tegration. It should be noted that the results of Ganapathi and Varadan (1996) do
not show such a close agreement with Chen’s results for a second reference case,
“Olson’s shell”. Ueda’s analysis (Ueda, 1979) is based on finite elements in a semi-
analytical approach, i.e. finite elements are used for the meridional (axial) direction
and for the circumferential direction of the shell a Fourier decomposition is used.
The difference between Ueda’s results and Chen’s results (for Chen’s shell) is smaller
than the disagreement between the present analysis and Chen’s results. It should
be noted that Ueda’s analysis does not include the double harmonic in the circum-
ferential direction. Summarizing, it can be concluded that there is a discrepancy
amongst the results of the different analyses which have appeared so far (including
the present one). This disagreement is only slight for the case of Chen’s shell, but
the differences are more clearly discernible in the case of Olson’s shell.

In Figs. 5.6 and 5.7 backbone curves for an orthotropic shell analysed by In
and Chia (1988b) (ICEZ shell) obtained from the Simplified Analysis based on the
method of averaging (Chapter 3) are compared with the present (Extended Analysis)
result for SS-3 boundary conditions. The data of this shell are:

- ICEZ shell: $R/h = 100$, $E_x/E_y = 3$, $G_{xy}/E_y = 0.5$, $\nu_{xy} = 0.25$, $\ell = 5$.

Response calculations obtained with the Simplified Analysis via numerical time-
integration are also shown in these figures, both for the analysis without and includ-
ing the double harmonic in the assumed radial deflection (Eq. (5.75) and Eq. (5.76),
respectively). Two shell lengths have been considered, $L/R = 0.5$ and $L/R = 1.0$.
The behaviour in the range of shell lengths considered is complicated by interactions
between the “primary” mode and the “secondary” vibration modes. This can be
elucidated by varying the shell length from $L/R = 1.0$ to $L/R = 0.5$. At $L/R = 1.0$, one can see the contribution of the familiar axisymmetric contraction mode to the response. Decreasing the shell length, at about $L/R = 0.625$ internal resonance occurs and the axisymmetric mode response jumps to the opposite sign (Fig. 5.8a). It can be concluded that the “primary” vibration mode exhibits a strong interaction with the axisymmetric mode $w_{a22}$ in this range of shell lengths. In addition, there is a strong interaction between the “primary” mode and the double harmonic (2n) mode associated with $w_{a22}$ (Fig. 5.8b), which is predominant in this case and results in a softening behaviour as can be seen in Figs. 5.6 and 5.7. The strong interaction of the “primary” mode with both the axisymmetric and the double harmonic asymmetric mode has been confirmed by results of the Level-1 type analysis using numerical time-integration, based on the 4-mode deflection in Eq. (5.76), which includes the double harmonic in the circumferential direction. Results of this analysis have also been included in Figs. 5.6 and 5.7. In Fig. 5.9, the results of the present (Extended) analysis and results of the Simplified Analysis without the double harmonic in the assumed radial deflection (Eq. (5.75)) are compared with other results which have appeared in the literature. Looking at the results displayed in Figs. 5.6 to 5.9, it can be concluded that the interactions between the “primary” mode and the “secondary” vibration modes play a significant role in the nonlinear behaviour of the ICEZ shell.
Figure 5.7: Backbone curves of orthotropic shell. Comparison between Simplified Analyses and Extended Analysis; ICEZ shell, $L/R = 1.0$. 

Figure 5.9: Nonlinear vibrations of orthotropic ICEZ shell. Comparison between solutions of $u/Ch_0$, $E_{sym}/Dep$ (maximum displacement [$u_{max}$]), Simplified Analysis (amplitude asymmetric mode [$A_1$]), and Extended Analysis (amplitude asymmetric mode [$A_2$]).
Single Mode and Coupled Mode vibrations of an isotropic shell

The first-order and second order modes corresponding to the single mode vibration of the ES2 shell are depicted in Fig. 5.10. The dynamic 2nd-order stresses are also shown in this figure.

The amplitude frequency curve for coupled mode free vibrations of two asymmetric modes which are circumferentially ninety degrees out-of-phase between each other, can be obtained via the approach, denoted earlier as the 'Coupled Mode' analysis in Section 5.7.2. The backbone curves for the coupled vibrations based on the b-factors in the expansion of Eqs. 5.60 and 5.61 are shown in Fig. 5.11. As stated earlier in Chapter 3, the combination of the two asymmetric modes leads to a reduction of the softening nonlinearity as compared to the single mode case.

The mixed 2nd-order mode $w_{23}$ (see Appendix G, Section G.6.1) contributes in the Couple Mode nonlinear vibrations. This mode is depicted in Fig. 5.12, where it is compared with the double harmonic second-order modes $w_{220}$ and $w_{/22}$ of the Single Mode case.

Effect of boundary conditions on linear vibrations and linear flutter of isotropic shell

The effect of boundary conditions on the natural frequency is illustrated for Olson's isotropic shell in Table 5.3 for different standard boundary conditions.
Figure 5.9: Nonlinear vibrations of orthotropic ICEZ shell. Comparison between solutions of Ju/Chia, El-Zaouk/Dym (maximum displacement $|w_{max}|$), Simplified Analysis (amplitude asymmetric mode $|A|$), and Extended Analysis (amplitude asymmetric mode $\xi$, used in the expansion of Eq. (5.7)); a) $L/R = 0.5$, b) $L/R = 1.0$. 
Figure 5.10: Top: Vibration mode of isotropic shell. Middle: Dynamic 2nd order modes. (a) 0th harmonic and (b) 2nd harmonic in time. Bottom: Dynamic 2nd order stresses. (a) 0th harmonic and (b) 2nd harmonic in time. ES2 shell.
5.9 Results and discussion

Figure 5.11: Backbone curve for Coupled Mode vibrations of isotropic shell. Comparison between Simplified Analysis (amplitude asymmetric mode $|\bar{A}|$) and Extended Analysis (amplitude asymmetric mode $\xi_v$ used in the expansion of Eq. (5.7)); ES2 shell.

Figure 5.12: Mixed second order mode for Coupled Mode nonlinear vibrations of isotropic shell. Comparison with second order modes of "single" (primary) mode vibration; ES2 shell.
Table 5.3: Effect of boundary conditions on natural frequency $f$ (Hz) of isotropic shell (Olson's shell).

<table>
<thead>
<tr>
<th>boundary condition</th>
<th>frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS-3</td>
<td>83.779</td>
</tr>
<tr>
<td>SS-4</td>
<td>150.31</td>
</tr>
<tr>
<td>C-3</td>
<td>84.443</td>
</tr>
<tr>
<td>C-4</td>
<td>150.71</td>
</tr>
</tbody>
</table>

The number of circumferential waves is $n = 10$. Using $W_{ex} = 0$ an elastic boundary condition parameter $k_n'$ (cf. Section 2.3.2) in the elastic edge restraint condition for $N_x$ and $u$ was determined such that the computed frequency agrees with the experimental value found by Olson (1967). In the present approach, the ‘reduced’ conditions for $N_x$ and $u$ are related through $N_x + k_n'(\frac{R}{n})^2 u_{yy} = 0$. The elastic boundary condition corresponding to the experimental value 131.2 Hz is obtained for $k_n' = 6.10 \cdot 10^4$ lb/in$^2$.

The effect of different boundary conditions on the aeroelastic stability of Olson's shell is given in Table 5.4 for the conditions stated in Barr and Stearman (1969) (see Appendix F).

Table 5.4: Effect of boundary conditions on critical freestream static pressure $p_\infty$ (lb/in$^2$) and corresponding flutter frequency of isotropic shell; Olson's shell. Ref.: Barr and Stearman (1969).

<table>
<thead>
<tr>
<th>boundary condition</th>
<th>$p_\infty$ (lb/in$^2$)</th>
<th>$\bar{\omega}$</th>
<th>$p_\infty$ (lb/in$^2$)</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS-3</td>
<td>0.38736</td>
<td>0.08552</td>
<td></td>
<td>0.4</td>
</tr>
<tr>
<td>SS-4</td>
<td>0.53999</td>
<td>0.08724</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C-3</td>
<td>0.40569</td>
<td>0.08573</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C-4</td>
<td>0.54289</td>
<td>0.08727</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Under these conditions, Barr and Stearman (1969) determined the critical freestream static pressure and the corresponding number of circumferential waves ($n = 24$) for simply supported boundary conditions. The present analysis code gives the possibility to investigate the effect of different boundary conditions on the aeroelastic stability. In the present example, the boundary condition has been varied while keeping the number of circumferential waves fixed at $n = 24$. The normalized frequency $\bar{\omega} = R\sqrt{(\rho h/A_{22})}$ corresponding to instability has also been tabulated in Table 5.4. The flutter mode shapes and corresponding stress functions are depicted in Fig. 5.13.
Effect of boundary conditions on nonlinear vibrations of isotropic shell

In Table 5.5 a comparison is made between the initial nonlinearity (reflected by the dynamic b-factors) for different boundary conditions. It is noted that b-factors correspond to a normalization with respect to the associated linear frequency $\omega_c$ (see Eq. (5.7)).

It is noted that the boundary conditions correspond to displacements relative to the ‘moving’ edge plane. A constraint for the displacement $u$ in the averaged sense would influence the solution via an edge effect for $w_u$ (due to a Poisson expansion), and via fundamental state deformations (rotations).
Effect of imperfections and loading on linearized vibrations of (anisotropic) shells

The effect of axisymmetric and asymmetric imperfections and of axial loading on the linearized vibration frequency of Booton’s anisotropic shell is shown for SS-3 boundary conditions in Fig. 5.14. The vibration mode corresponds to the lowest mode of the unloaded shell \((\ell = 6)\). The frequency was normalized with respect to the linear frequency of the unloaded perfect shell (denoted here as \(\omega_c\)), i.e. to \(\omega_c\) evaluated for the unloaded perfect shell. As stated earlier, it should be kept in mind that in general the lowest vibration mode does not correspond with the lowest buckling mode. Therefore, when the loading is increased, buckling can occur in another mode before the frequency of the mode considered becomes zero.

The axisymmetric imperfection has the following form

\[
\bar{W}/h = \xi_1 \cos \frac{2m\pi x}{L} + \xi_2 \{\bar{w}_1(x) \cos n\theta + \bar{w}_2(x) \sin n\theta\}
\]

with \(m = 1\). This form has also been used in the Simplified Analysis (Eq. (3.1)). The asymmetric imperfection is affine to the vibration mode,

\[
\bar{W}/h = \xi_2 \{\bar{w}_1(x) \cos n\theta + \bar{w}_2(x) \sin n\theta\}
\]

Comparing the presents results with the results obtained via the Simplified Analysis (Fig. 3.5), it is seen that the buckling load of the (initially) perfect shell from
the Extended Analysis is significantly lower than the buckling load from the Simplified Analysis. In the Extended Analysis, the prebuckling deformation is taken into account. The effect of the asymmetric imperfection in the Extended Analysis is relatively smaller than in the Simplified Analysis. In the Simplified Analysis (Fig. 3.5), the axisymmetric deformation and the buckling mode have a predetermined pattern, while in the Extended Analysis the axisymmetric deformation and the axial variation of the buckling mode are not constrained. The relation between the static state amplitudes $\xi$ and the fundamental state solution can be obtained from a Galerkin-type solution based on the expansions for the static state, Eqs. (5.5) and (5.6). The effect of an asymmetric imperfection is included in this procedure. The loading range for which this approach is valid remains to be established.

The ‘imperfect dynamic’ 2nd-order modes play an important role in the linearized vibration analysis of shells with asymmetric imperfections. They are depicted in Fig. 5.15.

**Initial postbuckling and nonlinear vibrations of (anisotropic) shells**

In Table 5.6, for SS-3 boundary conditions ‘static’ b-factors $b_s$ (Arbocz and Hol, 1989) for three load cases (axial compression, hydrostatic pressure, and torsion), and the ‘dynamic’ b-factor $b_d$ are tabulated for Booton’s anisotropic shell. The results of the present analysis are identical to the results of a recent version of the computational module for Koiter’s imperfection sensitivity theory ANILISA (Arbocz and Hol, 1989). It is noted that the static b-factor for a certain loading case can be related to the dynamic b-factor under the corresponding critical static loading.

As both previous investigations (Evensen, 1967; Chen and Babcock, 1975; Liu, 1988) and the present work have shown, an important contribution to the (softening) nonlinear behaviour stems from the interaction between the 2nd-order (compressive) membrane stresses in circumferential direction and the first order displacement mode. The dynamic 2nd-order modes are depicted in Fig. 5.16.
Figure 5.16: Top: Vibration mode of anisotropic shell. Middle: Dynamic 2nd order modes. (a) 0th harmonic and (b) 2nd harmonic in time. Bottom: Dynamic 2nd order stresses. (a) 0th harmonic and (b) 2nd harmonic in time. Booto's shell, $L/R = 1.414$. 
Table 5.6: Normalized buckling loads and vibration frequency (number of circumferential waves between parentheses), and static b-factors $b_s$ and dynamic b-factor $b_d$ for simply supported anisotropic shell. Booton's shell, $L/R = 1.414$.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Eigenvalue</th>
<th>$b_s$</th>
<th>$b_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>axial compression</td>
<td>0.39078 (6)</td>
<td>-0.27732</td>
<td></td>
</tr>
<tr>
<td>hydrostatic pressure</td>
<td>0.054537 (8)</td>
<td>-0.06724</td>
<td></td>
</tr>
<tr>
<td>clockwise torsion</td>
<td>-0.16316 (9)</td>
<td>-0.06724</td>
<td></td>
</tr>
<tr>
<td>counter-clockwise torsion</td>
<td>0.17155 (9)</td>
<td>-0.04471</td>
<td></td>
</tr>
<tr>
<td>vibration</td>
<td>0.19317 (6)</td>
<td>-0.14156</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.17: Influence of axisymmetric imperfection amplitude on dynamic b-factors. (a) ES2 shell, (b) Booton's shell.
The effect of axisymmetric imperfections of the form

\[ \bar{W}/h = \xi_1 \cos \frac{2\pi x}{L} \]

on the nonlinear vibrations is shown in Fig. 5.17 for both the ES2 shell and Booton's shell. The b-factors are compared with results obtained from the Simplified Analysis (Chapter 3).

In Fig. 5.18 a comparison is made between results of the present analysis, and the multi-mode results of Iu and Chia (1988a) and the finite element results of Ganapathi and Varadan (1995) for an orthotropic composite shell. In the latter two references the effects of transverse shear deformation and in-plane inertia are included. Further, the present analysis only accounts for the initial curvature of the amplitude-frequency curve. It should be noted also that the circumferential wave number for the modes considered here is relatively low \((n = 2 \text{ and } n = 4)\). Both the analysis of Iu and Chia (1988a) and the present analysis use Donnell-type governing equations, which may be less accurate for low circumferential wave numbers.

5.10 Conclusions

The linearized and nonlinear vibration behaviour of anisotropic cylindrical shells was modelled using a perturbation method to analyze the temporal behaviour. Using a Fourier decomposition in the circumferential direction of the shell, the shooting method was applied to solve the resulting spatial two-point boundary value problems. The specified boundary conditions at the shell edges can be satisfied rigorously.

To illustrate the capabilities of the module developed, results of a linearized and a nonlinear vibration analysis of a specific anisotropic shell were presented. The linearized vibration behaviour of an axially loaded imperfect shell was analysed. The loading range for which this approach is valid remains to be established. Results of a nonlinear vibration analysis of this anisotropic perfect shell were also presented. The related dynamic stability problem of linear flutter has been analysed via an extension of the approach used for (linearized) vibration problems. The effects of different boundary conditions on the nonlinear vibrations of a cylindrical shell have been investigated.

Results for a specific isotropic shell often used in the literature have been compared with results obtained using other methods. For this shell a good agreement was achieved between the present results and the results of a Simplified Analysis in which the effect of the double harmonic in the circumferential direction is included. The importance of including the double harmonic mode in the circumferential direction in the assumed shell response has been emphasized.

An extension of the theory from single mode analysis to multi-mode analysis is given in Appendix B. The result has been applied to the two-mode case of 'travelling wave' coupled mode nonlinear vibrations.

Restrictions of the present analysis stem firstly from the governing equations due to the shell theory used (Donnell-type equations). It should be noted that for composite shells it may be necessary to include the effect of transverse shear
Figure 5.18: Nonlinear vibrations of orthotropic GV shell. Comparison between solutions of Ganapathi/Varadan, Lu/Chia (maximum displacement $|w_{\text{max}}|$), and Extended Analysis for two types of boundary condition (amplitude asymmetric mode $\xi_c$); a) $n = 2$, b) $n = 4$. 
deformation and rotatory inertia in the formulation. Referring to the remarks on the accuracy and applicability of the Simplified Analyses in the two previous chapters, the following comments can be made:

- The perturbation theory used in the present approach is exact in an asymptotic sense. The influence of all "secondary" coordinates is automatically included in the perturbation procedure. In its original form, only the lowest order effects of the nonlinearity are taken into account. The analysis is a "single mode" analysis. In Appendix B, the theory is extended to a higher order analysis of the nonlinear vibrations of (perfect) structures.

Many aspects of the nonlinear vibration behaviour of (composite) shells warrant a further investigation, in particular, cases where modal interaction occurs (as in the cases of Olson's shell and the ICEZ shell). The multi-mode analysis presented in Appendix B can be applied to cases where modal interaction occurs for modes with coinciding frequency.

- The elastic boundary conditions are satisfied accurately by solving the resulting two-point boundary value problems numerically via the parallel shooting method. Both the linearized and the nonlinear vibration analysis have to be validated via other analyses (finite element analyses).
Chapter 6

Conclusions

In this final chapter the results of the present research on the nonlinear vibrations of anisotropic cylindrical shells will be given in relation with the objectives stated and background sketched in Chapter 1. Subsequently, the methods developed for the analysis of this problem will be summarized. This chapter will be concluded with recommendations for future investigations in this field.

- The principal objective of the present research was the development of semi-analytical methods for the nonlinear vibration analysis of anisotropic shells, in accordance with the three level “analysis strategy” outlined in Chapter 1. While evaluating the results of the corresponding analyses, the restrictions imposed by the underlying theory should be kept in mind.

Methods at two levels of complexity have been developed, namely, Level-1 (‘Simplified’) Analyses (both in combination with the method of averaging, and in combination with numerical time-integration) and a Level-2 (‘Extended’) Analysis based on a perturbation procedure. The methods have been implemented in FORTRAN computer codes. The analyses have been described in detail in Appendix G of this thesis.

The models can be used to investigate the effects of

- anisotropy
- initial geometric imperfections
- different boundary conditions
- static preloading

- Additional theoretical and experimental work in this area is required to find an explanation for the remaining discrepancies between the different theoretical results which have appeared in the literature (including the present one, see Chapter 5), and to resolve the differences between theory and experiment.

- The unexplained phenomena observed by Chen (1972) (beating responses), were recovered in the “transient simplified analysis”. Chaos-like phenomena were encountered for specific forcing frequencies and forcing amplitudes.
• The experiments of Gunawan (1998) for an isotropic shell under static axial loading were simulated numerically via the transient simplified analysis. The theoretical results were qualitatively in agreement with the experimental results. Softening behaviour and the occurrence of the companion mode were found both in the physical experiment and in the numerical simulation.

As stated in Chapter 1, the vibration research is strongly connected with the principal research topic in this group, buckling of cylindrical shells. The present research has also led to results which may be used in shell buckling and related fields:

• The programs which have been developed to investigate the vibration behaviour, include the capability of analysing static buckling, one of the main research topics in the Structures Group.

• The Level-1 model for transient analysis developed in Chapter 4 is capable of analysing not only nonlinear vibration problems, but also other dynamic stability problems of cylindrical shells. Results of parametric excitation and dynamic buckling problems for isotropic and anisotropic shells have been discussed.

Methods at two levels of complexity, a Level-1 analysis (Simplified Analysis) and a Level-2 analysis (Extended Analysis), have been developed. These methods can be characterized as follows:

• *Simplified Analysis*

The modeling level denoted as Simplified Analysis uses a small number of modes in a Galerkin-type procedure or variational procedure. The effects of important parameters, in particular, the effects of imperfections and a static loading in the previbration state consisting of axial compression, radial pressure, and torsion, are included. The simplified analysis using averaging is the simplest method to analyse the nonlinear vibration of shells. The essential role of the axisymmetric modes, which stems from their coupling with the “primary” asymmetric modes, has been accentuated. The use of axisymmetric modes in the assumed deflection mode is essential. Several shortcomings of this analysis have been identified. The method is inconsistent in the sense that the axisymmetric modes are not included as independent vibration modes. They are constrained by relating them *a priori* to the asymmetric modes via the circumferential periodicity condition. Further, for a consistent approximation of the higher order corrections to the amplitude-frequency relations, a higher order approximation of the method of averaging or numerical time-integration should be employed.

These disadvantages have been removed in a more consistent Level-1 method. In this formulation, the two axisymmetric modes in the assumed deflection function are incorporated as generalized coordinates, and numerical time-integration was used to describe the temporal behaviour.
The nonlinear modal interaction resulting in "travelling" wave vibrations in the circumferential direction of the shell has been included in the present formulation.

The double (2n-th) harmonic response in the circumferential direction of the shell may in certain cases strongly influence the results. Therefore a new 'Simplified Analysis' was developed, which included the double harmonic in the assumed displacement function.

The analyses confirm the results obtained earlier by Hol (1983) and Liu (1988), namely that an asymmetric imperfection which is affine to the vibration mode gives a decrease of the small amplitude vibration frequency.

- **Extended Analysis**

In the Extended Analysis the linearized and nonlinear vibration behaviour of anisotropic cylindrical shells was modelled using a perturbation method for the analysis of the temporal behaviour. Using a Fourier decomposition in the circumferential direction of the shell, the shooting method was applied to solve the resulting spatial two-point boundary value problems. In this way, the specified boundary conditions at the shell edges can be satisfied rigorously. 'Dynamic b-factors' are obtained via a perturbation procedure presented in a general form in Appendix B.

In the Extended Analysis, the possibility to specify an elastic boundary condition was implemented, which made it possible to attune the boundary condition such that the value of the experimental frequency is obtained.

The importance of the axisymmetric modes and the double harmonic in the circumferential direction have been confirmed via the Extended Analysis.

The perturbation method for multi-mode solutions presented in Appendix B, has been used in the Extended Analysis to analyse the Coupled Mode vibrations of a driven mode and a companion mode.

In the field of the nonlinear vibrations of shell structures there are many questions which warrant further investigations in this field. The nonlinear vibration behaviour of composite shells has not yet received much attention. Future investigations on nonlinear shell vibrations should combine appropriate semi-analytical methods, experimental work, and finite element approaches. The following recommendations for future research are given:

- **Improvements and extensions for the present approach**

Obvious improvements in the analysis can be made with regard to the basic assumptions adopted (Chapter 2). It is possible to remove the limitations of the Donnell theory by using more accurate shell equations. Further, the effect of in-plane inertia can be included, and a higher order shell theory may be used to account for transverse shear deformation and rotatory inertia.
The analytical formulation in Chapter 3 (a Level-1 Analysis in combination with averaging) can be extended to multi-mode cases. The method of multiple scales is suited to investigate modal interactions and internal resonance phenomena (Nayfeh et al., 1991; Chin and Nayfeh, 1996).

For the Level-1 Analysis using numerical time-integration in Chapter 4, it is suggested to analyse the chaotic behaviour of the coupled mode response of a driven and companion mode (characteristic of axisymmetric structures) in more detail using the concepts and tools of the theory of nonlinear dynamic systems (e.g. Thompson and Stewart, 1986; Moon, 1987).

The analyses presented in Chapter 4 can also be extended to multi-mode analyses, which take the coupling between different circumferential and axial harmonics into account. This may be useful in particular for dynamic buckling analyses and parametric excitation problems.

- **Experimental investigation**

  Experimental work has been carried out for isotropic shells in the Structures Group of the Faculty of Aerospace Engineering to validate the theoretical models developed. Recommendations for future experiments are given by Gunawan (1998).

- **Level-3 Analysis**

  Nonlinear shell vibration problems can be simulated in detail using finite elements (Level-3 analysis). Various approaches are possible. Finite elements have been used, in combination with a constrained version of the Lindstedt-Poincaré perturbation method describing the temporal behaviour, by Padovan (1980) to study the nonlinear vibrations of general structures. Finite elements for the spatial discretization were employed by Noor et al. (1993) in a reduced basis approach (using modes based on a perturbation analysis) to study the nonlinear vibrations of composite panels. Finally, modern computational resources have made it possible to employ a ‘brute force’ method to simulate the dynamic behaviour of shell structures by using finite elements in space and numerical integration in time (Ganapathi and Varadan, 1995, 1996).
Bibliography


BIBLIOGRAPHY


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Appendix A

Basic equations

A.1 Constitutive equations

The derivation of the constitutive equations in this appendix follows the presentation given by Arbocz and Hol (1989) (see also Ashton et al., 1969). Using the sign convention shown in Fig. 2.1, with \( W \) positive inward the numbering of the layers begins at the outer surface. The angle of rotation \( \theta_k \) \((k = 1, 2, \ldots, N)\) of the individual layers is defined with respect to the \( x \)-axis of the shell (Fig. 2.1). The shell reference surface coincides with the midsurface of the laminate. If the position of the \( k^{th} \) lamina is defined by \( h_{k-1} < z < h_k \) the total thickness of the laminate is

\[
h = \sum_{k=1}^{N} (h_k - h_{k-1}) \quad \text{(A.1)}
\]

We assume that each lamina may be considered as a homogeneous orthotropic medium in a plane stress state. The stress-strain relations for the \( k^{th} \) lamina can then be written as

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12k}
\end{bmatrix}_k =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}_k
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12k}
\end{bmatrix}_k \quad \text{(A.2)}
\]

where

\[
Q_{11} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}}
\]

\[
Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}}
\]

\[
Q_{12} = \nu_{21}\frac{E_{11}}{1 - \nu_{12}\nu_{21}}
\]

\[
Q_{66} = G_{12}
\]

(A.3)
The relations refer to the lamina principal axes \((1, 2)\). Transformation to the shell wall reference axes \((x, y)\) gives
\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}_k = \begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{12} & Q_{22} & Q_{26} \\
Q_{16} & Q_{26} & Q_{66}
\end{bmatrix}_k \begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{bmatrix}_k
\] (A.4)

where
\[
\begin{align*}
Q_{11} &= Q_{11}C^4 + 2(Q_{12} + 2Q_{66})C^2S^2 + Q_{22}S^4 \\
Q_{12} &= (Q_{11} + Q_{22} - 4Q_{66})C^2S^2 + Q_{12}(C^4 + S^4) \\
Q_{22} &= Q_{11}S^4 + 2(Q_{12} + 2Q_{66})C^2S^2 + Q_{22}C^4 \\
Q_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})C^2S^2 + Q_{66}(C^4 + S^4) \\
Q_{16} &= (Q_{11} - Q_{12} - 2Q_{66})CS + (Q_{12} - Q_{22} + 2Q_{66})C^3S \\
Q_{26} &= (Q_{11} - Q_{12} - 2Q_{66})CS^3 + (Q_{12} - Q_{22} + 2Q_{66})C^3S
\end{align*}
\] (A.5)

and \(C = \cos \theta_k, \ S = \sin \theta_k\).

Assuming that \(z/R \ll 1\), the stress resultants (see Fig. 2.2) acting at the shell midsurface are given by
\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}_k dz
\] (A.6)

and the moment resultants (Fig. 2.2) acting on the shell midsurface are defined as
\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}_k z dz
\] (A.7)

It is noted that for \(z/R \ll 1\), \(N_{xy} = N_{yx}\). According to the Kirchhoff-Love hypothesis for a thin shell, the strain at any layer can be written in terms of the strain and curvature of the midsurface as
\[
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{bmatrix}_k = \begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{bmatrix} + 2 \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}_k
\] (A.8)

Substituting these expressions into Eq. (A.4) and introducing the resulting relations into Eq. (A.7) and (A.8), followed by carrying out the indicated integrations gives the following constitutive equations...
A.1 Constitutive equations

\[
\begin{bmatrix}
N_x \\ N_y \\ N_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{25} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} +
\begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\] (A.9)

\[
\begin{bmatrix}
M_x \\ M_y \\ M_{xy} + M_{yz}
\end{bmatrix} =
\begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} +
\begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\] (A.10)

where

\[
A_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k (h_k - h_{k-1})
\] (A.11)

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{N} (\bar{Q}_{ij})_k (h_k^2 - h_{k-1}^2)
\] (A.12)

\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{N} (\bar{Q}_{ij})_k (h_k^3 - h_{k-1}^3)
\] (A.13)

for \(i, j = 1, 2, 6\). The constitutive equations (A.9) and (A.10) can be written in matrix form as

\[
\{N\} = A\{\epsilon\} + B\{\kappa\}
\] (A.14)

\[
\{M\} = B\{\epsilon\} + D\{\kappa\}
\] (A.15)

and after partial inversion as

\[
\{\epsilon\} = A^*\{N\} + B^*\{\kappa\}
\] (A.16)

\[
\{M\} = C^*\{N\} + D^*\{\kappa\}
\] (A.17)

where

\[
A^* = A^{-1}
\]

\[
B^* = -A^{-1}B
\]

\[
C^* = BA^{-1} = -B^*T
\]

\[
D^* = D - BA^{-1}B
\]

The stiffness parameters are nondimensionalized as follows

\[
\bar{A}_{ij} = \frac{1}{Eh} A_{ij}
\]

\[
\bar{B}_{ij} = \frac{2c}{Eh^2} B_{ij}
\]

\[
\bar{D}_{ij} = \frac{4c^2}{Eh^3} D_{ij}
\] (A.18)
Basic equations

(\ref{eq:19})

\[ \tilde{A}_{ij} = E h A_{ij} \]

\[ \tilde{B}_{ij} = \frac{2c}{h} B_{ij} \]

\[ \tilde{D}_{ij} = \frac{4c^2}{E h^3} D_{ij} \]

where

\[ c^2 = 3(1 - \nu^2) \]  

(A.20)

and the quantities \( E \) and \( \nu \) are (arbitrarily chosen) reference values.

A.2 Strain-displacement relations

Strain-displacement relations of an imperfect cylindrical shell valid for an 'intermediate' class of deformations (Brush and Almroth, 1975) can be written as

\[ \epsilon_x = u_{,x} + \frac{1}{2} \beta_x^2 + \beta_x \beta_y^* \]

\[ \epsilon_y = v_{,y} - \frac{W}{R} + \frac{1}{2} \beta_y^2 + \beta_y \beta_x^* \]

\[ \gamma_{xy} = u_{,y} + v_{,x} + \beta_y^2 \beta_x^* + \beta_x^2 \beta_y^* + \beta_x \beta_y \]

\[ \kappa_x = \beta_x,_{x} = -W_{,xx} \]

\[ \kappa_y = \beta_y,_{y} = -W_{,yy} + \left[ \frac{v_{,y}}{R} \right] \]

\[ \kappa_{xy} = \beta_x,_{y} + \beta_y,_{x} = -2W_{,xy} + \left[ \frac{v_{,x}}{R} \right] \]

(A.21)

where \( \beta_x^* = \beta_x - \gamma_{xx} = -W_{,x} \), \( \beta_y^* = \beta_y - \gamma_{yy} = -W_{,y} \), and \( \beta_{x,}^* = \beta_{x,} - \gamma_{xy} = -W_{,xy} \). They are the rotations of the middle surface, and the \( \beta_i \) are the rotations of the normal. In the present approach transverse shear deformation is neglected. If transverse shear strains \( \gamma_{xx} \) and \( \gamma_{yy} \) are included in the formulation (see Section A.4), the modified relations become

\[ \beta_x^* = \beta_x - \gamma_{xx} = -W_{,x} \]

\[ \beta_y^* = \beta_y - \gamma_{yy} = \left[ \frac{v_{,y}}{R} \right] - W_{,y} \]

For Donnell-type equations the terms in square brackets are neglected and the equations become:

\[ \epsilon_x = u_{,x} + \frac{1}{2} W_{,x}^2 + W_{,x} W_{,xx} \]

\[ \epsilon_y = v_{,y} - \frac{W}{R} + \frac{1}{2} W_{,y}^2 + W_{,y} W_{,yy} \]

\[ \gamma_{xy} = u_{,y} + v_{,x} + W_{,x} W_{,y} + W_{,x} W_{,yy} + W_{,xy} W_{,y} \]

\[ \kappa_x = -W_{,xx} \]

\[ \kappa_y = -W_{,yy} \]

\[ \kappa_{xy} = -2W_{,xy} \]

(A.22)
A.3 Equations of motion

The potential energy of an anisotropic shell under axial compression, radial pressure, and torsion is given by

$$
\Pi = \frac{1}{2} \int_0^{2\pi R} \int_0^L \left\{ N_z \varepsilon_x + N_y \varepsilon_y + N_{xy} \gamma_{xy} + M_x \kappa_x + M_y \kappa_y + \frac{M_{xy}}{2} \kappa_{xy} \right\} dx \, dy
$$

$$
- \int_0^{2\pi R} \left\{ \tilde{N}_x u + \tilde{N}_{xy} v + \tilde{M}_x W_{xx} + \tilde{M}_y W_{xy} \right\} \bigg|_{x=0} \, dy
$$

$$
- \int_0^{2\pi R} \left\{ \tilde{N}_x u + \tilde{N}_{xy} v + \tilde{M}_x W_{xx} + \tilde{M}_y W_{xy} \right\} \bigg|_{x=L} \, dy
$$

$$
- \int_0^{2\pi R} N_{x}^c \bigg|_{x=L} \int_0^L (u_{xx} - q W_{xx}) \, dx \, dy - \int_0^{2\pi R} \int_0^L p^c W \, dx \, dy
$$

$$
- \int_0^{2\pi R} N_{xy}^c \bigg|_{x=L} \int_0^L v_{xx} \, dx \, dy
$$

(A.23)

where $\tilde{N}_x$ and $\tilde{N}_{xy}$ are the specified circumferentially varying parts of the in-plane loads at the shell edges $x = 0$ and $x = L$, $M_z$ and $\tilde{H}$ are specified circumferentially varying parts of the out-of-plane loads, and where $N_{x}^c \bigg|_{x=L}$ is the applied axial compression (averaged in circumferential direction), $p^c$ is the applied external pressure, $N_{xy}^c \bigg|_{x=L}$ is the applied (averaged) counter-clockwise torsion and $q$ is the load eccentricity (positive inward) of the applied axial compression. The index ‘c’ refers to the conservativeness of the loading. Using (A.23) in combination with an expression for the work done by the applied nonconservative loads, and the appropriate expression for the kinetic energy (cf. Chapter 4), we can derive the equations of motion and the corresponding boundary conditions via Hamilton’s variational principle. The governing equations become

$$
N_{x,x} + N_{xy,y} = \{ \bar{\rho} h u_{,tt} \}
$$

(A.25)

$$
N_{xy,x} + N_{yy,y} + \frac{1}{R} [M_{y,y} + M_{xy,x} + N_y (W_{yy} + \bar{W}_{yy}) + N_{xy} (W_{yx} + \bar{W}_{yx})] = \{ \bar{\rho} h v_{,tt} \}
$$

(A.26)

$$
M_{x,xx} + (M_x + M_{y,x})_{xy} + M_{y,y,y} + \frac{1}{R} [N_y + N_x (W_{xx} + \bar{W}_{xx})]
$$

$$
+ 2 N_{xy} (W_{xy} + \bar{W}_{xy}) + N_y (W_{yy} + \bar{W}_{yy})
$$

$$
+ (N_{x,x} + N_{xy,y})(W_{xx} + \bar{W}_{xx}) + (N_{xy,x} + N_{y,y}) (W_{yy} + \bar{W}_{yy}) = -p + \bar{\rho} h W_{,tt}
$$

(A.27)
In this equation,

\[ \bar{\rho} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\rho_0) dz = \sum_{k=1}^{N} (\rho_0)_k (h_k - h_{k-1}) \]  

(A.28)

where \((\rho_0)_k\) is the mass density of the \(k^{th}\) layer. In-plane inertia is neglected (terms in curly braces are neglected), and Donnell-type equations are used (terms in square brackets are omitted). Introducing an Airy stress function \(F\) as \(N_z = F_{yy}, N_y = F_{xx}\) and \(N_{xy} = -F_{xy}\), (A.25) and (A.26) are identically satisfied and (A.27) becomes

\[ L_B^*(F) + L_D^*(W) = -\frac{1}{R} F_{xx} + L_{NL}(F, W + \bar{W}) + p - \bar{\rho} hW_{tt} \]  

(A.29)

The operators \(L_B^*, L_D^*\) and \(L_{NL}\) are defined in Eqs. (2.5) and (2.6). Another equation in \(W\) and \(F\) is obtained from the compatibility condition

\[ \varepsilon_{x,yy} + \varepsilon_{y,xx} - \gamma_{xy,xy} = -\frac{1}{R} W_{xx} - \frac{1}{2} L_{NL}(W, W + 2\bar{W}) \]  

(A.30)

Substitution of \(\{\varepsilon\}\) from (A.16) gives

\[ L_{A^*}(F) - L_{B^*}(W) = -\frac{1}{R} W_{xx} - \frac{1}{2} L_{NL}(W, W + 2\bar{W}) \]  

(A.31)

The operators \(L_{A^*}\) and \(L_{B^*}\) are defined in Eq. (2.5). The corresponding boundary conditions for circumferentially varying quantities at \(x = 0, L\) become:

\[ N_z = \bar{N}_z \text{ or } \delta u = 0 \]  

(A.32)

\[ N_{xy} = \bar{N}_{xy} \text{ or } \delta u = 0 \]  

(A.33)

\[ M_x = M_x \text{ or } \delta W_{xx} = 0 \]  

(A.34)

\[ M_{x,x} + (M_{xy} + M_{y,x})_{yy} + N_x (W_{xx} + \bar{W}_{xx}) + N_{xy} (W_{yy} + \bar{W}_{yy}) = \bar{H} \text{ or } \delta W = 0 \]  

(A.35)

### A.4 Transverse shear deformation

The Donnell-type formulation presented in the previous sections forms the basis of the analyses of the present thesis. In this section, the incorporation of transverse shear effects via a first-order transverse shear deformation theory is outlined (cf. Vinson and Chou, 1974). For this purpose, the kinematic relations are modified and the constitutive equations are extended by the introduction of the transverse shear angles \(\gamma_{xz}\) and \(\gamma_{yz}\). The stress-strain relation (A.2) is supplemented by

\[ \begin{bmatrix} \tau_{23} \\ \tau_{31} \end{bmatrix}_k = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix}_k \begin{bmatrix} \gamma_{23} \\ \gamma_{31} \end{bmatrix}_k \]  

(A.36)
where $Q_{44} = G_{23}$; $Q_{55} = G_{31}$.

The additional stress-strain relations for an orthotropic lamina, referring to the wall reference axes $(x', y')$ then become

$$
\begin{bmatrix}
\gamma_{yz} \\
\gamma_{yx}
\end{bmatrix}_k = 
\begin{bmatrix}
Q_{44} & Q_{45} \\
Q_{45} & Q_{55}
\end{bmatrix}_k
\begin{bmatrix}
\tau_{yx} \\
\tau_{xx}
\end{bmatrix}_k
$$

(A.37)

where

$$
\begin{align*}
\bar{Q}_{44} &= Q_{44}C^2 + Q_{55}S^2 \\
\bar{Q}_{55} &= Q_{44}S^2 + Q_{55}C^2 \\
\bar{Q}_{45} &= (Q_{55} - Q_{44})CS
\end{align*}
$$

where $C = \cos \theta_k$, and $S = \sin \theta_k$.

The stiffness coefficients $A_{ij}$ ($i, j = 4, 5$) may be determined in different ways.

Vinson and Chou (1974) assume that the transverse shear stresses are distributed parabolically across the laminate thickness, and they use the following weighting function for the integration over the thickness,

$$
f(z) = \frac{5}{4} \left[ 1 - \left( \frac{z}{h/2} \right)^2 \right]
$$

(A.38)

Multiplying the righthand side of Eq. (A.37) with this weighting function and substituting into the expression for the shear forces

$$
\begin{bmatrix}
Q_x \\
Q_y
\end{bmatrix}_k = 
\sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \begin{bmatrix}
\tau_{yx} \\
\tau_{xx}
\end{bmatrix}_k dz
$$

we obtain

$$
\begin{bmatrix}
Q_x \\
Q_y
\end{bmatrix} = 
\begin{bmatrix}
A_{55} & A_{45} \\
A_{45} & A_{44}
\end{bmatrix}
\begin{bmatrix}
\gamma_{xz} \\
\gamma_{yx}
\end{bmatrix}
$$

(A.40)

where

$$
A_{ij} = \frac{5}{4} \sum_{k=1}^{N} (\bar{Q}_{ij})_k [(h_k - h_{k-1}) - \frac{4}{3} \frac{(h_k^3 - h_{k-1}^3)}{h^2}]
$$

(A.41)

$i, j = 4, 5$. Alternatively, one may use

$$
A_{ij} = k_i k_j \sum_{k=1}^{N} (\bar{Q}_{ij})_k [(h_k - h_{k-1})]
$$

(A.42)

$i, j = 4, 5$, where $k_i$ are the shear correction factors. The strain-displacement relations are modified as follows:
Basic equations

\[ \beta_x = \beta_x - \gamma_{xx} = -W_{xx} \]  
\[ \beta_y = \beta_y - \gamma_{yy} = -W_{yy} \]  

(A.43)  

(A.44)

The potential energy expression has the following additional contribution:

\[ \frac{1}{2} \int_0^{2\pi} \int_0^L \{Q_x \gamma_{xx} + Q_y \gamma_{yy}\} dx dy \]  

(A.45)

Including rotatory inertia in the kinetic energy expression, the modified and additional Donnell-type equilibrium equations become (Vinson and Chou, 1974)

\[ N_{x,x} + N_{x,y,y} = \bar{\rho}h u_{tt} + Q\beta_{x,tt} \]  

(A.46)

\[ N_{x,y,x} + N_{y,y} = \bar{\rho}h v_{tt} + Q\beta_{y,tt} \]  

(A.47)

\[ M_{x,x} + M_{x,y,y} - Q_x = I_1 \beta_{x,tt} + Q u_{tt} \]  

(A.48)

\[ M_{x,y,x} + M_{y,y} - Q_y = I_1 \beta_{y,tt} + Q v_{tt} \]  

(A.49)

where

\[ Q = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\rho_0)_{k} z dz = \sum_{k=1}^{N} (\rho_0)_{k} (h_k^2 - h_{k-1}^2) \]  

(A.50)

\[ I = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} (\rho_0)_{k} z^2 dz = \sum_{k=1}^{N} (\rho_0)_{k} (h_k^3 - h_{k-1}^3) \]  

(A.51)

The effect of transverse shear deformation on buckling and vibration of anisotropic shells

In general, the effects of transverse shear deformation and rotatory inertia should be included for relatively thick shells. Moreover, for composite shells the transverse shear stiffness may be low with respect to the in-plane stiffness. In this case transverse shear deformation can affect the dynamic behaviour. It is noted by Ginsberg (1972) that it may also be necessary to include transverse shear deformation when “higher order” modes (i.e. modes with a large number of waves) contribute in the nonlinear behaviour. These modes may be secondary in the sense that they come into play for nonlinear (i.e. large amplitude) deflections. The inclusion of transverse shear deformation leads to a set of equation in the five unknowns \( u, v, W, \gamma_{xx}, \) and \( \gamma_{yy} \). As an example, the analysis for a simple set of equations, namely the linear buckling and vibration equations of a perfect anisotropic shell, will be outlined here.

For the buckling (or vibration) deflection of a perfect anisotropic shell we assume

\[ W/h = A \left\{ \frac{1}{2} \sin(\ell_n x - \ell_n y) + \frac{1}{2} \sin(\ell_n x + \ell_n y) \right\} \]  

(A.52)
where \( A \) is the normalized displacement amplitude and the definition of the wavenumber parameters \( \ell_m, \ell_p, \) and \( \ell_n \) can be found in the main text. Substituting the assumed displacement in the in-plane equilibrium equations and moment equilibrium equations the in-plane displacements and transverse shear angles are obtained in the following form:

\[
\begin{align*}
u &= U_1 \cos(\ell_m x - \ell_n y) + U_2 \cos(\ell_p x + \ell_n y) \\
\gamma_{xx} &= G_1 \cos(\ell_m x - \ell_n y) + G_2 \cos(\ell_p x + \ell_n y) \\
\gamma_{yz} &= H_1 \cos(\ell_m x - \ell_n y) + H_2 \cos(\ell_p x + \ell_n y)
\end{align*}
\]

where \( U_i, V_i, G_i \) and \( H_i \) (\( i = 1, 2 \)) are coefficients which depend linearly on the displacement amplitude \( A \). Substituting the displacements into the out-of-plane equilibrium equation and application of Galerkin’s method gives expressions for the buckling load of a composite shell. Analogously, an expression for the natural frequency of a (statically loaded) shell can be obtained.
Basic equations

The potential energy expression has the following additional assumptions:

\[ \frac{2}{3} \rho A \int \left( \frac{1}{2} (\mathbf{v} - \mathbf{v}_0)^2 + \frac{1}{2} (\mathbf{w} - \mathbf{w}_0)^2 \right) \, dV = \frac{1}{2} \int \mathbf{v} \times \mathbf{v} + \mathbf{w} \times \mathbf{w} \, dV \]  

(BM.A)

where

\[ \mathbf{v} = \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} (\mathbf{v}_0 + \mathbf{v}_1) \, dx \]

In general, the effects of transverse shear deformation and rotary inertia should be included in the analysis for thick shells. However, for composite shells the transverse shear stresses may be neglected with respect to the membrane stiffness. In the case of monocoque sections deformation shear effects are significant. It is noted by Cerrato (1972) that it may be necessary to include transverse shear deformation when higher order assumptions are made with large numbers of waves, or to coordinate the membrane behavior. These effects may be secondary in the sense that they may come into play for large amplitude deformations. The inclusion of transverse shear deformation is a clear-cut requirement in the free laminate, e.g., \( W, \gamma, \) and \( \phi \).

As an example, the analysis for a simple set of equations may be the linearized and simplified equations of a perfect laminate. The angle \( \phi \) will be neglected here.

For the buckling and vibration problems of a perfect laminate, we assume

\[ W = A \left( \frac{1}{2} (\mathbf{v} - \mathbf{v}_0)^2 + \frac{1}{2} (\mathbf{w} - \mathbf{w}_0)^2 + L \mathbf{v} \times \mathbf{w} \right) \]  

(A.52)
Appendix B

A perturbation method for imperfect structures

B.1 Introduction

In this appendix, a perturbation method will be used to analyse the nonlinear vibration behaviour of imperfect general structures. Rehfield (1973) introduced a perturbation method analogous to Koiter's initial postbuckling theory (Koiter, 1945) to investigate the nonlinear vibrations of general structures. Wedel-Heinen (1991) used this methodology to treat the small vibrations of statically loaded imperfect structures. In the present work, the effects of imperfections and of a nonlinear static state on the nonlinear vibrations of structures are included. Budiansky and Hutchinson (1964) and Budiansky (1967) extended the initial postbuckling theory in order to analyse the dynamic buckling of imperfection sensitive structures. Their functional notation, introduced in these studies and later adopted by several other authors, is also used in the present work. Fitch (1968) and Cohen (1968b) included the effect of a nonlinear prebuckling state in their initial postbuckling and imperfection sensitivity analysis.

B.2 Governing equations

Using D'Alembert's principle to introduce inertial loading, the variational equation of motion (dynamic equilibrium) can be formulated as follows:

\[ M(\ddot{u}) \cdot \delta u + \sigma \cdot \delta \epsilon = q \cdot \delta u \]  

(B.1)

where \( u \), \( \epsilon \) and \( \sigma \) denote the generalized displacement, strain and stress, respectively. These field variables can be interpreted as vector functions with variables appropriate to the problem. Further, \( M \) is the generalized mass operator, \( q \) is the applied load, and \( (\cdot) = \partial^2(\cdot)/\partial t^2 \). Eq. (B.1) is a statement of the principle of virtual work, where \( a \cdot b \) is the virtual work of stresses or loads \( a \) acting through strains or displacements \( b \), integrated over the entire structure for variations \( \delta u \) which are
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kinematically admissible. The mass operator is homogeneous, linear and has the property that

\[ M(u) \cdot v = M(v) \cdot u \]  (B.2)

for all \( u \) and \( v \). In addition, we have the strain-displacement relation

\[ \epsilon = L_1(u) + \frac{1}{2} L_2(u) + L_{11}(\bar{u}, u) \]  (B.3)

where \( \bar{u} \) is an initial geometric imperfection, and where \( L_1 \) and \( L_2 \) are homogeneous linear and quadratic functionals, respectively. The homogeneous bilinear functional \( L_{11} \) is defined by

\[ L_2(u + v) = L_2(u) + 2L_{11}(u, v) + L_2(v) \]  (B.4)

from which we have \( L_{11}(u, v) = L_{11}(v, u) \) and \( L_{11}(u, u) = L_{11}(u) \). The variation of the generalized strain becomes

\[ \delta \epsilon = L_1(\delta u) + L_{11}(u, \delta u) + L_{11}(\bar{u}, \delta u) \]  (B.5)

For linearly elastic structures, the constitutive equation can be written in the form

\[ \sigma = H(\epsilon) \]  (B.6)

where \( H \) is a homogeneous linear functional. The following reciprocity relation will be assumed to hold

\[ \sigma_1 \cdot \epsilon_2 = \sigma_2 \cdot \epsilon_1 \]  (B.7)

where the states indicated by ‘1’ and ‘2’ are arbitrary states of stress and strain.

### B.3 Static and dynamic analysis

In this section, the situation of a dynamic state on a nonlinear static state is analysed. Assuming that the structure is vibrating in a deformed (due to a static loading) configuration (the ‘static’ state), the variables can be written as a superposition of two states

\[ u = u_s + u_d; \quad \epsilon = \epsilon_s + \epsilon_d; \quad \sigma = \sigma_s + \sigma_d; \quad q = q_s + q_d; \]  (B.8)

where the indices ‘s’ and ‘d’ denote the static state and dynamic state, respectively. The static state response in turn is assumed to consist of a (nonlinear) fundamental (‘trivial’) state and a nonlinear buckling (‘nontrivial’ or ‘orthogonal’) state:

\[ u_s = u_0 + \bar{u}; \quad \epsilon_s = \epsilon_0 + \bar{\epsilon}; \quad \sigma_s = \sigma_0 + \bar{\sigma}; \quad q_s = q_0; \]  (B.9)
where the index '0' denotes the fundamental state and the tilde the (nonlinear) buckling state. The nonlinearity of the buckling state is taken into account via a perturbation expansion of the buckling variables. Substituting the expressions (B.8) and (B.9) into (B.1) gives the equations governing the equilibrium of the fundamental static state, the (nonlinear) buckling state, and the dynamic state, respectively.

**Fundamental state**

The equation governing equilibrium of the static fundamental state is:

\[ \sigma_0 \cdot \delta \epsilon_0 = q_0 \cdot \delta u \]  
\[ (B.10) \]

where the variation of the generalized strain becomes

\[ \delta \epsilon_0 = L_1(\delta u) + L_{11}(u_0, \delta u) + L_{11}(\bar{u}, \delta u) \]  
\[ (B.11) \]

and the constitutive equation is given by

\[ \sigma_0 = H(\epsilon_0) \]  
\[ (B.12) \]

**Buckling state**

For the nonlinear static buckling state the governing equations become:

\[ \sigma_s \cdot \delta \epsilon_s = 0 \]  
\[ (B.13) \]

where the variation of the strain is given by

\[ \delta \epsilon_s = L_1(\delta u) + L_{11}(u_s, \delta u) + L_{11}(\bar{u}, \delta u) \]  
\[ (B.14) \]

and

\[ \sigma_s = H(\epsilon_s) \]  
\[ (B.15) \]

**Dynamic state**

For the dynamic state we obtain the following equations:

\[ M(\ddot{u}_d) \cdot \delta u + \sigma_d \cdot \delta \epsilon_d + \sigma_s \cdot L_{11}(u_d, \delta u) = q_d \cdot \delta u \]  
\[ (B.16) \]

where

\[ \delta \epsilon_d = L_1(\delta u) + L_{11}(u_d, \delta u) + L_{11}(\bar{u} + u_0, \delta u) \]  
\[ (B.17) \]

and where

\[ \sigma_d = H(\epsilon_d) \]  
\[ (B.18) \]
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It is assumed that a single vibration mode is associated with the natural frequency. The multi-mode case will be discussed later. We will use the following initial imperfection:

\[ \bar{u} = \xi \hat{u} \]  

where \( \xi \) is the initial imperfection amplitude, and \( \hat{u} \) the imperfection mode. The following perturbation expansions will be assumed:

**Static buckling state**

\[ \bar{u} = \xi u_{x_1} + \xi^2 u_{x_2} + \xi^3 u_{x_3} + \ldots \]

\[ + \xi \left( \xi u_{x_{11}} + \xi^2 u_{x_{21}} + \xi^3 u_{x_{31}} + \ldots \right) \]

\[ + \ldots \]  

**Dynamic state**

\[ u_d = \xi_d u_{d_1} + \xi^2_d u_{d_2} + \xi^3_d u_{d_3} + \ldots \]

\[ + \xi \left( \xi_d u_{d_{11}} + \xi^2_d u_{d_{21}} + \xi^3_d u_{d_{31}} + \ldots \right) \]

\[ + \xi^2 \left( \xi_d u_{d_{22}} + \xi^2_d u_{d_{22}} + \xi^3_d u_{d_{23}} + \ldots \right) \]

\[ + \ldots \]  

where the effective imperfection amplitude \( \xi_d \) is defined by

\[ \xi_d = \xi + \xi \]  

It is noted that orthogonalizing \( u_{d_2}, u_{d_3} \) etc. in some sense with respect to \( u_{d_1} \), e.g. by

\[ M(u_{d_k}) \cdot u_{d_k} = 0, \quad k = 2, 3, \ldots \]  

defines \( u_{d_k} \) and makes the expansion unique. A similar remark can be made for the static state (cf. Fitch, 1968).

In the following, the imperfection mode is assumed to be affine to the linear static buckling response \( u_{s_1} \),

\[ \bar{u} = u_{s_1} \]  

The expansions for the strains, stresses, and the equilibrium equation of the different states can now be established.

**Static buckling state**

The expansion for the static strains becomes
B.3 Static and dynamic analysis

\[ \tilde{e} = \xi e_{s1} + \xi^2 e_{s2} + \xi^3 e_{s3} + \ldots \]
\[ + \xi^4(e_{s01} + \xi e_{s11} + \xi^2 e_{s21} + \xi^3 e_{s31} + \ldots) \quad (B.25) \]
\[ + \ldots \quad (B.26) \]

and the constitutive relation can be written as

\[ \bar{\sigma}_i = H(\epsilon_i), \quad i = 1, 2, 3, \ldots \quad (B.27) \]

Setting \( \delta u = u_{s1} \) in the equation governing equilibrium of the static nonlinear buckling state we obtain in a Galerkin-type solution the following relation between the amplitude parameter \( \xi \) and the fundamental state solution:

\[ \xi \left[1 - \frac{\lambda}{\lambda_C}\right] + A_s \xi^2 + B_s \xi^3 + \ldots + b_{01} \xi + \ldots = 0 \quad (B.28) \]

where the linear buckling load \( \lambda_C \) is obtained from the linearized eigenvalue problem

\[ -\lambda_C \sigma_{s1} e_{s1} + \sigma_{s1} \delta e_0 + \sigma_0 L_{11}(u_{s1}, \delta u) \quad (B.29) \]

\[ A_s = \frac{1}{\lambda_C \Delta_s} \left\{ \frac{3}{2} \sigma_{s1} \cdot L_{11}(u_{s1}, u_{s2}) \right\} \]
\[ B_s = \frac{1}{\lambda_C \Delta_s} \left\{ 2 \sigma_{s1} \cdot L_{11}(u_{s1}, u_{s2}) + \sigma_{s2} \cdot L_{11}(u_{s1}, u_{s1}) \right\} \]
\[ b_{01} = \frac{1}{\lambda_C \Delta_s} \left\{ 2 \sigma_0 \cdot L_{11}(u_{s1}, u_{s1}) + \sigma_{s1} \cdot L_{11}(u_{0}, u_{s1}) \right\} \quad (B.30) \]

where

\[ \Delta_s = \sigma_{s1} e_{s1} \quad (B.31) \]

The coefficient \( B_s \) contains the solution of the static second order state. The solution for \( u_{s2} \) is obtained from the second order term in the equation governing equilibrium of the static state, where \( \delta u \) is chosen orthogonal to \( u_{s1} \), in combination with the corresponding strain-displacement relation and constitutive equation for \( e_{s2} \). It is noted that the static buckling state analysis can be directly related to initial postbuckling and imperfection sensitivity analysis (Budiansky, 1967; Fitch, 1968). If \( u_{s1} \) is the buckling mode of the ‘perfect’ structure, small values of \( \xi \) result in singular perturbations to the relations between the fundamental state solution \( u_0 \) (or load parameter \( \lambda \)) and the response parameter \( \xi \) in the ‘perfect’ case (Budiansky, 1967).

**Dynamic state**

The expansion for the dynamic strains become
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\[ \dot{\epsilon}_d = \xi_d \epsilon_d + \xi_d^2 \epsilon_d^2 + \xi_d^3 \epsilon_d^3 + \ldots \]
\[ + (\epsilon_{d101} \dot{\xi} + \epsilon_{d101} \ddot{\xi}) + (\epsilon_{d201} \dot{\xi} + \epsilon_{d201} \ddot{\xi}) \xi + \ldots \]
\[ + (\epsilon_{d102} \dot{\xi}^2 + \epsilon_{d111} \dot{\xi} + \epsilon_{d102} \dddot{\xi}^2) + \ldots \]  
(B.32)

and the stress-strain relation can be written as

\[ \sigma_{d_0} = H(\epsilon_{d_0}) \]  
(B.33)

We consider resonance of the single “primary” mode \( u_{d_0} \). Assuming a periodic motion with unknown frequency \( \omega \), and introducing a new time scale \( \tau = \omega t \), the equation governing dynamic equilibrium for free vibration (\( q_0 = 0 \)) is integrated over one period and becomes:

\[
\int_0^{2\pi} \left[ \dot{\delta u} \{ \sigma_{d_0} L_1(\delta u) + \sigma_0 L_{11}(u_{d_0}, \delta u) + \sigma_{d_1} L_{11}(u_0, \delta u) - \omega^2 M(u_{d_0}) \delta u \} \right. \\
+ \xi_d \dot{\xi}_d \{ \sigma_{d10} L_1(\delta u) + \sigma_k L_{11}(u_{d10}, \delta u) + \sigma_{d_1} L_{11}(u_{d_1}, \delta u) \} \\
+ \xi_d^2 \dot{\xi}_d \{ \sigma_{d10} L_1(\dot{u}_d) + \sigma_0 L_{11}(u_{d10}, \dot{u}_d) + \sigma_{d_1} L_{11}(u_{d1}, \dot{u}_d) \} \\
+ \xi_d^3 \dot{\xi}_d \{ \sigma_{d10} L_1(\ddot{u}_d) + \sigma_0 L_{11}(u_{d10}, \ddot{u}_d) + \sigma_{d_1} L_{11}(u_{d1}, \ddot{u}_d) \} \\
+ \sigma_{d1} L_{11}(u_0, \delta u) - \omega^2 M(u_{d_0}) \delta u \right] d\tau = 0 \]  
(B.34)

The first order state equation forms an eigenvalue problem for the unknown natural square frequency \( \omega_0^2 \). By setting \( \delta u = u_{d_0} = u_{d_0} \cos \omega t \) and integrating from \( \tau = 0 \) to \( \tau = 2\pi \) we obtain the following expression for \( \omega_0^2 \):

\[ \sigma_{d_1} L_1(u_{d_0}) + \sigma_0 L_{11}(u_{d_0}, u_{d_1}) + \sigma_{d_1} L_{11}(u_0, u_{d_0}) - \omega_0^2 M(u_{d_0}) u_{d_0} = 0 \]  
(B.35)
Notice that this equation forms also the definition of the orthogonality between the nontrivial mode $u^j$ and the trivial mode $u_0$. Notice further that by setting $\delta u = u_n = \hat{u}_n \cos \omega t$ in the first order equation in (B.34), we obtain

$$\sigma_{d_1} L_{11}(u_{d_1}) + \sigma_0 L_{11}(u_{d_1}, (u_{d_n}) + \sigma_{d_1} L_{11}(u_0, (u_{d_n}) - \omega_0^2 M(u_{d_1})u_{d_n} = 0 \quad (B.36)$$

The relation of the frequency squared to the vibration amplitude and the effective imperfection amplitude is derived via a contraction procedure, i.e., the first order equilibrium equation at the critical point (which can be seen as an orthogonalization condition for the higher modes with respect to the critical mode) is used as a constraint to eliminate the third order variables. By using this constraint for $n = 2$ and $n = 3$, in combination with the reciprocity relation (B.7), the following expression for the expansion is obtained

$$\xi_d \left[1 - \frac{\omega^2}{\omega_0^2}\right] + A_d \xi_d^3 + B_d \xi_d^3$$
$$+ b_{101} \xi_d + b_{111} \xi_d \xi_d + b_{201} \xi_d^2 + b_{210} \xi_d \xi_d$$
$$+ b_{112} \xi_d^2 + b_{111} \xi_d \xi_d + b_{120} \xi_d^2 \xi_d = 0 \quad (B.37)$$

where the coefficients ('a-factor' and 'b-factors') are given by

$$A_d = \frac{1}{\omega_0^2 \Delta_d} \int_0^{2\pi} \frac{3}{2} \sigma_{d_1} \cdot L_{11}(u_{d_1}, u_{d_1}) d\tau$$
$$B_d = \frac{1}{\omega_0^2 \Delta_d} \int_0^{2\pi} \{2 \sigma_{d_1} \cdot L_{11}(u_{d_1}, u_{d_2}) + \sigma_{d_2} \cdot L_{11}(u_{d_1}, u_{d_1})\} d\tau$$
$$b_{110} = \frac{1}{\omega_0^2 \Delta_d} \int_0^{2\pi} \{\sigma_{s_1} \cdot L_{11}(u_{s_1}, u_{d_1}) + 2 \sigma_{d_1} \cdot L_{11}(u_{s_1}, u_{d_1})\} d\tau \quad (B.38)$$

and
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\[ b_{101} = \frac{1}{\omega_0^2 \Delta_d} \int_0^{2\pi} \left\{ H(L_{11}(u_{i1}, u_0)) \cdot L_{11}(u_{d1}, u_{d1}) - \sigma_{s1} \cdot L_{11}(u_{d1}, u_{d1}) \right\} d\tau \]

\[ b_{210} = \frac{1}{\omega_0^2 \Delta_d} \int_0^{2\pi} \left\{ \sigma_{s1} \cdot L_{11}(u_{d1}, u_{d1}) + \sigma_{d1} \cdot L_{11}(u_{d1}, u_{d1}) + \sigma_{d1} \cdot L_{11}(u_{d1}, u_{d1}) \right\} d\tau \]

\[ b_{201} = \frac{1}{\omega_0^2 \Delta_d} \int_0^{2\pi} \left\{ \sigma_{s1} \cdot L_{11}(u_{d1}, u_{d1}) - \sigma_{s1} \cdot L_{11}(u_{d1}, u_{d1}) \right\} d\tau \]

\[ b_{120} = \frac{1}{\omega_0^2 \Delta_d} \int_0^{2\pi} \left\{ \sigma_{s1} \cdot L_{11}(u_{d1}, u_{d1}) + \sigma_{d1} \cdot L_{11}(u_{d1}, u_{d1}) + \sigma_{d1} \cdot L_{11}(u_{d1}, u_{d1}) \right\} d\tau \]

\[ b_{111} = \frac{1}{\omega_0^2 \Delta_d} \int_0^{2\pi} \left\{ H(L_{11}(u_{i1}, u_0)) \cdot L_{11}(u_{d1}, u_{d1}) - \sigma_{s1} \cdot L_{11}(u_{d1}, u_{d1}) \right\} d\tau \]

\[ b_{102} = \frac{1}{\omega_0^2 \Delta_d} \int_0^{2\pi} \left\{ \sigma_{s2} \cdot L_{11}(u_{d1}, u_{d1}) + 2\sigma_{d1} \cdot L_{11}(u_{d1}, u_{d1}) \right\} d\tau \]

(B.39)

where

\[ \Delta_d = \int_0^{2\pi} M(u_1) \cdot u_1 d\tau \] (B.40)

Notice that to account for the nonlinearity of the fundamental state correctly, the expression for the strain-displacement relation should contain a term \( O(\dot{\xi}) \) (cf. Cohen, 1968b; Wedel-Heinen, 1991). To evaluate the \( B \)-term, we have to solve for the 2nd-order dynamic state, which can be found from the variational equation of motion for the second order terms

\[ -\omega^2 M(u_{d1}) \dot{\delta u} + \sigma_{d1} \delta e_0 + \sigma_{d1} L_{11}(u_{d1}, \delta u) = 0 \] (B.41)

where \( \delta u \) is chosen orthogonal to \( u_{d1} \), in combination with the corresponding strain-displacement relation and constitutive equation for \( e_{d1} \).

Similarly, to evaluate the 2nd-order terms \( b_{120}, b_{111}, \) and \( b_{102} \) which determine the imperfection sensitivity of the frequency, we have to solve for the imperfect dynamic state (\( \xi_2, \xi_2 \) state) from Eq. (B.34).

For a symmetric structure (i.e. the response of the structure is independent of the sign of the deformations), \( A_d = b_{101} = b_{110} = b_{201} = b_{210} = 0 \). Considering only the lowest order terms, the remaining 'b'-terms in (B.37) determine the nonlinear behaviour (the coefficient \( B_d \)) and the imperfection sensitivity of the vibrations (the coefficients \( b_{102}, b_{111}, \) and \( b_{101} \)). It is noted that an expansion for \( \omega^2 \) of the form (B.37) can also be assumed a priori with unknown coefficients \( A_d, B_d, \) etc., which are then obtained by equating the different perturbational terms in the equilibrium equation to zero. It is noted further that the expressions for the a-factor and b-factors will in general be evaluated under the affinity assumption.
where \( \hat{u}_{d_i} \) is the spatial mode of the linear vibration mode \( u_{d_i} = \hat{u}_{d_i} \cos \omega t \).

### B.4 Extensions of the theory

In this section, several extensions of the previous theory will be treated, namely the forced vibration analysis, a higher order expansion, and the multi-mode case.

#### Forced vibrations

The theory presented is easily extended to forced vibrations (Rehfield, 1974). Setting \( q_d = Qq_e \), where \( Q \) is the amplitude of the external excitation \( q_e \), the governing variational equilibrium equation is modified to

\[
M(\ddot{u}_d)\delta u + \sigma_d \delta \epsilon_0 + \sigma_0 L_{11}(u_{d_1}, \delta u) = Qq_e \cdot \delta u
\]

where

\[
\phi_0 = \frac{1}{\omega_0^2 \Delta_d} \int_{0}^{2\pi} q_0 \cdot u_{d_1} d\tau
\]

#### Higher order analysis

The preceding analysis gives information about the lowest order effects. By carrying more terms in the expansions, for a 'perfect' structure (i.e. without imperfections in the nontrivial mode), an amplitude-frequency relation of the following form can be obtained:

\[
\left( \frac{\omega}{\omega_0} \right)^2 = 1 + A_d \xi_d + B_d \xi_d^2 + C_d \xi_d^3 + D_d \xi_d^4
\]

where

\[
C_d = \frac{1}{\omega_0^3 \Delta_d} \int_{0}^{2\pi} \frac{1}{2} \left\{ \sigma_{d_1} \cdot L_{11}(u_{d_2}, u_{d_3}) + 4\sigma_{d_1} \cdot L_{11}(u_{d_2}, u_{d_3}) + 2\sigma_{d_3} \cdot L_{11}(u_{d_2}, u_{d_3}) \right\} d\tau
\]

\[
D_d = \frac{1}{\omega_0^3 \Delta_d} \int_{0}^{2\pi} \left\{ \sigma_{d_1} \cdot L_{11}(u_{d_2}, u_{d_3}) + 2\sigma_{d_1} \cdot L_{11}(u_{d_2}, u_{d_3}) + \sigma_{d_3} \cdot L_{11}(u_{d_2}, u_{d_3}) \right\} d\tau
\]
Multimode analysis

If a natural frequency corresponds to more than one vibration mode, a multi-mode analysis can be performed, analogous to the analysis by Byskov and Hutchinson (1977) for buckling mode interaction. Supposing that there are $M$ vibration modes, denoted by $u_i$, with corresponding natural frequency squared $\omega_i^2 = \omega_0^2 \xi_i$, and amplitude $\xi_i$ ($i = 1, 2, \ldots, M$), the displacement field for a 'perfect' unloaded structure can be written as

$$u_d = \xi_1 u_1 + \xi_2 u_{12} + \ldots$$

with corresponding expressions for the stress and strain fields

$$\sigma_d = \xi_1 \sigma_1 + \xi_2 \sigma_{12} + \ldots$$

$$\epsilon_d = \xi_1 \epsilon_1 + \xi_2 \epsilon_{12} + \ldots$$

We now obtain nonlinear amplitude-frequency relations of the following form

$$\xi_i [1 - \frac{\omega^2}{\omega_i^2}] + A_{ij} \xi_i \xi_j + B_{ijkl} \xi_i \xi_j \xi_k \xi_l = 0, \quad I = 1, 2, \ldots, M$$

where

$$A_{ij} = \frac{1}{\omega_i^2 \Delta t} \int_0^{2\pi} \frac{1}{2} [\sigma_i \cdot L_{11}(u_i, u_j) + 2\sigma_i \cdot L_{11}(u_j, u_i)] d\tau$$

$$B_{ijkl} = \frac{1}{\omega_i^2 \Delta t} \int_0^{2\pi} \frac{1}{2} [\sigma_i \cdot L_{11}(u_i, u_j) + \sigma_{ij} \cdot L_{11}(u_j, u_k)] d\tau$$

$$+ \sigma_i \cdot L_{11}(u_i, u_j) + \sigma_{ij} \cdot L_{11}(u_j, u_k) + 2\sigma_i \cdot L_{11}(u_j, u_{ij})] d\tau$$

and

$$\Delta t = \int_0^{2\pi} M(u_i) \cdot u_i d\tau$$
Appendix C

Details of the Extended Analysis

In this appendix, the relevant stages of the Extended Analysis are illustrated in detail. The perturbational terms of the periodicity condition are given for the static state, and the governing equations for the first-order static, dynamic, and flutter state, are shown. Furthermore, the governing equations of the second-order static state are worked out in detail. Finally, the reduced boundary conditions for the first order states are presented. In the notation in this appendix, the superscripts refering to the state, (') or ("), will be omitted.

C.1 Circumferential periodicity condition

The solution obtained with the Extended Analysis has to satisfy the circumferential periodicity condition

$$\int_0^{2\pi R} v_{,y} \, dy = 0$$  \hspace{1cm} (C.1)

where for a perfect shell

$$v_{,y} = \epsilon_y + \frac{W}{R} - \frac{1}{2} \frac{W_{,y}^2}{W}$$  \hspace{1cm} (C.2)

$$\epsilon_y = A_{12}^* N_x + A_{22}^* N_y + A_{20}^* N_{xy} + B_{21}^* \kappa_x + B_{22}^* \kappa_y + B_{20}^* \kappa_{xy}$$  \hspace{1cm} (C.3)

and

$$N_x = F_{,yy} ; \quad N_y = F_{,xx} ; \quad N_{xy} = -F_{,xy} ;$$  \hspace{1cm} (C.4)

$$\kappa_x = -W_{,xx} ; \quad \kappa_y = -W_{,yy} ; \quad \kappa_{xy} = -2W_{,xy} ;$$  \hspace{1cm} (C.5)

For the static state, substituting for $W$ and $F$ the assumed perturbation expansion yields after regrouping and ordering by powers of $\xi$
$v_{ry} = \frac{h}{cR} \{ (-\lambda A_{12}^* + cW_\nu) + (-\bar{p} \bar{A}_{22}^* + cW_p) + (-\bar{r} A_{26}^* + cW_l) \\
+ \bar{A}_{22} f_0^{\prime\prime} - \frac{h}{2R} \bar{B}_{21} w_0'' + cw_0 \}\ + \frac{h}{cR} \xi \{ [\bar{A}_{12}^* f_1'' A_{12}^* n^2 f_1 - \bar{A}_{26}^* n f_2'] \\
- \frac{h}{2R} (\bar{B}_{21}'' w_1'' - \bar{B}_{22}'' n^2 w_1 + 2\bar{B}_{26}'' n w_1') + cw_1 \} \cos \theta \\
+ [\bar{A}_{22} f_0'' - \bar{A}_{12}^* A_{12}^* n^2 f_1^2 - \bar{A}_{26}^* n f_1'] \\
+ \frac{h}{2R} (\bar{B}_{21}'' w_1'' - \bar{B}_{22}'' n^2 w_1 - 2\bar{B}_{26}'' n w_1') + cw_1 \sin \theta \}
+ \frac{h}{cR} \xi \{ (-\lambda A_{12}^* + cW_\nu) + (-\bar{r} A_{26}^* + cW_l) \\
+ [\bar{A}_{22} f_0'' - \bar{A}_{12}^* A_{12}^* n^2 f_1 - \bar{A}_{26}^* n f_1'] \\
+ \frac{h}{2R} (\bar{B}_{21}'' w_1'' - \bar{B}_{22}'' n^2 w_1 - 2\bar{B}_{26}'' n w_1') + cw_1 \} \cos \theta \\
+ \frac{cR}{4R} n^2 (w_1^2 + w_2^2) \cos 2n\theta \\
+ [\bar{A}_{22} f_0'' - \bar{A}_{12}^* A_{12}^* n^2 f_1 + \bar{A}_{26}^* n f_1'] \\
+ \frac{h}{2R} (\bar{B}_{21}'' w_1'' - \bar{B}_{22}'' n^2 w_1 - 2\bar{B}_{26}'' n w_1') + cw_1 \sin 2n\theta \}
+ \frac{h}{cR} \xi \{ -\bar{c}_1 [(w_1 w_3 + w_2 w_4) \cos n\theta - (w_2 w_3 - w_1 w_4) \sin n\theta \\
- (w_1 w_3 - w_2 w_4) \cos 3n\theta - (w_2 w_3 + w_1 w_4) \sin 3n\theta] \}
+ \frac{h}{cR} \xi \{ -\bar{c}_1 n^2 [w_3^2 + w_4^2 - 2w_2 w_4 \sin 4n\theta - (w_3^2 - w_4^2) \cos 4n\theta] \} \quad (C.6)

where $\theta = y/R$. Substituting this expression into equation (C.1) and carrying out the $y$-integration yields

$$\{( -\lambda \bar{A}_{12} + cW_\nu) + (-\bar{p} \bar{A}_{22} + cW_p) + (-\bar{r} \bar{A}_{26} + cW_l) \\
+ \bar{A}_{22} f_0'' - \frac{h}{2R} \bar{B}_{21} w_0'' + cw_0 \} \\
+ \frac{h}{cR} \xi \{ [\bar{A}_{12}^* f_1'' A_{12}^* n^2 f_1 - \bar{A}_{26}^* n f_2'] \\
- \frac{h}{2R} (\bar{B}_{21}'' w_1'' - \bar{B}_{22}'' n^2 w_1 + 2\bar{B}_{26}'' n w_1') + cw_1 \} \cos \theta \\
+ [\bar{A}_{22} f_0'' - \bar{A}_{12}^* A_{12}^* n^2 f_1^2 - \bar{A}_{26}^* n f_1'] \\
+ \frac{h}{2R} (\bar{B}_{21}'' w_1'' - \bar{B}_{22}'' n^2 w_1 - 2\bar{B}_{26}'' n w_1') + cw_1 \sin \theta \}
+ \frac{h}{cR} \xi \{ (-\lambda \bar{A}_{12} + cW_\nu) + (-\bar{r} \bar{A}_{26} + cW_l) \\
+ [\bar{A}_{22} f_0'' - \bar{A}_{12}^* A_{12}^* n^2 f_1 - \bar{A}_{26}^* n f_1'] \\
+ \frac{h}{2R} (\bar{B}_{21}'' w_1'' - \bar{B}_{22}'' n^2 w_1 - 2\bar{B}_{26}'' n w_1') + cw_1 \} \cos \theta \\
+ \frac{cR}{4R} n^2 (w_1^2 + w_2^2) \cos 2n\theta \\
+ [\bar{A}_{22} f_0'' - \bar{A}_{12}^* A_{12}^* n^2 f_1 + \bar{A}_{26}^* n f_1'] \\
+ \frac{h}{2R} (\bar{B}_{21}'' w_1'' - \bar{B}_{22}'' n^2 w_1 - 2\bar{B}_{26}'' n w_1') + cw_1 \sin 2n\theta \}
+ \frac{h}{cR} \xi \{ -\bar{c}_1 [(w_1 w_3 + w_2 w_4) \cos n\theta - (w_2 w_3 - w_1 w_4) \sin n\theta \\
- (w_1 w_3 - w_2 w_4) \cos 3n\theta - (w_2 w_3 + w_1 w_4) \sin 3n\theta] \}
+ \frac{h}{cR} \xi \{ -\bar{c}_1 n^2 [w_3^2 + w_4^2 - 2w_2 w_4 \sin 4n\theta - (w_3^2 - w_4^2) \cos 4n\theta] \} \quad (C.7)

Notice that the underlined terms vanish identically since they are equal to equations (5.18) and (C.35), respectively, with the constants $\bar{C}_1 = \bar{C}_2 = 0$ and $\bar{C}_3 = \bar{C}_4 = 0$. 

Details of the Extended Analysis
If one now lets

\[ W_\nu = \frac{\tilde{A}_{12}}{c} \lambda \quad (C.8) \]
\[ W_p = \frac{\tilde{A}_{22}}{c} \tilde{p} \quad (C.9) \]
\[ W_t = \frac{\tilde{A}_{26}}{c} \tau \quad (C.10) \]

and

\[ W_\nu^{(2)} = \frac{\tilde{A}_{12}}{c} \lambda^{(2)} \quad (C.11) \]
\[ W_t^{(2)} = \frac{\tilde{A}_{26}}{c} \tau^{(2)} \quad (C.12) \]

then the periodicity condition (C.1) is satisfied up to and including terms of the order \( \xi^3 \).

### C.2 First-order state problem

The equations governing the static and dynamic first order state (buckling problem, linear vibration problem, and flutter problem) can be written in the form

\[ L_{A^*}(\tilde{F}^{(1)}) - L_{B^*}(\tilde{W}^{(1)}) = -\frac{1}{R} \tilde{W}_{1xx}^{(1)} - \tilde{W}_{1yy}^{(1)} \tilde{W}^{(0)}_{1xx} \quad (C.13) \]
\[ L_{B^*}(\tilde{F}^{(1)}) + L_{D^*}(\tilde{W}^{(1)}) = \frac{1}{R} \tilde{F}_{1xx}^{(1)} + \tilde{F}_{1yy}^{(0)} \tilde{W}_{1yy}^{(1)} - 2 \tilde{F}_{1xy}^{(1)} \tilde{W}_{1xy}^{(1)} \]
\[ + \tilde{F}_{1yy}^{(0)} \tilde{W}_{1xx}^{(1)} + \tilde{F}_{1xy}^{(1)} \tilde{W}_{1xx}^{(0)} + \tilde{p}_{ax} - \tilde{\rho}h \tilde{W}_{1tt}^{(1)} \quad (C.14) \]

The first-order equations admit separable solutions of the form

\[ \tilde{W}^{(1)} = W^{(1)} e^{i\omega t} = h \{ \tilde{w}_1(x) \cos n\theta + \tilde{w}_2(x) \sin n\theta \} e^{i\omega t} \quad (C.15) \]
\[ \tilde{F}^{(1)} = F^{(1)} e^{i\omega t} = \frac{E R h}{c} \left\{ \tilde{f}_1(x) \cos n\theta + \tilde{f}_2(x) \sin n\theta \right\} e^{i\omega t} \quad (C.16) \]

where \( \theta = y/R \). Introduction into the governing equations, regrouping and equating coefficients of like trigonometric terms, gives the following set of four 4th-order differential equations in \( w_1, w_2, f_1, \) and \( f_2 \):

\[ \bar{A}_{22} f_1^{iv} - (2\bar{A}_{12} + \bar{A}_{66}) n^2 f_1'' + \bar{A}_{11} n^4 f_1 - 2\bar{A}_{26} n f_2''' + 2\bar{A}_{16} n^3 f_2 = \]
\[ - \frac{h}{R} \left\{ \bar{B}_{21} w_1^{iv} - (\bar{B}_{11} + \bar{B}_{22} - 2\bar{B}_{66}) n^2 w_1'' + \bar{B}_{12} n^4 w_1 + (2\bar{B}_{26} - \bar{B}_{61}) n w_2'' \right\} + cw_1'' - \frac{c^2 h}{R} n^2 w_1'' w_1 = 0 \quad (C.17) \]
\[ \bar{A}_{22} f''_2 - (2\bar{A}_{12} + \bar{A}_{66}) n^2 f''_2 + \bar{A}_{11} n^4 f_2 + 2\bar{A}_{26} n f'''_1 - 2\bar{A}_{16} n^3 f'_1 \]
\[ - \frac{h}{2R} \{ \bar{B}_{21}' w''_2 - (\bar{B}_{11}' + \bar{B}_{22}' - 2\bar{B}_{66}') n^2 w''_2 + \bar{B}_{16}' n^4 w_2 - (2\bar{B}_{26}' - \bar{B}_{61}') w'''_1 \]
\[ - (2\bar{B}_{16}' - \bar{B}_{62}') n^3 w'_1 \} + c w'_2 - \frac{ch}{R} n^2 w''_0 w_2 = 0 \quad (C.18) \]

\[ \frac{2R}{h} \{ \bar{B}_{21}' f''_1 - (\bar{B}_{11}' + \bar{B}_{22}' - 2\bar{B}_{66}') n^2 f''_1 \]
\[ + \bar{B}_{12}' n^4 f_1 + (2\bar{B}_{26}' - \bar{B}_{61}') n f'''_1 + (2\bar{B}_{16}' - \bar{B}_{62}') n^3 f'_1 \}
\[ + \bar{D}_{11}' w''_1 - 2(\bar{D}_{12}' + 2\bar{D}_{66}') n^2 w''_1 \]
\[ + \bar{D}_{22}' n^4 w_1 + 4\bar{D}_{16}' n w'''_1 - 4\bar{D}_{66}' n^3 w'_1 - \frac{4cR^2}{h^2} f''_1 \]
\[ + \frac{4cR}{h} \{ \lambda w''_1 - \bar{p} n^2 w_1 - 2n \bar{r} w'_1 - \bar{\rho} \bar{w}^2_0 w_1 + n^2 (f''_0 w_1 + w''_0 f_1) + \gamma \bar{p}_\infty M_\infty w'_1 \} = 0 \quad (C.19) \]

\[ \frac{2R}{h} \{ \bar{B}_{21}' f''_2 - (\bar{B}_{11}' + \bar{B}_{22}' - 2\bar{B}_{66}') n^2 f''_2 \]
\[ + \bar{B}_{12}' n^4 f_2 - (2\bar{B}_{26}' - \bar{B}_{61}') n f'''_2 + (2\bar{B}_{16}' - \bar{B}_{62}') n^3 f'_2 \}
\[ + \bar{D}_{11}' w''_2 - 2(\bar{D}_{12}' + 2\bar{D}_{66}') n^2 w''_2 \]
\[ + \bar{D}_{22}' n^4 w_2 + 4\bar{D}_{16}' n w'''_2 + 4\bar{D}_{66}' n^3 w'_2 - \frac{4cR^2}{h^2} f''_2 \]
\[ + \frac{4cR}{h} \{ \lambda w''_2 - \bar{p} n^2 w_2 - 2n \bar{r} w'_2 - \bar{\rho} \bar{w}^2_0 w_2 + n^2 (f''_0 w_2 + w''_0 f_2) + \gamma \bar{p}_\infty M_\infty w'_2 \} = 0 \]
\[ (C.20) \]

To be able to use the shooting method, the \( w''_1 \) term is eliminated from (C.17) and the \( f''_1 \) term from (C.19). Similarly, the \( w''_2 \) term is eliminated from (C.18) and the \( f''_2 \) term from (C.20). This finally results in the following equations:

\[ f''_1 = C_{17} f''_1 - C_{18} f_1 + C_{19} f'''_1 + C_{20} f'_1 \]
\[ + C_{21} w''_1 + C_{22} w_1 + C_{23} w'''_1 + C_{24} w'_1 \]
\[ + C_{26} w''_0 w_1 + C_{28} \bar{p} w_1 - C_{28} f''_0 w_1 \]
\[ + C_{31} \bar{p} \bar{w}^2_0 w_1 - C_{31} \gamma \bar{p}_\infty M_\infty w'_1 + C_{30} \bar{r} w'_2 - C_{31} \lambda w''_1 - C_{28} w''_0 f_1 \quad (C.21) \]

\[ f''_2 = C_{17} f''_2 - C_{18} f_2 - C_{19} f'''_2 + C_{20} f'_1 \]
\[ + C_{22} w''_2 - C_{23} w'''_2 - C_{24} w'_2 \]
\[ + C_{26} w''_0 w_2 + C_{28} \bar{p} w_2 - C_{28} f''_0 w_2 \]
\[ + C_{31} \bar{p} \bar{w}^2_0 w_2 - C_{31} \gamma \bar{p}_\infty M_\infty w'_2 + C_{30} \bar{r} w'_2 - C_{31} \lambda w''_2 - C_{28} w''_0 f_2 \quad (C.22) \]
C.3 Second-order state problem

\[ w_1'' = C_1 f_1'' + C_2 f_1 - C_3 f_2'' + C_4 f_2' \]
\[- C_5 f_1'' - C_6 w_1 - C_7 w_2'' + C_8 w_2' \]
\[- C_{10} w_0'' w_1 + C_{12} \bar{p} w_1 - C_{12} f_0'' w_1 \]
\[ + C_{15} \bar{\omega}_0^2 w_1 - C_{15} \gamma \bar{p}_\infty M_\infty w_1' + C_{14} \bar{\tau} w_2' - C_{15} \lambda w_1'' - C_{12} w_0'' f_1 \] (C.23)

\[ w_2'' = C_1 f_2'' + C_2 f_2 + C_3 f_1''' - C_4 f_1' \]
\[ + C_5 w_2'' - C_6 w_2 + C_7 w_1''' - C_8 w_1' \]
\[- C_{10} w_0'' w_2 + C_{12} \bar{p} w_2 - C_{12} f_0'' w_2 \]
\[ + C_{15} \bar{\omega}_0^2 w_2 - C_{15} \gamma \bar{p}_\infty M_\infty w_2' - C_{14} \bar{\tau} w_1' - C_{15} \lambda w_2'' - C_{12} w_0'' f_2 \] (C.24)

The constants \( C_1 - C_{31} \) are listed in Appendix D. This set of homogeneous differential equations with variable coefficients together with the appropriate boundary conditions listed in Section C.4 form an eigenvalue problem which is solved numerically. For the linearized buckling problem (\( \omega = \bar{p}_\infty = 0 \)), the loading parameter \( \lambda \), \( \bar{p} \) or \( \bar{\tau} \) is the eigenvalue parameter. For the linearized vibration problem (\( \bar{p}_\infty = 0 \)), the frequency parameter \( \bar{\omega}_0^2 \) is the eigenvalue parameter. In the linearized flutter problem \( \bar{p}_\infty \) is is gradually increased until the frequencies of two vibration modes coalesce.

C.3 Second-order state problem

The equations governing the static 2nd-order state are (omitting the indices corresponding to the static state)

\[ L_A (F^{(2)}) - L_B (W^{(2)}) = - \frac{1}{R} W_{yx}^{(2)} - \frac{h w_{0,xx} W_{yy}^{(2)}}{R} + \frac{h}{R} n^2 \{ w_1 w_{1,xx} + w_1 w_{1,xx} + w_2 w_{2,xx} + w_2 w_{2,xx} \} \]
\[ + \frac{1}{R} \frac{h}{2} \left\{ w_1 w_{1,xx} - w_1 w_{1,xx} - w_2 w_{2,xx} - w_2 w_{2,xx} \cos \theta \right\} + \left( w_1 w_{1,xx} + w_2 w_{2,xx} - 2 w_{1,xx} w_{2,xx} \right) \sin \theta \} \] (C.25)

\[ L_B (F^{(2)}) + L_D W^{(2)} = \frac{1}{R} F_{xx}^{(2)} + \frac{E h^2}{c} \frac{f_{0,xx} W_{xy}^{(2)}}{W_{yy}^{(2)}} + hw_{0,xx} W_{yy}^{(2)} + \frac{E h^2}{c} \frac{f_{0,xx} W_{yy}^{(2)}}{W_{yy}^{(2)}} \]
\[ - \frac{E h^3}{c R} \left\{ \lambda W_{xx}^{(2)} + \bar{p} W_{yy}^{(2)} - 2 \bar{\tau} W_{xy}^{(2)} \right\} - \frac{1}{E h^3} \frac{1}{2} \left\{ w_1 f_{1,xx} + 2 w_1 f_{1,xx} + w_1 w_1 f_{1,xx} + w_2 f_{2,xx} + 2 w_{2,xx} f_{2,xx} + w_{2,xx} f_{2} \right\} \]
\[- \frac{1}{E h^3} \left\{ w_1 f_{1,xx} - 2 w_1 f_{1,xx} + w_1 w_1 f_{1,xx} - (w_2 f_{2,xx} - 2 w_2 f_{2,xx} + w_{2,xx} f_{2}) \right\} \cos 2 \theta \]
\[ + \left( w_1 f_{1,xx} - 2 w_1 f_{1,xx} + w_1 w_1 f_{1,xx} - (w_2 f_{2,xx} - 2 w_2 f_{2,xx} + w_{2,xx} f_{2}) \right) \sin 2 \theta \} \] (C.26)
These equations admit separable solutions of the form

\[ W^{(2)} = h(W_{\nu}^{(2)} + W_{t}^{(2)}) + h \{ w_{\alpha,2}(x) + w_{\beta,2}(x) \cos 2n\theta + w_{\gamma,2}(x) \sin 2n\theta \} \quad (C.27) \]

\[ F^{(2)} = \frac{ERh^2}{c} \left[ f_{\alpha,2}(x) + f_{\beta,2}(x) \cos 2n\theta + f_{\gamma,2}(x) \sin 2n\theta \right] \]

\[ + \frac{ERh^2}{c} \left\{ -\frac{1}{2} \lambda^{(2)} y^2 - \tilde{r}^{(2)} x y \right\} \quad (C.28) \]

Substituting, regrouping and equating coefficients of like trigonometric terms yields the following system of 6 linear inhomogeneous ordinary differential equations with variable coefficients.

\[
\bar{A}_{\alpha}^{*} f_{\nu}^{*} - \frac{2R}{h} \bar{D}_{\alpha}^{*} w_{\alpha}^{*} + cw_{\alpha} = \frac{ch}{2R} n^2 (w_1 w''_1 + w'_1 w''_1 + w_2 w''_2 + w'_2 w''_2) \quad (C.29)
\]

\[
\bar{A}_{\beta}^{*} f_{\beta}^{*} - (2\bar{A}_{12}^{*} + \bar{A}_{66}^{*}) 4n^2 f'_\beta + \bar{A}_{11}^{*} 16n^4 f_\beta - 4 \bar{A}_{26}^{*} n f''_\gamma + 16 \bar{A}_{16}^{*} n^3 f'_\gamma
\]

\[- \frac{h}{2R} \{ \bar{B}_{\beta}^{*} w_{\beta}^{*} - (\bar{B}_{11}^{*} + \bar{B}_{22}^{*} - 2\bar{B}_{66}^{*}) 4n^2 w''_\beta + \bar{B}_{12}^{*} 16n^4 w_\beta \}
\]

\[ + (2\bar{B}_{26}^{*} - \bar{B}_{61}^{*}) 2n w''_\gamma - (2\bar{B}_{16}^{*} - \bar{B}_{62}^{*}) 8n^3 w'_\gamma \]

\[ + cw_{\gamma} - \frac{4ch}{R} n^2 w''_\gamma w_\gamma = \frac{ch}{2R} n^2 (w_1 w''_1 - w'_1 w''_1 - w_2 w''_2 - w'_2 w''_2) \quad (C.30)\]

\[
\bar{A}_{\gamma}^{*} f_{\gamma}^{*} - (2\bar{A}_{12}^{*} + \bar{A}_{66}^{*}) 4n^2 f''_\gamma + \bar{A}_{11}^{*} 16n^4 f_\gamma + 4 \bar{A}_{26}^{*} n f'''_\gamma - 16 \bar{A}_{16}^{*} n^3 f'_\gamma
\]

\[- \frac{h}{2R} \{ \bar{B}_{\gamma}^{*} w_{\gamma}^{*} - (\bar{B}_{11}^{*} + \bar{B}_{22}^{*} - 2\bar{B}_{66}^{*}) 4n^2 w''_\gamma + \bar{B}_{12}^{*} 16n^4 w_\gamma \}
\]

\[ - (2\bar{B}_{26}^{*} - \bar{B}_{61}^{*}) 8n w''_\gamma - (2\bar{B}_{16}^{*} - \bar{B}_{62}^{*}) 8n^3 w'_\gamma \]

\[ + cw_{\gamma} - \frac{4ch}{R} n^2 w''_\gamma w_\gamma = \frac{ch}{2R} n^2 (w_1 w''_1 + w_2 w''_2 - 2w'_1 w'_2) \quad (C.31)\]

\[
\bar{B}_{21}^{*} f_{\alpha}^{*} - (\bar{B}_{11}^{*} + \bar{B}_{22}^{*} - 2\bar{B}_{66}^{*}) 4n^2 f'_\beta
\]

\[ + B_{12}^{*} 16n^4 f_\beta + (2B_{26}^{*} - B_{61}^{*}) n f''_\gamma - (2B_{16}^{*} - B_{62}^{*}) 8n^3 f'_\gamma
\]

\[ + \frac{h}{2R} \{ \bar{D}_{\alpha}^{*} w_{\alpha}^{*} - 2(\bar{D}_{11}^{*} + 2\bar{D}_{66}^{*}) 4n^2 w''_\alpha + \bar{D}_{12}^{*} 16n^4 w_\alpha \}
\]

\[ + 8\bar{D}_{16}^{*} n w''_\gamma - 32\bar{D}_{26}^{*} n^3 w'_\gamma \]

\[ + 2c(\lambda w''_\alpha - 4n^2 w''_\gamma - 4n\tau w'_\gamma) \} + 8cn^2 (f''_\alpha w_\alpha + w''_\alpha f_\alpha) =
\]

\[- cn^2 (w_1 f''_1 + w_2 f''_2) + w''_1 f_1 + w''_2 f_2 + w'_1 f''_1 + w'_2 f''_2 + w'_1 f_1 + w'_2 f_2) \quad (C.32)\]

\[
\bar{B}_{21}^{*} f_{\gamma}^{*} - (\bar{B}_{11}^{*} + \bar{B}_{22}^{*} - 2\bar{B}_{66}^{*}) 4n^2 f'_\beta
\]

\[ + B_{12}^{*} 16n^4 f_\beta + (2B_{26}^{*} - B_{61}^{*}) n f''_\gamma - (2B_{16}^{*} - B_{62}^{*}) 8n^3 f'_\gamma
\]

\[ + \frac{h}{2R} \{ \bar{D}_{\gamma}^{*} w_{\gamma}^{*} - 2(\bar{D}_{12}^{*} + 2\bar{D}_{66}^{*}) 4n^2 w''_\gamma + \bar{D}_{22}^{*} 16n^4 w_\gamma \}
\]

\[ + 8\bar{D}_{16}^{*} n w''_\gamma - 32\bar{D}_{26}^{*} n^3 w'_\gamma \]

\[ + 2c(\lambda w''_\gamma - 4n^2 w''_\alpha - 4n\tau w'_\gamma) \} + 8cn^2 (f''_\gamma w_\gamma + w''_\gamma f_\gamma) =
\]

\[- cn^2 (w_1 f''_1 + w_2 f''_2) + w''_1 f_1 + w''_2 f_2 + w'_1 f''_1 + w'_2 f''_2 + w'_1 f_1 + w'_2 f_2) \quad (C.33)\]
C.3 Second-order state problem

\[ \vec{B}_{21}f_\gamma'' - (\vec{B}_{11} + \vec{B}_{22} + 2\vec{B}_{66})4n^2f_\gamma'' \\
+ \vec{B}_{12}16n^2f_\gamma - (2\vec{B}_{26} - \vec{B}_{61})nf_\beta'' + (2\vec{B}_{16} - \vec{B}_{62})8n^3f_\beta'' \\
+ \frac{h}{2R}\{\vec{D}_{11}w_{\gamma''} - 2(\vec{D}_{12} + 2\vec{D}_{66})4n^2w_{\gamma''} + \vec{D}_{22}16n^4w_{\gamma''} \\
- 8\vec{D}_{16}nw_{\beta''} + 32\vec{D}_{26}n^3w_{\beta'}\} - \frac{2cR}{h}f''_\gamma \\
+ 2c(\lambda w_{\gamma''} - 4n^2\bar{p}w_{\gamma''} + 4n\tau w_{\beta''}) + 8cn^2(f''_\gamma w_{\gamma} + w_{\gamma''}f_\gamma) = \\
- cn^2(w_1f_2'' - 2w_1f_3'' + w_2f_3'' + 2w_2f_3'' - w_1f_3'' + w_2f_3'') \tag{C.34} \]

Equation (C.29) can be integrated twice to yield

\[ f''_\alpha = \frac{h}{2R} \frac{\vec{B}_{22}w_{\alpha''}}{A_{22}} - \frac{c}{A_{22}}w_{\alpha} + c \frac{h n^2}{4R A_{22}}(w_1^2 + w_2^2) + \tilde{C}_5x + \tilde{C}_4 \tag{C.35} \]

where \( \bar{x} = x/R \) and the constants of integration \( \tilde{C}_5 \) and \( \tilde{C}_4 \) are identically equal to zero because of the periodicity condition (see Section C.1 for details). Eliminating \( f_\alpha \) between Eqs. (C.35) and (C.32) one obtains

\[ w_{\gamma''}'' + (D_1 - D_2\lambda)w_{\gamma''}'' - D_3w_{\gamma''} + D_4(w_1^2 + w_2^2) \\
- D_5(w_1w_{1''} + w_1w_{2''} + w_2w_{1''} + w_2w_{2''}) \\
- D_8(w_1f_1'' + 2w_1f_1'' + w_1f_1'' + 2w_2f_2'' + w_2f_2'' + w_2f_2'') \tag{C.36} \]

It is noted that since the axisymmetric loads are assumed to be specified at the fundamental state level, \( \lambda^{(2)} \) is implicitly assumed to be zero.

To be able to use the shooting method, the \( w_{\beta''}'' \) term is eliminated from (C.30) and the \( f_{\beta''}'' \) term from (C.33). Similarly, the \( w_{\gamma''}'' \) term is eliminated from (C.31) and the \( f_{\gamma''}'' \) term from (C.34). This finally results in the following equations

\[ f_{\beta''}'' = D_9f_{\beta''}'' - (D_{10} + D_{17}w_{0''})f_{\beta''} + D_{11}f_{\gamma''}'' + C_{20}f_{\gamma}'' \\
+ D_{12}f_{\gamma}'' - D_{12}f_{\gamma}'' - (D_{13} + D_{31}\lambda)w_{\beta''}'' \\
- (D_{14} + D_{18}w_{0''}w_{\gamma} - D_{17}(f_{\gamma}'' - \bar{p})w_{\gamma} - D_{15}w_{\beta''}'' + (D_{16} + D_{19})w_{\gamma}' \\
+ D_{32}(w_1w_{1''} - w_1w_{2''} + w_2w_{1''} + w_2w_{2''}) \\
- D_5(w_1f_1'' - 2w_1f_1'' + w_1f_1'' + 2w_2f_2'' + w_2f_2'' + w_2f_2'') \tag{C.37} \]

\[ f_{\gamma''}'' = D_9f_{\gamma''}'' - (D_{10} + D_{17}w_{0''})f_{\gamma} - D_{11}f_{\gamma''}'' + C_{20}f_{\gamma}' \\
+ D_{12}f_{\gamma}'' - D_{12}f_{\gamma}'' - (D_{13} + D_{31}\lambda)w_{\gamma}'' \\
- (D_{14} + D_{18}w_{0''}w_{\gamma} - D_{17}(f_{\gamma}'' - \bar{p})w_{\gamma} + D_{15}w_{\beta''}'' - (D_{16} + D_{19})w_{\gamma}'' \\
+ D_{32}(w_1w_{1''} + w_2w_{1''} - 2w_1w_{2''}) \\
- D_5(w_1f_1'' - 2w_1f_1'' + w_1f_1'' + 2w_2f_2'' + w_2f_2'' + w_2f_2'') \tag{C.38} \]
The equations admit separable solutions of the form

\begin{align}
    w_\beta'' &= -D_{20}f_\beta'' - (D_{21} + D_{22}w_0'')f_\beta \\
    &+ (D_{23} - D_{24}w_\beta)w_\beta'' - (D_{24} + D_{13}w_0'')w_\beta - D_{22}(f_\beta'' - \ddot{\tilde{p}})w_\beta \\
    &- D_{25}f_\beta'' - D_{25}f_\beta' + D_{25}w_\beta'' + (D_{28} + D_{29}\overline{\tau})w_\beta' \\
    &+ D_5(w_1w_\beta'' - w_1w_1' - w_2w_\beta + w_3w_\beta) \\
    &- D_8(w_1f_1'' - 2w_1f_1' + w_1w_1' - w_2f_2'' + 2f_2f_2' - w_2w_2')
\end{align}

(C.39)

\begin{align}
    w_\gamma'' &= -D_{20}f_\gamma'' - (D_{21} + D_{22}w_0'')f_\gamma \\
    &+ (D_{23} - D_{24}w_\gamma)w_\gamma'' - (D_{24} + D_{13}w_0'')w_\gamma - D_{22}(f_\gamma'' - \ddot{\tilde{p}})w_\gamma \\
    &- D_{25}f_\gamma'' + D_{25}f_\gamma' + D_{25}w_\gamma'' + (D_{28} + D_{29}\overline{\tau})w_\gamma' \\
    &+ D_5(w_1w_\gamma'' - w_1w_1' - w_2w_\gamma + w_3w_\gamma) \\
    &- D_8(w_1f_1'' - 2w_1f_1' + w_1w_1' - w_2f_2'' + 2f_2f_2' - w_2w_2')
\end{align}

(C.40)

where \( f_0'' \) is given by Eq. (5.18), and the constants \( D_1 - D_{32} \) are listed in Appendix D.

This set of inhomogeneous differential equations with variable coefficients together with the appropriate boundary conditions (Arbocz and Hol, 1989) form a response problem which is solved numerically.

### C.4 Boundary and symmetry conditions

The reduced boundary conditions for the fundamental state and first-order state are summarized in this appendix. The derivation can be found in Arbocz and Hol (1989).

#### C.4.1 Boundary conditions

**Fundamental state**

*simply supported:*

\begin{align}
    w_0 &= -(W_v + W_p + W_i) \\
    w_0'' &= B_1\lambda + B_2\overline{\tau}
\end{align}

(C.41)

*clamped:*

\begin{align}
    w_0 &= -(W_v + W_p + W_i) \\
    w_0' &= 0
\end{align}

(C.42)
First-order state

SS-1: \( N_x = -N_0; \quad N_{xy} = T_0; \quad W = 0; \quad M_z = -N_0q; \)

\[
\begin{align*}
f_1 &= f_2 = 0 \\
f_1' &= f_2' = 0 \\
w_1 &= w_2 = 0 \\
w_1'' &= -B_3 w_2' - B_4 f_2'' \\
w_2'' &= B_3 w_1' - B_4 f_2''
\end{align*}
\] (C.43)

SS-2: \( u = 0; \quad N_{xy} = T_0; \quad W = 0; \quad M_x = -N_0q; \)

\[
\begin{align*}
f_1''' &= B_{15} f_2'' - B_{16} f_2 + B_{12} w_1''' + B_{17} w_1' \\
f_2''' &= -B_{15} f_1'' - B_{16} f_1 + B_{12} w_2''' + B_{17} w_2' \\
f_1' &= f_2' = 0 \\
w_1 &= w_2 = 0 \\
w_1'' &= -B_3 w_2' - B_4 f_1'' \\
w_2'' &= B_3 w_1' - B_4 f_2''
\end{align*}
\] (C.44)

SS-3: \( N_x = -N_0; \quad v = 0; \quad W = 0; \quad M_z = -N_0q; \)

\[
\begin{align*}
f_1 &= f_2 = 0 \\
f_1'' &= B_1 f_1' - B_2 w_2' \\
f_2'' &= B_1 f_1' - B_2 w_1' \\
w_1 &= w_2 = 0 \\
w_1'' &= -B_{21} w_2' + B_{22} f_2' \\
w_2'' &= -B_{21} w_1' + B_{22} f_1'
\end{align*}
\] (C.45)
SS-4: \[ u = 0; \quad v = 0; \quad W = 0; \quad M_x = -N_0 q; \]

\[ f_1''' = B_{37} f_1' - B_{28} w_1' + B_{12} w_1''' - B_{29} f_2 \]
\[ f_2''' = B_{37} f_2' + B_{28} w_2' + B_{12} w_2''' + B_{29} f_1 \]
\[ f_1' = B_3 f_1 + B_1 f_2' + B_2 w_2' \]
\[ f_2' = B_3 f_1 - B_1 f_2' - B_2 w_1' \]
\[ w_1 = w_2 = 0 \]
\[ w_1'' = -B_{21} w_1' + B_{22} f_1' + B_{25} f_2 \]
\[ w_2'' = -B_{21} w_2' + B_{22} f_2' + B_{25} f_1 \]

(C.46)

C-1: \[ N_z = -N_0; \quad N_{xy} = T_0; \quad W = 0; \quad W_{xx} = 0; \]

\[ f_1 = f_2 = 0 \]
\[ f_1' = f_2' = 0 \]
\[ w_1 = w_2 = 0 \]
\[ w_1' = w_2' = 0 \]

(C.47)

C-2: \[ u = 0; \quad N_{xy} = T_0; \quad W = 0; \quad W_{xx} = 0; \]

\[ f_1''' = B_{137} f_2' - B_{18} f_2 + B_{12} w_1''' - B_{20} w_2'' \]
\[ f_2''' = -B_{137} f_1' + B_{18} f_1 + B_{12} w_2''' + B_{20} w_1'' \]
\[ f_1' = f_2' = 0 \]
\[ w_1 = w_2 = 0 \]
\[ w_1' = w_2' = 0 \]

(C.48)

C-3: \[ N_z = -N_0; \quad v = 0; \quad W = 0; \quad W_{xx} = 0; \]

\[ f_1 = f_2 = 0 \]
\[ f_1'' = B_{11} f_2' + B_{12} w_1' \]
\[ f_2'' = -B_{11} f_1' + B_{12} w_2' \]
\[ w_1 = w_2 = 0 \]
\[ w_1' = w_2' = 0 \]

(C.49)
C.4 Boundary and symmetry conditions

\[ C-4: \quad u = 0; \quad v = 0; \quad W = 0; \quad M_x = -N_0 q; \]

\[ f''_1 = B_{33} f'_1 - B_{34} f_2 + B_{12} w''_1 + B_{35} w''_2 \]
\[ f''_2 = B_{33} f'_2 + B_{34} f_1 + B_{12} w''_2 - B_{35} w''_1 \]
\[ f'_1 = B_{10} f_1 + B_{11} f'_2 + B_{12} w'_1 \]
\[ f'_2 = B_{10} f_2 - B_{11} f'_1 + B_{12} w'_2 \]
\[ w_1 = w_2 = 0 \]
\[ w'_1 = w'_2 = 0 \] \hspace{1cm} (C.50)

C.4.2 Symmetry conditions

Fundamental state

\[ w_0 = \text{symmetric w.r.t. } x = L/2: \]

\[ w'_0 = w''_0 = 0 \] \hspace{1cm} (C.51)

First-order state

\[ w_1 = \text{symmetric, } w_2 = \text{anti-symmetric w.r.t. } x = L/2: \]

\[ w_2 = f_2 = w'_2 = f''_2 = w''_1 = f''_1 = 0 \] \hspace{1cm} (C.52)

\[ w_2 = \text{anti-symmetric, } w_1 = \text{symmetric w.r.t. } x = L/2: \]

\[ w_1 = f_1 = w'_1 = f''_1 = w''_2 = f''_2 = 0 \] \hspace{1cm} (C.53)

The coefficients used in this appendix are listed in Appendix D.
Appendix D

Definition of constants

D.1 Constants used in the Extended Analysis

D.1.1 Fundamental state

The following constants are used in Eq. (5.63):

\[
\begin{align*}
\hat{C}_1 &= \frac{4cR}{\Delta h} \bar{B}_{21} \\
\hat{C}_2 &= \frac{4cR}{\Delta h} \bar{A}_{22} \\
\hat{C}_3 &= \frac{4c^2 R^2}{\Delta h} \bar{A}_{22}^3
\end{align*}
\]

D.1.2 First-order state

The following constants are used in Appendix C:

\[
\begin{align*}
C_1 &= \frac{2R}{\Delta h} \{[(\bar{A}_{22}'(\bar{B}_{11}' + \bar{B}_{22}' - 2\bar{B}_{66}') - \bar{B}_{21}')(2\bar{A}_{12}' + \bar{A}_{66}')]n^2 + \frac{2cR}{h} \bar{A}_{22}' \} \\
C_2 &= \frac{2R}{\Delta h} \{\bar{B}_{21}'\bar{A}_{11}' - \bar{B}_{12}'\bar{A}_{22}'\}n^4 \\
C_3 &= \frac{2R}{\Delta h} \{\bar{A}_{22}'(2\bar{B}_{26}' - \bar{B}_{61}') + 2\bar{B}_{21}'\bar{A}_{26}'\}n \\
C_4 &= \frac{2R}{\Delta h} \{\bar{A}_{22}'(2\bar{B}_{16}' - \bar{B}_{62}') + 2\bar{B}_{21}'\bar{A}_{16}'\}n^3
\end{align*}
\]
Definition of constants

\[ C_5 = \frac{1}{\Delta} \{ [2 \bar{A}_{22}(\bar{D}_{12} + 2\bar{D}_{66}) + \bar{B}_{21}(\bar{B}_{11} + \bar{B}_{22} - 2\bar{B}_{66})]n^2 + \frac{2cR}{\Delta} \bar{B}_{21} \} \]

\[ C_6 = \frac{1}{\Delta} \{ \bar{D}_{22}^* \bar{A}_{22} + \bar{B}_{12}^* \bar{B}_{21} \} n^4 \]

\[ C_7 = \frac{1}{\Delta} \{ 4\bar{D}_{16}^* \bar{A}_{22} + \bar{B}_{21}^* (2\bar{B}_{16} - \bar{B}_{61}) \} n \]

\[ C_8 = \frac{1}{\Delta} \{ 4\bar{D}_{26}^* \bar{A}_{22} + \bar{B}_{21}^* (2\bar{B}_{16} - \bar{B}_{62}) \} n^3 \]

\[ C_{10} = \frac{2c}{\Delta} n^2 \bar{D}_{21} \]

\[ C_{12} = \frac{4cR}{\Delta} n^2 \bar{A}_{22} \]

\[ C_{14} = \frac{8cR}{\Delta} n^2 \bar{B}_{21} \]

\[ C_{15} = \frac{4cR}{\Delta} n^2 \bar{A}_{22} \]

\[ C_{17} = \frac{1}{\Delta} \{ [\bar{D}_{11}^* (2\bar{A}_{12} + \bar{A}_{66}) + \bar{B}_{21}^* (\bar{B}_{11} + \bar{B}_{22} - 2\bar{B}_{66})]n^2 + \frac{2cR}{\Delta} \bar{B}_{21} \} \]

\[ C_{18} = \frac{1}{\Delta} \{ \bar{D}_{11}^* \bar{A}_{11} + \bar{B}_{12}^* \bar{B}_{21} \} n^4 \]

\[ C_{19} = \frac{1}{\Delta} \{ 2\bar{D}_{11}^* \bar{A}_{16} - \bar{B}_{21}^* (2\bar{B}_{16} - \bar{B}_{61}) \} n \]

\[ C_{20} = \frac{1}{\Delta} \{ -2\bar{D}_{11}^* \bar{A}_{16} + \bar{B}_{21}^* (2\bar{B}_{16} - \bar{B}_{62}) \} n^3 \]

\[ C_{21} = \frac{1}{\Delta} \frac{h}{2R} \{ [\bar{D}_{11}^* (\bar{B}_{11} + \bar{B}_{22} - 2\bar{B}_{66}) + 2\bar{B}_{21}^* (\bar{D}_{12} + 2\bar{D}_{66})]n^2 - c\bar{D}_{11} \} \]

\[ C_{22} = \frac{h}{2R} \frac{1}{\Delta} (-\bar{B}_{21}^* \bar{D}_{22} + \bar{B}_{12}^* \bar{D}_{11}^*) n^4 \]

\[ C_{23} = \frac{h}{2R} \frac{1}{\Delta} \{ \bar{D}_{11}^* (2\bar{B}_{16} - \bar{B}_{61}) - 4\bar{B}_{21}^* \bar{D}_{16} \} n \]

\[ C_{24} = \frac{h}{2R} \frac{1}{\Delta} \{ 4\bar{B}_{21}^* \bar{D}_{26} - \bar{D}_{11}^* (2\bar{B}_{16} - \bar{B}_{62}) \} n^3 \]

\[ C_{26} = \frac{c h}{\Delta} n^2 \bar{D}_{22}^* \]

\[ C_{28} = \frac{2c}{\Delta} n^2 \bar{B}_{21}^* \]

\[ C_{30} = \frac{4c}{\Delta} n \bar{B}_{21}^* \]

\[ C_{31} = \frac{2c}{\Delta} \bar{B}_{21}^* \]

where

\[ \Delta = \bar{A}_{22} \bar{D}_{11}^* + \bar{B}_{21}^* \]
D.1.3 Constants used in the static 2nd-order state equations

D.1.4 Constants used in the boundary conditions

The following constants are used in Eq. (5.73):

\[ \tilde{D}_1 = C_6 + C_{10}u''_0 + C_{12}(f''_0 - \tilde{p}) - C_{15}\tilde{\omega}_0^2 \]
\[ \tilde{D}_2 = C_8 + C_{14}\tau \]
\[ \tilde{D}_3 = C_5 - C_{15}\lambda \]
\[ \tilde{D}_4 = C_2 - C_{12}u''_0 \]
\[ \tilde{D}_5 = C_{22} + C_{26}u''_0 - C_{28}(f''_0 - \tilde{p}) + C_{31}\tilde{\omega}_0^2 \]
\[ \tilde{D}_6 = C_{24} + C_{30}\tau \]
\[ \tilde{D}_7 = C_{21} - C_{31}\lambda \]
\[ \tilde{D}_8 = C_{18} + C_{28}u''_0 \]

The following constants are used in Appendix C:

\[ D_1 = \frac{4c^2 R}{\Delta h} \tilde{B}_{21}^* \]
\[ D_2 = \frac{4c^2 R}{\Delta h} \tilde{A}_{22}^* \]
\[ D_3 = \frac{4c^2}{\Delta} \left( \frac{R}{h} \right)^2 \]
\[ D_4 = \frac{c^2 R}{\Delta h} n^2 \]
\[ D_5 = \frac{cn^2}{\Delta} \tilde{B}_{21}^* \]
\[ D_6 = \frac{h}{2R} \frac{\tilde{B}_{21}^*}{\tilde{A}_{22}^*} \]
\[ D_7 = \frac{c}{\Delta h} \tilde{A}_{22}^* \]
\[ D_8 = \frac{2cR}{h} \frac{n^2}{\Delta} \tilde{A}_{22}^* \]
\[
\frac{\left(\varepsilon_{g} \varepsilon_{r} Y + \varepsilon_{r} \varepsilon_{g} r\right)}{Y} = \varepsilon D
\]

\[
\frac{\varepsilon}{Y} = \varepsilon \mu \left(\varepsilon \varepsilon_{g} - \varepsilon_{r} \varepsilon_{g} \varepsilon_{r} Y \varepsilon_{r} \varepsilon_{g} r\right) \frac{Y}{Y} \frac{Y}{Y} = \varepsilon D
\]

\[
\left\{\frac{\varepsilon}{Y} \frac{\varepsilon}{Y} = \varepsilon \mu \left(\varepsilon \varepsilon_{g} - \varepsilon_{r} \varepsilon_{g} \varepsilon_{r} Y \varepsilon_{r} \varepsilon_{g} r\right) \frac{Y}{Y} \frac{Y}{Y} = \varepsilon D
\]

\[
\frac{\varepsilon}{Y} = \varepsilon \mu \left(\varepsilon \varepsilon_{g} - \varepsilon_{r} \varepsilon_{g} \varepsilon_{r} Y \varepsilon_{r} \varepsilon_{g} r\right) \frac{Y}{Y} \frac{Y}{Y} = \varepsilon D
\]

\[
\frac{\varepsilon}{Y} = \varepsilon \mu \left(\varepsilon \varepsilon_{g} - \varepsilon_{r} \varepsilon_{g} \varepsilon_{r} Y \varepsilon_{r} \varepsilon_{g} r\right) \frac{Y}{Y} \frac{Y}{Y} = \varepsilon D
\]

\[
\frac{\varepsilon}{Y} = \varepsilon \mu \left(\varepsilon \varepsilon_{g} - \varepsilon_{r} \varepsilon_{g} \varepsilon_{r} Y \varepsilon_{r} \varepsilon_{g} r\right) \frac{Y}{Y} \frac{Y}{Y} = \varepsilon D
\]

\[
\frac{\varepsilon}{Y} = \varepsilon \mu \left(\varepsilon \varepsilon_{g} - \varepsilon_{r} \varepsilon_{g} \varepsilon_{r} Y \varepsilon_{r} \varepsilon_{g} r\right) \frac{Y}{Y} \frac{Y}{Y} = \varepsilon D
\]

\[
\frac{\varepsilon}{Y} = \varepsilon \mu \left(\varepsilon \varepsilon_{g} - \varepsilon_{r} \varepsilon_{g} \varepsilon_{r} Y \varepsilon_{r} \varepsilon_{g} r\right) \frac{Y}{Y} \frac{Y}{Y} = \varepsilon D
\]

\[
\frac{\varepsilon}{Y} = \varepsilon \mu \left(\varepsilon \varepsilon_{g} - \varepsilon_{r} \varepsilon_{g} \varepsilon_{r} Y \varepsilon_{r} \varepsilon_{g} r\right) \frac{Y}{Y} \frac{Y}{Y} = \varepsilon D
\]

\[
\frac{\varepsilon}{Y} = \varepsilon \mu \left(\varepsilon \varepsilon_{g} - \varepsilon_{r} \varepsilon_{g} \varepsilon_{r} Y \varepsilon_{r} \varepsilon_{g} r\right) \frac{Y}{Y} \frac{Y}{Y} = \varepsilon D
\]

\[
\frac{\varepsilon}{Y} = \varepsilon \mu \left(\varepsilon \varepsilon_{g} - \varepsilon_{r} \varepsilon_{g} \varepsilon_{r} Y \varepsilon_{r} \varepsilon_{g} r\right) \frac{Y}{Y} \frac{Y}{Y} = \varepsilon D
\]

\[
\frac{\varepsilon}{Y} = \varepsilon \mu \left(\varepsilon \varepsilon_{g} - \varepsilon_{r} \varepsilon_{g} \varepsilon_{r} Y \varepsilon_{r} \varepsilon_{g} r\right) \frac{Y}{Y} \frac{Y}{Y} = \varepsilon D
\]
D.1 Constants used in the Extended Analysis

\[ D_{25} = \frac{2R}{\Delta h} \{ A_{22}^* (2B_{26}^* - \bar{B}_{62}^*) + 2B_{21}^* A_{26}^* \} 2n \]
\[ D_{26} = \frac{2R}{\Delta h} \{ A_{22}^* (2\bar{B}_{16}^* - \bar{B}_{62}^*) + 2\bar{B}_{21}^* A_{16}^* \} 8n^3 \]
\[ D_{27} = \frac{2n}{\Delta} \{ 4A_{22}^* \bar{D}_{16}^* + \bar{B}_{21}^* (2\bar{B}_{26}^* - \bar{B}_{26}^*) \} \]
\[ D_{28} = \frac{8n^3}{\Delta} \{ 4A_{22}^* \bar{D}_{26}^* + \bar{B}_{21}^* (2\bar{B}_{16}^* - \bar{B}_{62}^*) \} \]
\[ D_{29} = \frac{8cR}{\Delta h} 2n^2 \bar{A}_{22}^* \]
\[ D_{30} = \frac{ct}{4R} A_{22}^* \]
\[ D_{31} = \frac{2c}{\Delta} \bar{A}_{22}^* \]
\[ D_{32} = \frac{ct}{2R} \bar{D}_{11}^* \]

where

\[ \Delta = A_{22}^* \bar{D}_{11}^* + \bar{B}_{21}^2 \]

D.1.4 Constants used in the boundary conditions

The following constants are used in Appendix C:

\[ B_1 = \frac{1}{\Delta_2} \{ \bar{q} - \frac{2R}{h} (\bar{B}_{21}^* A_{12}^* - \bar{B}_{11}^* ) \} \]
\[ B_3 = 2n \frac{\bar{D}_{16}^*}{D_{11}^*} \]
\[ B_4 = 2n \frac{\bar{B}_{11}^*}{R \bar{D}_{11}^*} \]
\[ B_9 = 2n^2 \frac{2}{h} \frac{R \bar{B}_{11}^*}{D_{11}^*} \]
\[ B_{10} = n^2 \frac{\bar{A}_{12}^*}{A_{22}^*} \]
\[ B_{12} = \frac{h}{2R} \frac{B_{21}^*}{A_{22}^*} \]
\[ B_{13} = \frac{h}{R} \frac{\bar{B}_{21}^*}{A_{22}^*} \]
\[ B_{14} = n^2 \frac{\bar{A}_{16}^*}{A_{22}^*} \]
\[ B_{15} = 2n \frac{\bar{A}_{26}^*}{A_{22}^*} \]
\[ B_{16} = n^2 \frac{\bar{B}_{16}^*}{A_{22}^*} \]
\[ B_{17} = n^2 \frac{(\bar{A}_{16}^* + \bar{A}_{66}^*)}{A_{22}^*} \]
\[ B_{18} = \frac{n}{2R} \frac{\bar{A}_{26}^*}{A_{22}^*} (2\bar{B}_{66}^* - \bar{B}_{22}^*) \]
\[ B_{19} = \frac{1}{A_{22}^*} \{ \frac{h}{2R} n^2 (2\bar{B}_{66}^* - \bar{B}_{22}^*) - c \} \]
\[ B_{20} = \frac{h}{2R} \frac{n}{A_{22}^*} (2\bar{B}_{66}^* - \bar{B}_{22}^*) \]
Definition of constants

\[ B_1 = \frac{1}{\Delta_B} (B_{11} + B_5 B_{12}) \]
\[ B_2 = \frac{1}{\Delta_B} (B_{13} - B_3 B_{12}) \]
\[ B_5 = \frac{1}{\Delta_B} (B_{10} - B_6 B_{12}) \]
\[ B_{15} = B_{15} - B_4 B_{20} \]
\[ B_{16} = B_{18} - B_4 B_{20} \]
\[ B_{21} = B_3 - B_4 B_2 \]
\[ B_{17} = B_{19} - B_3 B_{20} \]
\[ B_{25} = B_6 - B_4 B_5 \]
\[ B_{18} = B_8 - B_4 B_1 \]
\[ B_{27} = B_{17} - B_{15} B_1 - B_{20} B_{22} \]
\[ B_{28} = B_{19} - B_{19} B_2 - B_{29} B_{20} \]
\[ B_{29} = B_{18} - B_{19} B_5 - B_{20} B_{25} \]
\[ B_{30} = B_{17} - B_{11} B_{15} \]
\[ B_{32} = B_{18} - B_{16} B_{15} \]

\[ B_{33} = B_{17} - B_{11} B_{15} \]
\[ B_{35} = B_{20} + B_{12} B_{15} \]

\[ B_{34} = B_{18} - B_{16} B_{15} \]

\[ B = . \]
\[ B_{13} = B_{13} - B_5 B_{12} \]
\[ B_{19} = B_{19} - B_5 B_{12} \]
\[ B_{22} = B_8 - B_4 B_1 \]
\[ B_{27} = B_{17} - B_{15} B_1 - B_{20} B_{22} \]
\[ B_{29} = B_{18} - B_{15} B_5 - B_{20} B_{25} \]
\[ B_{30} = B_{17} - B_{11} B_{15} \]
\[ B_{32} = B_{18} - B_{16} B_{15} \]

\[ \Delta_B = 1 + B_4 B_{12} \]
\[ \Delta_2 = \frac{\bar{D}_{11} B_{21}}{A_{22}} \]

and where

\[ \bar{q} = \frac{4 c R}{h^2} q \]

is the nondimensionalized axial load eccentricity (positive inward).
Appendix E

Parallel Shooting Method

E.1 Introduction

Free vibration and buckling problems of circular cylindrical shells are often reduced to homogeneous boundary value problems (eigenvalue problems) for ordinary differential equations. The eigenvalue parameters are the square frequency and the loading, respectively. The solution of these problems can be based on the numerical integration of corresponding initial value problems. Kalnins (1964) presented a method for the free vibration of shells of revolution which consists of systematically evaluating a characteristic determinant. The eigenvalue parameter is increased in small steps. For each trial value of the eigenvalue parameter $\lambda$ one has to obtain the solution vectors of the homogeneous system of differential equations to compute the characteristic determinant. Vanishing of the determinant is, for simple eigenvalues, revealed by a sign change. This method will be referred to as determinant plotting. After a zero has been detected, the eigenvalue can be determined accurately by the secant method. The method has been employed by Booton (1976) in the bifurcation buckling problem of anisotropic cylinders. Another approach was presented by Cohen (1965). This method is a generalization of the well-known Stodola method for beams and will be referred to as the mode iteration method. Substituting an estimate of the eigenmode into the governing equations yields an inhomogeneous boundary value problem giving an improved eigenmode. The corresponding eigenvalue is computed from the Rayleigh quotient. In each iteration step, one has to compute a particular solution vector. The complementary solution vectors have to be calculated only in the first step. It can be proven that this method is convergent for the lowest eigenvalue $\lambda_1$. The rate of convergence depends on the ratio $\lambda_2/\lambda_1$, where $\lambda_2$ is the next smallest eigenvalue. One can speed up the rate of convergence by the method of eigenvalue shifting. Each new reference value for the eigenvalue correction requires the computation of the complementary solution vectors. It is possible to compute higher eigenvalues by orthogonalization with respect to lower modes and/or eigenvalue shifting. The mode iteration method was used by Cohen (1968a) to compute the buckling load of shells of revolution including a nonlinear prebuckling state (i.e. the prebuckling state depends nonlinearly on the load parameter). In this case, the critical eigenvalue has to be approached by a sequence
of linearized problems. The geometrical interpretation of the method is that the
eigenvalue (stability) problem of the (fictitious) equilibrium states is solved on the
tangent to the nonlinear load-deformation curve, at an assumed load $A$ below the
critical load. For loads in the vicinity of $A_1$, the corresponding fictitious states are
good approximations to the neighbouring nonlinear states. Consequently, by incre­
menting $A$ towards the critical load, one obtains fictitious critical loads which are
increasingly accurate approximations to the critical load of the nonlinear problem.
For each $A$, the method of successive approximations can be used to obtain the
corresponding fictitious critical load. This procedure has also been employed by Ar­
bocz and Hol (1989) in the imperfection sensitivity analysis of anisotropic cylindrical
shells.

E.2 Fundamentals of the Shooting Method

In the basic versions of the shooting method (see e.g. Keller, 1968; Hall and Watt,
1976; Ascher et al., 1988), an inhomogeneous boundary value problem is converted
into a sequence of initial value problems which are solved by numerical integra­
tion. Guesses for the unknown boundary values are iteratively adjusted until all
prescribed boundary conditions are satisfied. In this way, the boundary value prob­
lem has been reduced to the solution of a system of (nonlinear) equations for the
unknown boundary values. Thus in general, the shooting procedure consists of two
steps: 1) numerical integration of corresponding initial value problems with initial
guesses for the unknowns, 2) solution of a linear algebraic system for a correction
of the unknowns. These two steps can be repeated in an iterative procedure until
convergence has been achieved. To avoid the problems caused by a rapid growth
of the initial value solutions, one often has to employ parallel shooting. In this
modification, the growth of the solutions is controlled by dividing the range of in­
tegration into a number of smaller intervals. It can be difficult to estimate the
unknown boundary values. This problem may be overcome by perturbing a simpler
problem in stages into the original problem. The method of shooting and matching
can also be applied to (nonlinear) eigenvalue problems, i.e., to problems in which a
coefficient in the differential equation or boundary conditions has to be determined
such that a (nontrivial) solution exists. The eigenvalue can be treated as a param­
eter in the shooting procedure; in other words, the eigenvalue is an unknown in the
(Newton-type) iteration scheme. Two approaches can be distinguished. The first
approach is an obvious extension of the determinant plotting method (i.e. moni­
toring the determinant for a varying value of the eigenvalue parameter). Thurston
(1978) presented a Newton-type root-finding procedure for lambda matrices, i.e.,
for matrices of which the determinant has to equal zero. The problem is reduced
to a sequence of linear algebraic eigenvalue problems. The only eigenvalues (i.e.,
eigenvalue corrections) of interest are those which are small in absolute value. This
method can be started when a zero of the determinant has been detected. Another
approach is given by Keller (1968), who described a general shooting method which
simultaneously yields an eigenvalue and the corresponding eigenfunction. Apart
from the unknown boundary values one has the eigenvalue as a parameter in the Newton-type iteration scheme. For standard eigenvalue problems the eigenfunction is determined to within a multiplicative constant. The eigenfunction can be made unique by adjoining some kind of normalization condition. This gives an additional equation for the eigenvalue. One can also fix one of the inhomogeneous conditions at a boundary and in this way normalize the solution. The method is applicable to general (nonlinear) eigenvalue problems. Close to the root, the Newton-type iteration procedure converges rapidly (quadratically). To ensure that the iterations will converge to the desired root one has to supply an initial guess which is close enough to this root.

E.3 Details of Parallel Shooting

In this section, the shooting procedure is illustrated for an eigenvalue problem, defined on the interval $0 < x < 1$, and formulated as follows:

$$\frac{d \mathbf{Y}}{dx} = \mathbf{f}(x; \mathbf{Y}; \Lambda) = \mathbf{A}(x; \Lambda) \mathbf{Y}$$

(E.1)

where $\mathbf{Y}$ is the $N$-dimensional vector of dependent variables, $\mathbf{f}$ is an $N$-dimensional vector function, and $\Lambda$ is an eigenvalue parameter.

To make the eigenfunctions unique, to within a sign (for simple eigenvalues), they can be normalized by

$$\int_0^1 \mathbf{Y} \cdot \mathbf{Y} dx = C$$

(E.2)

where $C$ is a (positive) constant. This condition can be written as the differential equation

$$\frac{d \mathbf{Y}_{N+1}}{dx} = \mathbf{Y} \cdot \mathbf{Y} = f_{N+1}(x, \mathbf{Y}), \quad 0 \leq x \leq 1$$

(E.3)

with the boundary conditions

$$\mathbf{Y}_{N+1}(x = 0) = 0$$

(E.4)

$$\mathbf{Y}_{N+1}(x = 1) = C$$

(E.5)

By adding the normalization condition to the homogeneous boundary value problem the eigenvalue can be treated as one of the unknown parameters in the shooting method. In general multiple shooting is employed over $2N$ intervals (see Fig. E.1), with starting points at $x = x_{2j}$, and matching points at $x = x_{2j+1}$. For simplicity, parallel shooting over 2 intervals will be considered (‘double shooting’ version). The following 2 associated initial value problems are introduced:
Parallel Shooting Method

Figure E.1: Parallel shooting over $2N$ intervals

**Forward integration:** $0 \leq x \leq x_1$

\[
\frac{d\bar{U}}{dx} = \hat{f}(x, \bar{U}; \Lambda)
\]

\[
\bar{U}(x = 0) = \begin{bmatrix} s_0 \\ 0 \end{bmatrix}
\]

where $s_0$ contains $N/2$ independent parameters $s_i$. The $s_i$ are assembled in the $N/2$ dimensional vector of initial guesses $s$.

**Backward integration:** $x_1 \leq x \leq 1$

\[
\frac{d\bar{V}}{dx} = \hat{f}(x, \bar{V}; \Lambda)
\]

\[
\bar{V}(x = 1) = \begin{bmatrix} s_2 \\ C \end{bmatrix}
\]

where $s_0$ contains $N/2$ independent parameters $t_i$ which are assembled in the initial guess vector $t$. The extended vectors $\bar{U}$ and $\bar{V}$ are defined by

\[
\bar{U} = \begin{bmatrix} U \\ U_{N+1} \end{bmatrix} = \begin{bmatrix} U \\ N_U \end{bmatrix}
\]

\[
\bar{V} = \begin{bmatrix} V \\ V_{N+1} \end{bmatrix} = \begin{bmatrix} V \\ N_V \end{bmatrix}
\]

and

\[
\hat{f} = \begin{bmatrix} f \\ f_{N+1} \end{bmatrix}
\]
E.3 Details of Parallel Shooting

Introducing a new vector function \( \hat{\Phi} \) the solutions must satisfy the matching condition and normalization condition

\[
\hat{\Phi}(\hat{S}) = \begin{bmatrix} U[x = x_1] - V[x = x_1] \\ N_U[x = x_1] - N_V[x = x_1] \end{bmatrix} - \gamma = 0 \tag{E.13}
\]

where

\[
\hat{\Phi} = \begin{bmatrix} \Phi \\ \Phi_{N+1} \end{bmatrix} \tag{E.14}
\]

\[
\hat{S} = \begin{bmatrix} S \\ \Lambda \end{bmatrix} = \begin{bmatrix} s \\ t \Lambda \end{bmatrix} \tag{E.15}
\]

and

\[
\gamma = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{E.16}
\]

If one has initial guesses which are sufficiently close to the eigenvalue and eigenfunction, one can use Newton's method to solve (E.13). The following iteration scheme is used:

\[
\hat{S}^{\nu+1} = \hat{S}^{\nu} + \Delta \hat{S}^{\nu}, \quad \nu = 0, 1, \ldots \tag{E.17}
\]

where \( \Delta \hat{S} \) is the solution of the \((N + 1)\)st-order linear system

\[
\frac{\partial \hat{\Phi}}{\partial \hat{S}} (\hat{S}^{\nu}) \Delta \hat{S}^{\nu} = -\hat{\Phi}(\hat{S}^{\nu}) \tag{E.18}
\]

In order to solve for the components of the Jacobian

\[
\hat{J} = \frac{\partial \hat{\Phi}}{\partial \hat{S}} = \begin{bmatrix} \frac{\partial \Phi_1}{\partial S_1} & \ldots & \frac{\partial \Phi_1}{\partial S_N} & \frac{\partial \Phi_1}{\partial \Lambda} \\ \vdots & \ddots & \vdots & \ddots \\ \frac{\partial \Phi_N}{\partial S_1} & \ldots & \frac{\partial \Phi_N}{\partial S_N} & \frac{\partial \Phi_N}{\partial \Lambda} \\ \frac{\partial \Phi_{N+1}}{\partial S_1} & \ldots & \frac{\partial \Phi_{N+1}}{\partial S_N} & \frac{\partial \Phi_{N+1}}{\partial \Lambda} \end{bmatrix} \tag{E.19}
\]

the following new vectors are introduced

\[
\hat{W}_i = \begin{bmatrix} W_i \\ W_{i+N+1} \end{bmatrix} = \frac{\partial \hat{U}}{\partial \hat{S}_i} \quad i = 1, 2, \ldots, \frac{N}{2} \tag{E.20}
\]
\[ Z_i = \begin{bmatrix} Z_i \\ Z_{i+1} \end{bmatrix} = \frac{\partial V}{\partial S_i} \quad i = \frac{N}{2} + 1, \frac{N}{2} + 2, \ldots, N \quad (E.21) \]

These vectors are found by solving the following variational equations, obtained by differentiating (E.7) and (E.9) with respect to the parameters \( S_i \):

**Forward integration:** \( 0 \leq x \leq x_1 \)

\[ \frac{d\mathbf{W}_i}{dx} = \frac{\partial \mathbf{f}(x, \mathbf{U}; \Lambda)}{\partial \mathbf{U}} \mathbf{W}_i \quad i = 1, 2, \ldots, \frac{N}{2} \quad (E.22) \]

**Backward integration:** \( x_1 \leq x \leq 1 \)

\[ \frac{d\mathbf{Z}_i}{dx} = \frac{\partial \mathbf{f}(x, \mathbf{U}; \Lambda)}{\partial \mathbf{V}} \mathbf{Z}_i \quad i = \frac{N}{2} + 1, \frac{N}{2} + 2, \ldots, N \quad (E.23) \]

with initial conditions

\[ \mathbf{W}_i(0) = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial \mathbf{U}_i} \\ 0 \end{bmatrix} \quad i = 1, 2, \ldots, \frac{N}{2} \quad (E.24) \]

and

\[ \mathbf{Z}_i(1) = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial \mathbf{V}_i} \\ 0 \end{bmatrix} \quad i = \frac{N}{2} + 1, \frac{N}{2} + 2, \ldots, N \quad (E.25) \]

In addition one must solve the following inhomogeneous variational equations with homogeneous boundary conditions, obtained by differentiating (E.7) and (E.9) with respect to the eigenvalue parameter \( \Lambda \):

**Forward integration:** \( 0 \leq x \leq x_1 \)

\[ \frac{d}{dx} \frac{\partial \mathbf{U}}{\partial \Lambda} = \frac{\partial \mathbf{f}(x, \mathbf{U}; \Lambda)}{\partial \mathbf{U}} \frac{\partial \mathbf{U}}{\partial \Lambda} + \frac{\partial \mathbf{f}(x, \mathbf{U}; \Lambda)}{\partial \Lambda} \quad (E.26) \]

**Backward integration:** \( x_1 \leq x \leq 1 \)

\[ \frac{d}{dx} \frac{\partial \mathbf{V}}{\partial \Lambda} = \frac{\partial \mathbf{f}(x, \mathbf{U}; \Lambda)}{\partial \mathbf{U}} \frac{\partial \mathbf{U}}{\partial \Lambda} + \frac{\partial \mathbf{f}(x, \mathbf{U}; \Lambda)}{\partial \Lambda} \quad (E.27) \]

The components of the matrix \( \mathbf{J}' = \frac{\partial \mathbf{f}}{\partial \mathbf{U}} = \frac{\partial \mathbf{f}}{\partial \mathbf{V}} \) can be calculated directly:
The Jacobian matrix $\tilde{J}$ has the following form

$$
\tilde{J} = \begin{bmatrix}
\frac{\partial U}{\partial \alpha}[x_1] & -\frac{\partial V}{\partial \alpha}[x_1] & \frac{\partial U}{\partial \alpha}[x_1] - \frac{\partial V}{\partial \alpha}[x_1]
\end{bmatrix}
$$

(E.29)

### E.4 Properties of $W(\bar{x})$

In the shooting procedure for the buckling and vibration problem of composite shells (Eq. (5.69)), the initial value problem for $W(\bar{x})$ can be written in the following partitioned form (cf. Booton, 1976)

$$
\begin{bmatrix}
W'_{11}(\bar{x}) & W'_{12}(\bar{x}) \\
W'_{21}(\bar{x}) & W'_{22}(\bar{x})
\end{bmatrix}
= \begin{bmatrix}
O & A_{12}(\bar{x}) \\
-A_{12}(\bar{x}) & O
\end{bmatrix}
\begin{bmatrix}
W_{11}(\bar{x}) & W_{12}(\bar{x}) \\
W_{21}(\bar{x}) & W_{22}(\bar{x})
\end{bmatrix}
$$

(E.30)

with the initial conditions (if multiple shooting is employed)

$$
\begin{bmatrix}
W_{11}(\bar{x} = \bar{x}_{2j}) & W_{12}(\bar{x} = \bar{x}_{2j}) \\
W_{21}(\bar{x} = \bar{x}_{2j}) & W_{22}(\bar{x} = \bar{x}_{2j})
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
$$

(E.31)

where $\bar{x}_{2j}$ are the starting points for the integration ($j = 1, \ldots, N - 1$ if $2N$ is the number of intervals). For edge intervals ($j = 0$ or $j = N$) the analysis is similar. Two uncoupled initial value problems have been obtained,

$$
W'_{11} = A_{12}W_{21}
$$

(E.32)

$$
W'_{21} = -A_{12}W_{11}
$$

(E.33)

$$
W_{11}(\bar{x}_{2j}) = I
$$

(E.34)

$$
W_{21}(\bar{x}_{2j}) = 0
$$

(E.35)

and
Comparing the two initial value problems, it is seen that

\[ W_{11} = W_{22} \]  \hspace{1cm} (E.40)

and

\[ W_{21} = -W_{12} \]  \hspace{1cm} (E.41)

Consequently, \( W(x) \) can be written as follows:

\[
W(x) = \begin{bmatrix} W_{11}(x) & -W_{21}(x) \\ W_{21}(x) & W_{11}(x) \end{bmatrix}
\]  \hspace{1cm} (E.42)

Therefore only eight solutions of the variational equations are required.
Appendix F

Shell Data

In this appendix the data of the shells are given, which have been used in the calculations.

**Booton's shell:** an anisotropic shell used in static stability investigations (Booton, 1976; Arbocz and Hol, 1989). The data are given in Table F.1.

<table>
<thead>
<tr>
<th>shell geometry</th>
<th>radius ( R = 2.67 ) in.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>length ( L = 3.776 ) in.  (or ( L = 5.34 ) in.)</td>
</tr>
<tr>
<td>laminate geometry</td>
<td>3 layers (the numbering starts at the outside of the shell)</td>
</tr>
<tr>
<td></td>
<td>layer thickness ( h_1 = h_2 = h_3 = 0.0089 ) in.</td>
</tr>
<tr>
<td></td>
<td>layer orientation ( \theta_1 = 30 ) deg, ( \theta_2 = 0 ) deg, ( \theta_3 = -30 ) deg</td>
</tr>
<tr>
<td>layer properties</td>
<td>composite material: glass-epoxy</td>
</tr>
<tr>
<td></td>
<td>modulus of elasticity 1-direction ( E_{11} = 0.583 \cdot 10^7 ) psi</td>
</tr>
<tr>
<td></td>
<td>modulus of elasticity 2-direction ( E_{22} = 0.242 \cdot 10^7 ) psi</td>
</tr>
<tr>
<td></td>
<td>major Poisson's ratio ( \nu_{12} = 0.363 )</td>
</tr>
<tr>
<td></td>
<td>shear modulus 12-plane ( G_{12} = 0.668 \cdot 10^6 ) psi</td>
</tr>
</tbody>
</table>

**ES2-shell:** isotropic shell earlier used in forced nonlinear vibration analysis by Evensen (1967). The vibration mode is characterized by \( m \) axial half waves and \( \ell \) circumferential full waves. In Chapter 3 the following data are used: \( \epsilon = (\ell^2 h/R)^2 = 0.01 \), \( \xi = \pi R/\ell \) = 0.1, \( \nu = 0.3 \).

**EZ-shell:** orthotropic shell used by El-Zaouk and Dym (1973): \( E_y/E_x = 1 \), \( \nu_{xy} = 0.3 \), and \( G_{xy}/E_x = \frac{\gamma}{1-\nu_{xy}\nu_{yz}} \), where \( E_x, E_y \) etc. are the usual stiffness parameters for an orthotropic shell, and \( \gamma \) is the shear ratio parameter.

**Olson's shell:** isotropic (copper) shell (Olson, 1965). In the linear vibration analysis the following data are used: \( L = 15.4 \) in., \( R = 8 \) in., \( h = 0.004 \) in., \( \nu = 0.35 \), \( E = 16 \cdot 10^6 \) lb/in.\(^2\), \( p = 8.33 \cdot 10^{-4} \) lb sec\(^2\)/in.\(^4\) (cf. Evensen and Olson, 1967).
In the nonlinear vibration analysis: $L = 15\frac{3}{4}$ in., $R = 8$ in., $h = 0.0044$ in., $\nu = 0.30$ (cf. Chen, 1972). In the flutter analysis (Extended Analysis): $L = 16$ in., $R = 8$ in., $h = 0.004$ in., $\nu = 0.33$ (cf. Barr and Stearman, 1969).

**Chen’s shell:** isotropic shell (Chen, 1972): $R = 4$ in. $h = 0.01$ in. $L = 8$ in., $\nu = 0.31$.

**Bogdanovich’ shell:** isotropic shell (Bogdanovich, 1993): $R = 1$ m, $R/h = 100$, $L/R = 2$, $E = 4 \cdot 10^{10}$ N/m$^2$, $\nu = 0.3$, $\rho = 2.5 \cdot 10^3$ kg/m$^3$.

**Gunawan’s shell:** isotropic (aluminum) shell (Gunawan, 1998): $R = 125$ mm, $L = 240$ mm, $h = 0.253$ mm, $\rho = 2700$ kg/m$^3$, $E = 72000 \cdot 10^6$ N/m$^2$.

**WN-shell:** isotropic shell used by Watawala and Nash (1982): $h/R = 1/720$, $L/R = 2/3$, $\nu = 0.272$.

**GV-shell:** orthotropic shell used by Ganapathi and Varadan (1995): A two-layered $[0/90]$ cross-ply laminate. The layer properties are: $E_{11}/E_{22} = 40$, $G_{12}/E_{22} = 0.5$, $G_{13}/E_{22} = 0.5$, $G_{23}/E_{22} = 0.33$, $\nu_{12} = 0.25$. The geometry parameters are: $L/R = 1$, $R/h = 20$, $h = 1.0$.

**ICEZ-shell:** orthotropic (glass-epoxy) shell used by Fu and Chia (1988b): $R/h = 100$, $E_x/E_y = 3$, $G_{xy}/E_y = 0.5$, $\nu_{xy} = 0.25$, $\ell = 5$. 
Appendix G

Derivation of equations using REDUCE

G.1 REDUCE notation for Simplified Analysis

The REDUCE notation (and FORTRAN notation) which corresponds to the notation used in the text for the Simplified Analysis is given in the following tables:

<table>
<thead>
<tr>
<th>REDUCE</th>
<th>text</th>
</tr>
</thead>
<tbody>
<tr>
<td>lm, lp</td>
<td>$\ell_m, \ell_p$</td>
</tr>
<tr>
<td>ln, ll</td>
<td>$\ell_n, \ell_l$</td>
</tr>
<tr>
<td>h</td>
<td>$h$</td>
</tr>
<tr>
<td>r</td>
<td>$R$</td>
</tr>
</tbody>
</table>

Table G.1: REDUCE notation. Wave number parameters.

<table>
<thead>
<tr>
<th>FORTRAN</th>
<th>text</th>
</tr>
</thead>
<tbody>
<tr>
<td>NCIRC</td>
<td>$n$</td>
</tr>
<tr>
<td>LCIRC</td>
<td>$\ell$</td>
</tr>
</tbody>
</table>

Table G.2: FORTRAN notation: Wave numbers.
Table G.3: REDUCE notation. FORTRAN notation between parentheses if different from REDUCE notation.

<table>
<thead>
<tr>
<th>REDUCE</th>
<th>text</th>
</tr>
</thead>
<tbody>
<tr>
<td>anxref</td>
<td>$N_{x,t}$</td>
</tr>
<tr>
<td>r</td>
<td>$R$</td>
</tr>
<tr>
<td>h</td>
<td>$h$</td>
</tr>
<tr>
<td>c</td>
<td>$c$</td>
</tr>
<tr>
<td>w0pp</td>
<td>$w_0''$</td>
</tr>
<tr>
<td>wbarpp</td>
<td>$\bar{w}_0''$</td>
</tr>
<tr>
<td>n (an)</td>
<td>$n$</td>
</tr>
<tr>
<td>ambdao</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>pbare</td>
<td>$\bar{\rho}$</td>
</tr>
<tr>
<td>rcapt</td>
<td>$\tilde{\tau}$</td>
</tr>
<tr>
<td>omgno2</td>
<td>$\bar{\omega}_2^2$</td>
</tr>
<tr>
<td>omgpn2</td>
<td>$\bar{\omega}_2^2$</td>
</tr>
</tbody>
</table>

Table G.4: REDUCE notation. Stiffness matrices.

<table>
<thead>
<tr>
<th>REDUCE</th>
<th>text</th>
</tr>
</thead>
<tbody>
<tr>
<td>a1ls, a12s, a16s</td>
<td>$A_{11}, A_{12}, A_{16}$</td>
</tr>
<tr>
<td>a2ls, a22s, a26s</td>
<td>$A_{21}, A_{22}, A_{26}$</td>
</tr>
<tr>
<td>a61s, a62s, a66s</td>
<td>$A_{61}, A_{62}, A_{66}$</td>
</tr>
<tr>
<td>b1ls, b12s, b16s</td>
<td>$B_{11}, B_{12}, B_{16}$</td>
</tr>
<tr>
<td>b2ls, b22s, b26s</td>
<td>$B_{21}, B_{22}, B_{26}$</td>
</tr>
<tr>
<td>b61s, b62s, b66s</td>
<td>$B_{61}, B_{62}, B_{66}$</td>
</tr>
<tr>
<td>d1ls, d12s, d16s</td>
<td>$D_{11}, D_{12}, D_{16}$</td>
</tr>
<tr>
<td>d2ls, d22s, d26s</td>
<td>$D_{21}, D_{22}, D_{26}$</td>
</tr>
<tr>
<td>d61s, d62s, d66s</td>
<td>$D_{61}, D_{62}, D_{66}$</td>
</tr>
</tbody>
</table>
The REDUCE notation which corresponds to the notation used in the text for the Extended Analysis is given in the following tables:
Derivation of equations using REDUCE

<table>
<thead>
<tr>
<th>REDUCE</th>
<th>text</th>
</tr>
</thead>
<tbody>
<tr>
<td>f0sn(l,0,-1), f0sn(0,1,1)</td>
<td>$f_{n,1}^s$, $f_{n,2}^s$</td>
</tr>
<tr>
<td>f0sn(l,2,1), f0sn(2,1,-1)</td>
<td>$f_{n,3}^s$, $f_{n,4}^s$</td>
</tr>
<tr>
<td>f0cn(1,-1,-2)</td>
<td>$f_{0}^s$</td>
</tr>
<tr>
<td>f0cn(0,1,1)</td>
<td>$f_{0}^s$</td>
</tr>
<tr>
<td>fnl(l,0,0,0,-1), fsl(0,1,0,0,1)</td>
<td>$f_{n-\ell,1}^s$, $f_{n-\ell,2}^s$, $f_{n-\ell,3}^s$</td>
</tr>
<tr>
<td>fnl(l,2,0,0,1), fml(2,1,0,0,-1)</td>
<td>$f_{n+\ell,1}^s$, $f_{n+\ell,2}^s$, $f_{n+\ell,3}^s$</td>
</tr>
<tr>
<td>fcnl(0,0,0,0,1), fcml(0,1,0,0,1)</td>
<td>$f_{c-\ell,1}^c$, $f_{c-\ell,2}^c$, $f_{c-\ell,3}^c$</td>
</tr>
<tr>
<td>fcnl(1,0,0,0,1), fcml(1,1,0,0,1)</td>
<td>$f_{c+\ell,1}^c$, $f_{c+\ell,2}^c$, $f_{c+\ell,3}^c$</td>
</tr>
<tr>
<td>fcnpl(0,0,0,0,1), fnpl(0,1,0,0,1)</td>
<td>$f_{\ell,1}^c$, $f_{\ell,2}^c$, $f_{\ell,3}^c$</td>
</tr>
<tr>
<td>fcnpl(1,0,0,0,1), fnpl(1,1,0,0,1)</td>
<td>$f_{\ell,1}^c$, $f_{\ell,2}^c$, $f_{\ell,3}^c$</td>
</tr>
<tr>
<td>fs2l(1,-1,0,0,-1), fcs2l(1,-1,0,0,1)</td>
<td>$f_{c\ell}^c$, $f_{c\ell}^c$, $f_{c\ell}^c$</td>
</tr>
<tr>
<td>fco(l,1,1,0,0)</td>
<td>$f_{c}^c$</td>
</tr>
</tbody>
</table>

Table G.7: REDUCE notation. Coefficients of stress function.
### Table G.8: REDUCE notation. FORTRAN notation between parentheses if different from REDUCE notation.

<table>
<thead>
<tr>
<th>REDUCE</th>
<th>text</th>
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<td>eref</td>
<td>$E$</td>
</tr>
<tr>
<td>r</td>
<td>$R$</td>
</tr>
<tr>
<td>h</td>
<td>$h$</td>
</tr>
<tr>
<td>c</td>
<td>$c$</td>
</tr>
<tr>
<td>wcpp</td>
<td>$w''_0$</td>
</tr>
<tr>
<td>wbarcpp</td>
<td>$\bar{w}''$</td>
</tr>
<tr>
<td>n</td>
<td>$n$</td>
</tr>
<tr>
<td>ambdao</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>pbare</td>
<td>$\bar{p}$</td>
</tr>
<tr>
<td>rcapt</td>
<td>$\bar{L}$</td>
</tr>
<tr>
<td>omgno2</td>
<td>$\omega^2$</td>
</tr>
<tr>
<td>omgpn2</td>
<td>$\bar{\omega}^2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>REDUCE</th>
<th>text</th>
</tr>
</thead>
<tbody>
<tr>
<td>ambd2</td>
<td>$\lambda^{(2)}$</td>
</tr>
<tr>
<td>rcap2</td>
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<tr>
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<tr>
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<tr>
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<td>$M_\infty$</td>
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<tr>
<td>gamma</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>pinf</td>
<td>$p_\infty$</td>
</tr>
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</table>

### Table G.9: REDUCE notation. Coefficients of stress function.

<table>
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<tr>
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</tr>
<tr>
<td>cf1p(ix,i0,j0)</td>
<td>$c_{j_{11}}$</td>
</tr>
<tr>
<td>cf1p2(ix,i0,j0)</td>
<td>$c_{j_{12}}$</td>
</tr>
<tr>
<td>cf1p3(ix,i0,j0)</td>
<td>$c_{j_{13}}$</td>
</tr>
<tr>
<td>cf2(ix,i0,0.2)</td>
<td>$c_{j_{20}}$</td>
</tr>
<tr>
<td>cf2p(ix,i0,0.2)</td>
<td>$c_{j_{21}}$</td>
</tr>
<tr>
<td>cf2p2(ix,i0,0.2)</td>
<td>$c_{j_{22}}$</td>
</tr>
<tr>
<td>cf2p3(ix,i0,0.2)</td>
<td>$c_{j_{23}}$</td>
</tr>
</tbody>
</table>

<table>
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<th>text</th>
</tr>
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<tr>
<td>cw1p(ix,i0,j0)</td>
<td>$c_{j_{31}}$</td>
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<tr>
<td>cw1p2(ix,i0,j0)</td>
<td>$c_{j_{32}}$</td>
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<td>cw1p3(ix,i0,j0)</td>
<td>$c_{j_{33}}$</td>
</tr>
<tr>
<td>cw2(ix,i0,0.0)</td>
<td>$c_{j_{40}}$</td>
</tr>
<tr>
<td>cw2p(ix,i0,0.0)</td>
<td>$c_{j_{41}}$</td>
</tr>
<tr>
<td>cw2p2(ix,i0,0.0)</td>
<td>$c_{j_{42}}$</td>
</tr>
<tr>
<td>cw2p3(ix,i0,0.0)</td>
<td>$c_{j_{43}}$</td>
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</table>
G.3 Simplified Analysis - Static State

The Simplified Analysis for the static state outlined in Chapter 3 relies on derivations via the symbolic manipulation package REDUCE. The analysis has been coded in the FORTRAN computer program SISIAN (Simplified Static Analysis). In this section, details of the derivations will be presented.

A particular solution for the stress function $F_p$ of the static state is:

$$
\begin{align*}
\hat{F}_p &= f^0_{mn}(1,0,-1)\sin(\ell_m x - \ell_n y) + f^0_{mn}(0,1,1)\sin(\ell_n x + \ell_m y) \\
&+ f^0_{mn}(1,2,1)\sin(\ell_m + 2\ell_n x + \ell_n y) + f^0_{mn}(2,1,-1)\sin(2\ell_n x - \ell_m y) \\
&+ f^0_{mn}(1,1,-2)\cos(\ell_m x - \ell_n y) + f^0_{01}(1,1)\cos(\ell_n x + \ell_m y)
\end{align*}
$$ (G.1)

In REDUCE notation,

$$
\hat{F}_p := \begin{align*}
&+ f^0_{sn}(1,0,-1)\sin(lm x - ln y) \\
&+ f^0_{sn}(0,1,1)\sin(lp x + ln y) \\
&+ f^0_{sn}(1,2,1)\sin(lm x + 2lp x + ln y) \\
&+ f^0_{sn}(2,1,-1)\sin(2lm x + lp x - ln y) \\
&+ f^0_{sn}(1,1,-2)\cos(lm x - lp x - 2ln y) + f^0_{01}(1,1)\cos(lm x + lp x)
\end{align*}
$$

The coefficients $f^0_{sn}$ of the different goniometric terms $g_i$ of the right-hand side have been obtained using the COEFFN operator in REDUCE. The coefficients $g_i$, in REDUCE notation, are:

$$
\begin{align*}
\end{align*}
$$

The constant in-plane stresses $\bar{N}_{xy}$ and $\bar{N}_{x}^2$ can be related to the averaged in-plane loads prescribed at the shell edges. The circumferential stress $N_{p}$ is obtained from the periodicity condition. Details about the constant in-plane stresses will be discussed later in this section.

The assumed stress function is substituted into the compatibility equation of the static state. The right-hand side of the compatibility equation can be written in the form $\sum_a g_i f_i$. In a REDUCE notation the right-hand side (denoted in REDUCE as "rhs") becomes:

$$
\begin{align*}
\text{rhs} := \\
&+ asn(1,0,-1)\sin(lm x - ln y) \\
&+ asn(0,1,1)\sin(lp x + ln y) \\
&+ asn(1,2,1)\sin(lm x + 2lp x + ln y) \\
&+ asn(2,1,-1)\sin(2lm x + lp x - ln y) \\
&+ acn(1,1,-2)\cos(lm x - lp x - 2ln y) \\
&+ ac0(1,1)\cos(lm x + lp x)
\end{align*}
$$

This expression can be written in the form $\sum_i f^i g_i$. The unknown coefficients $f^i$ (where $f^i = f^0_{sn}(1,0,-1), f^i = f^0_{sn}(0,1,1)$ etc.) of the goniometric terms $g_i = \sin(\ell_m x - \ell_n y)$, $g_o = \sin(\ell_n x + \ell_m y)$, etc.) have to be determined.

The stress function $\hat{F}$ can now be written as

$$
\hat{F} = \frac{1}{2}\hat{N}_x^2 y^2 + \frac{1}{2}\hat{N}_y^2 x^2 - \hat{N}_{xy} xy + \hat{F}_p
$$ (G.2)

The coefficients $\alpha_i^2$ of the different goniometric terms $g_i$ of the right-hand side have been obtained using the COEFFN operator in REDUCE. The coefficients $\alpha_i^2$, in REDUCE notation, are:

$$
\begin{align*}
\end{align*}
$$
The coefficients $a_i$ depend on the displacement amplitudes $\xi_1$ and $\xi_2$. The left-hand side of the compatibility equation can be written in the form $\sum_i a_i^0 g_i$, in REDUCE notation:

\[
lhs := (a_i^{0}(1,0,-1) \sin(lm+x-ln*y) + a_i^{0}(0,1,1) \sin(lp+x-ln*y) + a_i^{0}(2,1,-1) \sin(2lm+x-lp-x-ln*y) + a_i^{0}(1,1) \cos(lm-x-lp+2lm+ln)y) + a_i^{0}(1,1) \cos(lm+l)x)\]

The coefficients $a_i^0$ of the goniometric terms $g_i$ of the left-hand side are determined via the COEFFN operator. The coefficients $a_i^0$ in REDUCE notation, are:

\[
\begin{align*}
ALSN(1,0,-1) &:= (A11S^{LN} + 2A12S^{LM} + 2A12S^{LN} + A22S^{LM} + 4A26S^{LN} + 2A66S^{LM} + 2A26S^{LM}) + F0SN(1,0,-1) \\
ALSN(0,1,1) &:= (A11S^{LN} + 2A12S^{LM} + 2A12S^{LN} + A22S^{LM} + 4A26S^{LN} + 2A66S^{LM} + 2A26S^{LM}) + F0SN(0,1,1) \\
ALSN(1,2,1) &:= (2A12S + A66S)(LM + 2LP) + 3A26S^{LN} + 2A26S^{LN} - 2A26S^{LM} + A66S^{LM} + 2A26S^{LM}) + F0SN(1,2,1) \\
ALSN(2,1,-1) &:= (2A12S + A66S)(LM + 2LP) + 3A26S^{LN} + 2A26S^{LN} - 2A26S^{LM} + A66S^{LM} + 2A26S^{LM}) + F0SN(2,1,-1) \\
ALSN(1,-1,-2) &:= (2A12S + A66S)(LM + 2LP) + 3A26S^{LN} + 2A26S^{LN} - 2A26S^{LM} + A66S^{LM} + 2A26S^{LM}) + F0SN(1,-1,-2) \\
ALSN(1,1) &:= A26S^{LM} + A66S^{LM} + 4F0SN(1,1) \\
\end{align*}
\]

The coefficients $a_i^0$ of the goniometric terms $g_i$ of the lefthand side are determined via the COEFFN operator. The coefficients $a_i^0$ in REDUCE notation, are:

\[
\begin{align*}
A_lS_{(1,0,-1)} &:= (A11S^{LN} + 2A12S^{LM} + 2A12S^{LN} + A22S^{LM} + 4A26S^{LN} + 2A66S^{LM} + 2A26S^{LM}) + F0SN(1,0,-1) \\
A_lS_{(0,1,1)} &:= (A11S^{LN} + 2A12S^{LM} + 2A12S^{LN} + A22S^{LM} + 4A26S^{LN} + 2A66S^{LM} + 2A26S^{LM}) + F0SN(0,1,1) \\
A_lS_{(1,2,1)} &:= (2A12S + A66S)(LM + 2LP) + 3A26S^{LN} + 2A26S^{LN} - 2A26S^{LM} + A66S^{LM} + 2A26S^{LM}) + F0SN(1,2,1) \\
A_lS_{(2,1,-1)} &:= (2A12S + A66S)(LM + 2LP) + 3A26S^{LN} + 2A26S^{LN} - 2A26S^{LM} + A66S^{LM} + 2A26S^{LM}) + F0SN(2,1,-1) \\
A_lS_{(1,-1,-2)} &:= (2A12S + A66S)(LM + 2LP) + 3A26S^{LN} + 2A26S^{LN} - 2A26S^{LM} + A66S^{LM} + 2A26S^{LM}) + F0SN(1,-1,-2) \\
A_lS_{(1,1)} &:= A26S^{LM} + A66S^{LM} + 4F0SN(1,1) \\
\end{align*}
\]

The coefficients $a_i^0$ depend on the unknown coefficients of the stress function $f_i$, $a_i^0 = a_i f_i$. The coefficients $a_i$ will be determined via REDUCE.

The coefficients of the goniometric terms of the right-hand side of the compatibility equation depend on the displacement amplitudes $\xi_1$ and $\xi_2$ and can be written in the general form

\[
a_i^0 = a_{i1}^0 \xi_1 + a_{i2}^0 \xi_1^2 + a_{i3}^0 \xi_1^3 + a_{i4}^0 \xi_1^4
\]

Written out in detail,

\[
\begin{align*}
a_{i1}^0(1,0,-1) &= a_{i12}^0(1,0,-1) \xi_1 + a_{i11}^0(1,0,-1) \xi_1^2 + a_{i10}^0(1,0,-1) \xi_1^3 + a_{i1-1}^0(1,0,-1) \xi_1^4 \\
a_{i2}^0(0,1,1) &= a_{i12}^0(0,1,1) \xi_1 + a_{i11}^0(0,1,1) \xi_1^2 + a_{i10}^0(0,1,1) \xi_1^3 + a_{i1-1}^0(0,1,1) \xi_1^4 \\
a_{i3}^0(1,2,1) &= a_{i12}^0(1,2,1) \xi_1 + a_{i11}^0(1,2,1) \xi_1^2 + a_{i10}^0(1,2,1) \xi_1^3 + a_{i1-1}^0(1,2,1) \xi_1^4 \\
a_{i4}^0(2,1,-1) &= a_{i12}^0(2,1,-1) \xi_1 + a_{i11}^0(2,1,-1) \xi_1^2 + a_{i10}^0(2,1,-1) \xi_1^3 + a_{i1-1}^0(2,1,-1) \xi_1^4 \\
a_{i5}^0(1,-1,-2) &= a_{i12}^0(1,-1,-2) \xi_1 + a_{i11}^0(1,-1,-2) \xi_1^2 + a_{i10}^0(1,-1,-2) \xi_1^3 + a_{i1-1}^0(1,-1,-2) \xi_1^4 \\
a_{i6}^0(2,1,1) &= a_{i12}^0(2,1,1) \xi_1 + a_{i11}^0(2,1,1) \xi_1^2 + a_{i10}^0(2,1,1) \xi_1^3 + a_{i1-1}^0(2,1,1) \xi_1^4 \\
a_{i7}^0(1,1) &= a_{i12}^0(1,1) \xi_1 + a_{i11}^0(1,1) \xi_1^2 + a_{i10}^0(1,1) \xi_1^3 + a_{i1-1}^0(1,1) \xi_1^4 \\
\end{align*}
\]

The coefficients of the stress function $f_i$ are obtained by equating the coefficients of corresponding goniometric terms $g_i$ on the lefthand side and the righthand side of the compatibility equation,

\[
\begin{align*}
f_i &= (1/a_i) (a_{i1}^0 \xi_1 + a_{i2}^0 \xi_1^2 + a_{i3}^0 \xi_1^3 + a_{i4}^0 \xi_1^4) \\
\end{align*}
\]

Written out in detail,

\[
\begin{align*}
f_{i1} &= f_{i1} \xi_1 + f_{i2} \xi_1^2 + f_{i3} \xi_1^3 + f_{i4} \xi_1^4 \\
f_{i2} &= f_{i1} \xi_1 + f_{i2} \xi_1^2 + f_{i3} \xi_1^3 + f_{i4} \xi_1^4 \\
f_{i3} &= f_{i1} \xi_1 + f_{i2} \xi_1^2 + f_{i3} \xi_1^3 + f_{i4} \xi_1^4 \\
f_{i4} &= f_{i1} \xi_1 + f_{i2} \xi_1^2 + f_{i3} \xi_1^3 + f_{i4} \xi_1^4 \\
f_{i5} &= f_{i1} \xi_1 + f_{i2} \xi_1^2 + f_{i3} \xi_1^3 + f_{i4} \xi_1^4 \\
f_{i6} &= f_{i1} \xi_1 + f_{i2} \xi_1^2 + f_{i3} \xi_1^3 + f_{i4} \xi_1^4 \\
f_{i7} &= f_{i1} \xi_1 + f_{i2} \xi_1^2 + f_{i3} \xi_1^3 + f_{i4} \xi_1^4 \\
\end{align*}
\]
The procedure to evaluate the coefficients $f_{ij}$ is coded in FORTRAN. The coefficients $a_i$ and $a_j$ are determined via REDUCE. REDUCE has an option to generate FORTRAN code. The coefficients of the amplitudes on the righthand side of the equation, $a_{ij}$, are denoted as "B" in the FORTRAN program. The FORTRAN code generated by REDUCE becomes:

$$A = \text{LN}^*4\text{A1S}^2 + \text{LN}^*3\text{LM}\text{A1S}^2 + \text{LN}^*2\text{LM}^*2\text{A1S}^2 + \text{LN}^*\text{LM}^*3\text{A1S}^2 + \text{LM}^*4\text{A2S}$$
$$B = (\text{H}^*2\text{LM}^*2\text{LM}^*2\text{LH} + \text{LM}^*2\text{LM}^*3\text{A2S})/4.$$

$$f_{0\alpha}(1,0,-1) = f_{0\alpha12}(1,0,-1)\text{xi1}^0\text{xi1}^0 + f_{0\alpha2}(0,1,0)-f_{0\alpha12}(0,1,0)\text{xi1}^0$$
$$f_{0\alpha}(0,1,1) = f_{0\alpha12}(0,1,1)\text{xi1}^0\text{xi1}^0 + f_{0\alpha1}(0,1,1)-f_{0\alpha12}(0,1,1)\text{xi1}^0$$
$$f_{0\alpha}(1,2,1) = f_{0\alpha12}(1,2,1)\text{xi1}^0\text{xi1}^0 + f_{0\alpha1}(1,2,1)-f_{0\alpha12}(1,2,1)\text{xi1}^0$$
$$f_{0\alpha}(2,1,-1) = f_{0\alpha12}(2,1,-1)\text{xi1}^0\text{xi1}^0 + f_{0\alpha1}(2,1,-1)-f_{0\alpha12}(2,1,-1)\text{xi1}^0$$
$$f_{0\alpha}(1,-1,2) = f_{0\alpha2}(1,-1,2)\text{xi1}^0\text{xi1}^0 + f_{0\alpha1}(1,-1,2)-f_{0\alpha2}(1,-1,2)\text{xi1}^0$$
$$f_{0\alpha}(1,1) = f_{0\alpha2}(1,1)\text{xi1}^0\text{xi1}^0 + f_{0\alpha1}(1,1)-f_{0\alpha2}(1,1)\text{xi1}^0$$. 

The procedure to evaluate the coefficients $f_{ij}$ is coded in FORTRAN. The coefficients $a_i$ and $a_j$ are determined via REDUCE. REDUCE has an option to generate FORTRAN code. The coefficients of the amplitudes on the righthand side of the equation, $a_{ij}$, are denoted as "B" in the FORTRAN program. The FORTRAN code generated by REDUCE becomes: 

$$A = \text{LM}^*4\text{A1S}^2 + \text{LM}^*3\text{LM}\text{A1S}^2 + \text{LM}^*2\text{LM}^*2\text{A1S}^2 + \text{LM}^*\text{LM}^*3\text{A1S}^2 + \text{LM}^*4\text{A2S}$$
$$B = (\text{H}^*2\text{LM}^*2\text{LM}^*2\text{LH} + \text{LM}^*2\text{LM}^*3\text{A2S})/4.$$

$F_{0\alpha12}(1,0,-1) = B/A$

$$f_{0\alpha}(1,0,-1) = f_{0\alpha12}(1,0,-1)\text{xi1}^0\text{xi1}^0 + f_{0\alpha1}(1,0,-1)-f_{0\alpha12}(1,0,-1)\text{xi1}^0$$
$$f_{0\alpha}(0,1,1) = f_{0\alpha12}(0,1,1)\text{xi1}^0\text{xi1}^0 + f_{0\alpha1}(0,1,1)-f_{0\alpha12}(0,1,1)\text{xi1}^0$$
$$f_{0\alpha}(1,2,1) = f_{0\alpha12}(1,2,1)\text{xi1}^0\text{xi1}^0 + f_{0\alpha1}(1,2,1)-f_{0\alpha12}(1,2,1)\text{xi1}^0$$
$$f_{0\alpha}(2,1,-1) = f_{0\alpha12}(2,1,-1)\text{xi1}^0\text{xi1}^0 + f_{0\alpha1}(2,1,-1)-f_{0\alpha12}(2,1,-1)\text{xi1}^0$$
$$f_{0\alpha}(1,-1,2) = f_{0\alpha2}(1,-1,2)\text{xi1}^0\text{xi1}^0 + f_{0\alpha1}(1,-1,2)-f_{0\alpha2}(1,-1,2)\text{xi1}^0$$
$$f_{0\alpha}(1,1) = f_{0\alpha2}(1,1)\text{xi1}^0\text{xi1}^0 + f_{0\alpha1}(1,1)-f_{0\alpha2}(1,1)\text{xi1}^0$. 

The procedure to evaluate the coefficients $f_{ij}$ is coded in FORTRAN. The coefficients $a_i$ and $a_j$ are determined via REDUCE. REDUCE has an option to generate FORTRAN code. The coefficients of the amplitudes on the righthand side of the equation, $a_{ij}$, are denoted as "B" in the FORTRAN program. The FORTRAN code generated by REDUCE becomes: 

$$A = \text{LM}^*4\text{A1S}^2 + \text{LM}^*3\text{LM}\text{A1S}^2 + \text{LM}^*2\text{LM}^*2\text{A1S}^2 + \text{LM}^*\text{LM}^*3\text{A1S}^2 + \text{LM}^*4\text{A2S}$$

$F_{0\alpha12}(1,0,-1) = B/A$

$$A = \text{LM}^*4\text{A1S}^2 + \text{LM}^*3\text{LM}\text{A1S}^2 + \text{LM}^*2\text{LM}^*2\text{A1S}^2 + \text{LM}^*\text{LM}^*3\text{A1S}^2 + \text{LM}^*4\text{A2S}$$

$F_{0\alpha12}(0,1,1) = B/A$

$$A = \text{LM}^*4\text{A1S}^2 + \text{LM}^*3\text{LM}\text{A1S}^2 + \text{LM}^*2\text{LM}^*2\text{A1S}^2 + \text{LM}^*\text{LM}^*3\text{A1S}^2 + \text{LM}^*4\text{A2S}$$

$F_{0\alpha12}(1,2,1) = B/A$
The circumferential periodicity condition is used to establish a relation between the constant stress $N^*$, the amplitudes $\xi_0$ and $\xi_2$ and the other constant stresses $N^*$ and $N^*_{xy}$. The periodicity condition states that the expression

$$\int_0^{2\pi} \tilde{\varepsilon}_y \, dy = 0$$  \hspace{1cm} (G.19)$$

is satisfied. Using the expression for the circumferential strain $\tilde{\varepsilon}_y$, substituting for $W$, $\tilde{W}$ and $F$, and carrying out the integration in $y$-direction yields

$$A_1^2 N_{x}^* + A_2^2 N_{x}^* + A_4^2 N_{xy}^* + h(-\frac{1}{8})\eta_2^2 \xi_2^2 - (\frac{1}{4})h_2^2 \xi_2^2 \xi_2^4 + \xi_0 / R = 0$$ \hspace{1cm} (G.20)$$

The constant in-plane stresses $N^*_{xy}$ and $N^*_{xy}$ are equal to the prescribed averaged stress resultants at the shell edge,

$$N^*_{xy} = -\tilde{N}_0, \, N^*_{xy} = \tilde{T}_0$$ \hspace{1cm} (G.21)$$

which, together with the expression for $N^*_{xy}$ from Eq. (G.20), can be used in the expression for the stress function $F$.

The given imperfection mode, the assumed radial deflection of the static state and the solution obtained for the stress function $F$ are substituted into the static out-of-plane equilibrium equation. The terms in the resulting expression which give a contribution in the Galerkin procedure can be written as follows:

$$\epsilon = \epsilon_0 + \epsilon_{02} \cos(2m\pi x/L) + \epsilon_{m} \sin(m \pi x - \ell_m y) + \epsilon_{y} \sin(\ell_p x + \ell_y y)$$ \hspace{1cm} (G.22)$$
The coefficients $c_{ij}$, $e_{sm}$, and $e_{sp}$ depend on the displacement amplitudes $\xi_1$ and $\xi_2$ and can be written in the general form

\[ c_{ij} = \sum_{j} \sum_{j} e^{ij}(i,j) \xi_1^{j} \xi_2^{j} \]  
\[ e_{sm} = \sum_{j} \sum_{j} e^{sm}(i,j) \xi_1^{j} \xi_2^{j} \]  
\[ e_{sp} = \sum_{j} \sum_{j} e^{sp}(i,j) \xi_1^{j} \xi_2^{j} \]

where $i$ and $j$ are the exponents of $\xi_1$ and $\xi_2$, respectively. The coefficients $e^{ij}(i,j)$, $e^{sm}(i,j)$, and $e^{sp}(i,j)$ are derived via REDUCE and are evaluated numerically via a FORTRAN program. In the corresponding FORTRAN code, the coefficients are stored in the two-dimensional arrays $E02$, $ESM$, and $ESP$.

The contributions $e^{ij}(i,j)$, $e^{sm}(i,j)$, and $e^{sp}(i,j)$ consist of several contributions,

\[ e^{ij}(i,j) = e^{ij}(i,j) - e^{ij}(i,j) - e^{ij}(i,j) - e^{ij}(i,j) \]  
\[ e^{sm}(i,j) = e^{sm}(i,j) - e^{sm}(i,j) - e^{sm}(i,j) - e^{sm}(i,j) \]  
\[ e^{sp}(i,j) = e^{sp}(i,j) - e^{sp}(i,j) - e^{sp}(i,j) - e^{sp}(i,j) \]

namely the contributions of the linear part of the equilibrium equation, and the contributions stemming from the nonlinear parts of the equation:

1. The contributions stemming from the linear part of the equilibrium equation:
   - $e^{ij}(i,j)$, $e^{sm}(i,j)$, and $e^{sp}(i,j)$ are denoted as $E02(i,j)$, $ESM(i,j)$, and $ESP(i,j)$, respectively.
   - $E02(i,j) = e^{ij}(i,j) - e^{ij}(i,j) - e^{ij}(i,j) - e^{ij}(i,j)$
   - $ESM(i,j) = e^{sm}(i,j) - e^{sm}(i,j) - e^{sm}(i,j) - e^{sm}(i,j)$
   - $ESP(i,j) = e^{sp}(i,j) - e^{sp}(i,j) - e^{sp}(i,j) - e^{sp}(i,j)$

These contributions are listed here in FORTRAN notation:

1. The contributions stemming from the linear part of the equilibrium equation ($e^{ij}(i,j)$, $e^{sm}(i,j)$, and $e^{sp}(i,j)$) are denoted as $E02(I,J)$, $ESM(I,J)$, and $ESP(I,J)$, respectively:

   \[ E02(0,1) = -(2.*((2.*B16S-B62S)*LM*LN**3+(B22S-2.*B66S))*LM**2+2.*LM*2*B21S*LN*LN+4*B11S+2*B12S)*F0SN12(1,0,0) \]
   \[ E02(1,0) = 0. \]
   \[ E02(0,0) = 0. \]

2. The contributions stemming from the nonlinear part $\tilde{N}_1(\tilde{W}_{xx} + \tilde{W}_{yy} + \tilde{W}_{xy})$

   \[ (e^{ij}(i,j), e^{sm}(i,j), e^{sp}(i,j)) \] are denoted as $E02(I,J)$, $ESM(I,J)$, and $ESP(I,J)$, respectively:

   \[ E02(0,1) = -(2.*((2.*B16S-B62S)*LM*LN**3+(B22S-2.*B66S))*LM**2+2.*LM*2*B21S*LN*LN+4*B11S+2*B12S)*F0SN12(1,0,0) \]
   \[ E02(1,0) = 0. \]
   \[ E02(0,0) = 0. \]
ESM(0, 3) = F0CN2S(1,(-1),(-2)) • H*LM*2*LP*2
ESM(0, 2) = (F0CN2S(1,(-1),(-2)) • XIB2*FOCN2(1,(-1),(-2)) • LM*2*LP*2) * H*LM*2
ESM(1, 1) = (F0SN12(2,1,(-1)) • XI1B1*FOSN12(0,1,1) * (LM*LP*2) * XIB1 + 2*F0CN2(1,(-1),(-2)) • XIB2*FOSN12(0,1,1) * (LM*LP*2) * XIB2)
ESM(2, 1) = (F0SN12(2,1,(-1)) • XIB2) • H*LM*2

3. The contributions stemming from the nonlinear part \( \tilde{\mathcal{N}}_{ij}(W_{ij} W_{jk}) \)

\( (e_{ij}^{(2)}(i,j), e_{ij}^{(m)}(i,j), e_{ij}^{(P)}(i,j)) \) are denoted as E02(I,J), ESM(I,J), and ESP(I,J), respectively:

ESM(0, 3) = -(((LM+LP)*2*FC02S(1,1)-(LM-LP)**2*FOCN2S(1,(-1),(-2)))*H*LM*2)/4
ESM(0, 2) = (((FOCN2S(1,(-1),(-2))*XIB2+FOCN2(1,(-1),(-2))) • (LH-LP)*2-FOCN2S(1,1)) • (LM+LP)**2)*H*LM*2)/4
ESM(0, 1) = -(((LM+LP)*2*FC02(1,1)*XIB2-(LM-LP)**2*FOCN2(1,(-1),(-2)))*XIB2+2*F0CN2(1,(-1),(-2)) • LH)**2)*H*LM*2)/4
ESM(1, 1) = -((LM+LP)*2*FC01(1,1) • XIB1)*H*LM*2)/4
ESM(0, 0) = -((LM+LP)*2*FC01(1,1) • XIB1)*H*LM*2)/4

E02(0, 2) = (((2*(LM+LP)*2*FOSN2(2,1,(-1))+(LM+2*LP)**2*FOSN2(1,2,1))) • (LM+2*LP)**2)*H*LM*2)/4
E02(1, 2) = (((2*(LM+LP)*2*FOSN2(2,1,(-1))) • (LM+2*LP)**2*FOSN2(1,2,1)) • (LM+2*LP)**2)*H*LM*2)/4
E02(0, 1) = (((2*(LM+LP)*2*FOSN2(2,1,(-1))) • (LM+2*LP)**2*FOSN2(1,2,1)) • (LM+2*LP)**2)*H*LM*2)/4
E02(1, 1) = (((2*(LM+LP)*2*FOSN2(2,1,(-1))) • (LM+2*LP)**2*FOSN2(1,2,1)) • (LM+2*LP)**2)*H*LM*2)/4
E02(0, 0) = -((LM+LP)*2*FOSN2(1,2,1) * (LM+2*LP)**2)*H*LM*2)/4

By application of the Galerkin procedure, a set of three nonlinear algebraic equations in the unknown amplitudes $\xi_0$, $\xi_1$, and $\xi_2$ is obtained. The Galerkin procedure, involves the evaluation of integrals over the shell surface. The weighting functions used in the Galerkin method are the assumed displacement modes,

$$
\frac{\partial W}{\partial \xi_0} = h
$$

$$
\frac{\partial W}{\partial \xi_1} = h \cos(2m \pi x/L)
$$

$$
\frac{\partial W}{\partial \xi_2} = h \sin(m \pi x/L) \cos\left(\frac{(n/R)(y - \gamma x)}{L}\right)
$$

The first equation (the equation corresponding to the weighting function $\frac{\partial W}{\partial \xi_0}$) becomes

$$
pR = -\tilde{N}_s
$$

Substituting for $\tilde{N}_s$ the expression which can be obtained from Eq. (G.20) yields $\tilde{\xi}_0$ in terms of $\tilde{\xi}_2$ and the applied loads $\lambda$, $p$, and $\gamma$,

$$
\tilde{\xi}_0 = C_{f,2} \tilde{\xi}_2 + C_{f,1} \tilde{\xi}_1 + C_{f,0}
$$

The constant coefficients $C_{f,2}$, $C_{f,1}$, and $C_{f,0}$ are defined as follows, in FORTRAN notation (CF2, CF1, and CFO, respectively),

$$
CF2 = \frac{LK*\sigma_{2}/8.0 * R}{HREF}
$$

$$
CF1 = \frac{LN*\sigma_{2}/4.0 * R}{HREF*XIB2}
$$

$$
CF0 = \frac{(R/HREF)*A22S*(PBARE * ANXREF)}{\sigma_{2}} + \frac{(R/HREF)*A12S*(ALABDA * ANXREF)}{\sigma_{2}} + \frac{(R/HREF)*A26S*(-RCAPT * ANXREF)}{\sigma_{2}}
$$

The second and third equation resulting from the Galerkin procedure are functions of the unknowns $\xi_1$ and $\xi_2$ and can be written as

$$
\sum \sum a_{1}(i,j)\xi_1 \xi_2 = 0
$$

$$
\sum \sum a_{2}(i,j)\xi_1 \xi_2 = 0
$$

The coefficients $c_{q2}(i,j)$, $c_{m}(i,j)$, and $c_{p}(i,j)$ are denoted, in FORTRAN notation, as $E02\{I,J\}$, $ESM\{I,J\}$, and $ESP\{I,J\}$, respectively.
where $i = 0, 1$ and $j = 0, 1, 2$ for $\xi_1$ and $i = 0, 1, 2$ and $j = 0, 1, 2, 3$ for $\xi_2$. Written out in detail:

\begin{align*}
\xi_0 &= C_{f1} \xi_1^2 + C_{f2} \xi_2^2 + C_{f2} \\
C_{10} \xi_1 + C_{01} \xi_2 + C_{11} \xi_1 \xi_2 + C_{00} \xi_1^2 + C_{12} \xi_1 \xi_2^2 + C_{00} = 0 \\
D_{10} \xi_1 + D_{01} \xi_2 + D_{20} \xi_1^2 + D_{11} \xi_1 \xi_2 + D_{02} \xi_2^2 + D_{21} \xi_1 \xi_2 + D_{00} = 0
\end{align*}

(G.35) 

(G.36) 

(G.37)

The coefficients of these equations have been coded in the FORTRAN subroutine ALGFUN. The coefficients $C_{ij}$ of the second equation (the equation corresponding to the weighting function $\partial W/\partial \xi_1$) are stored in the array $A_1$, and the coefficients $D_{ij}$ of the third equation (corresponding to $\partial W/\partial \xi_2$) in the array $A_2$. The indices $i$ and $j$ of the two-dimensional arrays $A_1(i,j)$ and $A_2(i,j)$ refer to the exponents $i$ and $j$ of the corresponding term $\xi_1^i \xi_2^j$. Introducing the following coefficients, in FORTRAN notation,

\begin{align*}
CSM &= 0.5 \cdot \text{HREF} \\
CSP &= 0.5 \cdot \text{HREF} \\
C02 &= \text{HREF}
\end{align*}

the coefficients of the algebraic equations $C_{ij}$ and $D_{ij}$ are defined as follows, in FORTRAN notation:

\begin{align*}
A_1(0,0) &= 0.5 \cdot C02 \cdot E02(0,0) \\
A_1(1,0) &= 0.5 \cdot C02 \cdot E02(1,0) \\
A_1(0,1) &= 0.5 \cdot C02 \cdot E02(0,1) \\
A_1(1,1) &= 0.5 \cdot C02 \cdot E02(1,1) \\
A_1(0,2) &= 0.5 \cdot C02 \cdot E02(0,2) \\
A_1(1,2) &= 0.5 \cdot C02 \cdot E02(1,2) \\
A_2(0,0) &= 0.5 \cdot CSM \cdot ESM(0,0) + 0.5 \cdot CSP \cdot ESP(0,0) \\
A_2(1,0) &= 0.5 \cdot CSM \cdot ESM(1,0) + 0.5 \cdot CSP \cdot ESP(1,0) \\
A_2(0,1) &= 0.5 \cdot CSM \cdot ESM(0,1) + 0.5 \cdot CSP \cdot ESP(0,1) \\
A_2(1,1) &= 0.5 \cdot CSM \cdot ESM(1,1) + 0.5 \cdot CSP \cdot ESP(1,1) \\
A_2(0,2) &= 0.5 \cdot CSM \cdot ESM(0,2) + 0.5 \cdot CSP \cdot ESP(0,2) \\
A_2(1,2) &= 0.5 \cdot CSM \cdot ESM(1,2) + 0.5 \cdot CSP \cdot ESP(1,2) \\
A_2(0,3) &= 0.5 \cdot CSM \cdot ESM(0,3) + 0.5 \cdot CSP \cdot ESP(0,3)
\end{align*}
G.4 Simplified Analysis - Steady State Vibration

The Simplified Analysis for the steady state vibrations outlined in Chapter 3 relies on derivations via the symbolic manipulation package REDUCE. The analysis has been coded in the FORTRAN computer program SILVANA (Simplified Large amplitude Vibration ANalysis including Anisotropy). In this section, details of the derivations will be presented.

A particular solution for the stress function $F_p$ of the static state is:

$$F_p = f_{0,0,-1}^{(1,0,0,0,0)} \sin(\ell_{m} x - \ell_{n} y) + f_{0,0,1}^{(1,0,0,0,0)} \sin(\ell_{m} x + \ell_{n} y) + f_{0,0,1}^{(1,0,0,0,0)} \sin(\ell_{2m} x - \ell_{2n} y) \ldots (G.38)$$

In REDUCE notation,

$$fd := \left( +f_{0,0,-1}^{(1,0,0,0,0)} \sin(l_{m} x - l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{2m} x - l_{2n} y) + f_{0,0,1}^{(1,0,0,0,0)} \sin(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{2m} x - l_{2n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{3m} x - l_{3n} y) + f_{0,0,1}^{(1,0,0,0,0)} \sin(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{2m} x - l_{2n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{3m} x - l_{3n} y) \right)$$

$$fd := \left( +f_{0,0,-1}^{(1,0,0,0,0)} \cos(l_{m} x - l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{2m} x - l_{2n} y) + f_{0,0,1}^{(1,0,0,0,0)} \cos(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{2m} x - l_{2n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{3m} x - l_{3n} y) + f_{0,0,1}^{(1,0,0,0,0)} \cos(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{2m} x - l_{2n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{3m} x - l_{3n} y) \right)$$

$$fd := \left( +f_{0,0,-1}^{(1,0,0,0,0)} \sin(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{m} x - l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{2m} x + l_{2n} y) + f_{0,0,1}^{(1,0,0,0,0)} \sin(l_{m} x - l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{2m} x + l_{2n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{3m} x + l_{3n} y) + f_{0,0,1}^{(1,0,0,0,0)} \sin(l_{m} x - l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{2m} x + l_{2n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{3m} x + l_{3n} y) \right)$$

$$fd := \left( +f_{0,0,-1}^{(1,0,0,0,0)} \cos(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{m} x - l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{2m} x + l_{2n} y) + f_{0,0,1}^{(1,0,0,0,0)} \cos(l_{m} x - l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{2m} x + l_{2n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{3m} x + l_{3n} y) + f_{0,0,1}^{(1,0,0,0,0)} \cos(l_{m} x - l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{2m} x + l_{2n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{3m} x + l_{3n} y) \right)$$

$$fd := \left( +f_{0,0,-1}^{(1,0,0,0,0)} \sin(l_{m} x - l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{2m} x - l_{2n} y) + f_{0,0,1}^{(1,0,0,0,0)} \sin(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{2m} x - l_{2n} y) \right)$$

$$fd := \left( +f_{0,0,-1}^{(1,0,0,0,0)} \cos(l_{m} x - l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{2m} x - l_{2n} y) + f_{0,0,1}^{(1,0,0,0,0)} \cos(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{2m} x - l_{2n} y) \right)$$

$$fd := \left( +f_{0,0,-1}^{(1,0,0,0,0)} \sin(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{m} x - l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \sin(l_{2m} x + l_{2n} y) + f_{0,0,1}^{(1,0,0,0,0)} \sin(l_{m} x - l_{n} y) \right)$$

$$fd := \left( +f_{0,0,-1}^{(1,0,0,0,0)} \cos(l_{m} x + l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{m} x - l_{n} y) + f_{1,0,0,0,1}^{(1,0,0,0,0)} \cos(l_{2m} x + l_{2n} y) + f_{0,0,1}^{(1,0,0,0,0)} \cos(l_{m} x - l_{n} y) \right)$$

In REDUCE notation,
This expression can be written in the form $\sum f_i g_i$. The unknown coefficients $f_i$ ($f_1 = f_1(0,0,0,-1)$, $f_2 = f_2(0,1,0,0,-1)$ etc.) of the goniometric terms $g_i$ ($g_1 = \sin(\xi_0 - \xi_0 y)$, $g_2 = \sin(\xi_0 + \xi_0 y)$, etc.) have to be determined.

The stress function $F$ can now be written as

$$F = \frac{1}{2} N_x y^2 + \frac{1}{2} N_y x^2 - N_z x y + F_p \quad (G.39)$$

The constant in-plane stresses $N_x$ and $N_y$ are equal to the prescribed average stress resultants at the shell edges, hence $N_x$ and $N_y$ are equal to zero. The circumferential stress $N_z$ is assumed to be zero. The background of this assumption will be discussed later in this section.

The assumed stress function is substituted into the compatibility equation of the dynamic state. The right-hand side of the compatibility equation can be written in the form $\sum a_j g_j$. In a REDUCE notation the right-hand side (denoted in REDUCE as "rhs") becomes

$$\text{rhs} := (\ldots)$$

The coefficients $a_j$ of the different goniometric terms $g_j$ of the right-hand side have been obtained using the COEFFN operator in REDUCE. The coefficients $a_j$, in REDUCE notation, are:

$$a_{nm} := \text{as}_{nm}(0,0,0,-1) \cdot \sin(\xi \cdot \xi y) + \ldots$$

The coefficients $a_j$ of the different goniometric terms $g_j$ of the right-hand side have been obtained using the COEFFN operator in REDUCE. The coefficients $a_j$, in REDUCE notation, are:
The coefficients \( a_j \) depend on the displacement amplitudes \( A, B \) and \( C \). The left-hand side of the compatibility equation can be written in the form \( \sum_j a_j g_j \) in REDUCE notation, as:

\[
\text{lhs} :=
\]

\[
+ \text{alsn}(1,0,0,0,-1) \sin(lm-x-lp+y)
+ \text{alsn}(0,1,0,0,1) \sin(lp-x-ly)
+ \text{alsn}(1,2,0,0,1) \sin(lm+2*lp-x-ly)
+ \text{alsn}(2,1,0,0,1) \sin(2*lm+lp-x-ly)
+ \text{alci}(1,0,0,0,-1) \cos(lm-x-lp+y)
+ \text{alci}(0,1,0,0,1) \cos(lp-x-ly)
+ \text{alci}(1,2,0,0,1) \cos(lm+2*lp-x-ly)
+ \text{alci}(2,1,0,0,1) \cos(2*lm+lp-x-ly)
+ \text{alsm}(0,0,0,0,1) \sin(ly-y-lp+y)
+ \text{alsm}(1,1,0,0,1) \sin(lm+lp-y-lp+y)
+ \text{alsm}(0,0,0,0,1) \cos(ly-y-lp+y)
+ \text{alsm}(1,1,0,0,1) \cos(lm+lp-y-lp+y)
+ \text{als2}(1,-1,0,0,-1) \sin(lnx-lpx-lny-ll*y)
+ \text{als2}(2,0,0,0,-1) \sin(2*lnx-lpx-lny-ll*y)
+ \text{als2}(0,2,0,0,1) \sin(2*lnx+lpx+lny+ll*y)
+ \text{als2}(1,-1,0,0,-1) \cos(lnx-lpx-lny-ll*y)
+ \text{als2}(2,0,0,0,-1) \cos(2*lnx-lpx-lny-ll*y)
+ \text{als2}(0,2,0,0,1) \cos(2*lnx+lpx+lny+ll*y)
+ \text{als}(1,0,0,0,-1) \sin(lnx-lpx-lny-ll*y)
+ \text{als}(2,0,0,0,-1) \sin(2*lnx-lpx-lny-ll*y)
+ \text{als}(0,2,0,0,1) \sin(2*lnx+lpx+lny+ll*y)
+ \text{als}(1,0,0,0,-1) \cos(lnx-lpx-lny-ll*y)
+ \text{als}(2,0,0,0,-1) \cos(2*lnx-lpx-lny-ll*y)
+ \text{als}(0,2,0,0,1) \cos(2*lnx+lpx+lny+ll*y)
+ \text{als0}(1,1,0,0,0) \cos(lm*x+lp*x)
; \]

The coefficients \( a_j \) of the goniometric terms \( g_j \) of the left-hand side are determined via the COEFEF operator. The coefficients \( a_j \) in REDUCE notation, are:

\[
\text{alsn}(1,0,0,0,-1) := \text{alsn}(1,0,0,0,-1) \cdot \sin(lm-x-lp+y)
+ \text{alsn}(0,1,0,0,1) \cdot \sin(lp-x-ly)
+ \text{alsn}(1,2,0,0,1) \cdot \sin(lm+2*lp-x-ly)
+ \text{alsn}(2,1,0,0,1) \cdot \sin(2*lm+lp-x-ly)
+ \text{alci}(1,0,0,0,-1) \cdot \cos(lm-x-lp+y)
+ \text{alci}(0,1,0,0,1) \cdot \cos(lp-x-ly)
+ \text{alci}(1,2,0,0,1) \cdot \cos(lm+2*lp-x-ly)
+ \text{alci}(2,1,0,0,1) \cdot \cos(2*lm+lp-x-ly)
+ \text{alsm}(0,0,0,0,1) \cdot \sin(ly-y-lp+y)
+ \text{alsm}(1,1,0,0,1) \cdot \sin(lm+lp-y-lp+y)
+ \text{alsm}(0,0,0,0,1) \cdot \cos(ly-y-lp+y)
+ \text{alsm}(1,1,0,0,1) \cdot \cos(lm+lp-y-lp+y)
+ \text{als2}(1,-1,0,0,-1) \cdot \sin(lnx-lpx-lny-ll*y)
+ \text{als2}(2,0,0,0,-1) \cdot \sin(2*lnx-lpx-lny-ll*y)
+ \text{als2}(0,2,0,0,1) \cdot \sin(2*lnx+lpx+lny+ll*y)
+ \text{als2}(1,-1,0,0,-1) \cdot \cos(lnx-lpx-lny-ll*y)
+ \text{als2}(2,0,0,0,-1) \cdot \cos(2*lnx-lpx-lny-ll*y)
+ \text{als2}(0,2,0,0,1) \cdot \cos(2*lnx+lpx+lny+ll*y)
+ \text{als}(1,0,0,0,-1) \cdot \sin(lnx-lpx-lny-ll*y)
+ \text{als}(2,0,0,0,-1) \cdot \sin(2*lnx-lpx-lny-ll*y)
+ \text{als}(0,2,0,0,1) \cdot \sin(2*lnx+lpx+lny+ll*y)
+ \text{als}(1,0,0,0,-1) \cdot \cos(lnx-lpx-lny-ll*y)
+ \text{als}(2,0,0,0,-1) \cdot \cos(2*lnx-lpx-lny-ll*y)
+ \text{als}(0,2,0,0,1) \cdot \cos(2*lnx+lpx+lny+ll*y)
+ \text{als0}(1,1,0,0,0) \cdot \cos(lm*x+lp*x)
; \]

The coefficients \( a_j \) of the goniometric terms \( g_j \) of the left-hand side are determined via the COEFEF operator. The coefficients \( a_j \) in REDUCE notation, are:
The coefficients $a_i$ depend on the unknown coefficients of the stress function $f_i$. 

\[ a_{lcnm}(1,1,0,0,-1) := \text{alls}(fcmn(1,1,0,0,-1) \times (ll**4 + 4*ll**3*ln + 6*ll**2*ln**2 + 4*ll*ln**3 + ln**4)) + 2*a12s*fcnp(1,-1,0,0,-1) \times (ll*ln + ln*lp + 4*ln*lp**2 + 2*ln**2*lp + ln**3 + ln**2*lp**2) \times \]

\[ a_{lcnpl}(2,0,0,0,-1) := \text{alls}(fcnp(2,0,0,0,-1) \times (ll**4 + 4*ll**3*ln + 6*ll**2*ln**2 + 4*ll*ln**3 + ln**4)) + 2*a12s*fcnp(2,0,0,0,-1) \times (ll*ln + ln*lp + 4*ln*lp**2 + 2*ln**2*lp + ln**3 + ln**2*lp**2) \times \]

\[ a_{lc0d}(1,1,0,0) := a_{22s}(fc0(l,1,0,0)) \times (ll**4 + 4*ll**3*ln + 6*ll**2*ln**2 + 4*ll*ln**3 + ln**4) + 2*a12s*fcnp(1,1,0,0,-1) \times (ll*ln + ln*lp + 4*ln*lp**2 + 2*ln**2*lp + ln**3 + ln**2*lp**2) \times \]

\[ a_{lc0d}(1,0,0) := a_{22s}(fc0(l,1,0,0)) \times (ll**4 + 4*ll**3*ln + 6*ll**2*ln**2 + 4*ll*ln**3 + ln**4) + 2*a12s*fcnp(1,1,0,0,-1) \times (ll*ln + ln*lp + 4*ln*lp**2 + 2*ln**2*lp + ln**3 + ln**2*lp**2) \times \]
The coefficients $a_i$ will be determined via REDUCE.

The coefficients $a_i$ of the goniometric terms of the right-hand side of the compatibility equation depend on the displacement amplitudes $C_1$, $A$ and $B$, and can be written in the general form

$$a_i' = a_{i1}C_1 + a_{i2}C_1A + a_{i3}C_1B + a_{i4}A + a_{i5}B + a_{i6}A^2 + a_{i7}B^2 + a_{i8}AB$$  \hspace{1cm} (G.40)

Written out in detail, in a REDUCE notation,

$$\begin{align*}
\text{as}_{11}(1,0,0,0,-1) &= \text{asn}(1,0,0,0,-1)\cdot \text{cl}\quad \text{as}_{12}(0,1,0,0,1) &= \text{asn}(0,1,0,0,1)\cdot \text{cl}\quad \text{as}_{13}(1,2,0,0,1) &= \text{asn}(1,2,0,0,1)\cdot \text{cl}\quad \\
\text{as}_{14}(2,1,0,0,-1) &= \text{asn}(2,1,0,0,-1)\cdot \text{cl}\quad \\
\text{as}_{15}(0,1,0,0,1) &= \text{asn}(0,1,0,0,1)\cdot \text{cl}\quad \text{as}_{16}(0,2,0,0,1) &= \text{asn}(0,2,0,0,1)\cdot \text{cl}\quad \\
\text{as}_{17}(1,2,0,0,1) &= \text{asn}(1,2,0,0,1)\cdot \text{cl}\quad \text{as}_{18}(1,2,0,0,1) &= \text{asn}(1,2,0,0,1)\cdot \text{cl}
\end{align*}$$

The coefficients $a_i$ will be determined via REDUCE.

The unknown coefficients of the stress function $\hat{f}$ are obtained by equating the coefficients of corresponding goniometric terms $g_i$ on the left-hand side and the right-hand side of the compatibility equation,

$$\begin{align*}
\hat{f}_1 &= \left(\frac{1}{a_1}\right)\left[a_{i1}C_1 + a_{i2}C_1A + a_{i3}C_1B + a_{i4}A + a_{i5}B + a_{i6}A^2 + a_{i7}B^2 + a_{i8}AB\right]
\end{align*}$$  \hspace{1cm} (G.43)

Written out in detail, in REDUCE notation,

$$\begin{align*}
\text{fs}_{11}(1,0,0,0,-1) &= \text{fs}_{12}(0,1,0,0,1)\cdot \text{cl}\quad \text{fs}_{13}(1,2,0,0,1) &= \text{fs}_{14}(2,1,0,0,-1)\cdot \text{cl}\quad \\
\text{fs}_{15}(0,1,0,0,1) &= \text{fs}_{16}(0,2,0,0,1)\cdot \text{cl}\quad \text{fs}_{17}(1,2,0,0,1) &= \text{fs}_{18}(1,2,0,0,1)\cdot \text{cl}
\end{align*}$$

which gives for the coefficients of the stress function,

$$\begin{align*}
\hat{f}_1 &= \left(\frac{1}{a_1}\right)\left[\hat{f}_1C_1 + \hat{f}_2C_1A + \hat{f}_3C_1B + \hat{f}_4A + \hat{f}_5B + \hat{f}_6A^2 + \hat{f}_7B^2 + \hat{f}_8AB\right]
\end{align*}$$  \hspace{1cm} (G.44)

Written out in detail, in REDUCE notation,

$$\begin{align*}
\text{fs}_{11}(1,0,0,0,-1) &= \text{fs}_{12}(0,1,0,0,1)\cdot \text{cl}\quad \text{fs}_{13}(1,2,0,0,1) &= \text{fs}_{14}(2,1,0,0,-1)\cdot \text{cl}\quad \\
\text{fs}_{15}(0,1,0,0,1) &= \text{fs}_{16}(0,2,0,0,1)\cdot \text{cl}\quad \text{fs}_{17}(1,2,0,0,1) &= \text{fs}_{18}(1,2,0,0,1)\cdot \text{cl}
\end{align*}$$

The coefficients $a_i$ will be determined via REDUCE.

The coefficients $a_i$ are obtained by equating the coefficients of corresponding goniometric terms $g_i$ on the left-hand side and the right-hand side of the compatibility equation,

$$\begin{align*}
\hat{f}_1 &= \left(\frac{1}{a_1}\right)\left[a_{i1}C_1 + a_{i2}C_1A + a_{i3}C_1B + a_{i4}A + a_{i5}B + a_{i6}A^2 + a_{i7}B^2 + a_{i8}AB\right]
\end{align*}$$  \hspace{1cm} (G.41)

The coefficients $a_i$ are obtained by equating the coefficients of corresponding goniometric terms $g_i$ on the left-hand side and the right-hand side of the compatibility equation,
The procedure to evaluate the coefficients $f_j$ is coded in FORTRAN. The coefficients $a_i$ and $a_{i,j}$ are determined via REDUCE. REDUCE has an option to generate FORTRAN code. The FORTRAN code becomes:

\begin{align*}
A &= \text{LN}^{*}4\*\text{A11S}+2.\*\text{LM}^{*}4\*\text{A123S}+
\text{LN}^{*}2\*\text{LM}^{*}2\*\text{A66S}+2.\*\text{LM}^{*}2\*\text{A12S}+
\text{LM}^{*}2\*\text{LL}^{*}2\*\text{A26S}+2.\*\text{LM}^{*}2\*\text{LL}^{*}2\*\text{A22S}+
\text{LM}^{*}2\*\text{LL}^{*}2\*\text{A66S}+2.\*\text{LL}^{*}2\*\text{A26S}+
\text{LM}^{*}2\*\text{LL}^{*}2\*\text{A66S}+2.\*\text{LL}^{*}2\*\text{A26S}+
\text{LM}^{*}2\*\text{LL}^{*}2\*\text{A22S}+\text{LM}^{*}2.\*\text{LL}^{*}2\*\text{A26S}.
\end{align*}

\begin{align*}
B &= (\text{H}^{*}4\*\text{A12S}+2.\*\text{LM}^{*}2\*\text{A26S}+2.\*\text{LM}^{*}2\*\text{A12S}+
\text{LM}^{*}2\*\text{LL}^{*}2\*\text{A26S}+2.\*\text{LM}^{*}2\*\text{LL}^{*}2\*\text{A22S}+
\text{LM}^{*}2\*\text{LL}^{*}2\*\text{A66S}+2.\*\text{LM}^{*}2\*\text{LL}^{*}2\*\text{A26S}.
\end{align*}

\begin{align*}
\text{FSNCl}(1,0,0,0,1) &= B/A
\end{align*}

\begin{align*}
\text{FSNCl}(0,1,0,0,1) &= B/A
\end{align*}

\begin{align*}
\text{FSNLAC}(1,0,0,0,1) &= B/A
\end{align*}

\begin{align*}
\text{FSNLAC}(0,1,0,0,1) &= B/A
\end{align*}
Derivation of equations using REDUCE.
In the present approach, the amplitudes of the axisymmetric modes, \( C_0 \) and \( C_1 \), are expressed in terms of the amplitudes of the asymmetric vibration modes, \( A \) and \( B \). These relations between the axisymmetric and the asymmetric vibration amplitudes are based on assumptions concerning the stresses which are constant in circumferential direction. The circumferential periodicity condition is used to establish a relation between \( C_0 \) and the amplitudes of the asymmetric modes \( A \) and \( B \), assuming implicitly that the constant circumferential stress \( N_{z_0} \) is equal to zero. The periodicity condition states that the expression

\[
\int_0^{2\pi} \overline{\varepsilon}_y \, dy = 0 \quad (G.44)
\]

is satisfied. Using the expression for the circumferential strain \( \overline{\varepsilon}_y \), substituting for \( W, F, \overline{W}, \) and \( F \), and carrying out the integration in \( y \)-direction yields the following expression, denoted as "\( \text{vyd00} \)" in REDUCE notation,

\[
\text{vyd00} := a_{12s} \cdot n_x \star \text{star} + a_{22s} \cdot n_y \star \text{star} + a_{26s} \cdot n_x \star \text{ystar} + h \cdot C - \frac{1}{8} \cdot a \cdot a \cdot h \cdot n \cdot n \cdot x \cdot l \cdot 0 \cdot 2 - \frac{1}{4} \cdot a \cdot n \cdot n \cdot x \cdot i \cdot 2 \cdot - \frac{1}{8} \cdot a \cdot a \cdot h \cdot n \cdot n \cdot x \cdot b \cdot 2 \cdot l \cdot 2 \cdot l \cdot 2 - c \cdot 0 \cdot r \cdot e \cdot (-1))
\]

which has to be equal to zero. The constant in-plane stresses \( \tilde{N}_{z_0} \) and \( \tilde{N}_{z_y} \) are equal to the prescribed averaged stress resultants at the shell edge,

\[
\tilde{N}_z = -\tilde{N}_{z_0} = 0, \quad \tilde{N}_{z_y} = \tilde{T}_0 = 0 \quad (G.45)
\]

The constant stress in the circumferential direction can now be written as

\[
\tilde{N}_y = \frac{1}{C_N} \left\{ A^2 + B^2 + 2A \cdot B \cdot c \cdot N \cdot \left( \xi_2 + \xi_2 \right) - c \cdot D \right\} \quad (G.46)
\]
where \( C_N = A_N^2 / (1 + \ell R) \) and \( \epsilon = 8/5R \ell R \). Assuming that \( \tilde{N}^* \) is equal to zero, the relation between \( C_0 \) and \( A \) and \( B \) becomes

\[
C_0 = \left( 1 / \epsilon \right) \left\{ A^2 + B^2 + 2 \delta_{n,1} A(\xi_2 + \xi_3) \right\} \quad (G.47)
\]

where \( \epsilon \) is defined in the main text. The axisymmetric amplitude \( C_1 \) is also expressed in terms of the \( A \) and \( B \). Inspection of the coefficients of the stress function \( FC0(1,1,0,0) \) shows that, if \( B_{11}^* \) is equal to zero, a relation between the amplitude of the axisymmetric mode \( C_1 \) and the amplitudes of the asymmetric modes \( A \) and \( B \) can be chosen such that the coefficient \( \tilde{C} \) of the \( \cos \ell m + \ell p X \) term of the stress function vanishes,

\[
C_1 = \left( -1 / \epsilon \right) \left\{ A^2 + B^2 + 2 \delta_{n,1} A(\xi_2 + \xi_3) \right\} \quad (G.48)
\]

A more general approach, in which the amplitudes of the axisymmetric terms \( C_0 \) and \( C_1 \) are used as generalized coordinates in the formulation, is given in the next section.

The given imperfection mode, the assumed radial deflections of the static state and the dynamic state, and the solutions obtained for the stress functions \( F \) and \( F \) are substituted into the dynamic out-of-plane equilibrium equation.

The terms in the resulting expression which give a contribution in the Galerkin procedure can be written as follows:

\[
\epsilon = \epsilon_0 + 2 \epsilon_2 \cos(2\pi \ell x / L) + \epsilon_{2m} \sin(\ell m x - \ell y) + \epsilon_{2p} \sin(\ell p x + \ell y) + \epsilon_{2m} \cos(\ell m x - \ell y) + \epsilon_{2p} \cos(\ell p x + \ell y) \quad (G.49)
\]

The coefficients \( \epsilon_0, \epsilon_2, \epsilon_{2m}, \epsilon_{2p}, \epsilon_{2m}, \) and \( \epsilon_{2p} \) depend on the displacement amplitudes \( C_1, A \) and \( B \).

Application of Galerkin's method leads to a coupled set of nonlinear ordinary differential equations in the unknown vibration amplitudes \( A \) and \( B \).

For the goniometric terms in the out-of-plane equilibrium equation which will give a contribution in the Galerkin procedure, the coefficients of the terms \( A^{i}B^{j} \), where \( i \) and \( j \) are the exponents of \( A \) and \( B \), respectively, have been derived via REDUCE and are evaluated numerically via a FORTRAN program.

The coefficients of the goniometric terms in the out-of-plane equilibrium equation consist of the contributions of the linear and nonlinear parts. This has been coded in FORTRAN as follows.

\[
\begin{align*}
e^{00}(i,j) &= \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} \\
e^{02}(i,j) &= \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} \\
e^{2m}(i,j) &= \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} \\
e^{2p}(i,j) &= \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} \\
e^{00}(i,j) &= \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} - \epsilon^{00}_{i,j} \\
e^{02}(i,j) &= \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} - \epsilon^{02}_{i,j} \\
e^{2m}(i,j) &= \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} - \epsilon^{2m}_{i,j} \\
e^{2p}(i,j) &= \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} - \epsilon^{2p}_{i,j} \\
(50)
\end{align*}
\]

These contributions are listed here in FORTRAN notation. First the coefficients \( C_2 \) (\( C_2 \) in FORTRAN notation) and \( C_1 \) (\( C_1 \) in FORTRAN notation) are defined as follows, in FORTRAN notation:
The contributions which stem from the linear part of the equilibrium equation (e_{i,j}^{c_0}, e_{i,j}^{c_1}, e_{i,j}^{c_2}, and e_{i,j}^{c_3}) are denoted as E00, E02, ESM, and ESP, respectively:

\[ \text{ESH}(3,0) = \frac{-((2 \times B16S - B62S) \times LMM \times LLL \times \text{LL}^3 - (B22S - 2 \times B66S) \times LMM \times \text{LL}^2 \times 2 - (B66S \times LMM \times \text{LL}^2 \times 2 - 2 \times B66S \times LMM \times \text{LL}^4))}{R \times (LMM \times 2)} \]

\[ \text{ESH}(1,0) = \frac{-((2 \times B16S - B62S) \times LMM \times LLL \times \text{LL}^3 - (B22S - 2 \times B66S) \times LMM \times \text{LL}^2 \times 2 - (B66S \times LMM \times \text{LL}^2 \times 2 - 2 \times B66S \times LMM \times \text{LL}^4))}{R \times (LMM \times 2)} \]

\[ \text{ESH}(2,0) = \frac{-((2 \times B16S - B62S) \times LMM \times LLL \times \text{LL}^3 - (B22S - 2 \times B66S) \times LMM \times \text{LL}^2 \times 2 - (B66S \times LMM \times \text{LL}^2 \times 2 - 2 \times B66S \times LMM \times \text{LL}^4))}{R \times (LMM \times 2)} \]

\[ \text{ESH}(0,3) = \frac{-((2 \times B16S - B62S) \times LMM \times LLL \times \text{LL}^3 - (B22S - 2 \times B66S) \times LMM \times \text{LL}^2 \times 2 - (B66S \times LMM \times \text{LL}^2 \times 2 - 2 \times B66S \times LMM \times \text{LL}^4))}{R \times (LMM \times 2)} \]

\[ \text{ESM}(3,0) = \frac{-((2 \times B16S - B62S) \times LMM \times LLL \times \text{LL}^3 - (B22S - 2 \times B66S) \times LMM \times \text{LL}^2 \times 2 - (B66S \times LMM \times \text{LL}^2 \times 2 - 2 \times B66S \times LMM \times \text{LL}^4))}{R \times (LMM \times 2)} \]

\[ \text{ESM}(1,0) = \frac{-((2 \times B16S - B62S) \times LMM \times LLL \times \text{LL}^3 - (B22S - 2 \times B66S) \times LMM \times \text{LL}^2 \times 2 - (B66S \times LMM \times \text{LL}^2 \times 2 - 2 \times B66S \times LMM \times \text{LL}^4))}{R \times (LMM \times 2)} \]

\[ \text{ECP}(2,1) = \frac{-((2 \times B16S - B62S) \times LMM \times LLL \times \text{LL}^3 - (B22S - 2 \times B66S) \times LMM \times \text{LL}^2 \times 2 - (B66S \times LMM \times \text{LL}^2 \times 2 - 2 \times B66S \times LMM \times \text{LL}^4))}{R \times (LMM \times 2)} \]

\[ \text{ECP}(1,1) = \frac{-((2 \times B16S - B62S) \times LMM \times LLL \times \text{LL}^3 - (B22S - 2 \times B66S) \times LMM \times \text{LL}^2 \times 2 - (B66S \times LMM \times \text{LL}^2 \times 2 - 2 \times B66S \times LMM \times \text{LL}^4))}{R \times (LMM \times 2)} \]
Derivation of equations using REDUCE

2. The contributions stemming from the nonlinear part \( \hat{N}_x(W_{xx} + W_{xy}) \)

\( c_{ij}^{(0)}(t), c_{ij}^{(1)}(t), c_{ij}^{(2)}(t), c_{ij}^{(3)}(t), c_{ij}^{(4)}(t) \)

are denoted as \( E(i,j), E(0,j), E_{i,j}, E_{i,j}, E_{i,j}, E_{i,j} \) respectively.
The contributions stemming from the nonlinear part $N_y(W_{xy} + W_{yy})$
($e^{p_0}_{00}(i,j), e^{p_1}_{00}(i,j), e^{m_0}_{00}(i,j), e^{m_1}_{00}(i,j), e^{p_0}_{10}(i,j), e^{p_1}_{10}(i,j), e^{m_0}_{10}(i,j), e^{m_1}_{10}(i,j))$ are denoted as $E00(I,J), E02(I,J), ESM(I,J), ESP(I,J), ECM(I,J),$ and $ECP(I,J)$, respectively.

C NYO

$E02(0,0)=((F0SN12(2,1,(-1))*XI01*XI02+F0SN1(2,1,(-1))*XI01+F0SN2(2,1,(-1))*XI02)*LM+2*LP)*XI02+2*NL+2*DELNL)/4.

$E00(1,0)=((F0SN12(1,0,(-1))*XI01*XI02+F0SN12(0,1,1)*XI01*XI02+F0SN1(2,1,(-1))*XI01+F0SN2(2,1,(-1))*XI02)*LM+2*LP)*XI02+2*NL+2*DELNL)/4.

$E00(2,0)=((F0SN12(1,0,(-1))*XI01*XI02+F0SN12(0,1,1)*XI01*XI02+F0SN1(2,1,(-1))*XI01+F0SN2(2,1,(-1))*XI02)*LM+2*LP)*XI02+2*NL+2*DELNL)/4.

The contributions stemming from the nonlinear part $N_y(W_{xy} + W_{yy})$
($e^{p_0}_{00}(i,j), e^{p_1}_{00}(i,j), e^{m_0}_{00}(i,j), e^{m_1}_{00}(i,j), e^{p_0}_{10}(i,j), e^{p_1}_{10}(i,j), e^{m_0}_{10}(i,j), e^{m_1}_{10}(i,j))$ are denoted as $E00(I,J), E02(I,J), ESM(I,J), ESP(I,J), ECM(I,J),$ and $ECP(I,J)$, respectively.

C NYO

$E02(0,0)=((F0SN12(2,1,(-1))*XI01*XI02+F0SN1(2,1,(-1))*XI01+F0SN2(2,1,(-1))*XI02)*LM+2*LP)*XI02+2*NL+2*DELNL)/4.

$E00(1,0)=((F0SN12(1,0,(-1))*XI01*XI02+F0SN12(0,1,1)*XI01*XI02+F0SN1(2,1,(-1))*XI01+F0SN2(2,1,(-1))*XI02)*LM+2*LP)*XI02+2*NL+2*DELNL)/4.

$E00(2,0)=((F0SN12(1,0,(-1))*XI01*XI02+F0SN12(0,1,1)*XI01*XI02+F0SN1(2,1,(-1))*XI01+F0SN2(2,1,(-1))*XI02)*LM+2*LP)*XI02+2*NL+2*DELNL)/4.

The contributions stemming from the nonlinear part $N_y(W_{xy} + W_{yy})$
($e^{p_0}_{00}(i,j), e^{p_1}_{00}(i,j), e^{m_0}_{00}(i,j), e^{m_1}_{00}(i,j), e^{p_0}_{10}(i,j), e^{p_1}_{10}(i,j), e^{m_0}_{10}(i,j), e^{m_1}_{10}(i,j))$ are denoted as $E00(I,J), E02(I,J), ESM(I,J), ESP(I,J), ECM(I,J),$ and $ECP(I,J)$, respectively.

C NYO

$E02(0,0)=((F0SN12(2,1,(-1))*XI01*XI02+F0SN1(2,1,(-1))*XI01+F0SN2(2,1,(-1))*XI02)*LM+2*LP)*XI02+2*NL+2*DELNL)/4.

$E00(1,0)=((F0SN12(1,0,(-1))*XI01*XI02+F0SN12(0,1,1)*XI01*XI02+F0SN1(2,1,(-1))*XI01+F0SN2(2,1,(-1))*XI02)*LM+2*LP)*XI02+2*NL+2*DELNL)/4.

$E00(2,0)=((F0SN12(1,0,(-1))*XI01*XI02+F0SN12(0,1,1)*XI01*XI02+F0SN1(2,1,(-1))*XI01+F0SN2(2,1,(-1))*XI02)*LM+2*LP)*XI02+2*NL+2*DELNL)/4.

The contributions stemming from the nonlinear part $N_y(W_{xy} + W_{yy})$
($e^{p_0}_{00}(i,j), e^{p_1}_{00}(i,j), e^{m_0}_{00}(i,j), e^{m_1}_{00}(i,j), e^{p_0}_{10}(i,j), e^{p_1}_{10}(i,j), e^{m_0}_{10}(i,j), e^{m_1}_{10}(i,j))$ are denoted as $E00(I,J), E02(I,J), ESM(I,J), ESP(I,J), ECM(I,J),$ and $ECP(I,J)$, respectively.
ESM(1,0)=((F0CN2S(1,(-1),(-2))*XI02+F0CN2(1,(-1),(-2)))*(LM-LP)*LN*LP*DELNL*XI02+LH*NXYP)*H/LL
E3P(1,0)=-((F0CN2S(1,(-1),(-2))*XI02+F0CN2(1,(-1),(-2)))*(LM-LP)*LN*LP*DELNL*XI02+NXYP)*LP)*H/LL
ECM(0,1)=-((F0CN2S(1,(-1),(-2))*XI02+F0CN2(1,(-1),(-2)))*(LH-LP)*LN*LP*DELNL*XI02-LM*NXYP)*H/LL
ECP(0,1)=-((F0CN2S(1,(-1),(-2))*XI02+F0CN2S(1,(-1),(-2)))*(LM-LP)*LN*LP*DELNL*XI02-NXYP)*LP)*H/LL
E02(1,0)=(-((F0SN12(2,1,(-1))*XI01*XI02+F0SN12(2,1,(-1)))*XI01#XI02+LH*XI02)*LM*LM+LP*LP)*H/LL
E00(1,0)=-((F0SN12(1,0,(-1))*XI01*XI02+F0SN12(0,1,(-1)))*LH*XI01+LH*XI02)*LM*LM+LP*LP)*H/LL

The contributions stemming from the nonlinear part $\tilde{x}_e$ are denoted as $E_00(I,J), E_02(I,J), E_{SN}(I,J), E_{CP}(I,J), E_{CM}(I,J), E_{EC}(I,J)$, respectively.

$E_{3M}(5,0)=((F_{SNC}(2,1,(-1)),(LM+LP))#(H+2)*C2)/4.
E_{SH}(4,0)=(-((F_{SN}(2,1,(-1)),(LM+LP))#(H+2)*C2-FSNC(0,1,(-1))#(LM+LP))#(H+2)*C2)/4.
E_{SM}(3,0)-((((F_{SLAC}(2,1,(-1))-F_{SLAC}(0,1,1))#(LM+LP))#(H+2)*C2)/4.
E_{SM}(3,2)-((F_{SLAC}(2,1,(-1))-F_{SLAC}(0,1,1))#(LM+LP))#(H+2)*C2)/2.
E_{3P}(4,0)=-((F_{SN}(1,2,1),LM+LP)#(H+2)*C2)/4.
E_{3P}(3,0)-((((F_{SLAC}(1,2,1)-F_{SLAC}(1,0,(-1))#(LM+LP))#(H+2)*C2)/4.
E_{3P}(2,0))-((F_{SN}(1,2,1),LM+LP))#(H+2)*C2)/4.
E_{3P}(1,0)=(-((F_{SN}(1,2,1))#(LM+LP))#(H+2)*C2)/4.

C NEX

$E_{SM}(5,0)=((F_{SLAC}(2,1,(-1))-F_{SLAC}(0,1,1))#(LM+LP))#(H+2)*C2)/4.
E_{SN}(4,0)=(-((F_{SN}(2,1,(-1))LM+LP))#(H+2)*C2)/4.
E_{SN}(3,0)=(-((F_{SN}(2,1,(-1)))#(LM+LP))#(H+2)*C2)/4.
E_{SN}(2,0)=((F_{SN}(2,1,(-1)))#(LM+LP))#(H+2)*C2)/4.
E_{SN}(1,0)=((F_{SN}(2,1,(-1)))#(LM+LP))#(H+2)*C2)/2.

$E_{SN}(0,2)=-((F_{SN}(2,1,(-1)))#(LM+LP))#(H+2)*C2)/4.

5. The contributions stemming from the nonlinear part $\tilde{x}_e$ are denoted as $E_{00}(I,J), E_{02}(I,J), E_{SN}(I,J), E_{CP}(I,J), E_{CM}(I,J)$, and ECP(I,J), respectively.
Derivation of equations using REDUCE
E00(1,2)=(((FSLAC(l,0,(-l))^L*L^2+FSLAC(0,l,1)•LP^2)*(XIB2+XI02)•LL*C2-(FCNMLACl,1,1)+FCNHLA(1,1,(-1)))*(LM+LP)*H^2*(LN-LL)C2+FSNC(l,0,(-1))H^2*LN*C2+FSNC(0,1,1)•LN*L^2•C2+FCLBC(l,0,(-l))H^2*LL*^2)*H*DELNL)/4.

E00(0,2)=(((FSNC{1,0,(-1)}LM^2+FSNC(0,1,1)•LP^2)•(XIB2+XI02)•LN^2*C2+FCLB(1,0,(-1))LH^2*LL^2-FCLB(0,1,1)•LP^2*LL^2)*H)/4.

E00(0,4)=C(FCLBC(1,0,(-1))LM^2-FCLBC(0,1,1)•LP^2)H*LL/L2*C2)/4.

6. The contributions stemming from the nonlinear part \( \tilde{N}_y(W_{yy}+W_{yy}) \) are denoted as \( E_{00(I,J)}, E_{02(I,J)}, E_{SM(I,J)}, E_{SP(I,J)}, ECM(I,J), \) and \( ECP(I,J), \) respectively.
The contributions stemming from the nonlinear part $2N_{xy}(W_{xy} + W_{yx})$
$E_{00}(I,J), E_{02}(I,J), E_{SM}(I,J), E_{SP}(I,J), E_{CM}(I,J),$ and $E_{CP}(I,J),$ respectively.

7. The contributions stemming from the nonlinear part $2N_{xy}(W_{xy} + W_{yx})$
$e_{yy}^{(0)}(i,j), e_{yz}^{(0)}(i,j), e_{zz}^{(0)}(i,j), e_{yy}^{(1)}(i,j), e_{yz}^{(1)}(i,j), e_{zz}^{(1)}(i,j), e_{yy}^{(2)}(i,j), e_{yz}^{(2)}(i,j), e_{zz}^{(2)}(i,j))$ are denoted as $E_{00}(IJ), E_{02}(IJ), E_{SM}(IJ), E_{SP}(IJ), E_{CM}(IJ),$ and $E_{CP}(IJ),$ respectively.
The coefficients $e^{(i,j)}$, $e^{m(i,j)}$, and $e^{p(i,j)}$ are denoted in FORTRAN notation, as $E00(i,j)$ $E02(i,j)$ $ESM(i,j)$ and $ESP(i,j)$, respectively.

By application of the Galerkin procedure, a set of two nonlinear differential equations in time for the unknown amplitudes $A$ and $B$ is obtained. The Galerkin procedure involves the evaluation of integrals over the shell surface. The weighting functions used in the Galerkin method are the assumed displacement modes,

\[
\begin{align*}
\pi_{ij} = & M \sin \xi \cos \eta (y - \tau_k x) + (M||)^1 A + (M||)^2 \sin (\eta - \xi x)}  \\
\frac{d^A}{dA} = & h \sin \frac{m \pi x}{L} \cos \frac{n \pi y}{L} \\
\frac{d^B}{dB} = & h \sin \frac{m \pi y}{L} \cos \frac{n \pi x}{L}
\end{align*}
\]

The equations resulting from the Galerkin procedure can be written as follows,

\[
\begin{align*}
\frac{\partial^A}{\partial A} + C_{10} A + C_{11} B + C_{12} A B + C_{13} A^2 + C_{14} B^2 = c_{15} \cos \omega t & (G.53) \\
\frac{\partial^B}{\partial B} + C_{20} A + C_{21} B + C_{22} A^2 + C_{23} B^2 = 0 & (G.54)
\end{align*}
\]

The coefficients of these equations have been coded in FORTRAN. The coefficients $\gamma_{ij}$ and $c_{ij}$ of the first equation (the equation corresponding to first weighting function in Eq. 3.35) are stored in the array $D1(i,j)$ and the coefficients $\delta_{ij}$ and $d_{ij}$ of the second equation (corresponding to the second weighting function) in the array $D2(i,j)$. The indices $i$ and $j$ of the twodimensional arrays $D1(i,j)$ and $D2(i,j)$ refer to the exponents $i$ and $j$ of the corresponding term $A^i B^j$. Introducing the following coefficients, in FORTRAN notation,
the coefficients of the differential equations $c_{ij}$ and $d_{ij}$ are defined as follows, in FORTRAN notation:

\[
\begin{align*}
D1(1,0) &= 0.5 \cdot CASM \cdot ESM(1,0) + 0.5 \cdot CASP \cdot ESP(1,0) \\
D2(1,0) &= 0.5 \cdot CASM \cdot ESM(1,0) + 0.5 \cdot CASP \cdot ESP(1,0) \\
D1(2,0) &= 0.5 \cdot CASM \cdot ESM(2,0) + 0.5 \cdot CASP \cdot ESP(2,0) \\
D2(2,0) &= 0.5 \cdot CASM \cdot ESM(2,0) + 0.5 \cdot CASP \cdot ESP(2,0) \\
D1(3,0) &= 0.5 \cdot CASM \cdot ESM(3,0) + 0.5 \cdot CASP \cdot ESP(3,0) \\
D2(3,0) &= 0.5 \cdot CASM \cdot ESM(3,0) + 0.5 \cdot CASP \cdot ESP(3,0) \\
D1(4,0) &= 0.5 \cdot CASM \cdot ESM(4,0) + 0.5 \cdot CASP \cdot ESP(4,0) \\
D2(4,0) &= 0.5 \cdot CASM \cdot ESM(4,0) + 0.5 \cdot CASP \cdot ESP(4,0) \\
D1(1,1) &= 0.5 \cdot CASM \cdot ESM(1,1) + 0.5 \cdot CASP \cdot ESP(1,1) \\
D2(1,1) &= 0.5 \cdot CASM \cdot ESM(1,1) + 0.5 \cdot CASP \cdot ESP(1,1) \\
D1(2,1) &= 0.5 \cdot CASM \cdot ESM(2,1) + 0.5 \cdot CASP \cdot ESP(2,1) \\
D2(2,1) &= 0.5 \cdot CASM \cdot ESM(2,1) + 0.5 \cdot CASP \cdot ESP(2,1) \\
D1(3,1) &= 0.5 \cdot CASM \cdot ESM(3,1) + 0.5 \cdot CASP \cdot ESP(3,1) \\
D2(3,1) &= 0.5 \cdot CASM \cdot ESM(3,1) + 0.5 \cdot CASP \cdot ESP(3,1) \\
D1(4,1) &= 0.5 \cdot CASM \cdot ESM(4,1) + 0.5 \cdot CASP \cdot ESP(4,1) \\
D2(4,1) &= 0.5 \cdot CASM \cdot ESM(4,1) + 0.5 \cdot CASP \cdot ESP(4,1) \\
D1(1,2) &= 0.5 \cdot CASM \cdot ESM(1,2) + 0.5 \cdot CASP \cdot ESP(1,2) \\
D2(1,2) &= 0.5 \cdot CASM \cdot ESM(1,2) + 0.5 \cdot CASP \cdot ESP(1,2) \\
D1(2,2) &= 0.5 \cdot CASM \cdot ESM(2,2) + 0.5 \cdot CASP \cdot ESP(2,2) \\
D2(2,2) &= 0.5 \cdot CASM \cdot ESM(2,2) + 0.5 \cdot CASP \cdot ESP(2,2) \\
D1(3,2) &= 0.5 \cdot CASM \cdot ESM(3,2) + 0.5 \cdot CASP \cdot ESP(3,2) \\
D2(3,2) &= 0.5 \cdot CASM \cdot ESM(3,2) + 0.5 \cdot CASP \cdot ESP(3,2) \\
D1(4,2) &= 0.5 \cdot CASM \cdot ESM(4,2) + 0.5 \cdot CASP \cdot ESP(4,2) \\
D2(4,2) &= 0.5 \cdot CASM \cdot ESM(4,2) + 0.5 \cdot CASP \cdot ESP(4,2) \\
D1(1,3) &= 0.5 \cdot CASM \cdot ESM(1,3) + 0.5 \cdot CASP \cdot ESP(1,3) \\
D2(1,3) &= 0.5 \cdot CASM \cdot ESM(1,3) + 0.5 \cdot CASP \cdot ESP(1,3) \\
D1(2,3) &= 0.5 \cdot CASM \cdot ESM(2,3) + 0.5 \cdot CASP \cdot ESP(2,3) \\
D2(2,3) &= 0.5 \cdot CASM \cdot ESM(2,3) + 0.5 \cdot CASP \cdot ESP(2,3) \\
D1(3,3) &= 0.5 \cdot CASM \cdot ESM(3,3) + 0.5 \cdot CASP \cdot ESP(3,3) \\
D2(3,3) &= 0.5 \cdot CASM \cdot ESM(3,3) + 0.5 \cdot CASP \cdot ESP(3,3) \\
D1(4,3) &= 0.5 \cdot CASM \cdot ESM(4,3) + 0.5 \cdot CASP \cdot ESP(4,3) \\
D2(4,3) &= 0.5 \cdot CASM \cdot ESM(4,3) + 0.5 \cdot CASP \cdot ESP(4,3) \\
D1(1,4) &= 0.5 \cdot CASM \cdot ESM(1,4) + 0.5 \cdot CASP \cdot ESP(1,4) \\
D2(1,4) &= 0.5 \cdot CASM \cdot ESM(1,4) + 0.5 \cdot CASP \cdot ESP(1,4) \\
\end{align*}
\]

Subsequently, the method of averaging has been used to reduce the set of ordinary differential equations in time to a set of (nonlinear) algebraic equations in the averaged vibration amplitudes $A$ and $B$:

\[
\begin{align*}
(a_{10} - a_{13} \Omega^2) \dot{A} + (a_{01} - a_{03} \Omega^2) \dot{B} + (a_{12} - a_{14} \Omega^2) \dot{B}^2 \\
&+ \alpha_{30} A^3 + \alpha_{23} A B^2 + \alpha_{14} A B^4 = G_{int} \quad (G.55) \\
(b_{01} - b_{03} \Omega^2) \dot{B} + (b_{20} - b_{23} \Omega^2) \dot{A} B + (b_{03} - b_{05} \Omega^2) \dot{B}^3 \\
&+ \beta_{41} A^4 + \beta_{23} A^2 B^2 + \beta_{05} B^5 = 0 \quad (G.56)
\end{align*}
\]

Introducing the following constants, in FORTRAN notation,

\[
\begin{align*}
E00A1 &= 0.5 \cdot RH0BAR \cdot HREF * 2 \cdot 2.0 \cdot C2 \\
E00A2 &= 0.5 \cdot RH0BAR \cdot HREF * 2 \cdot 2.0 \cdot C2 \\
E00A3 &= 0.5 \cdot RH0BAR \cdot HREF * 2 \cdot 2.0 \cdot C2 \\
E00B1 &= 0.5 \cdot RH0BAR \cdot HREF * 2 \cdot 2.0 \cdot C2 \\
E00B2 &= 0.5 \cdot RH0BAR \cdot HREF * 2 \cdot 2.0 \cdot C2 \\
E00A0 &= 0.5 \cdot RH0BAR \cdot HREF * 2 \cdot DELNL + C1 \\
E02A1 &= -0.5 \cdot RH0BAR \cdot HREF * 2 \cdot 2.0 \cdot C2 \\
\end{align*}
\]
the following coefficients are obtained for the algebraic equations (3.40) and (3.40),
where \( a_i, b_j, b_i, \) and \( j_k \) are denoted as \( A1E(I,J), A1H(I,J), A2E(I,J), \) and \( A2I(I,J), \) respectively, in the FORTRAN notation:

C -- first equation

\[
\begin{align*}
A1E(1,0) & = 0.5 \times D1(1,0) \\
A1H(1,0) & = 0.5 \times D1A0 \\
A1E(3,0) & = 3.0 / 8.0 \times D1(3,0)
\end{align*}
\]
G.5 Simplified Analysis - Transient Vibration

The Simplified Analysis for the transient vibrations outlined in Chapter 4 relies on derivations via the symbolic manipulation package REDUCE. The analysis has been coded in the FORTRAN computer program SILVANA (Simplified Large amplitude Vibration Analysis including Anisotropy). For the nonlinear vibration case, the coefficients of the stress function are determined in the case of the Steady State vibration analysis. In the present case, the axisymmetric vibration amplitudes $C_0$ and $C_1$ are generalized coordinates in the formulation.

The solution for $F$ becomes

$$F = F_p + F^*$$

where the complementary solution $F^*$, corresponding to stress resultants constant over the shell, can be written as

$$F^* = \frac{1}{2} N_2 y^2 + \frac{1}{2} N_2 x^2 - N_3 x y$$

The corresponding complementary in-plane displacements $\tilde{u}^*$ and $\tilde{v}^*$, are assumed, for a fixed end at $x = 0$, as

$$\tilde{u}^* = -\left(\frac{C_1 h}{L}\right) x, \quad \tilde{v}^* = -\left(\frac{C_2 h}{L}\right) x$$

where the introduced generalized coordinates $C_0$ and $C_1$ can be related to the spatially constant stress resultants via the boundary conditions for averaged in-plane stress resultants and the periodicity condition:

$$\int_0^{2\pi R} \int_0^L \tilde{u}_{xy} \, dx \, dy = -2 \pi R C_0 = f_{1,lin}(N_2, N_2, N_3) + f_{1,nl}(q_i)$$

$$\int_0^{2\pi R} \tilde{v}_{xy} \, dx \, dy = 0 = f_{2,lin}(N_2, N_2, N_3) + f_{2,nl}(q_i)$$

$$\int_0^{2\pi R} \tilde{v}_{xy} \, dx \, dy = -2 \pi R C_1 = f_{3,lin}(N_2, N_2, N_3) + f_{3,nl}(q_i)$$

where $f_{1,lin}$ are linear functions of the constant stresses, and $f_{1,nl}$ are nonlinear functions of the generalized coordinates $q_i$. The functions are obtained by using the kinematic relations (A.22) in combination with the partially inverted constitutive equations (A.16).

In REDUCE notation,

$$\tilde{u}_{xy} := a_1 s * N_2 + a_2 s * N_3 + a_1 s * N_3 + a_2 s * N_2 + h**2 * c - l/16 * a**2 * l**2 - l/16 * b**2 * l**2 - l/8 * q_1**2 * l**2$$

$$\tilde{v}_{xy} := a_1 s * N_2 + a_2 s * N_3 + a_1 s * N_3 + a_2 s * N_2 + h**2 * c$$

Inverting the relations (4.15) to (4.17) we can obtain the unknown (spatially) constant stress resultants as functions of $C_0$, $C_1$ and $q_i$, gives, in REDUCE notation,

$$N_2 := \frac{1}{16} * (4 * (2 * (x_0 + xib) + cl) * (lm + lp) + 2 * c + 2 * cl * (lm + lp) + 2 * c) * f_{1,nl}(q_i)$$

$$N_3 := \frac{1}{16} * (4 * (2 * (x_0 + xib) + cl) * (lm + lp) + 2 * c + 2 * cl * (lm + lp) + 2 * c) * f_{2,nl}(q_i)$$

$$N_4 := \frac{1}{16} * (4 * (2 * (x_0 + xib) + cl) * (lm + lp) + 2 * c + 2 * cl * (lm + lp) + 2 * c) * f_{3,nl}(q_i)$$

In REDUCE notation,
Substitution of the given imperfection mode, the assumed radial deflections of the 
static state and the solutions for the stress functions \( F \) and \( F \) into the energy expres­
sion, leads to a coupled set of nonlinear ordinary differential equations in the unknown 
vibration amplitudes of the following form:

\[
\ddot{q}_i + c_i \dot{q}_i + k_i q_i + \sum_j \sum_k a_{ij,k} \dot{q}_j q_k + \sum_j \sum_k b_{ij,k} \dot{q}_j q_k q_l = g_i \quad (G.63)
\]

where \( k_i, a_{ij,k}, \) and \( b_{ij,k} \) are coefficients, which can in general depend on time, and \( g_i \) is the forcing term. Further, the damping parameter \( \zeta_i \) can be defined as

\[
\zeta_i = \frac{c_i}{2k_i} \quad (G.64)
\]

In the present case, the six equations correspond to the generalized coordinates \( C_a, C_t, \) \( C_0, C_1, A \) and \( B, \) respectively.

The first two equations correspond to \( C_a \) and \( C_t, \) respectively. In the case that the 
in-plane vibration modes are not used as generalized coordinates (if the inertia of the 
in-plane modes is neglected) \( C_a \) and \( C_t \) can be eliminated.

For the first equation we introduce the following REDUCE notation,

<table>
<thead>
<tr>
<th>REDUCE</th>
<th>REDUCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ddot{q}_1 )</td>
<td>cdadot</td>
</tr>
<tr>
<td>( \dot{q}_1 )</td>
<td>cadot</td>
</tr>
<tr>
<td>( \Omega_{C_0} )</td>
<td>omegaC</td>
</tr>
<tr>
<td>( g_{10} )</td>
<td>-eq10</td>
</tr>
<tr>
<td>( g_{10}^2 )</td>
<td>-eq10X</td>
</tr>
<tr>
<td>( \zeta_1 )</td>
<td>eqld</td>
</tr>
</tbody>
</table>

These coefficients, include the damping coefficient \( eqld \) and the loading terms \( eq10 \) and \( eq10t. \) The equation can be written, in REDUCE notation, as

\[
cdadot :=
\begin{align*}
-\text{eq10} \\
-\text{eq10t} \times \cos(\omegaC \cdot t) \\
-\text{eqld} + \text{cadot}
\end{align*}
\]

\[
\text{+eq1lm}(1,0,0) \times \text{cad} \\
\text{+eq1lm}(0,1,0) \times \text{ctd} \\
\text{+eq1lm}(0,0,1) \times \text{cd} \\
\text{+eq1lf}(1,0,0) \times \text{cl} \\
\text{+eq1lf}(0,1,0) \times \text{ad} \\
\text{+eq1lf}(0,0,1) \times \text{bd}
\end{align*}
\]

\[
\text{+eq1lm}(0,0,0) \times \text{cad} \times \text{cl} \\
\text{+eq1lm}(0,0,0,1) \times \text{cad} \times \text{ad} \\
\text{+eq1lm}(0,0,0,1) \times \text{cad} \times \text{bd} \\
\text{+eq1lm}(0,0,1,0,0) \times \text{ctd} \times \text{cl} \\
\text{+eq1lm}(0,0,1,0,0) \times \text{ctd} \times \text{ad} \\
\text{+eq1lm}(0,0,1,0,0) \times \text{ctd} \times \text{bd}
\end{align*}
\]

\[
\text{+eq1qfs}(1,1,0,0,0) \times \text{cl} \times \text{ad} \\
\text{+eq1qfs}(1,0,1,0,0) \times \text{cl} \times \text{bd} \\
\text{+eq1qfs}(1,0,1,0,0) \times \text{ad} \times \text{bd} \\
\text{+eq1qfs}(2,0,0,0,0) \times \text{cl} \times \text{ad} \times \text{bd} \\
\text{+eq1qfs}(2,0,0,0,0) \times \text{ad} \times \text{bd} \times \text{bd} \\
\text{+eq1qfs}(2,1,0,0,0) \times \text{cl} \times \text{ad} \times \text{bd} \\
\text{+eq1qfs}(2,1,0,0,0) \times \text{ad} \times \text{bd} \times \text{bd}
\end{align*}
\]
The other five equations are written in a similar form (replacing eqlO, eqlOt, ..., eqlcs, by eq20, eq20t, ..., eq2cs, etc.). The coefficient of these equations have been converted from REDUCE notation to FORTRAN code. These coefficients are defined as follows, in FORTRAN notation:

eqld = (6.*sqrt(omg2rl)*cbarca*h*pi*r)/(omg2rl*(2.*all*r*rhobar+3.*ainass))
eq2d = (6.*eref*h*pi*r**2*rcapcp)/(c+omg2rl*(3.*ainass+2.*all*h*r*rhobar))
eq3d = (sqrt(omg2rl)*(ainf*cbar*h+gainma*pinf))/(ainf*b*rhobar)
eq4d = (sqrt(omg2rl)*(ainf*cbar*b+gainma*pinf))/(ainf*b*rhobar)
eq5d = (sqrt(omg2rl)*(ainf*cbar*h+gainma+pinf))/(ainf*h*rhobar)
eq6d = (sqrt(omg2rl)*(ainf*cbar*h+gainma+pinf))/(ainf*h*rhobar)
eql0t = (-6.*alabcp*eref*h*pi)/((2.*all*h*r*rhobar+3.*ainass)*all*omg2rl)
eq20t = (6.*eref*h*pi*r**2*rcapcp)/(c+omg2rl*(3.*ainass+2.*all*h*r*rhobar))
eq30t = (-eref*pbarcp)/(c+omg2rl*(3.*ainass+2.*all*h*r*rhobar))
eq40t = (-qqcpcl)/(h**2*omg2rl*rhobar)
eq50t = (-qqcpa)/(h**2*omg2rl*rhobar)
eq60t = (-qqcpb)/(h**2*omg2rl*rhobar)
eql0 = (-6.*alabdy*eref*h*pi)/(2.*all*h*r*rhobar)
eqlm(l,0,0,l,0,0) = (-3.*al2*a22s+2.*a22s*a22s+2.*a22s*a22s)*pi*r/(4.*(2.*all*h*r*rhobar+3.*ainass)*all*omg2rl)
eqlm(l,0,0,0,l,0,0) = (-3.*al2*a22s+2.*a22s*a22s+2.*a22s*a22s)*pi*r/(4.*(2.*all*h*r*rhobar+3.*ainass)*all*omg2rl)
eqlm(l,0,0,0,0,l) = (-3.*al2*a22s+2.*a22s*a22s+2.*a22s*a22s)*pi*r/(4.*(2.*all*h*r*rhobar+3.*ainass)*all*omg2rl)
eq2cfs(2,1,0,0,0,0)=0.
eq2cfs(2,0,1,0,0,0)=0.
eq2cfs(0,2,1,0,0,0)=0.
eq2cfs(0,0,2,1,0,0)=0.
eq2cfs(3,0,0,0,0,0)=0.
eq2cfs(0,3,0,0,0,0)=0.
eq2cfs(0,0,3,0,0,0)=0.
eq2cfs(2,0,0,1,0,0)=0.
eq2cfs(2,0,0,0,1,0)=0.
eq2cfs(2,0,0,0,0,1)=0.
eq2cfs(0,2,0,1,0,0)=0.
eq2cfs(0,2,0,0,1,0)=0.
eq2cfs(0,2,0,0,0,1)=0.
eq2cfs(0,0,2,1,0,0)=0.
eq2cfs(0,0,2,0,1,0)=0.
eq2cfs(0,0,2,0,0,1)=0.
eq2cfs(0,0,0,2,1,0)=0.
eq2cfs(0,0,0,2,0,1)=0.
eq2cfs(0,0,0,1,2,0)=0.
eq2cfs(0,0,0,1,0,1)=0.
eq2cfs(0,0,0,0,2,0)=0.
eq2cfs(0,0,0,0,1,1)=0.
eq2cfs(0,0,0,0,0,2)=0.
eq2cfs(1,2,0,0,0,0)=0.
eq2cfs(1,0,2,0,0,0)=0.
eq2cfs(2,1,0,0,0,0)=0.
eq2cfs(2,0,1,0,0,0)=0.
eq2cfs(0,2,1,0,0,0)=0.
eq2cfs(0,0,2,1,0,0)=0.
eq2cfs(0,0,0,2,0,0)=0.
eq2cfs(0,0,0,0,0,0)=0.
eq2cfs(2,0,0,2,0,0)=0.
eq2cfs(2,0,0,0,2,0)=0.
eq2cfs(2,0,0,0,0,2)=0.
eq2cfs(0,2,0,2,0,0)=0.
eq2cfs(0,2,0,0,2,0)=0.
eq2cfs(0,2,0,0,0,2)=0.
eq2cfs(0,0,2,2,0,0)=0.
eq2cfs(0,0,2,0,2,0)=0.
eq2cfs(0,0,2,0,0,2)=0.
eq2cfs(0,0,0,2,2,0)=0.
eq2cfs(0,0,0,2,0,2)=0.
eq2cfs(0,0,0,0,2,2)=0.
eq2cfs(0,0,0,0,2,0)=0.
eq2cfs(0,0,0,0,0,2)=0.
eq2cfs(0,0,0,0,0,0)=0.
eq2cfs(2,0,2,0,0,0)=0.
eq2cfs(2,0,0,2,0,0)=0.
eq2cfs(2,0,0,0,2,0)=0.
eq2cfs(2,0,0,0,0,2)=0.
eq2cfs(1,2,0,2,0,0)=0.
eq2cfs(1,2,0,0,2,0)=0.
eq2cfs(1,2,0,0,0,2)=0.
eq2cfs(1,0,2,2,0,0)=0.
eq2cfs(1,0,2,0,2,0)=0.
eq2cfs(1,0,2,0,0,2)=0.
eq2cfs(1,0,0,2,2,0)=0.
eq2cfs(1,0,0,2,0,2)=0.
eq2cfs(1,0,0,0,2,2)=0.
eq2cfs(1,0,0,0,2,0)=0.
eq2cfs(1,0,0,0,0,2)=0.
eq2cfs(1,0,0,0,0,0)=0.
eq2cfs(0,2,2,0,0,0)=0.
eq2cfs(0,2,0,2,0,0)=0.
eq2cfs(0,2,0,0,2,0)=0.
eq2cfs(0,2,0,0,0,2)=0.
eq2cfs(0,0,2,2,0,0)=0.
eq2cfs(0,0,2,0,2,0)=0.
eq2cfs(0,0,2,0,0,2)=0.
eq2cfs(0,0,0,2,2,0)=0.
eq2cfs(0,0,0,2,0,2)=0.
eq2cfs(0,0,0,0,2,2)=0.
eq2cfs(0,0,0,0,2,0)=0.
eq2cfs(0,0,0,0,0,2)=0.
eq2cfs(0,0,0,0,0,0)=0.
eq2cfs(0,0,0,0,0,0)=0.
eq2cfs(0,0,0,0,0,0)=0.
eq2cfs(0,0,0,0,0,0)=0.
eq2cfs(0,0,0,0,0,0)=0.
eq2cfs(0,0,0,0,0,0)=0.
eq2cfs(0,0,0,0,0,0)=0.
eq2cfs(0,0,0,0,0,0)=0.
eq2cfs(0,0,0,0,0,0)=0.
eq2cfs(0,0,0,0,0,0)=0.
eq2cfs(0,0,0,0,0,0)=0.
...
The six differential equations can now be written in the following form:

\[
\begin{align*}
\text{caddot} := & -\text{eq10} - \text{eq10t} \cdot \cos(\text{omegca} \cdot \text{tm}) - \text{eq1id} \cdot \text{cadot} \\
& + \text{eq1lm}(l,0,0) \cdot \text{cad} + \text{eq1lm}(0,l,0) \cdot \text{ctd} + \text{eq1lm}(0,0,l) \cdot \text{ctd} \\
& + \text{eq1f}(1,0,0) \cdot \text{cld} + \text{eq1f}(0,1,0) \cdot \text{ad} + \text{eq1qfs}(2,0,0,0,0,0) \cdot \text{cld}^2 - \text{eq1qfs}(0,2,0,0,0,0) \cdot \text{ad}^2 - \text{eq1qfs}(0,0,2,0,0,0) \cdot \text{bd}^2 \\
& + \text{eq1f}(2,0,1,0,0,0) \cdot \text{ad} \cdot \text{rhobar} \end{align*}
\]

\[
\begin{align*}
\text{ctddot} := & -\text{eq20} - \text{eq20t} \cdot \cos(\text{omegct} \cdot \text{tm}) - \text{eq2d} \cdot \text{ctdot} \\
& + \text{eq2lm}(1,0,0) \cdot \text{cad} + \text{eq2lm}(0,1,0) \cdot \text{ctd} + \text{eq2lm}(0,0,1) \cdot \text{ctd} \\
& + \text{eq2f}(1,0,0) \cdot \text{eld} + \text{eq2f}(0,1,0) \cdot \text{ad} + \text{eq2qfs}(2,0,0,0,0,0) \cdot \text{cld}^2 - \text{eq2qfs}(0,2,0,0,0,0) \cdot \text{ad}^2 - \text{eq2qfs}(0,0,2,0,0,0) \cdot \text{bd}^2 \\
& + \text{eq2f}(2,0,1,0,0,0) \cdot \text{ad} \cdot \text{rhobar} \end{align*}
\]

\[
\begin{align*}
& + \text{eq2lm}(3,0,0,0,0,0) \text{eq2f}(3,0,0,0,0,0) \text{eq2f}(0,3,0,0,0,0) \text{eq2qfs}(0,3,0,0,0,0) \text{eq2f}(0,0,3,0,0,0) \text{eq2qfs}(0,0,3,0,0,0) \text{eq2f}(0,0,0,3,0,0) \text{eq2qfs}(0,0,0,3,0,0) \text{eq2f}(0,0,0,0,3,0) \text{eq2qfs}(0,0,0,0,3,0) \text{eq2f}(0,0,0,0,0,3) \text{eq2qfs}(0,0,0,0,0,3) \text{eq2f}(0,0,0,0,0,0) \text{eq2qfs}(0,0,0,0,0,0)
\end{align*}
\]
\begin{align*}
  c\ddot{q} & = -c_30 \cdot \cos(\omega c_0 t) + c_4 \cdot c_\dot{q}\nonumber \\
  + & c_1 m(l,0,0) \cdot c_\dot{a} + c_1 m(0,l,0) \cdot c_\dot{d} + c_2 m(0,0,1) \cdot c_\dot{c} 
onumber \\
  + & c_1 f(1,0,0) \cdot c_\dot{a} + c_1 f(0,1,0) \cdot c_\dot{d} 
onumber \\
  + & c_3 q f_s(2,0,0,0,0,0) \cdot c_\dot{a}^2 + c_3 q f_s(0,2,0,0,0,0) \cdot c_\dot{d}^2 + c_3 q f_s(0,0,2,0,0,0) \cdot b^2 
\end{align*}

\begin{align*}
  c_\ddot{d} & = -c_7 \cdot \cos(\omega c_1 t) + c_4 \cdot c_\dot{d}\nonumber \\
  + & c_1 m(l,0,0) \cdot c_\dot{a} + c_1 m(0,l,0) \cdot c_\dot{d} + c_2 m(0,0,1) \cdot c_\dot{c} 
onumber \\
  + & c_1 f(1,0,0) \cdot c_\dot{a} + c_1 f(0,1,0) \cdot c_\dot{d} + c_2 q f_s(1,2,0,0,0,0) \cdot c_\dot{a} \cdot c_\dot{d} + c_2 q f_s(1,0,2,0,0,0) \cdot c_\dot{a}^2 + c_2 q f_s(0,0,2,0,0,0) \cdot c_\dot{d}^2 + c_2 q f_s(0,2,0,0,0,0) \cdot b^2 
\end{align*}

\begin{align*}
  \ddot{a} & = -c_5 \cdot \cos(\omega a t) + c_4 \cdot \ddot{a}
\end{align*}
\[ \begin{align*}
+\text{eq6cfs}(0,2,0,0,0)\cdot\text{bd}^2 & \\
+\text{eq6cfs}(2,1,0,0,0)\cdot\text{c1d}^2 & \\
+\text{eq6cfs}(0,1,2,0,0)\cdot\text{ad}^2 & \\
+\text{eq6cfs}(0,3,0,0,0)\cdot\text{ad}^3 & \\
) ;
\]

\[
\text{bd}\dot{\text{dot}} := \\
( \\
-\text{eq60} \\
-\text{eq60t} \cos(\text{omegbtms}) \\
-\text{eq6d} = \text{bd}\dot{\text{dot}} \\
+\text{eq6lf}(0,0,1)\cdot\text{bd} \\
+\text{eq6qnf}(1,0,0,0,0)\cdot\text{cad} & \\
+\text{eq6qnf}(1,0,0,0,1)\cdot\text{ctd} & \\
+\text{eq6qnf}(0,0,1,0,0)\cdot\text{c0d} & \\
+\text{eq6qfs}(1,0,1,0,0)\cdot\text{c1d} & \\
+\text{eq6qfs}(0,1,1,0,0)\text{ad} & \\
+\text{eq6cfs}(2,0,1,0,0)\text{c1d}^2 & \\
+\text{eq6cfs}(0,2,1,0,0)\text{ad}^2 & \\
+\text{eq6cfs}(0,2,1,0,0)\cdot\text{bd}^2 \\
+\text{eq6cfs}(0,3,0,0,0)\cdot\text{bd}^3 & \\
) ;
\]
G.6

Extended Analysis

In this section, the derivations for the Extended Analysis are shown. T h e analysis has
been coded in the FORTRAN computer program BIANCA (Bifurcation ANalysis of
Cylinders including Anisotropy).

G.6.1

Governing equations

'I he governing equations, the equilibrium equation and the compatibility equation, for
the first order dynamic state, second order dynamic state, imperfect dynamic state,
and coupled mode dynamic state have been derived via REDUCE.
The equations in REDUCE notation are as follows,
dynamic s t a t e S
dynamic f i r s t - o r d e r

state$

equileqdlOO := b l l s » d f ( f f c l , x , 2 , y , 2 ) + b l 2 s « d f ( f f c l , y . 4 ) + 2»bl6s*df(
f f c l , x , y , 3 ) + dlls*dfCw»cl,x,4) + d l 2 s * d f ( w w c l , x , 2 , y , 2 ) + 2*dl6s»
d f ( » w c l , x , 3 , y ) + b 2 1 s * d f ( f f c l , x , 4 ) + b 2 2 s * d f ( f f c l , x , 2 , y , 2 ) + 2»b26s«
d f ( f f c l , x , 3 , y ) + d 2 1 s » d f ( ™ c l , x , 2 , y , 2 ) + d22s«df(wwcl,y,4) + 2»d26s«
d f ( u w c l , x , y , 3 ) - b 5 1 s * d f ( f f c l , x , 3 , y ) - b 6 2 s * d f ( f f c l , x , y , 3 ) - 2»b66s*
d f ( f f c l , x , 2 , y , 2 ) + 2 * d 6 1 s » d f ( w c l , x , 3 , y ) + 2»d62s*df(uwcl,x,y,3) + 4»
d66s«df(w»cl,x,2,y,2) - df(f0,x,2)*df(irecl,y,2)«c»*(-l)»eref*h»»2»r df(ffcl,x,2)*r»«(-l) - df(ffcl,y,2)«df(wO,x,2)»h - df(ffcl,y,2)«df(
wbar,x,2)*h + dfCwwcl,tm,2)*h*rhobar - 2*dfCwwcl,x,y)*c**(-l)*erGf
*h**2*r**C-l)*rcapt + df(wwcl,x,2)*alambda*c**(-l)*erGf*h**2*r*+C-1)
+ df(wwcl,x)*aminf*gamma*pinf + df(wwcl,y,2)*c+*(-l)*Gref*h**2*pbare*
r»«(-l)$
compeqdlOO := a l l s » d f (f f c l ,y ,4) + 2*al2s«df (ff c l , x , 2 , y , 2 ) - 2«al6s»df
( f f c l , x , y . 3 ) + a 2 2 s t d f ( f f c l , x , 4 ) - 2 « a 2 6 s * d f ( f f c l , x , 3 , y ) + a66s*df(
f f c l , x , 2 , y , 2 ) - b l l s * d f ( u u c l , x , 2 , y , 2 ) - b l 2 s * d f ( u u c l , y , 4 ) - 2*bl6s
» d f ( » w c l , x , y , 3 ) - b 2 1 s » d f ( ™ c l , x , 4 ) - b 2 2 s * d f ( u w c l , x , 2 , y , 2 ) - 2*b26s*
df(wwcl,x,3,y) + b61s*df(wwcl,x,3,y) + b62s*df(wwcl,x,y,3) + 2*b66s*
d f ( u » c l , x , 2 , y , 2 ) + df(w0,x,2)«df(i™cl,y,2)«h + df(wbar,x,2)«df(uucl,y
,2)»h + d f ( u u c l , x , 2 ) * r » » ( - l ) $
dynamic second-order s t a t e !
aquileqd200 := b l l s t d f (f f c 2 , x , 2 , y , 2 ) + bl2s»df (f f c2,y ,4) + 2*bl6s»df(
f f c 2 , x , y , 3 ) + d l l s t d f (»uc2,x,4) + dl2s«df ( u u c 2 , x , 2 , y ,2) + 2*dl6s»
df ( u u c 2 , x , 3 , y ) + b21s*df (f f c 2 , x , 4 ) + b22s»df (f f c 2 , x , 2 , y ,2) + 2»b26s»
df ( f f c 2 , x , 3 , y ) + d21s«df ( u u c 2 , x , 2 , y , 2 ) + d22s«df (uuc2,y,4) + 2«d26s*

df ( u u c 2 , x , y , 3 ) - b61s»df ( f f c 2 , x , 3 , y ) - b62s*df (f f c 2 , x , y ,3) - 2«b66s»
d f ( f f c 2 , x , 2 , y , 2 ) + 2*d61s*df(uuc2,x,3,y) + 2»d62s*df(uuc2,x,y,3) + 4»
d66s»df(u¥c2,x,2,y,2) - df(f0,x,2)«df(uuc2,y,2)*c*»(-l)*eref•h»*2*r +
2*df(ffcl,x,y)»df(uucl.x.y) - df(ffcl,x,2)»df(uucl,y,2) - df(ffcl,y,
2)«df(uucl,x,2) - df(ffc2,x,2)»r*»(-l) - df(ffc2,y,2)*df(u0,x,2)»h
- df(ffc2,y,2)*df(ubar,x,2)*h + df(wuc2,tm,2)*h+rhobar - 2*df(uuc2,x
,y)*c**(-l)*eref*h*+2*r**(-l)*rcapt + df(uwc2,x,2)*alambda*c**(-l)
*Gref*h**2*r**(-l) + df(wuc2,x)*aminf*gamma*pinf + df(uuc2,y,2)*c**(
-l)*Gref*h**2*pbare*r+*(-l) - h*omgn21*omgpn2*rhobar*u¥Cl$
compeqd200 :- alls«df (f f c2,y,4) + 2*al2s»df (f f c2,x.2 ,y,2) - 2tal6s*df
(ffc2,x,y,3) + a22s»df(ffc2,x,4) - 2«a26s»df(ffc2,x,3,y) + a66s»df{
ffc2,x,2,y,2) - blls«df (uuc2,x,2,y,2) - bl2stdf (uuc2,y ,4) - 2»bl6s
•df (uuc2,x,y.3) - b21s»df (uuc2,x,4) - b22s»df (uuc2,x,2.y,2) - 2»b26s*
df (uuc2,x,3,y) + b61s*df (uwc2,x,3,y) + b62s*df (wuc2,x,y ,3) + 2*b66s*
df(uuc2,x,2,y,2) + df(uO,x,2)*df(uuc2,y,2)*h + df(wbar,x,2)*df(wuc2,y
,2)»li - df(uucl,x,y)*»2 + df(uucl,x,2)»df(uucl,y,2) + df(uuc2,x,2)
•r*»(-l)$
imperfect dynamic state$
equileqdllO := blls*df(ffcU,x,2,y,2) + bl2s«df(ffell,y,4) + 2*bl6s«
df(ffcll,x,y,3) + dlls*df(uucll,x,4) + dl2s«df(uucll,x,2,y,2) + 2*
dl6s»df(uucll,x,3,y) + b21s»df(ffell,x,4) + b22s»df(ffell,x,2,y,2) +
2*b26s*df(ffcll,x,3,y) + d21s*df(uucll,x,2,y,2) + d22s*df(uucll,y,4)
+ 2«d26s*df(uucll,x,y,3) - b61s*df(ffell,x,3,y) - b62s»df(ffell,x,y,3
) - 2»b66s»df(ffcll,x,2,y,2) • 2«d61s»df(uucll,x,3,y) + 2*d62s»df(
uucll,x,y,3) + 4»d66s*df(»ucll,x,2,y.2) - df(f0,x,2)«df(uucll,y,2)
• c*»(-l)*oref*h«»2»r + 2»df(ff 1,x,y)»df(uucl,x,y) - df(ff 1,x,2)«df(
uucl,y,2) - df(ffl,y,2)»df(uucl,x,2) + 2*df(ffcl,x,y)«df(uul,x,y)
- df(ffcl.x,2)«df(uul,y,2) - df(ffcl,y,2)*df(uul,x,2) - df(ffell,x,2)
»r**(-l) - df(ffcll,y,2)»df(u0,x,2)*h - df(ffcll,y,2)»df(ubar,x,2)*h
+ df(uucll,tm,2)*h*rhobar - 2*df(uucll,x,y)*c**(-l)*eref*h**2*r**(-l)
*rcapt + df(uwcll,x,2)*alambda*c**(-l)*eref*h**2*r**(-l) + df(uwcll,x
)*aminf*ganmia*pinf + df(uucll,y,2)*c**(-l)*erGf*h**2*pbare*r**(-l)$
compeqdllO := alls»df(ffell,y,4) + 2*al2s»df(ffell,x,2,y,2) - 2»al6s*
df(ffcll,x,y,3) + a22s»df(ffcll,x,4) - 2*a26stdf(ffell,x,3,y) + a66s*
df(ffcll,x,2,y,2) - blls»df(U¥cll,x,2,y,2) - bl2s*df(uucll,y,4) - 2*
bl6s*df(u¥cll,x,y,3) - b21s*df(uucll,x,4) - b22s«df(uucll,x,2,y,2) 2»b26s*df(uucll,x,3,y) + b61s»df(uucll,x,3,y) + b62s*df(uucll,x,y,3)
+ 2»b66s*df(uucll,x,2,y,2) + df(uO,x,2)*df(uucll,y,2)*h + df(ubar,x,2
)*df(uucll,y,2)*h - 2*df(uul,x,y)*df(uucl.x.y) + df(uul,x,2)*df(uucl,
y,2) • df(uul,y,2)*df(uucl,x,2) + df(uucll,x,2)tr»»(-l)$


mixed dynamic second-order state

\[ \text{equivdcl2} := \text{bl}1\text{as} \text{df}(\text{ffcl}2, x, 2, y, 2) + \text{bl}2\text{as} \text{df}(\text{ffcl}2, y, 2) + 2 \text{bl}6\text{as} \text{df}(\text{ffcl}2, x, 2, y, 3) + \text{dl}1\text{as} \text{df}(\text{wc}12, x, 2, y, 2) + 2 \text{dl}6\text{as} \text{df}(\text{wc}12, x, 3, y, 3) + \text{bl}1\text{as} \text{df}(\text{ffcl}2, x, 2, y, 2) + \text{bl}2\text{as} \text{df}(\text{ffcl}2, x, 2, y, 3) + \text{bl}6\text{as} \text{df}(\text{ffcl}2, x, 3, y, 3) + 2 \text{dl}1\text{as} \text{df}(\text{wc}12, x, 2, y, 3) + 2 \text{dl}6\text{as} \text{df}(\text{wc}12, x, 3, y, 3) + 2 \text{bl}6\text{as} \text{df}(\text{ffcl}2, x, 2, y, 3) + 2 \text{dl}6\text{as} \text{df}(\text{wc}12, x, 2, y, 3) + 2 \text{bl}6\text{as} \text{df}(\text{ffcl}2, x, 3, y, 3) + 2 \text{dl}6\text{as} \text{df}(\text{wc}12, x, 3, y, 3) + 2 \text{bl}6\text{as} \text{df}(\text{ffcl}2, x, 2, y, 3) + 2 \text{dl}6\text{as} \text{df}(\text{wc}12, x, 2, y, 3) + 2 \text{bl}6\text{as} \text{df}(\text{ffcl}2, x, 3, y, 3) + 2 \text{dl}6\text{as} \text{df}(\text{wc}12, x, 3, y, 3) + 2 \text{bl}6\text{as} \text{df}(\text{ffcl}2, x, 2, y, 3) + 2 \text{dl}6\text{as} \text{df}(\text{wc}12, x, 2, y, 3) + 2 \text{bl}6\text{as} \text{df}(\text{ffcl}2, x, 3, y, 3) + 2 \text{dl}6\text{as} \text{df}(\text{wc}12, x, 3, y, 3)
\]

The equations of the dynamic first-order state admit separable solutions of the form

**Dynamic first-order state**

\[ \dot{W}^{(1)} = W^{(1)} \cos \omega t = h \left( \dot{\psi}_1(x) \cos \omega t + \dot{\psi}_2(x) \sin \omega t \right) \cos \omega t \] (G.65)

\[ \dot{F}^{(1)} = F^{(1)} \cos \omega t = \frac{ERh^2}{c} \left( \dot{f}_1(x) \cos \omega t + \dot{f}_2(x) \sin \omega t \right) \cos \omega t \] (G.66)

where \( \omega = \gamma / R \) and \( n \) is the number of circumferential waves.

The effect of asymmetric imperfections on the small amplitude vibrations is governed by the equations of the 'dynamic imperfect' state (\( \xi \text{,}_\text{e}-\text{terms} \)). They admit separable solutions of the form

**Dynamic imperfect state (\( \xi \text{,}_\text{e}-\text{terms} \))**

\[ \dot{W}^{(\text{II})} = \dot{W}^{(\text{I})} + \dot{W}^{(\text{II})} \]

\[ \dot{F}^{(\text{II})} = \frac{ERh^2}{c} \left\{ \dot{f}_1(x) \cos \omega t + \dot{f}_2(x) \sin \omega t \right\} \cos \omega t \] (G.67)

\[ \dot{F}^{(\text{II})} = \frac{ERh^2}{c} \left\{ \dot{f}_1(x) \cos \omega t + \dot{f}_2(x) \sin \omega t \right\} \cos \omega t \] (G.68)

and the initial nonlinearity of the large amplitude vibrations is governed by the equations of the dynamic second-order state (\( \xi \text{,}_\text{e}-\text{terms} \)), which admit separable solutions of the form

**Dynamic 2nd-order state (\( \xi \text{,}_\text{e}-\text{terms} \))**

\[ \dot{W}^{(\text{II})} = \dot{W}^{(\text{I})} + \dot{W}^{(\text{II})} \]

\[ \frac{ERh^2}{c} \left\{ \dot{f}_1(x) \cos \omega t + \dot{f}_2(x) \sin \omega t \right\} \cos \omega t \] (G.69)

\[ \dot{F}^{(\text{II})} = \frac{ERh^2}{c} \left\{ \dot{f}_1(x) \cos \omega t + \dot{f}_2(x) \sin \omega t \right\} \cos \omega t \] (G.70)

The Coupled Mode vibration case involves an asymmetric mode, the driven mode, and a mode which is circumferentially 90 degrees out-of-phase with respect to this mode, the companion mode. The expansion for the displacement is assumed as

\[ \dot{W} = \xi_1 \dot{W}^{(\text{I})} + \xi_2 \dot{W}^{(\text{II})} + \xi_1 \xi_2 \dot{W}^{(\text{III})} + \ldots \] (G.71)

where the driven mode and the companion mode can be written as, respectively,
The equations governing the mixed coupled mode state admit solutions of the form

\[\hat{W}^{(1)}(t) = W^{(1)}(t) \cos \omega t\]  
(G.72)

\[\hat{W}^{(2)}(t) = W^{(1)}(t) \sin \omega t\]  
(G.73)

The assumptions for the assumed deflections, are substituted into the governing differential equations. For every problem type, a set of four differential equations can be derived. Each equation of this set can be written in a general form.

In the following, the indices \(i_0\) and \((i_1,j_1)\) indicate the problem type:

- \(i_0, j_0 = (10,0)\): 1st-order state, static
- \(i_0, j_0 = (20,0)\): 2nd-order state, static
- \(i_0, j_0 = (110,1)\): 1st-order state, dynamic
- \(i_0, j_0 = (120,0)\): 2nd-order state, dynamic, 0th-harmonic in time
- \(i_0, j_0 = (120,2)\): 2nd-order state, dynamic, 2nd-harmonic in time
- \(i_0, j_0 = (112,2)\): mixed 2nd-order state, dynamic, 2nd-harmonic in time
- \(i_0, j_0 = (111,1)\): xit-xiv state, dynamic, 1st-harmonic in time

The coefficients in the differential equations, are determined via REDUCE and are listed in a FORTRAN notation.

For the first-order static state:

\[c_{10} = 0.\]

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For the first-order dynamic state the coefficients become:

\[ c_{il-110} = 0. \]

\[ c_{jl+1} = 0. \]

\[ c_{2p2}(ix,2,1) = 2.*a22s*d26s*an**3*divl**(-1)+2.*b21s*d61s*an**3*divl**(-1) \]

\[ c_{2p3}(ix,2,1) = -2.*as2s*d16s*an**3*divl**(-1)-2.*a22s*d61s*an**3*divl**(-1) \]

\[ c_{il}(ix,1,1) = 0. \]

\[ c_{lp}(ix,2,1) = 0. \]

\[ c_{lp2}(ix,2,1) = 0. \]

\[ c_{lp3}(ix,2,1) = 0. \]

\[ c_{l}(ix,2,1) = 0. \]

\[ c_{lp3}(ix,2,2) = 0. \]

\[ c_{lp}(ix,2,2) = 0. \]

\[ c_{lp2}(ix,2,2) = 0. \]

\[ c_{lp3}(ix,2,2) = 0. \]

\[ c_{l}(ix,2,2) = 0. \]

\[ c_{lp2}(ix,2,2) = 0. \]

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\[ c_{lp}(ix,2,2) = 0. \]

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\[ c_{lp3}(ix,2,2) = 0. \]

\[ c_{l}(ix,2,2) = 0. \]

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\[ c_{lp3}(ix,2,2) = 0. \]

\[ c_{l}(ix,2,2) = 0. \]

\[ c_{lp3}(ix,2,2) = 0. \]

\[ c_{l}(ix,2,2) = 0. \]

\[ c_{lp2}(ix,2,2) = 0. \]

\[ c_{lp}(ix,2,2) = 0. \]

\[ c_{lp2}(ix,2,2) = 0. \]

\[ c_{lp3}(ix,2,2) = 0. \]

\[ c_{l}(ix,2,2) = 0. \]

\[ c_{lp2}(ix,2,2) = 0. \]

\[ c_{lp3}(ix,2,2) = 0. \]

\[ c_{l}(ix,2,2) = 0. \]

For the first-order dynamic state the coefficients become:

\[ C_{11} = 110. \]

\[ C_{ji} = 1. \]

\[ C_{il} = 0. \]

\[ C_{jl} = 0. \]

\[ C_{il-110} = 0. \]

\[ C_{jl+1} = 0. \]

\[ C_{2p2}(ix,2,1) = 2.*a22s*d26s*an**3*divl**(-1)+2.*b21s*d61s*an**3*divl**(-1) \]

\[ C_{2p3}(ix,2,1) = -2.*a22s*d16s*an**3*divl**(-1)-2.*a22s*d61s*an**3*divl**(-1) \]

\[ C_{il}(ix,1,1) = 0. \]

\[ C_{lp}(ix,2,1) = 0. \]

\[ C_{lp2}(ix,2,1) = 0. \]

\[ C_{lp3}(ix,2,1) = 0. \]

\[ C_{l}(ix,2,1) = 0. \]

\[ C_{lp}(ix,2,2) = 0. \]

\[ C_{lp2}(ix,2,2) = 0. \]

\[ C_{lp3}(ix,2,2) = 0. \]

For the first-order dynamic state the coefficients become:

\[ C_{11} = 110. \]

\[ C_{ji} = 1. \]

\[ C_{il} = 0. \]

\[ C_{jl} = 0. \]

\[ C_{il-110} = 0. \]

\[ C_{jl+1} = 0. \]

\[ C_{il}(ix,1,1) = 0. \]

\[ C_{jl}(ix,2,1) = 0. \]

\[ C_{il}(ix,2,1) = 0. \]

\[ C_{jl}(ix,2,1) = 0. \]

\[ C_{il-110} = 0. \]

\[ C_{jl+1} = 0. \]

\[ C_{il}(ix,1,1) = 0. \]

\[ C_{jl}(ix,2,1) = 0. \]

\[ C_{il}(ix,2,1) = 0. \]

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\[ C_{il}(ix,1,1) = 0. \]

\[ C_{jl}(ix,2,1) = 0. \]

\[ C_{il}(ix,2,1) = 0. \]

\[ C_{jl}(ix,2,1) = 0. \]
Derivation of equations using REDUCE
For the second-order static state:

\[ C_{il} = 20. \]

\[ C_{jl} = 0. \]

\[ C_{ui} = 0. \]

\[ C_{ulp(i,j,l)} = 4. a_{22s} b_{61s} a_{ii} c_{divl} (-1) \gamma_{pinf}^{r}. \]

\[ c_{2p} = 16. b_{21s} d_{22s} a_{ii} c_{divl} (-1) \gamma_{opp} + p_{bare} - 16. a_{22s} b_{61s} a_{ii} c_{divl} (-1) \gamma_{ref}^{h} r. \]

\[ c_{2p} = 16. \]
\[
c_c2p3(lx,1,2) = 0.
c_f1(lx,1,2) = 0.
c_f1p(lx,1,2) = \alpha_1 s d_1 s + a m + 3 d_i v_1 (1) \cdot \alpha_1 s d_1 s + a m + 3 d_i v_1 (1) - 16.\]
\[
\beta_2 s + b_2 s + a m + 3 d_i v_1 (1) - 8. \cdot b_2 s + b_2 s + a m + 3 d_i v_1 (1) - 16.\]
\[
\beta_1 s + b_2 s + a m + 3 d_i v_1 (1) - 8. \cdot b_2 s + b_2 s + a m + 3 d_i v_1 (1) - 16.\]
\[
\beta_2 s + b_2 s + a m + 3 d_i v_1 (1) - 8. \cdot b_2 s + b_2 s + a m + 3 d_i v_1 (1) - 16.\]
\[
c_c2p1(lx,1,2) = 0.
c_c2p1p(lx,1,2) = -8. \cdot a_2 b_2 s + b_2 s + a m + 3 d_i v_1 (1) \cdot \alpha_1 s d_1 s + a m + 3 d_i v_1 (1) - 16.\]
\[
\alpha_1 s d_1 s + a m + 3 d_i v_1 (1) - 8. \cdot b_2 s + b_2 s + a m + 3 d_i v_1 (1) - 16.\]
\[
\alpha_1 s d_1 s + a m + 3 d_i v_1 (1) - 8. \cdot b_2 s + b_2 s + a m + 3 d_i v_1 (1) - 16.\]
\[
c_c2p1p2(lx,1,2) = 0.
c_c2p1p3(lx,1,2) = 0.
\]
For the imperfect dynamic state:

\[ C_{11} = 1. \]
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\[
cw2p2(\text{i}, 1, 2) = -4. \times \text{blls} \times \text{dlls} \times \text{an}^2 \times \text{c} \times \text{divl}^{-1} \times \text{eref}^{-\text{l}} \times \text{h}^{-\text{l}} \times \text{r}^{-\text{l}} - 4. \times \text{dlls} \times \text{blls}^2 \times \text{ail} \times \text{c} \times \text{divl}^{-1} \times \text{eref}^{-\text{l}} \times \text{h}^{-\text{l}} \times \text{r}^{-\text{l}} - 4. \times \text{dlls} \times \text{blls} \times \text{ail}^2 \times \text{c} \times \text{divl}^{-1} \times \text{eref}^{-\text{l}} \times \text{r}^{-\text{l}} + 16. \times \text{blls} \times \text{dlls} \times \text{an}^2 \times \text{c} \times \text{divl}^{-1} \times \text{eref}^{-\text{l}} \times \text{h}^{-\text{l}} \times \text{r}^{-\text{l}} - 4. \times \text{blls} \times \text{dlls} \times \text{an} \times \text{divl}^{-1} \times \text{eref}^{-\text{l}} \times \text{h}^{-\text{l}} \times \text{r}^{-\text{l}} - 16. \times \text{blls} \times \text{dlls} \times \text{an} \times \text{divl}^{-1} \times \text{eref}^{-\text{l}} \times \text{h}^{-\text{l}} \times \text{r}^{-\text{l}} - b21s \times \text{divl}^{-1} \times \text{h}.
\]

\[
cw2p3(\text{i}, 1, 2) = 0.
\]

\[
cw2p3(\text{i}, 1, 2) = 0.
\]

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cw2p3(\text{i}, 1, 2) = 0.
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cw2p3(\text{i}, 1, 2) = 0.
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cw2p3(\text{i}, 1, 2) = 0.
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cw2p3(\text{i}, 1, 2) = 0.
\]

\[
cw2p3(\text{i}, 1, 2) = 0.
\]
The second-order dynamic state (0th harmonic in time):

\[ c_{il} = 120. \]
\[ c_{j1} = 0. \]
\[ u_{l}^{(i,l,l)} = 0. \]
\[ v_{l}^{(i,l,l)} = 0. \]
\[ w_{l}^{(i,l,l)} = 0. \]
\[ x_{l}^{(i,l,l)} = 0. \]
\[ y_{l}^{(i,l,l)} = 0. \]
\[ z_{l}^{(i,l,l)} = 0. \]

Derivation of equations using REDUCE:

\[ c_{il}^{(i,l,l)} = 120. \]
\[ c_{j1}^{(i,l,l)} = 0. \]
\[ u_{l}^{(i,l,l)} = 0. \]
\[ v_{l}^{(i,l,l)} = 0. \]
\[ w_{l}^{(i,l,l)} = 0. \]
\[ x_{l}^{(i,l,l)} = 0. \]
\[ y_{l}^{(i,l,l)} = 0. \]
\[ z_{l}^{(i,l,l)} = 0. \]
Derivation of equations using REDUCE
Dynamic second-order state (2nd harmonic in time):

\[
c_{1}(x,1,1)=0.4 \cdot f_{cpp}(110,1,2) \cdot u_{c}(110,1,2) - 1.4 \cdot f_{cpp}(110,1,1) \cdot u_{c}(110,1,1) - 1.4 \cdot f_{cpp}(110,1,1) \cdot u_{c}(110,1,1) - 1.4 \cdot f_{cpp}(110,1,2) \cdot u_{c}(110,1,2)
\]

\[
c_{2}(x,1,1)=1.6 \cdot a_{16s} \cdot b_{21s} \cdot a_{n}^2 \cdot c_{n}^2 \cdot e_{ref}^2 \cdot (1 - e_{ref}) \cdot h \cdot r \cdot (1 - a_{22s} \cdot d_{26s} \cdot a_{n}^2 \cdot e_{ref}^2 \cdot (1 - e_{ref}) \cdot h \cdot r)
\]

\[
c_{3}(x,1,1)=0.4 \cdot a_{22s} \cdot d_{16s} \cdot a_{n}^2 \cdot e_{ref}^2 \cdot (1 - e_{ref}) \cdot h \cdot r
\]
Derivations of equations using REDUCE
G.6.2 Boundary conditions

The boundary conditions at $x = 0$ can be written in the following form

\[ B_{11} Y_1(0) = 0 \quad (G.76) \]

\[ B_{11} Y_2(0) = 0 \quad (G.77) \]

where $B_{11}$ is a $4 \times 8$ coefficient matrix and $Y = \{Y_1, Y_2\}^T$, and similar expressions can be given for the boundary conditions at $x = L$. The $8 \times 16$ matrix with coefficients, in FORTRAN notation, is defined as follows:

\[
\begin{align*}
B_{c1}(y, 1) &= c_{w1}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 2) &= c_{w2}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 3) &= c_{w3}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 4) &= c_{w4}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 5) &= c_{w5}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 6) &= c_{w6}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 7) &= c_{w7}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 8) &= c_{w8}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 9) &= c_{w9}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 10) &= c_{w10}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 11) &= c_{w11}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 12) &= c_{w12}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 13) &= c_{w13}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 14) &= c_{w14}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 15) &= c_{w15}(y, 1, \text{ID, IE}) \\
B_{c1}(y, 16) &= c_{w16}(y, 1, \text{ID, IE}) \\
\end{align*}
\]
The coefficients of the boundary condition matrix BCXMAT are, in FORTRAN notation:

\[ \text{ix = 1: first order state} \]
\[ \text{ix = 2: second order state} \]

- if IEQ = 1 sin \( n \theta \) term
- if IEQ = 1 cos \( n \theta \) term

The coefficients of the boundary condition matrix BCXMAT are, in FORTRAN notation:
\[ \text{cf2p}(ix,12,1) = a_{26s} \cdot \text{an} \cdot c_{\cdot\cdot\cdot}(-1) \cdot \text{eref} \cdot h \cdot 2 \cdot r \cdot r_{\cdot\cdot\cdot}(-1) \]
\[ \text{cf2p2}(ix,12,1) = 0. \]
\[ \text{cf2p3}(ix,12,1) = 0. \]
\[ \text{cu2}(ix,12,1) = 0. \]
\[ \text{cw2}(ix,12,1) = 0. \]
\[ \text{cw2p}(ix,12,1) = 0. \]
\[ \text{cw2p2}(ix,12,1) = 0. \]
\[ \text{cw2p3}(ix,12,1) = 0. \]

\[ \text{cw1}(ix,13,1) = h \]
\[ \text{cw1p}(ix,13,1) = 0. \]
\[ \text{cw1p2}(ix,13,1) = 0. \]
\[ \text{cw1p3}(ix,13,1) = 0. \]
\[ \text{cw2}(ix,13,1) = 0. \]
\[ \text{cw2p}(ix,13,1) = 0. \]
\[ \text{cw2p2}(ix,13,1) = 0. \]
\[ \text{cw2p3}(ix,13,1) = 0. \]

\[ \text{cw2}(ix,13,2) = h \]
\[ \text{cw2p}(ix,13,2) = 0. \]
\[ \text{cw2p2}(ix,13,2) = 0. \]
\[ \text{cw2p3}(ix,13,2) = 0. \]
\[ \text{cf1}(ix,13,2) = 0. \]
\[ \text{cf1p}(ix,13,2) = 0. \]
\[ \text{cf1p2}(ix,13,2) = 0. \]
\[ \text{cf1p3}(ix,13,2) = 0. \]
\[ \text{cf2}(ix,13,2) = 0. \]
\[ \text{cf2p}(ix,13,2) = 0. \]
\[ \text{cf2p2}(ix,13,2) = 0. \]
\[ \text{cf2p3}(ix,13,2) = 0. \]

\[ \text{cf1}(ix,14,1) = 0. \]
\[ \text{cf1p}(ix,14,1) = 0. \]
\[ \text{cf1p2}(ix,14,1) = 0. \]
\[ \text{cf1p3}(ix,14,1) = 0. \]
\[ \text{cf2}(ix,14,1) = 0. \]
\[ \text{cf2p}(ix,14,1) = 0. \]
\[ \text{cf2p2}(ix,14,1) = 0. \]
\[ \text{cf2p3}(ix,14,1) = 0. \]
\[ \text{c02}(ix,14,1) = 0. \]
\[ \text{cw2}(ix,14,2) = 0. \]
\[ \text{cw2p}(ix,14,2) = 0. \]
\[ \text{cw2p2}(ix,14,2) = 0. \]
\[ \text{cw2p3}(ix,14,2) = 0. \]

\[ \text{cw1}(ix,15,1) = h \]
\[ \text{cw1p}(ix,15,1) = 0. \]
\[ \text{cw1p2}(ix,15,1) = 0. \]
\[ \text{cw1p3}(ix,15,1) = 0. \]
\[ \text{cw2}(ix,15,1) = 0. \]
\[ \text{cw2p}(ix,15,1) = 0. \]
\[ \text{cw2p2}(ix,15,1) = 0. \]
\[ \text{cw2p3}(ix,15,1) = 0. \]

\[ \text{cw2}(ix,15,2) = h \]
\[ \text{cw2p}(ix,15,2) = 0. \]
\[ \text{cw2p2}(ix,15,2) = 0. \]
\[ \text{cw2p3}(ix,15,2) = 0. \]
\[ \text{cf2p2}(ix,18,1) = 0. \]
\[ \text{cf2p3}(ix,18,1) = 0. \]
\[ \text{co2p}(ix,18,1) = 0. \]
\[ \text{cwl}(ix,18,2) = 2. * \text{dl16s} * \text{an} * \text{hr} **(-2) \]
\[ \text{cwlp}(ix,18,2) = 0. \]
\[ \text{cwlp3}(ix,18,2) = 0. \]
\[ \text{cw2}(ix,18,2) = - \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw2p}(ix,18,2) = 0. \]
\[ \text{cw2p2}(ix,18,2) = - \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw2p3}(ix,18,2) = 0. \]
\[ \text{cf1}(ix,18,2) = 0. \]
\[ \text{cf1p}(ix,18,2) = - \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cf1p2}(ix,18,2) = 0. \]
\[ \text{cf1p3}(ix,18,2) = 0. \]
\[ \text{cf2}(ix,18,2) = - \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cf2p}(ix,18,2) = 0. \]
\[ \text{cf2p2}(ix,18,2) = - \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cf2p3}(ix,18,2) = 0. \]
\[ \text{cf3}(ix,18,2) = \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cf3p}(ix,18,2) = 0. \]
\[ \text{cf3p2}(ix,18,2) = 0. \]
\[ \text{cf3p3}(ix,18,2) = 0. \]
\[ \text{cw3}(ix,18,2) = \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw3p}(ix,18,2) = 0. \]
\[ \text{cw3p2}(ix,18,2) = 0. \]
\[ \text{cw3p3}(ix,18,2) = 0. \]
\[ \text{cw4}(ix,18,2) = \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw4p}(ix,18,2) = 0. \]
\[ \text{cw4p2}(ix,18,2) = 0. \]
\[ \text{cw4p3}(ix,18,2) = 0. \]
\[ \text{cw5}(ix,18,2) = \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw5p}(ix,18,2) = 0. \]
\[ \text{cw5p2}(ix,18,2) = 0. \]
\[ \text{cw5p3}(ix,18,2) = 0. \]
\[ \text{cw6}(ix,18,2) = \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw6p}(ix,18,2) = 0. \]
\[ \text{cw6p2}(ix,18,2) = 0. \]
\[ \text{cw6p3}(ix,18,2) = 0. \]
\[ \text{cw7}(ix,18,2) = \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw7p}(ix,18,2) = 0. \]
\[ \text{cw7p2}(ix,18,2) = 0. \]
\[ \text{cw7p3}(ix,18,2) = 0. \]
\[ \text{cw8}(ix,18,2) = \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw8p}(ix,18,2) = 0. \]
\[ \text{cw8p2}(ix,18,2) = 0. \]
\[ \text{cw8p3}(ix,18,2) = 0. \]
\[ \text{cw9}(ix,18,2) = \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw9p}(ix,18,2) = 0. \]
\[ \text{cw9p2}(ix,18,2) = 0. \]
\[ \text{cw9p3}(ix,18,2) = 0. \]
\[ \text{cw10}(ix,18,2) = \text{dl}16s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw10p}(ix,18,2) = 0. \]
\[ \text{cw10p2}(ix,18,2) = 0. \]
\[ \text{cw10p3}(ix,18,2) = 0. \]

\[ \text{cw11}(ix,11,1) = -8. * \text{b}66s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw11p}(ix,11,1) = 0. \]
\[ \text{cw11p2}(ix,11,1) = 0. \]
\[ \text{cw11p3}(ix,11,1) = 0. \]
\[ \text{cw12}(ix,11,1) = -8. * \text{b}66s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw12p}(ix,11,1) = 0. \]
\[ \text{cw12p2}(ix,11,1) = 0. \]
\[ \text{cw12p3}(ix,11,1) = 0. \]
\[ \text{cw13}(ix,11,1) = -8. * \text{b}66s * \text{an} * \text{hr} **(-2) \]
\[ \text{cw13p}(ix,11,1) = 0. \]
\[ \text{cw13p2}(ix,11,1) = 0. \]
\[ \text{cw13p3}(ix,11,1) = 0. \]

\[ \text{IF} \ ( \text{ICASE(2)} = 20) \ \text{OR} \ \text{ICASE(2)} = 22) \ \text{THEN} \]
\[ \text{co2}(ix,11,1) = -8. * \text{b}66s * \text{an} * \text{hr} **(-2) \]
\[ \text{co2p}(ix,11,1) = 0. \]
\[ \text{co2p2}(ix,11,1) = 0. \]
\[ \text{co2p3}(ix,11,1) = 0. \]
\[ \text{ENDIF} \]

\[ \text{IF} \ ( \text{ICASE(2)} = 11) \ \text{THEN} \]
\[ \text{co2}(ix,11,1) = -8. * \text{b}66s * \text{an} * \text{hr} **(-2) \]
\[ \text{co2p}(ix,11,1) = 0. \]
\[ \text{co2p2}(ix,11,1) = 0. \]
\[ \text{co2p3}(ix,11,1) = 0. \]

\[ \text{IF} \ ( \text{ICASE(2)} = 20) \ \text{OR} \ \text{ICASE(2)} = 22) \ \text{THEN} \]
\[ \text{co2}(ix,11,1) = -8. * \text{b}66s * \text{an} * \text{hr} **(-2) \]
\[ \text{co2p}(ix,11,1) = 0. \]
\[ \text{co2p2}(ix,11,1) = 0. \]
\[ \text{co2p3}(ix,11,1) = 0. \]

\[ \text{IF} \ ( \text{ICASE(2)} = 11) \ \text{THEN} \]
\[ \text{co2}(ix,11,1) = -8. * \text{b}66s * \text{an} * \text{hr} **(-2) \]
\[ \text{co2p}(ix,11,1) = 0. \]
\[ \text{co2p2}(ix,11,1) = 0. \]
\[ \text{co2p3}(ix,11,1) = 0. \]
c02(ix,11,2)=1./2.*an**2*h**2*r**2*(-3)*wc(110,1,1).
  .2)*ucp(10,0,1)+wc(110,1,1)*ucp(10,0,2)+wc(10,0,2)*ucp(110,1,1).
  .1)*ucp(10,0,1)+wc(110,1,1))
cw10(ix,11)=1./4.*an**2*h**2*r**2*(-3)*wc(110,1,1).
  .2)*ucp(10,0,2)+wc(110,1,1)*ucp(10,0,1)+wc(10,0,2)*ucp(110,1,1).
  .1))
ENDIF

IF ( (ICASE(2) .EQ. 20) .OR. (ICASE(2) .EQ. 22) ) THEN
  c02(ix,12,1)=an**2*h**2*r**2*(-2)*(-l./8.*uc(110,1,2).
  .2)+.1./8.*wc(110,1,1)**2)
c02(ix,12,2)=1./4.*wc(110,1,2)*wc(110,1,1)*.2)*an**2*h**2*r**2*(-2)
ENDIF

IF ( (ICASE(2) .EQ. 11) THEN
  c02(ix,12,1)=an**2*h**2*r**2*(-2)*(-1./2.*wc(110,1,1).
  .2)*uc(10,0,2)+1./2.*wc(110,1,1)*wc(10,0,1)
  .1))
c02(ix,12,2)=1./4.*wc(110,1,2)*wc(110,1,1)*.2)*an**2*h**2*r**2*(-2)
ENDIF
where

\[ \text{cw1}(ix,13,2) = 0. \]
\[ \text{cwlp}(ix,13,2) = 0. \]
\[ \text{cwlp2}(ix,13,2) = 0. \]
\[ \text{cwlp3}(ix,13,2) = 0. \]
\[ \text{cw2}(ix,13,2) = b. \]
\[ \text{cw2p}(ix,13,2) = 0. \]
\[ \text{cw2p2}(ix,13,2) = 0. \]
\[ \text{cw2p3}(ix,13,2) = 0. \]
\[ \text{culp}(ix,13,2) = 0. \]
\[ \text{culp2}(ix,13,2) = 0. \]
\[ \text{culp3}(ix,13,2) = 0. \]
\[ \text{cf1}(ix,13,2) = 0. \]
\[ \text{cf1p}(ix,13,2) = 0. \]
\[ \text{cf1p2}(ix,13,2) = 0. \]
\[ \text{cf1p3}(ix,13,2) = 0. \]
\[ \text{cf2}(ix,13,2) = 0. \]
\[ \text{cf2p}(ix,13,2) = 0. \]
\[ \text{cf2p2}(ix,13,2) = 0. \]
\[ \text{cf2p3}(ix,13,2) = 0. \]
\[ \text{c02}(ix,13,2) = 0. \]
\[ \text{c02p}(ix,13,2) = 0. \]
\[ \text{c02p2}(ix,13,2) = 0. \]
\[ \text{c02p3}(ix,13,2) = 0. \]
\[
\begin{align*}
\text{cfl}(ix,15,2) &= 0. \\
\text{cfip}(ix,15,2) &= 0. \\
\text{cfip2}(ix,15,2) &= 0. \\
\text{cfip3}(ix,15,2) &= 0. \\
\text{cf2}(ix,15,2) &= 4. \ast a \ast n \ast 2 \ast c \ast (-1) \ast \text{erof} \ast h \ast 2 \ast r \ast (-1) \\
\text{cf2p}(ix,15,2) &= 0. \\
\text{cf2p2}(ix,15,2) &= 0. \\
\text{cf2p3}(ix,15,2) &= 0. \\
\text{c02}(ix,15,2) &= 0. \\
\text{cwl}(ix,15) &= 0. \\
\text{cwlp}(ix,15) &= 0. \\
\text{cwlp2}(ix,15) &= 0. \\
\text{cwl0}(ix,15) &= 0. \\
\text{cw1}(ix,16,1) &= 0. \\
\text{cw1p}(ix,16,1) &= 0. \\
\text{cw1p2}(ix,16,1) &= 0. \\
\text{cw1p3}(ix,16,1) &= 0. \\
\text{cw2}(ix,16,1) &= 0. \\
\text{cw2p}(ix,16,1) &= 0. \\
\text{cw2p2}(ix,16,1) &= 0. \\
\text{cw2p3}(ix,16,1) &= 0. \\
\text{cf1}(ix,16,1) &= 0. \\
\text{cf1p}(ix,16,1) &= 0. \\
\text{cf1p2}(ix,16,1) &= 0. \\
\text{cf1p3}(ix,16,1) &= 0. \\
\text{cf2}(ix,16,1) &= -2. \ast a \ast n \ast c \ast (-1) \ast \text{erof} \ast h \ast 2 \ast r \ast (-1) \\
\text{cf2p}(ix,16,1) &= 0. \\
\text{cf2p2}(ix,16,1) &= 0. \\
\text{cf2p3}(ix,16,1) &= 0. \\
\text{c02}(ix,16,1) &= 0. \\
\text{cw1}(ix,18,1) &= 0. \\
\text{cw1p}(ix,18,1) &= 0. \\
\text{cw1p2}(ix,18,1) &= 0. \\
\text{cw1p3}(ix,18,1) &= 0. \\
\text{cw2}(ix,18,1) &= 0. \\
\text{cw2p}(ix,18,1) &= 0. \\
\text{cw2p2}(ix,18,1) &= 0. \\
\text{cw2p3}(ix,18,1) &= 0. \\
\text{cf1}(ix,18,1) &= 4. \ast b \ast 1 \ast a \ast n \ast 2 \ast c \ast (-1) \ast \text{erof} \ast h \ast 2 \ast r \ast (-1) \\
\text{cf1p}(ix,18,1) &= 0. \\
\text{cf1p2}(ix,18,1) &= 0. \\
\text{cf1p3}(ix,18,1) &= 0. \\
\text{cf2}(ix,18,1) &= 2. \ast a \ast n \ast c \ast (-1) \ast \text{erof} \ast h \ast 2 \ast r \ast (-1) \\
\text{cf2p}(ix,18,1) &= 0. \\
\text{cf2p2}(ix,18,1) &= 0. \\
\text{cf2p3}(ix,18,1) &= 0. \\
\end{align*}
\]
cf2p(ix,18,2)=0.
cf2p2(ix,18,2)=-b21s*c**(-1)*eref*h**2*r**(-1).
cf2p3(ix,18,2)=0.
c02(ix,18,2)=0.
cval(ix,18)=a22s**(-1)*b21s*h*r**(-1).
cvalp(ix,18)=0.
cvalp2(ix,18)=-d1s*h*r**(-2) - a22s**(-1)*b21s*h*r**(-2).
cvalp3(ix,18)=0.
c0al(ix,18)=(-1./4.)*a22s**(-1)*b21s*an**2*h*r**2**r**(-2).
c0al2(ix,18)=0.
c0al3(ix,18)=0.

IF ( (ICASE(2) .Eq. 20) .OR. (ICASE(2) .Eq. 22) ) THEN
c0al0(ix,18)=-1./8.*a22s**(-1)*b21s*an**2*h*r**2**r**(-2).
ENDIF

IF (ICASE(2) .Eq. 11) THEN
c0al0(ix,18)=1./4. *a22s**(-1)*b21s*an**2*h*r**2**r**(-2).
.grid*10,0,2)*2*uc(10,1,1)**2).
ENDIF

IF (ICASE(2) .Eq. 23) THEN
i1=i12.
j1=2.

cv1(ix,11,1,1)=4.*an**2*h*r**2**r**(-3)*w0p
cv1p(ix,11,1,1)=4.*an**2*h*r**2**r**(-3)*b66s*an**2*h*r**2**r**(-3) + 8.*b66s*an**2*h*r**2**r**(-3) + h*r**(-2)
cv1p0(ix,11,1,1)=0.
cv1p3(ix,11,1,1)=4.*an**2*h*r**2**r**(-3)*w0p
cv2(ix,11,1,1)=6.*b62s*an**3*h*r**(-3)
cv2p(ix,11,1,1)=0.
cv2p0(ix,11,1,1)=-8.*b62s*an**3*h*r**(-3)
cv2p2(ix,11,1,1)=0.
cv2p3(ix,11,1,1)=4.*b62s*an**2*h*r**(-3)
cf1(ix,11,1,1)=4.*an**2*h*r**2**r**(-3)*w0p
cf1p(ix,11,1,1)=4.*an**2*h*r**2**r**(-3)*b66s*an**2*h*r**2**r**(-3) + 8.*b66s*an**2*h*r**2**r**(-3) + h*r**(-2)
cf1p0(ix,11,1,1)=0.
cf1p3(ix,11,1,1)=4.*an**2*h*r**2**r**(-3)*w0p
cf2(ix,11,1,1)=6.*b62s*an**3*h*r**(-3)
cf2p(ix,11,1,1)=0.
cf2p0(ix,11,1,1)=-8.*b62s*an**3*h*r**(-3)
cf2p2(ix,11,1,1)=0.
cf2p3(ix,11,1,1)=4.*b62s*an**2*h*r**(-3)
cf3(ix,11,1,1)=4.*b62s*an**2*h*r**(-3)*w0p
cf3p(ix,11,1,1)=4.*b62s*an**2*h*r**(-3)
cf3p0(ix,11,1,1)=0.
cf3p3(ix,11,1,1)=4.*b62s*an**2*h*r**(-3)*w0p.

ENDIF
\[
\begin{align*}
\text{cf2}(i,14.,1.) &= 0. \\
\text{cf2p}(i,14.,1.) &= 0. \\
\text{cf2p2}(i,14.,1.) &= 0. \\
\text{cf2p3}(i,14.,1.) &= 0. \\
\text{co2}(i,14.,1.) &= 0. \\
\text{cw2}(i,14.,1.) &= 0. \\
\text{cw2p}(i,14.,1.) &= 0. \\
\text{cw2p2}(i,14.,1.) &= 0. \\
\text{cw2p3}(i,14.,1.) &= 0. \\
\text{cflp}(i,15.,1.) &= 0. \\
\text{cf1p2}(i,15.,1.) &= 0. \\
\text{cf1p3}(i,15.,1.) &= 0. \\
\text{cf2}(i,15.,1.) &= 0. \\
\text{cf2p}(i,15.,1.) &= 0. \\
\text{cf2p2}(i,15.,1.) &= 0. \\
\text{cf2p3}(i,15.,1.) &= 0. \\
\text{co2}(i,15.,1.) &= 0. \\
\text{cw2}(i,15.,1.) &= 0. \\
\text{cw2p}(i,15.,1.) &= 0. \\
\text{cw2p2}(i,15.,1.) &= 0. \\
\text{cw2p3}(i,15.,1.) &= 0. \\
\text{cf1}(i,15.,1.) &= 0. \\
\text{cf1p}(i,15.,1.) &= 0. \\
\text{cf1p2}(i,15.,1.) &= 0. \\
\text{cf1p3}(i,15.,1.) &= 0. \\
\text{co2}(i,16.,1.) &= 0. \\
\text{cw2}(i,16.,1.) &= 0. \\
\text{cw2p}(i,16.,1.) &= 0. \\
\text{cw2p2}(i,16.,1.) &= 0. \\
\text{cw2p3}(i,16.,1.) &= 0. \\
\text{cw2}(i,16.,2.) &= 0. \\
\text{cw2p}(i,16.,2.) &= 0. \\
\text{cw2p2}(i,16.,2.) &= 0. \\
\text{cw2p3}(i,16.,2.) &= 0. \\
\text{cf1}(i,16.,2.) &= 0. \\
\text{cf1p}(i,16.,2.) &= 0. \\
\text{cf1p2}(i,16.,2.) &= 0. \\
\text{cf1p3}(i,16.,2.) &= 0. \\
\text{co2}(i,16.,2.) &= 0. \\
\text{cw2}(i,16.,2.) &= 0. \\
\text{cw2p}(i,16.,2.) &= 0. \\
\text{cw2p2}(i,16.,2.) &= 0. \\
\text{cw2p3}(i,16.,2.) &= 0. \\
\text{cw2}(i,16.,2.) &= 0. \\
\text{cw2p}(i,16.,2.) &= 0. \\
\text{cw2p2}(i,16.,2.) &= 0. \\
\text{cw2p3}(i,16.,2.) &= 0. \\
\end{align*}
\]
culp3(ix,16.,2.)=0.
cw2p(ix,16.,2.)=0.
cw2p2(ix,16.,2.)=0.
cw2p(i,16.,2.)=0.
cw2p3(ix,16.,2.)=0.
cf1(ix,16.,2.)=0.
cf1p(ix,16.,2.)=-2.*an*c**(-1)*eref*h**2*r**(-1)
cf1p2(ix,16.,2.)=0.
cf1p3(ix,16.,2.)=0.
cf2(ix,16.,2.)=0.
cf2p(ix,16.,2.)=0.
cf2p2(ix,16.,2.)=0.
cf2p3(ix,16.,2.)=0.
c02(ix,16.,2.)=0.
cw1(ix,16.,2.)=0.
cwal(ix,16.,2.)=0.
cwalp(ix,16.,2.)=0.
cwlp2(ix,16.,2.)=0.
cwalp3(ix,16.,2.)=0.
cwal0(ix,16.,2.)=0.
cw1p3(ix,16.,2.)=0.
cw2p(ix,16.,2.)=0.
cw2p2(ix,16.,2.)=0.
cw2p3(ix,16.,2.)=0.
cfl(ix,16.,2.)=0.
cflp(ix,16.,2.)=2.*an*c**(-1)*eref*h**2*r**(-1)
cflp2(ix,16.,2.)=0.
cflp3(ix,16.,2.)=0.
cf2(ix,16.,2.)=0.
cf2p(ix,16.,2.)=0.
cf2p2(ix,16.,2.)=0.
cf2p3(ix,16.,2.)=0.
c02(ix,16.,2.)=0.
cw1(ix,16.,2.)=0.
cwal(ix,16.,2.)=0.
cwalp(ix,16.,2.)=0.
cwalp2(ix,16.,2.)=0.
cwalp3(ix,16.,2.)=0.
cwal0(ix,16.,2.)=0.
Derivation of equations using REDUCE
Acknowledgement

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Curriculum Vitae

The author was born on February 14, 1965, in 's-Hertogenbosch, the Netherlands. In 1983 he graduated from pre-university education (VWO) at the Stedelijk Gymnasium in Leiden. In the same year he started his studies at the Delft University of Technology, Faculty of Aerospace Engineering. The work on shell buckling described in his graduation thesis was carried out in the Structures Group of Professor Arbocz. After his graduation in 1989, the author became a PhD student / research assistant under the supervision of Professor Arbocz until January 1995, carrying out research on shell vibrations for a PhD degree and participating in other research projects. After this period he continued working on his PhD thesis, which was not yet finished by that time. From October 1996 through March 1998 the author was employed as a research associate in a project funded by the Dutch Technology Foundation STW, at the University of Nijmegen, Faculty of Mathematics and Informatics. From the summer of 1998 he worked as a research associate in the Materials Group of the Faculty of Aerospace Engineering in Delft, in the Glare Research Program (GRP) project, first part-time (combining this with writing his PhD thesis), and from March 1999 through April 2000 full-time. Since May 2000 the author is an assistant professor in the Structures Group of the Faculty of Aerospace Engineering in Delft.
Stellingen behorende bij het proefschrift:
Nonlinear Vibrations of Anisotropic Cylindrical Shells
door E.L. Jansen

1. De verplaatsing in omtreksrichting behorend bij de aangenomen radiale trillingsvormen van een cylinderschaal met een asymmetrische imperfectie dient te voldoen aan de voorwaarde van periodiciteit in omtreksrichting.
The circumferential displacement corresponding to the assumed radial vibration modes of a cylindrical shell with an asymmetric imperfection has to satisfy the circumferential periodicity condition.

2. De invloed van imperfecties met de vorm van de trillingsvorm op de frequentie wijkt in het algemeen kwalitatief af van de invloed van imperfecties met een andere vorm dan de trillingsvorm.
The influence of imperfections with the shape of the vibration mode on the frequency is in general qualitatively different from the influence of imperfections with a shape different from the vibration mode.

3. Bij de beschrijving van “single mode” trillingen van een cylinderschaal onder parametrische excitatie dienen ook axiaal-symmetrische trillingsvormen in de verplaatsingsfunctie te worden opgenomen.
When modelling the “single mode” vibrations of a cylindrical shell under parametric excitation axisymmetric vibration modes have to be included in the displacement function.

4. Bij de beschrijving van het niet-lineaire trillingsgedrag van algemene constructies verdient het consistent in rekening brengen van de niet-lineaire statische toestand bijzondere aandacht.
In the formulation of the nonlinear vibration behaviour of general structures special attention should be given to taking into account the nonlinear static state.

5. Het verschijnsel dat de kniklast van een composiet cylinderschaal met een ongebalanceerd laminaat afhankelijk is van de richting van de aangebrachte torsiö-belasting, komt voort uit een lineair effect in de knik-toestand.
The phenomenon that the buckling load of a composite cylindrical shell with an unbalanced laminate depends on the direction of the applied torque, stems from a linear effect in the buckling state.

6. De in de literatuur gepubliceerde resultaten over de niet-lineaire trillingen van cylinderschalen lopen meer uiteen dan de kwalificaties van de verschillen door de auteurs suggereren.
There is a larger discrepancy between the results published in the literature about the nonlinear vibrations of cylindrical shells than the qualifications of the differences by the authors suggest.